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Approximation Algorithms for the Maximum Leaf Spanning Tree Problem on Acyclic Digraphs

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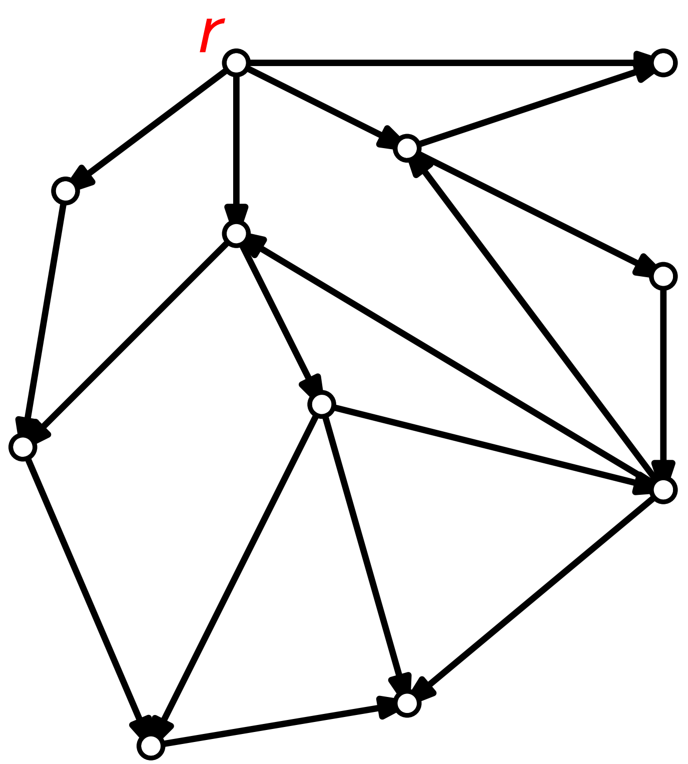
WAOA '11

Lehrstuhl für Informatik I
Universität Würzburg, Germany

Problem Definition

Given a **digraph** G with **root** r,
find an r-rooted **spanning tree**
with the **maximum number of leaves**.



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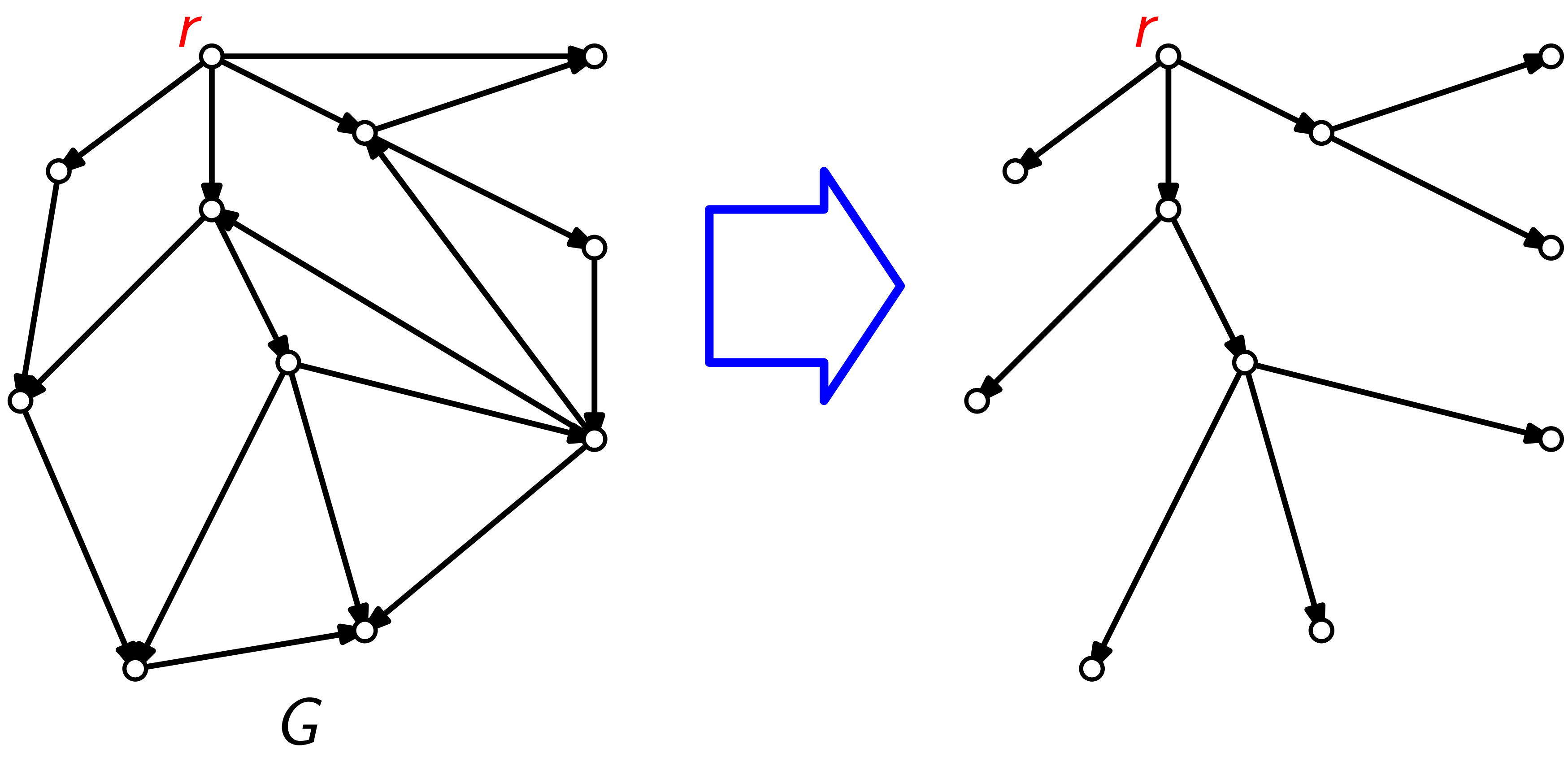

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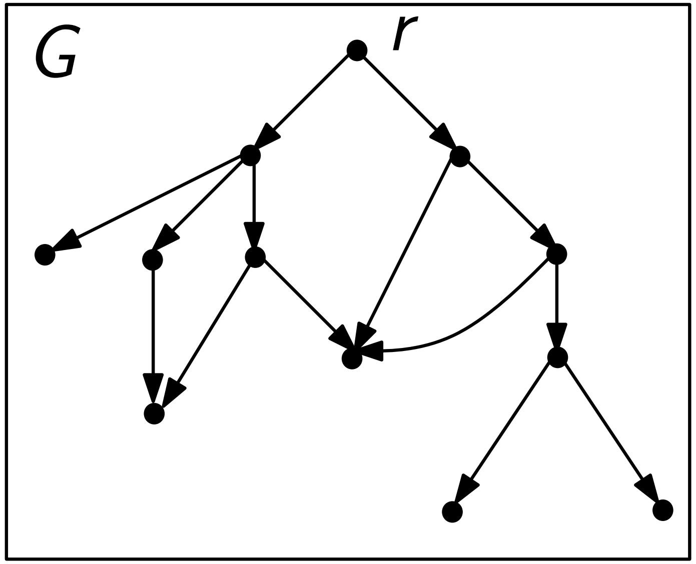

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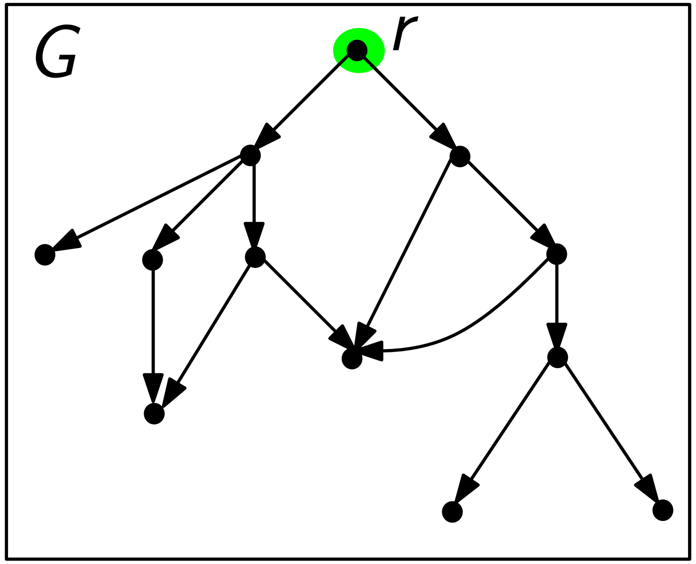

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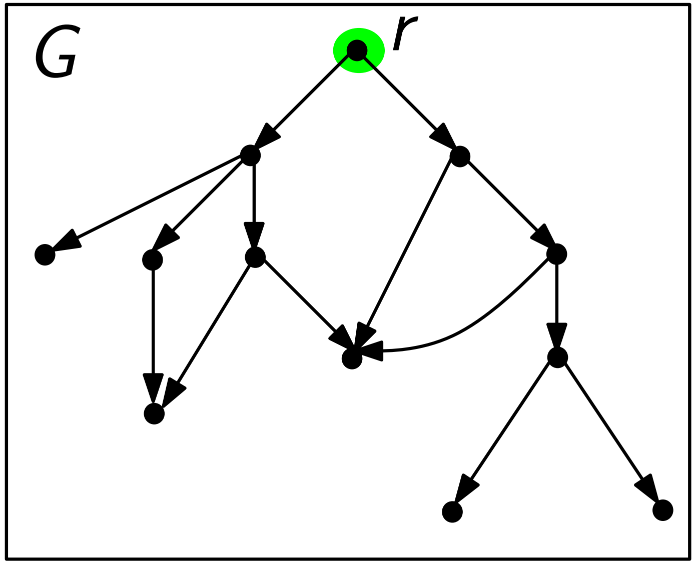

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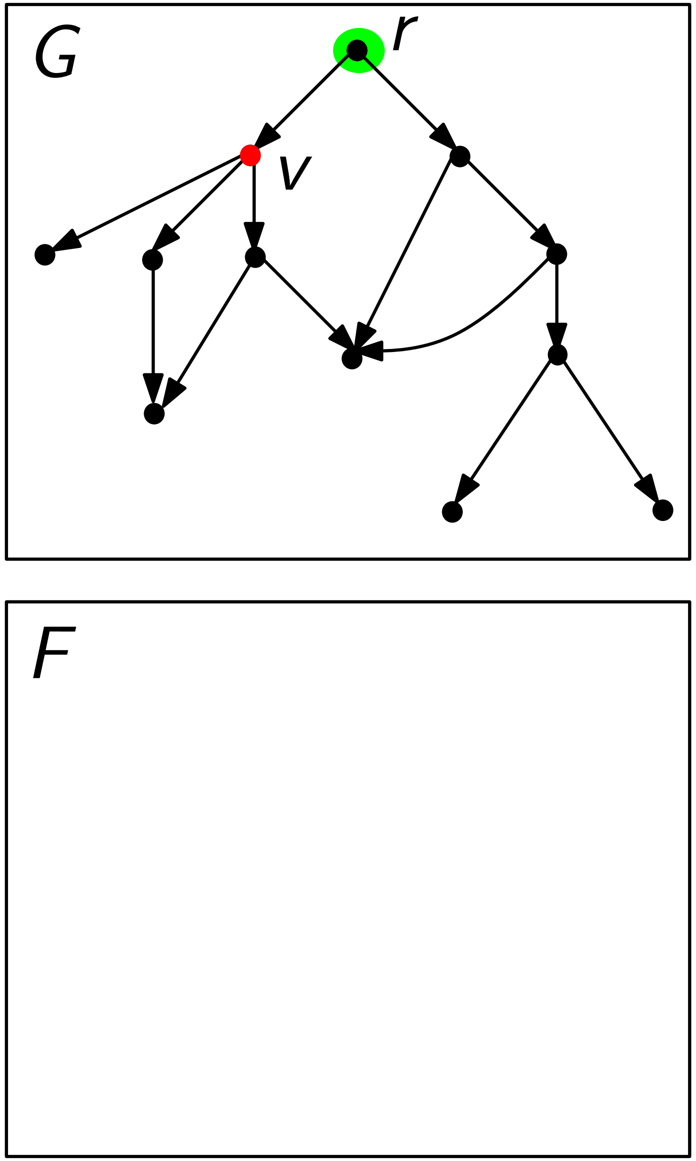

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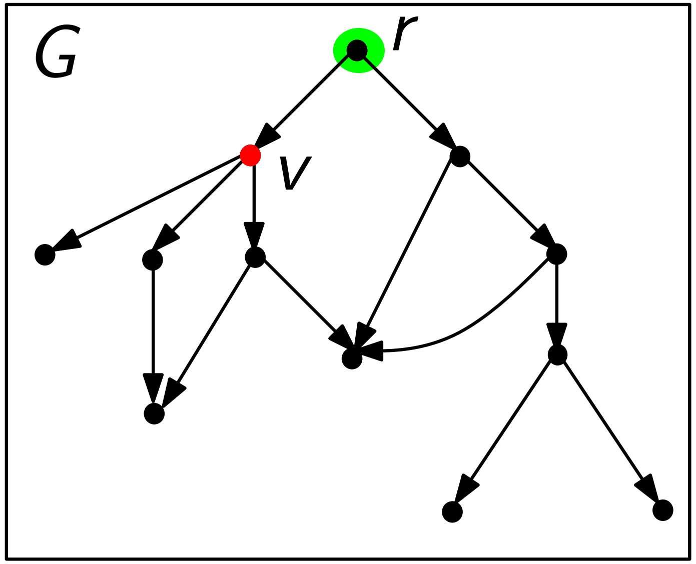

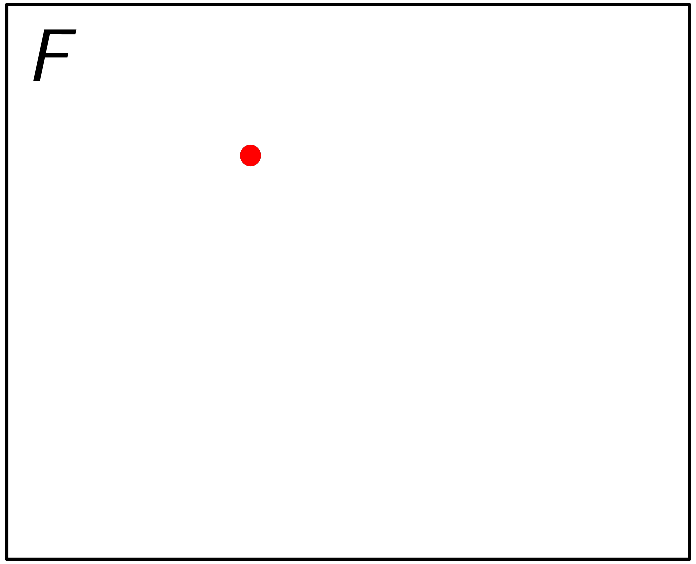

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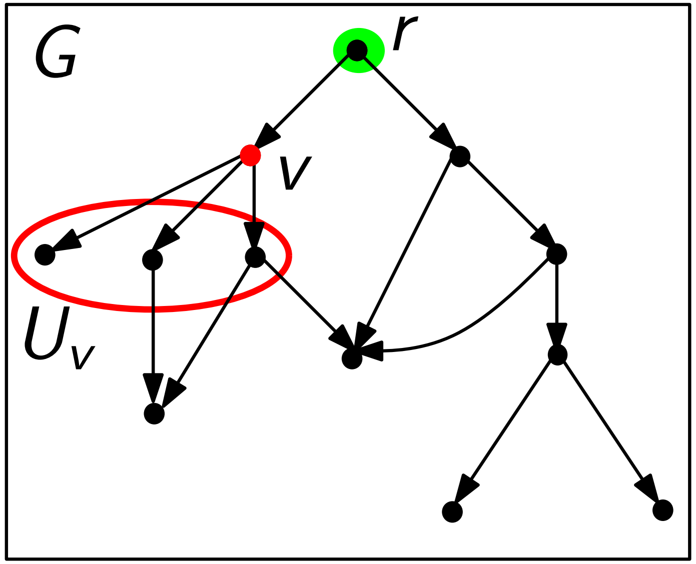

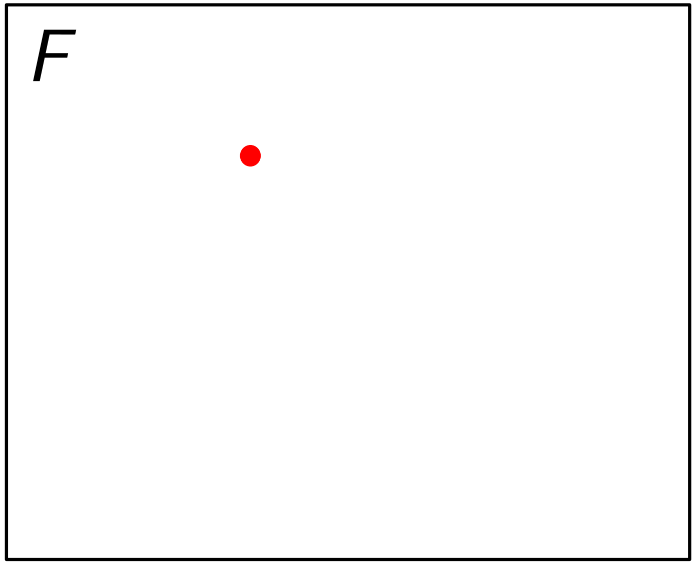

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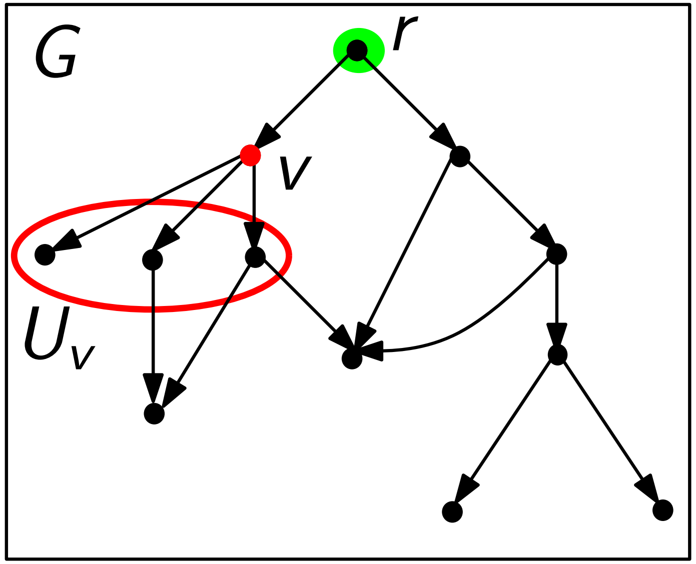

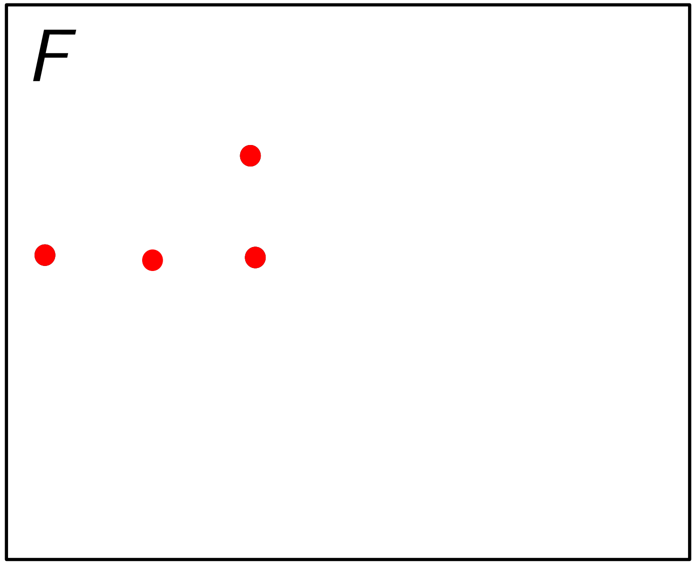

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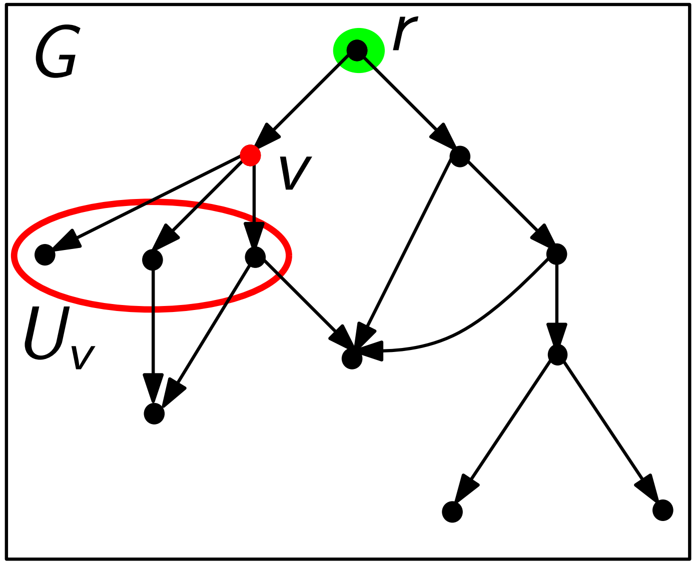

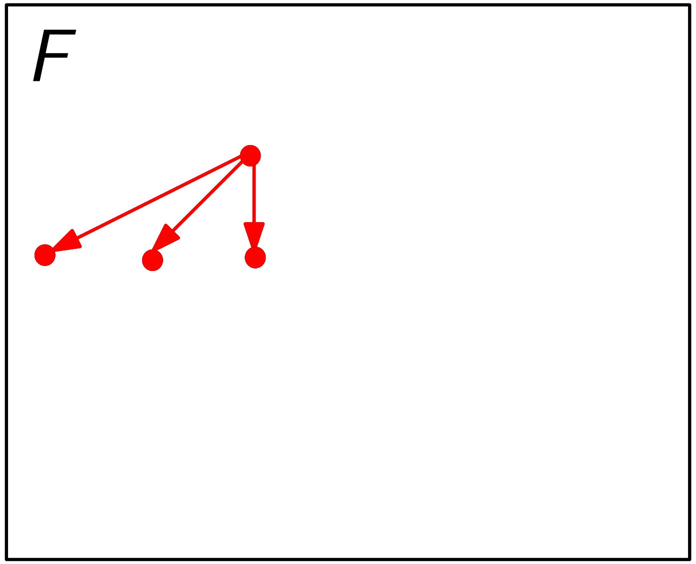

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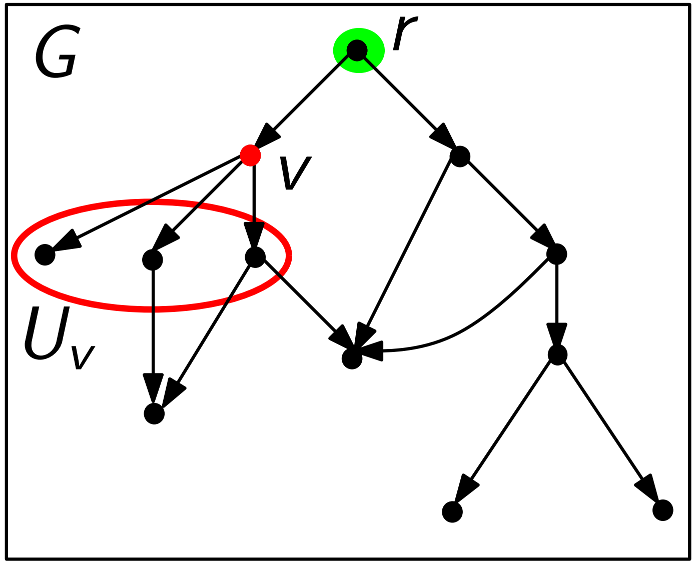

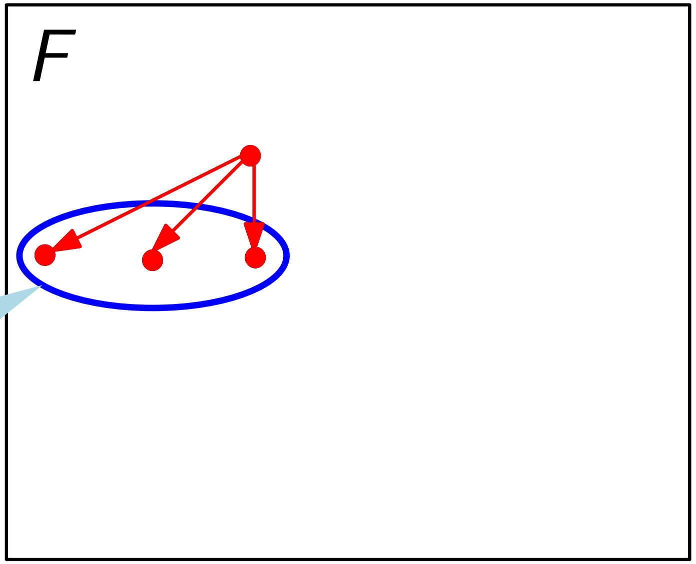

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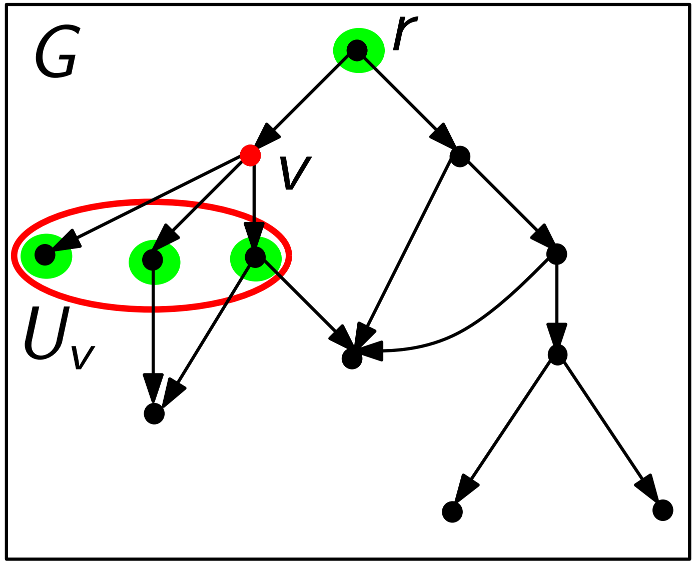

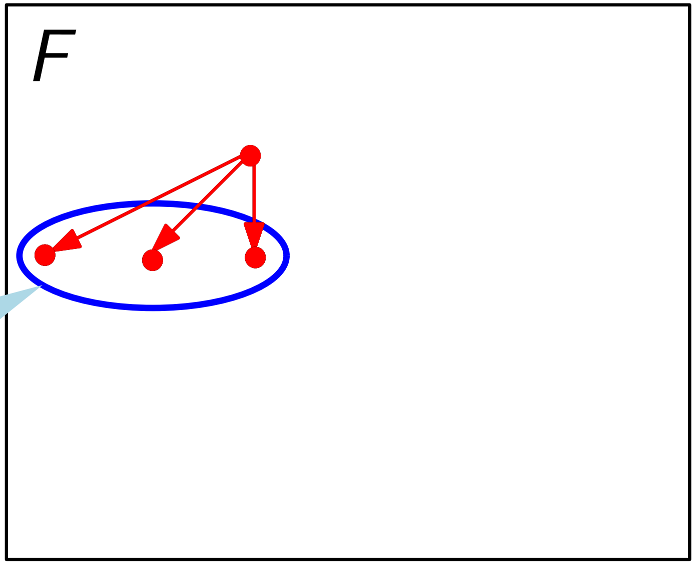

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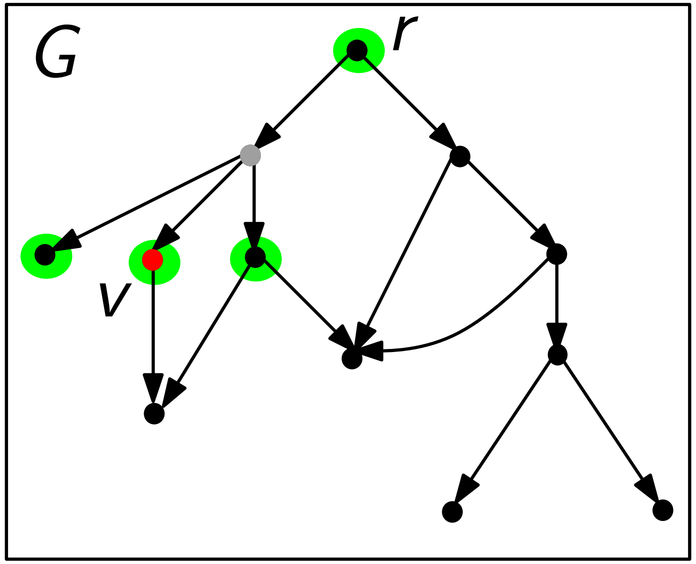

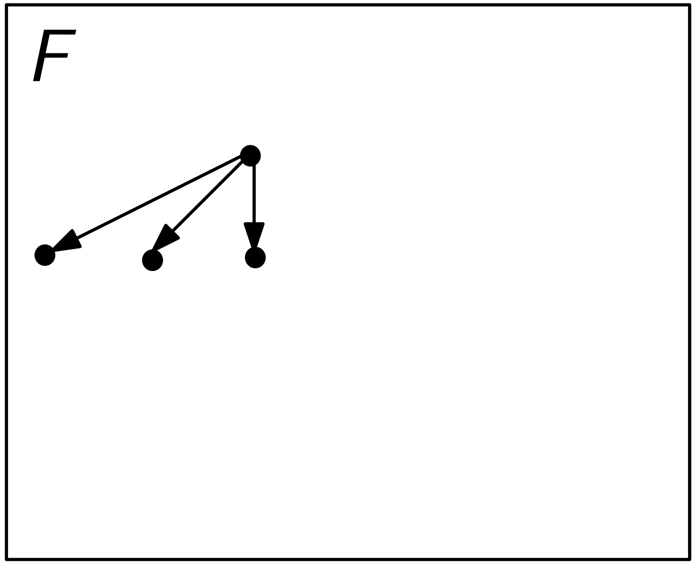

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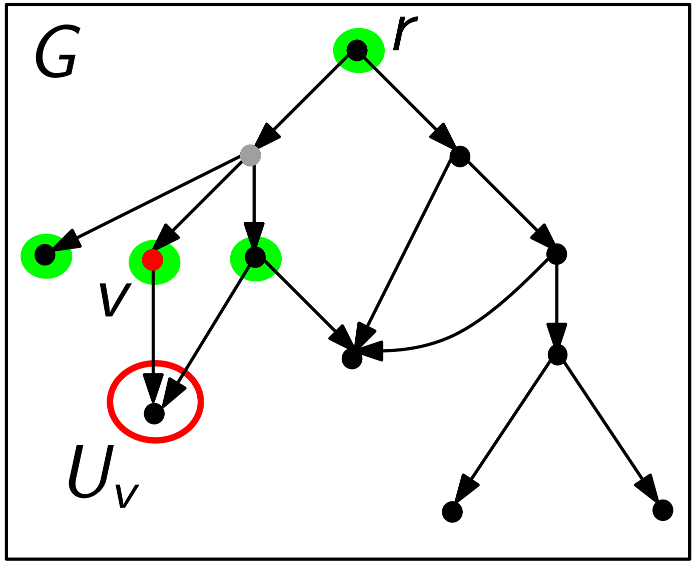

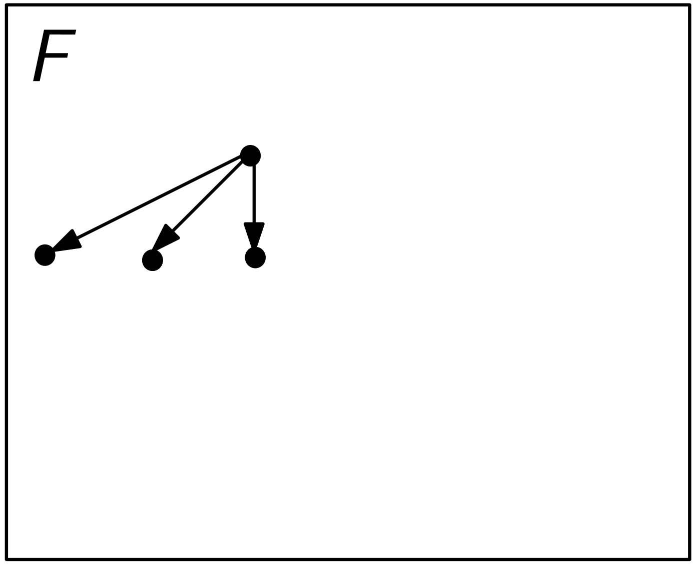

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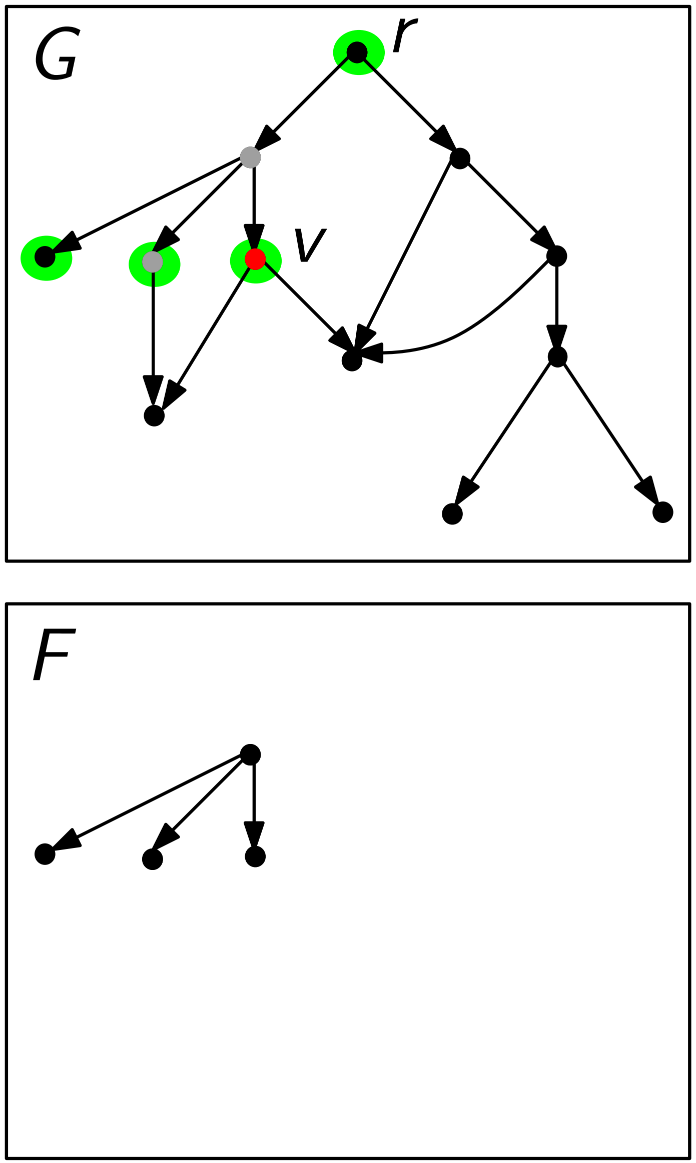

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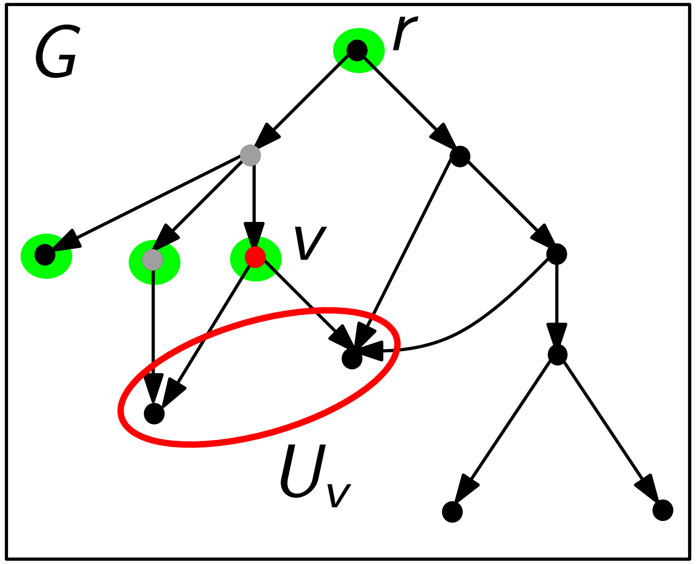

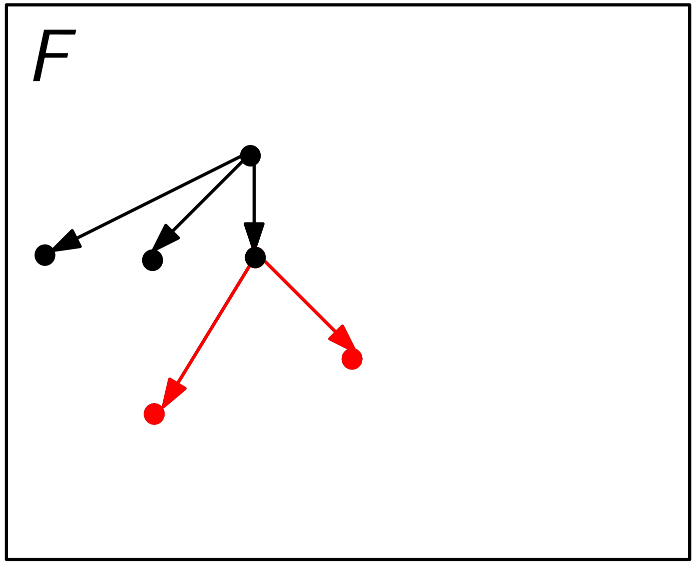

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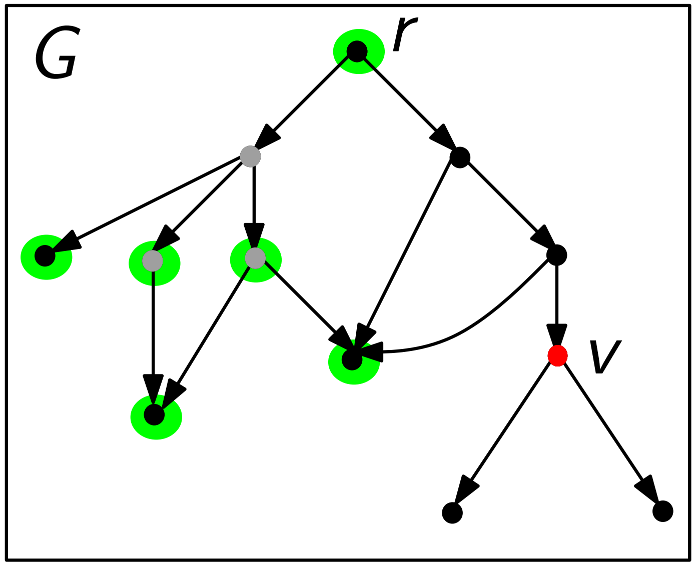

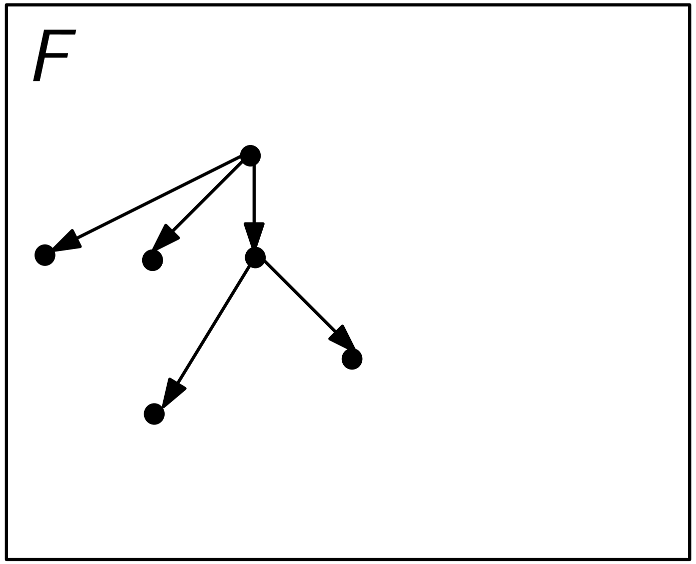

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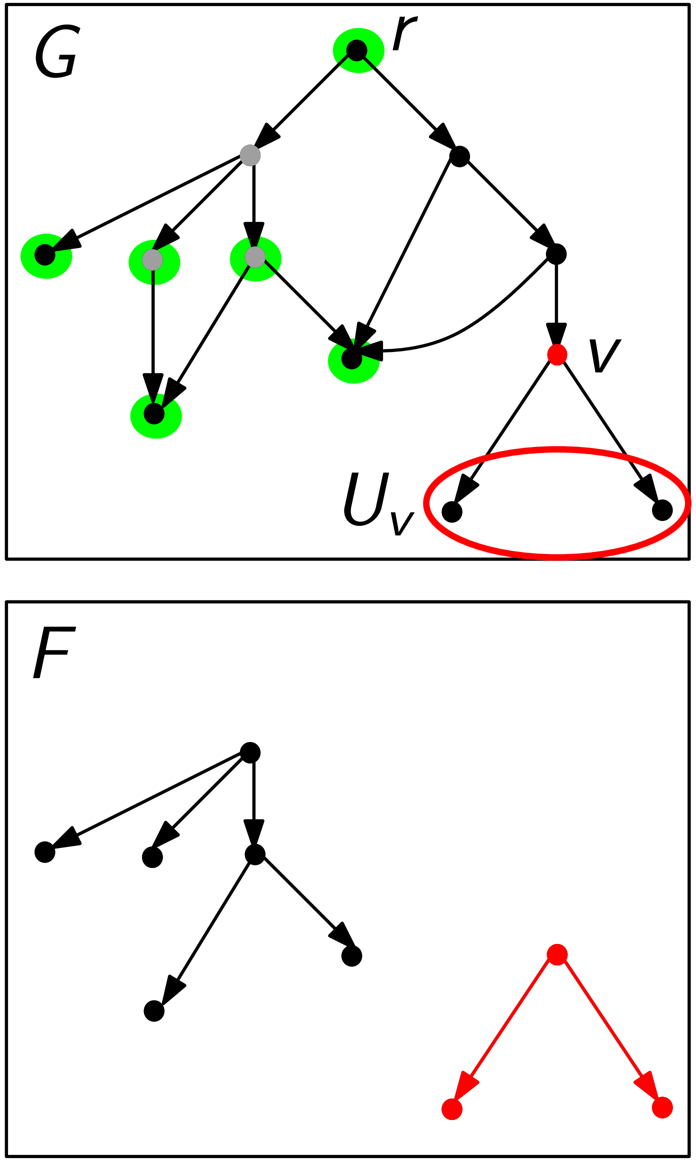

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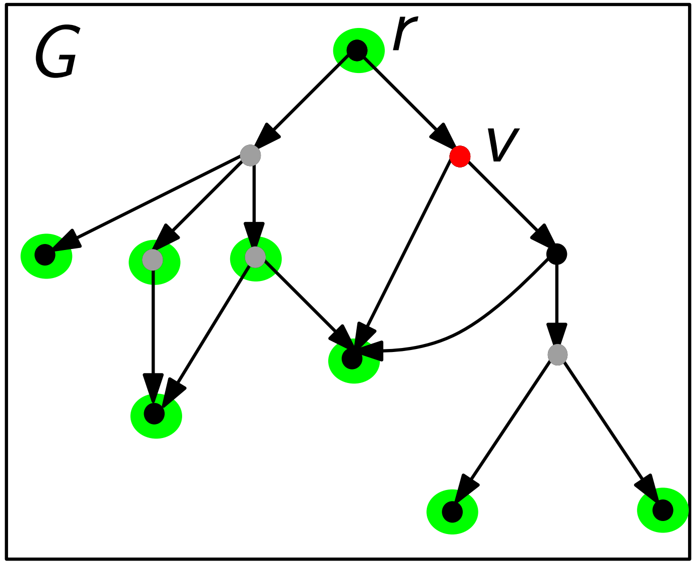

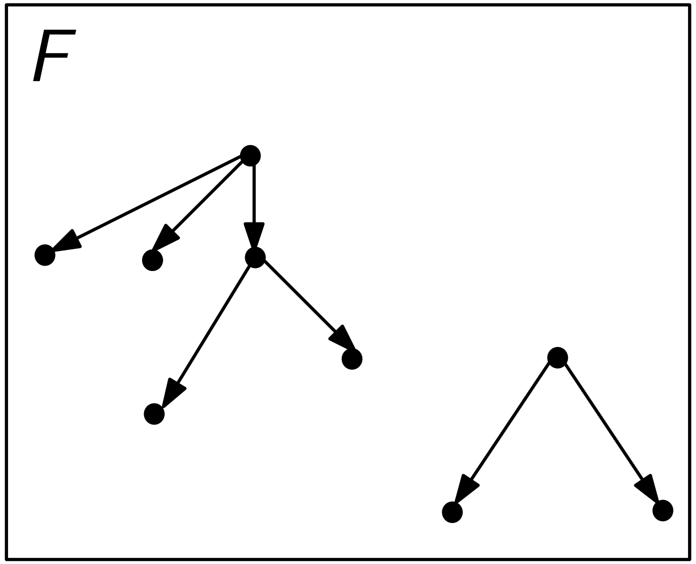

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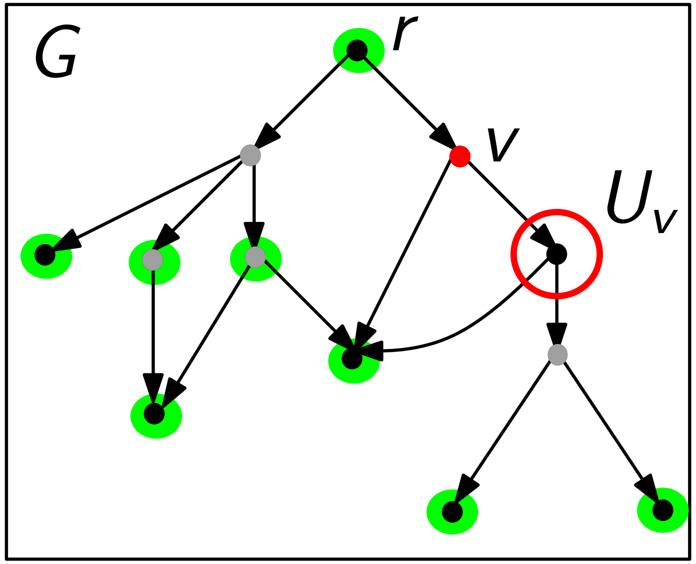

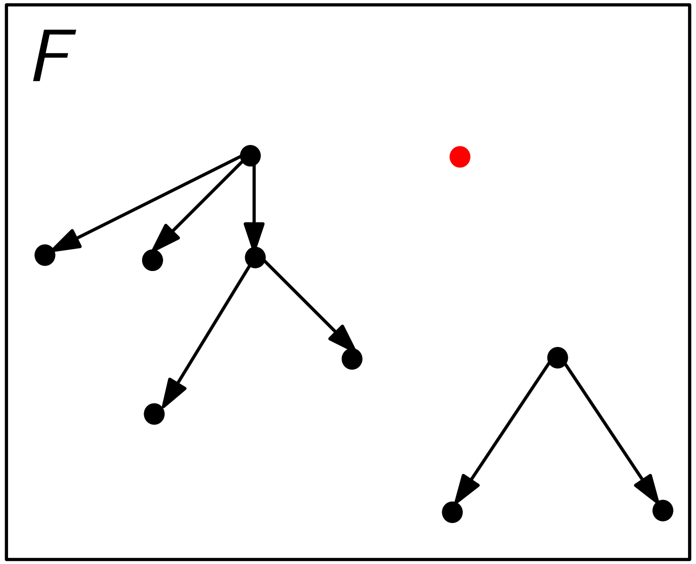

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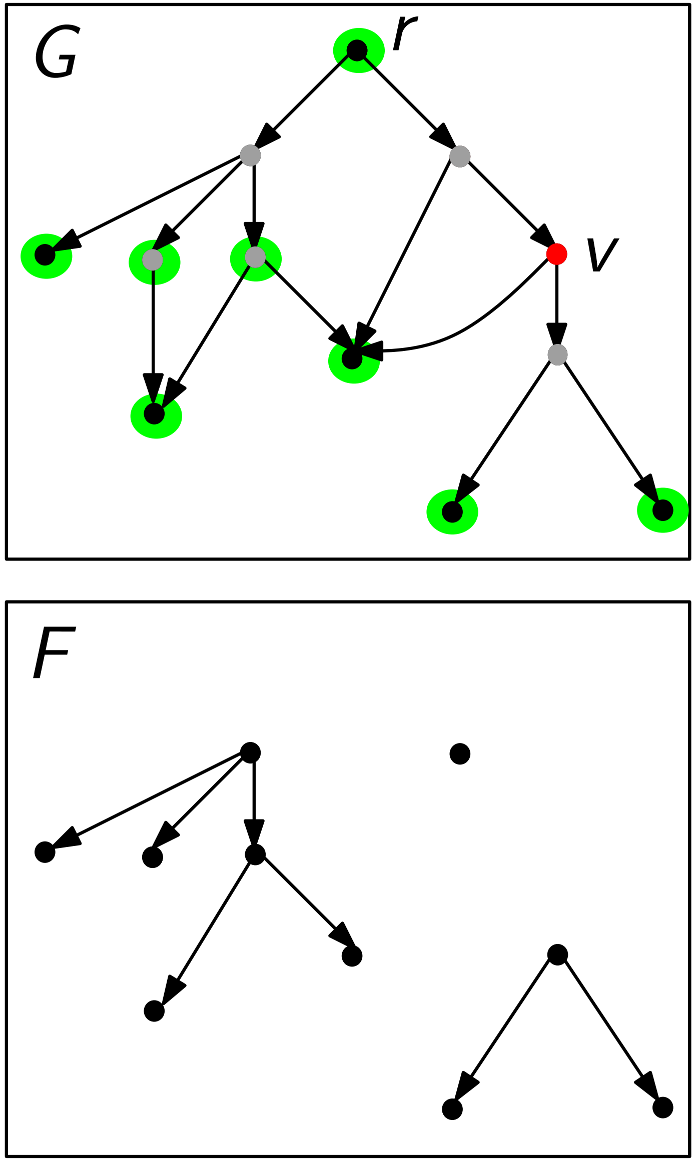

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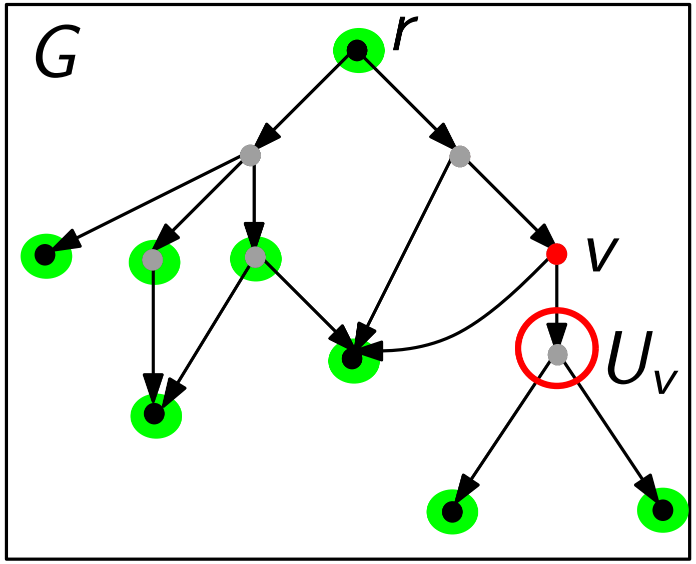

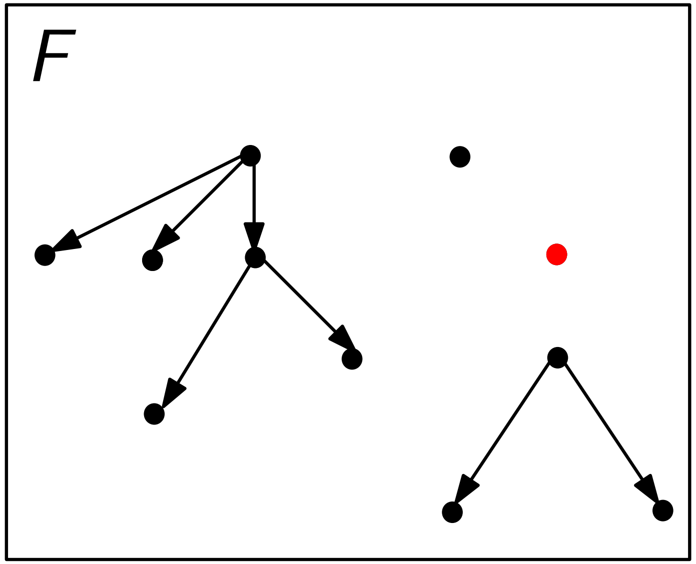

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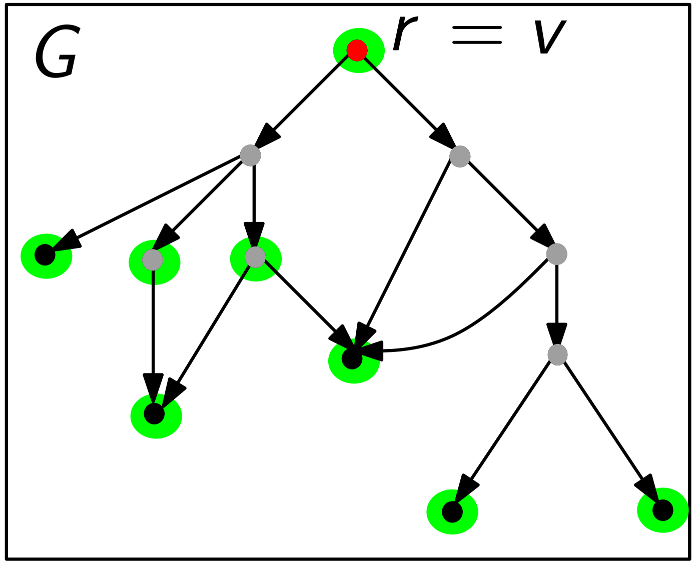

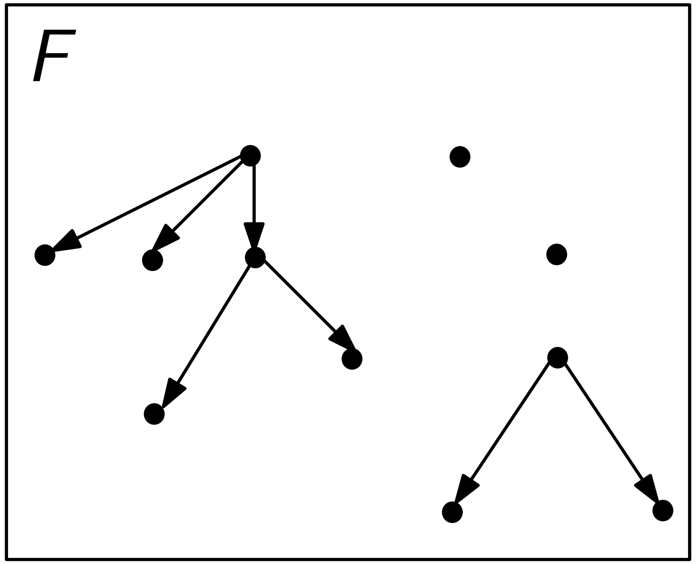

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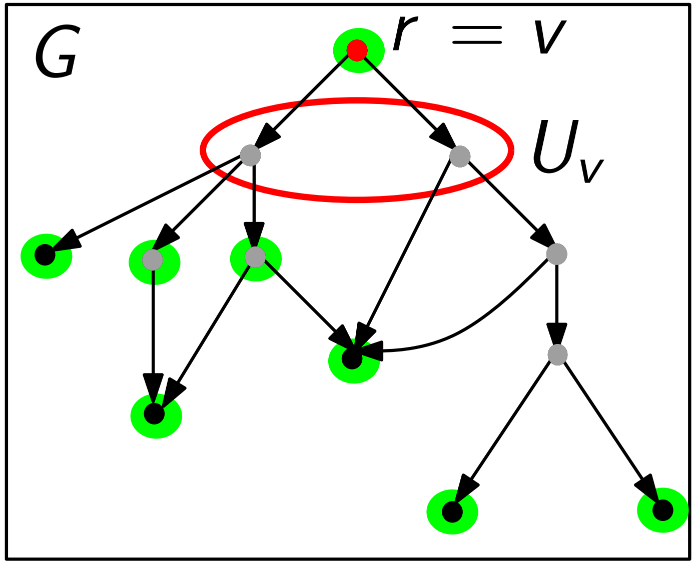

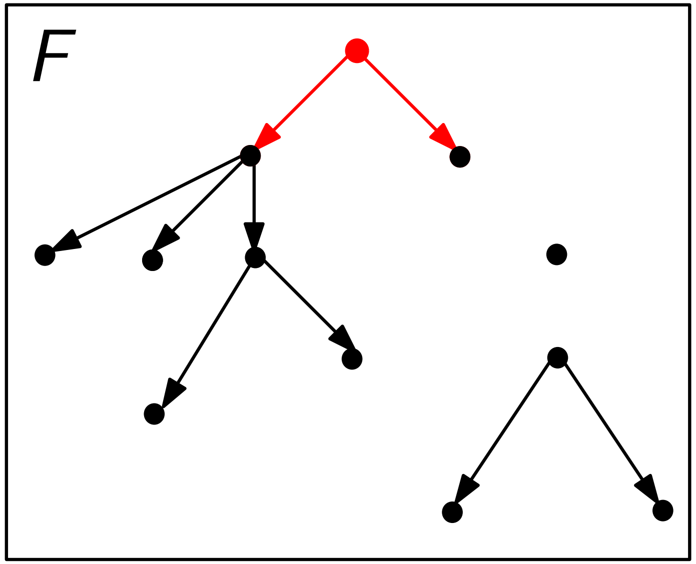

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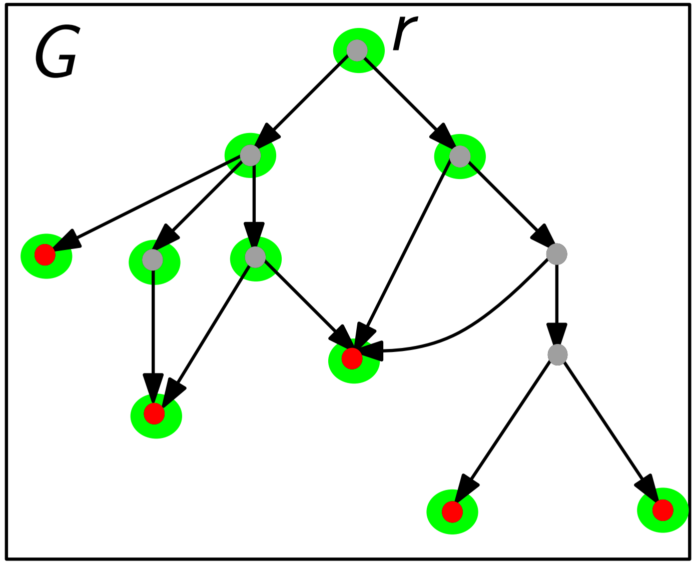

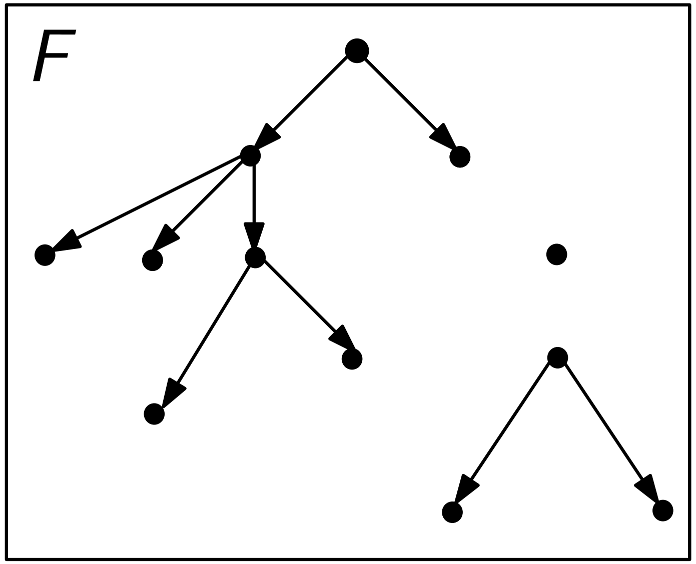

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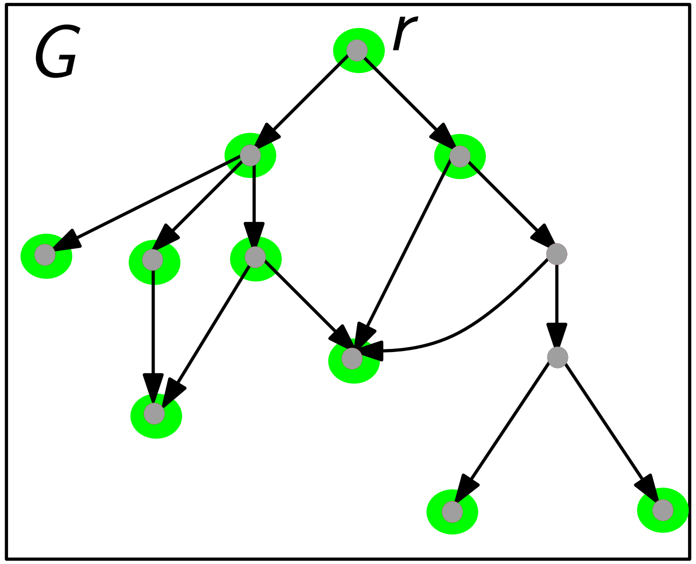

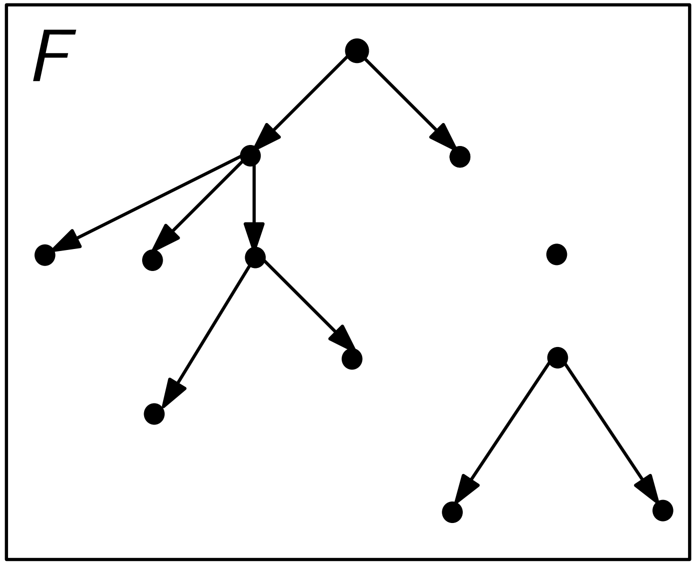

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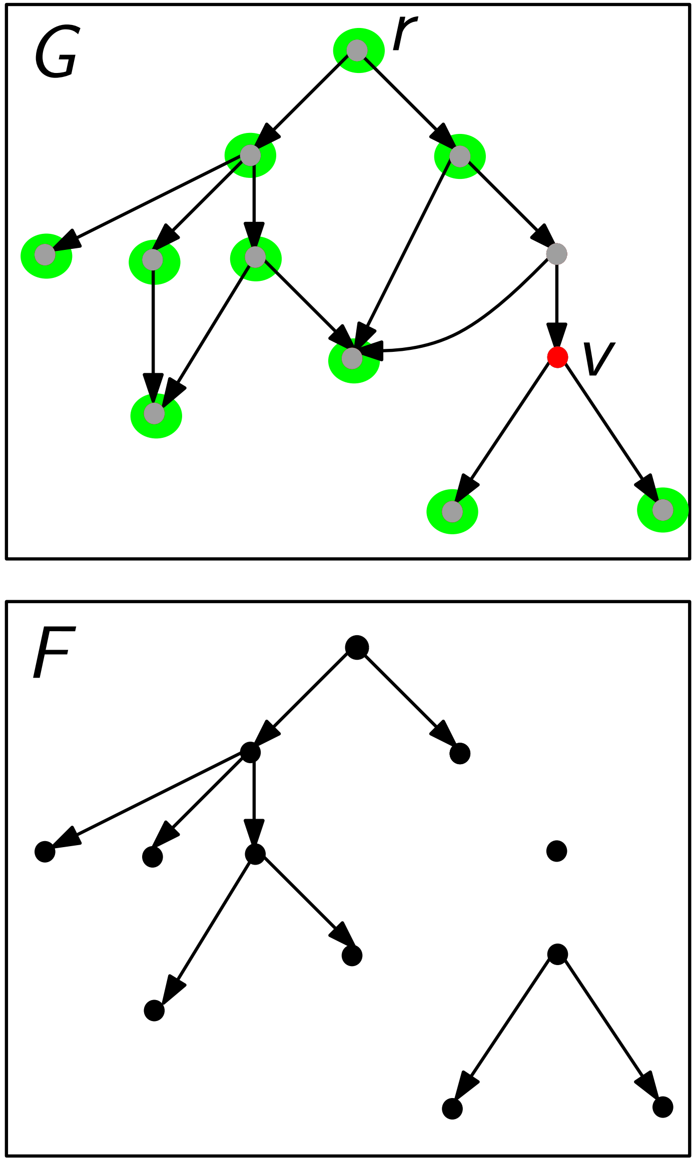

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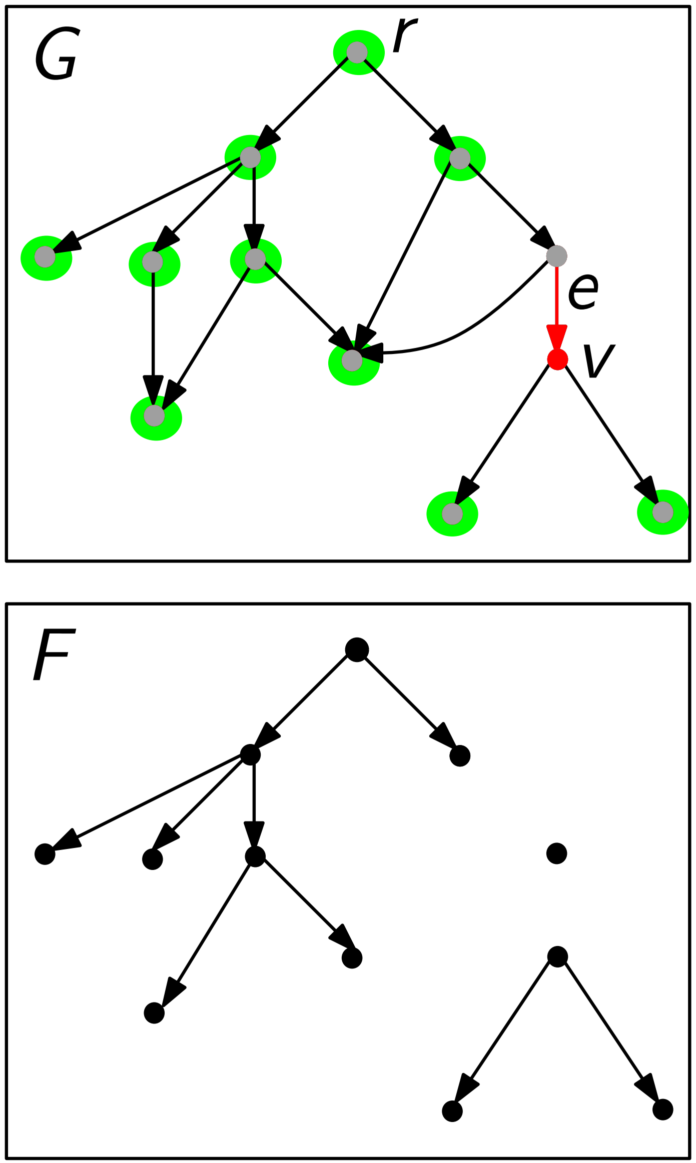

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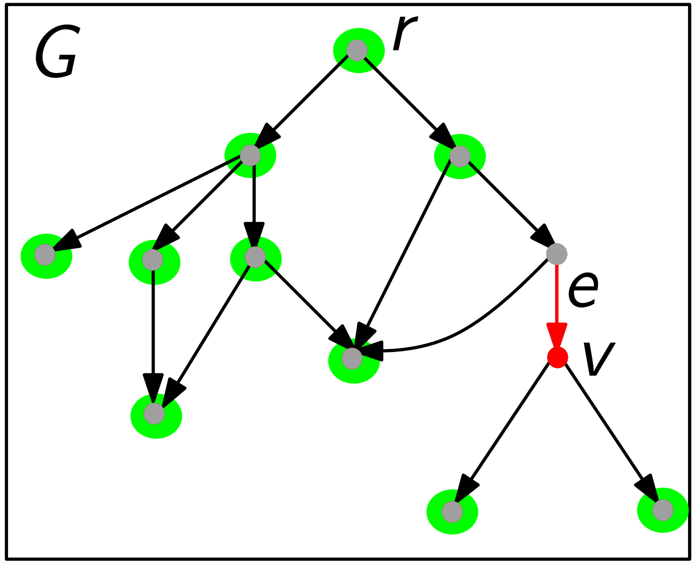

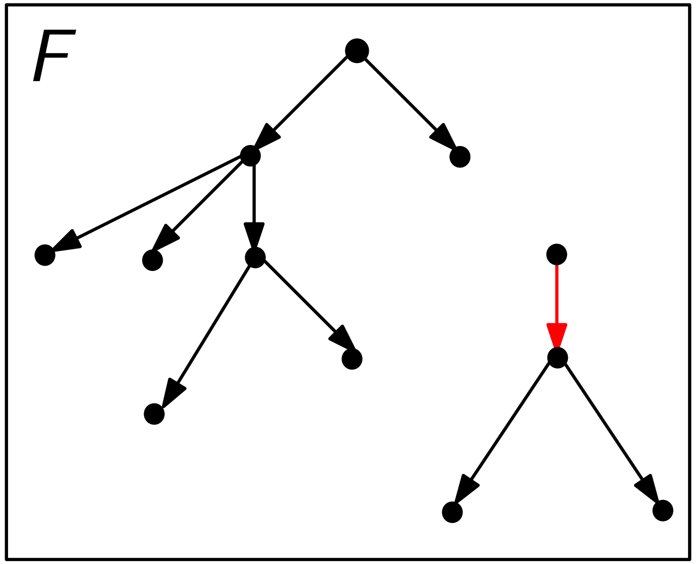

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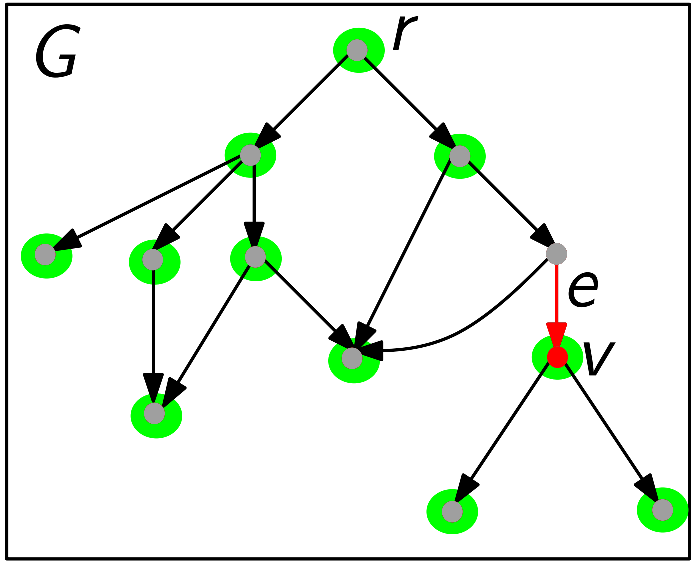

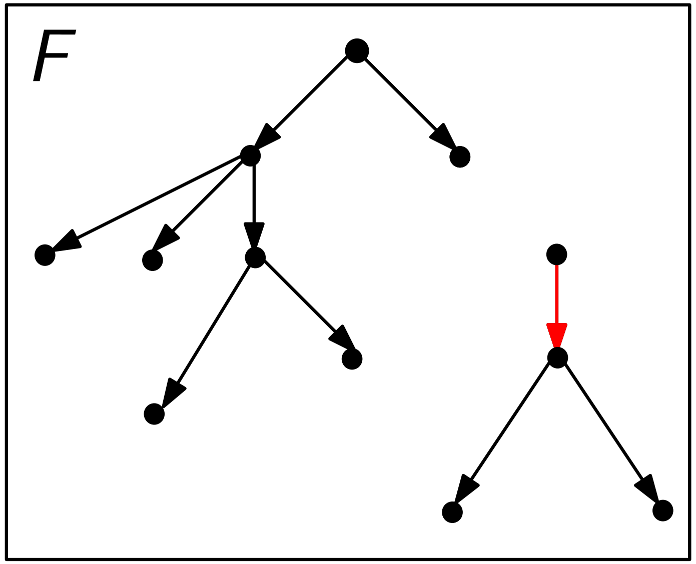

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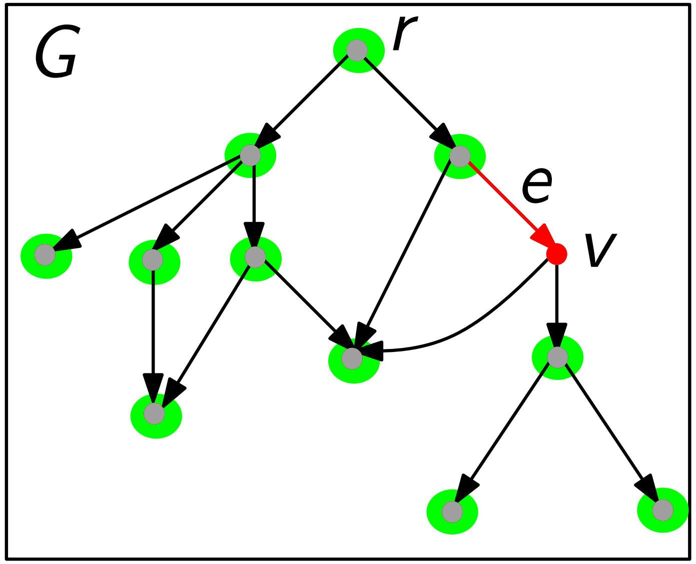

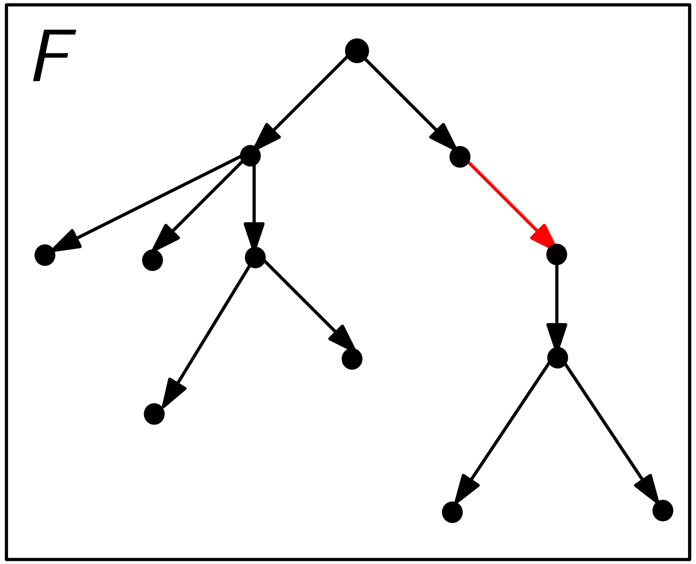

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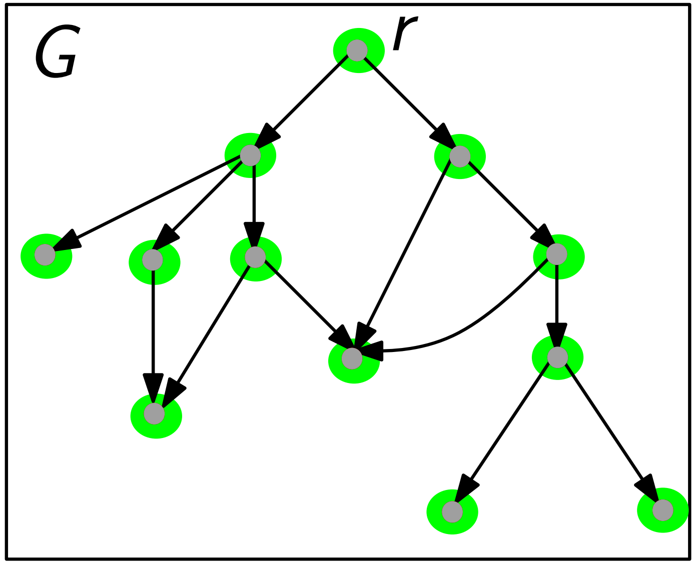

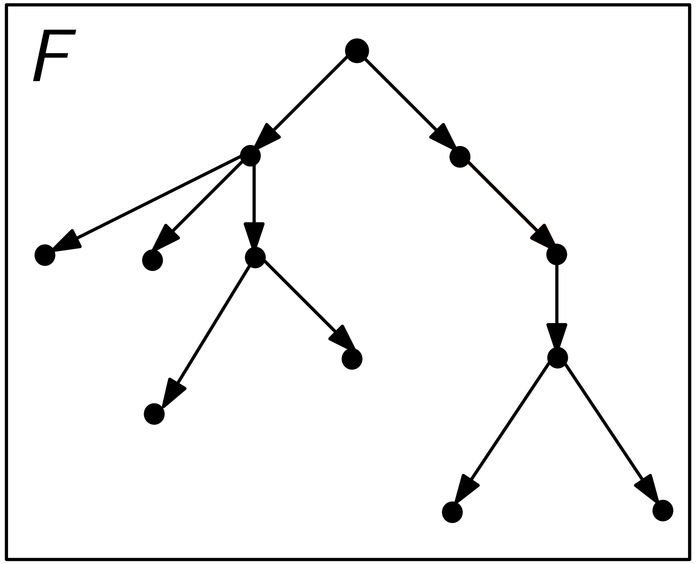

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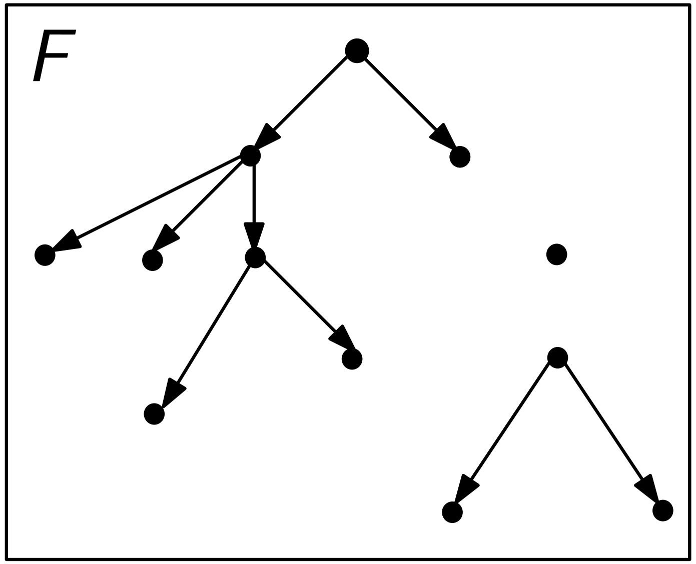
Observation. Procedure `expand` yields forest F.



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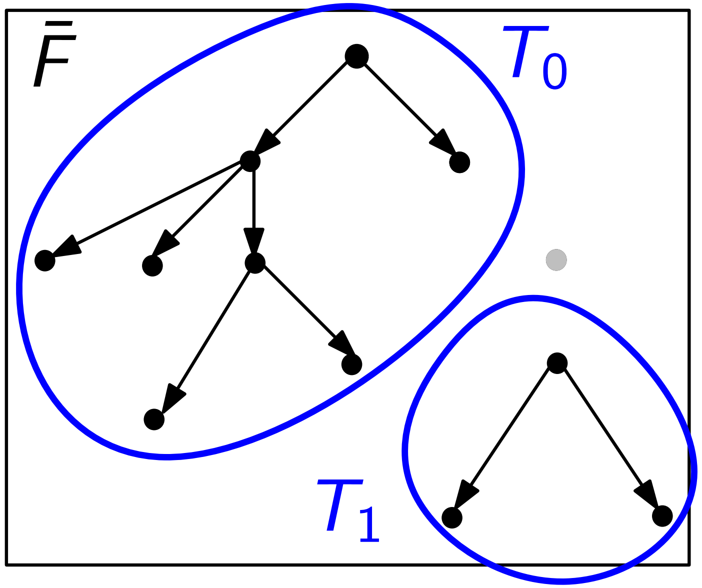

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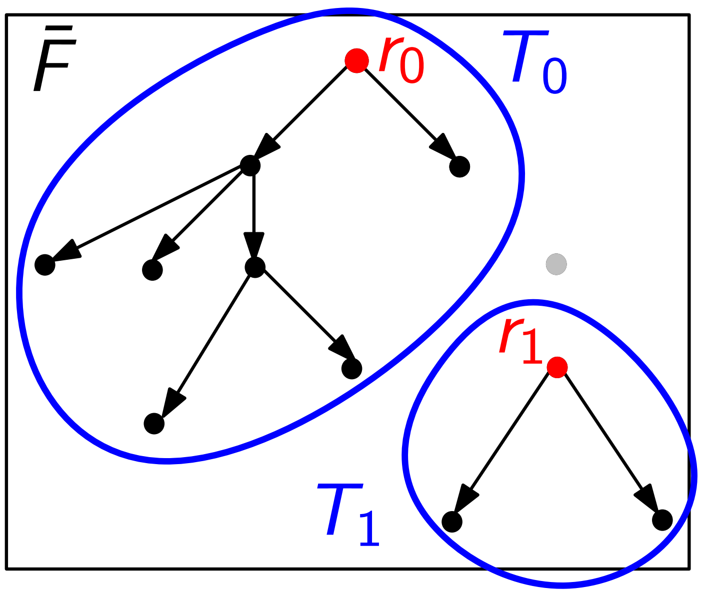

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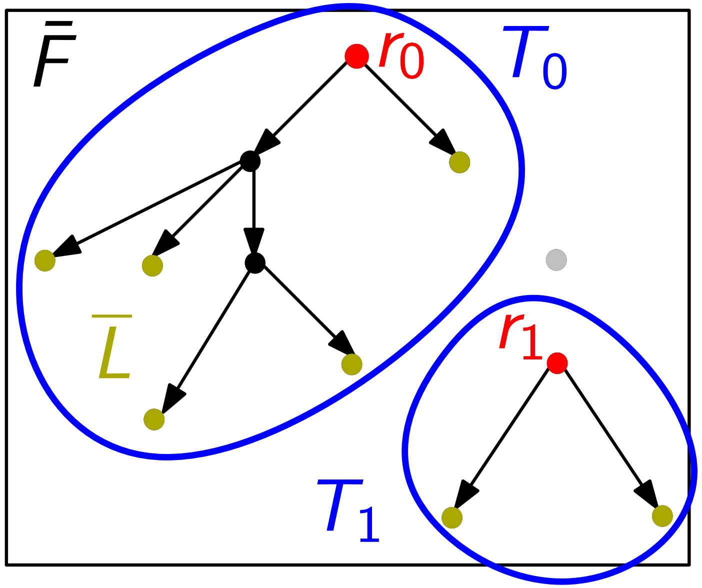

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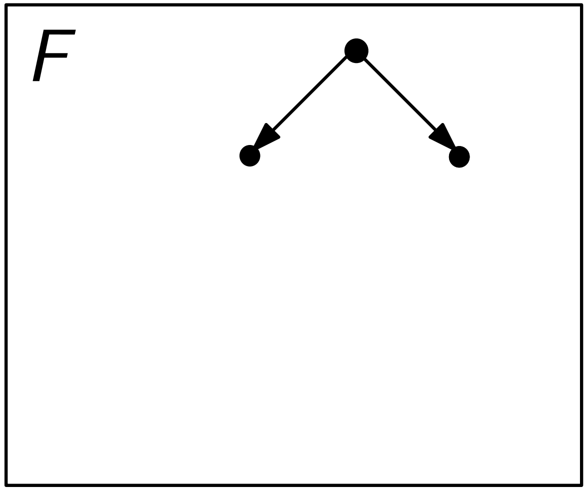

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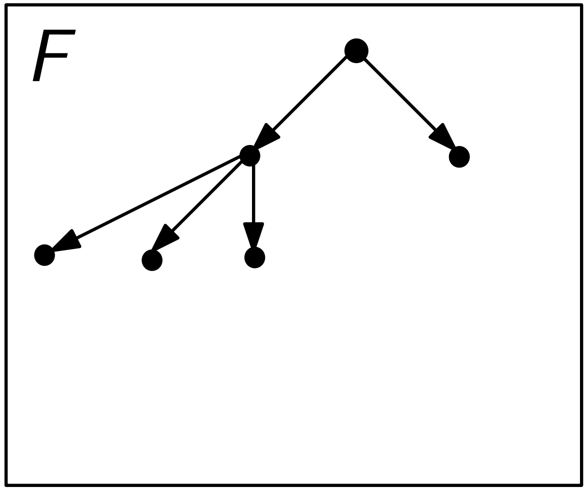

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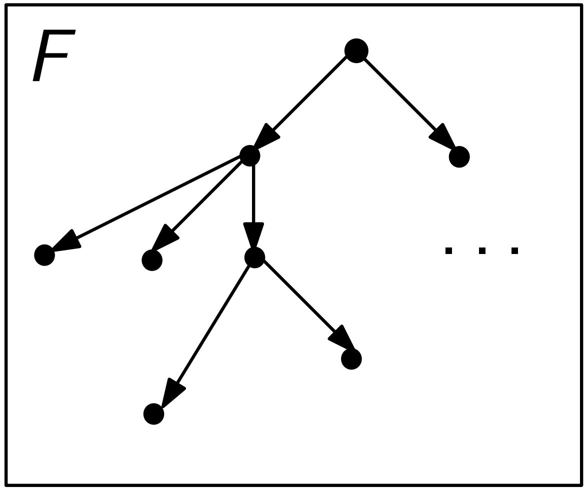

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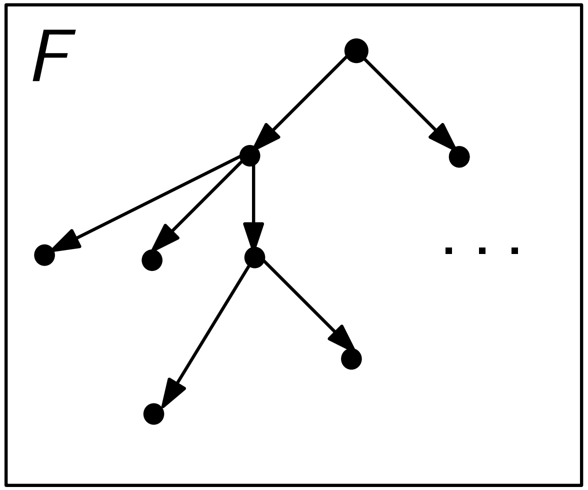

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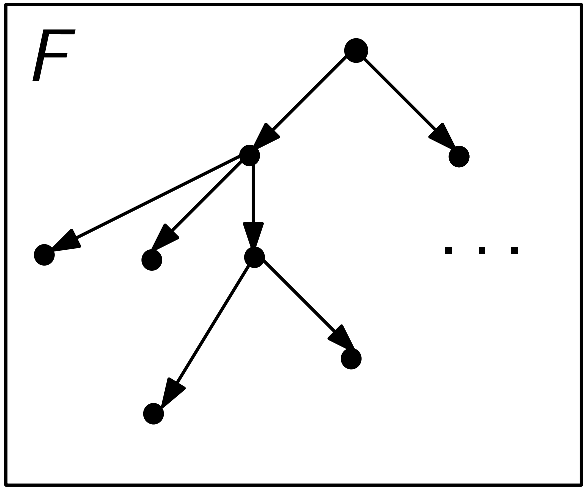

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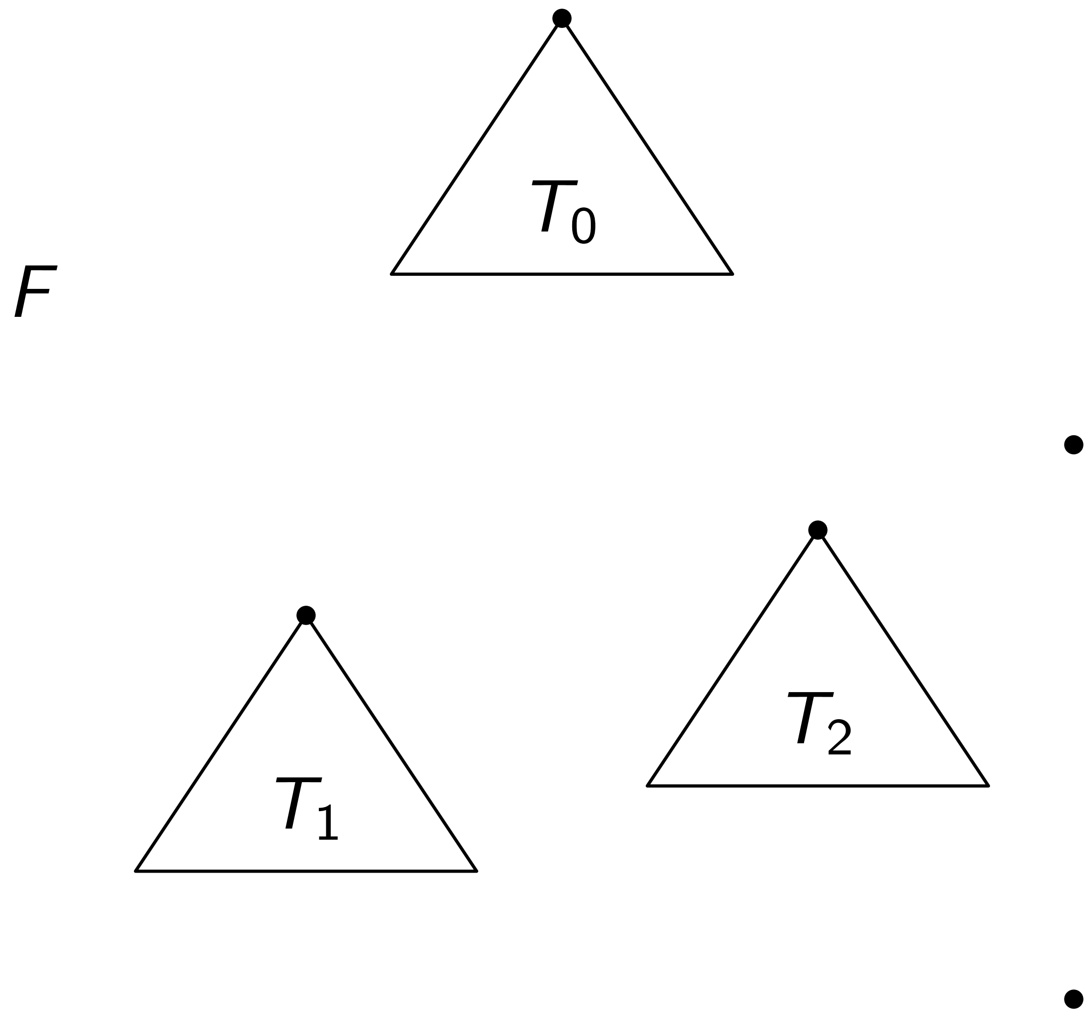

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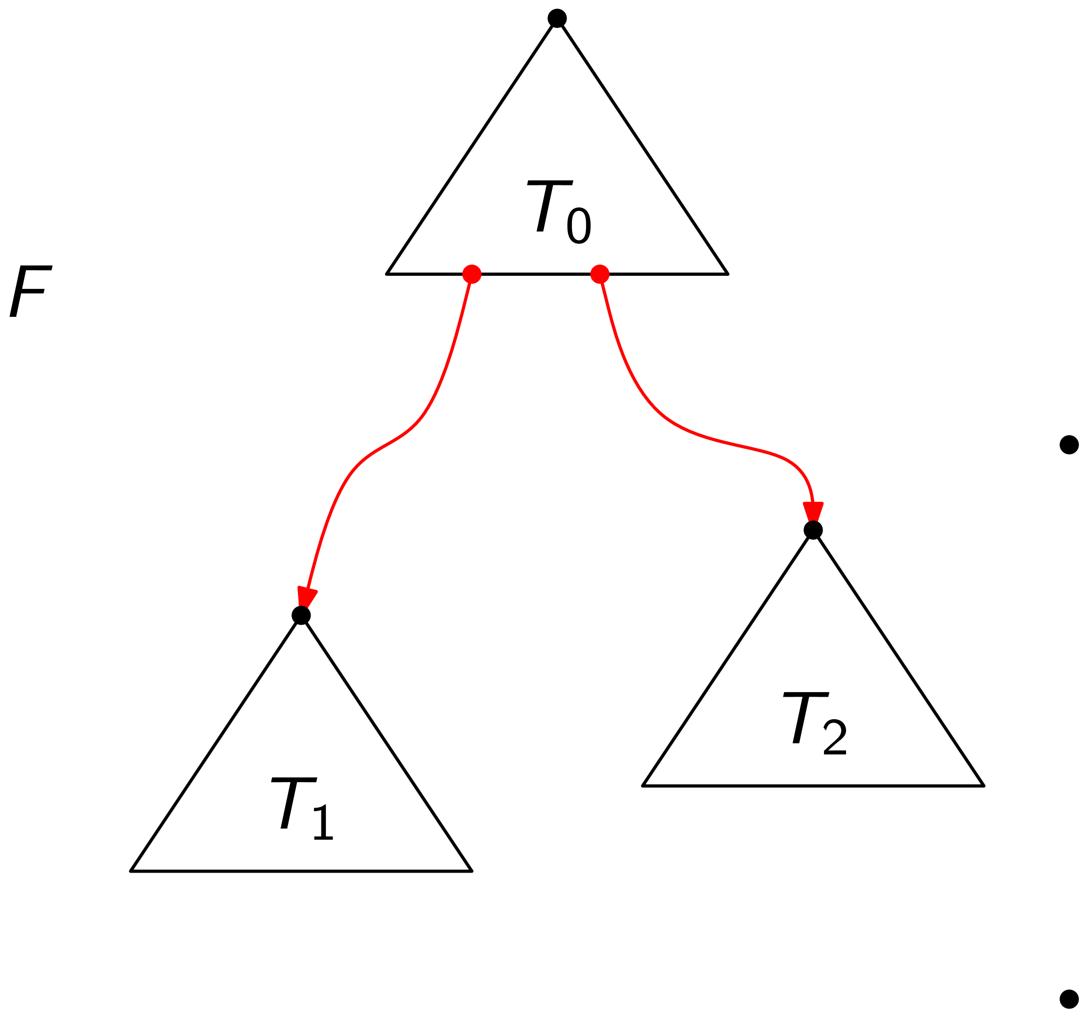

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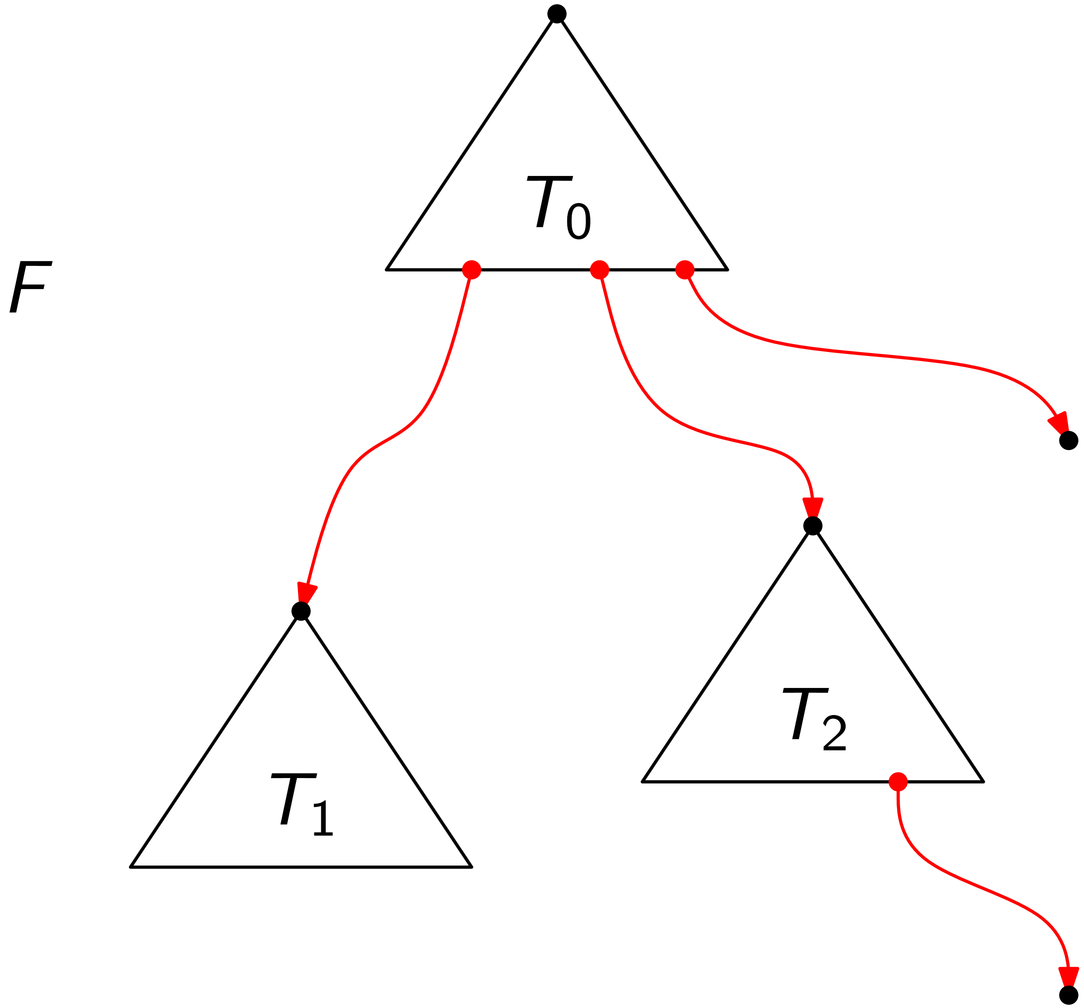

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show the uniqueness

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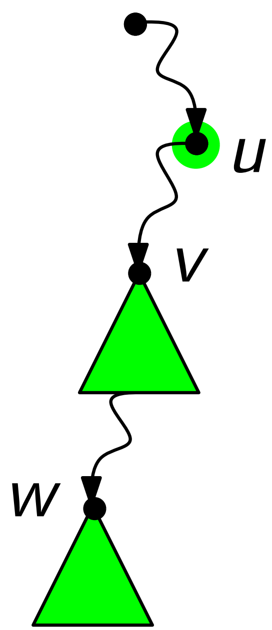

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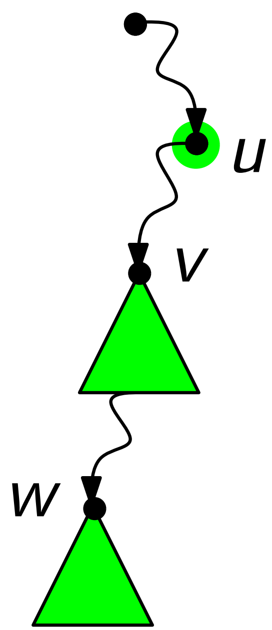

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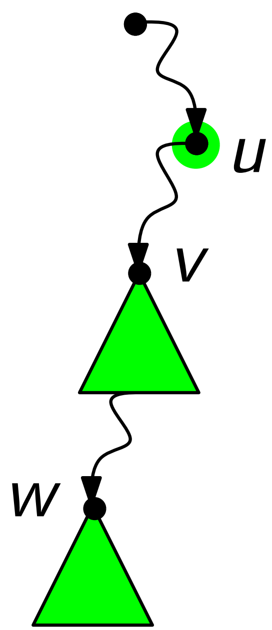

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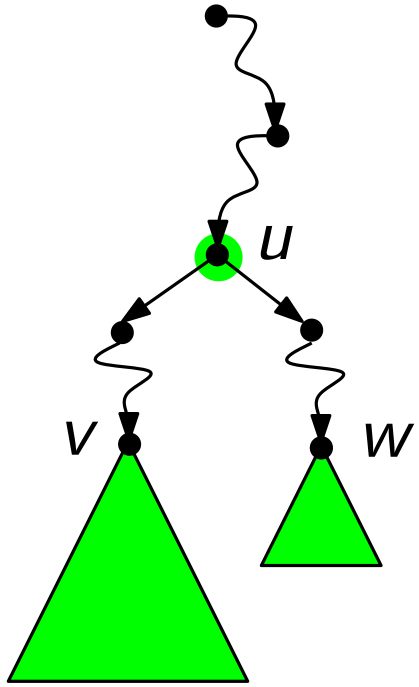

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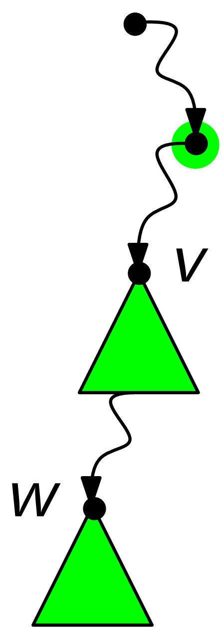

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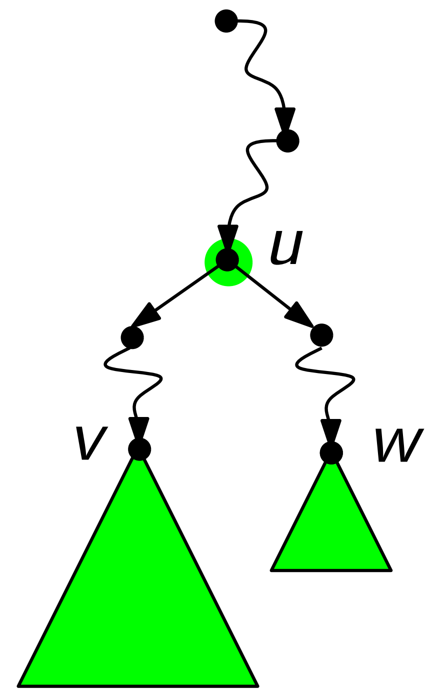

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