Coloring Mixed and Directional Interval Graphs

GD 2022, Tokyo

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UNIVERSITÄT PASSAU

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Framework for layered graph drawing by Sugiyama, Tagawa, and Toda (1981).

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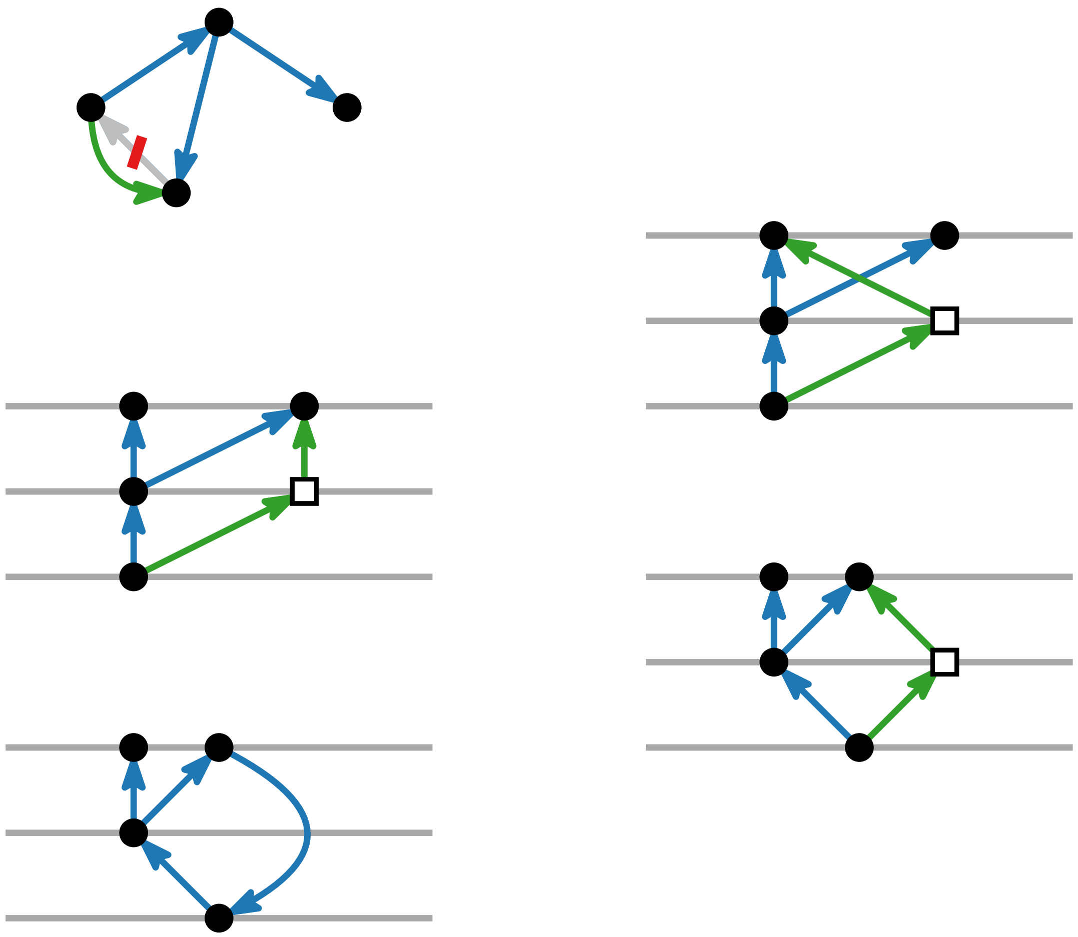
1. cycle elimination
2. layer assignment
3. crossing minimization
4. node placement
5. edge routing



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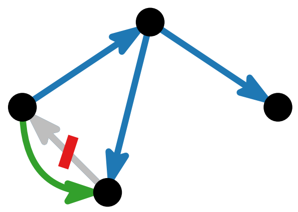
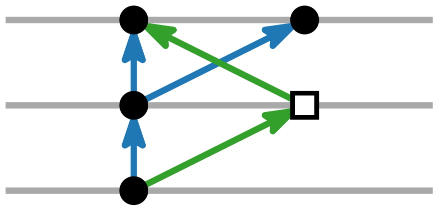
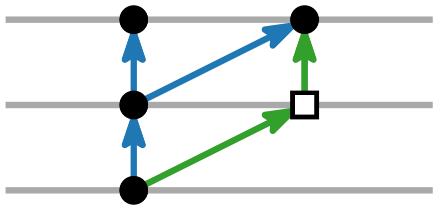


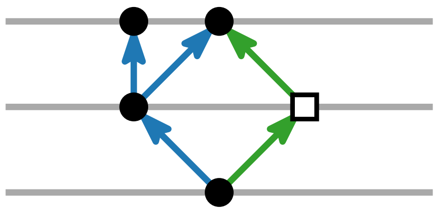
2. layer assignment



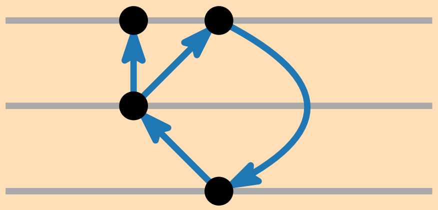
3. crossing minimization



4. node placement



5. edge routing



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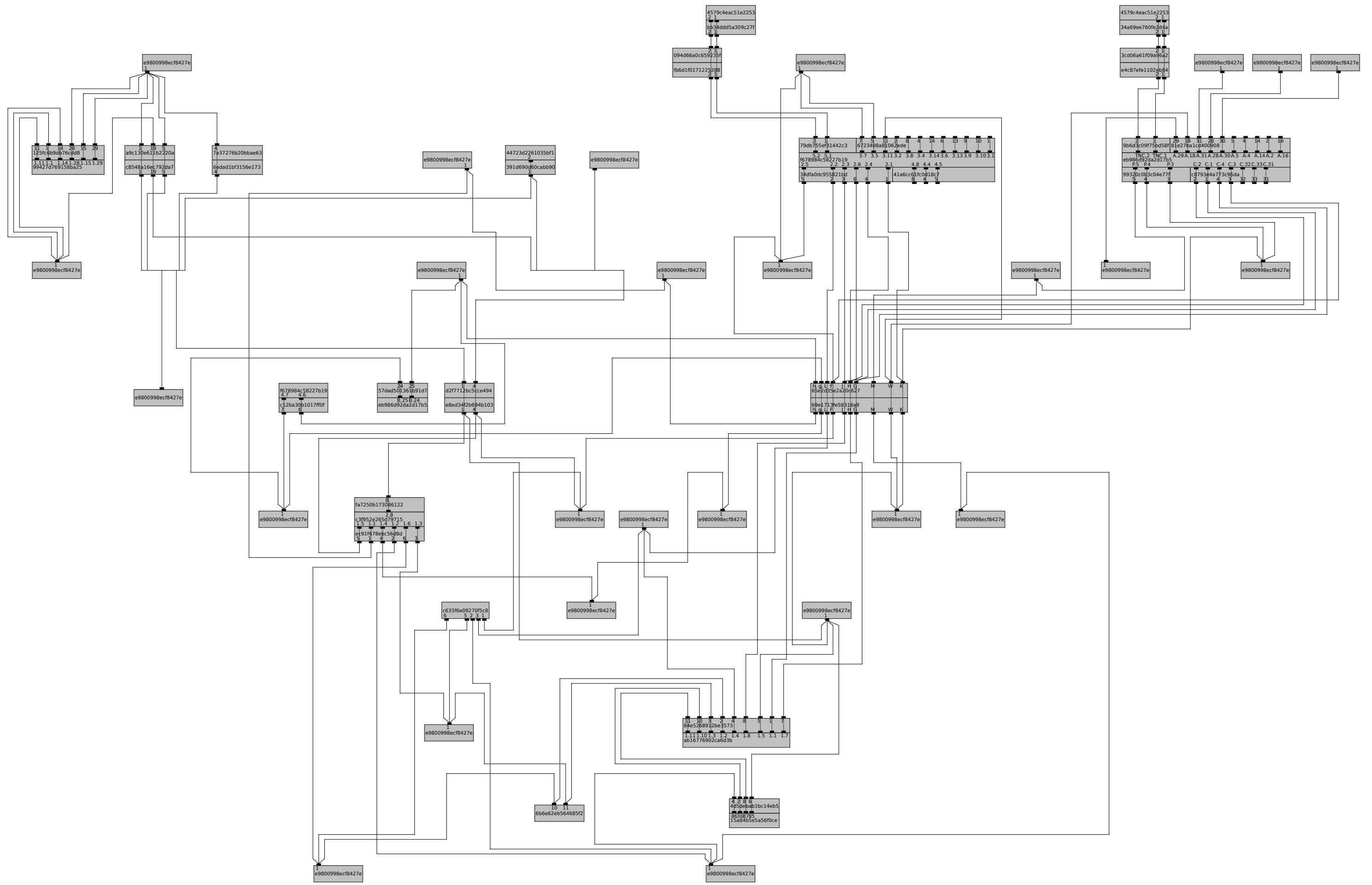
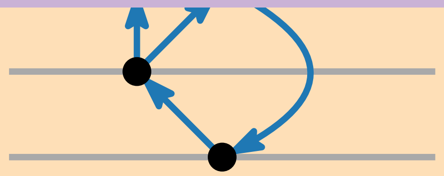


cable plan

[Walter, Z., Baumeister, Wolff; GD'20, CGTA'22]



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Motivation – Layered Orthogonal Edge Routing

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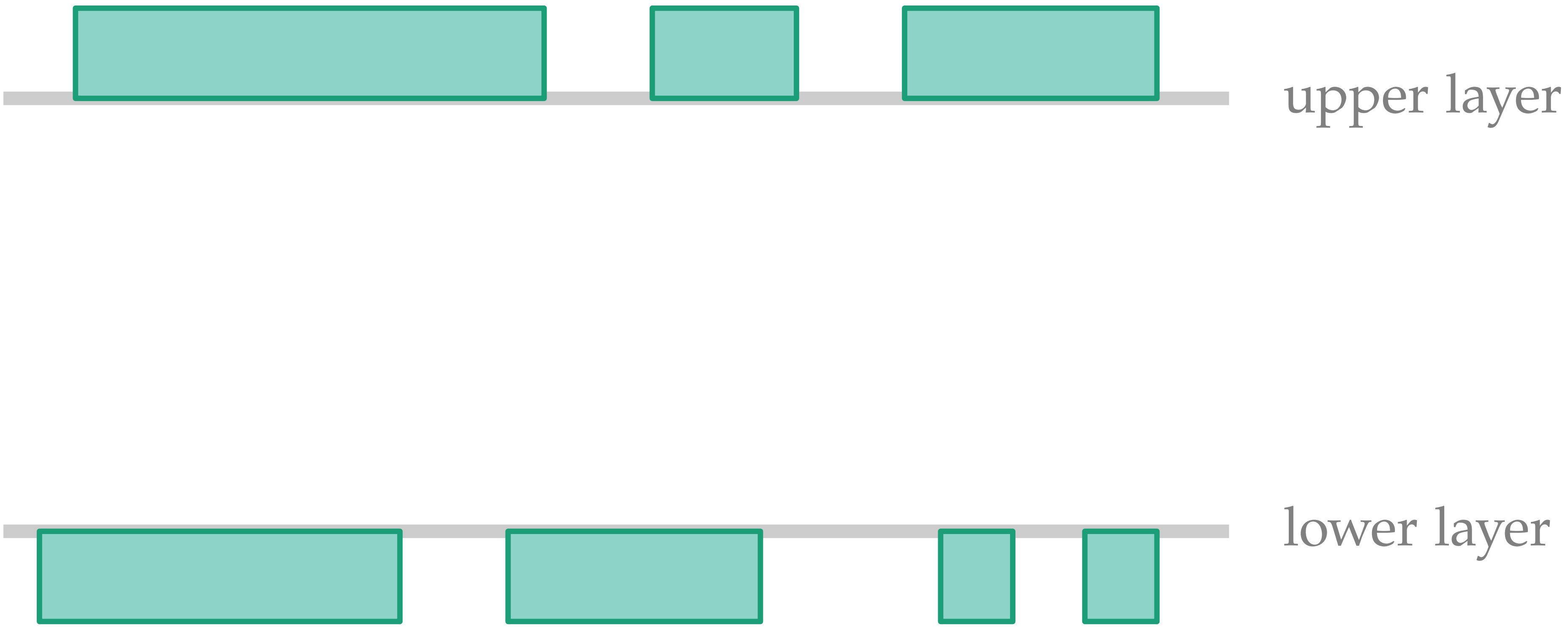
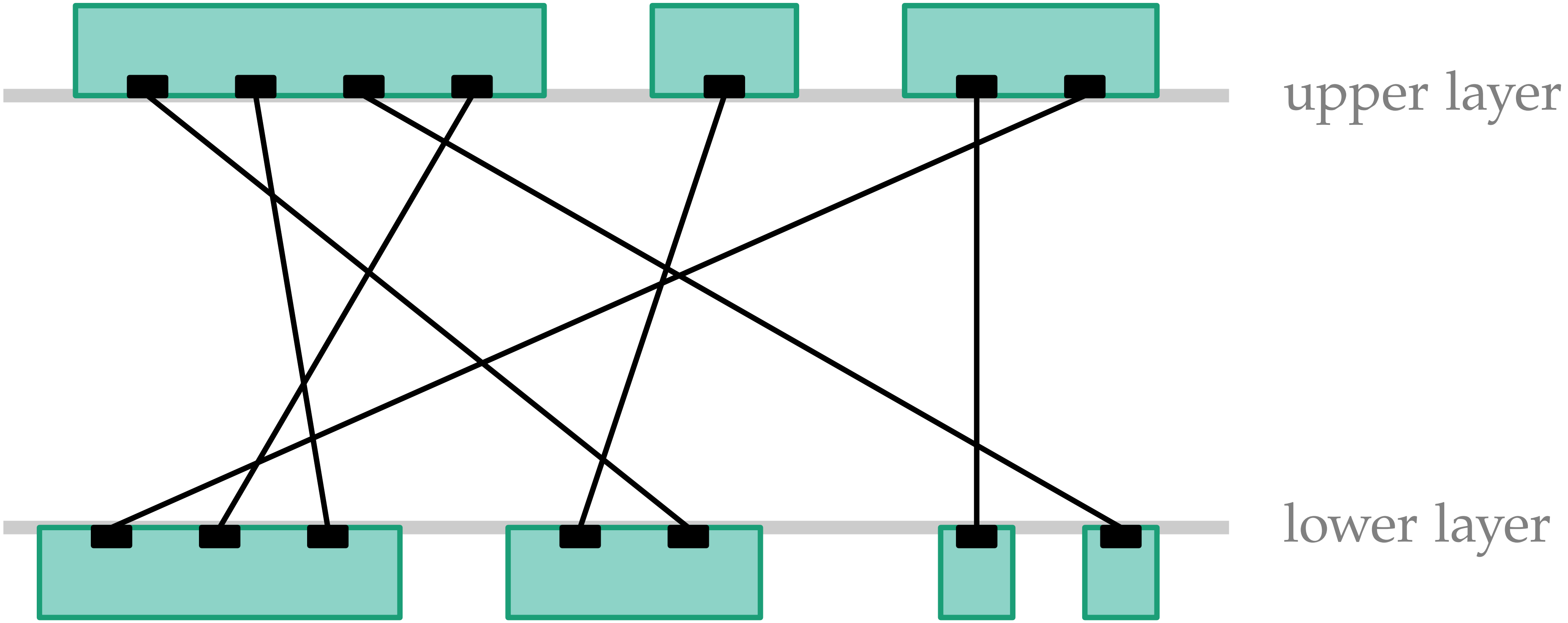


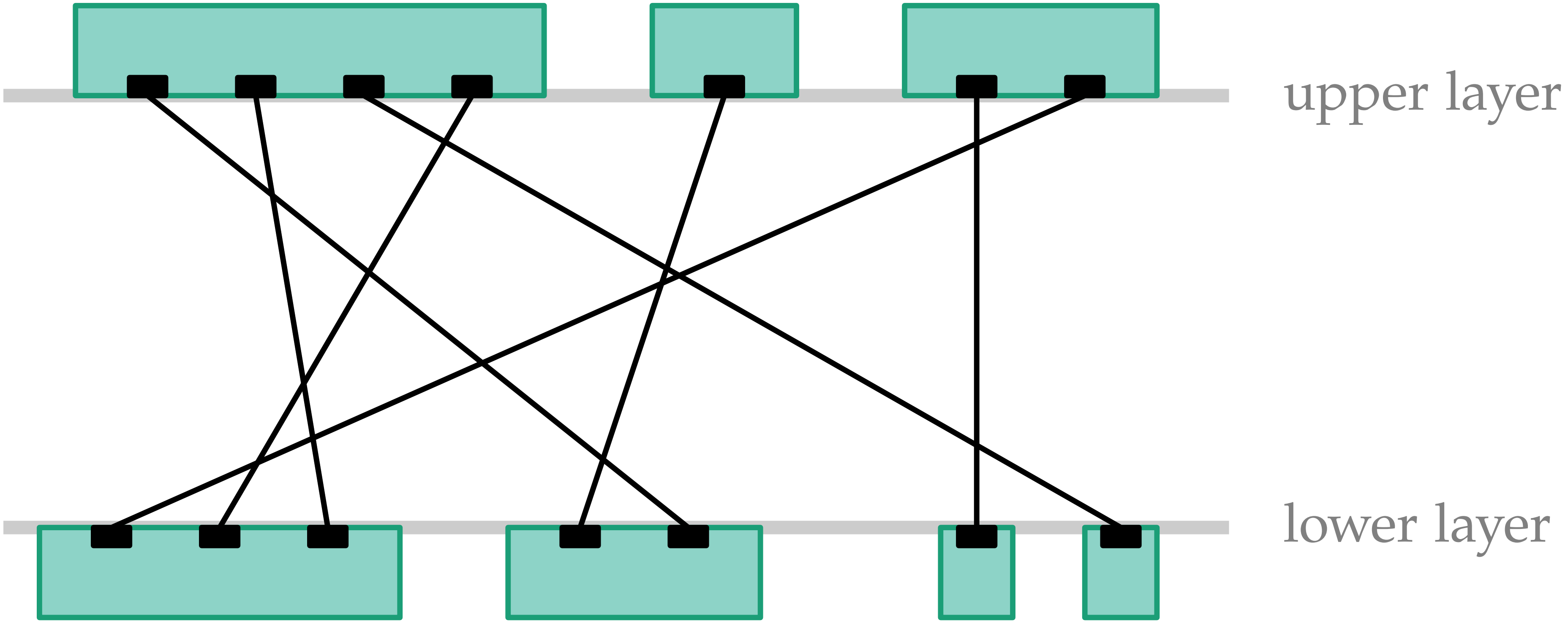
Motivation – Layered Orthogonal Edge Routing

- it suffices to consider each pair of consecutive layers individually
- positions of vertices are fixed
- no two edges share a common end point (vertices have distinct ports)



Motivation – Layered Orthogonal Edge Routing

- draw each edge with at most two vertical and one horizontal line segments



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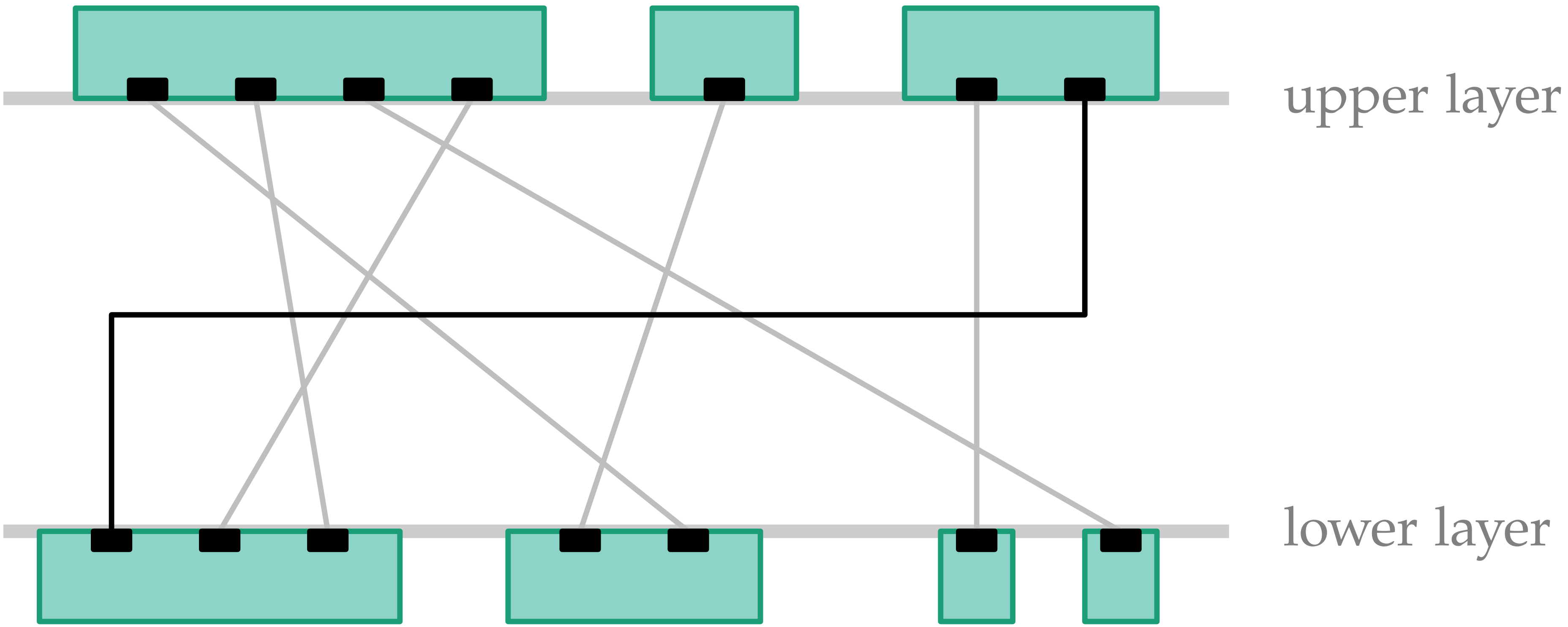

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- draw each edge with at most two vertical and one horizontal line segments
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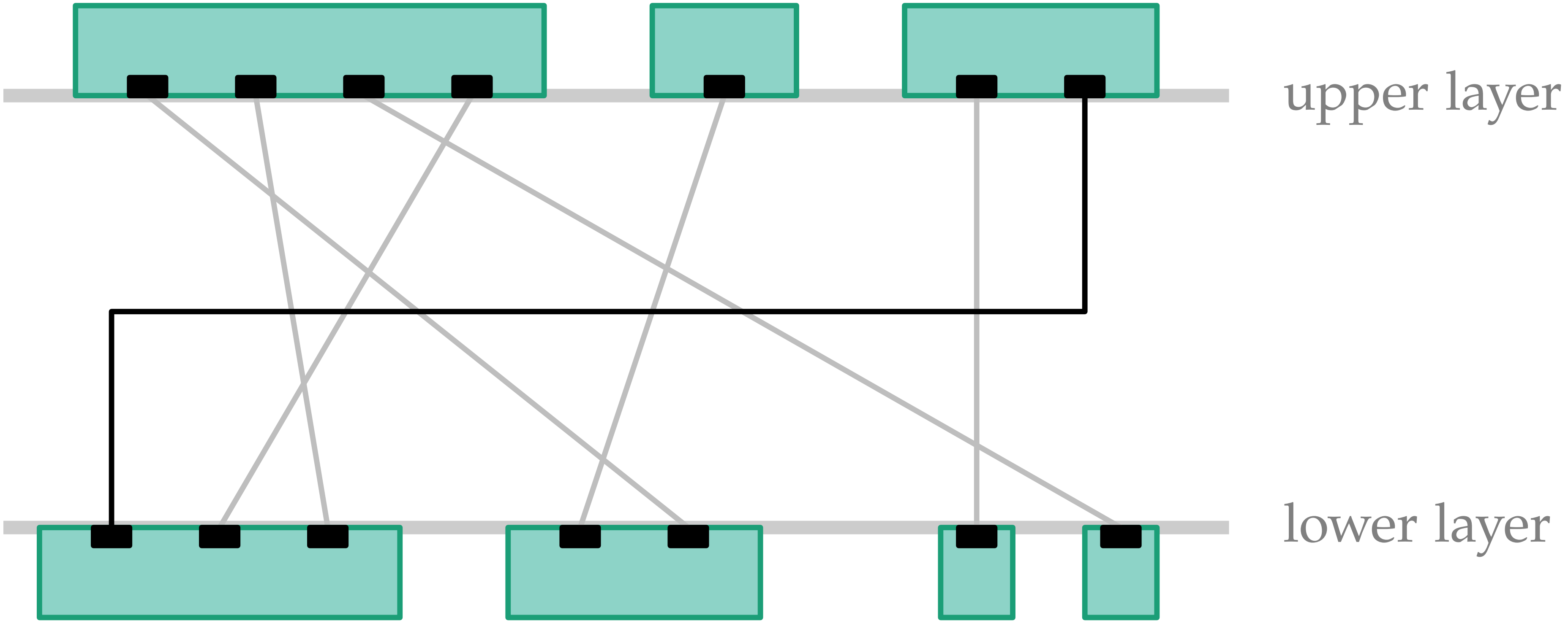

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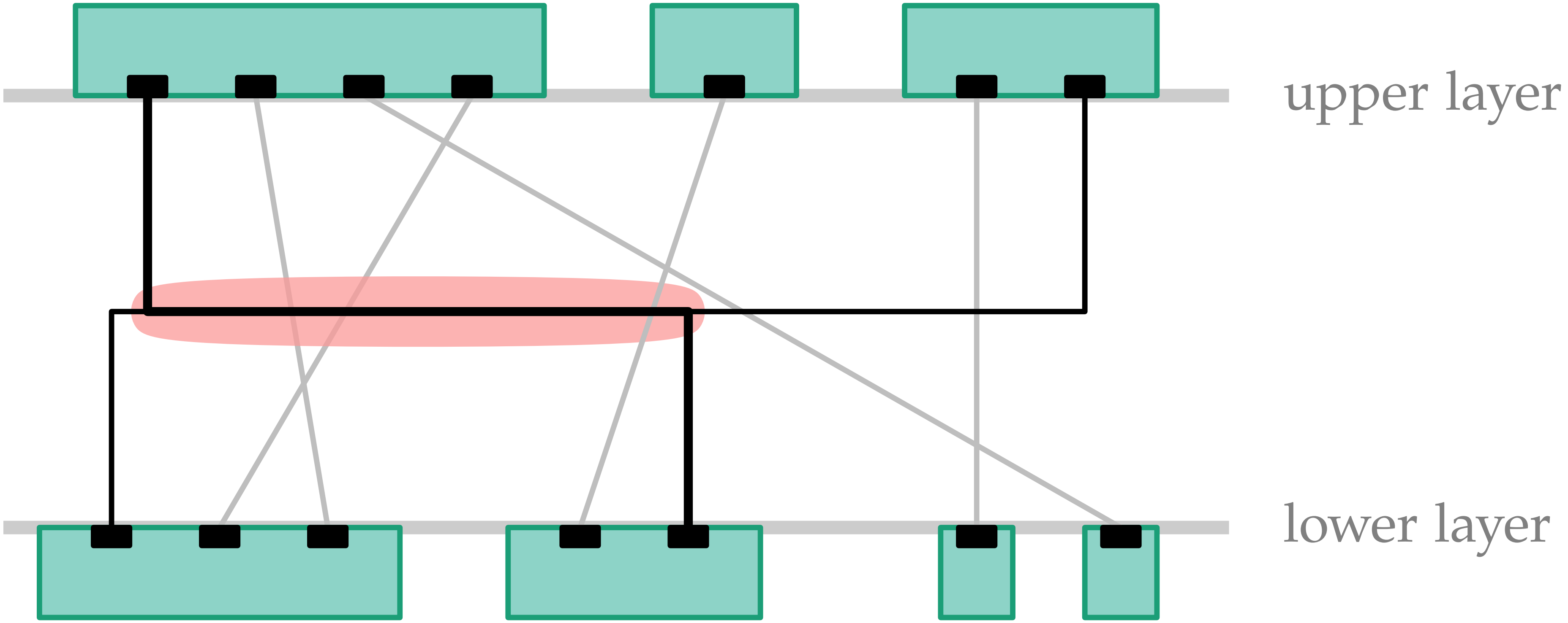

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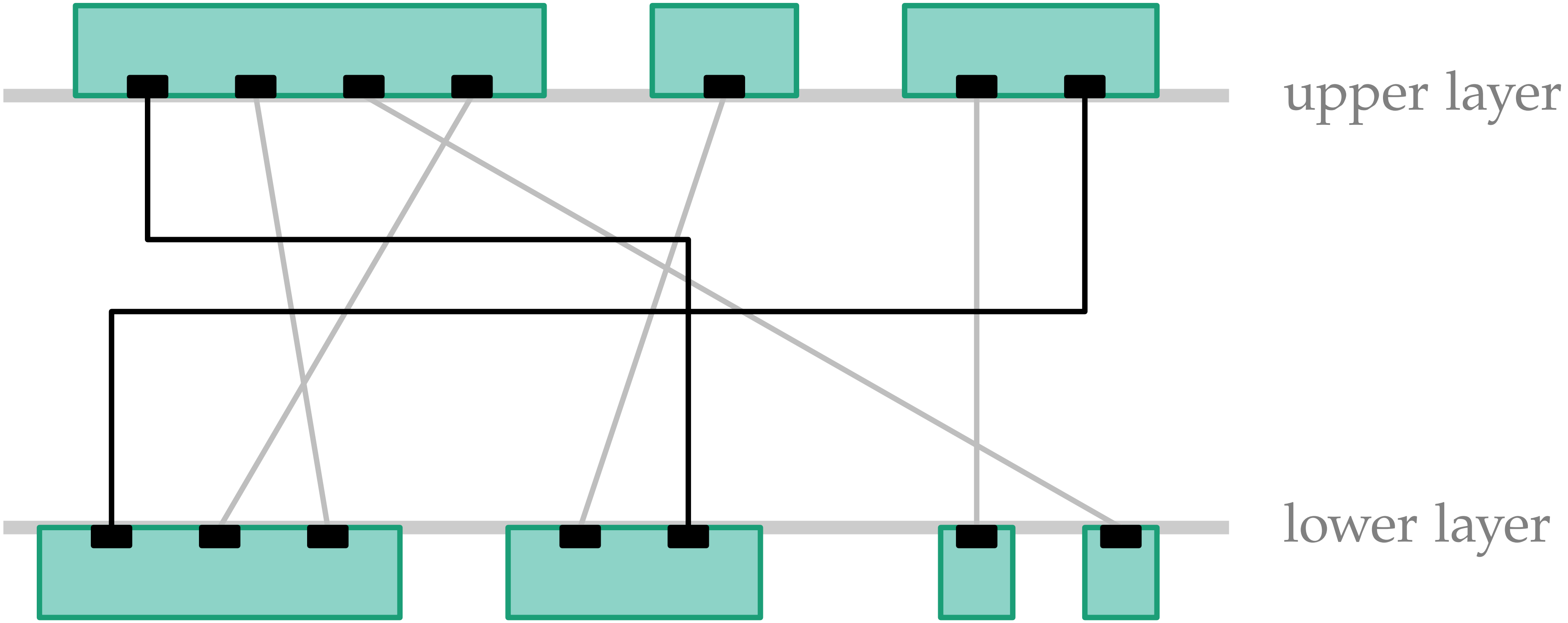

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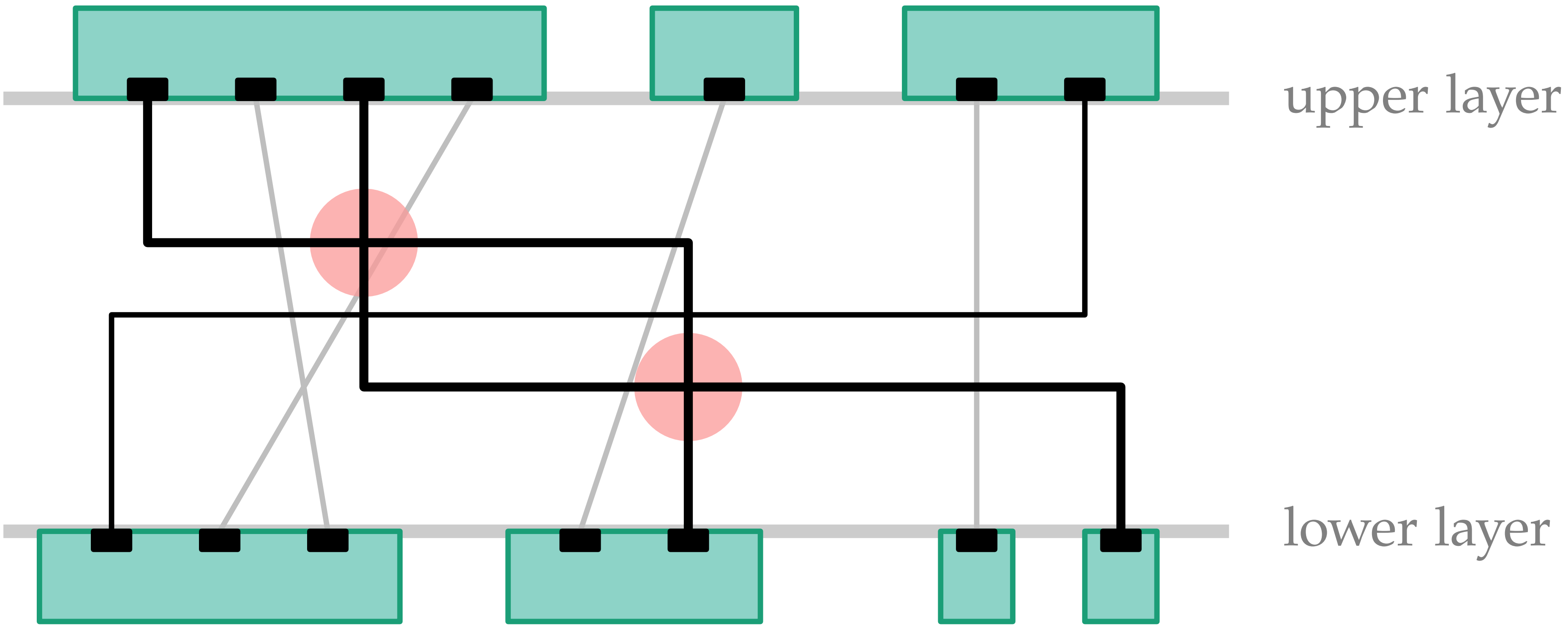

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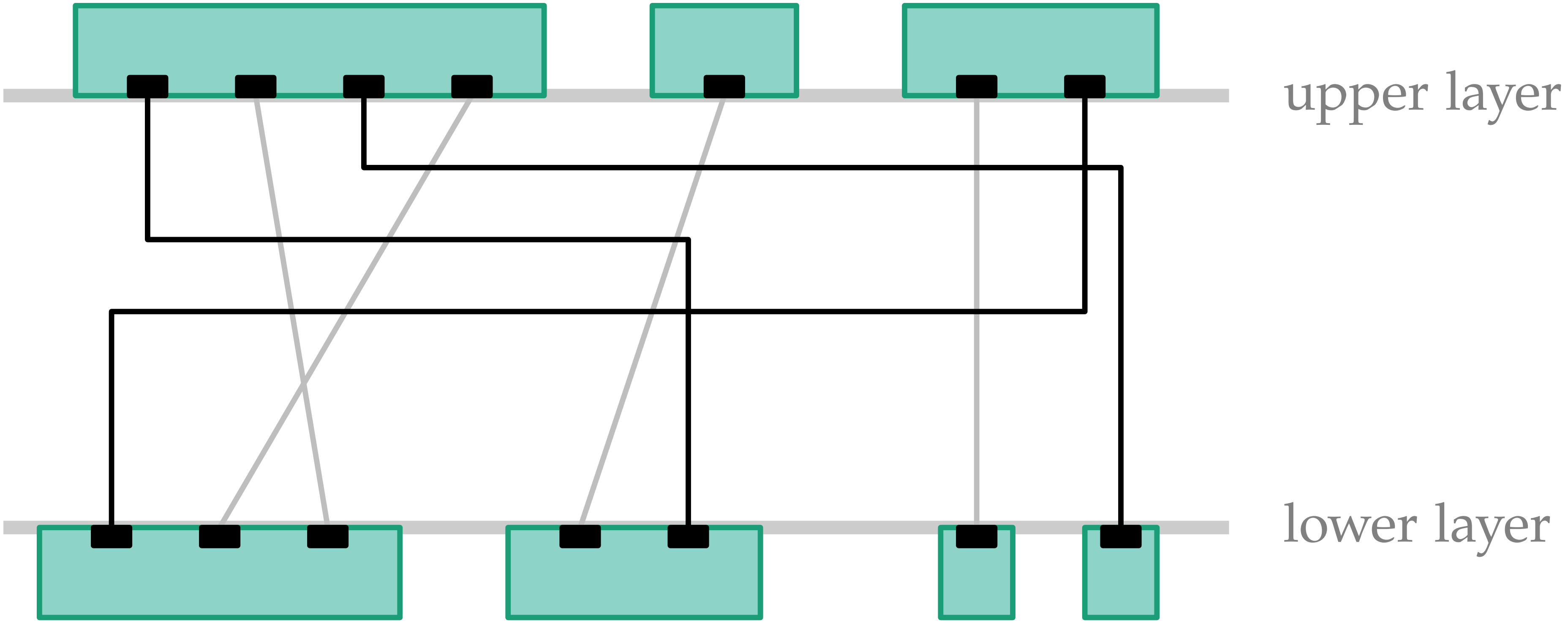

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- draw each edge with at most two vertical and one horizontal line segments
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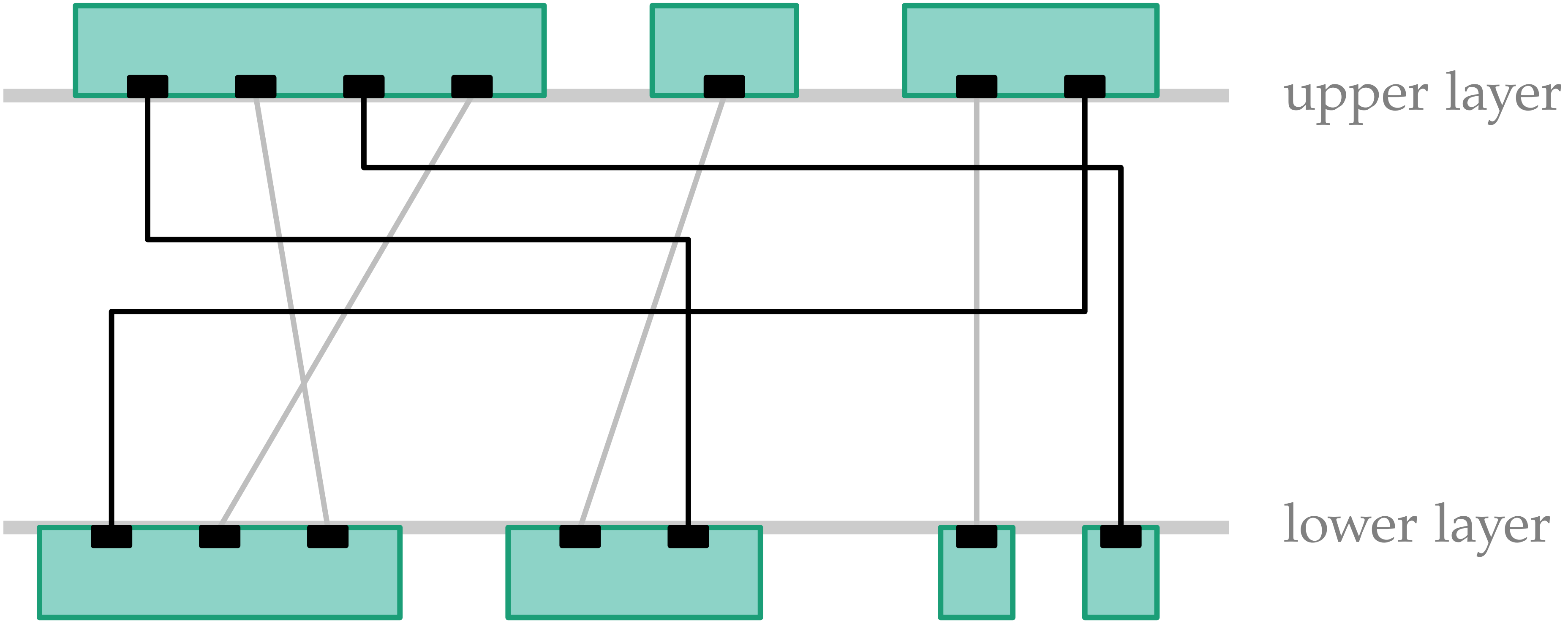

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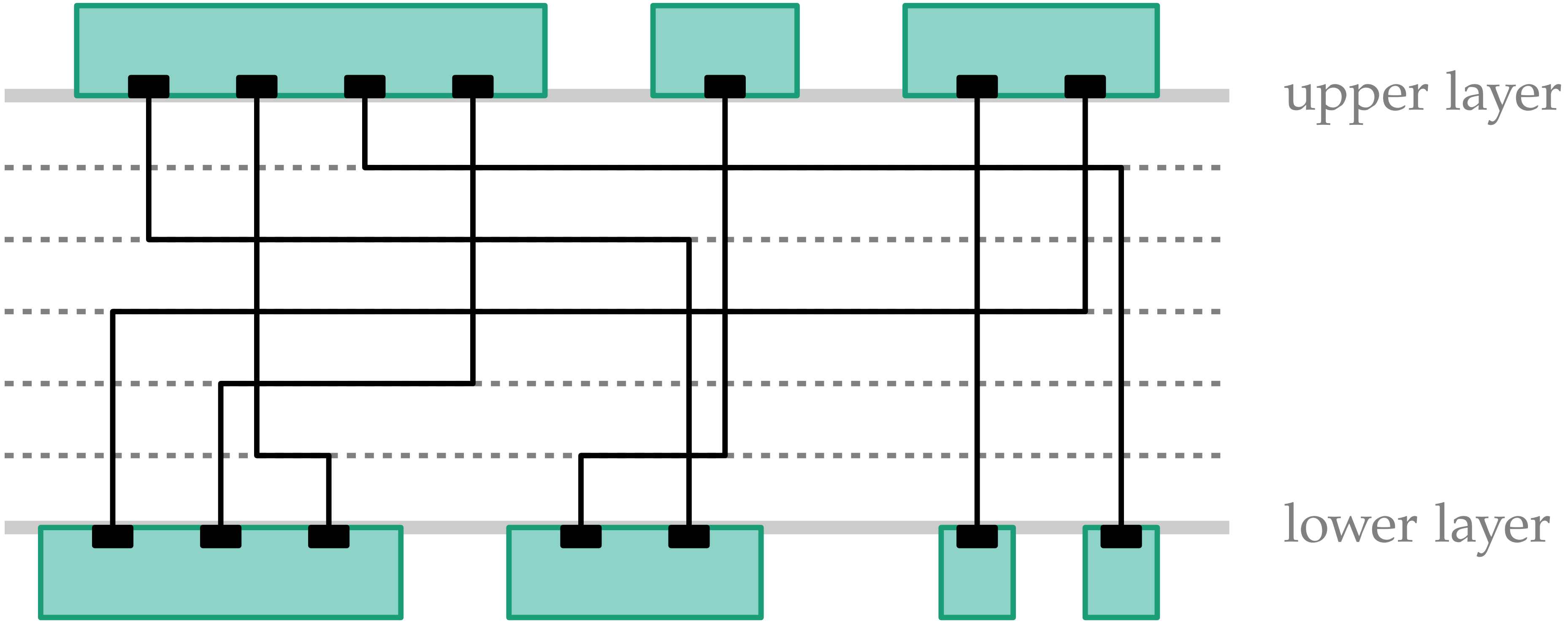

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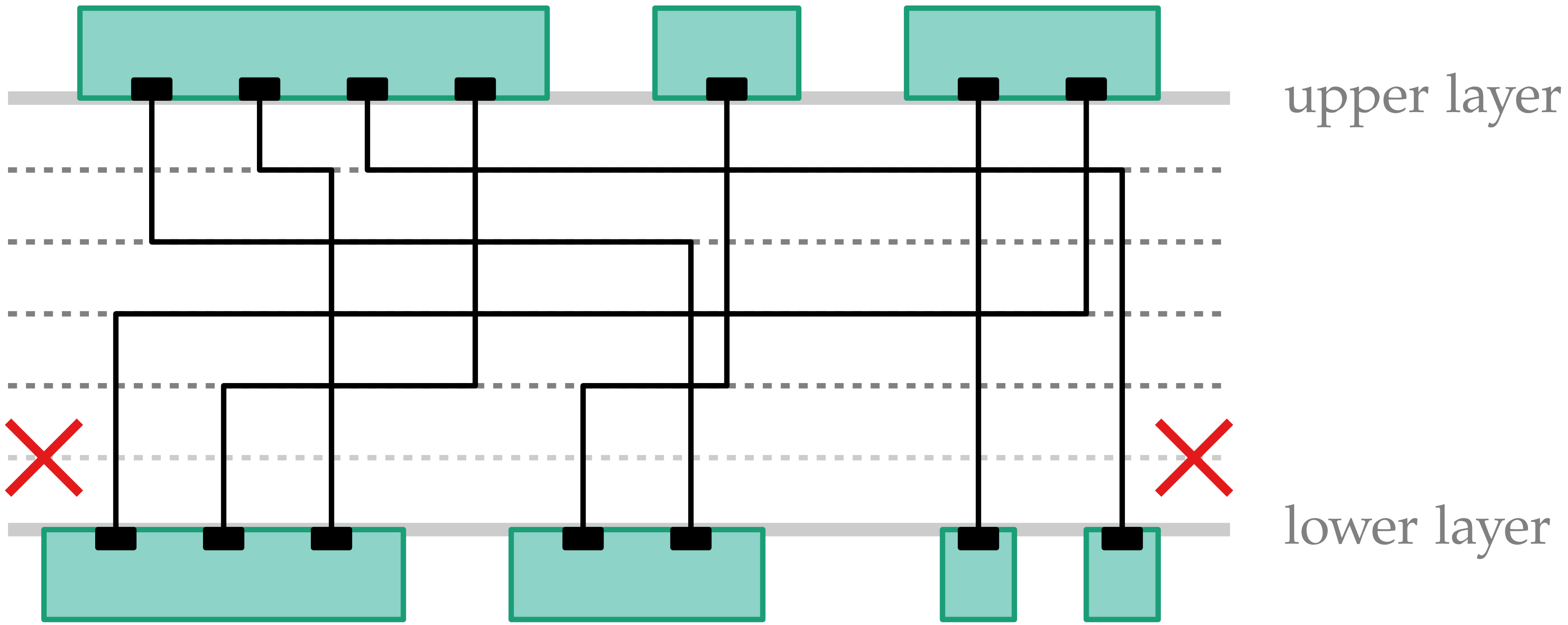
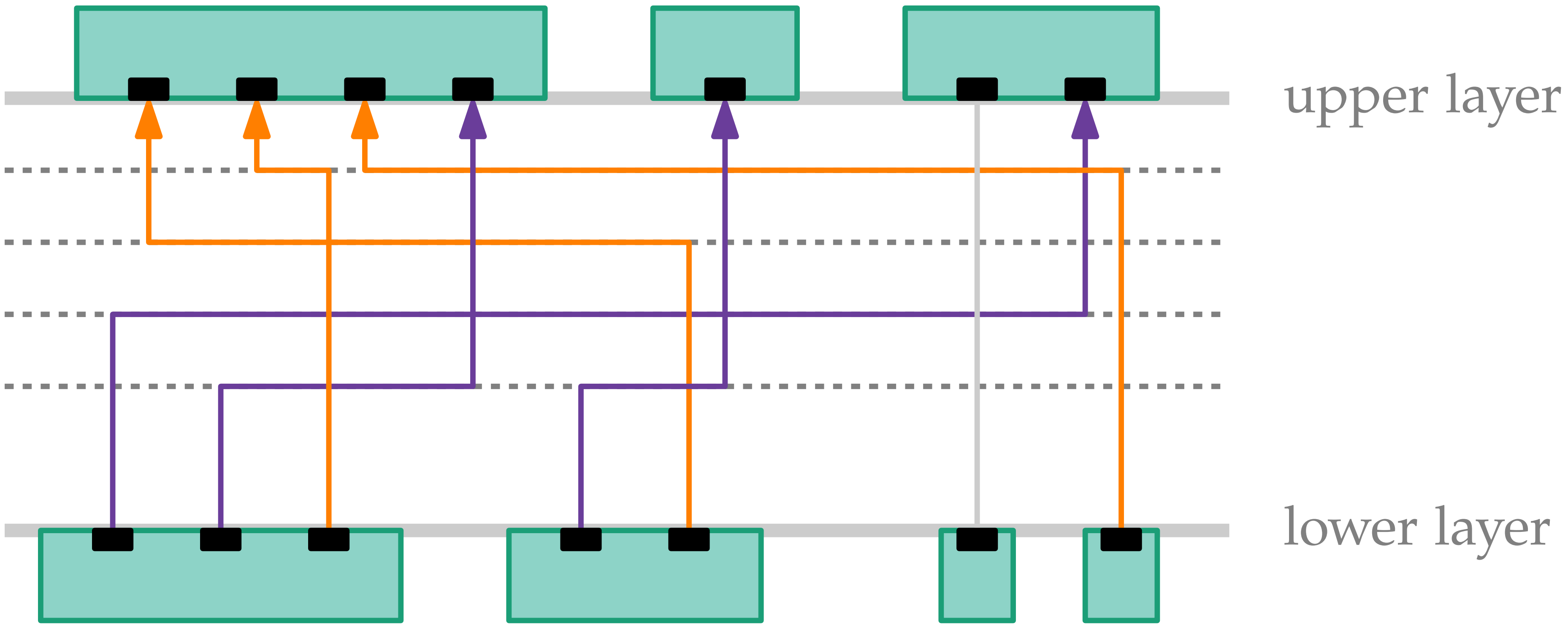


Motivation – Layered Orthogonal Edge Routing

- distinguish between *left-going* and *right-going* edges



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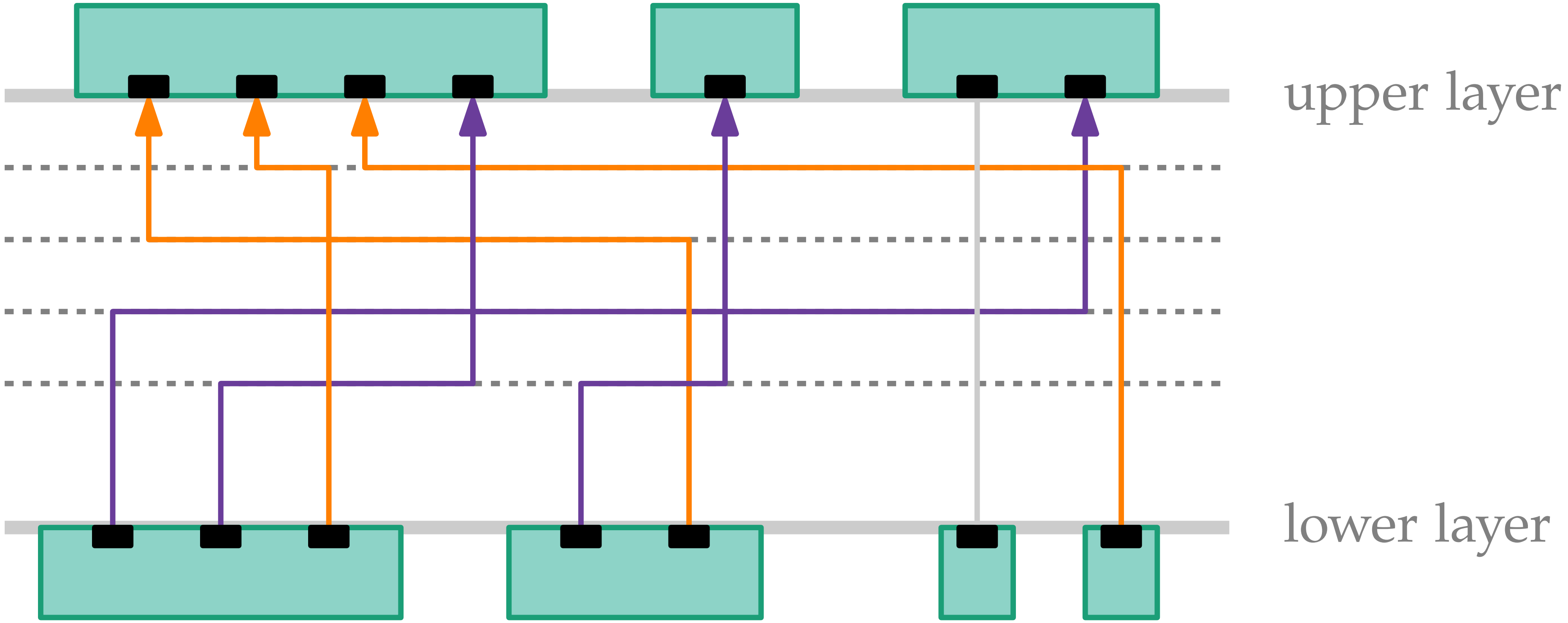
- distinguish between *left-going* and *right-going* edges
- only edges going in the same direction and overlapping partially in x-dimension can cross twice



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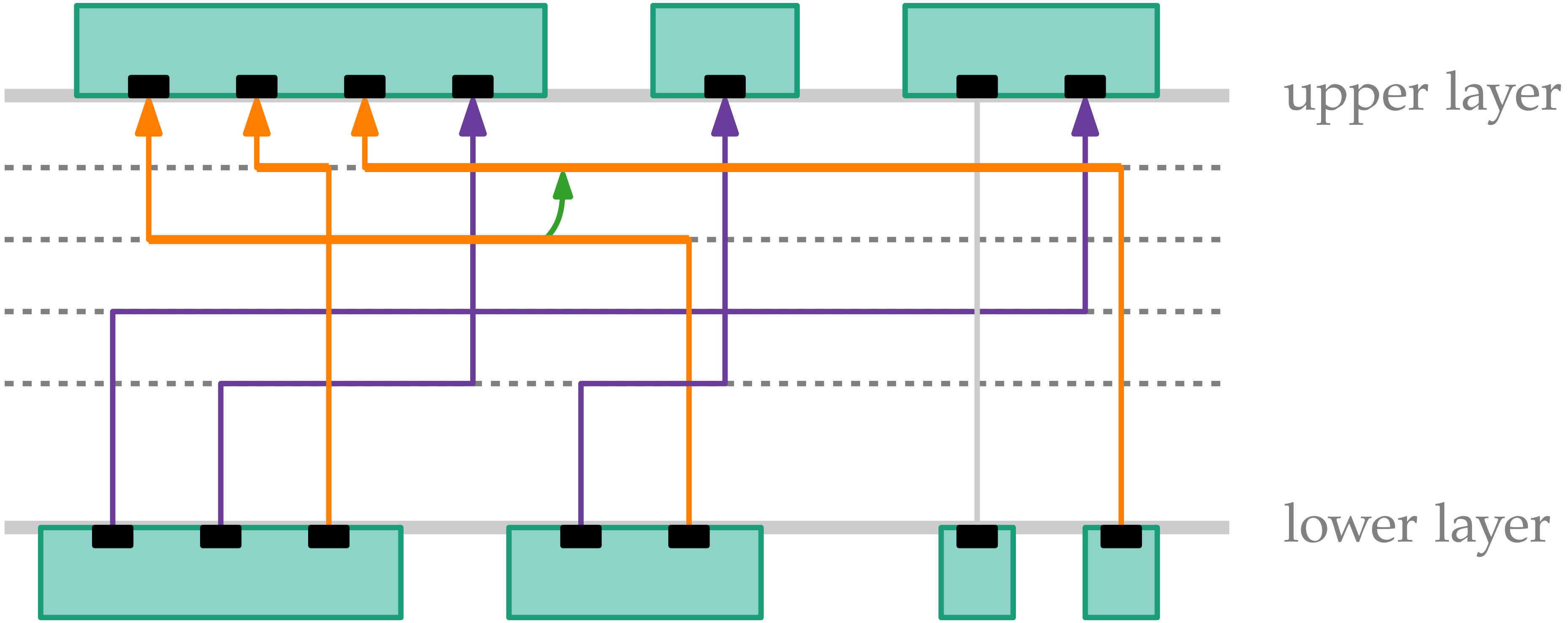
  $\Rightarrow$ induce a vertical order for the horizontal middle segments



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Interval representation: set of intervals

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Directional interval graph:

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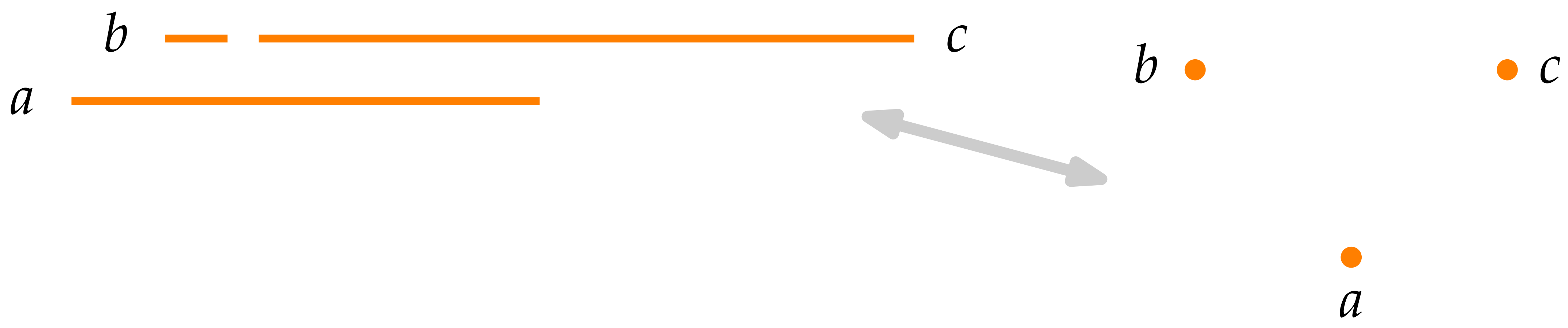

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Directional interval graph:

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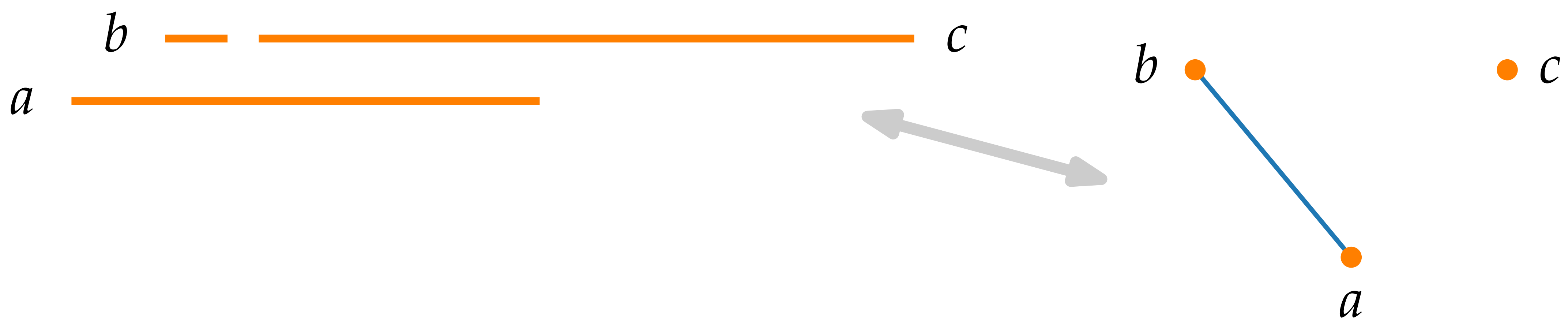

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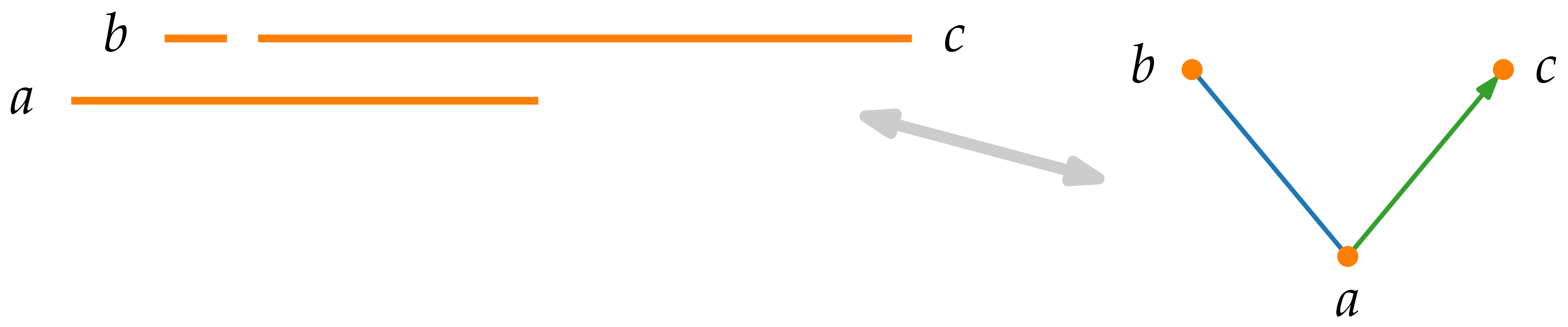

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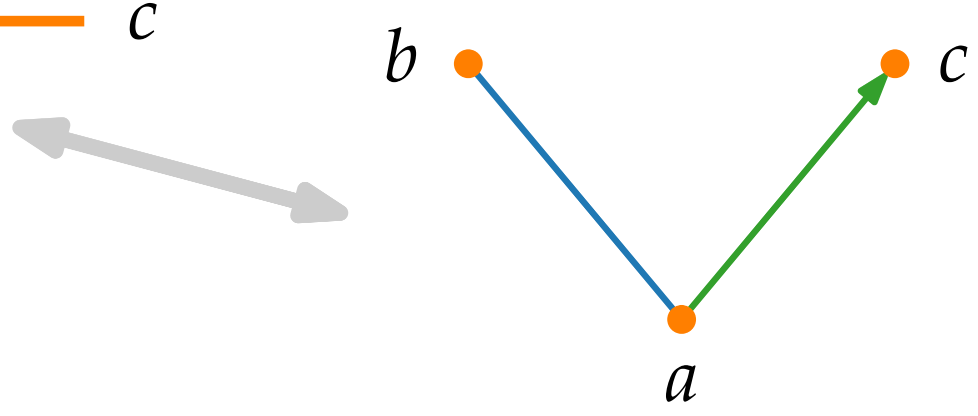

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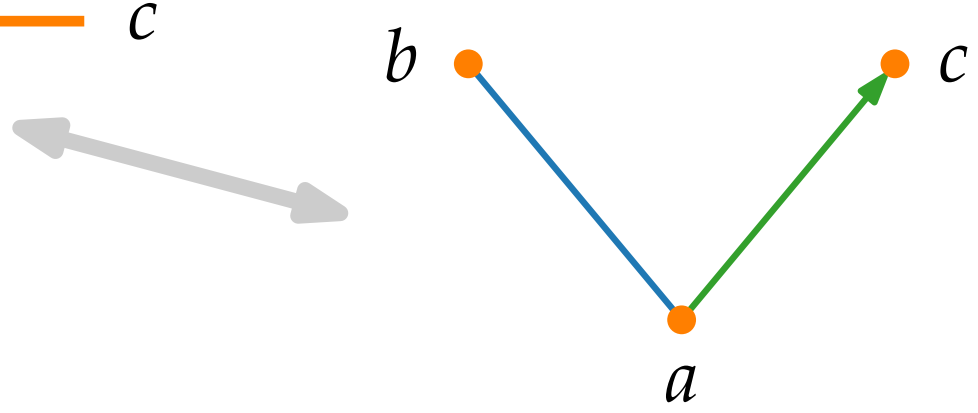

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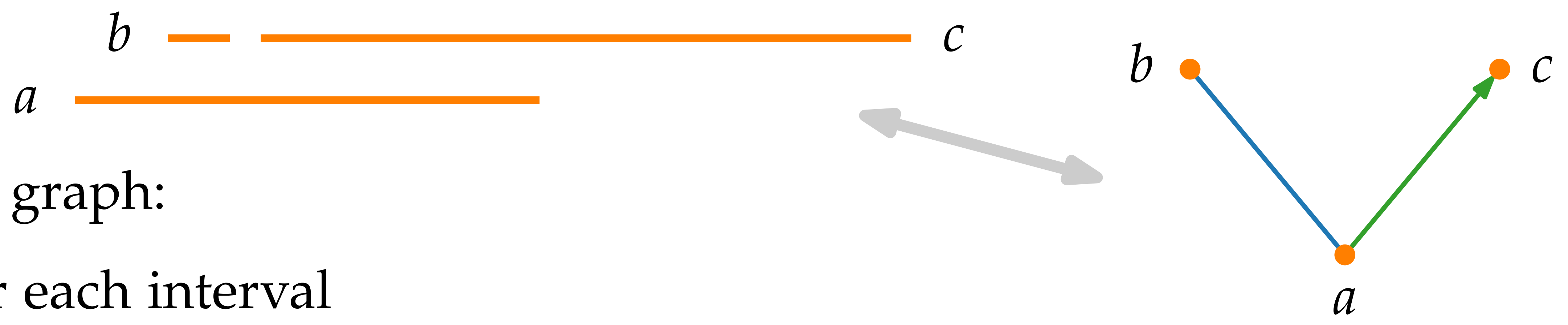

Mixed interval graph:

- vertex for each interval
- for each two overlapping intervals: undirected or arbitrarily directed edge

Coloring Mixed Graphs

Find a graph coloring $c\colon V \to \mathbb{N}$ such that:

[Sotskov, Tanaev '76; Hansen, Kuplinsky, de Werra '97]

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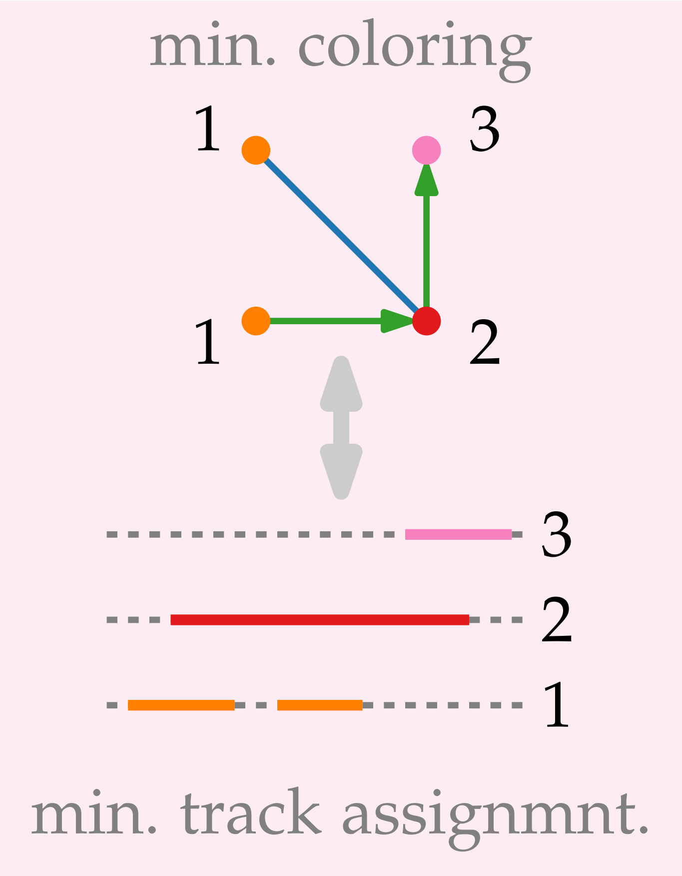

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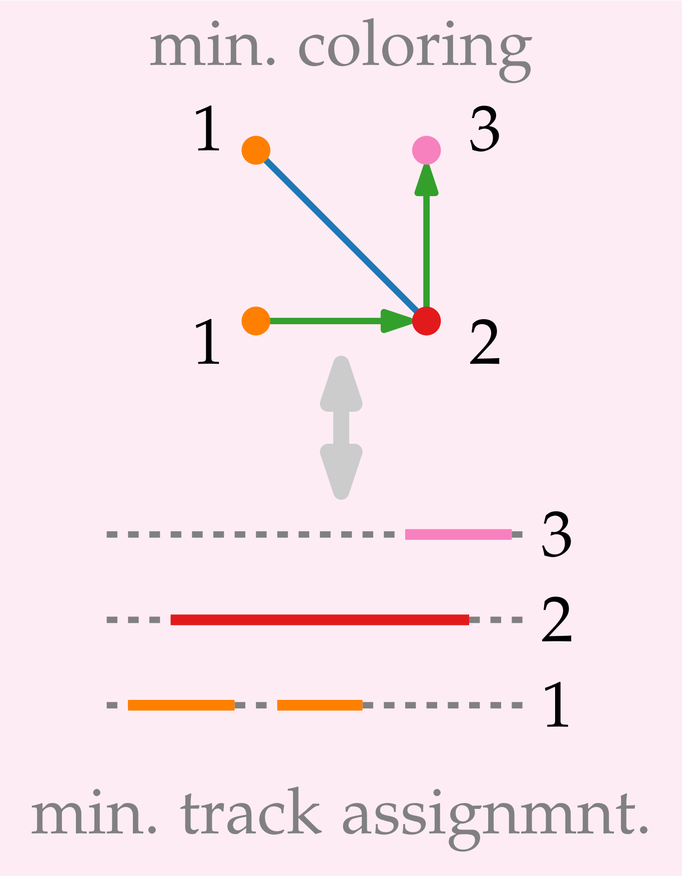

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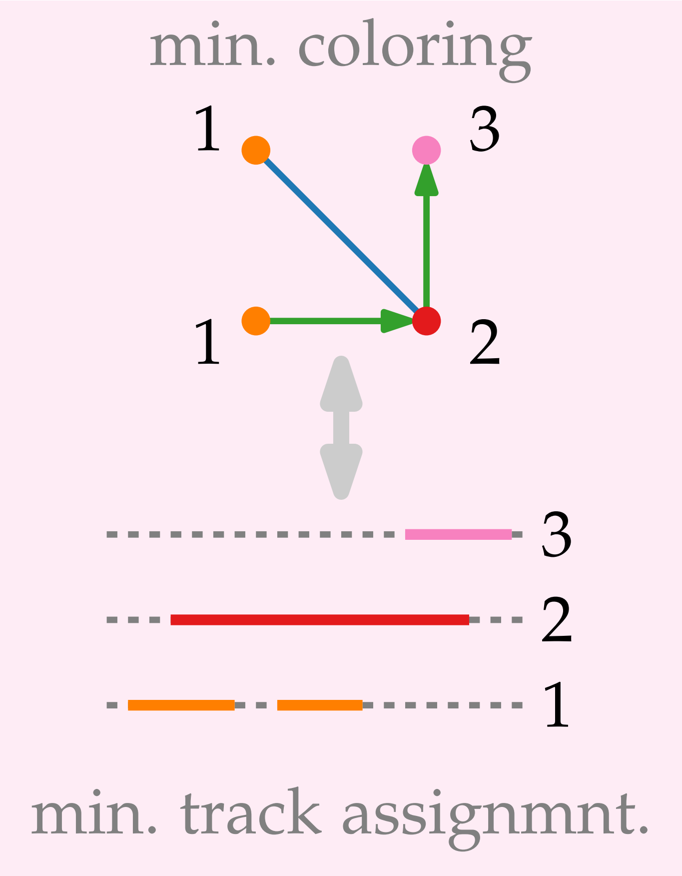

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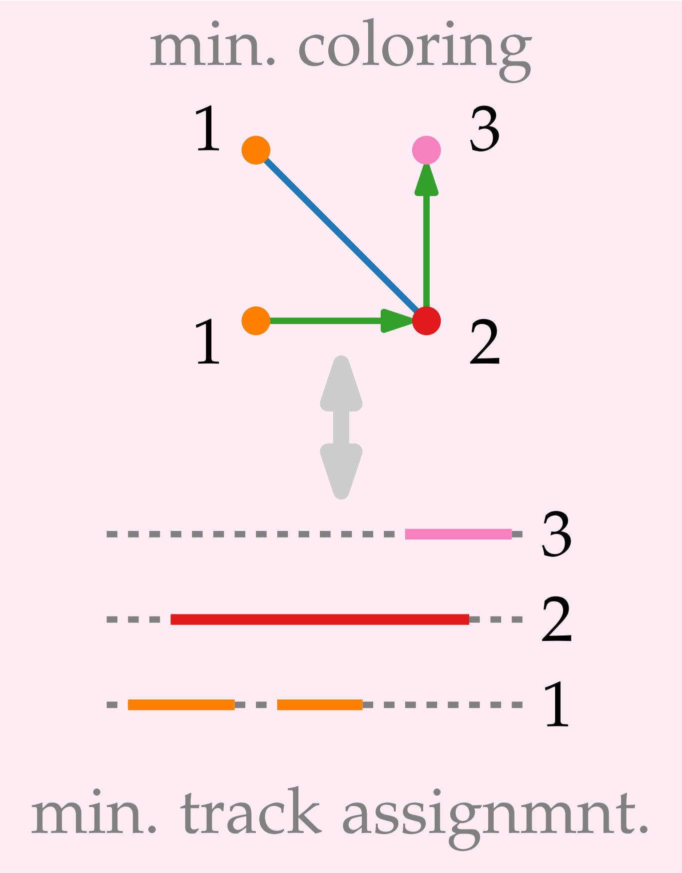

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agenda for this talk

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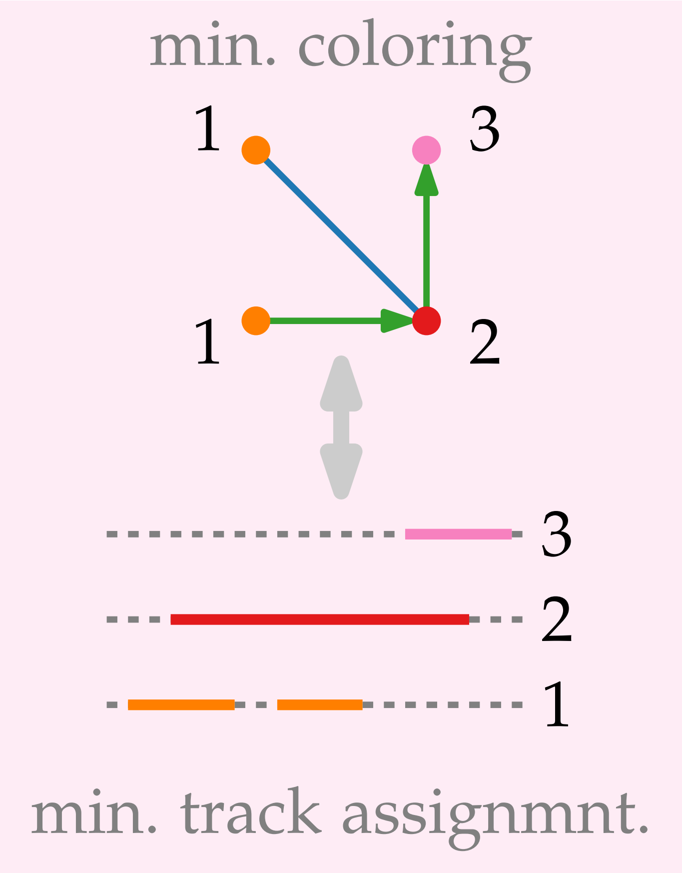

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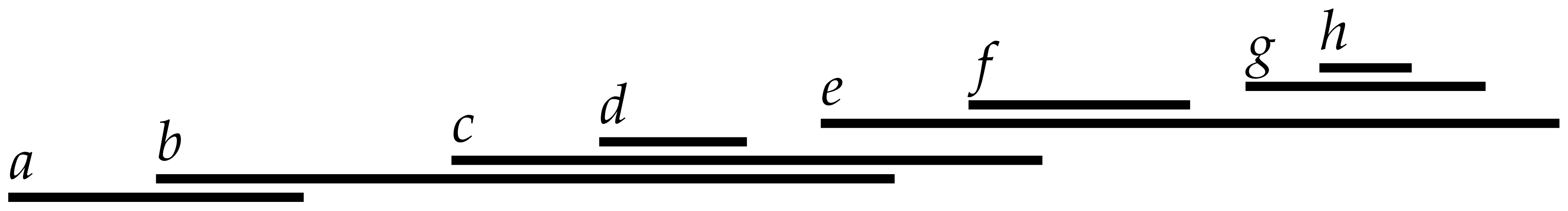

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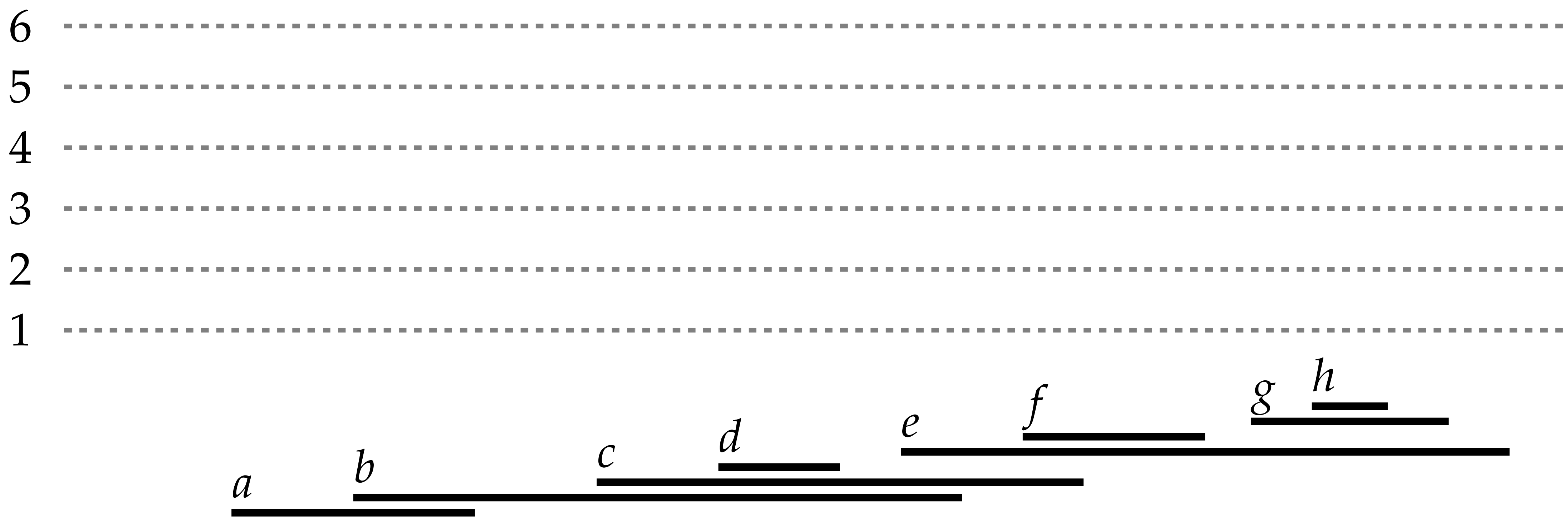

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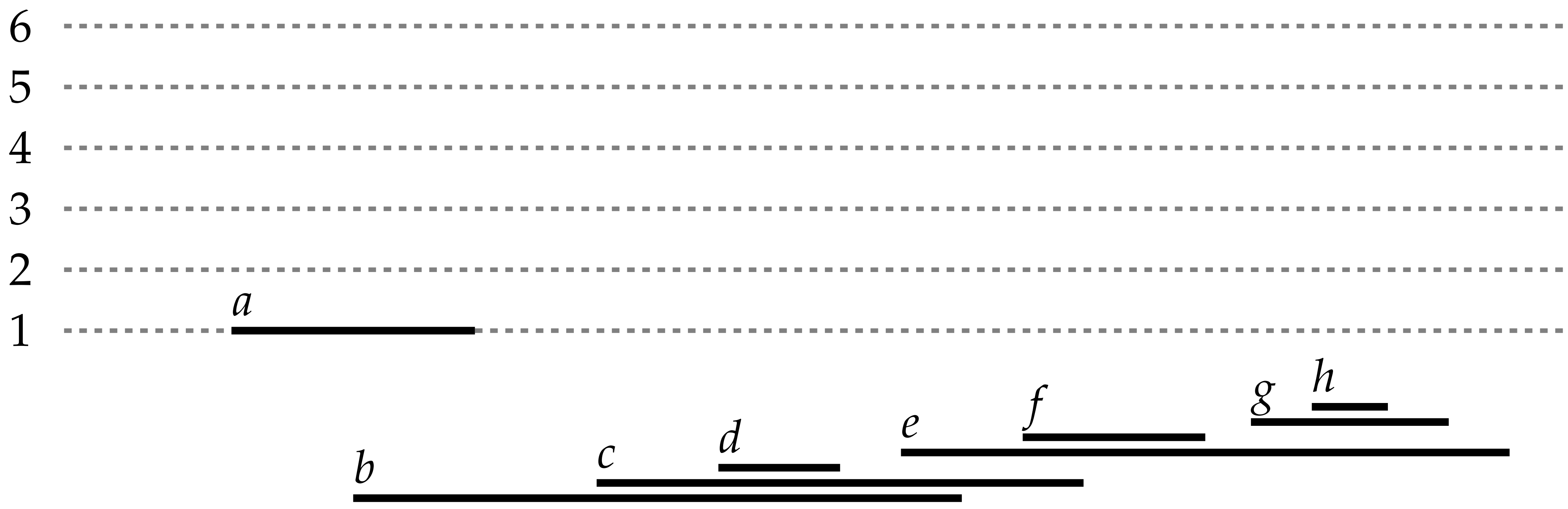

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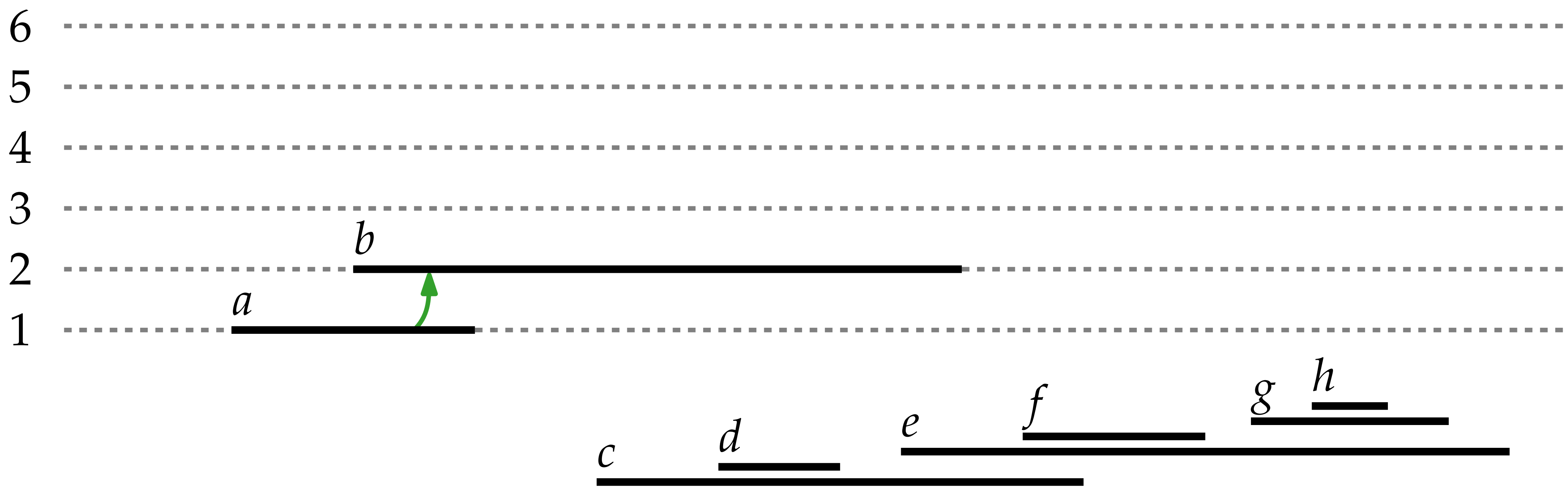

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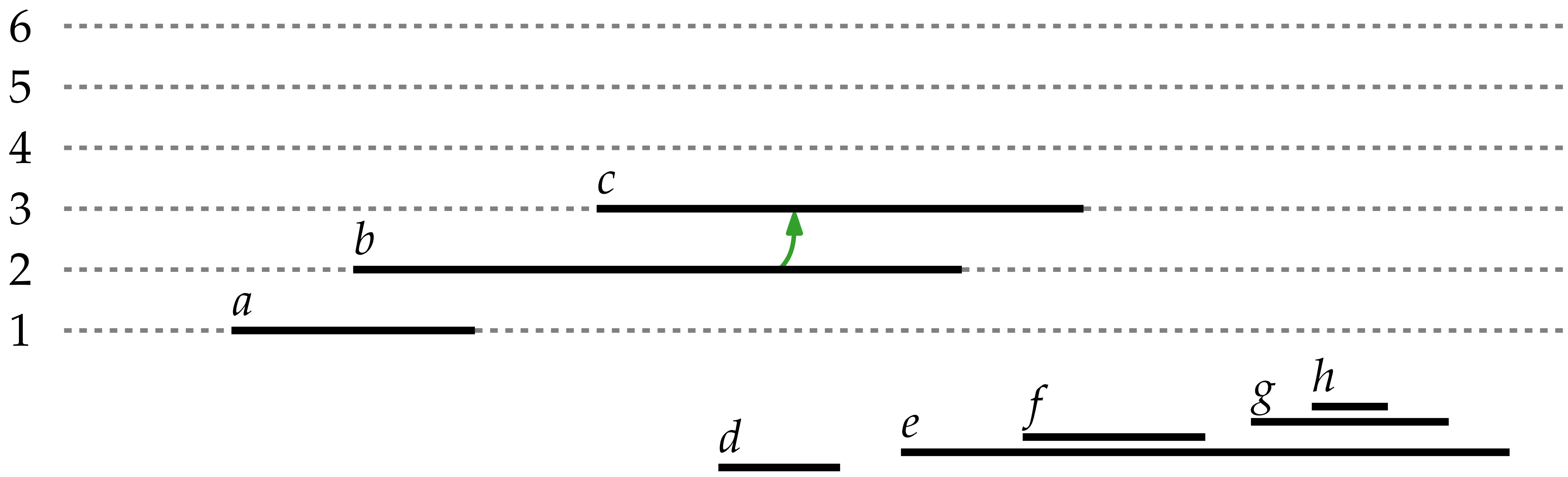

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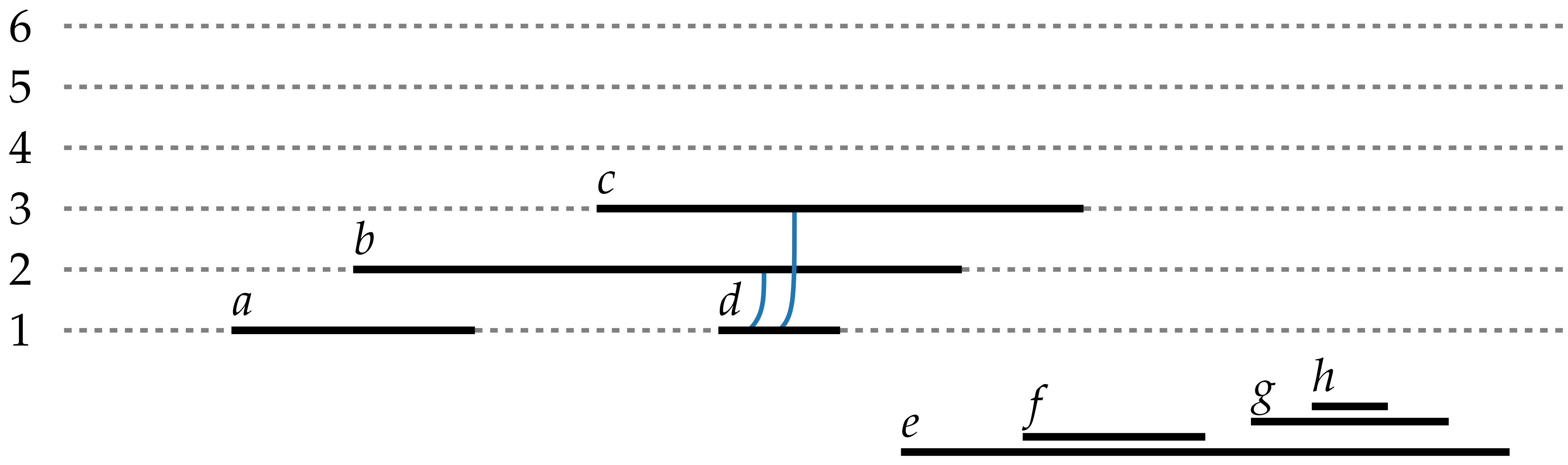

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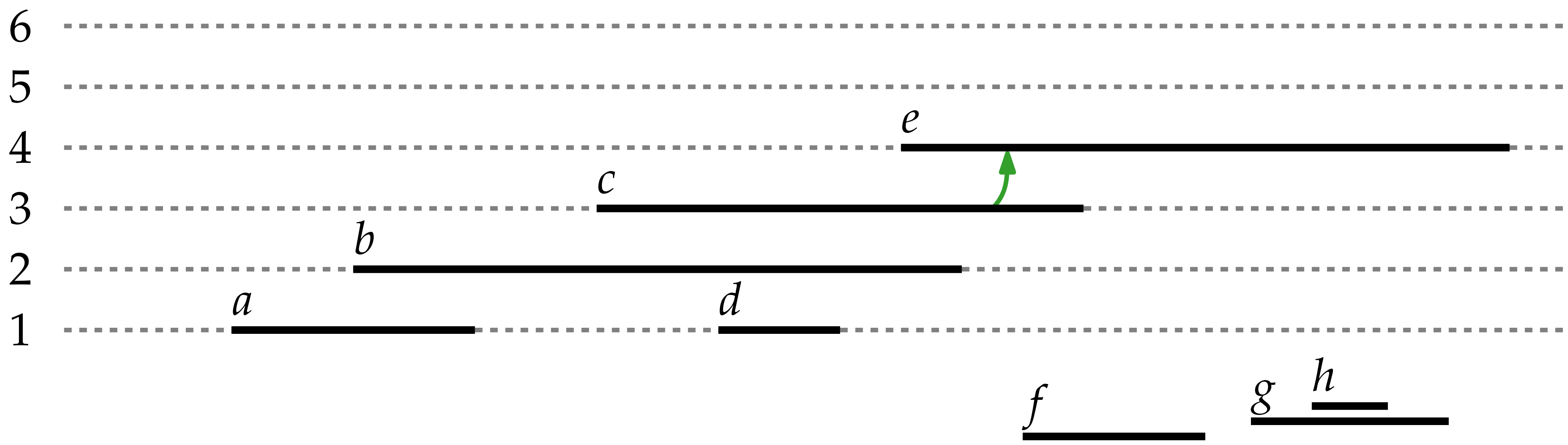

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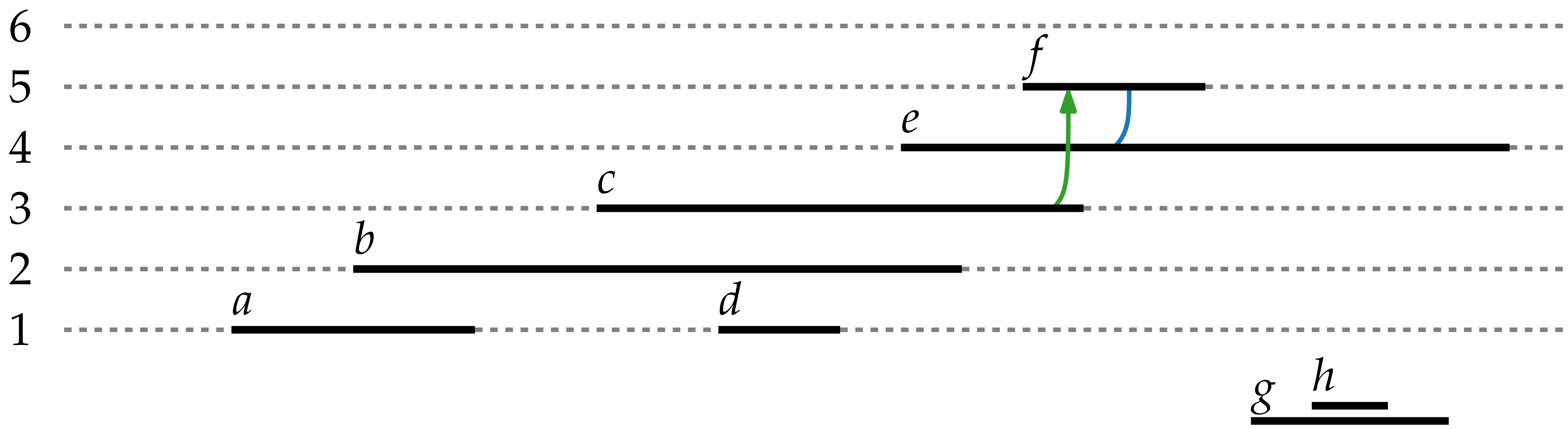

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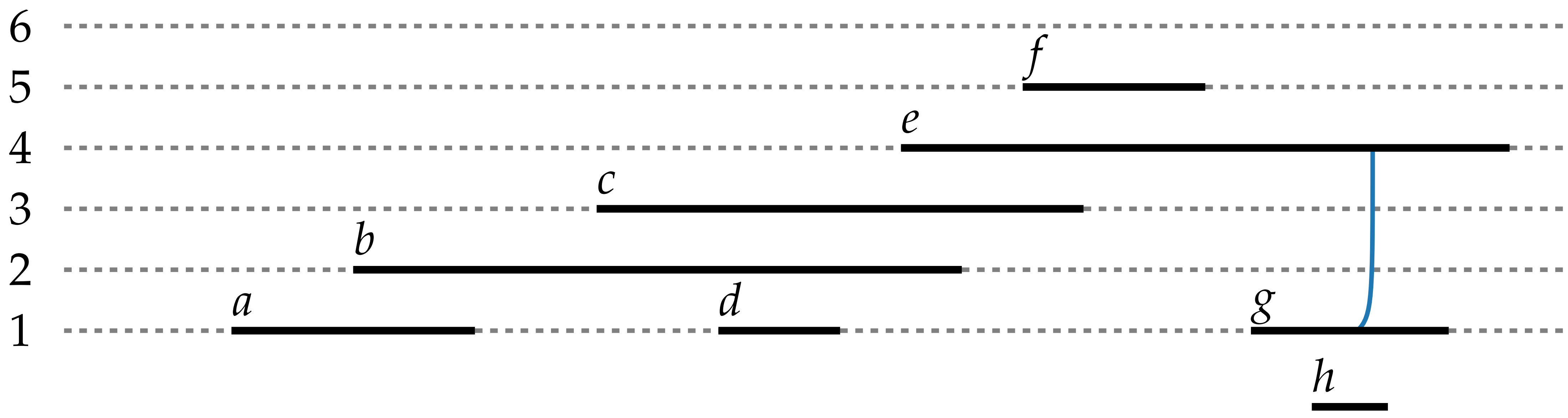

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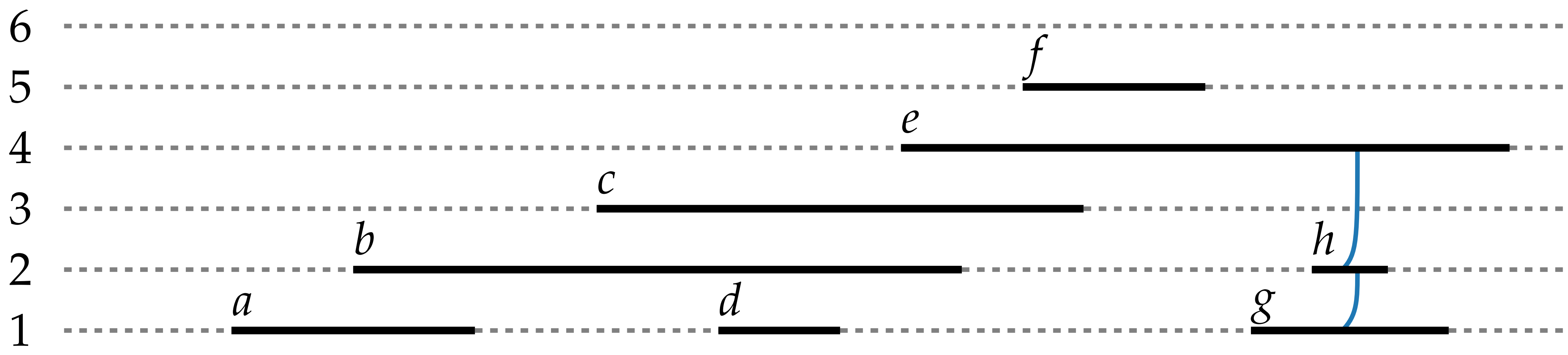

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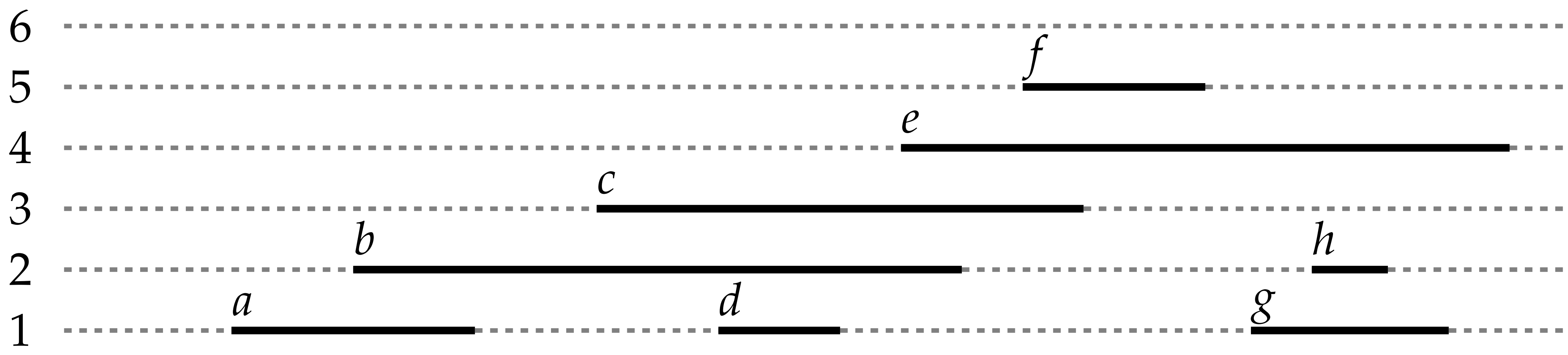

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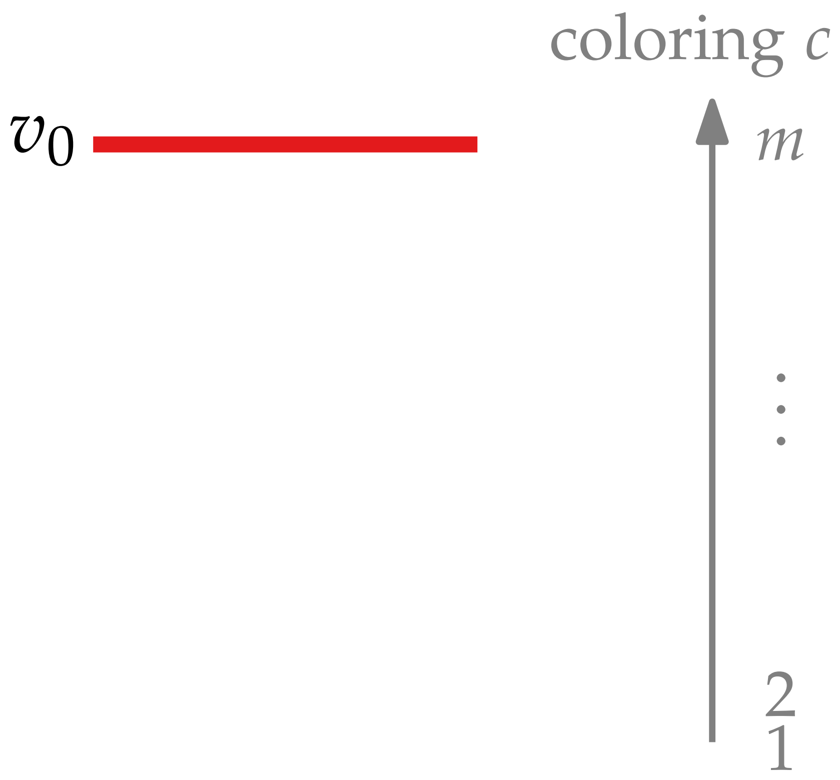

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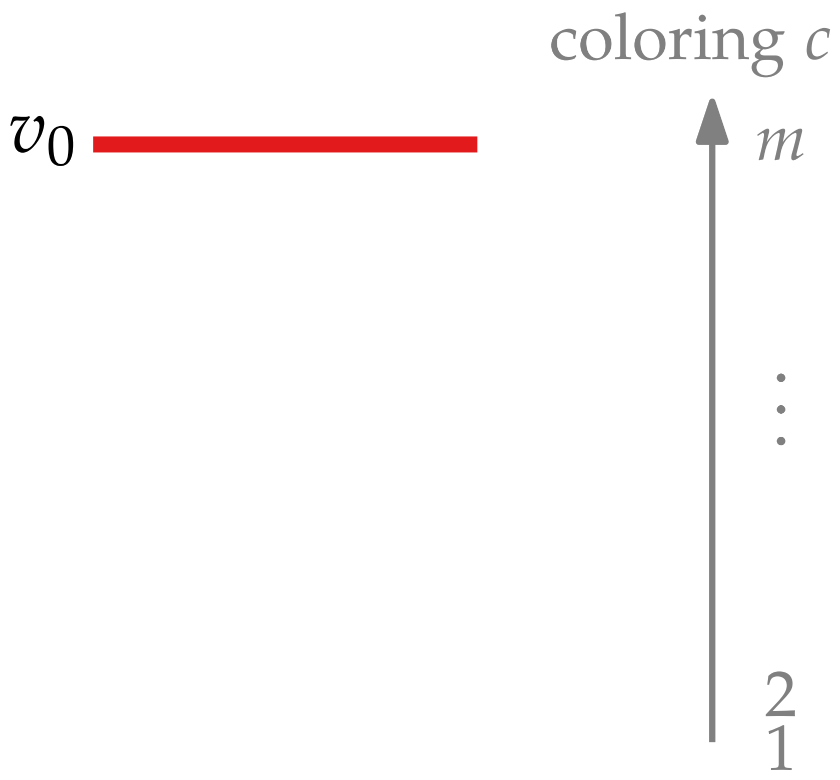

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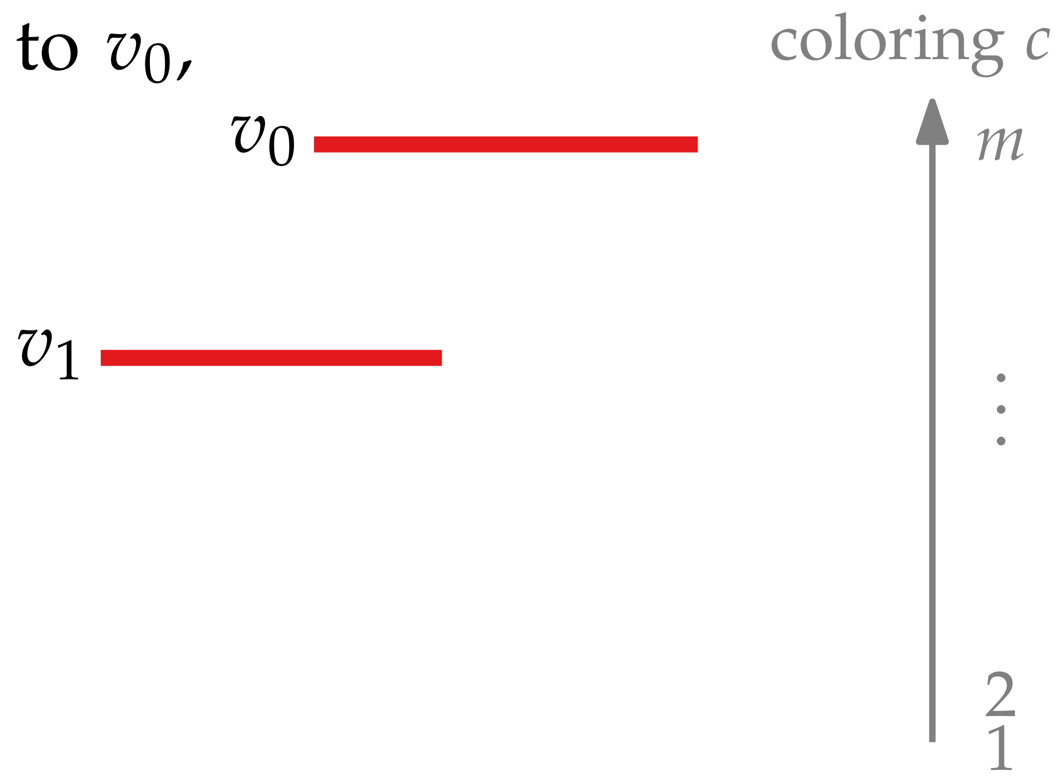

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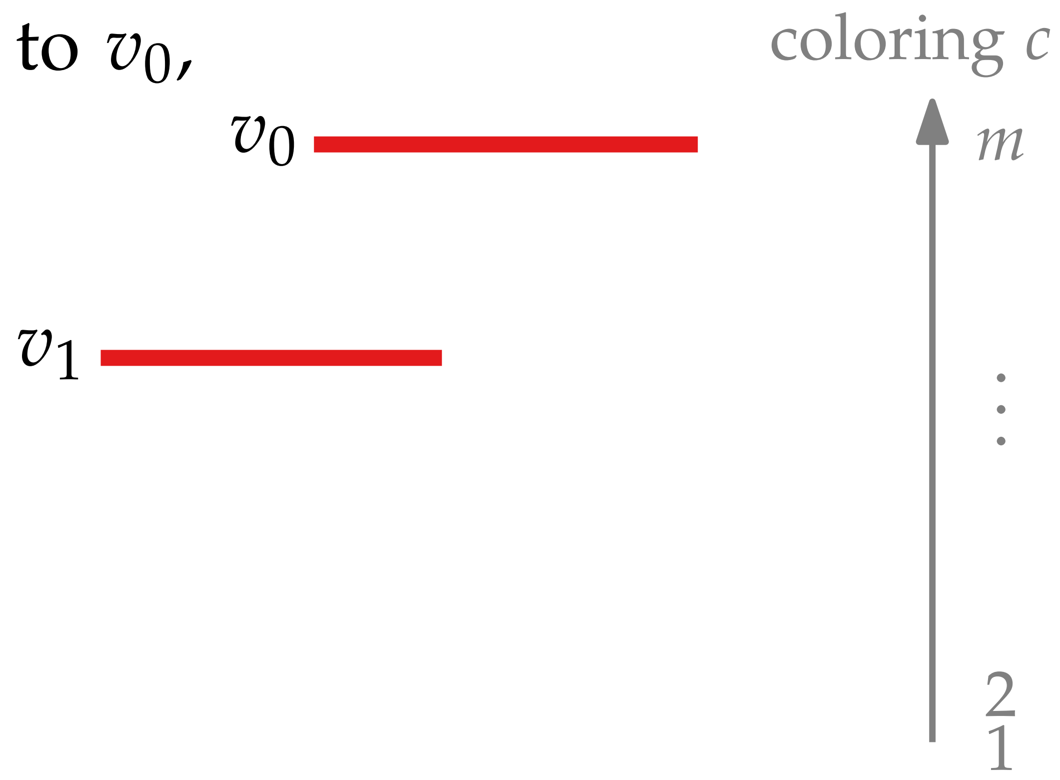

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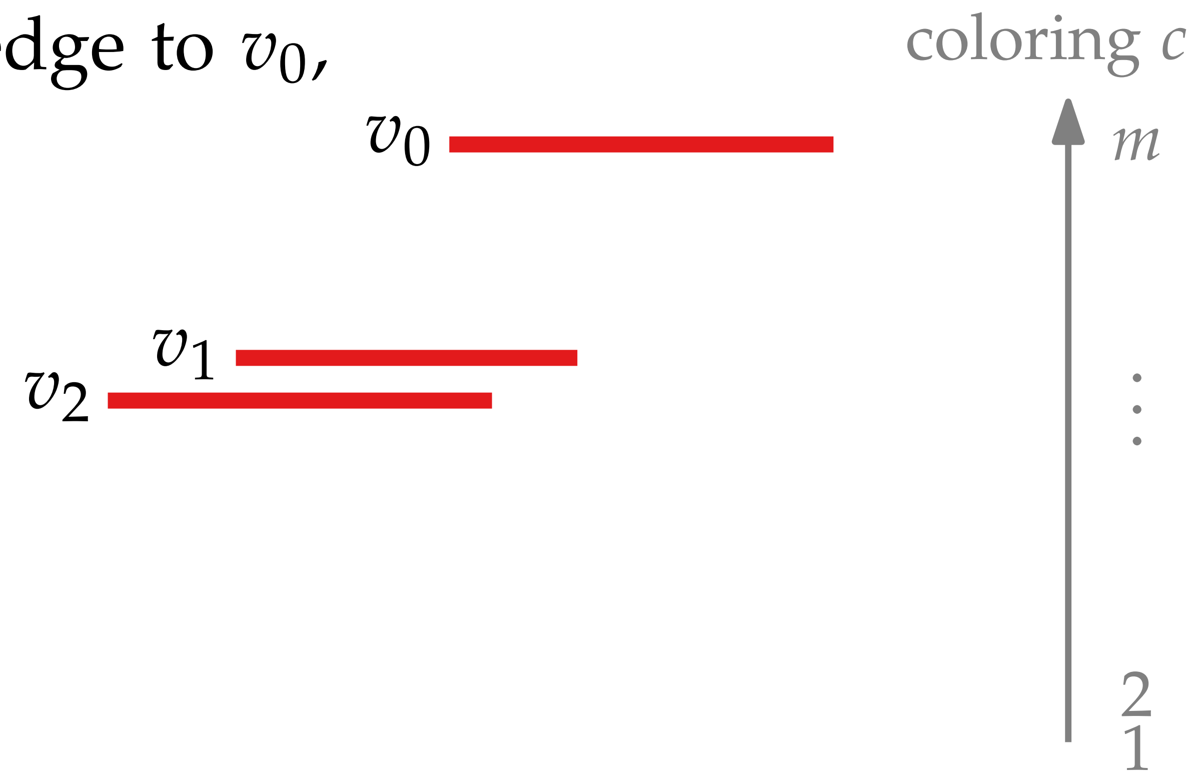

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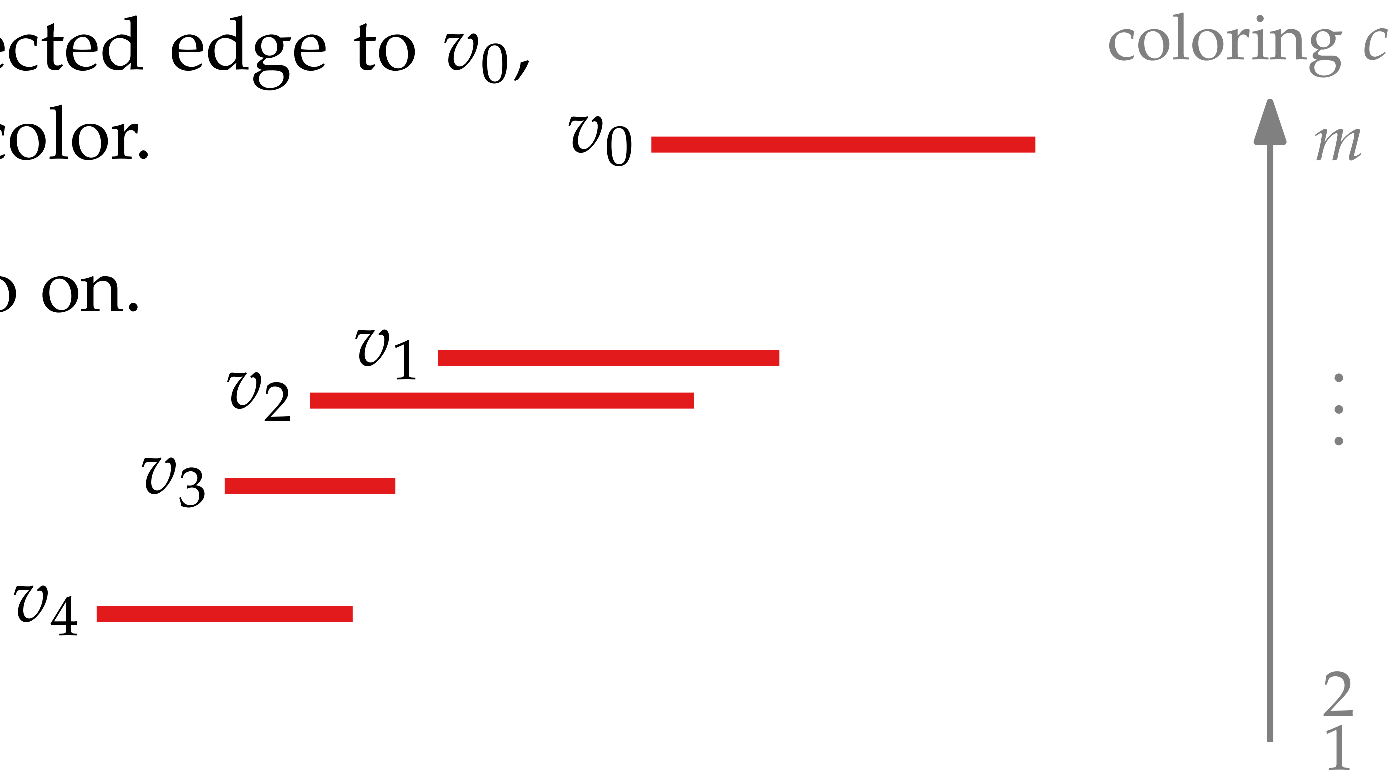

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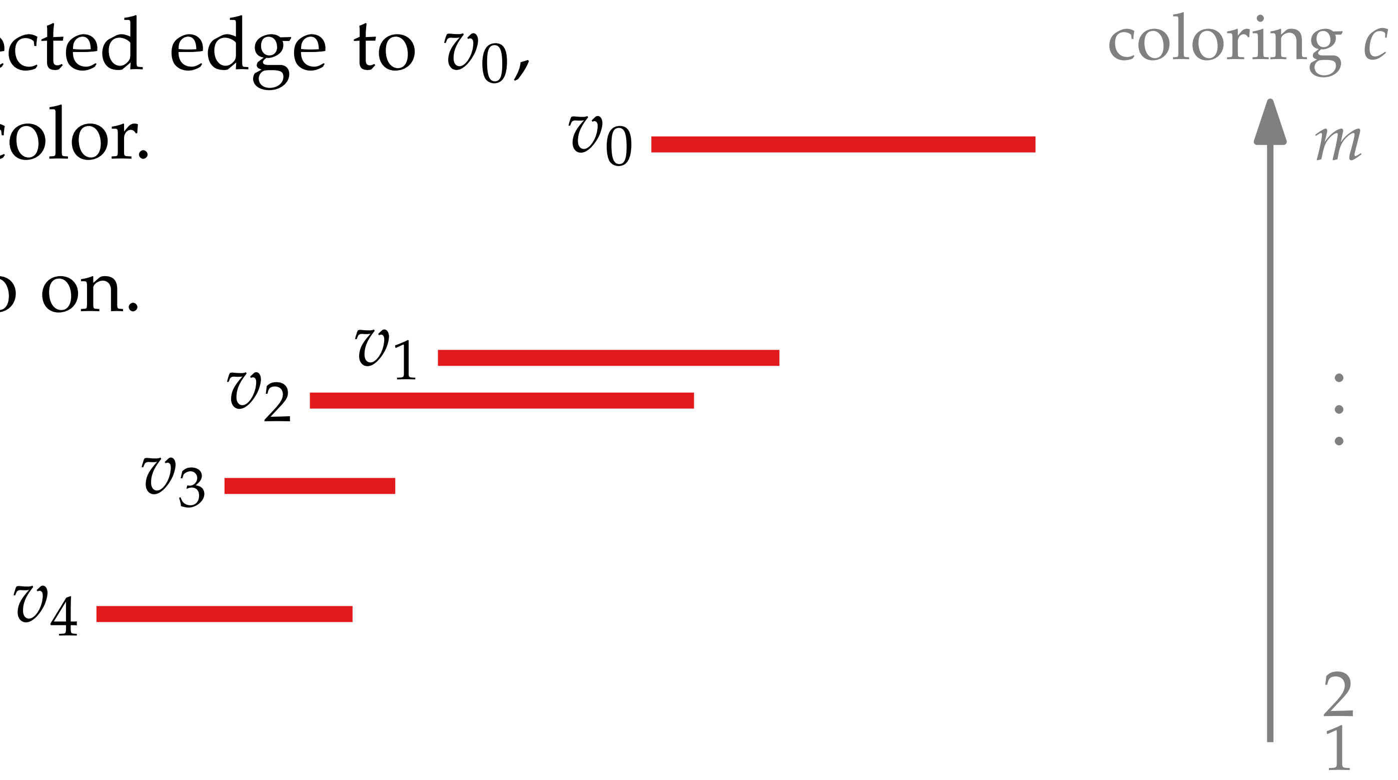

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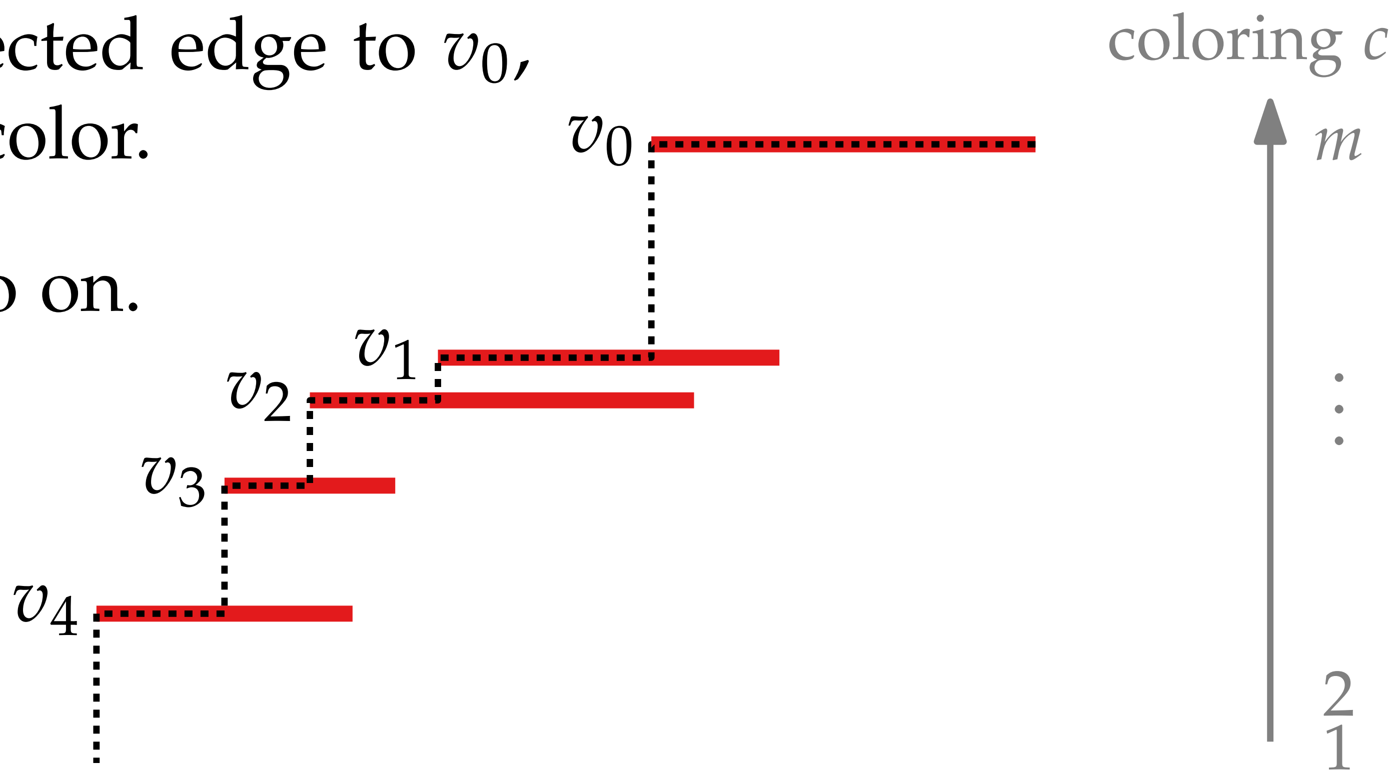

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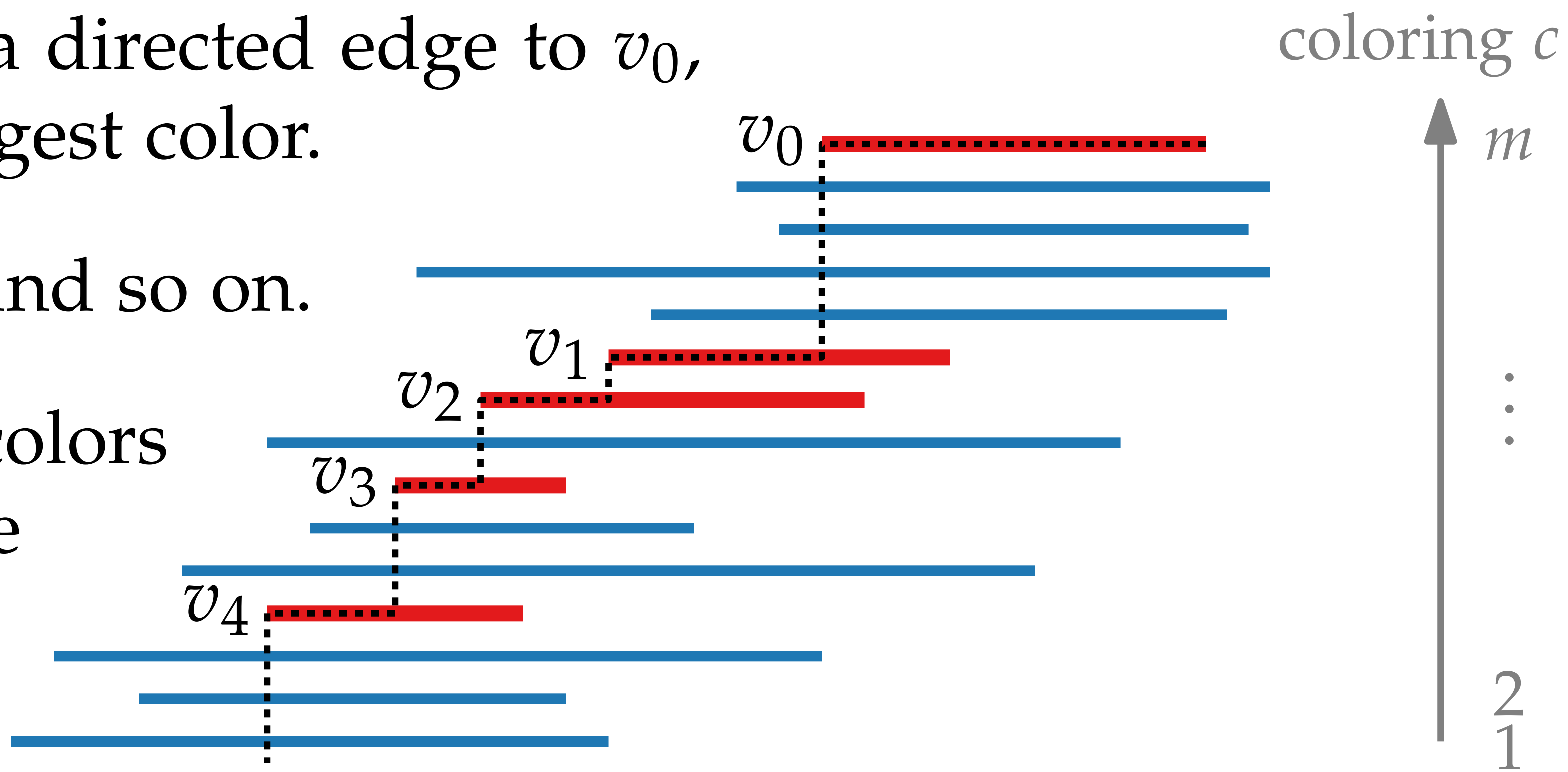

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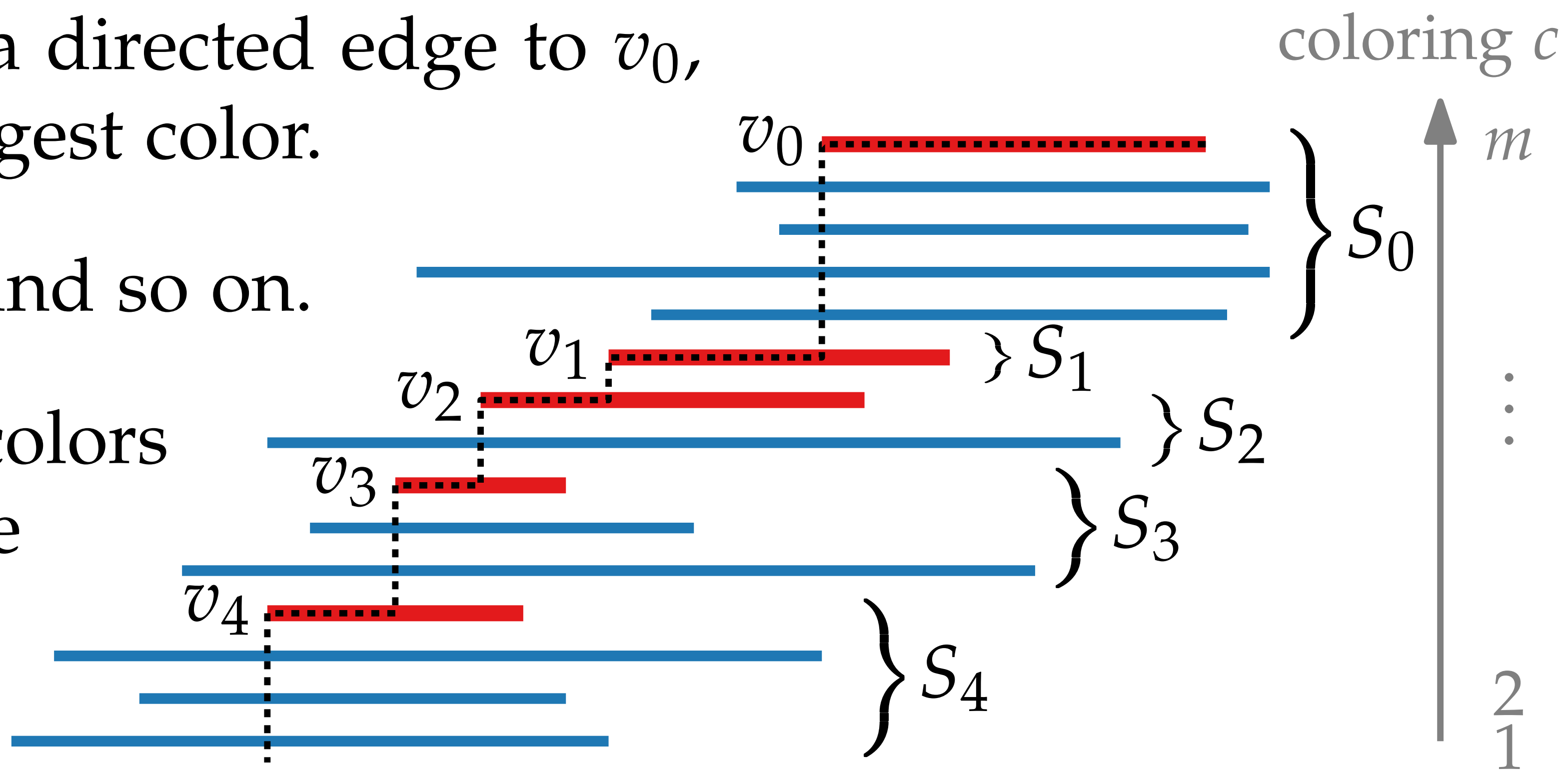

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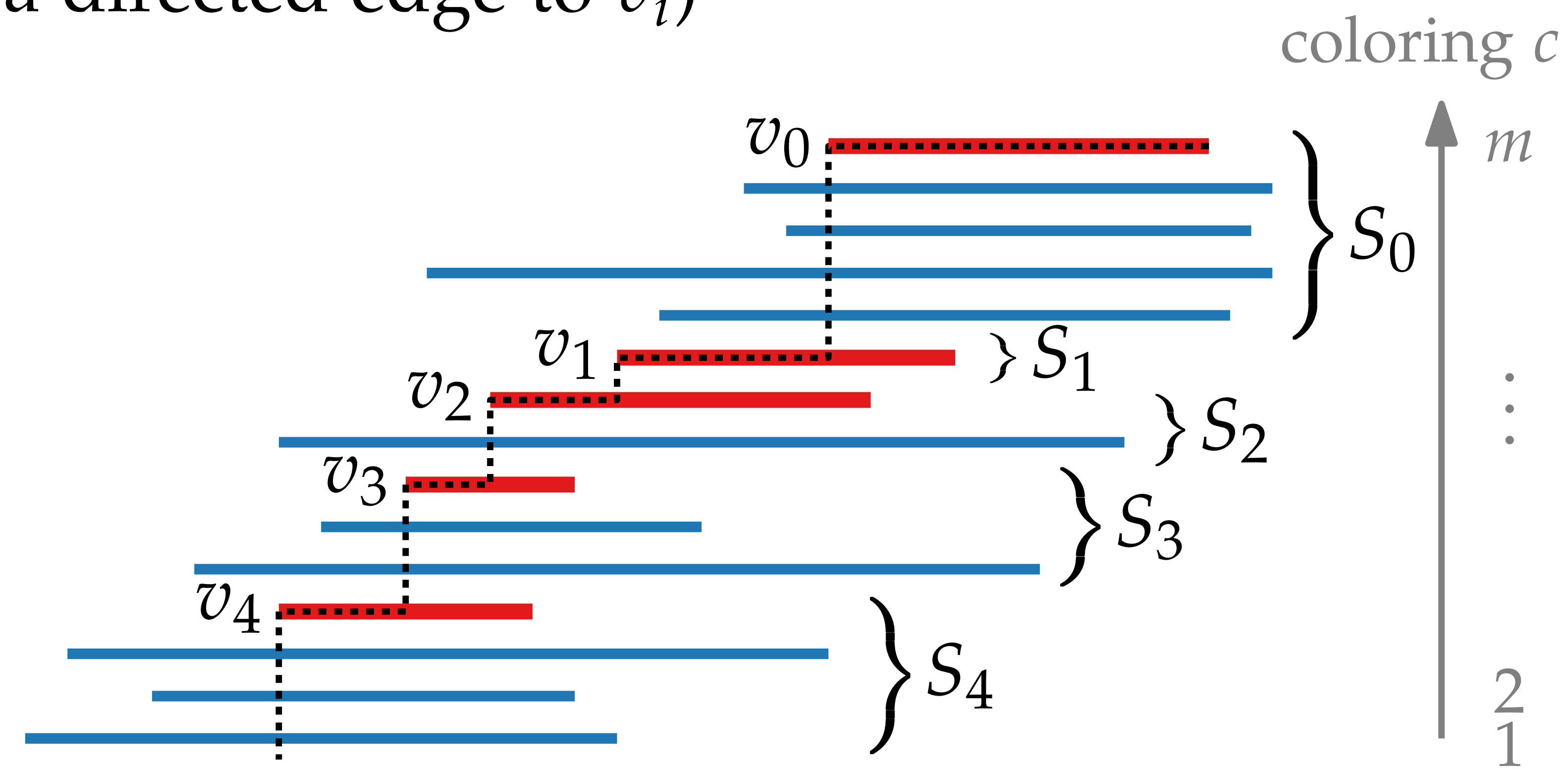

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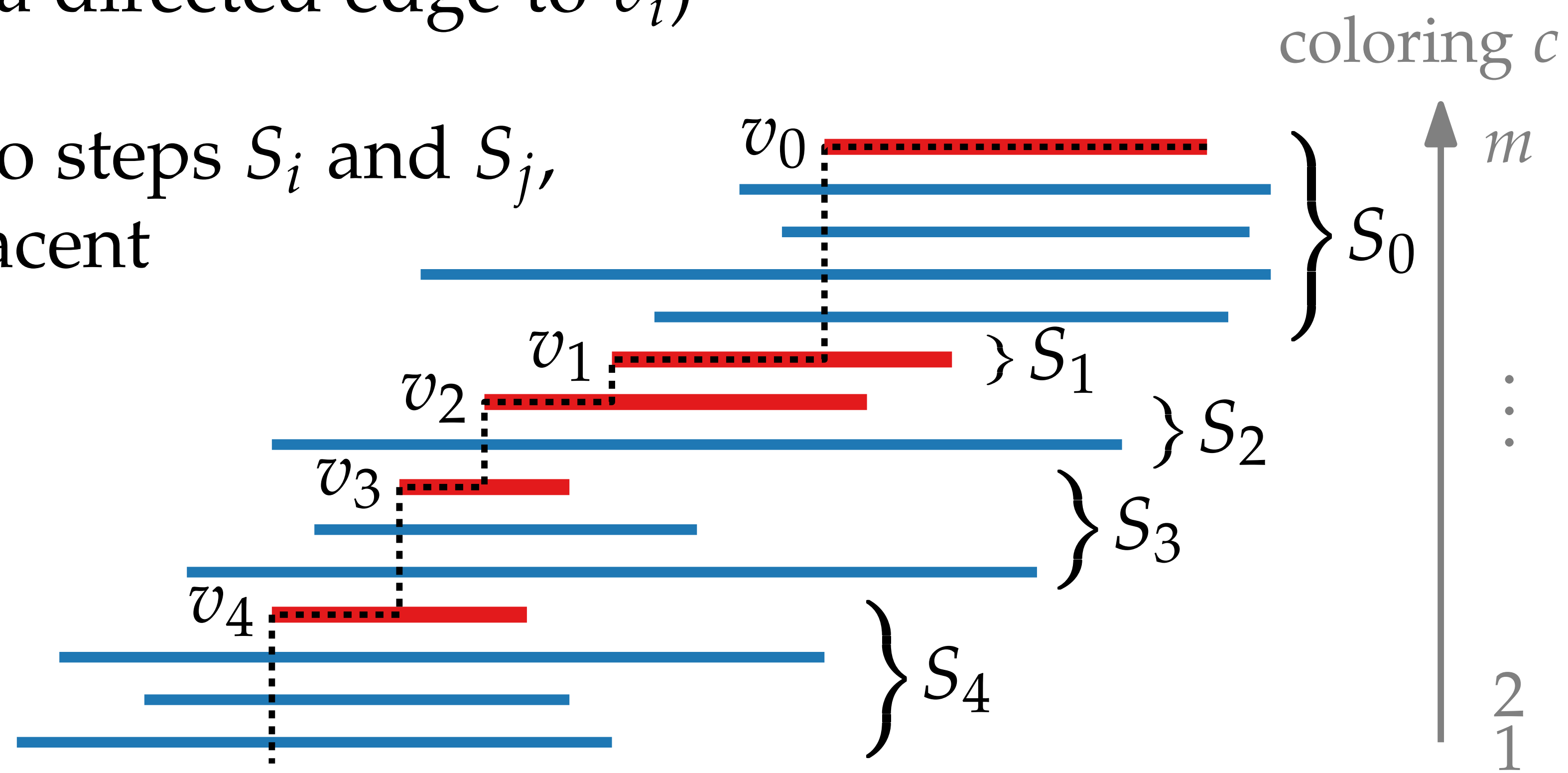

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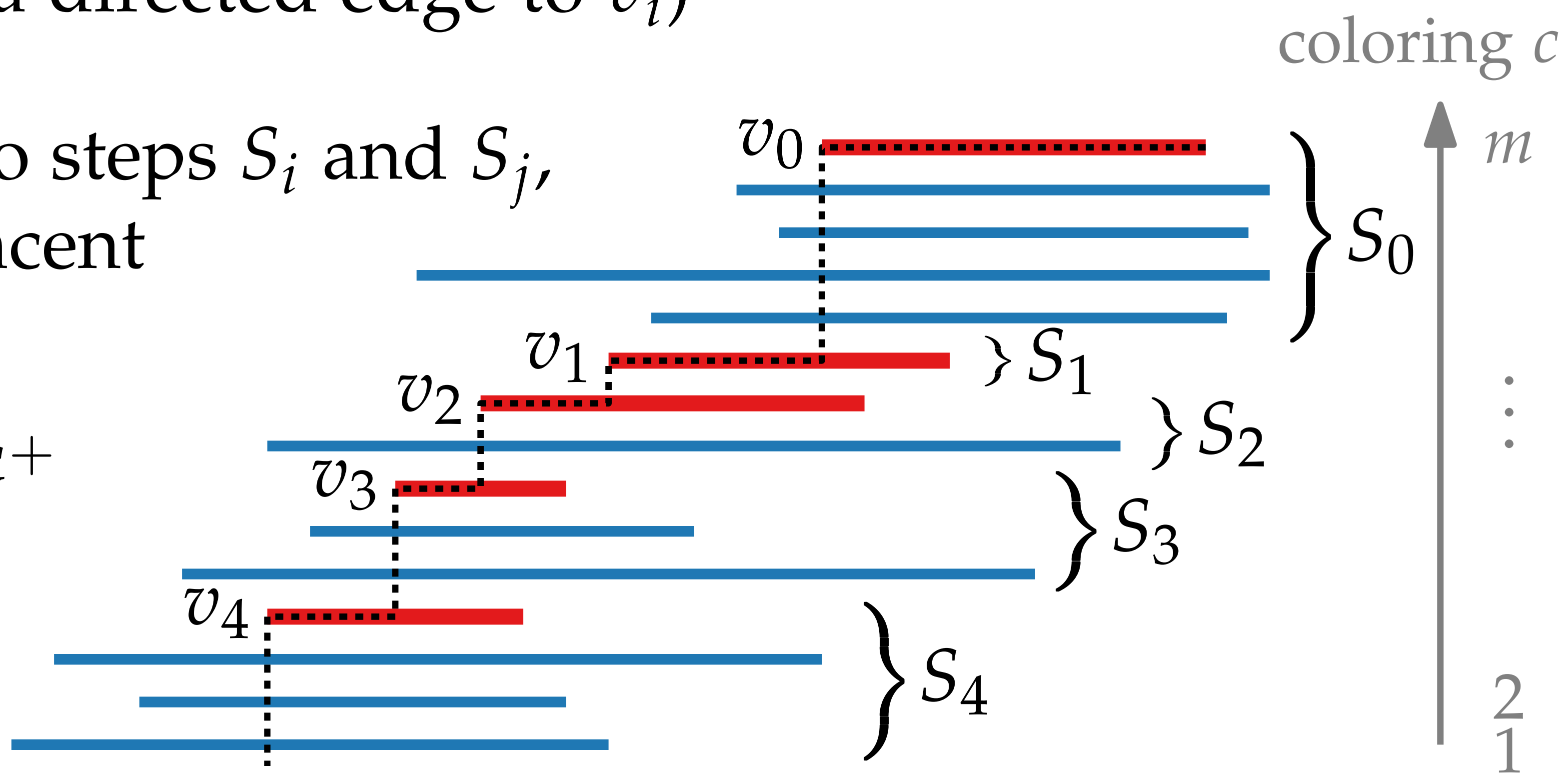

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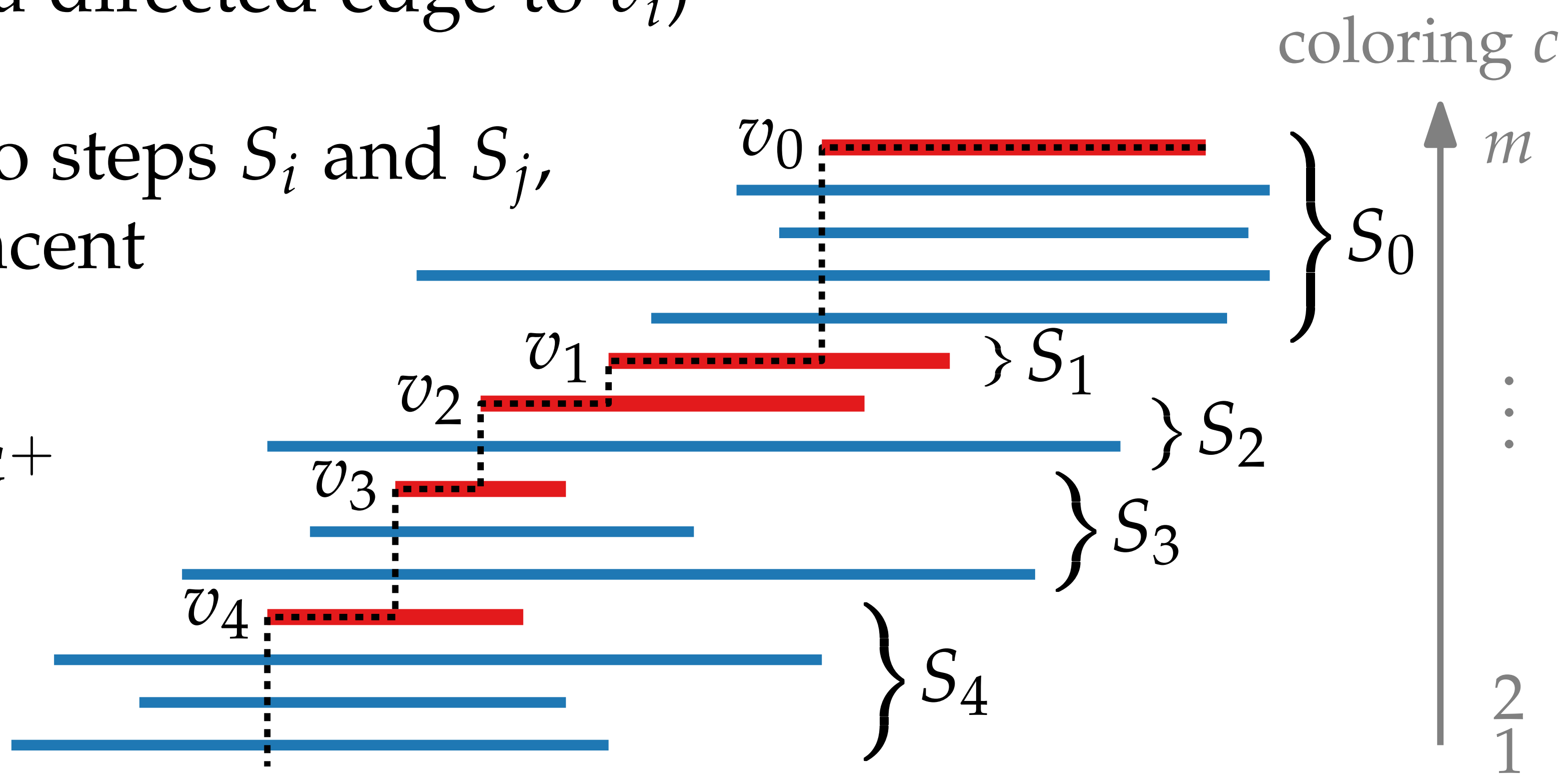

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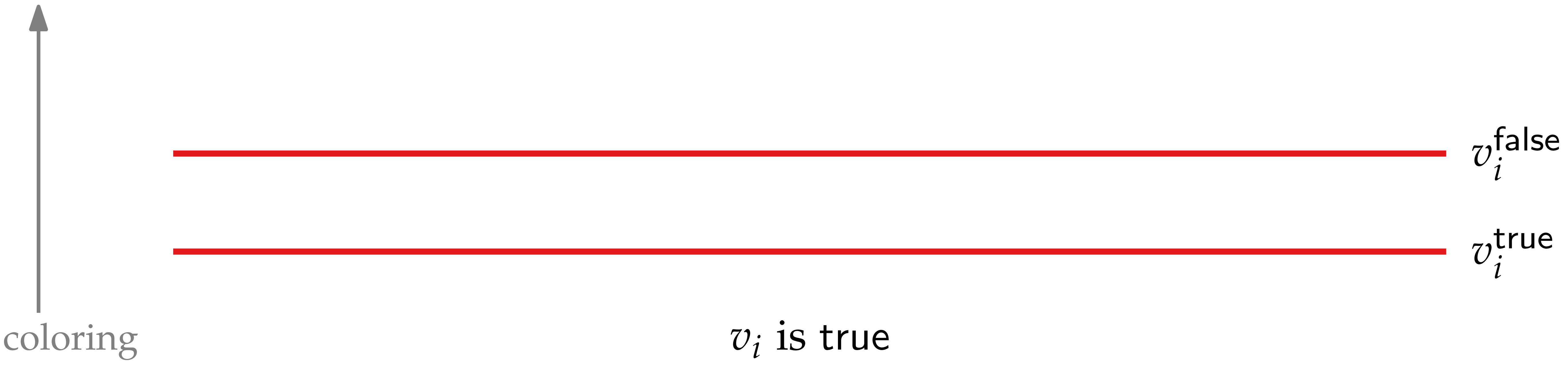

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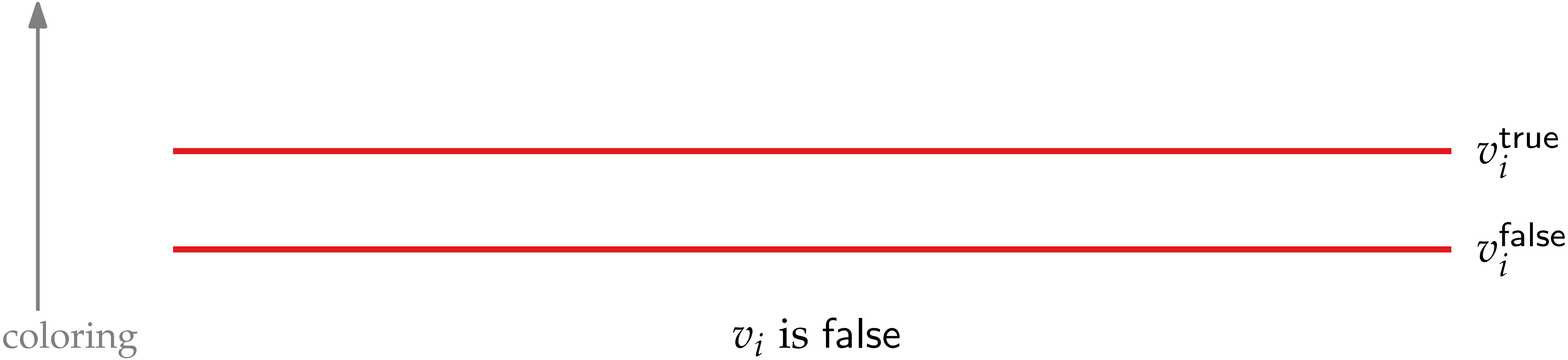

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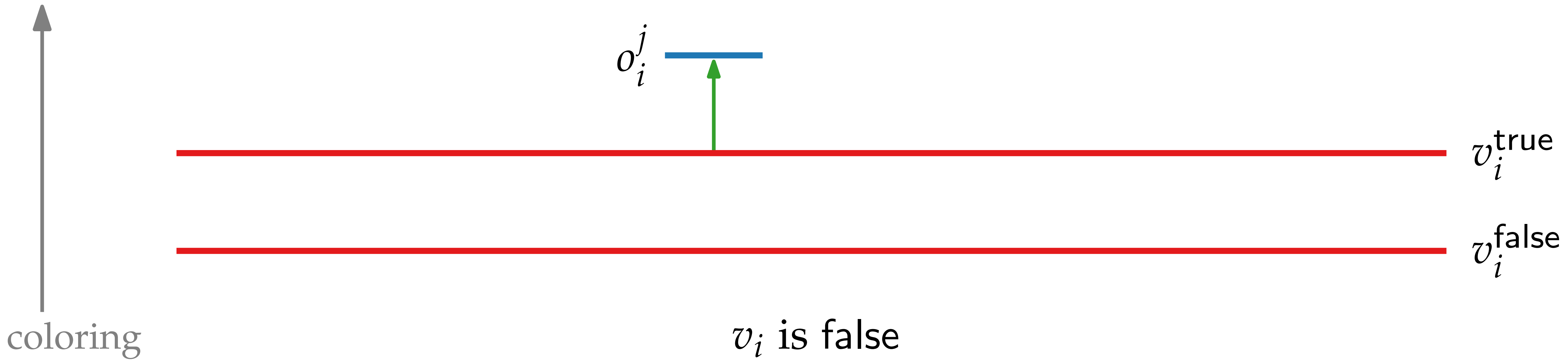

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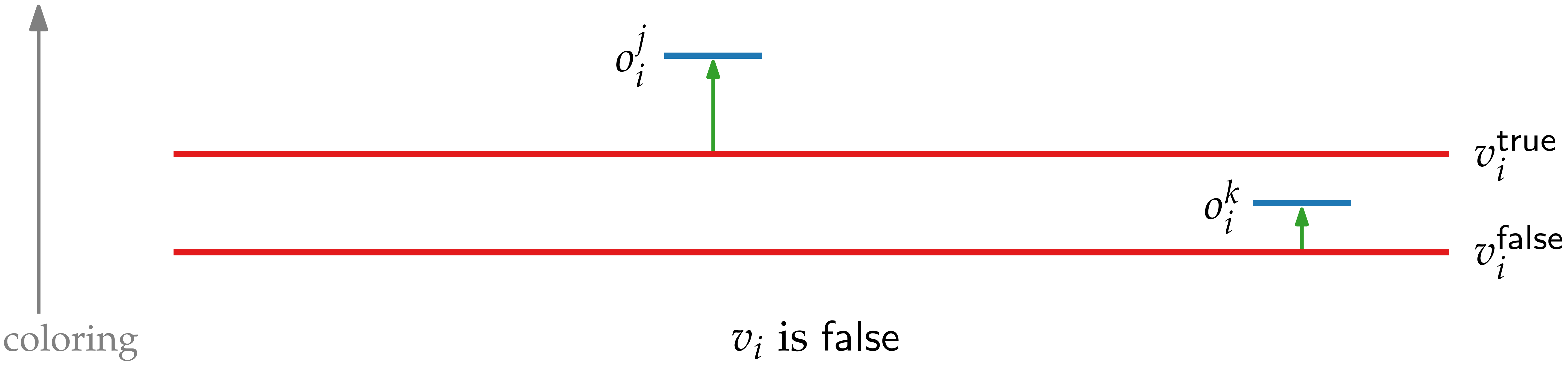

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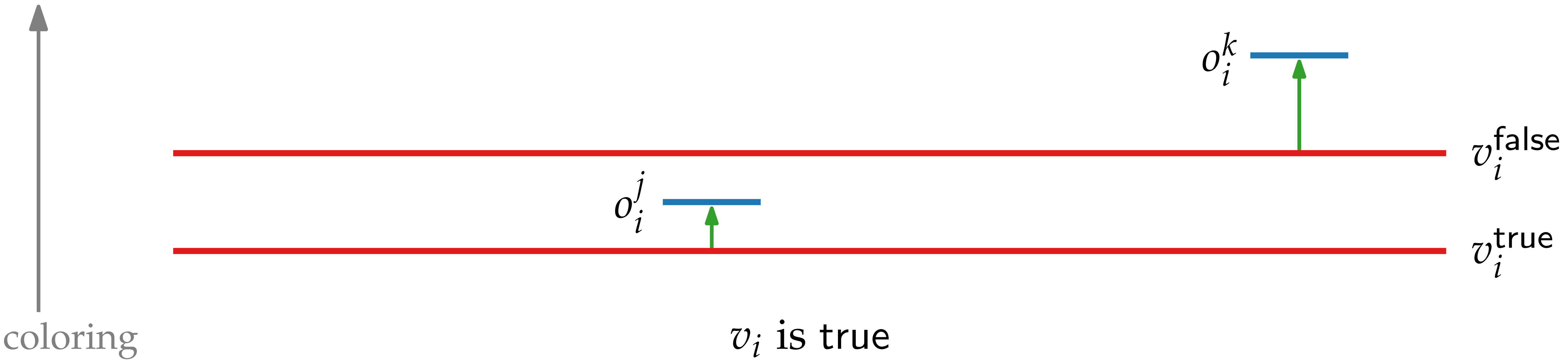

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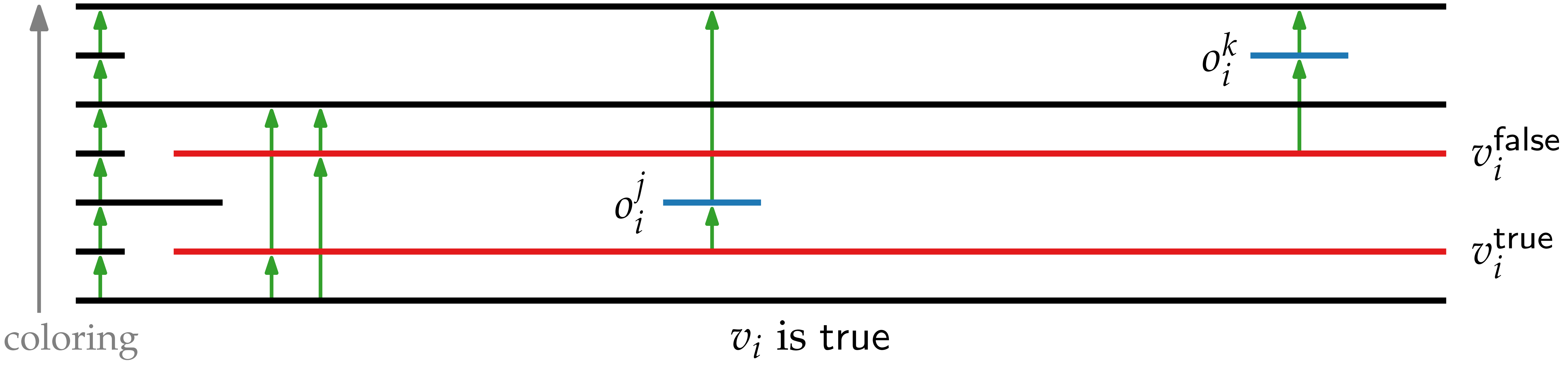

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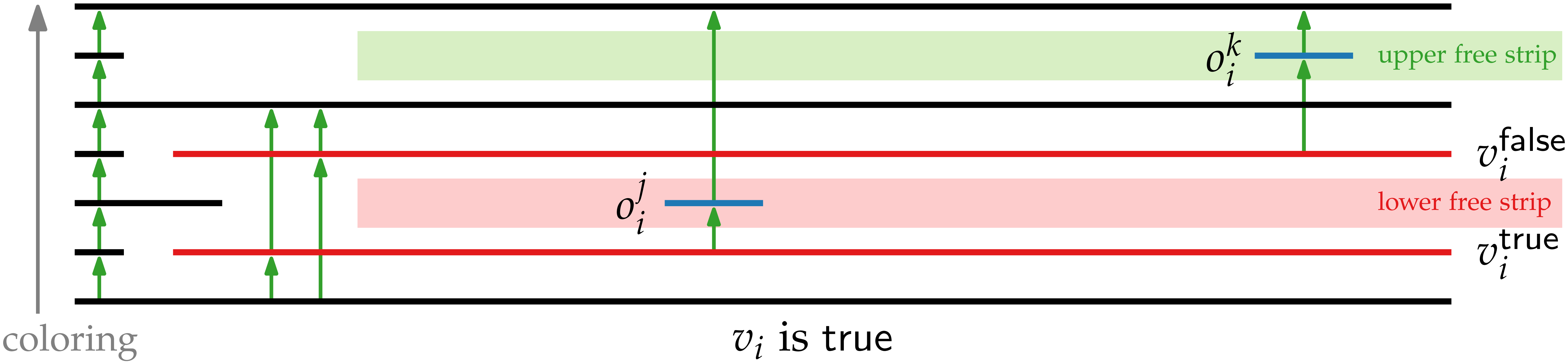

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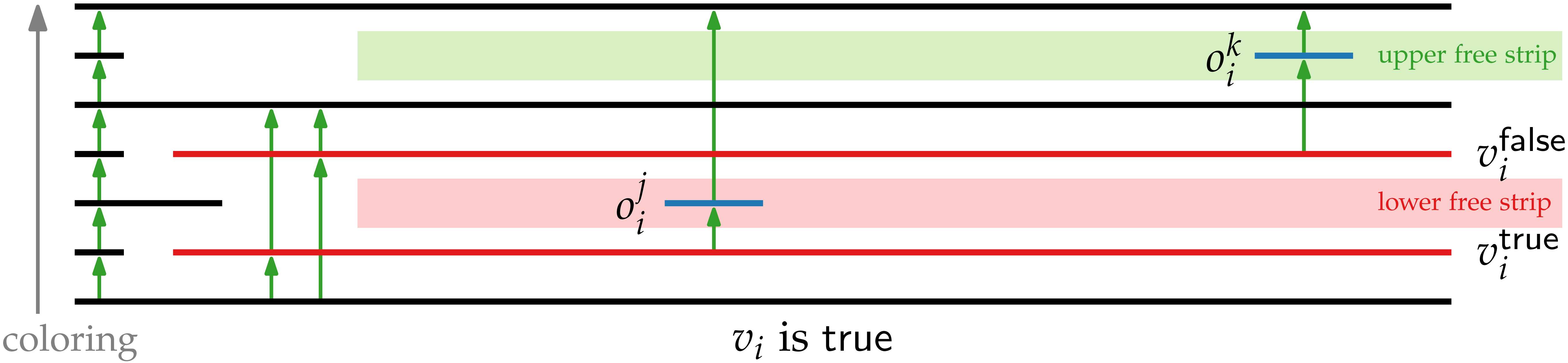

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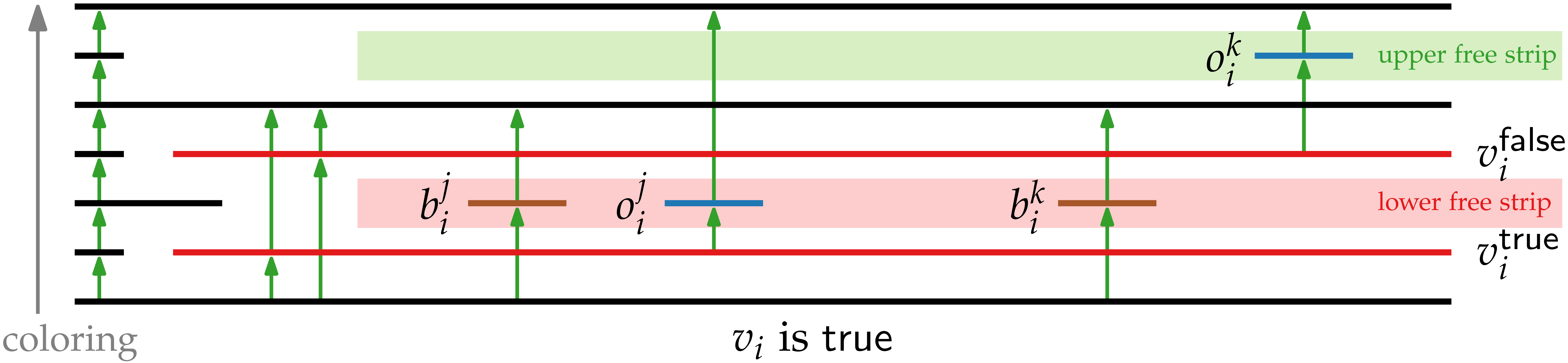

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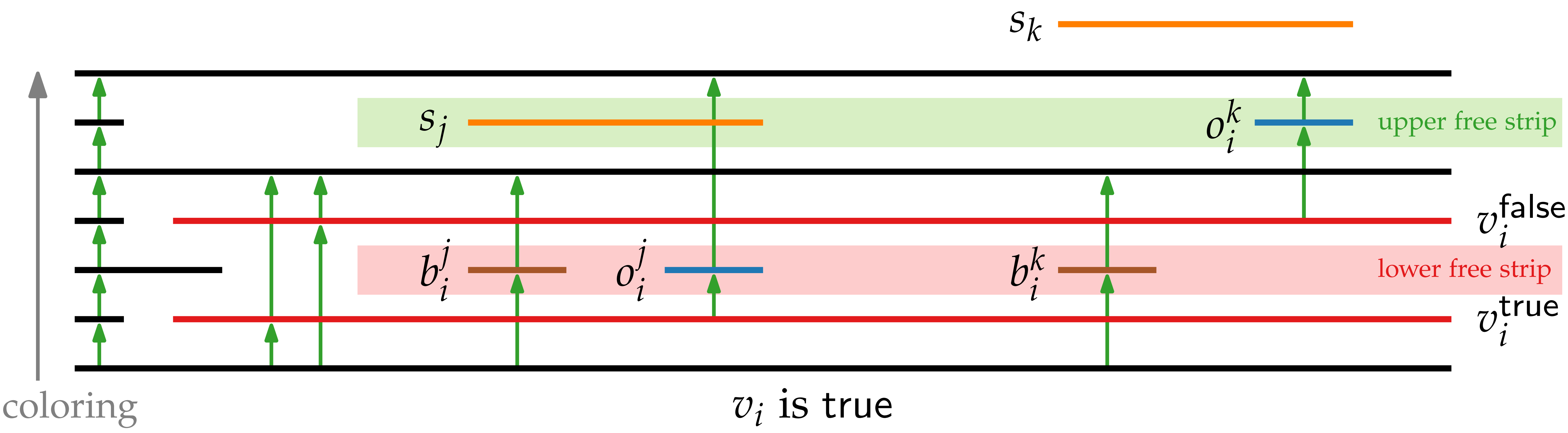

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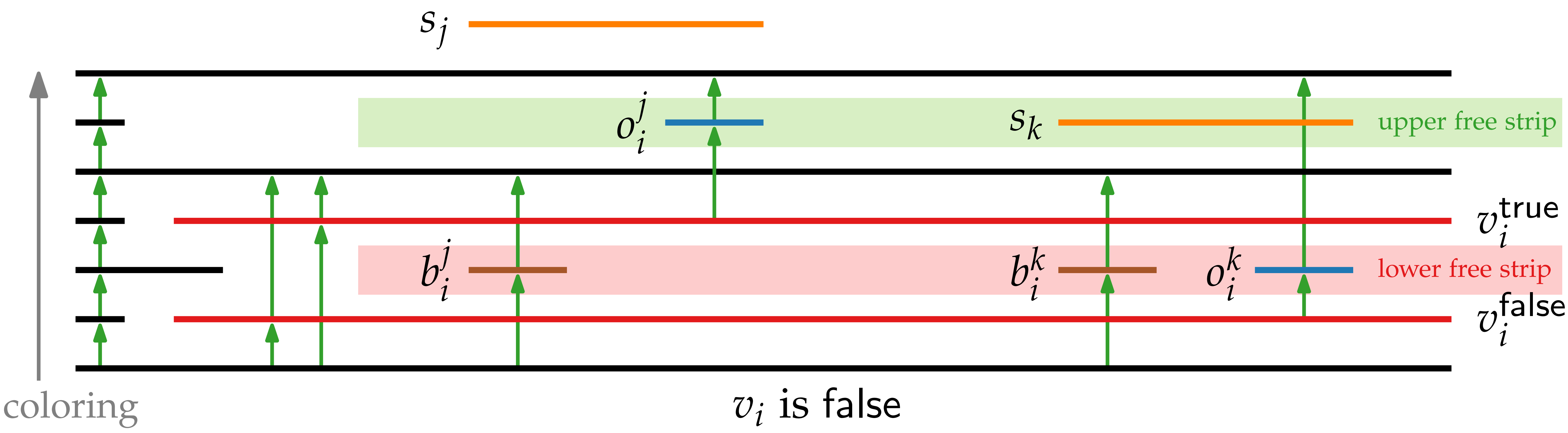

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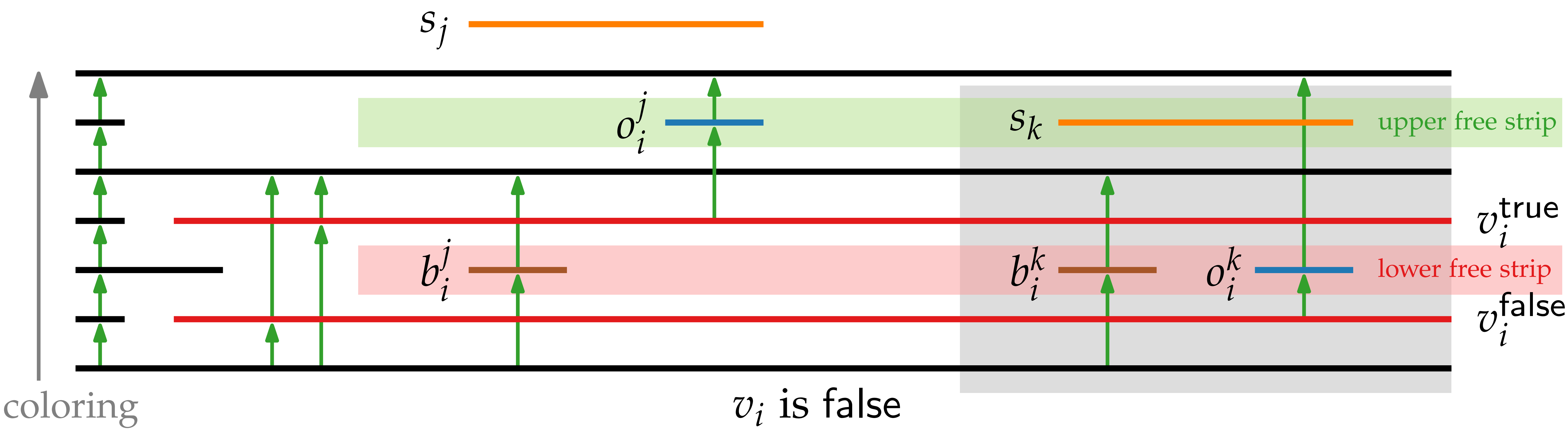

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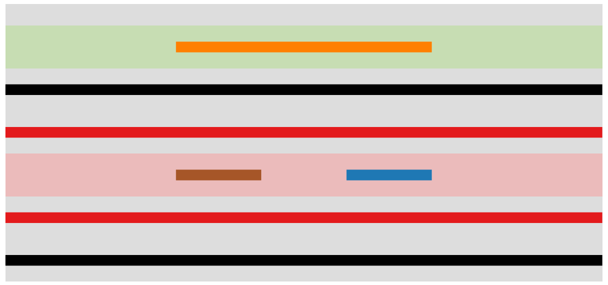

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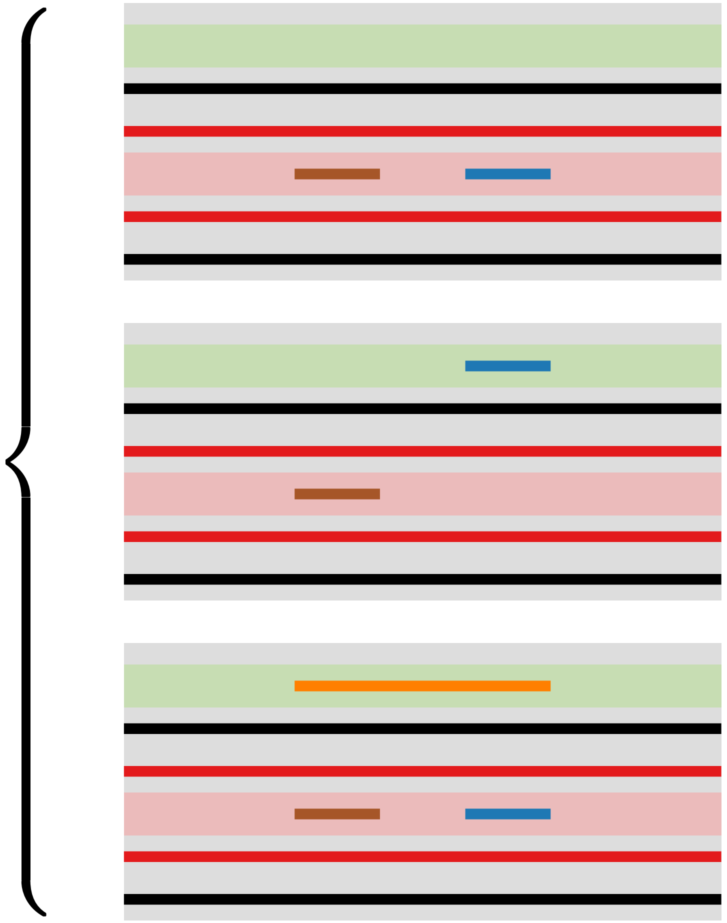
($n :=$ # variables)



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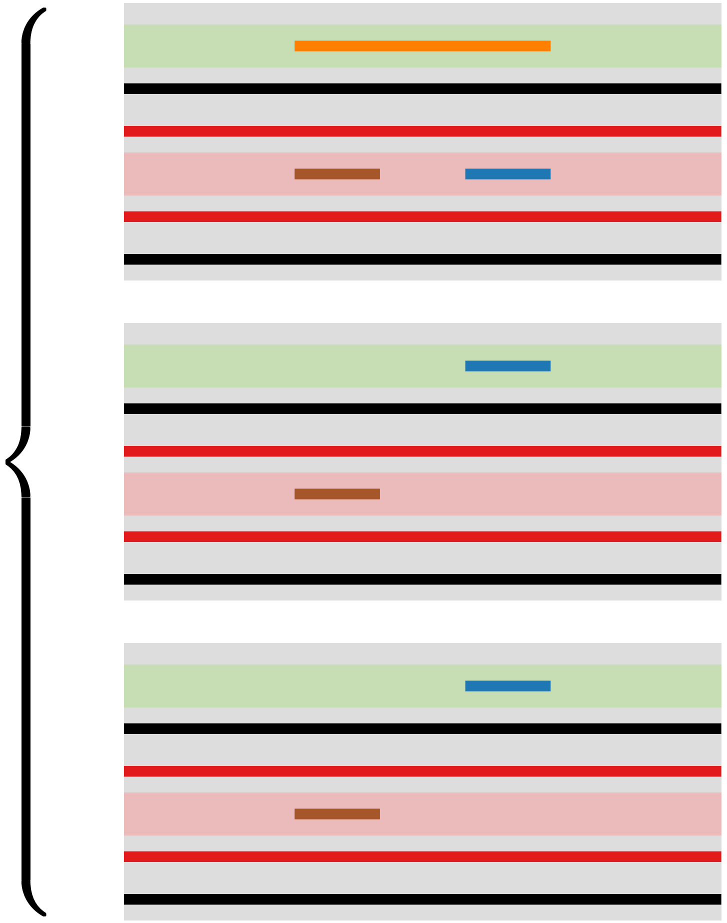

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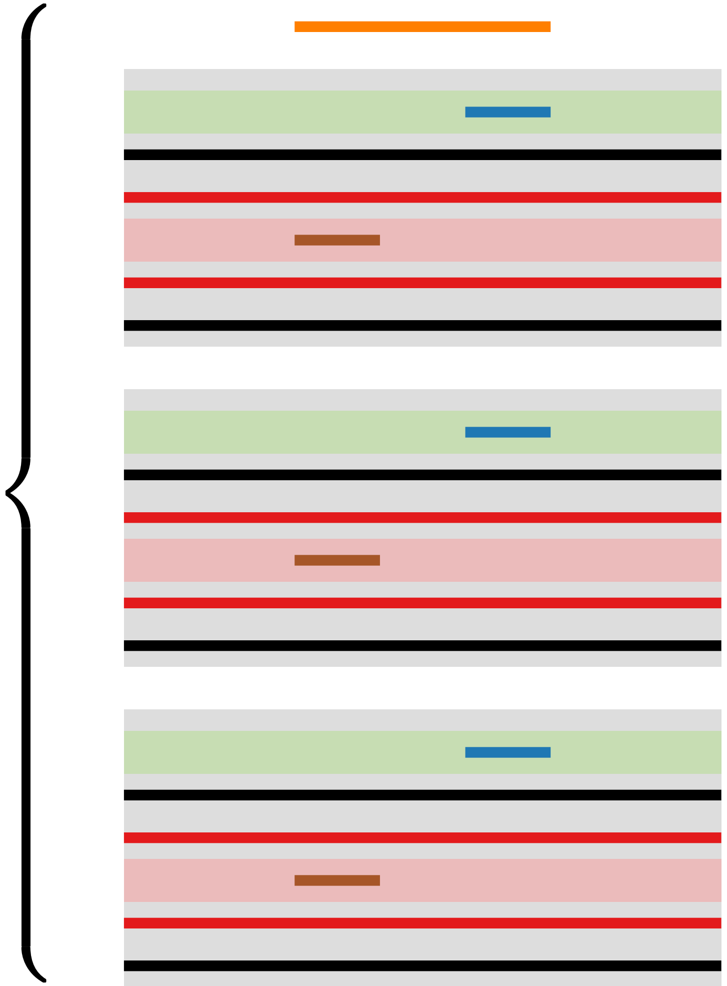

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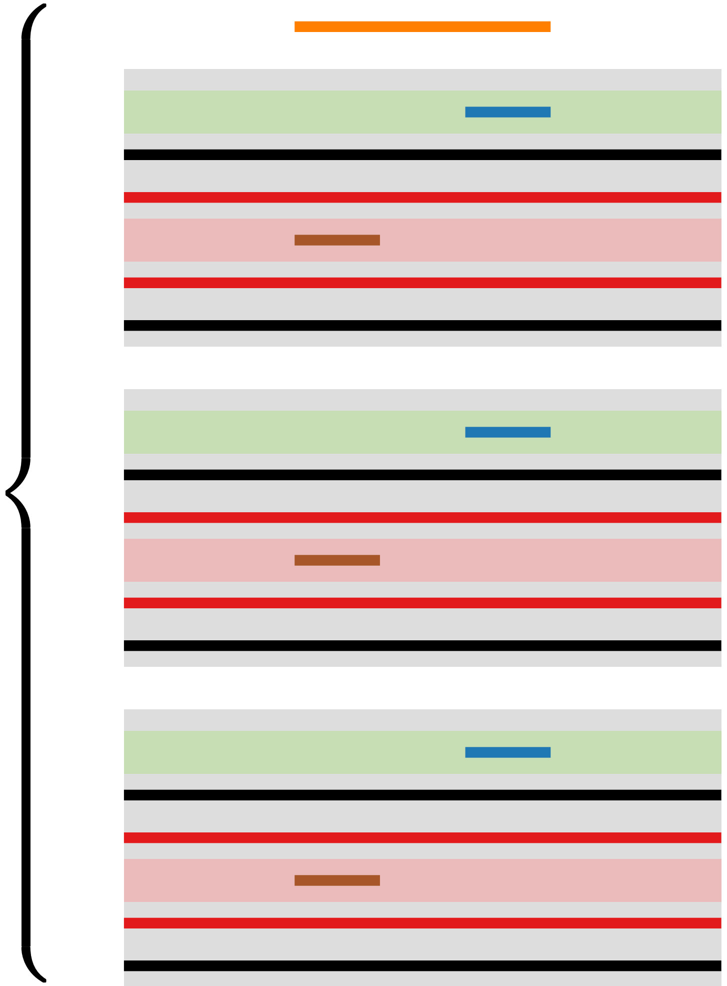
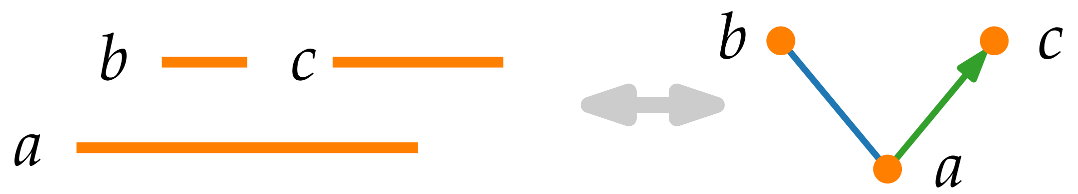


Φ is satisfiable $\Leftrightarrow$ G_Φ admits a coloring with $6n$ colors

$\square$

Conclusion and Open Problems



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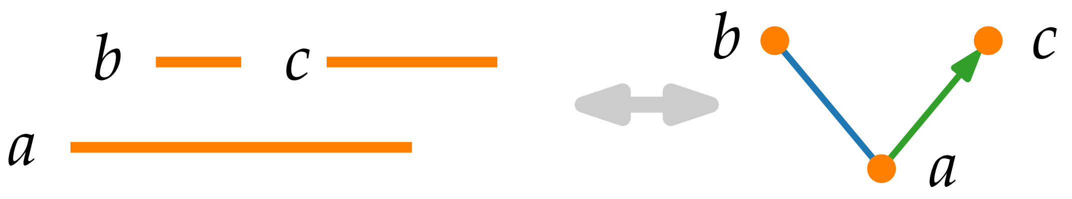

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- A simple greedy algorithm colors these graphs optimally in $O(n \log n)$ time.
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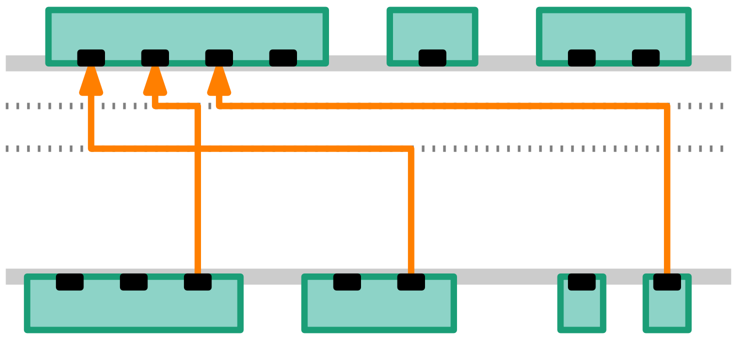
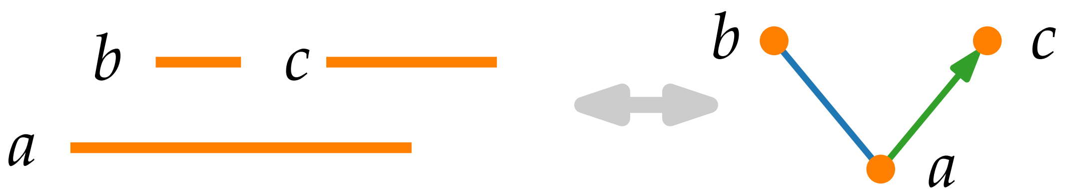
- In layered graph drawing, this corresponds to routing "left-going" edges orthogonally to the fewest horizontal tracks. (Symmetrically "right-going".)



Conclusion and Open Problems



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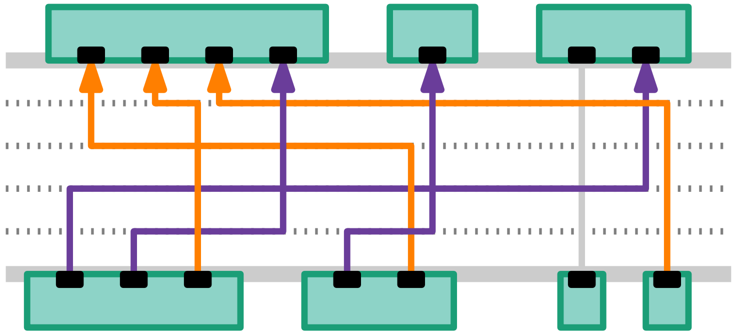
$\Rightarrow$ Combining the drawings of left-going and right-going edges yields a 2-approximation for the number of tracks. (bidirectional interval graphs)



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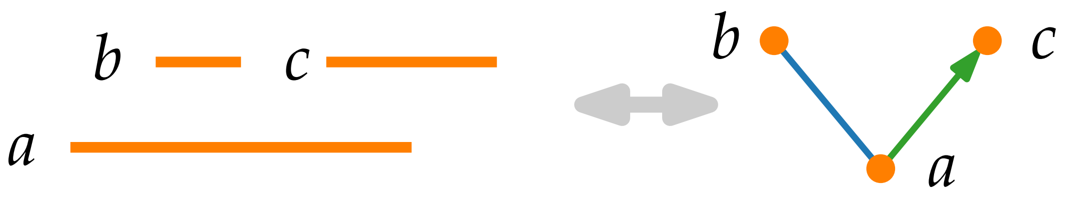

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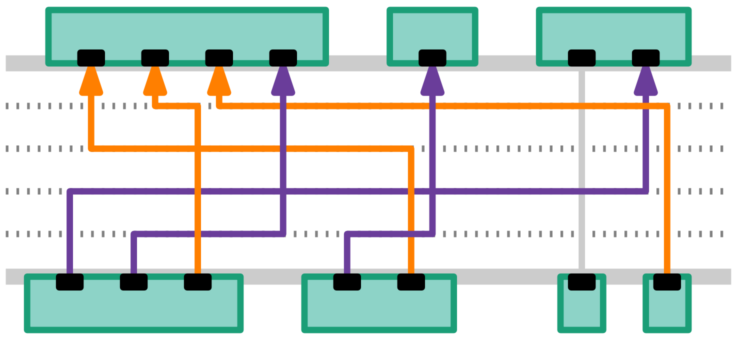

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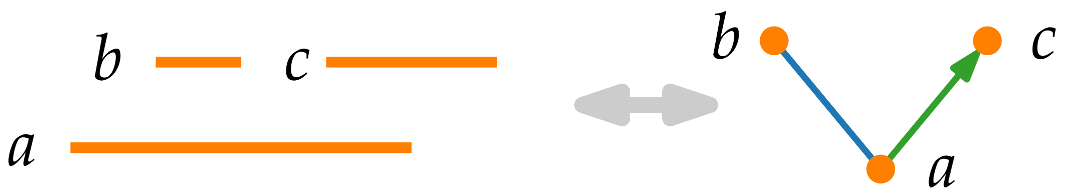

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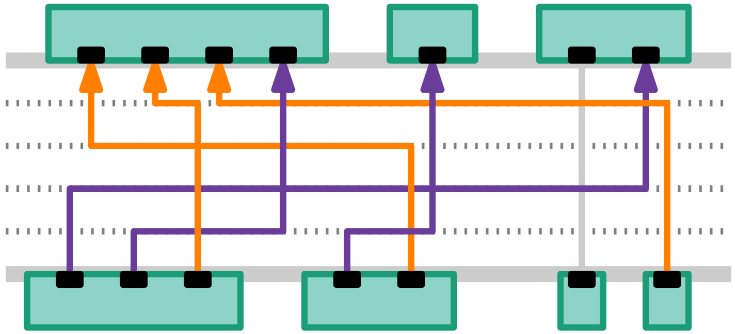

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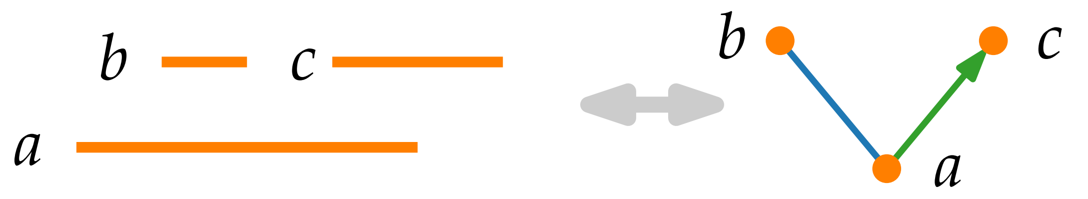

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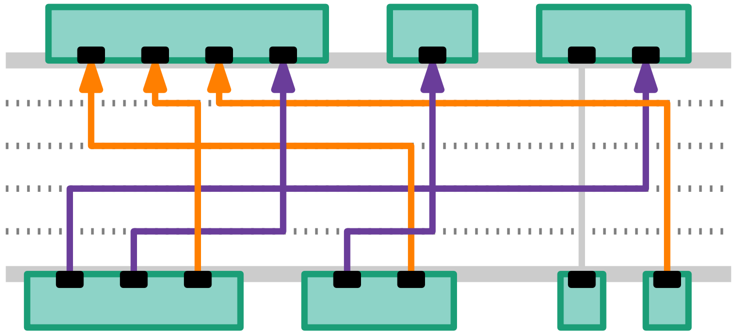

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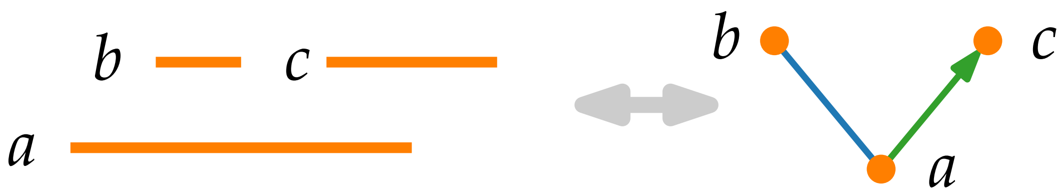

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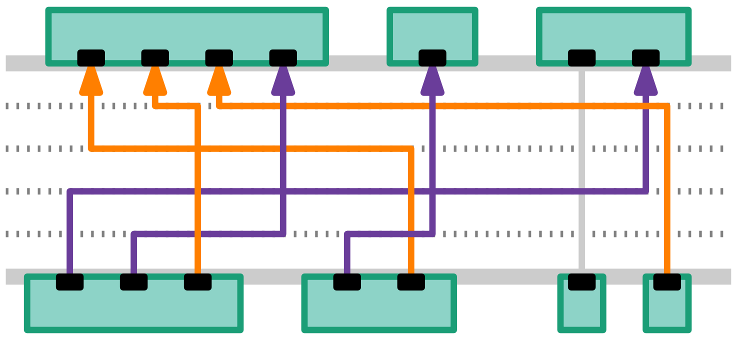

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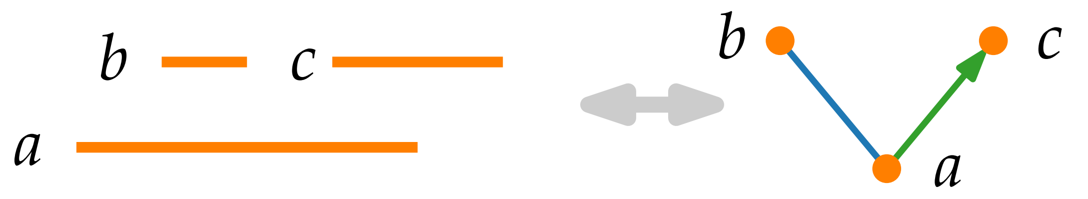

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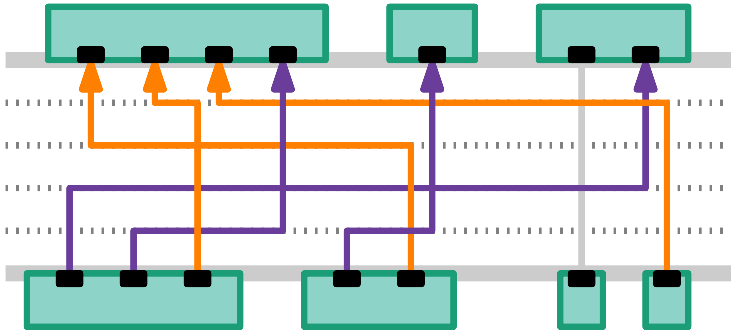

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