Augmenting the Connectivity of Planar and Geometric Graphs

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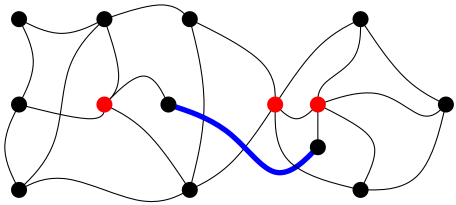
Universität Karlsruhe TU Eindhoven

Augmentation Problems

2-Vertex Connectivity Augmentation (**VCA**):

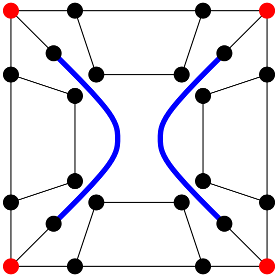
Given a graph $G = (V, E)$,
find a set of vertex pairs E' of minimal cardinality
such that
$G' = (V, E \cup E')$ is biconnected.



Augmentation Problems

Planar 2-Vertex Connectivity Augmentation (**PVCA**):

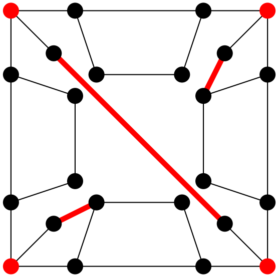
Given a planar graph $G = (V, E)$,
find a set of vertex pairs E' of minimal cardinality
such that
$G' = (V, E \cup E')$ is biconnected and planar.



Augmentation Problems

Planar 2-Vertex Connectivity Augmentation (**geometric PVCA**):

Given a plane geometric graph $G = (V, E)$,
find a set of vertex pairs E' of minimal cardinality
such that
$G' = (V, E \cup E')$ is biconnected and plane geometric.



Graph Type	2-Vertex Connectivity
general	**VCA**
planar	**PVCA**
plane geometric	**geometric PVCA**

Graph Type	2-Vertex Connectivity	2-Edge Connectivity
general	**VCA**	
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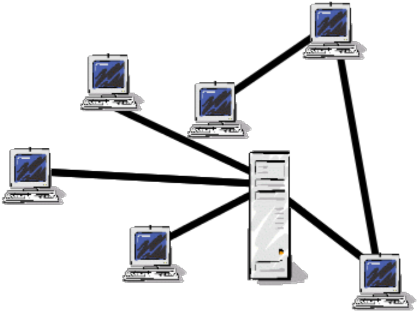
Graph Type	2-Vertex Connectivity	2-Edge Connectivity
general	**VCA**	**ECA**
planar	**PVCA**	**PECA**
plane geometric	**geometric PVCA**	**geometric PECA**

Applications

- Network design:



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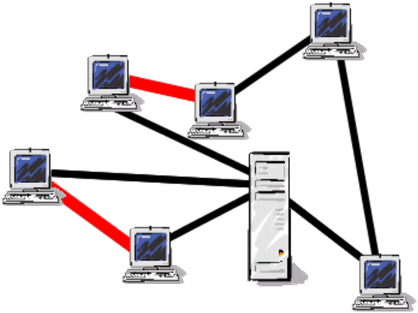

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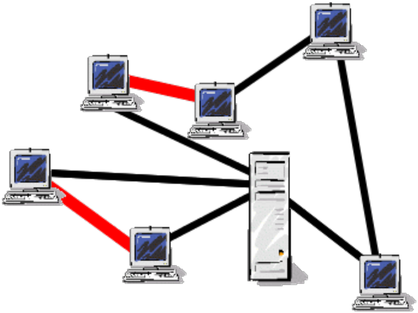

- Graph drawing

Previous work

- Without planarity constraint [Eswaran, Tarjan '76]
  solvable in $O(n)$ time
- PVCA is NP-hard [Bodlaender, Kant '91]

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- 5/3-approximation for PVCA [Fialko, Mutzel '98]

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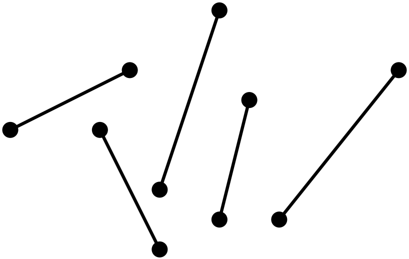
Open problem: Is PECA NP-hard?

Previous work

Problem CONNECTSIMPLEPOLYGON:
Given a set of non-crossing line segments in the plane.
Can we connect them to a simple polygon?



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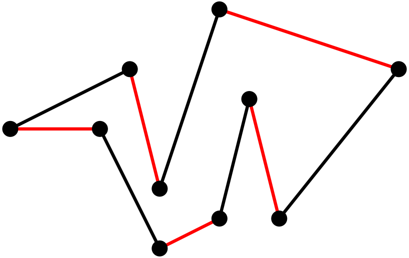

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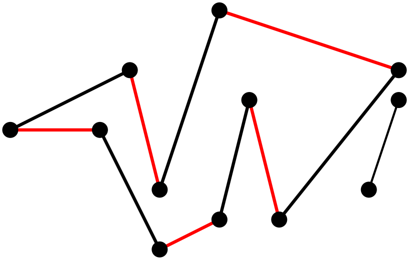

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Abellanas et al.: [Abellanas, García, Hurtado, Tejel, Urrutia '08]

- geometric PECA needs at most $5n/6$ edges
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Conjecture: in general $2n/3$ edges suffice, for trees $n/2$.

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Fact: [Tóth '08]
~~Conjecture~~: in general $2n/3$ edges suffice, for trees $n/2$.

Overview

1. Convex geometric graphs
2. Complexity
3. s–t path augmentation

Convex geometric graphs

Theorem

Geometric PVCA *and geometric* PECA *can be solved in linear time for connected convex geometric graphs.*

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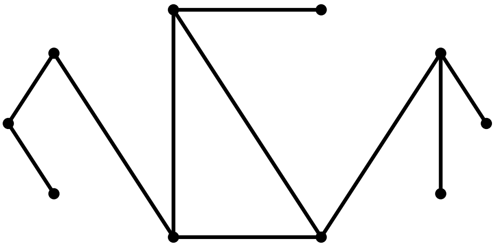

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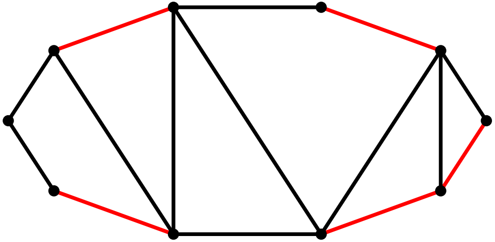

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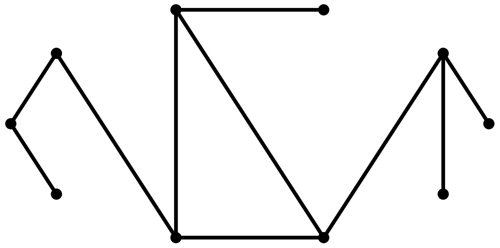

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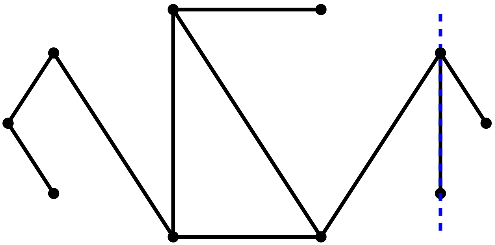

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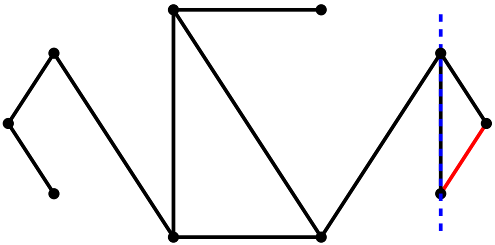

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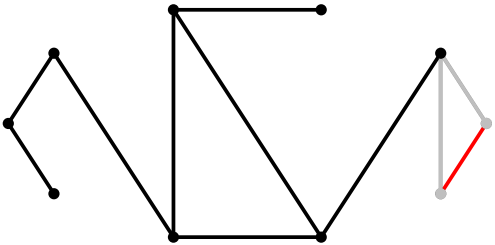

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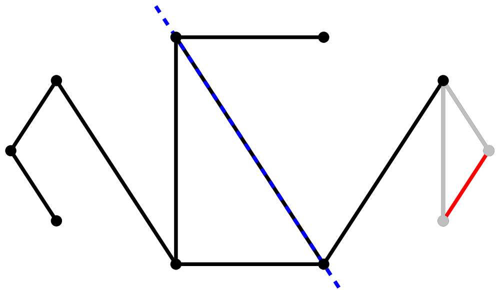

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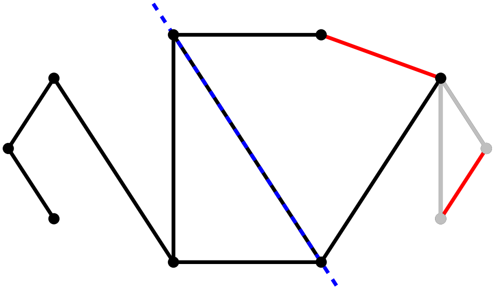

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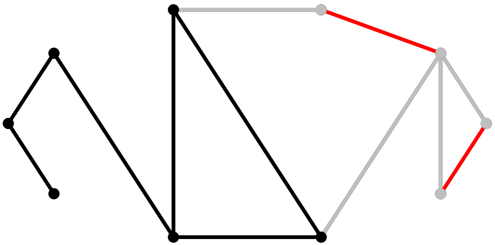

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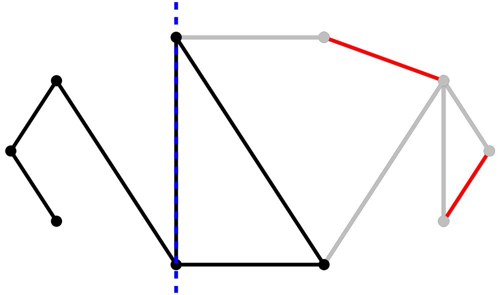

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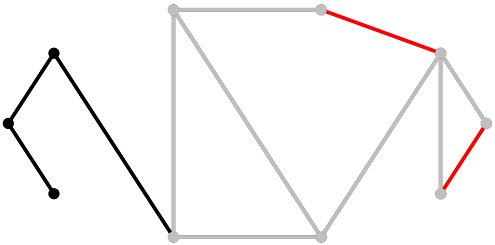

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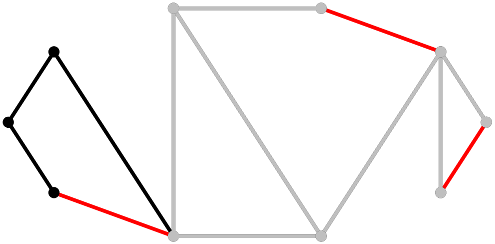

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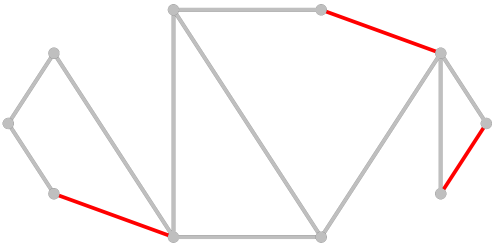

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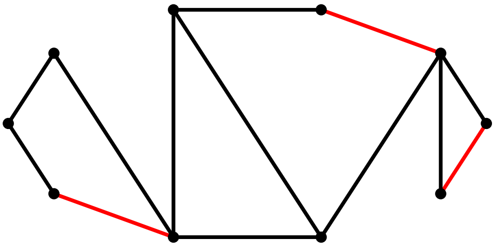

Overview

Complexity of PECA

Theorem

PECA *is NP-hard.*

Complexity of PECA

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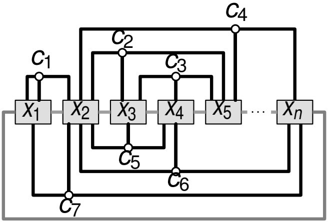
Proof:

- gadget proof
- reduction from PLANAR3SAT



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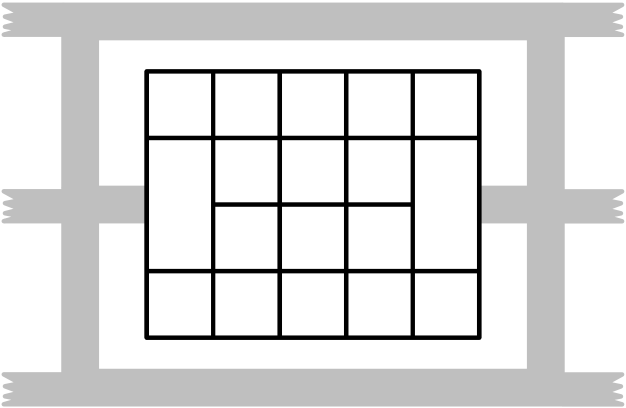

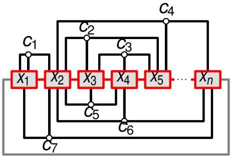

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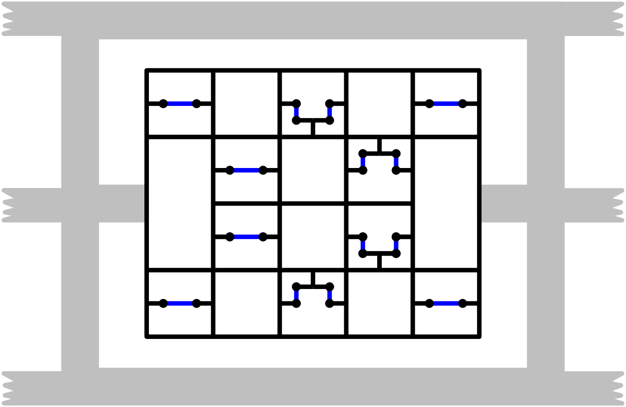

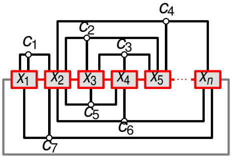

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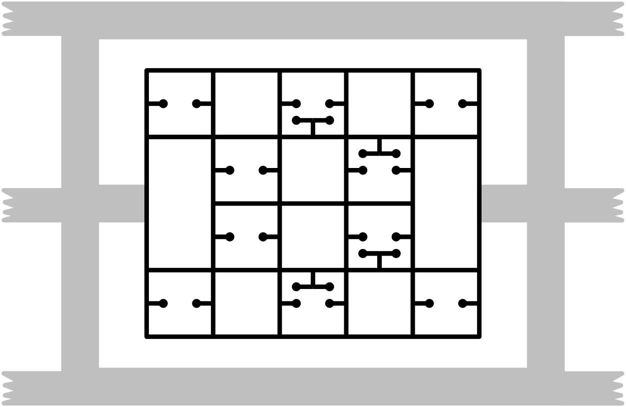

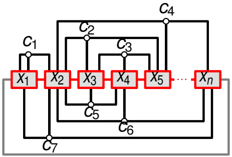

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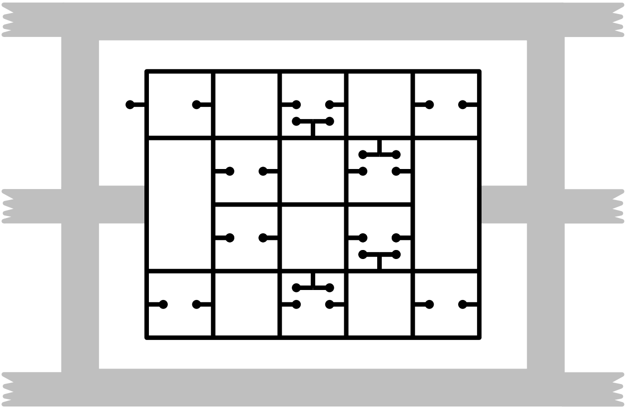

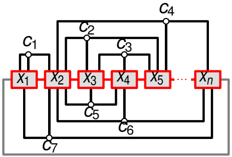

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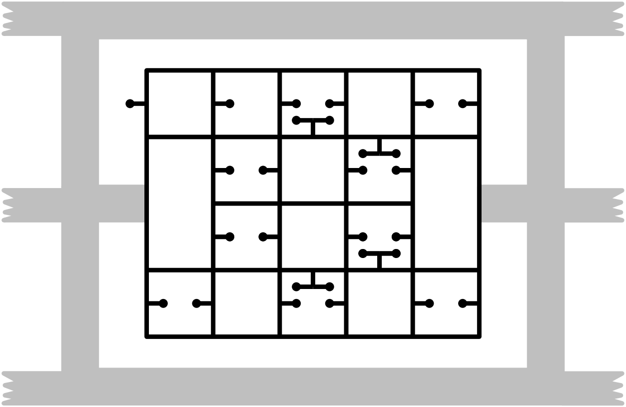

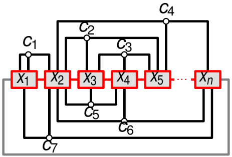

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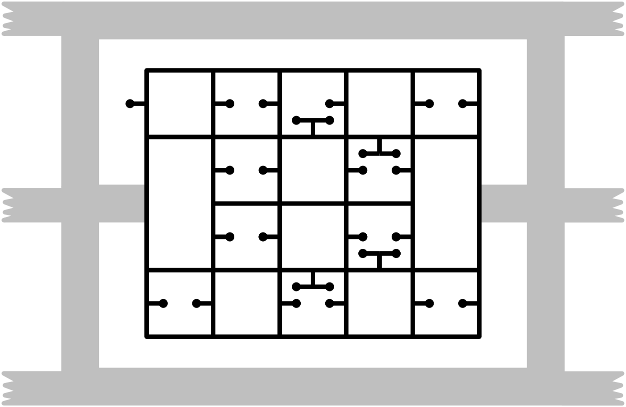

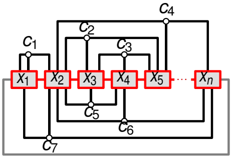

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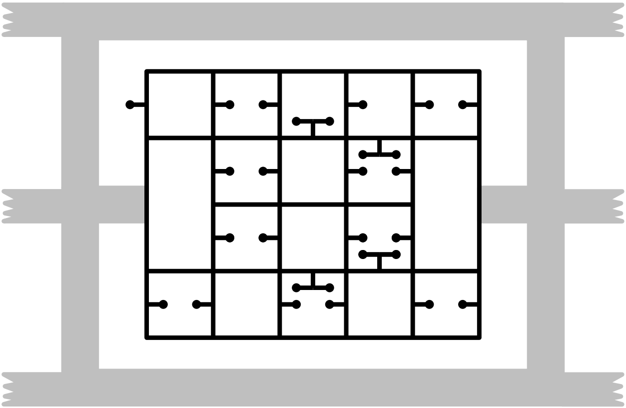

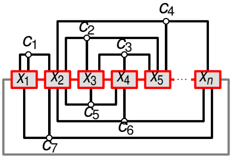

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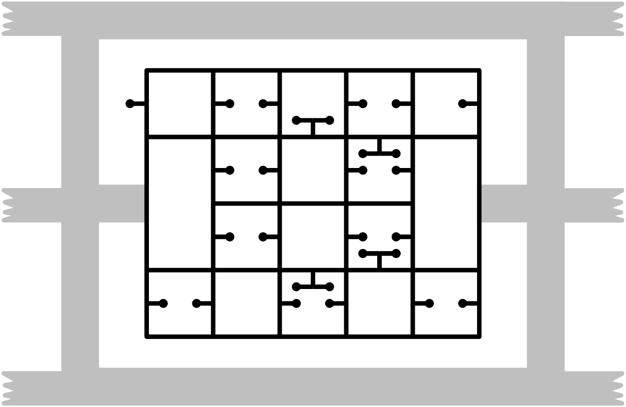

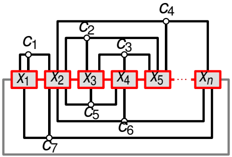

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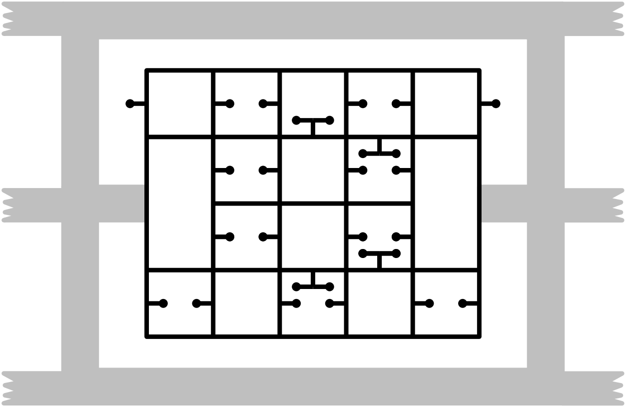

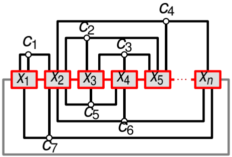

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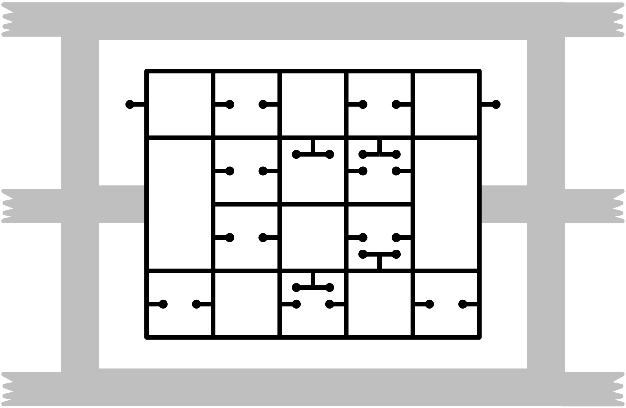

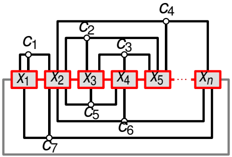

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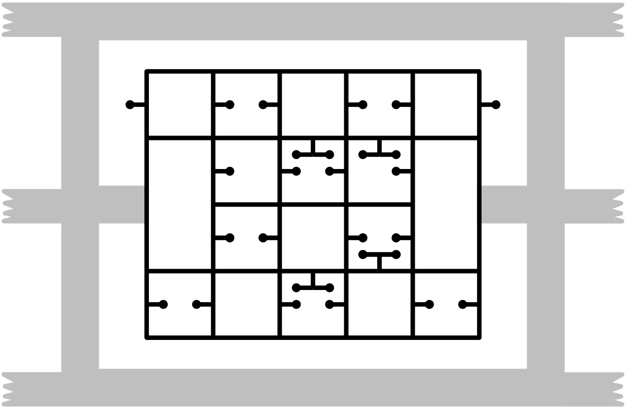

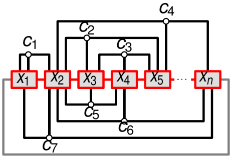

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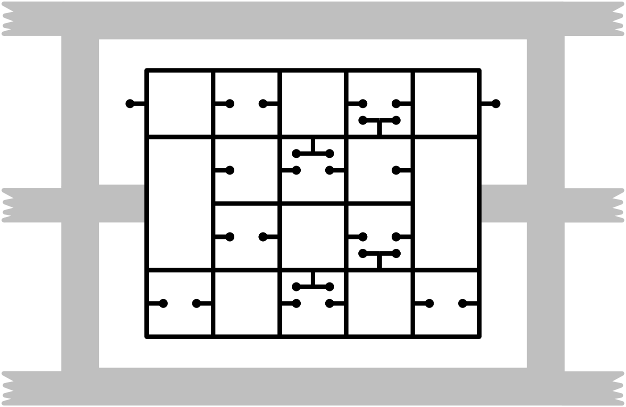

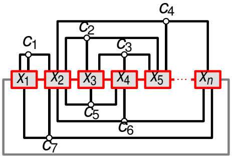

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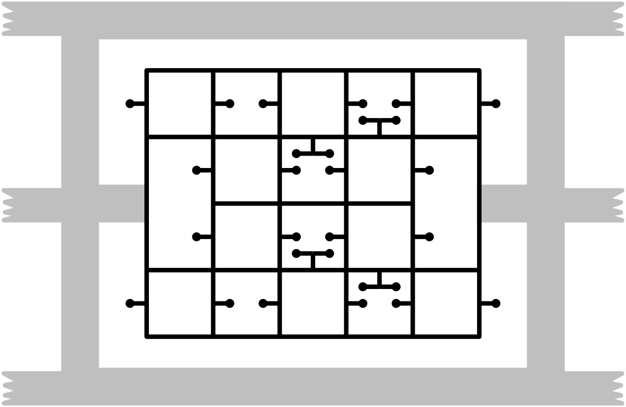

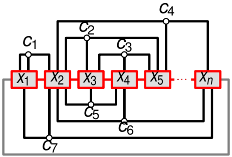

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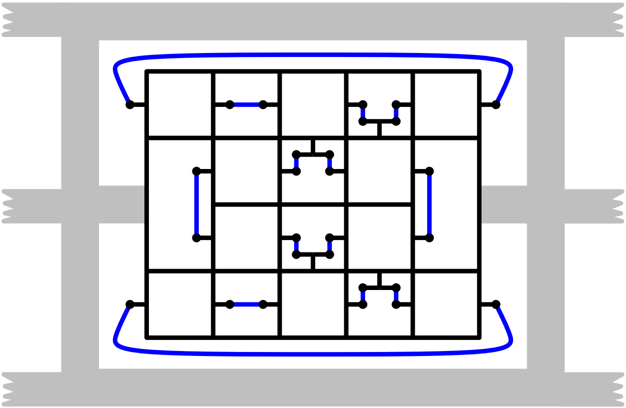

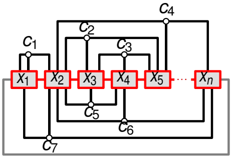

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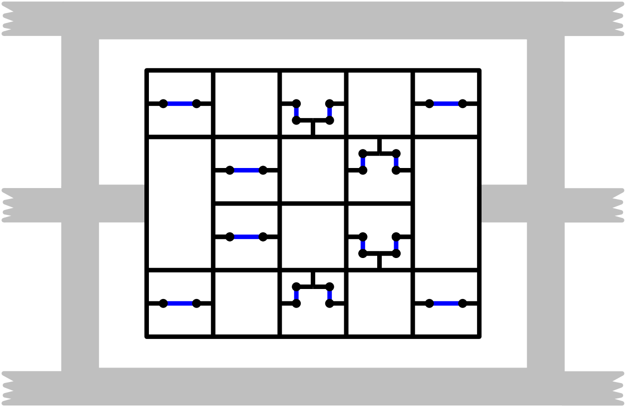

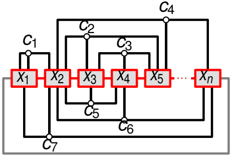

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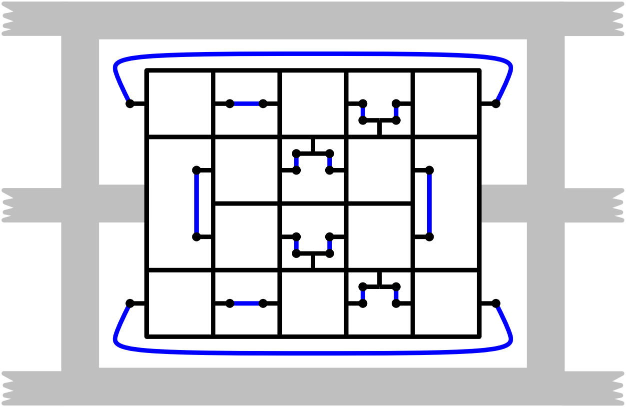
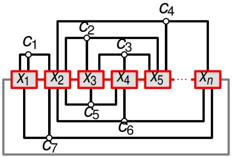
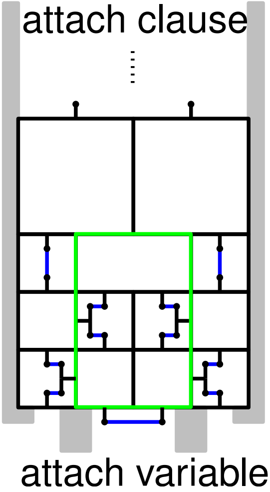




Pipe


attach clause
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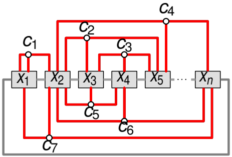
c_4
c_2
c_1
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x_2
x_3
x_4
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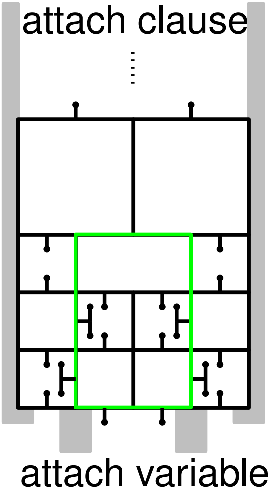
Pipe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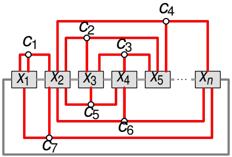
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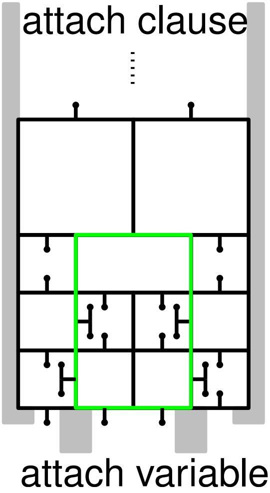
Pipe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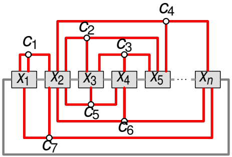
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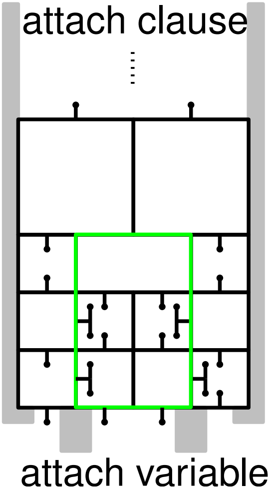
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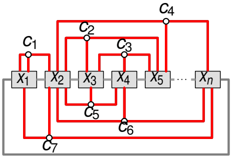
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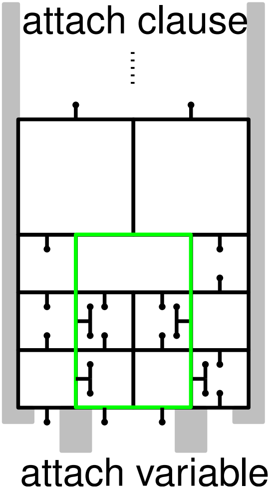
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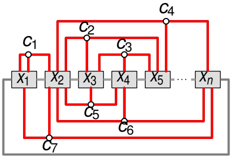
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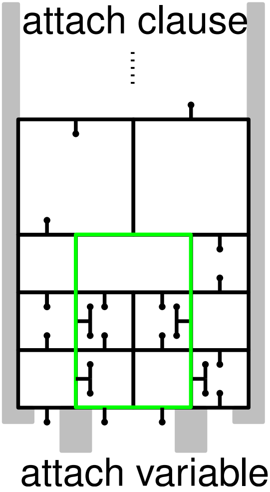
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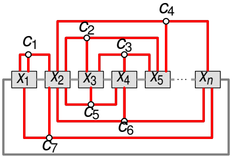
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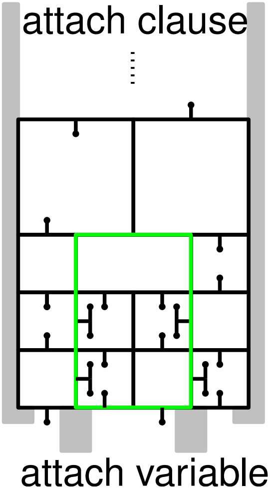
Pipe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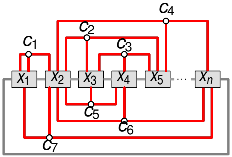
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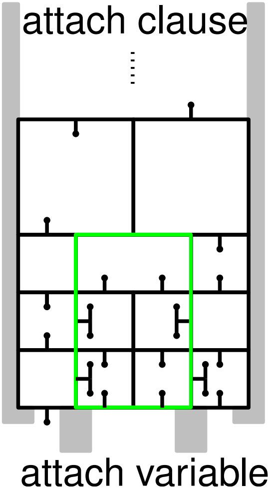
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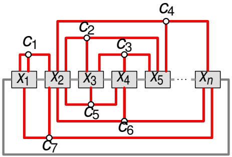
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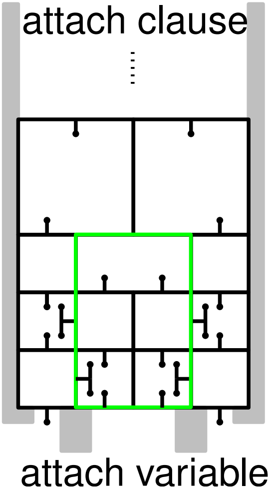
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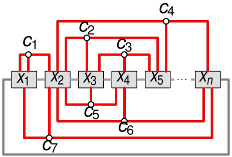
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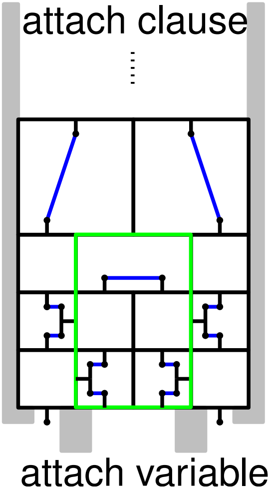
Pipe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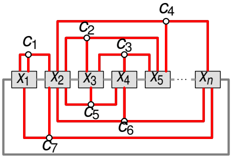
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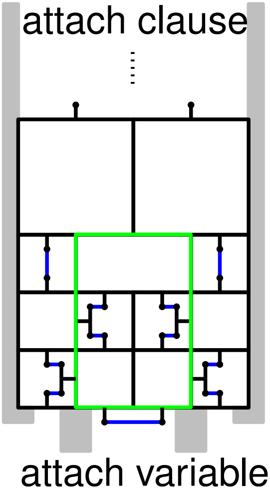
Pipe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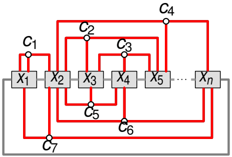
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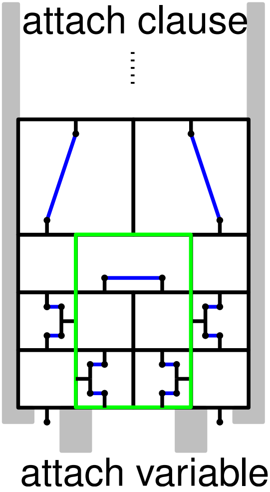
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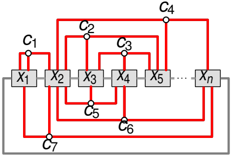
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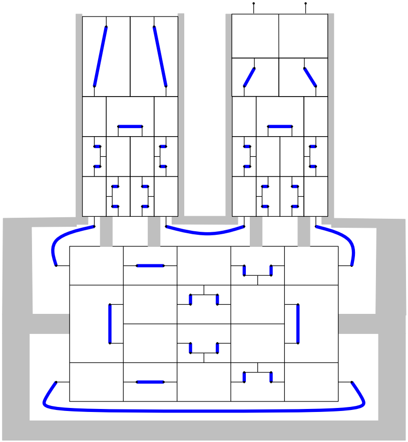
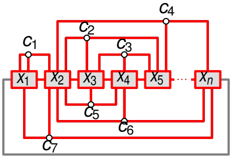
Literals





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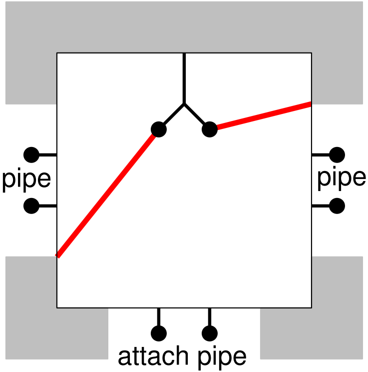

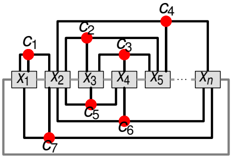

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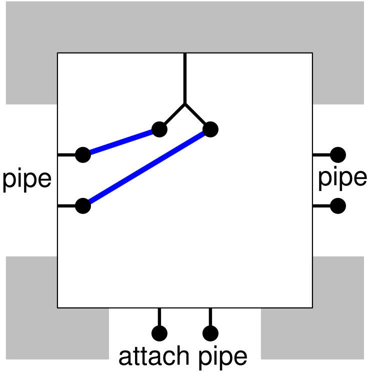

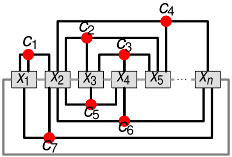

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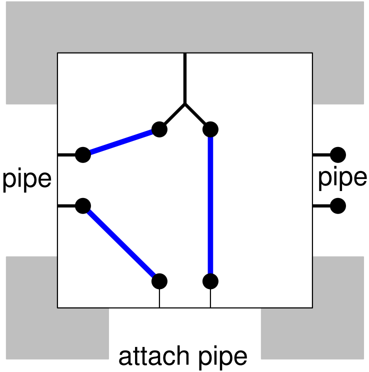

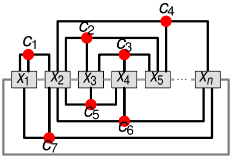

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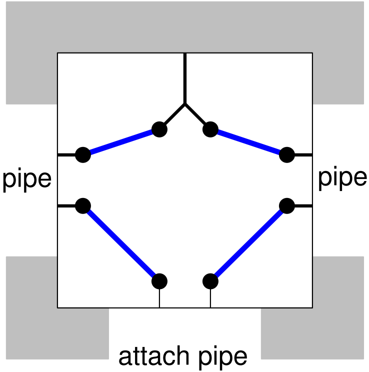

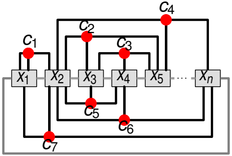

Complexity of geometric PVCA / geometric PECA

We conclude:

Theorem

PECA *is NP-hard.*

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yet another gadget proof ;-)

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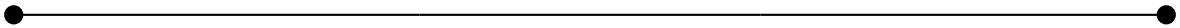

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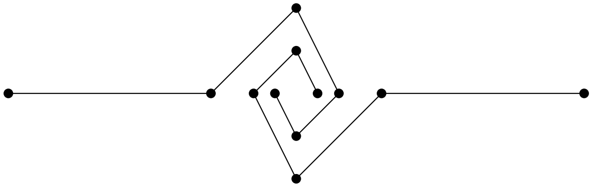

Overview

1. Convex geometric graphs
2. Complexity
3. s–t path augmentation

Geometric path augmentation

Problem: s–t k-ConnAug

Given: connected plane geometric graph $G = (V, E)$ and two vertices s and t in G.

Find: Minimal set of vertex pairs E', such that

- $G' = (V, E \cup E')$ is plane and
- G' contains k edge-disjoint s–t paths.

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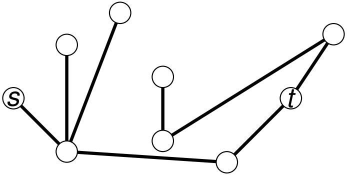
example for $k = 2$:



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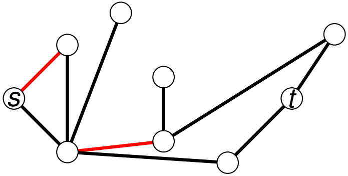

Worst-Case analysis for *s–t* 2-Aug

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$G = (V, E)$ a plane connected geometric graph, $s, t \in V$, $n = |V|$. Then

1. *G has an s–t 2-Aug of size $n/2$.*
2. *Such an s–t 2-Aug can be computed in $O(n)$ time.*

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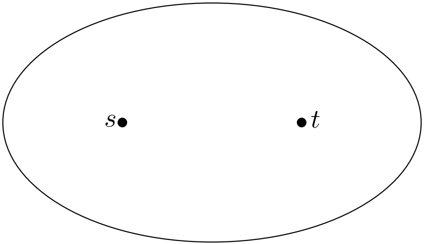
Proof:

1. Compute any triangulation T of G.
2. Find an s–t path π with $|\pi| \leq n/2$ in T.
3. Compute an augmentation from π with $\leq |\pi|$ edges.

Triangulation contains s–t path of length $\leq n/2$.

Consider growing neighborhoods $N^i(s)$, $N^i(t)$.



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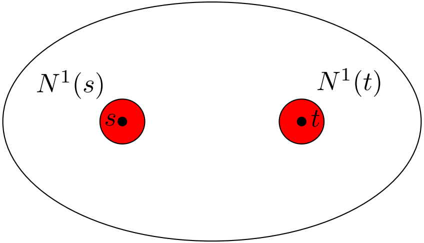

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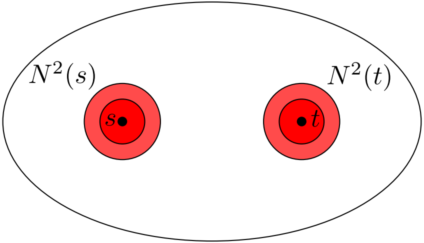

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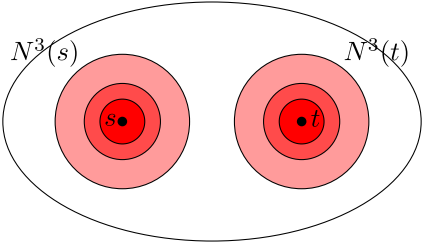

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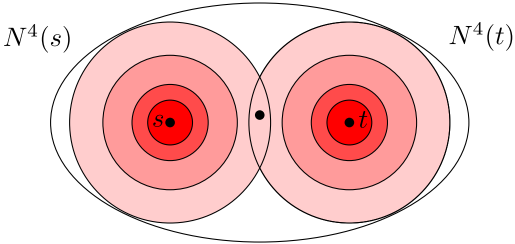

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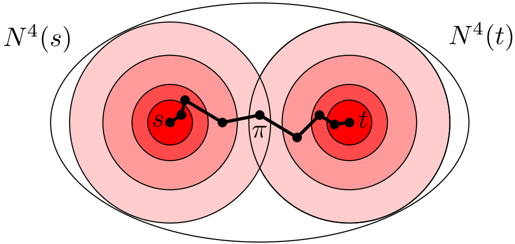

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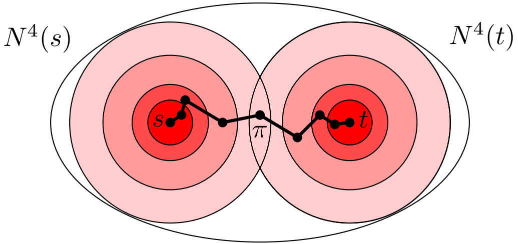

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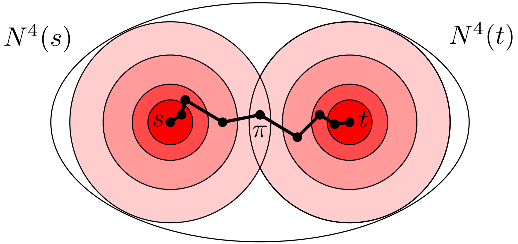

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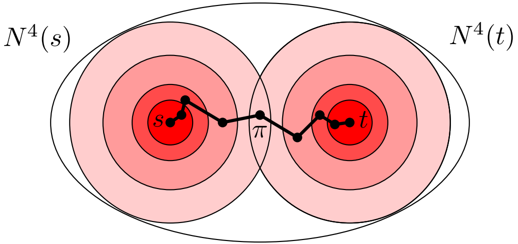

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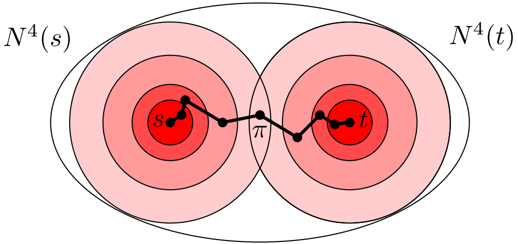

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$\Rightarrow$ after $k \lessapprox n/4$ steps the whole graph is covered by $N^k(s) \cup N^k(t)$.

How to construct an augmentation from π

Consider each edge e of π

1. e is not in G $\Rightarrow$ add e to G.

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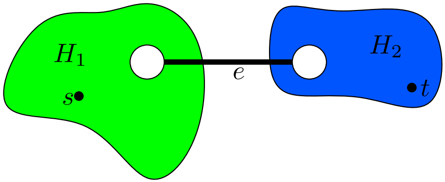

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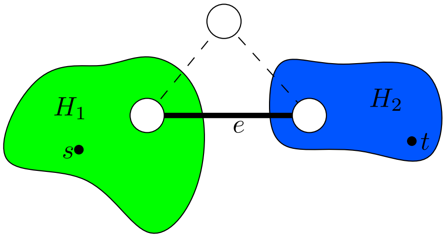

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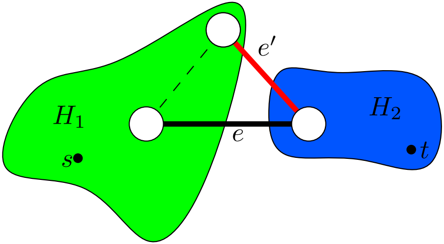

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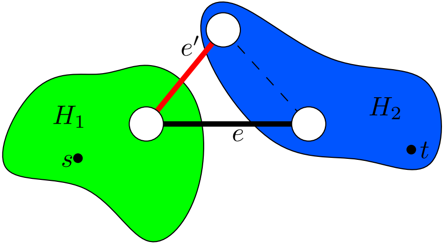

$\Rightarrow$ add edge *e*′.

Worst-Case analysis for *s–t* 2-Aug

Theorem

$G = (V, E)$ a plane connected geometric graph, $s, t \in V$, $n = |V|$. Then

1. *G has an s–t 2-Aug of size $n/2$.*
2. *Such an s–t 2-Aug can be computed in $O(n)$ time.*

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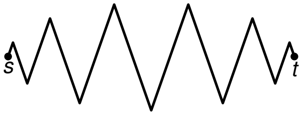
1 is worst-case optimal: [Abellanas et al. '08]



Results for Higher Connectivity

– 3 edge-disjoint s–t paths

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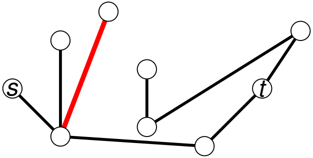

Open Questions

- Can we approximate geometric PVCA and geometric PECA?
- Is geometric *s*–*t* 2-Aug NP-hard?
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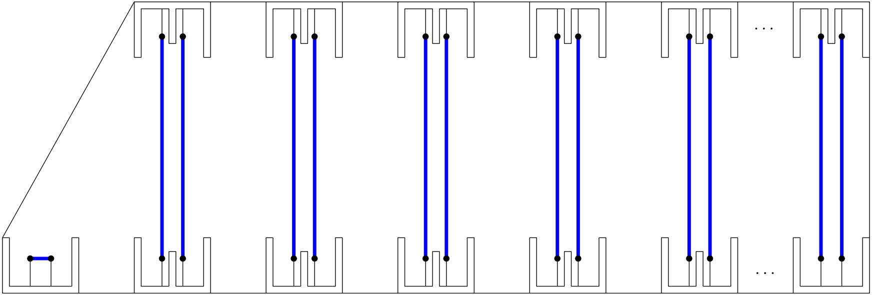
Complexity of geometric PVCA and PECA

Theorem

Geometric PVCA *and geometric* PECA *are NP-complete.*

Variable



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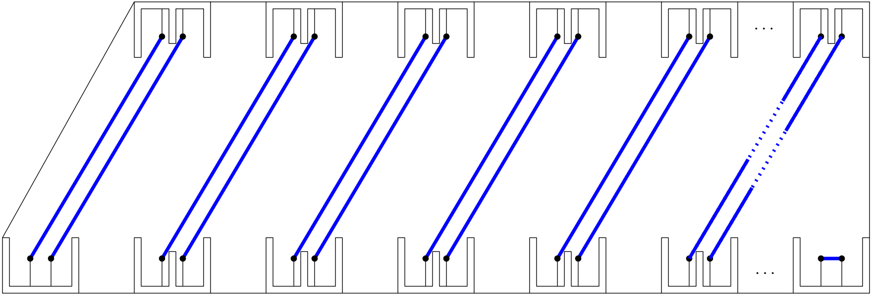
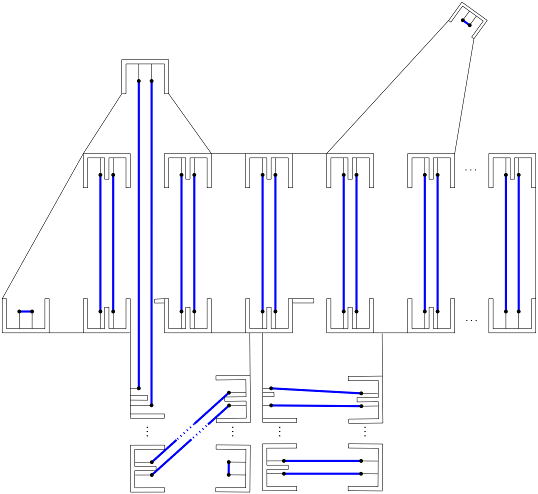


Literals



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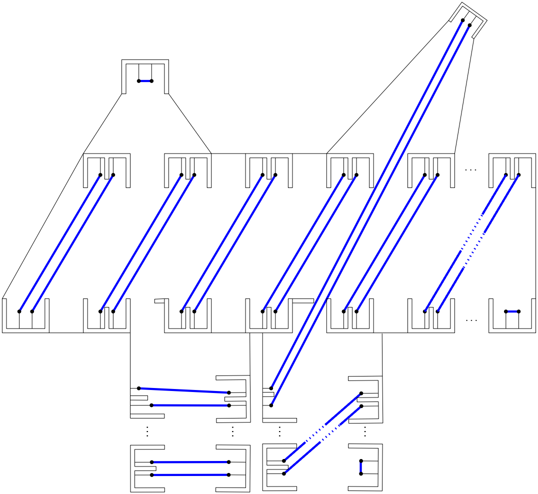
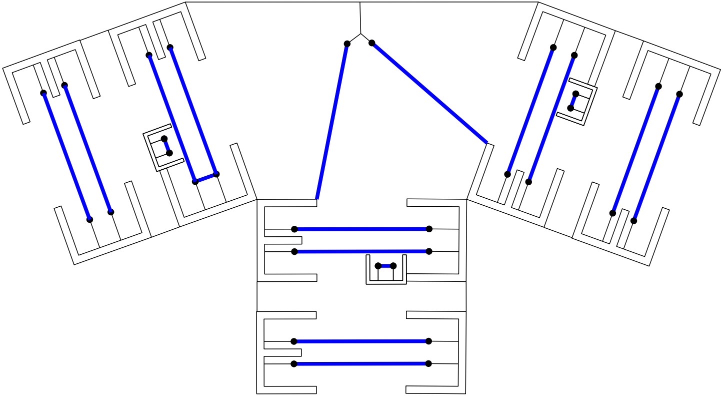


Clause



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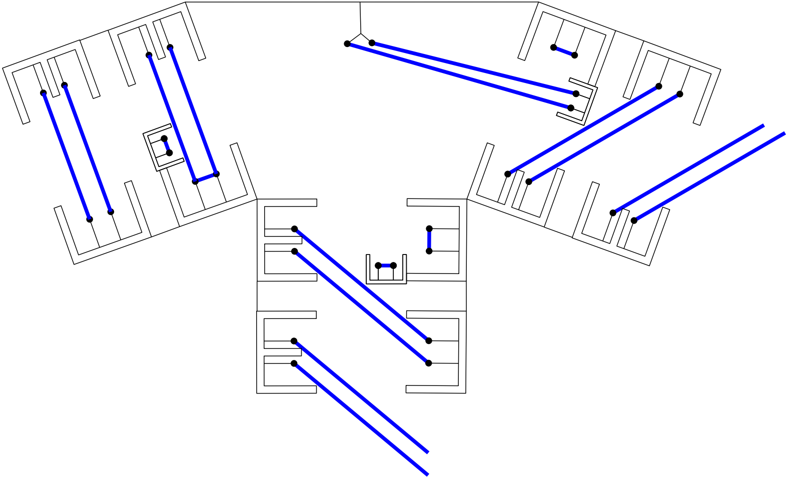

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