# Snapping Graph Drawings to the Grid Optimally

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# Introduction

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**Our question:**

What about topological equivalence?

# Intuition

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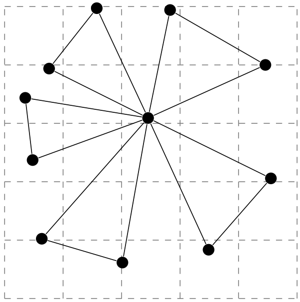

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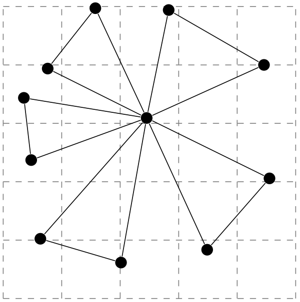

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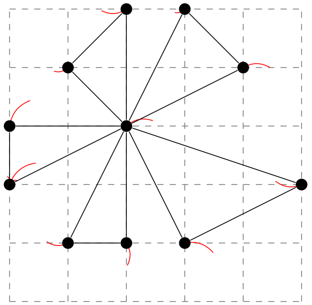

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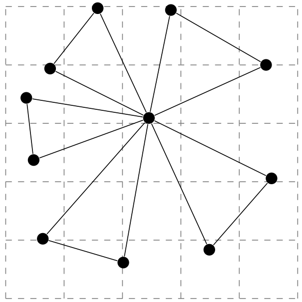

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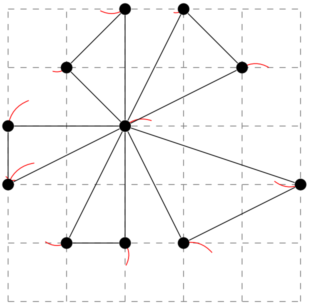

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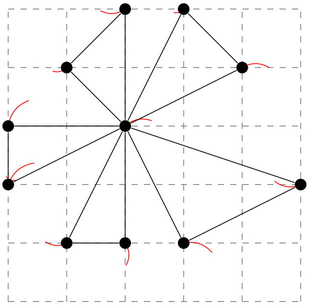

- Snap-rounding already is topologically equivalent.

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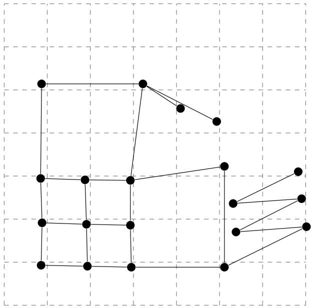

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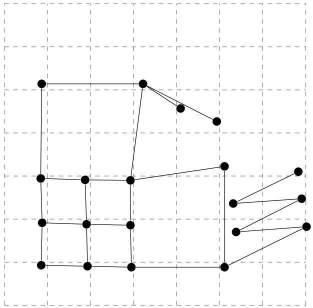

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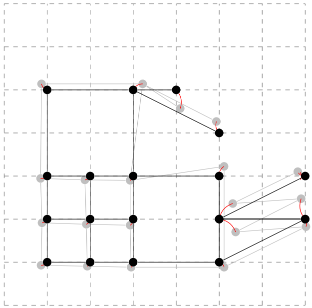

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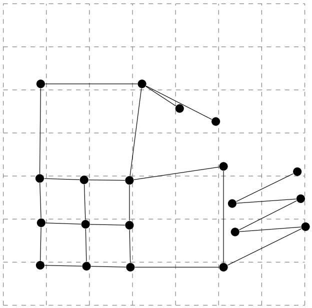

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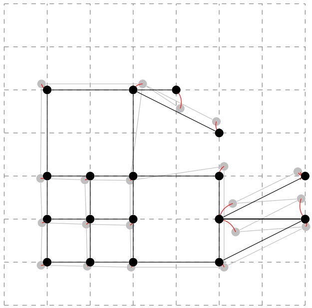

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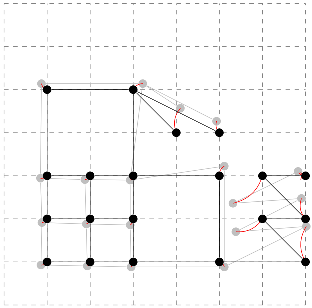

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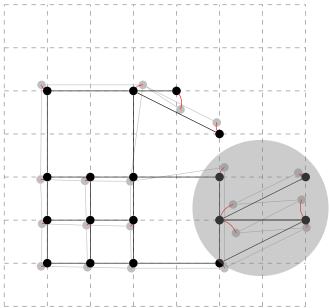

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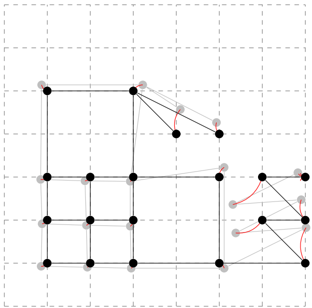

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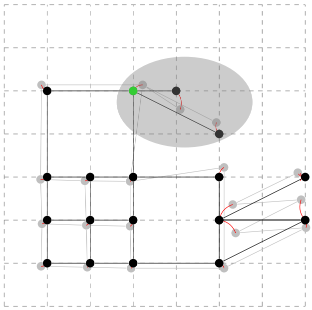

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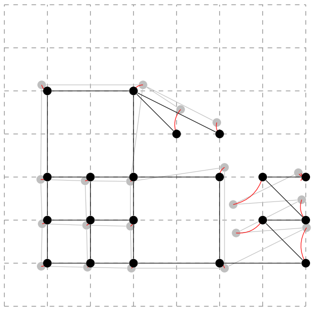

- Snap-rounding alters incidences and forces edges to collapse.
- Rounding to the nearest grid point changes the embedding of the upper-left vertex.

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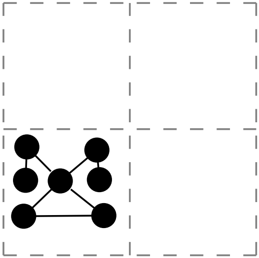

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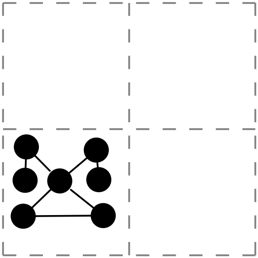

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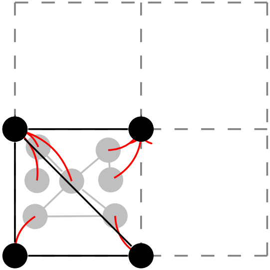

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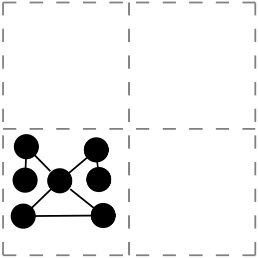

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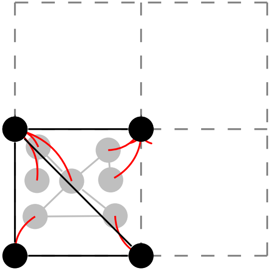

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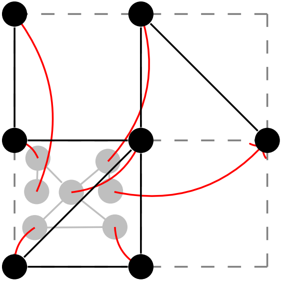

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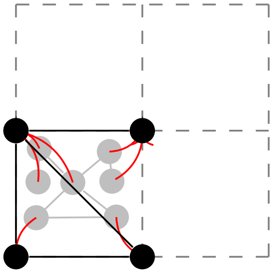

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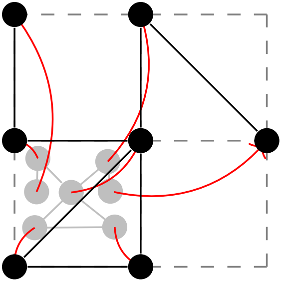

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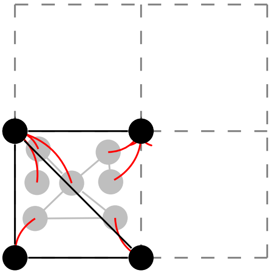

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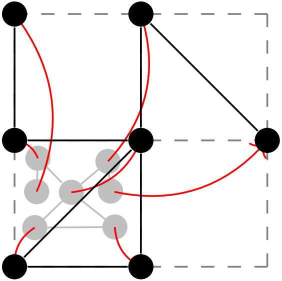

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- "Rounding" dense structures with no features collapsing is closely related to creating minimum-area drawings.

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*Graph $G = (V, E)$ with given embedding,*
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- Movement is measured in Manhattan-distance.

# Our Contribution

- $\mathcal{NP}$-hardness proof for Topologially Safe Snapping.

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# The $\mathcal{NP}$-hardness proof

# NP-Hardness

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- For reduction, consider a decision variant:

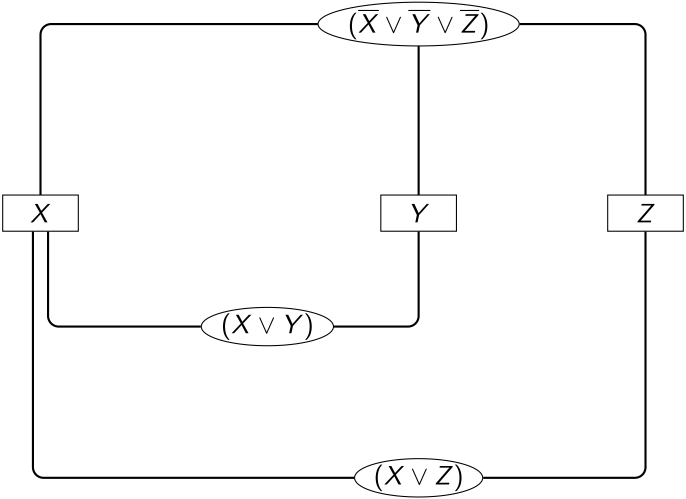
**Problem (Cost-bound TOPOLOGIALLY SAFE SNAPPING)**

*Graph $G = (V, E)$ with given embedding,*
*bounding box $B = [0, X_{\mathrm{max}}] \times [0, Y_{\mathrm{max}}]$, cost-bound $c_{\mathrm{min}} \in \mathbb{R}^+$.*

---

*Can $G$ be rounded to integer coordinates within $B$, preserving the given embedding with total movement of $c_{\mathrm{min}}$?*

# PM3SAT-formula



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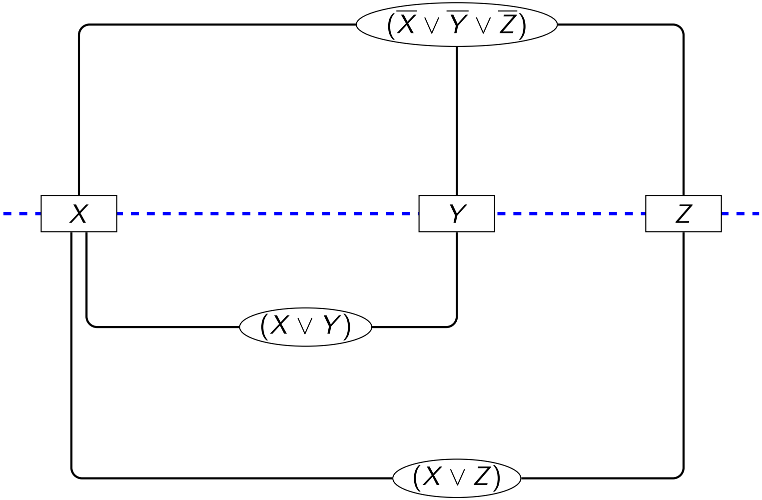

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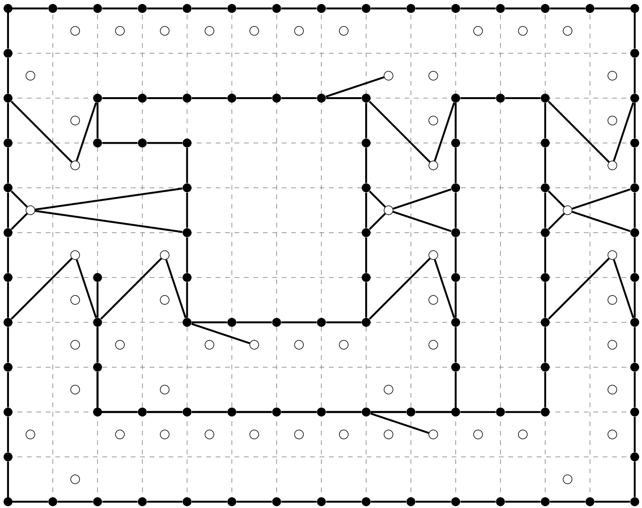

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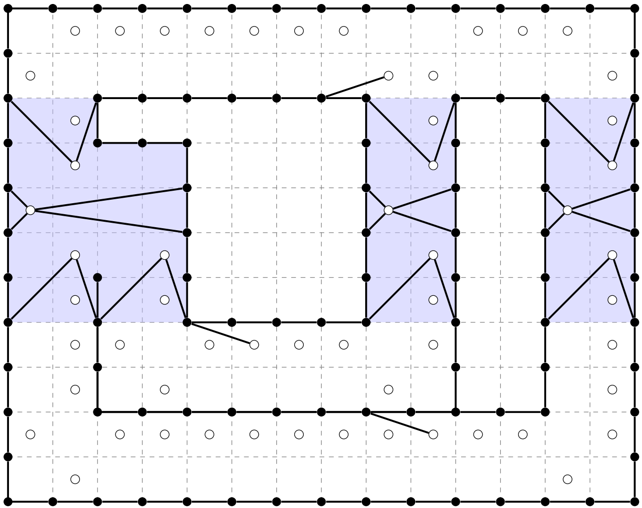

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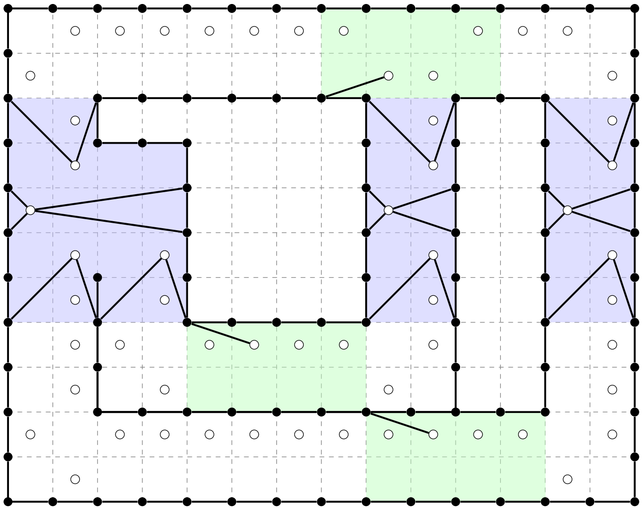

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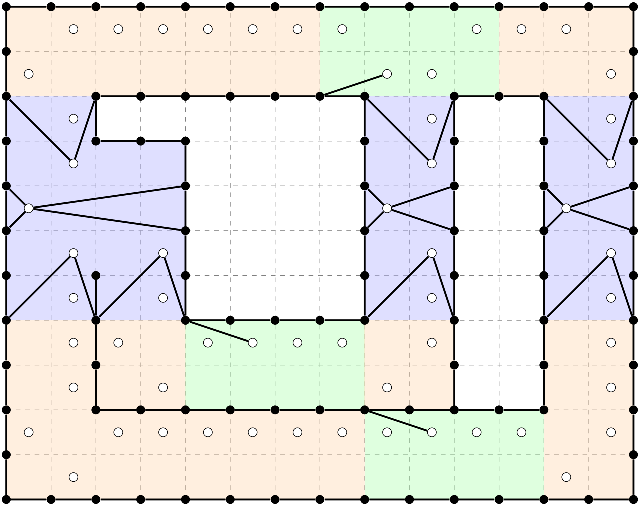

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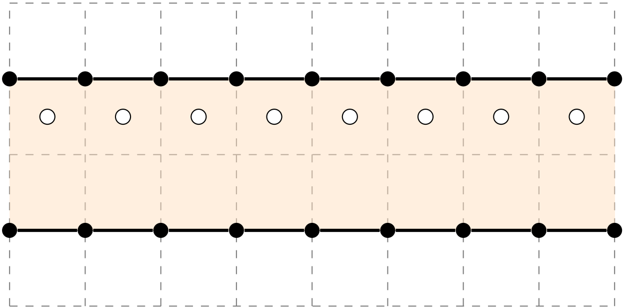

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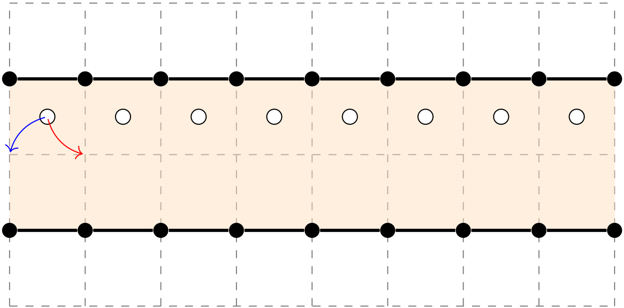

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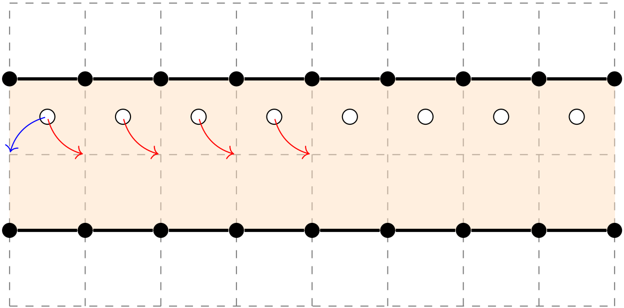

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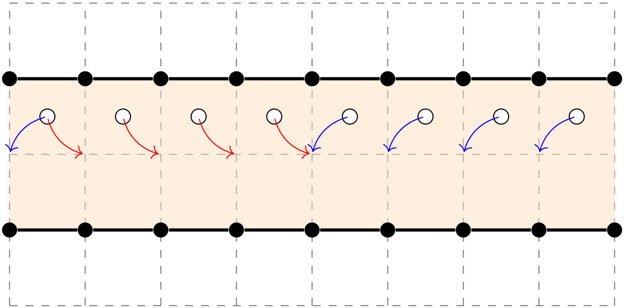

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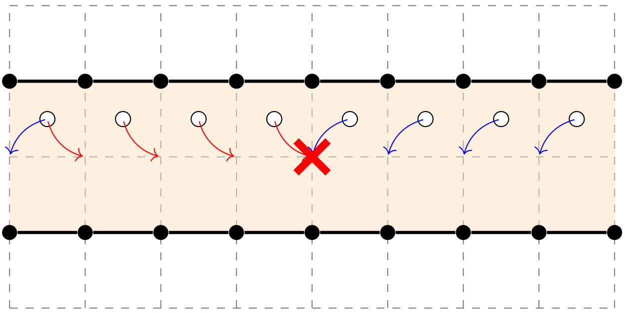
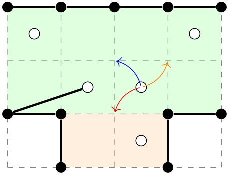
- Edges form tunnels that transmit pushes.
- Topological safety ensures consistency of transmission.



# Clauses



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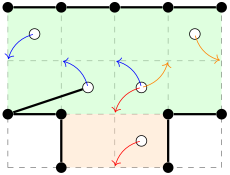

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- Following one arrow, rounding generates pushes.

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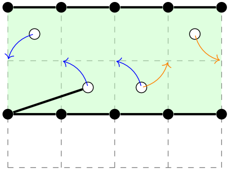

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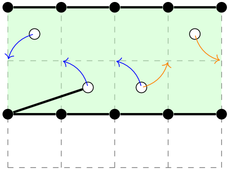

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- All-unnegated gadgets are constructed mirroring at a horizontal line.

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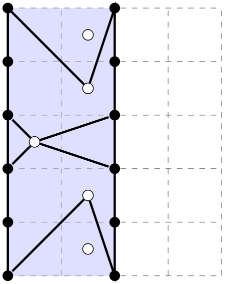

- Has tunnel connections for negated and unnegated occurances.

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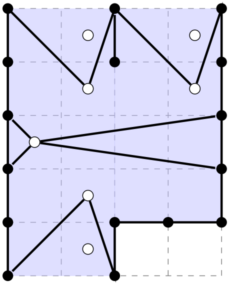

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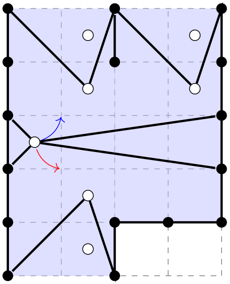

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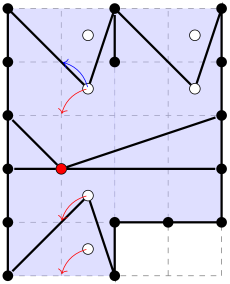

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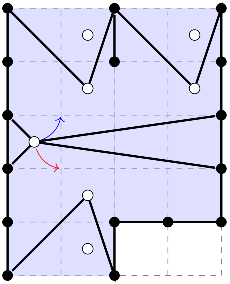

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- Moving the assignment vertex up equals a TRUE-assignment, FALSE otherwise.

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- Cost-bound $c_{\min}$ equals number of white vertices.
- If total movement cost equals $c_{\min}$, truth-assignment is obtained from assignment vertices.
- If the formula is unsatisfiable, at least one black vertex has to be moved $\Rightarrow$ $c_{\min}$ is exceeded.

# Other results

**Corollary**

TOPOLOGIALLY SAFE SNAPPING *is also $\mathcal{NP}$-hard when using* ***Euclidean*** *distance.*

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**Corollary**

***Euclidean*** TOPOLOGIALLY SAFE SNAPPING *with the objective to minimize* ***maximum*** *movement is $\mathcal{APX}$-hard.*

# Integer Linear Program

# Overview

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- Constraint: distinct vertex coordinates.

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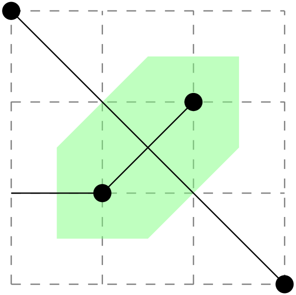
- **Idea:** every edge has some $D_{\text{min}}$-neighborhood that only incident edges are allowed to intersect.



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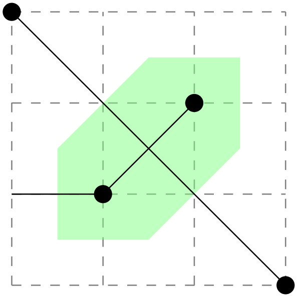

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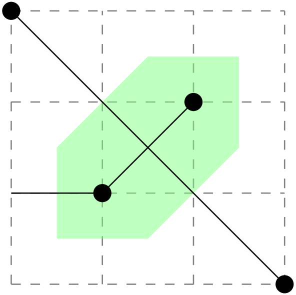

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- According to bounding box size:

$$D_{\min} = \frac{1}{\max\{X_{\max}, Y_{\max}\} + 1}$$

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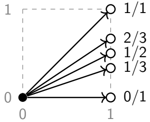

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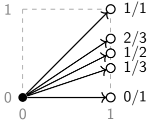


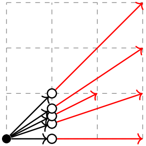

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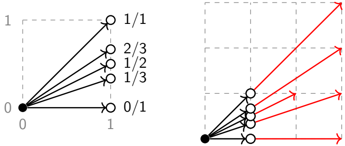


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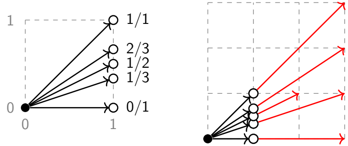


- Inside $[-k, k] \times [-k, k]$ area, there are $\Theta(k^2)$ directions to consider.
- Consider them to be ordered counter-clockwise.

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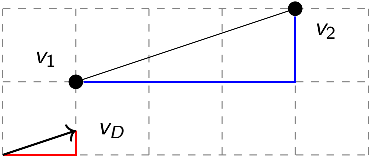

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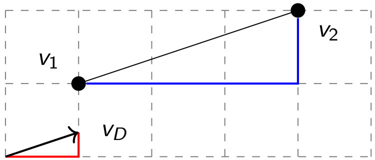


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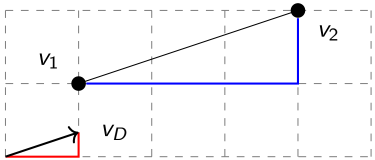


- Map edges to directions and compare the ordering of those directions to the given embedding.

# Integer Linear Program

**Theorem**

*This ILP solves* TOPOLOGIALLY SAFE SNAPPING.

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- We use delayed constraint generation to iteratively improve our model.
- We generate most constraints on demand: first iteration is simple rounding (with unique coordinates).

# Experimental Evaluation

# The Setup

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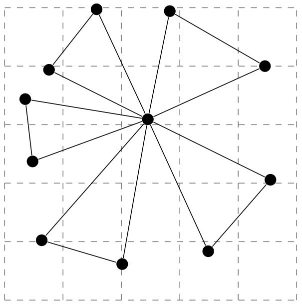
- Numbers of rows & columns before CPLEX presolving.
- Runtime in wall-clock time.
- For delayed constraint generation, time is accumulated total.

# The Good



| | Full | Delayed |
|---|---|---|
| rows | | |
| cols | | |
| time | | |

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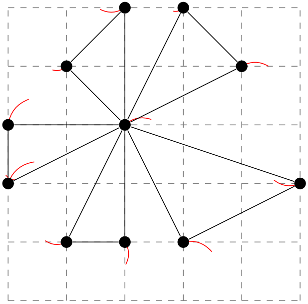

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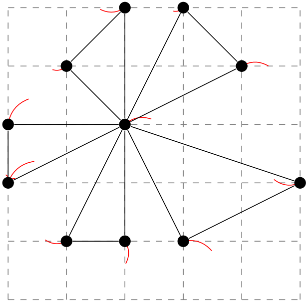

- Even small examples take several seconds to solve.

| | Full | Delayed |
|---|---|---|
| rows | 42 699 | |
| cols | 11 300 | |
| time | 10.6 s | |

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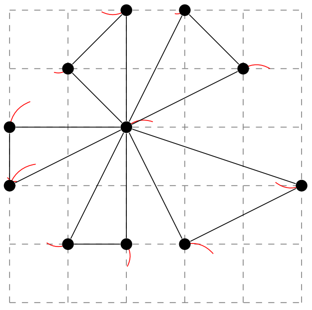

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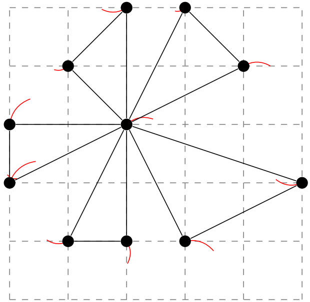
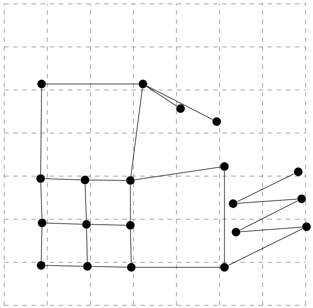


- Even small examples take several seconds to solve.
- This is a very simple example!
- Delayed constraint generation gives speed-up.

| | Full | Delayed |
|---|---|---|
| rows | 42 699 | 88 |
| cols | 11 300 | 110 |
| time | 10.6 s | 0.5 s |

# The Bad



| | Full | Delayed |
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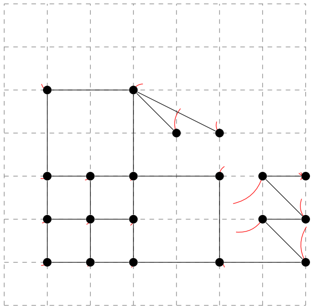

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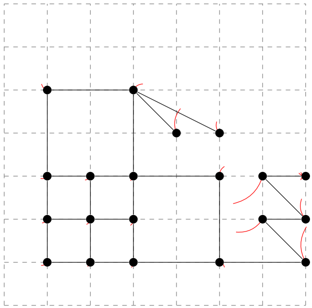

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|---|---|---|
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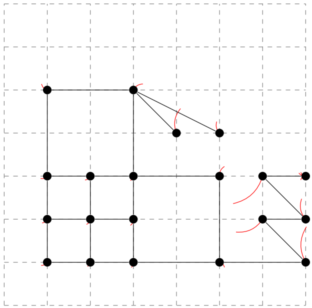

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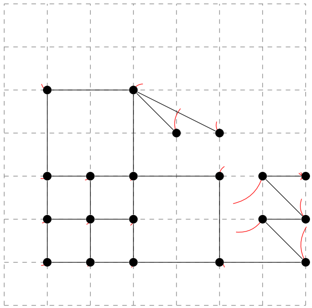

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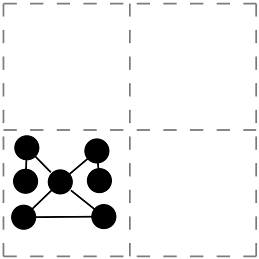
- ...waiting more than 3 minutes is too long for a graph on 20 vertices!

| | Full | Delayed |
|---|---|---|
| rows | 323 441 | 15 161 |
| cols | 82 816 | 4 044 |
| time | † | 211.6 s |

# The Ugly



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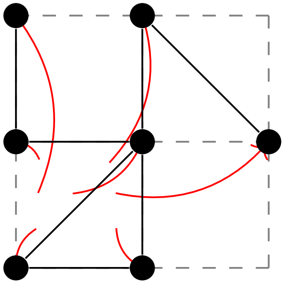

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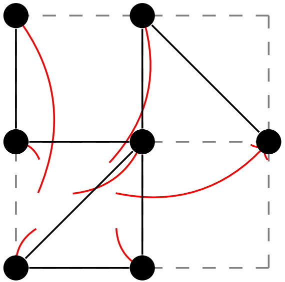

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| | Full | Delayed |
|---|---|---|
| rows | 2 603 | |
| cols | 916 | |
| time | 4.8 s | |

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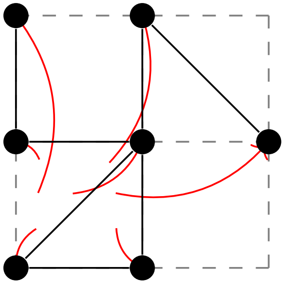

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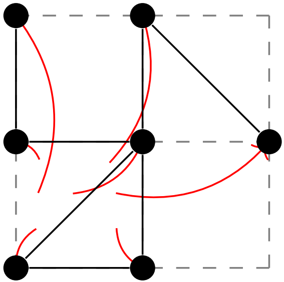

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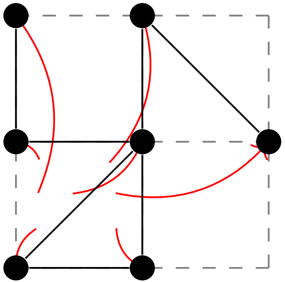

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- Rounding this graph is very similar to finding a minimum-area drawing, which is also $\mathcal{NP}$-hard.

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|---|---|---|
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- Find some heuristic algorithm.
- Questions about approximability remain open.