Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Chair for INFORMATICS I Efficient Algorithms and Knowledge-Based Systems

Institute for Informatics

# On the Weak Line Cover Numbers

**Oksana Firman** Alexander Wolff

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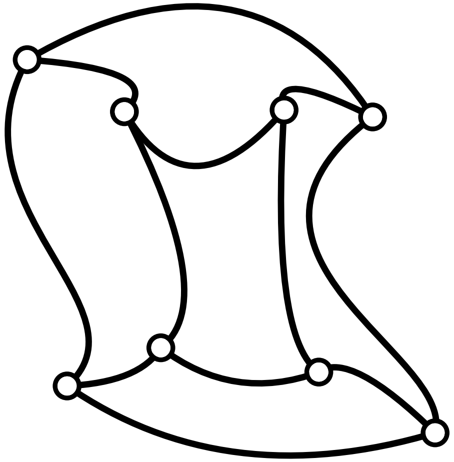
Alexander Ravsky

Pidstryhach Institute for Applied Problems of Mechanics and Mathematics,
National Academy of Science of Ukraine, Lviv, Ukraine

# Visual complexity of a drawing of a graph

Given: graph $G$



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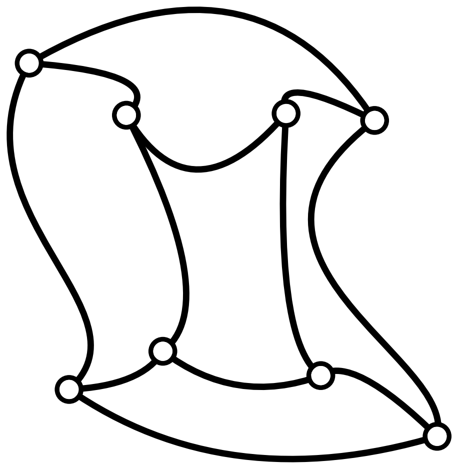
Given: graph $G$

Use as few objects as possible to draw $G$



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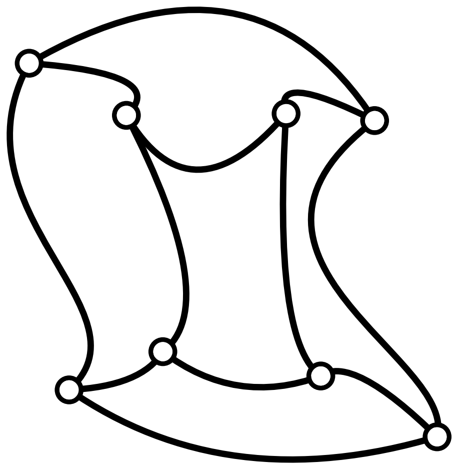
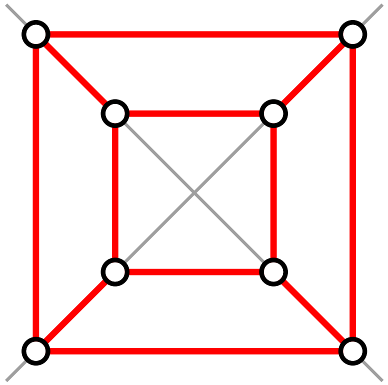


Use as few objects as possible to draw $G$

**12** segments



[Dujmović et al., CGTA'07]

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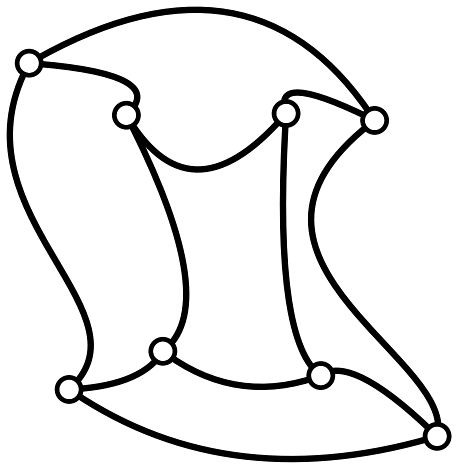

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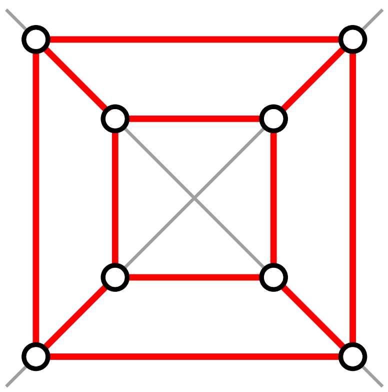
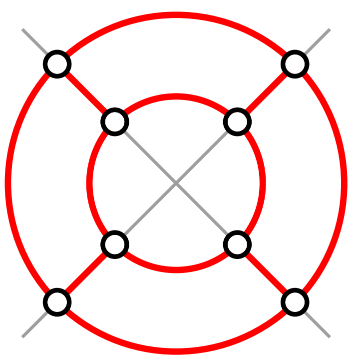


[Dujmović et al., CGTA'07]

**6** arcs



[Schulz, JGAA'15]

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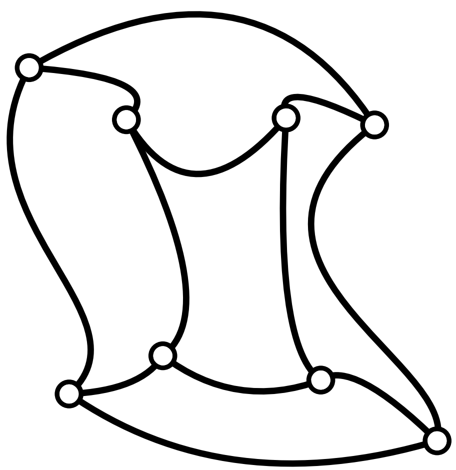

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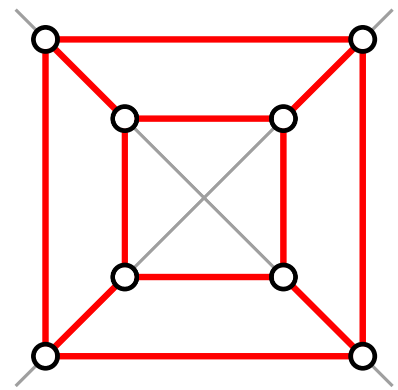
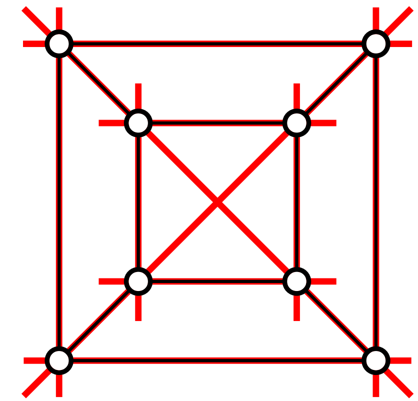


[Dujmović et al., CGTA'07]

**10** straight lines



[Chaplick et al., GD'16]

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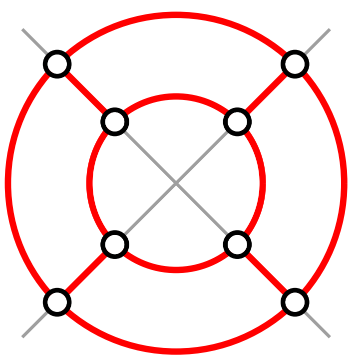

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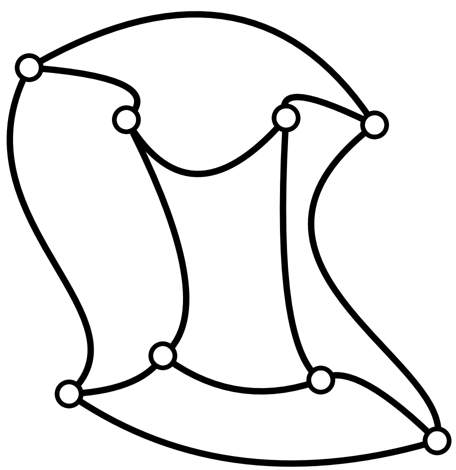

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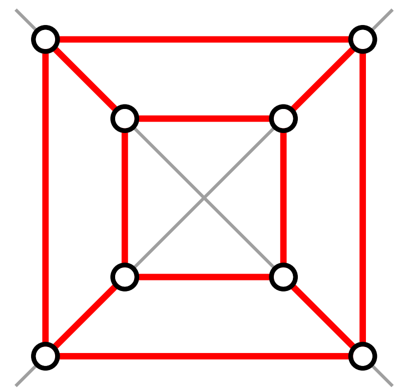

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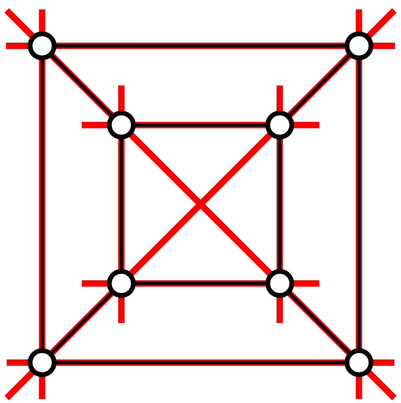

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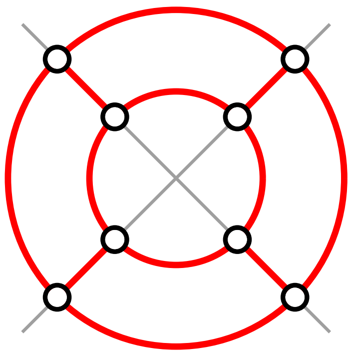
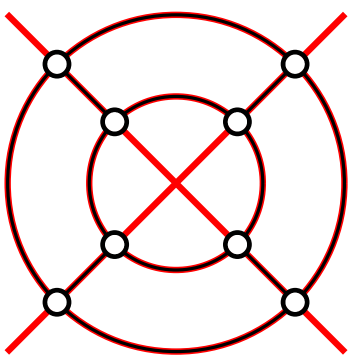


[Schulz, JGAA'15]

**4** circles



[Kryven et al., CALDAM'18]

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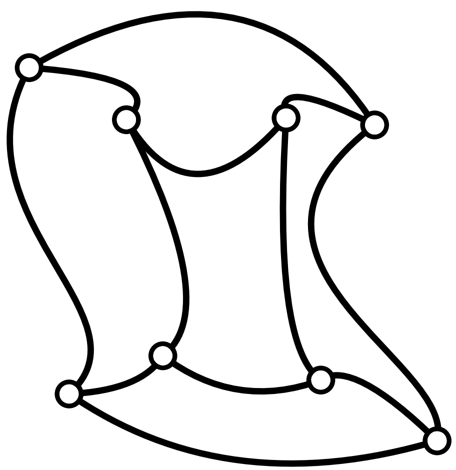

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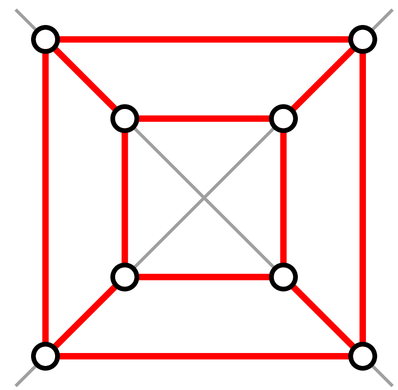

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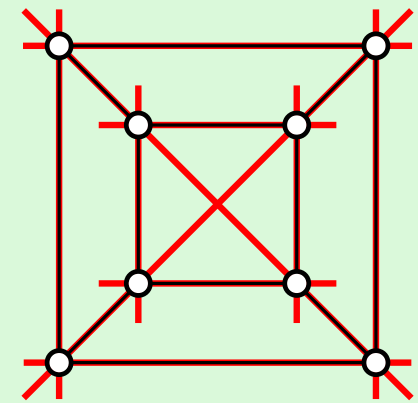

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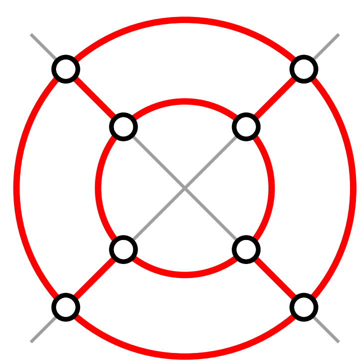

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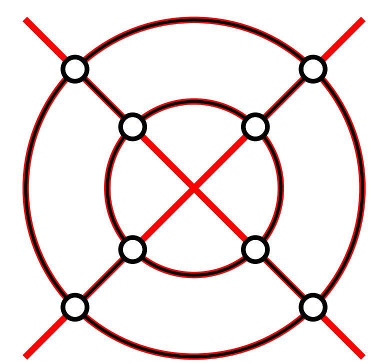

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# Affine cover numbers

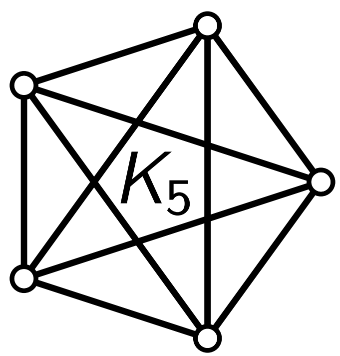
[Chaplick et al., GD'16]

Let $G$ be a graph and let $1 \leq m < d$.

All drawings are straight-line and crossing-free.

**Def.** The *m-dimensional affine cover number* $\rho^m_d(G)$ is the minimum number of $m$-dimensional planes in $\mathbb{R}^d$ such that the vertices and the edges of a drawing of $G$ are contained in the union of these planes.

$\rho^2_3(K_5) =$



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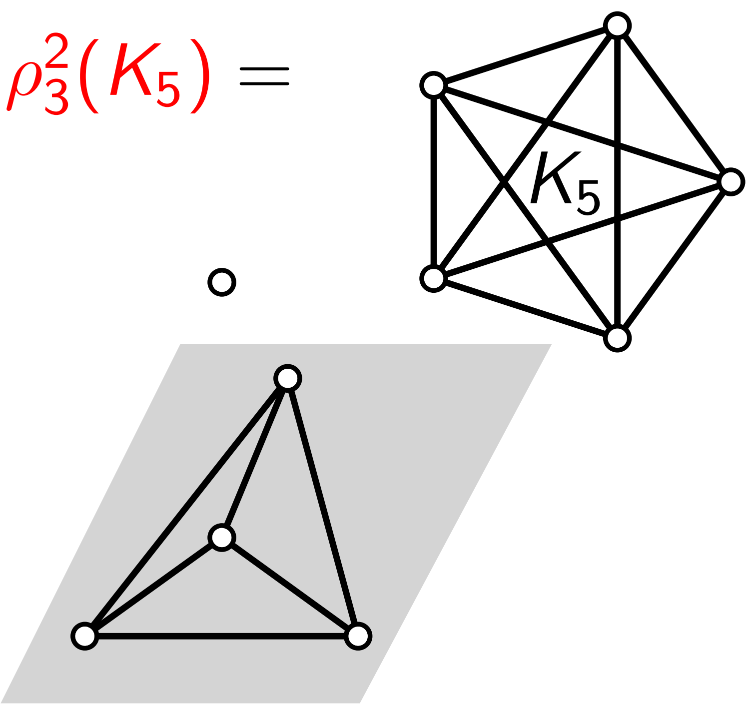

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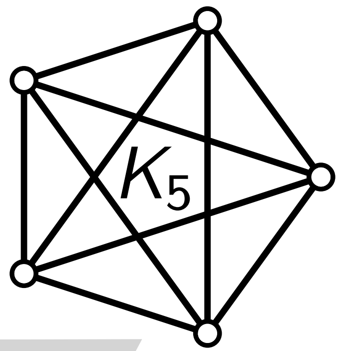


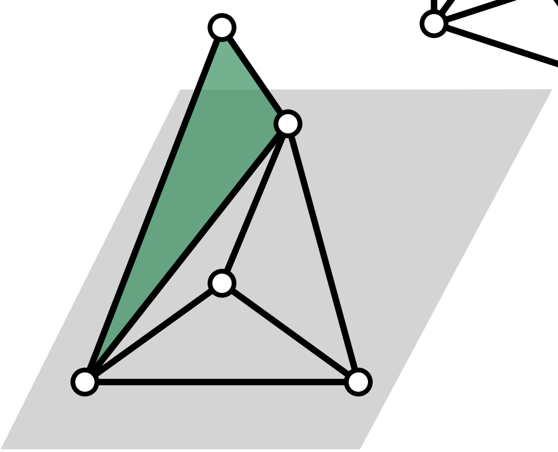

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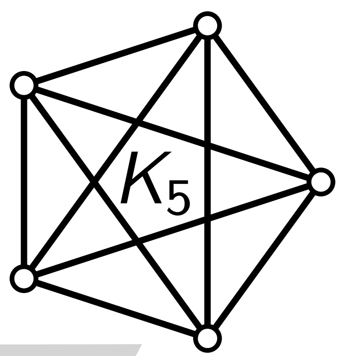


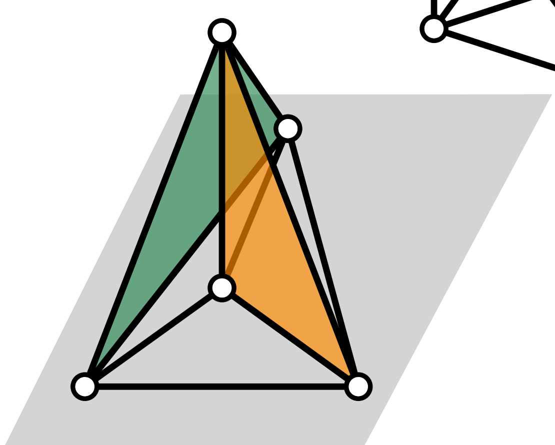

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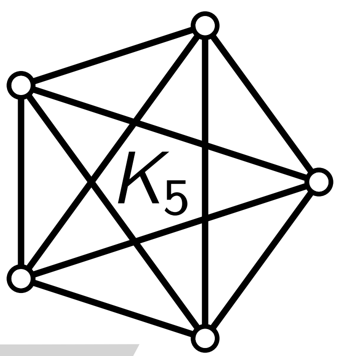


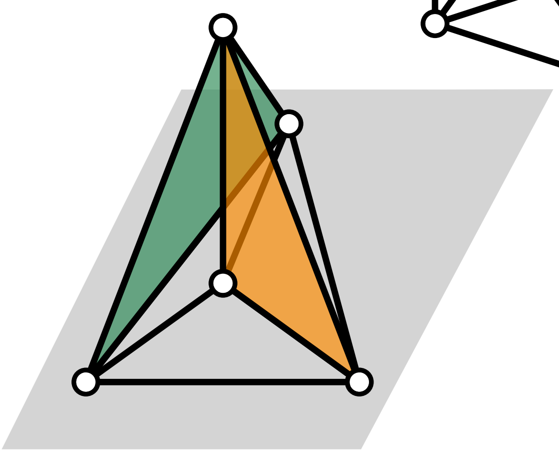

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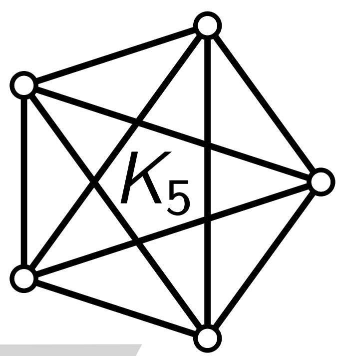
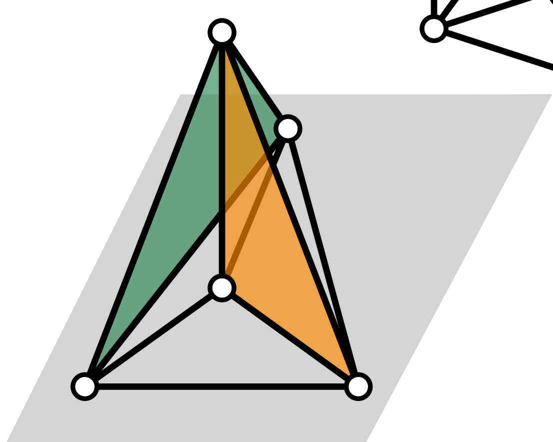
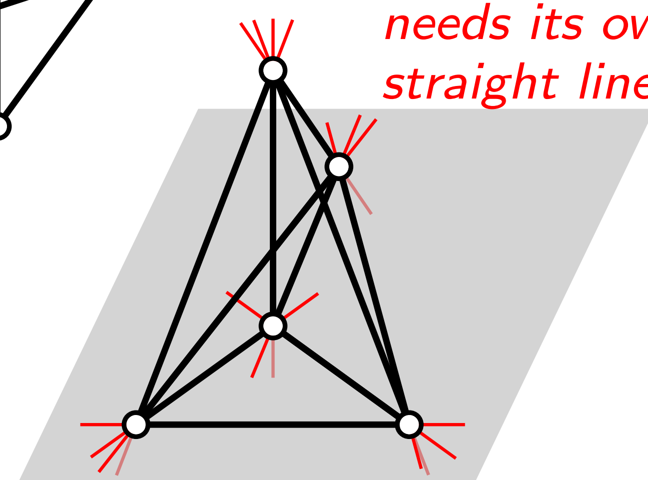


$\rho^1_3(K_5) = 10$

*Each edge needs its own straight line.*



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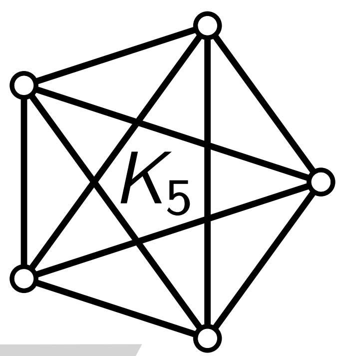


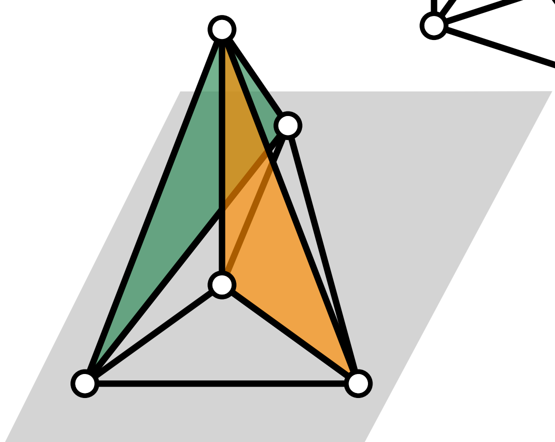

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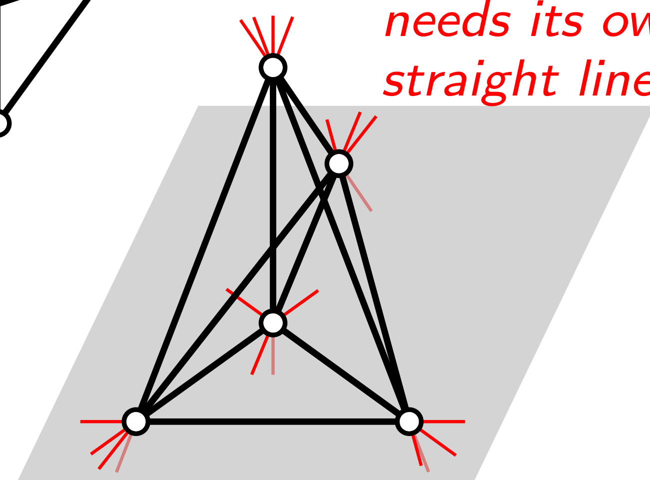
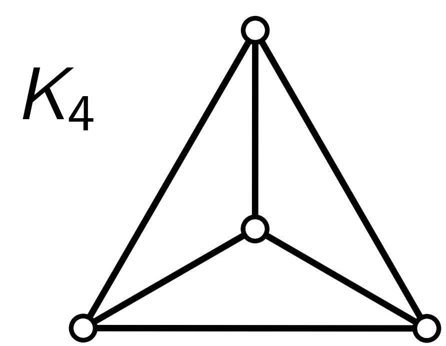


$\rho_2^1(K_4) =$



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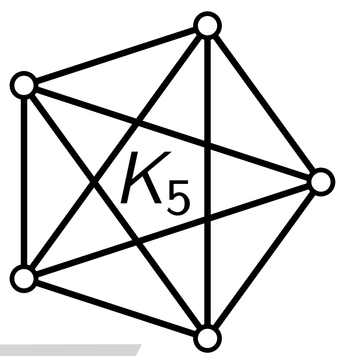


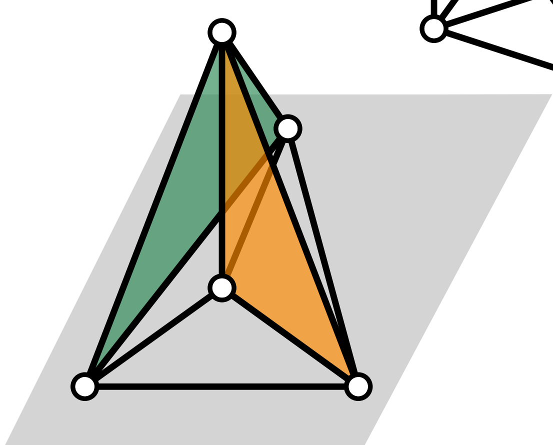

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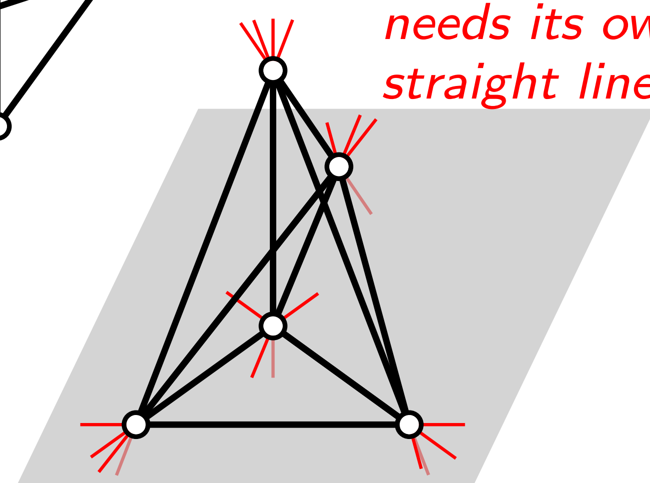
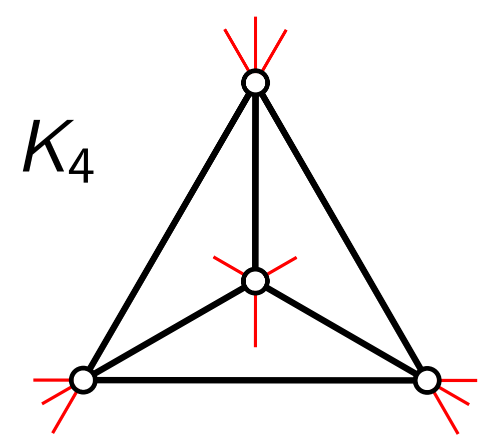


$\rho^1_2(K_4) = 6$



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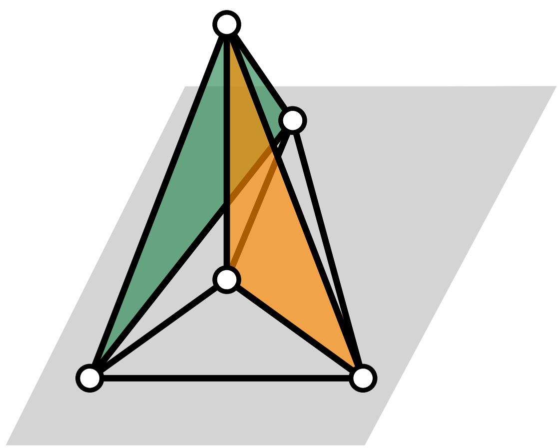

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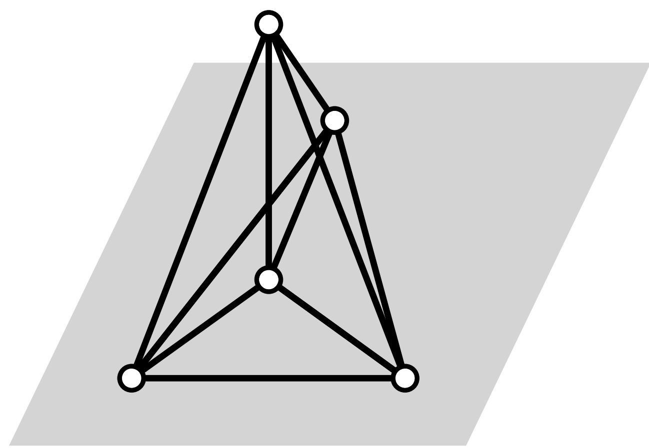

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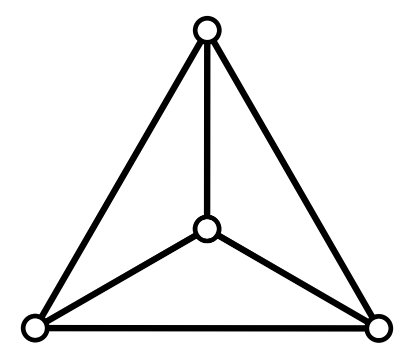

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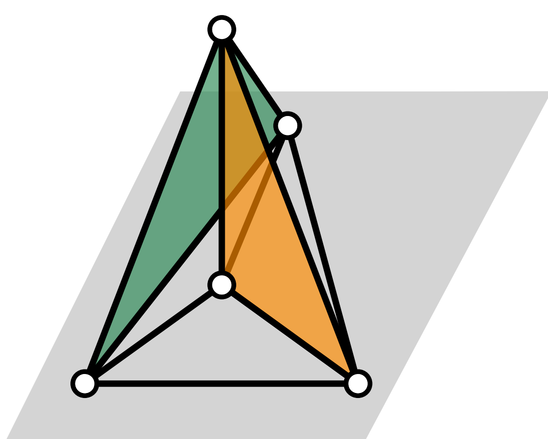

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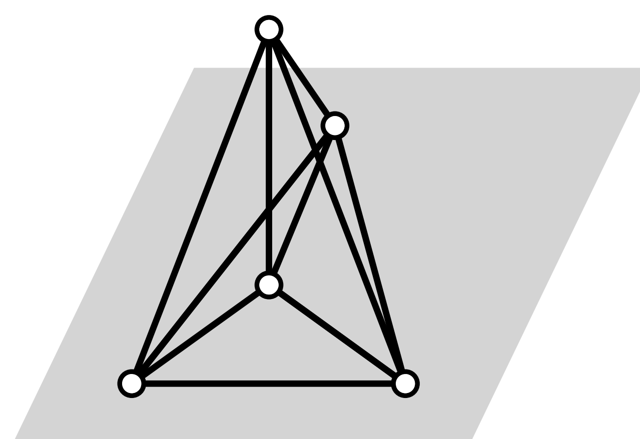

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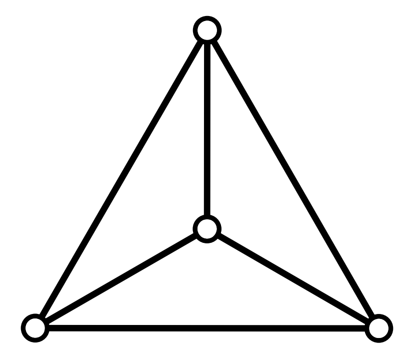

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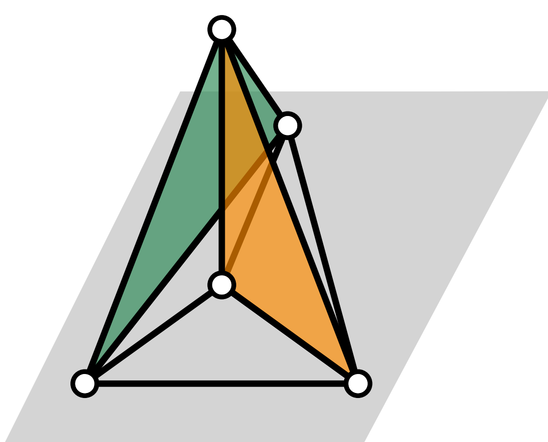

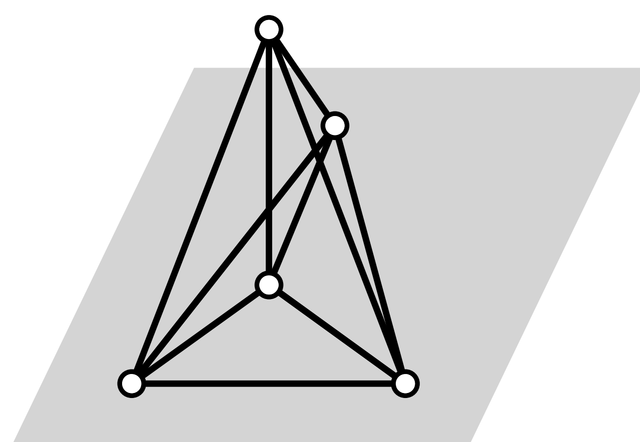

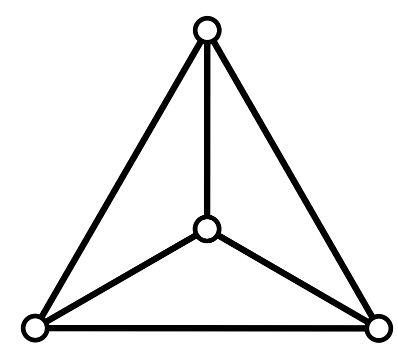

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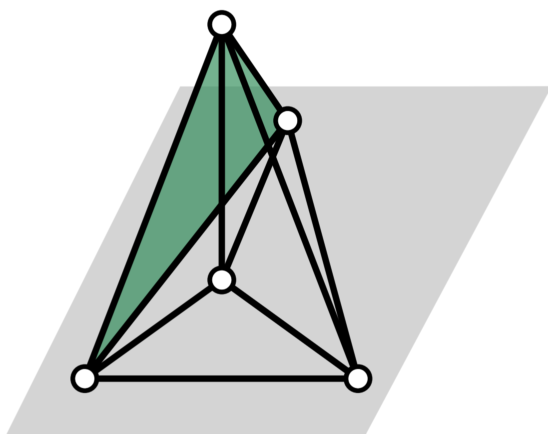
$\rho_3^2(K_5) = 3$
$\pi_3^2(K_5) = 2$



$\rho_3^1(K_5) = 10$
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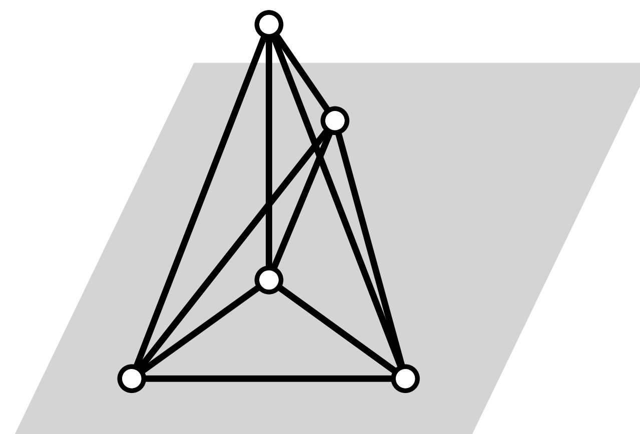

$\rho_2^1(K_4) = 6$
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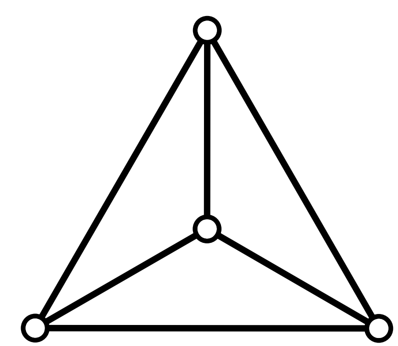

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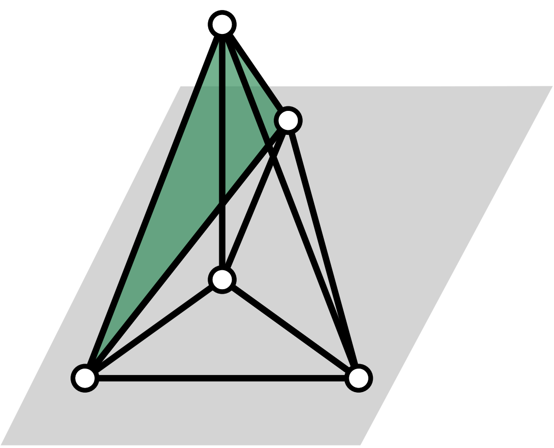
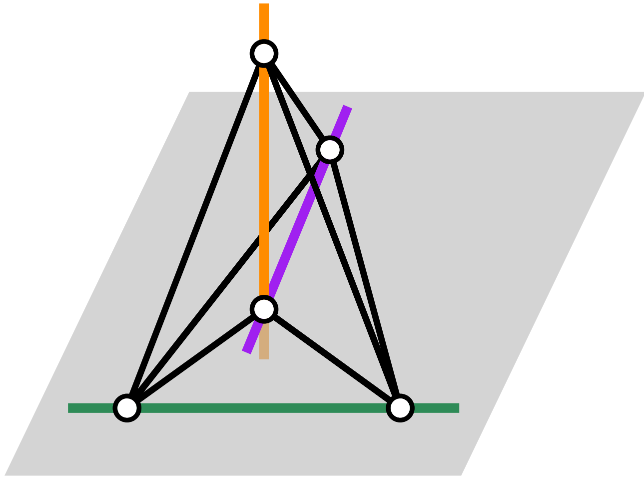
$\rho_3^2(K_5) = 3$

$\pi_3^2(K_5) = 2$



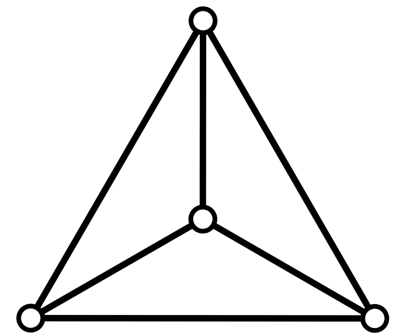
$\rho_3^1(K_5) = 10$

$\pi_3^1(K_5) = 3$



$\rho_2^1(K_4) = 6$

$\pi_2^1(K_4) =$



# Affine cover numbers

[Chaplick et al., GD'16]

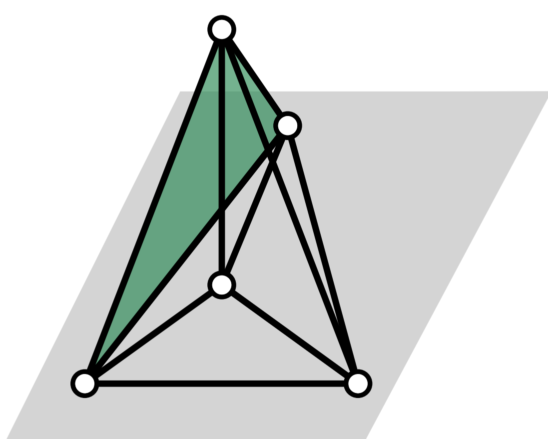
Let $G$ be a graph and let $1 \leq m < d$.

All drawings are straight-line and crossing-free.

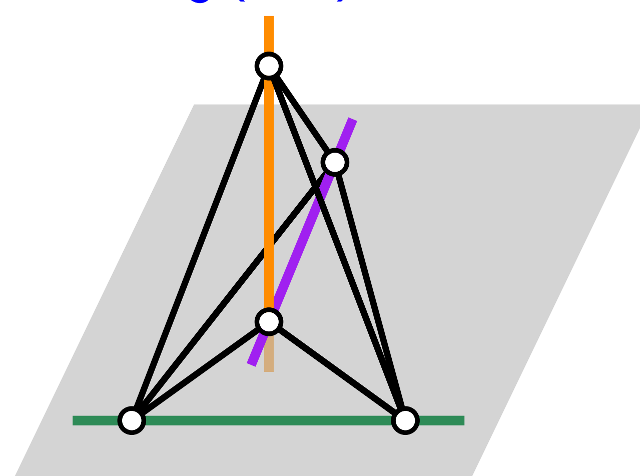
weak

**Def.** The *m-dimensional affine cover number* $\pi_d^m(G)$ ~~$\rho_d^m(G)$~~ is the minimum number of $m$-dimensional planes in $\mathbb{R}^d$ such that the vertices ~~and the edges~~ of a drawing of $G$ are contained in the union of these planes.

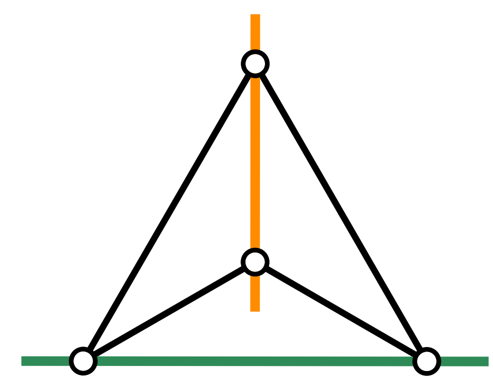
$\rho_3^2(K_5) = 3$
$\pi_3^2(K_5) = 2$



$\rho_3^1(K_5) = 10$
$\pi_3^1(K_5) = 3$



$\rho_2^1(K_4) = 6$
$\pi_2^1(K_4) = 2$



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Let $G$ be a graph and let $1 \leq m < d$.

All drawings are straight-line and crossing-free.

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$\rho_3^2$ $\rho_3^1$ $\rho_2^1$

$\pi_3^2$ $\pi_3^1$ $\pi_2^1$

# Affine cover numbers

[Chaplick et al., GD'16]

Let $G$ be a graph and let $1 \leq m < d$.

All drawings are straight-line and crossing-free.

weak

$\pi_d^m(G)$

**Def.** The *m-dimensional affine cover number* ~~$\rho_d^m(G)$~~ is the minimum number of $m$-dimensional planes in $\mathbb{R}^d$ such that the vertices ~~and the edges~~ of a drawing of $G$ are contained in the union of these planes.

$\rho_3^2$ $\rho_3^1$ $\rho_2^1$

$\pi_3^2$ $\pi_3^1$ $\pi_2^1$

**Thm.** Collapse of the Multidimensional Affine Hierarchy

For any integers $1 \leq l < 3 \leq d$ and for any graph $G$, it holds that $\pi_d^l(G) = \pi_3^l(G)$ and $\rho_d^l(G) = \rho_3^l(G)$.

# Complexity of affine cover numbers

[Chaplick et al., WADS'17]

Let $G$ be a graph and let $1 \leq m < d$.

All drawings are straight-line and crossing-free.

**Def.** The weak *m-dimensional affine cover number* ~~$\rho_d^m(G)$~~ $\pi_d^m(G)$ is the minimum number of $m$-dimensional planes in $\mathbb{R}^d$ such that the vertices ~~and the edges~~ of a drawing of $G$ are contained in the union of these planes.

$\rho_3^2$ $\rho_3^1$ $\rho_2^1$

$\pi_3^2$ $\pi_3^1$ $\pi_2^1$

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| | | | | | |
|---|---|---|---|---|---|
| $\rho_3^2$ | **NP-hard** | $\rho_3^1$ | **NP-hard** | $\rho_2^1$ | **NP-hard** |
| $\pi_3^2$ | **NP-hard** | $\pi_3^1$ | **NP-hard** | $\pi_2^1$ | |

**Thm.** Collapse of the Multidimensional Affine Hierarchy

For any integers $1 \leq l < 3 \leq d$ and for any graph $G$, it holds that $\pi_d^l(G) = \pi_3^l(G)$ and $\rho_d^l(G) = \rho_3^l(G)$.

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Let $G$ be a graph and let $1 \leq m < d$.

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| | | | | | |
|---|---|---|---|---|---|
| $\rho_3^2$ | **NP-hard** | $\rho_3^1$ | **NP-hard** | $\rho_2^1$ | **NP-hard** |
| $\pi_3^2$ | **NP-hard** | $\pi_3^1$ | **NP-hard** | $\pi_2^1$ | **? OPEN** |

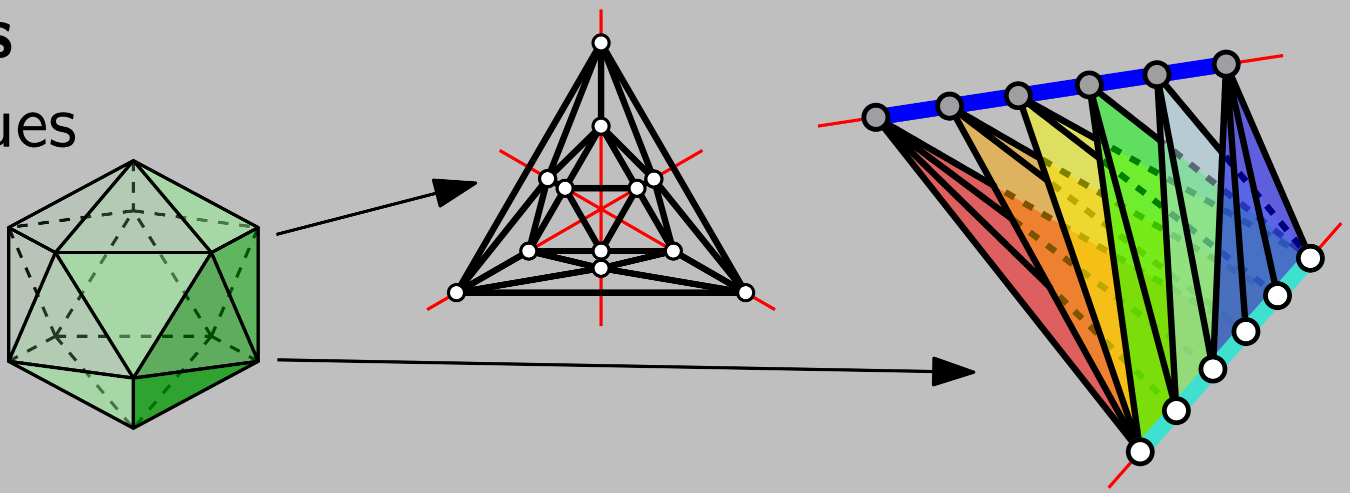
**Thm.** Collapse of the Multidimensional Affine Hierarchy

For any integers $1 \leq l < 3 \leq d$ and for any graph $G$, it holds that $\pi_d^l(G) = \pi_3^l(G)$ and $\rho_d^l(G) = \rho_3^l(G)$.

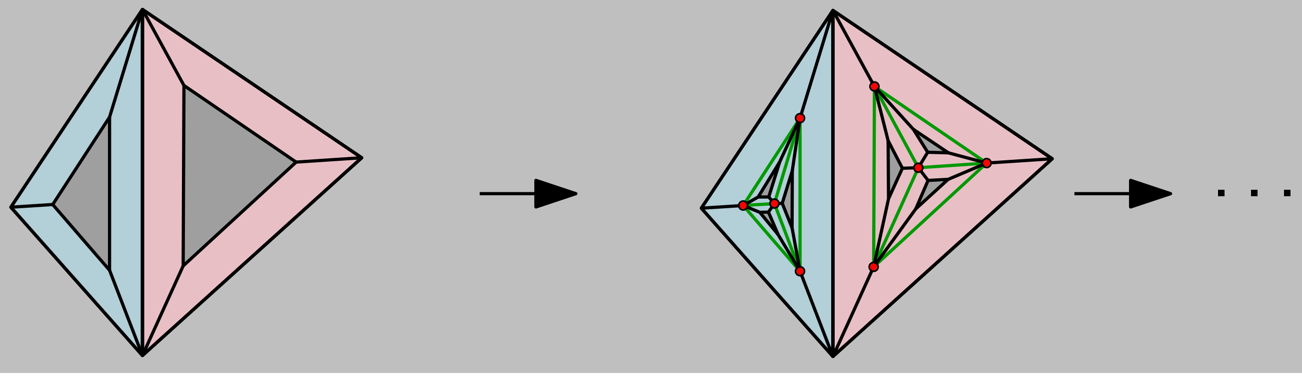
# Overview

- Notation

- **Platonic solids**
  $\pi_2^1$- and $\pi_3^1$-values



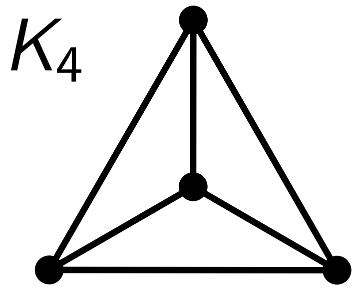
- **Main contribution**
  Infinite family of planar graphs with unbounded $\pi_2^1$-value



# Notation

*Linear vertex arboricity* lva($G$) of a graph $G$:
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
such that every $V_i$ induces a linear forest.

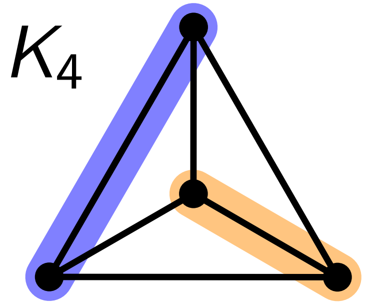
lva($K_4$) $=$



# Notation

*Linear vertex arboricity* $\mathrm{lva}(G)$ of a graph $G$:
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
such that every $V_i$ induces a linear forest.

$\mathrm{lva}(K_4) = 2$



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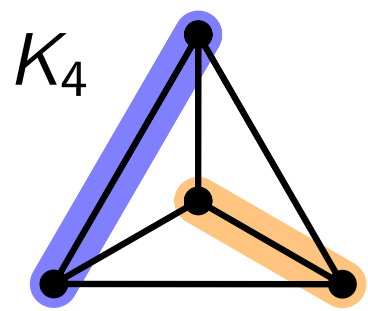
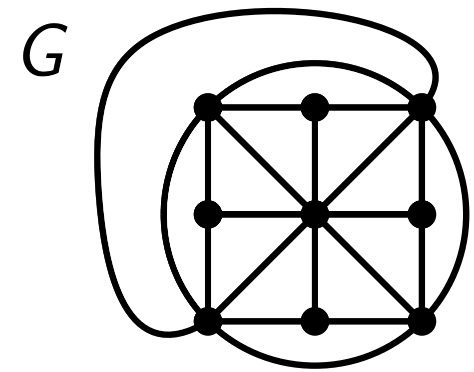


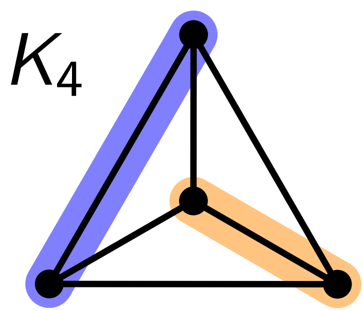

$\mathrm{lva}(G) =$



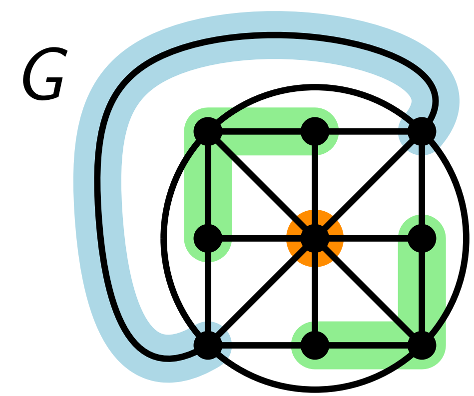
# Notation

*Linear vertex arboricity* lva$(G)$ of a graph $G$:
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
such that every $V_i$ induces a linear forest.

lva$(K_4) = 2$



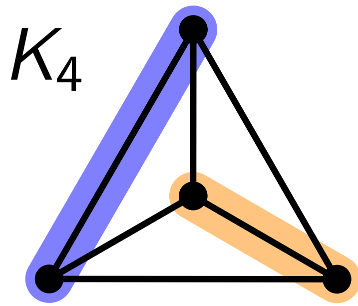

lva$(G) = 3$



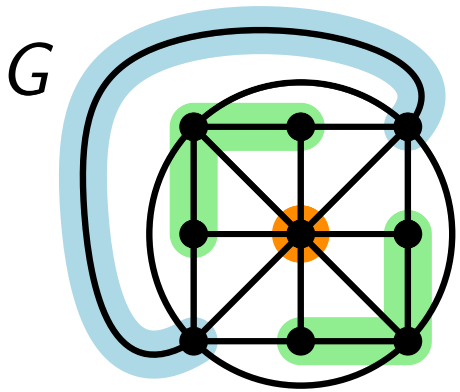
# Notation

*Linear vertex arboricity* lva($G$) of a graph $G$:
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
such that every $V_i$ induces a linear forest.

lva($K_4$) $= 2$




lva($G$) $= 3$




*Treewidth* tw($G$) of a graph $G$:

*upper bound*

tw($G$) $\leq k$
if $G$ is a subgraph of a $k$-tree.

# Notation

*Linear vertex arboricity* $\mathrm{lva}(G)$ of a graph $G$:
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
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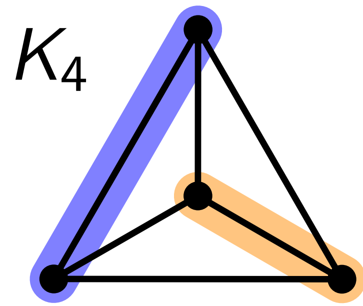
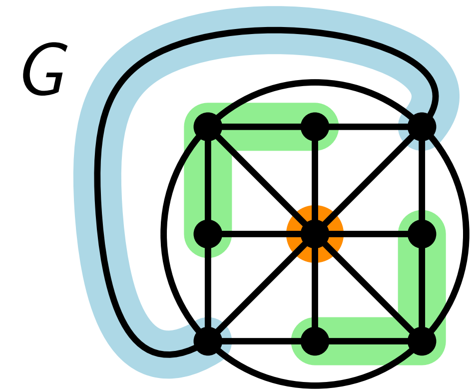
$\mathrm{lva}(K_4) = 2$




$\mathrm{lva}(G) = 3$

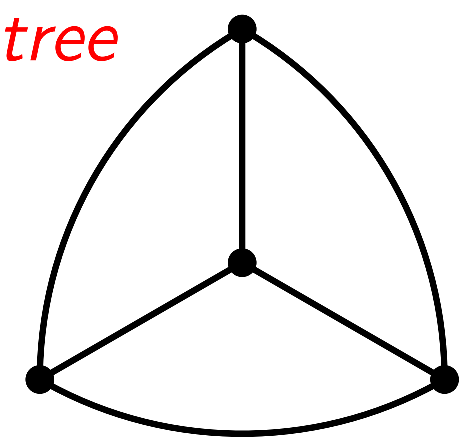



*Treewidth* $\mathrm{tw}(G)$ of a graph $G$:

*upper bound*

$\mathrm{tw}(G) \leq k$
if $G$ is a subgraph of a $k$-tree.

3-*tree*



start
with
$K_4 \ldots$

# Notation

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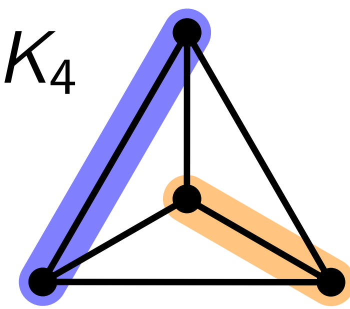
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
such that every $V_i$ induces a linear forest.

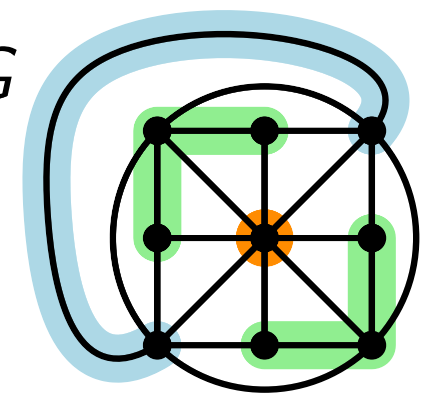
lva($K_4$) $= 2$

$K_4$



lva($G$) $= 3$

$G$

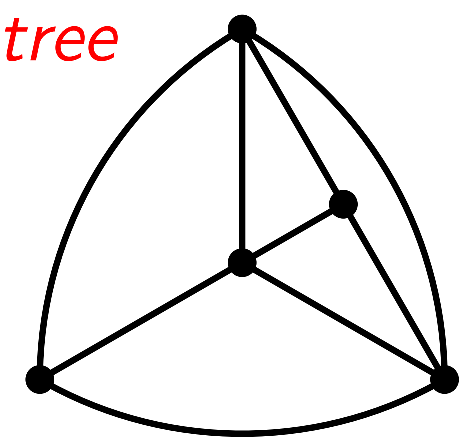


*Treewidth* tw($G$) of a graph $G$:

*upper bound*

tw($G$) $\leq k$
if $G$ is a subgraph of a $k$-tree.

3-*tree*



start
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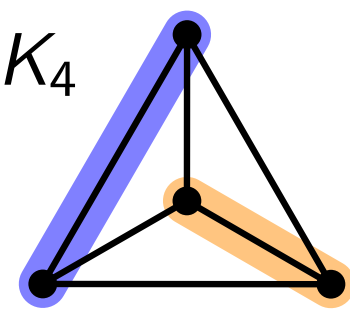
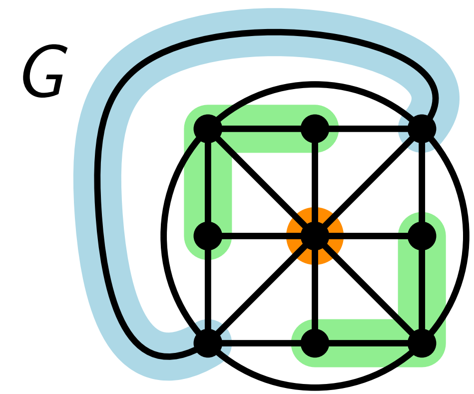
$\text{lva}(K_4) = 2$




$\text{lva}(G) = 3$




*Treewidth* tw($G$) of a graph $G$:

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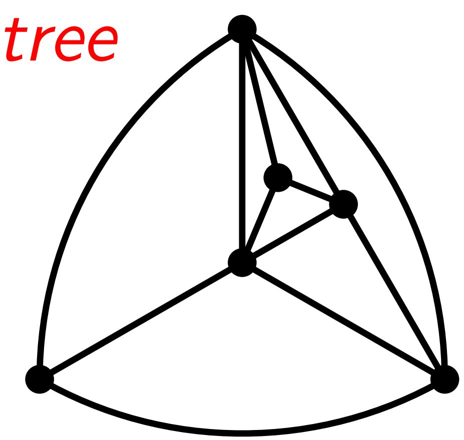
3-*tree*



start
with
$K_4 \ldots$

# Notation

*Linear vertex arboricity* lva($G$) of a graph $G$:
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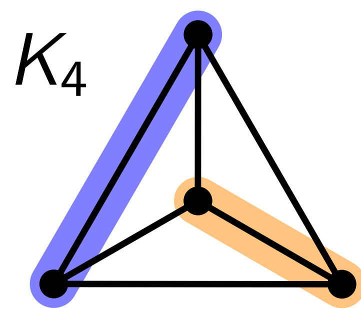
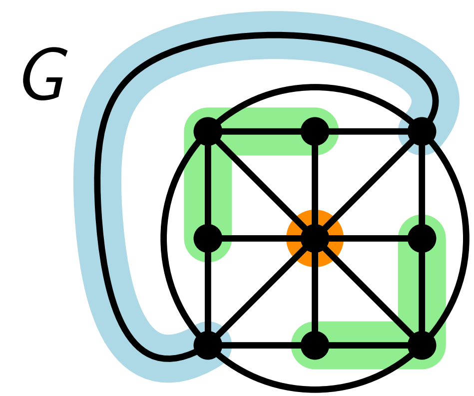
lva($K_4$) $= 2$




lva($G$) $= 3$




*Treewidth* tw($G$) of a graph $G$:

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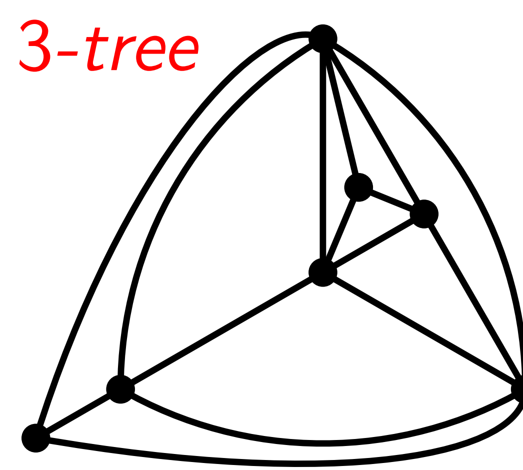


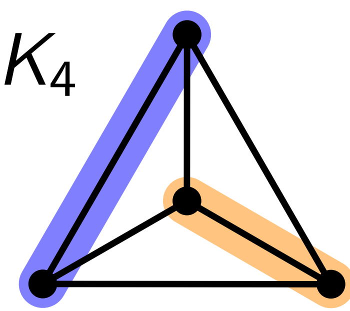

start
with
$K_4 \ldots$

# Notation

*Linear vertex arboricity* lva($G$) of a graph $G$:
smallest size $r$ of a partition of $V(G) = V_1 \cup \cdots \cup V_r$
such that every $V_i$ induces a linear forest.

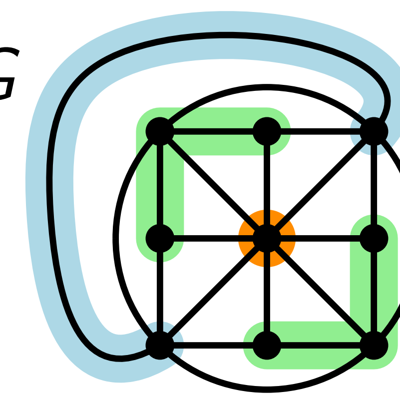
lva($K_4$) $= 2$

$K_4$



lva($G$) $= 3$

$G$



*Treewidth* tw($G$) of a graph $G$:

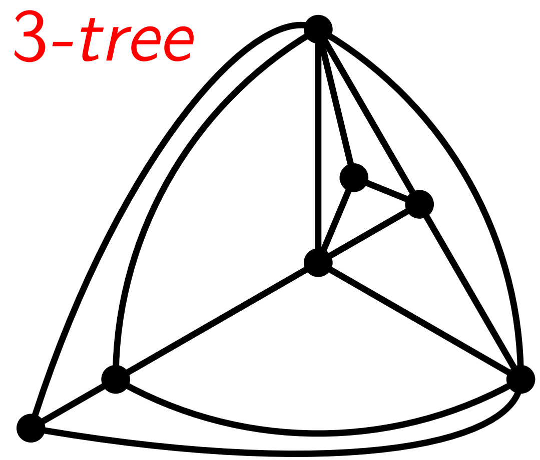
*upper bound*

$\mathrm{tw}(G) \le k$
if $G$ is a subgraph of a $k$-tree.

*lower bound*

$\mathrm{tw}(G) \ge \mathrm{mindeg}(G)$.

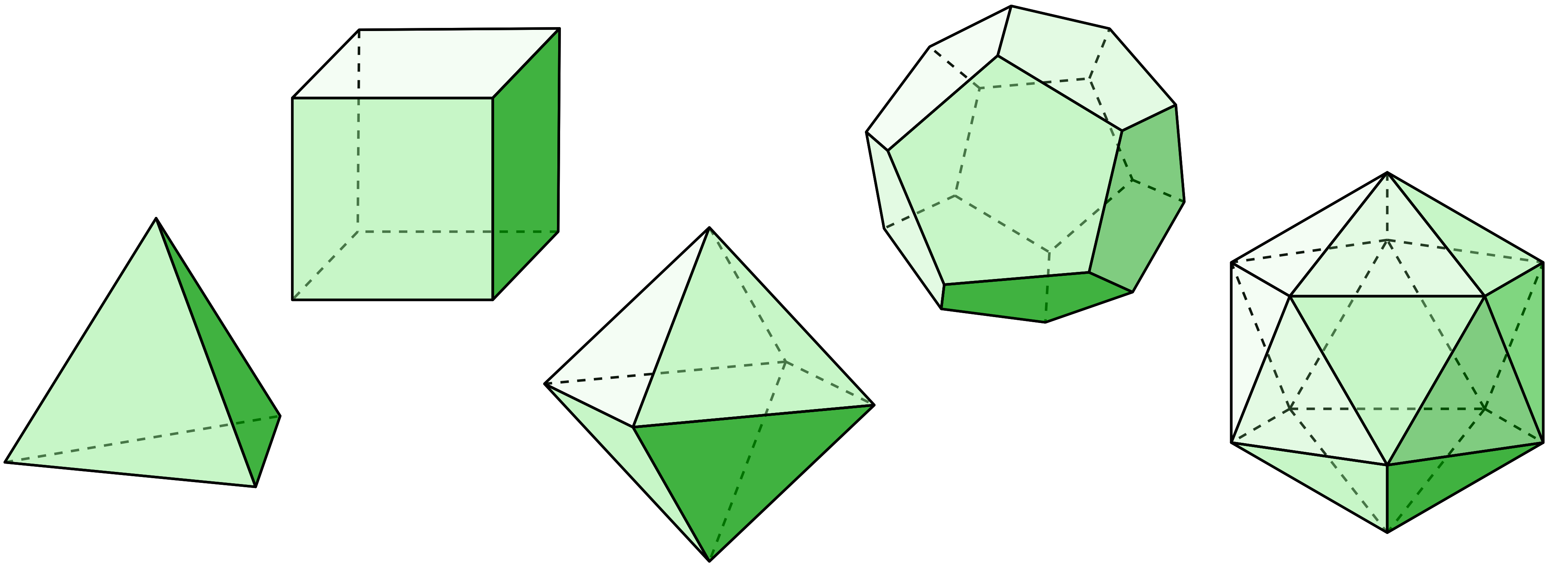
3-*tree*



start with $K_4 \ldots$

# Platonic solids

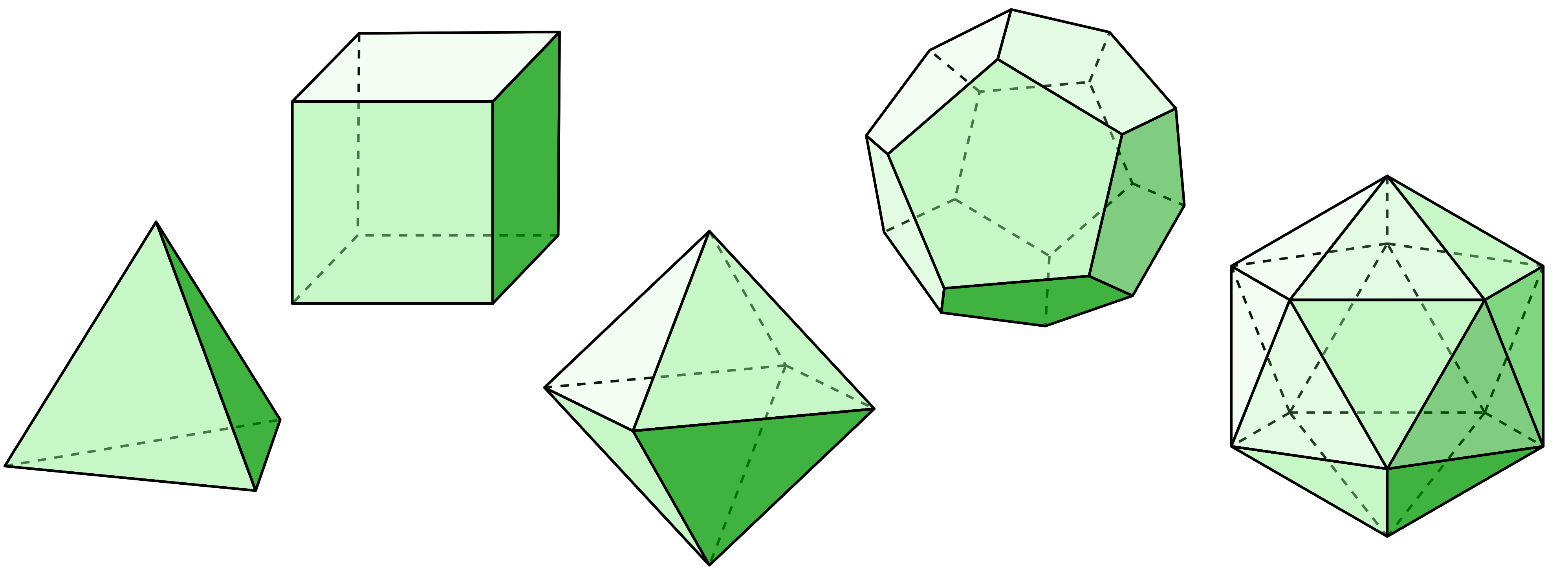
| $G = (V, E)$ | $\lvert V \rvert$ | $\lvert E \rvert$ | $\lvert F \rvert$ | $\rho_2^1(G)$ | $\rho_3^1(G)$ | $\pi_2^1(G)$ | $\pi_3^1(G)$ |
|---|---|---|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | | | | |
| cube | 8 | 12 | 6 | | | | |
| octahedron | 6 | 12 | 8 | | | | |
| dodecahedron | 20 | 30 | 12 | | | | |
| icosahedron | 12 | 30 | 20 | | | | |



# Platonic solids

[Kryven et al., CALDAM'18]

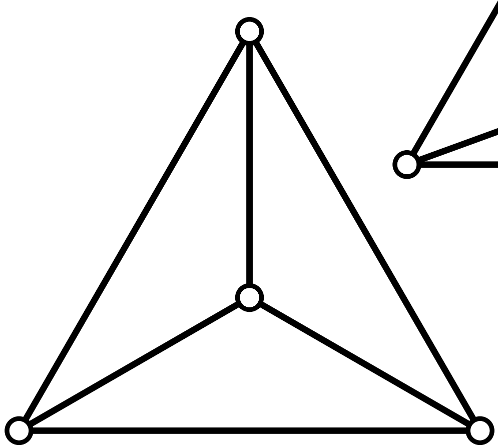
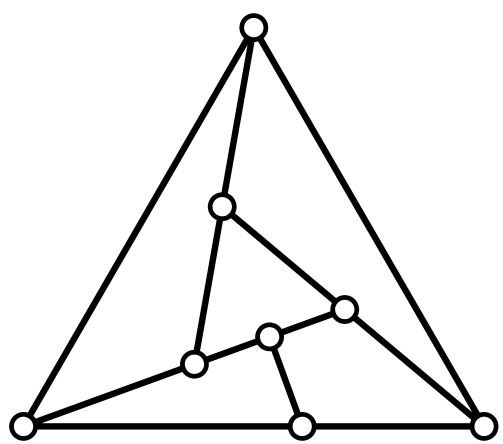
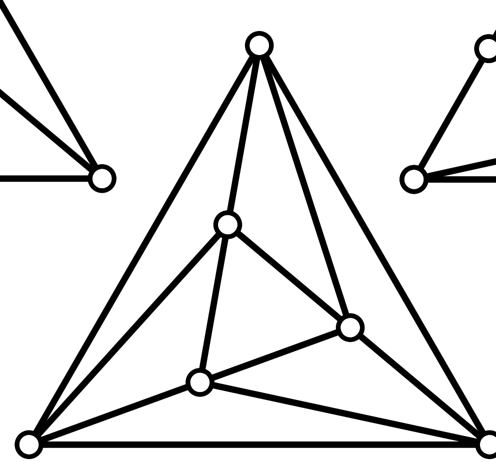
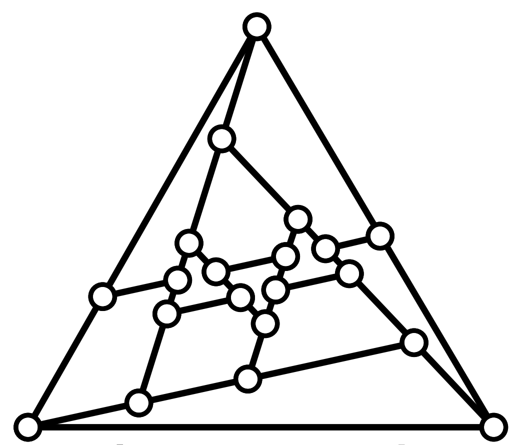
| $G = (V, E)$ | $\lvert V\rvert$ | $\lvert E\rvert$ | $\lvert F\rvert$ | $\rho_2^1(G)$ | $\rho_3^1(G)$ | $\pi_2^1(G)$ | $\pi_3^1(G)$ |
|---|---|---|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | | | | |
| cube | 8 | 12 | 6 | | | | |
| octahedron | 6 | 12 | 8 | | | | |
| dodecahedron | 20 | 30 | 12 | | | | |
| icosahedron | 12 | 30 | 20 | | | | |



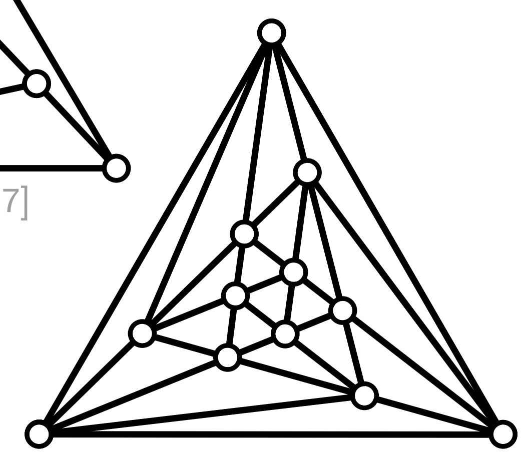
# Platonic solids

[Kryven et al., CALDAM'18]

| $G = (V, E)$ | $\lvert V \rvert$ | $\lvert E \rvert$ | $\lvert F \rvert$ | $\rho_2^1(G)$ | $\rho_3^1(G)$ | $\pi_2^1(G)$ | $\pi_3^1(G)$ |
|---|---|---|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | 6 | 6 | | |
| cube | 8 | 12 | 6 | 7 | 7 | | |
| octahedron | 6 | 12 | 8 | 9 | 9 | | |
| dodecahedron | 20 | 30 | 12 | $9 \ldots 10$ | $9 \ldots 10$ | | |
| icosahedron | 12 | 30 | 20 | $13 \ldots 15$ | $13 \ldots 15$ | | |

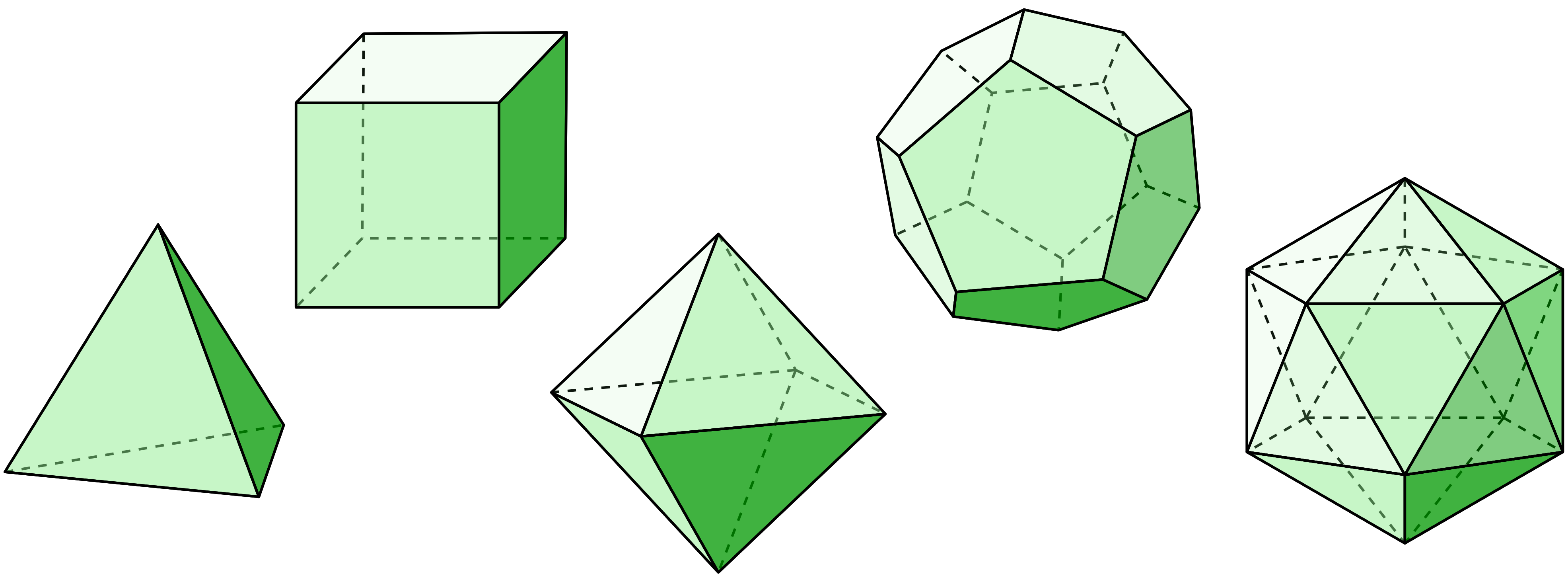
[Scherm, B.Th.'17]



# Platonic solids

[Kryven et al., CALDAM'18]

| $G = (V, E)$ | $\|V\|$ | $\|E\|$ | $\|F\|$ | $\rho^1_2(G)$ | $\rho^1_3(G)$ | $\pi^1_2(G)$ | $\pi^1_3(G)$ |
|---|---|---|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | 6 | 6 | | |
| cube | 8 | 12 | 6 | 7 | 7 | | |
| octahedron | 6 | 12 | 8 | 9 | 9 | | |
| dodecahedron | 20 | 30 | 12 | $9 \dots 10$ | $9 \dots 10$ | | |
| icosahedron | 12 | 30 | 20 | $13 \dots 15$ | $13 \dots 15$ | | |



# Platonic solids

[Kryven et al., CALDAM'18]

| $G=(V,E)$ | $\lvert V\rvert$ | $\lvert E\rvert$ | $\lvert F\rvert$ | $\rho_2^1(G)$ | $\rho_3^1(G)$ | $\pi_2^1(G)$ | $\pi_3^1(G)$ |
|---|---|---|---|---|---|---|---|
| tetrahedron | 4 | 6 | 4 | 6 | 6 | | |
| cube | 8 | 12 | 6 | 7 | 7 | | |
| octahedron | 6 | 12 | 8 | 9 | 9 | | |
| dodecahedron | 20 | 30 | 12 | 9...10 | 9...10 | | |
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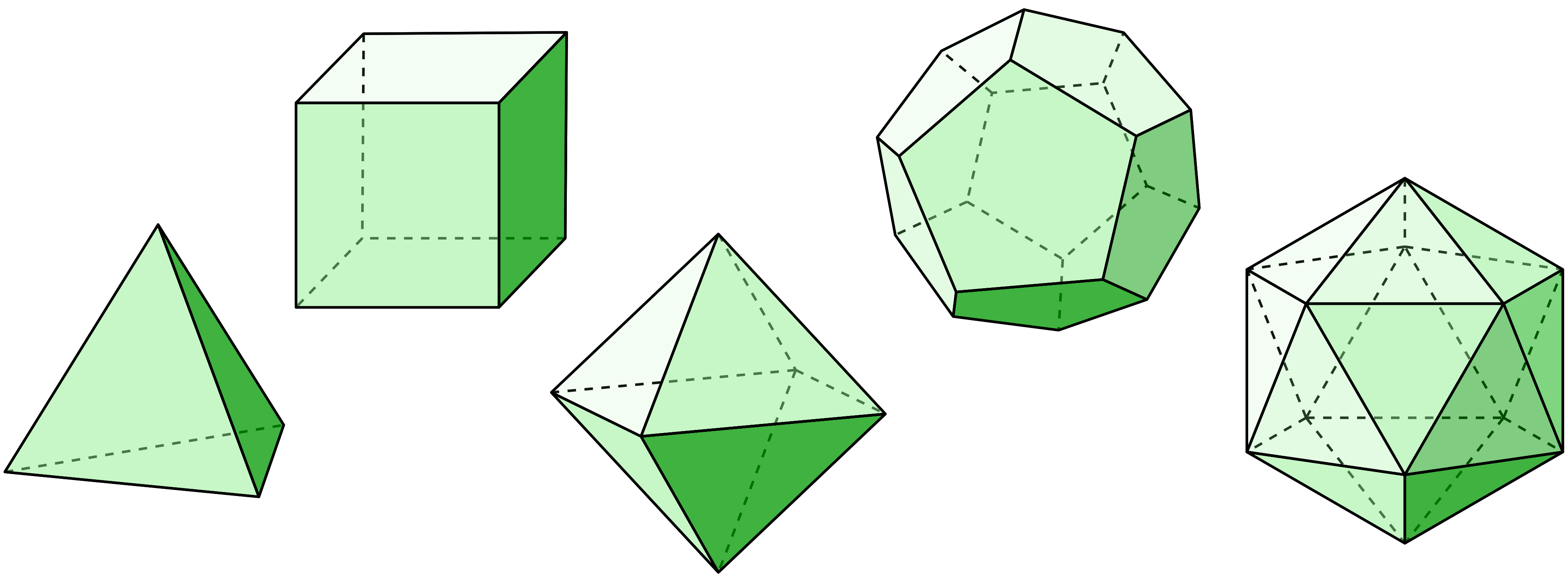

# Platonic solids

[Kryven et al., CALDAM'18]

| $G = (V, E)$ | $\|V\|$ | $\|E\|$ | $\|F\|$ | $\rho^1_2(G)$ | $\rho^1_3(G)$ | $\pi^1_2(G)$ | $\pi^1_3(G)$ |
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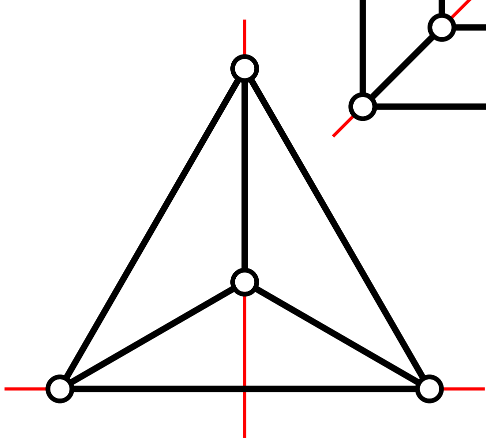
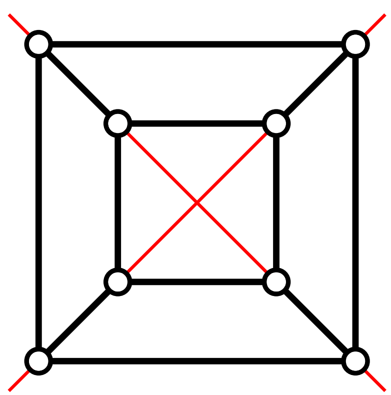
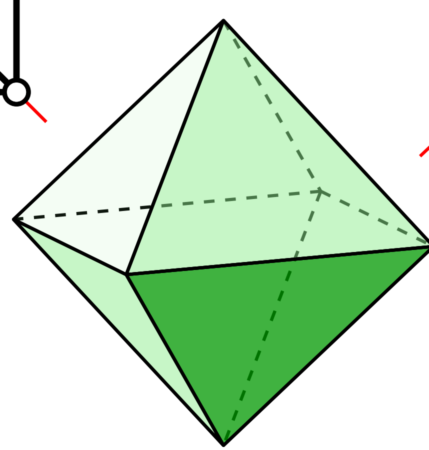
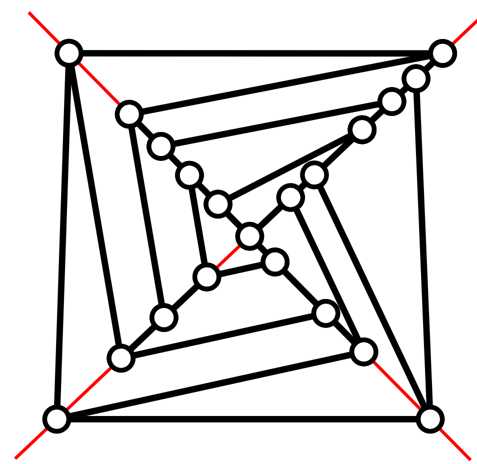
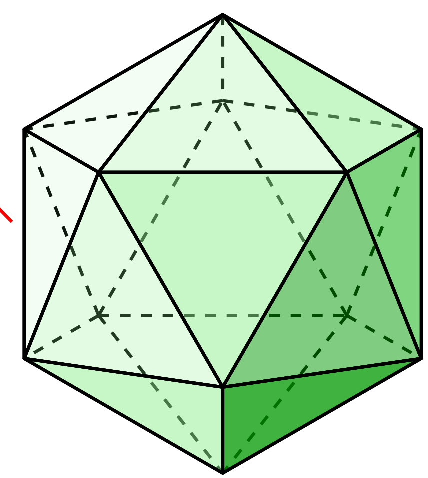

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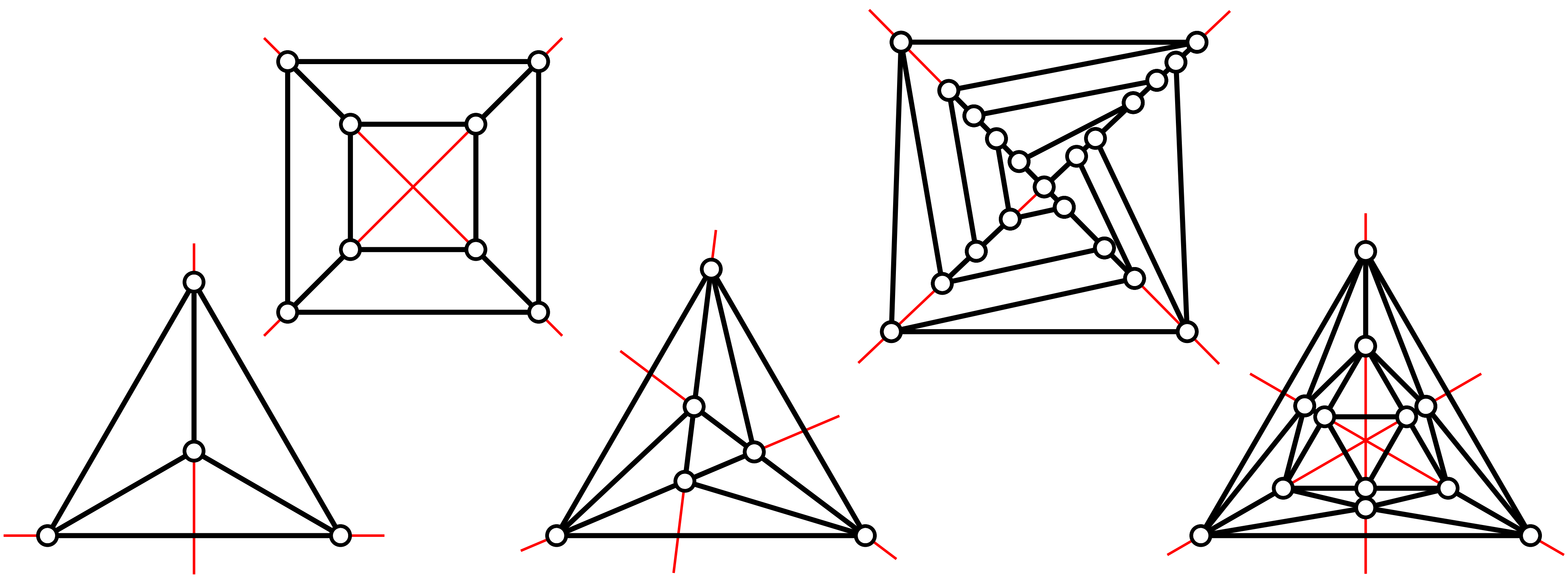

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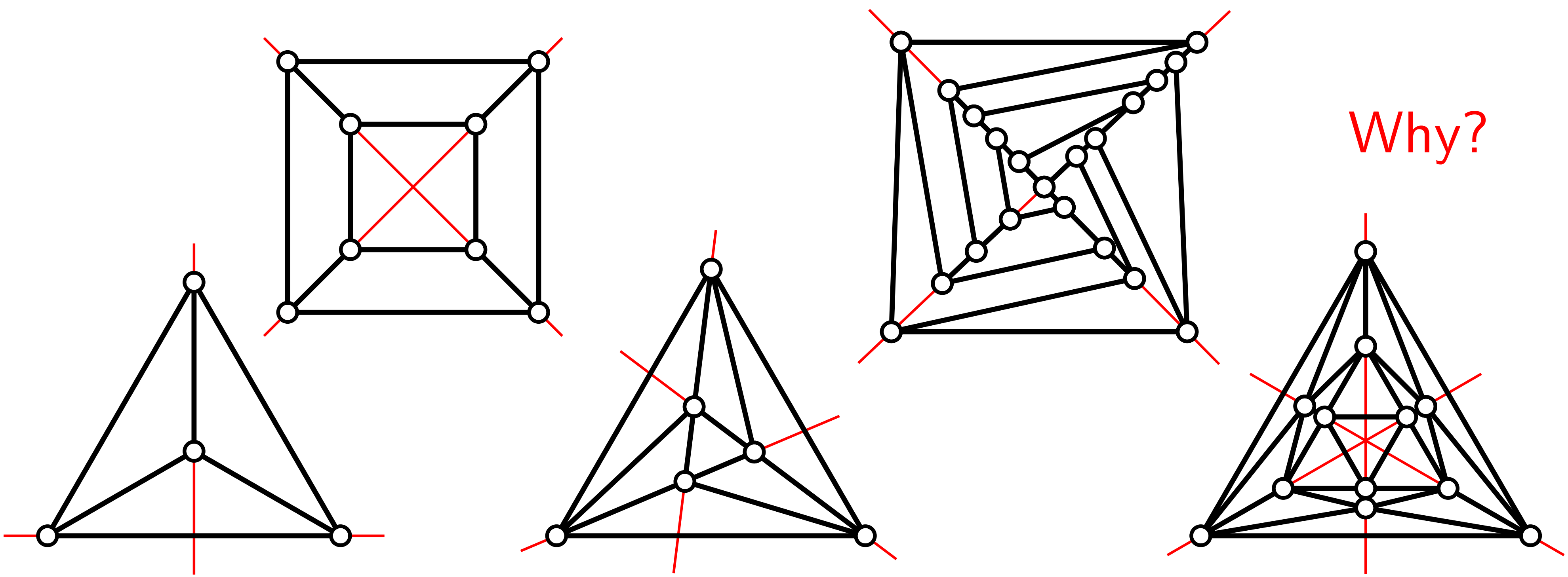

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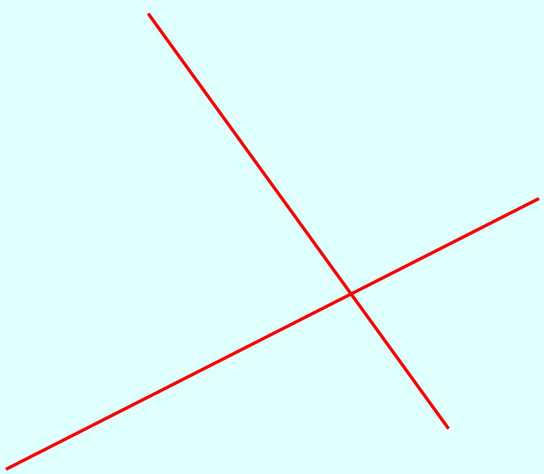
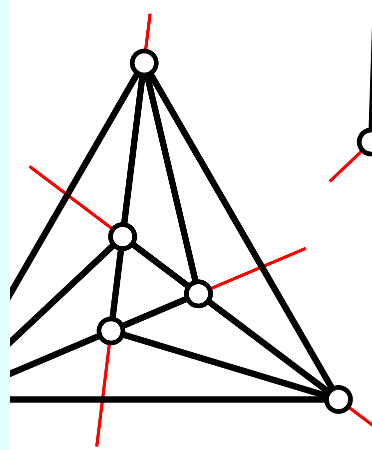
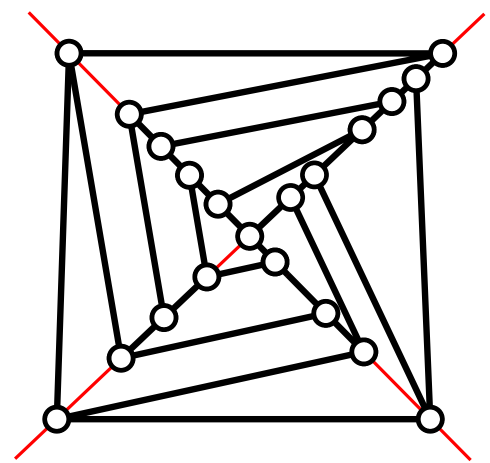
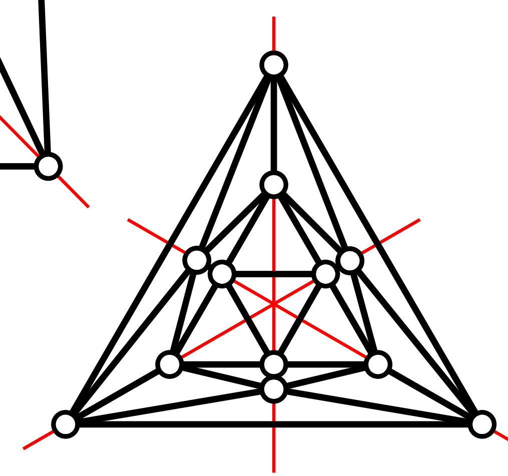
Proof







Why?



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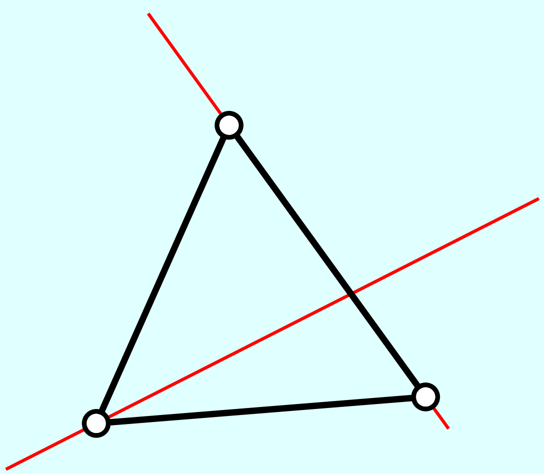

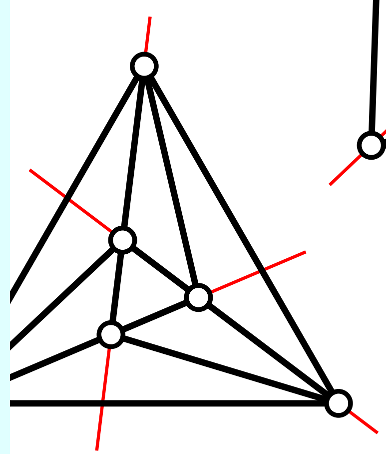

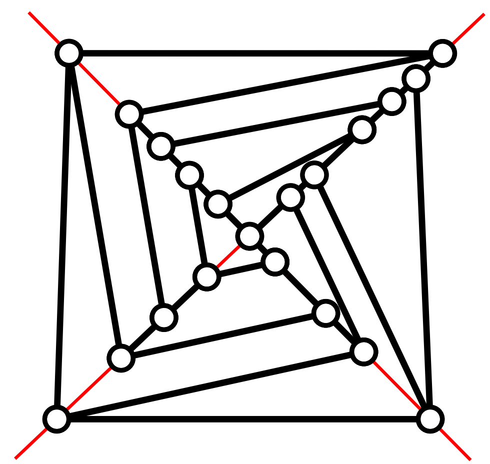

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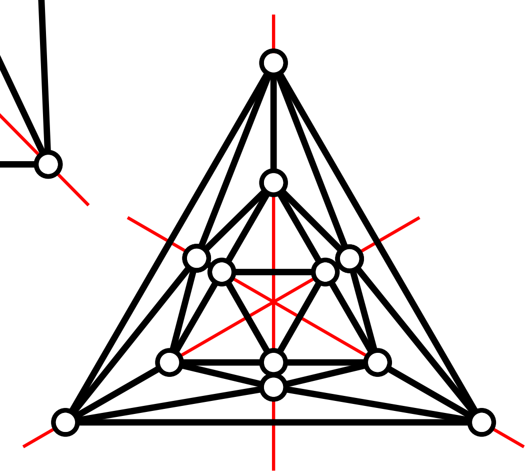

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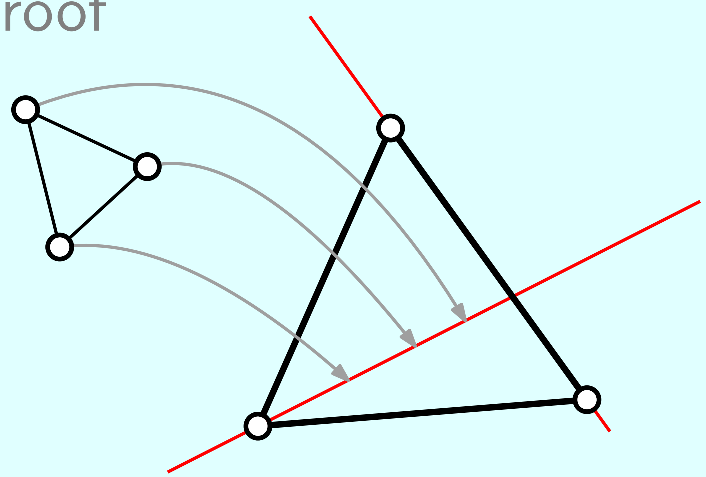

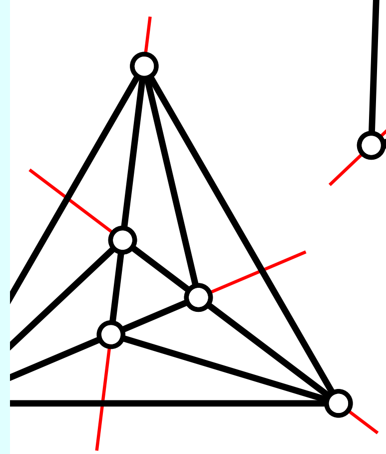

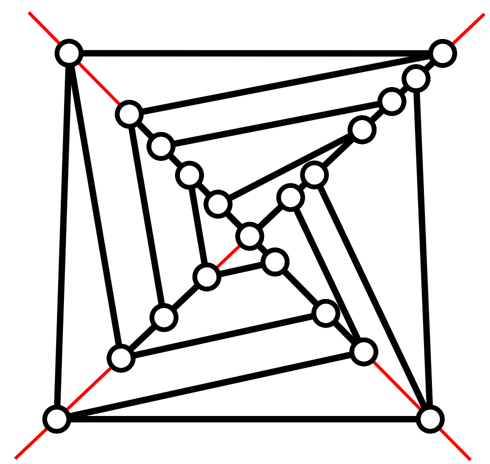

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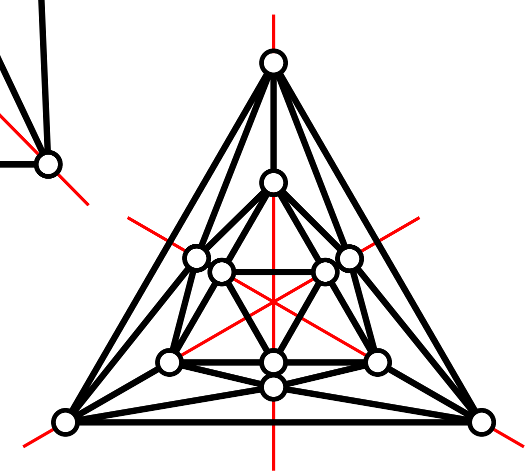

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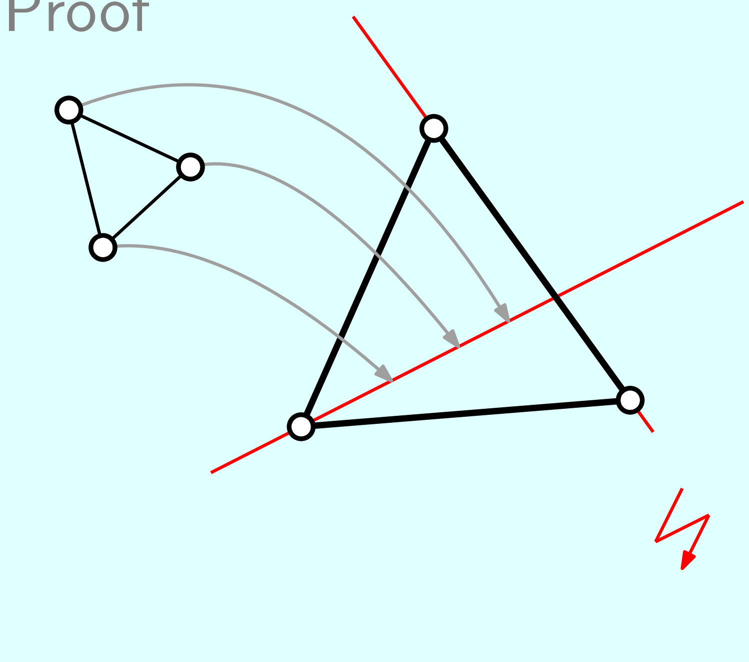

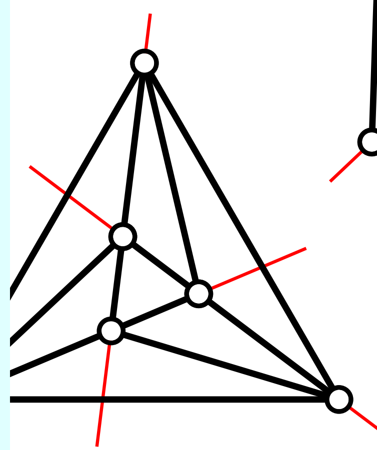

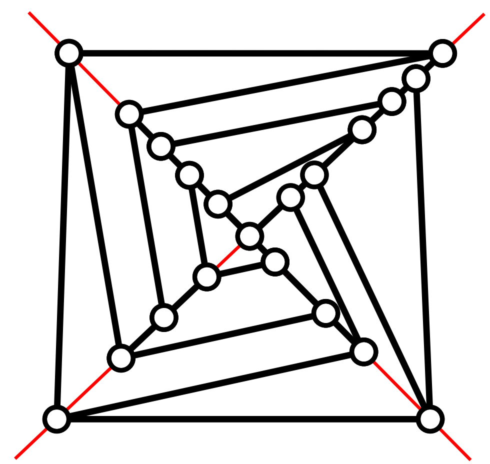

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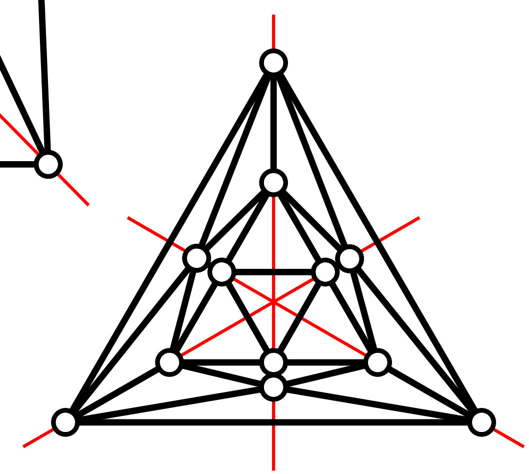

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$$\pi^1_3(G) = \mathrm{lva}(G)$$

[Chaplick et al., GD'16]

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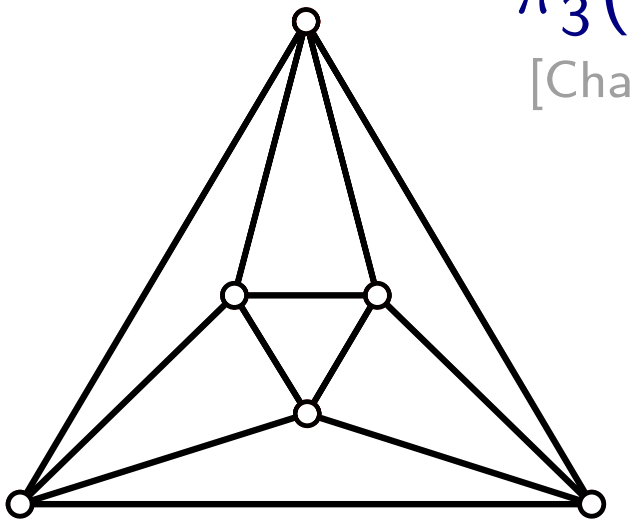

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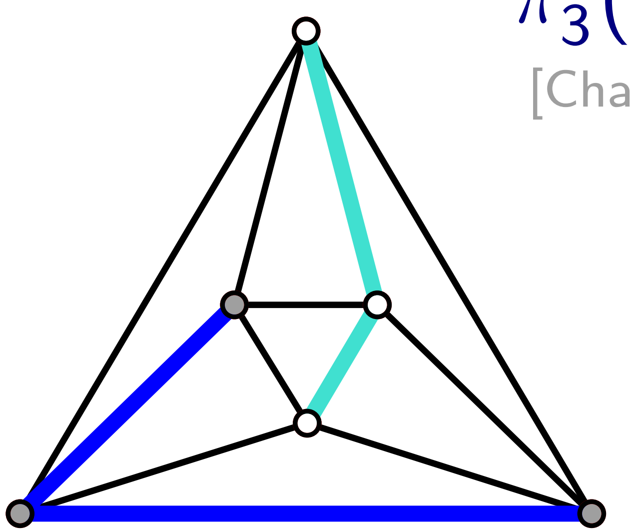

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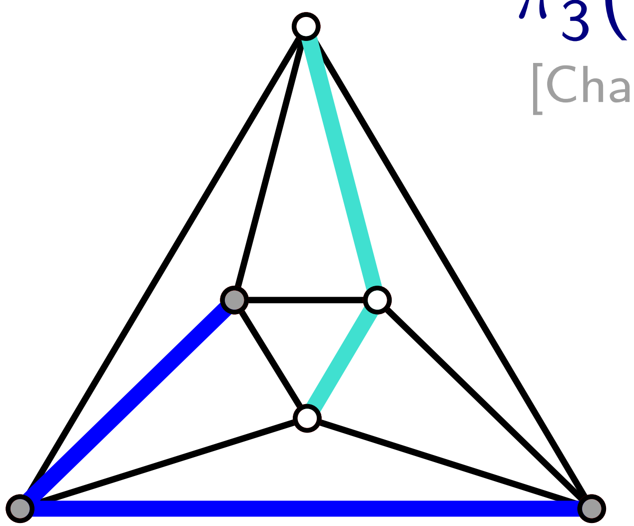

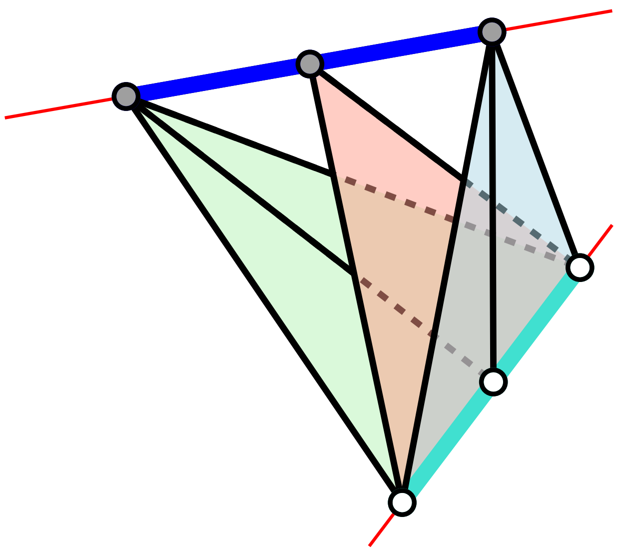

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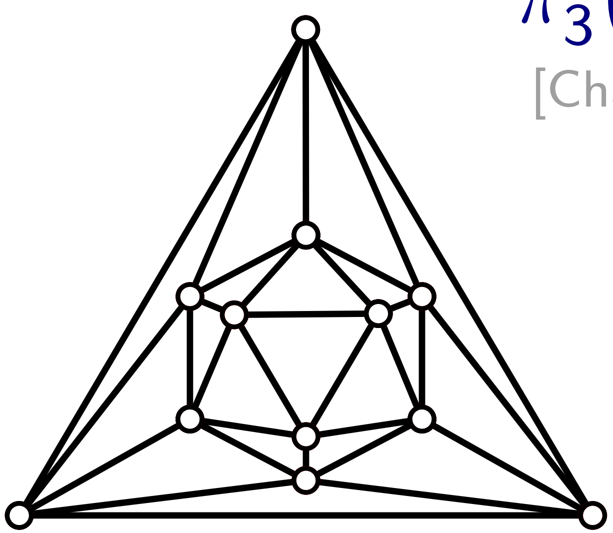

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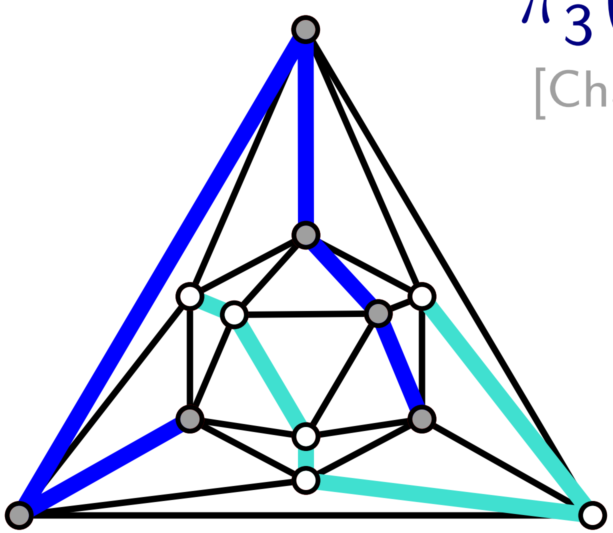

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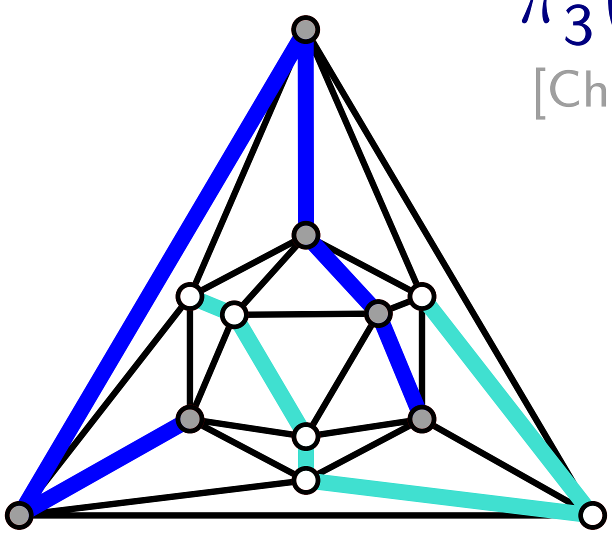

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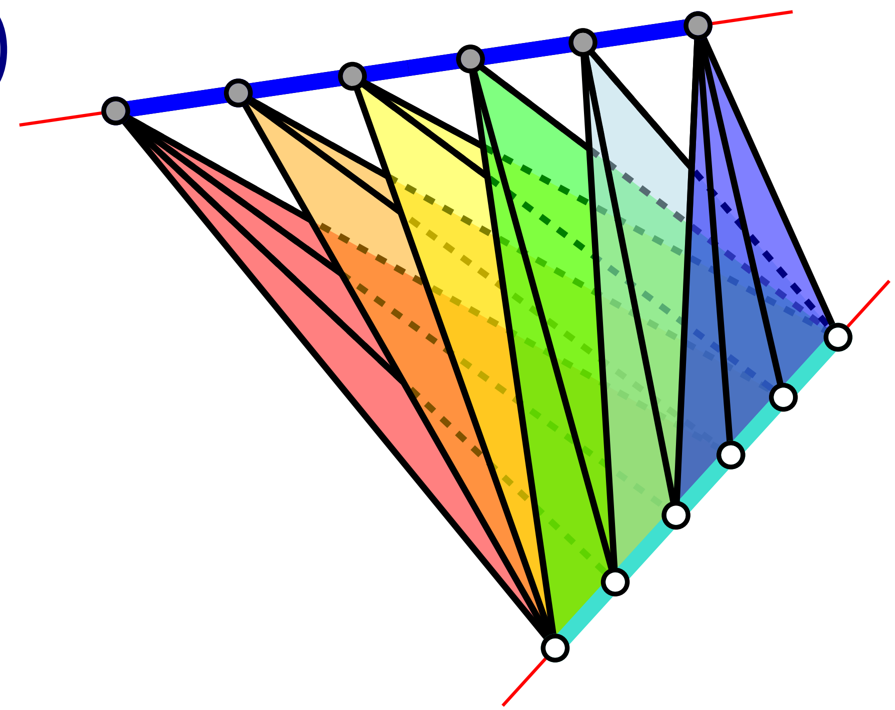

# Motivation

$\pi_2^1$ How large can it be?

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$\pi_2^1$

How large can it be?

Q: Is the $\pi_2^1$-value unbounded for some graph families?

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$\pi_{\frac{1}{2}}$

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Yes!

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[Ravsky and Verbitsky, WG'11]
[Da Lozzo et al., GD'16]

- $\mathrm{tw}(G_i) = 5$

# Motivation

$\pi^1_2$ How large can it be?

Q: Is the $\pi^1_2$-value unbounded for some graph families?

Yes!

[Ravsky and Verbitsky, WG'11]
[Da Lozzo et al., GD'16]

- $\text{tw}(G_i) = 5$

[Chaplick et al., GD'16]

- $\text{maxdeg}(G_i) \leq 12$
  $\pi^1_2(G_i) \geq n^{0.01}$

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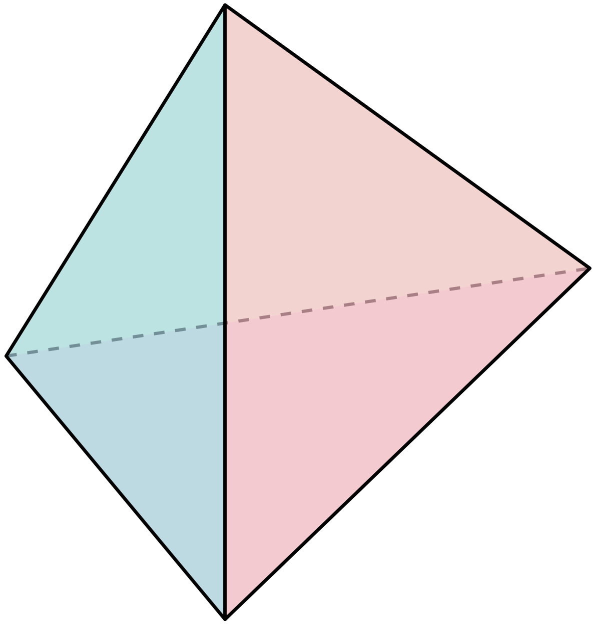
  $\pi_2^1(G_i) \geq n^{0.01}$

New!

- $\text{tw}(G_i) = 3$
  $\text{maxdeg}(G_i) = 6$
  $\pi_2^1(G_i) \in \Omega(\log n_i)$

# Main contribution

We construct an infinite family of graphs



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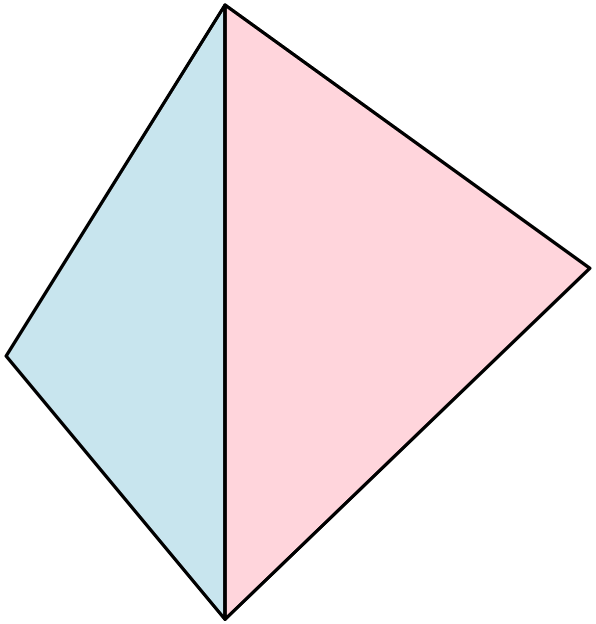

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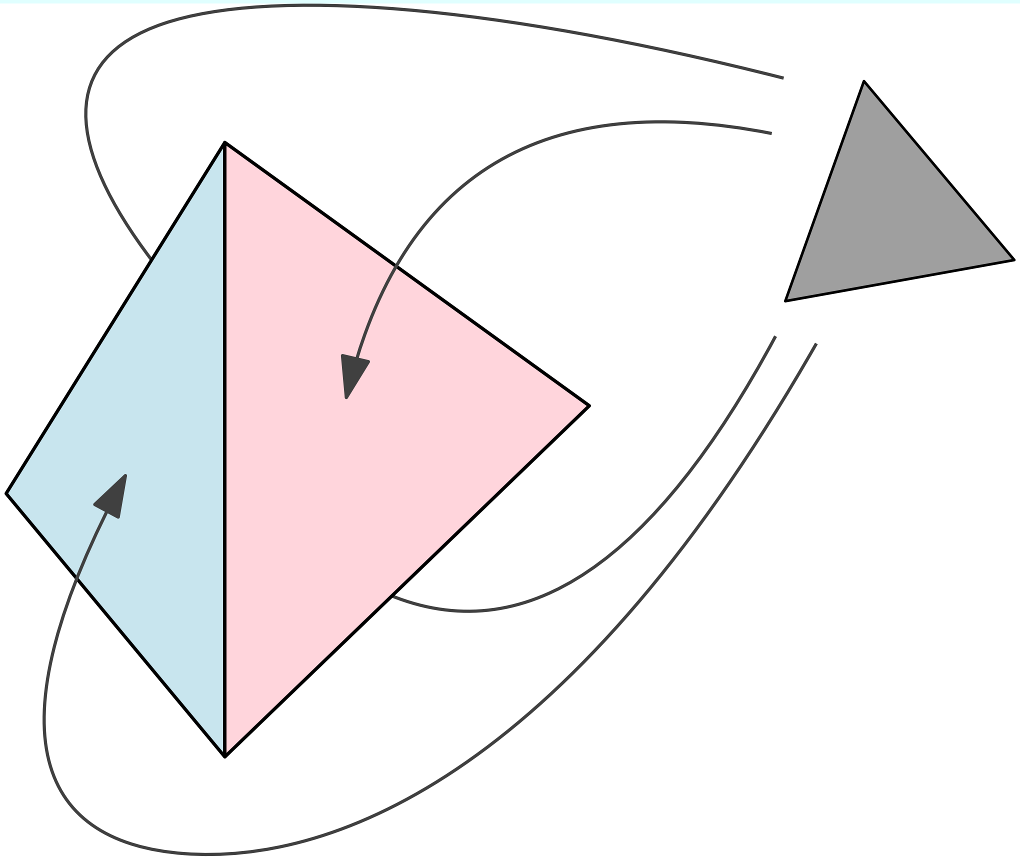
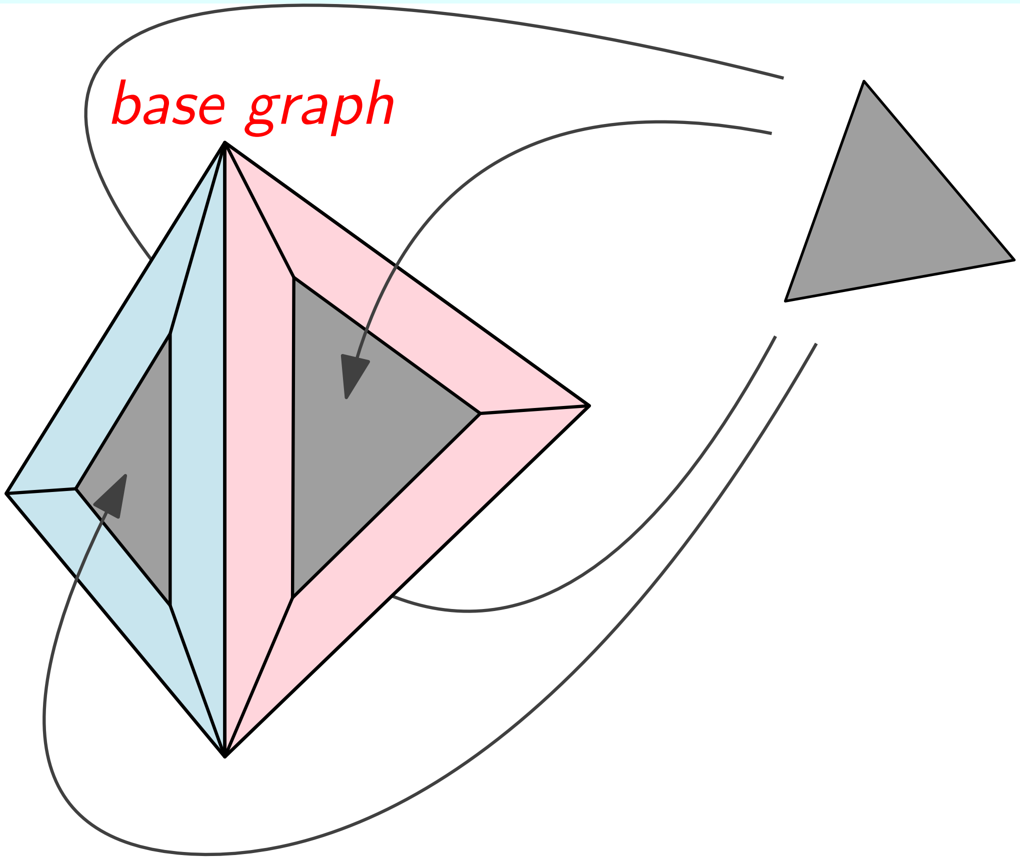


# Main contribution

We construct an infinite family of graphs



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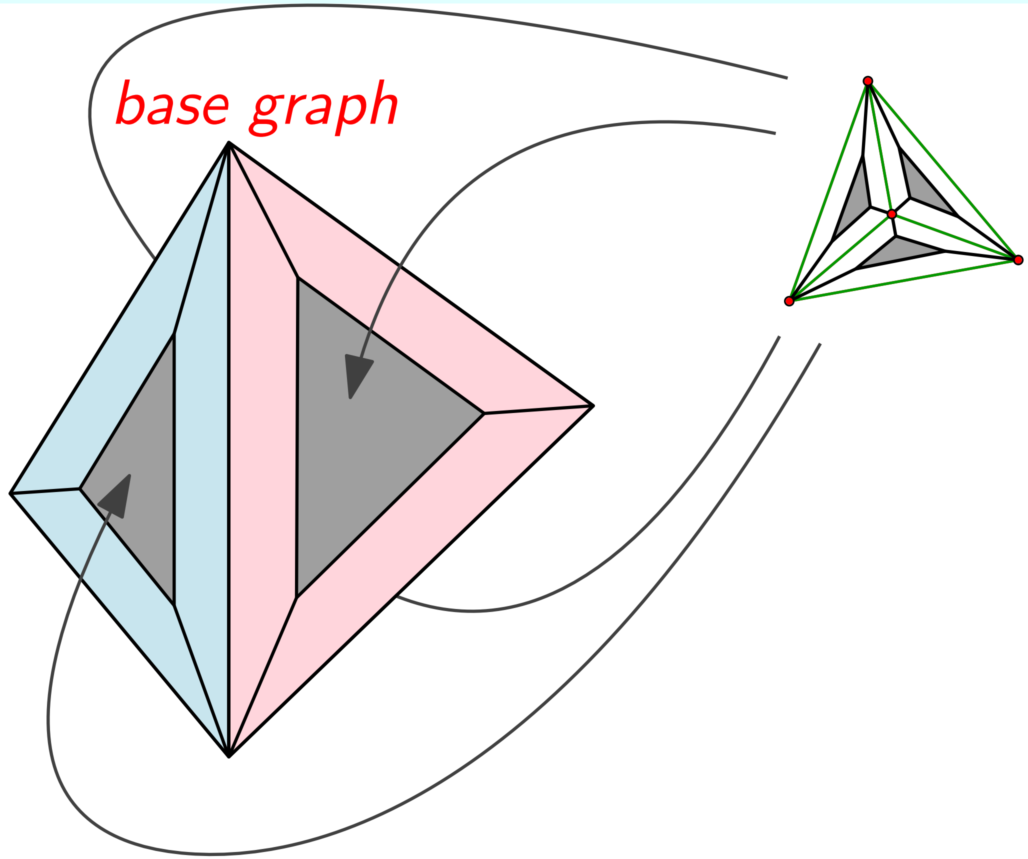

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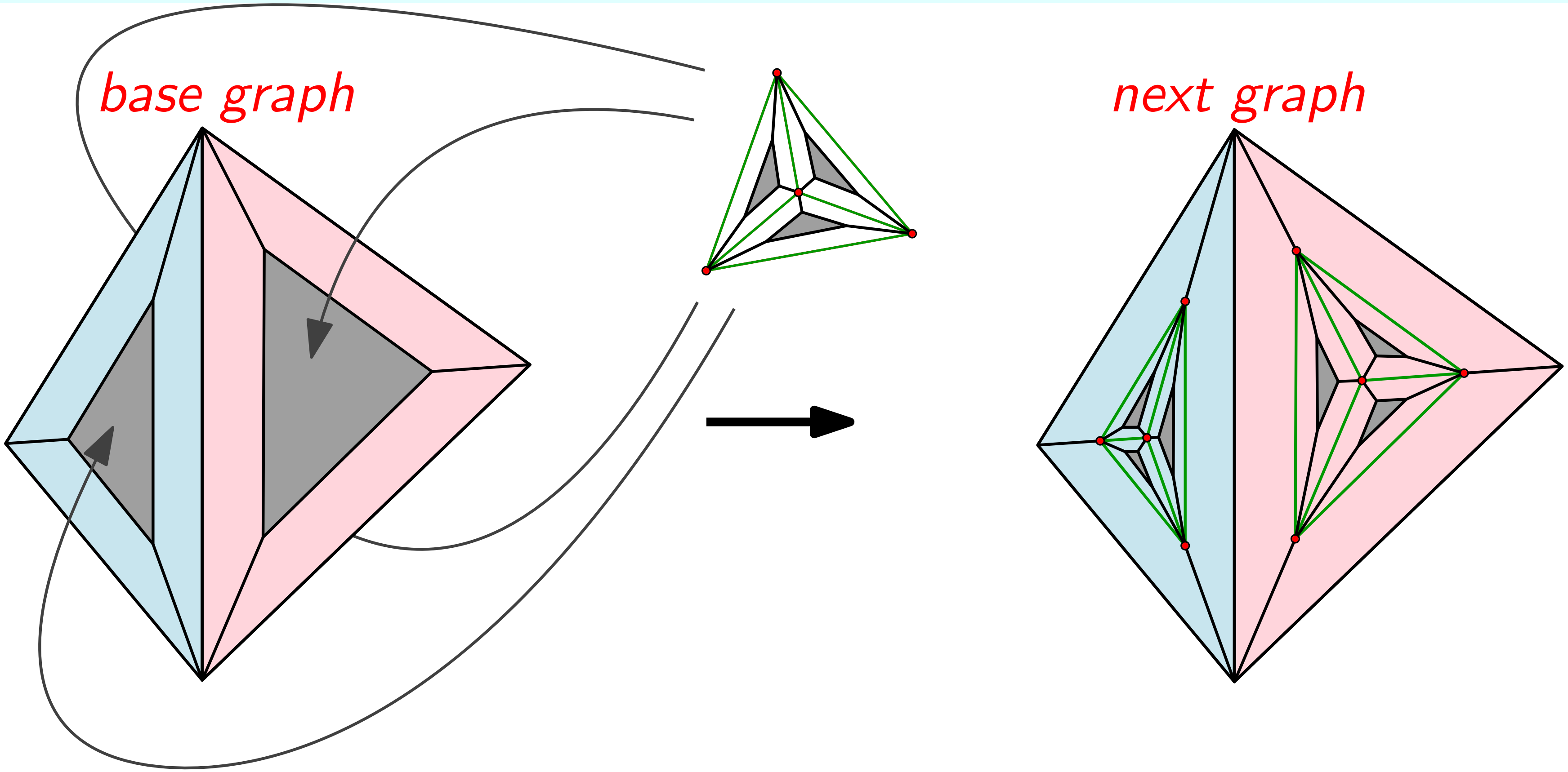

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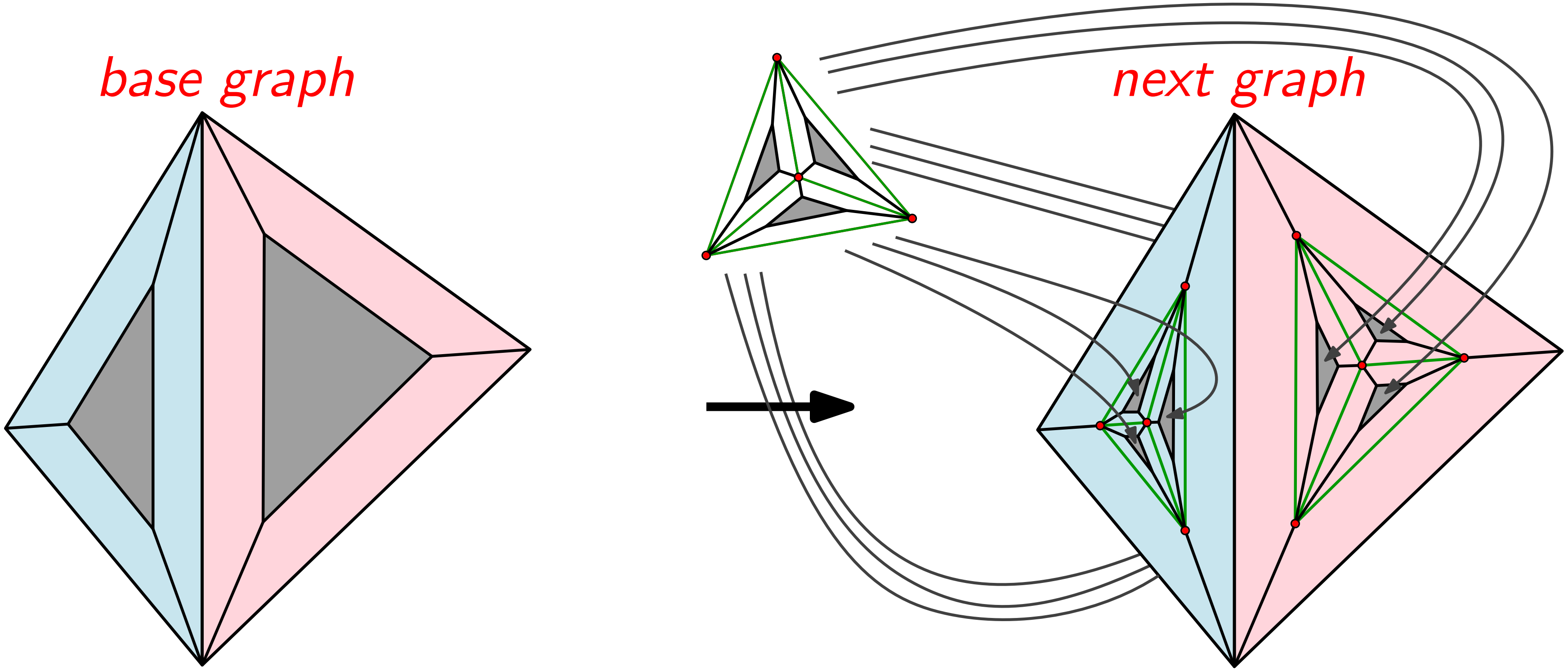

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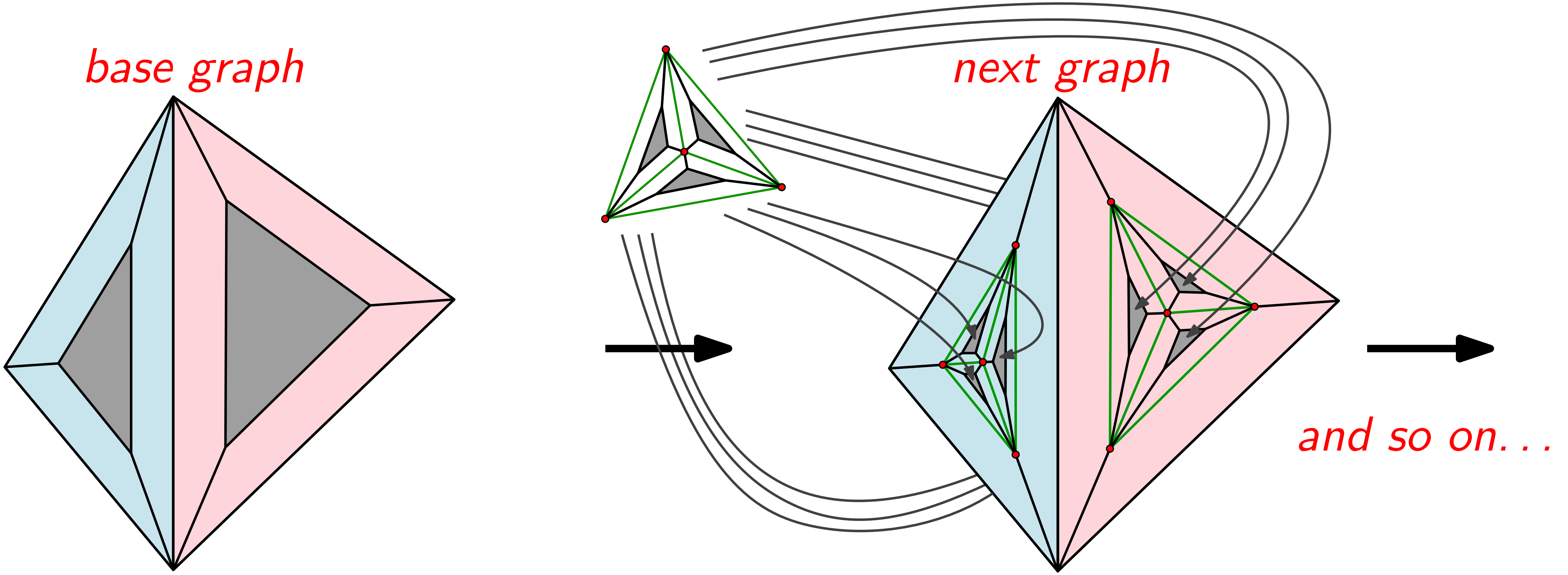

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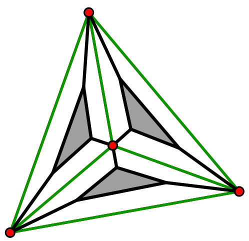

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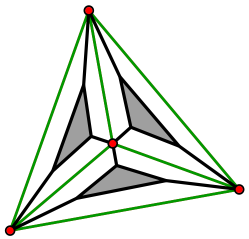
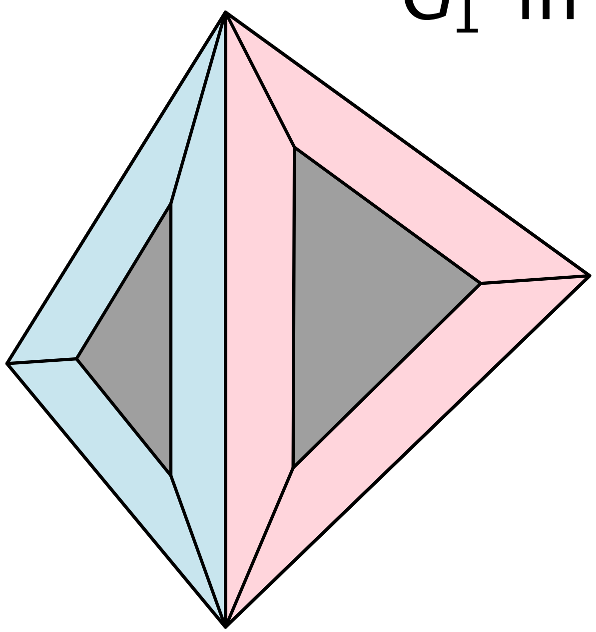


?

$G_1$ in 3D



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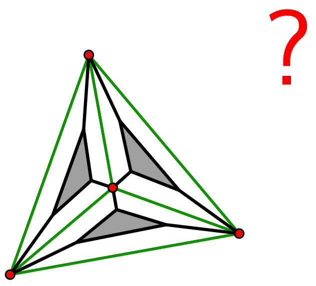


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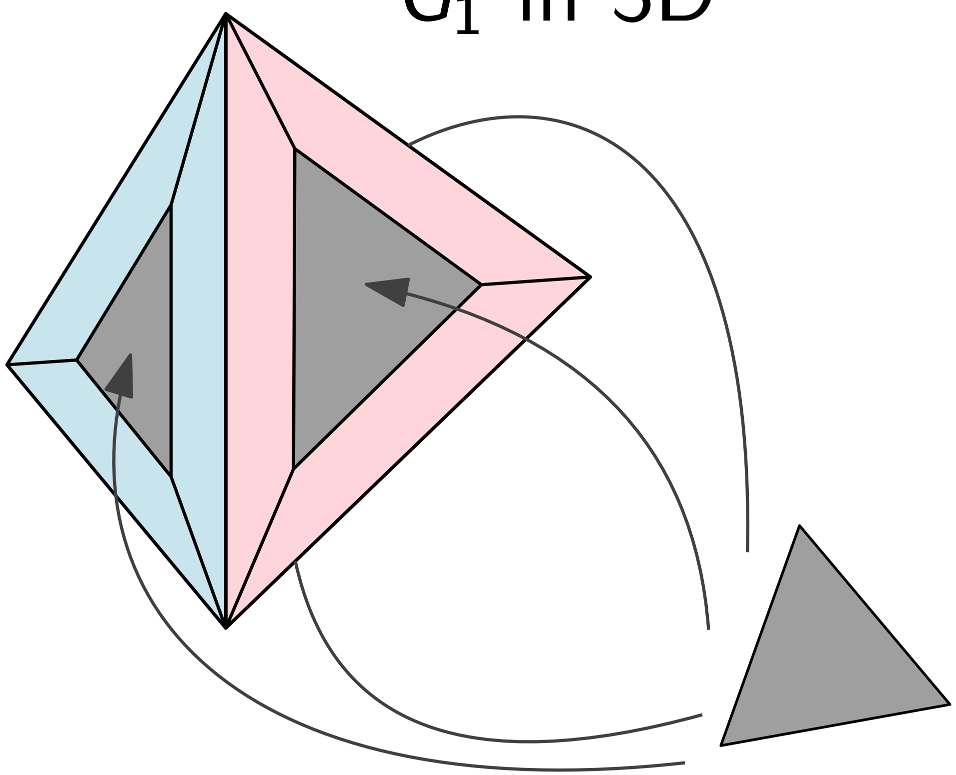

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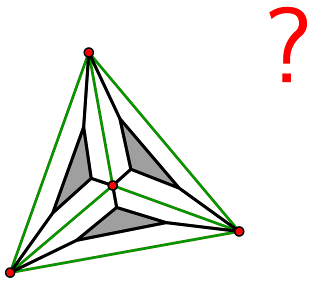


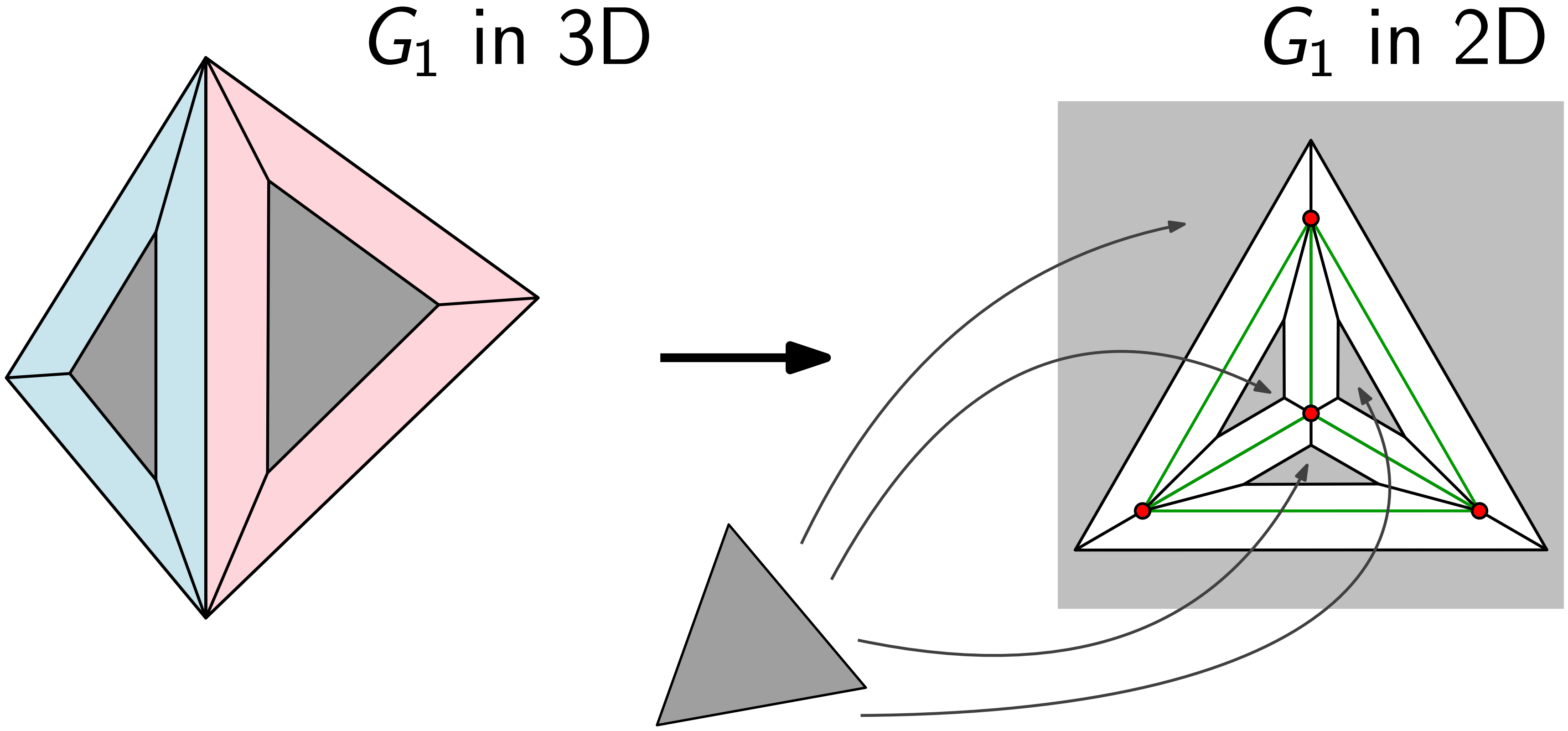

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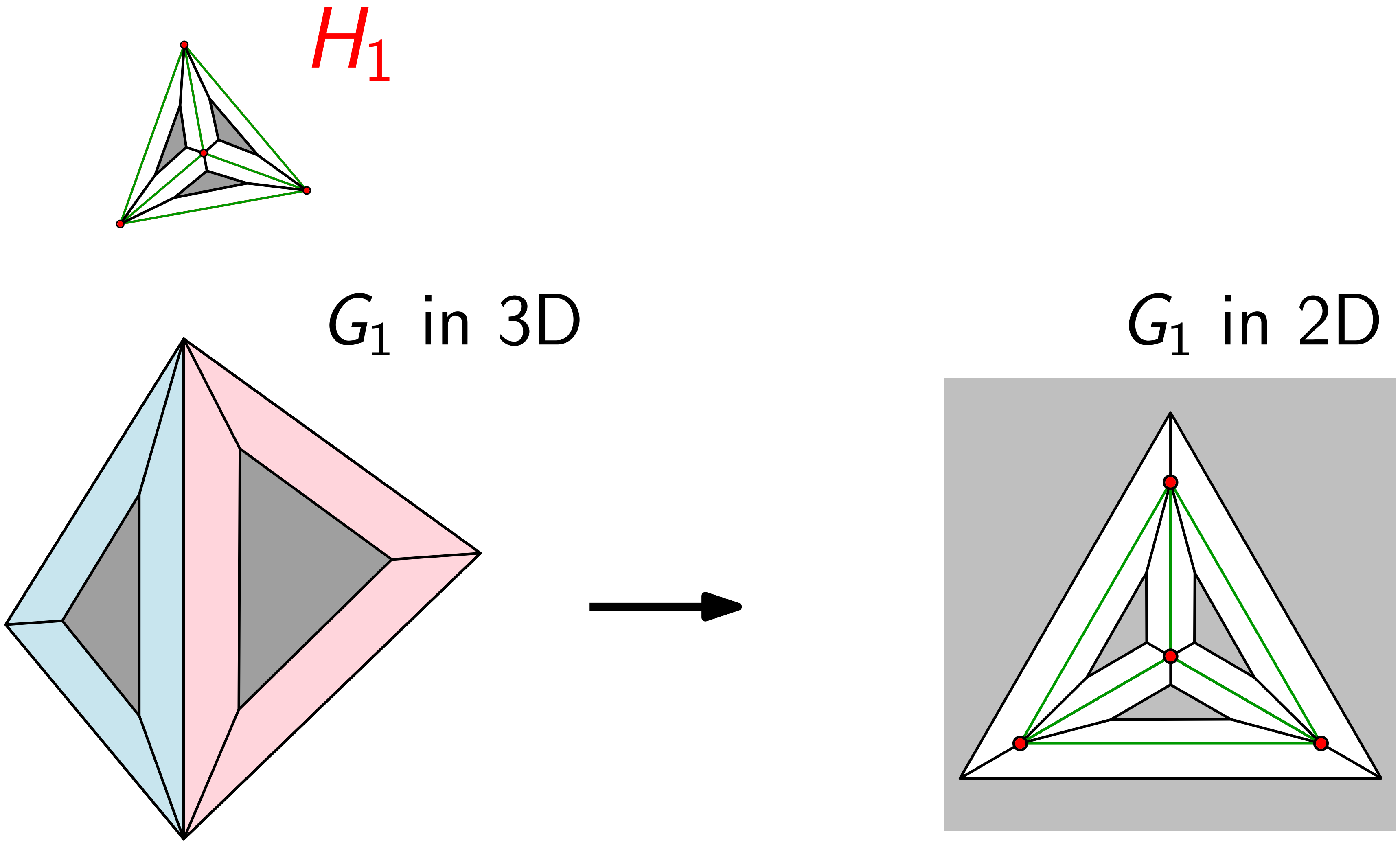

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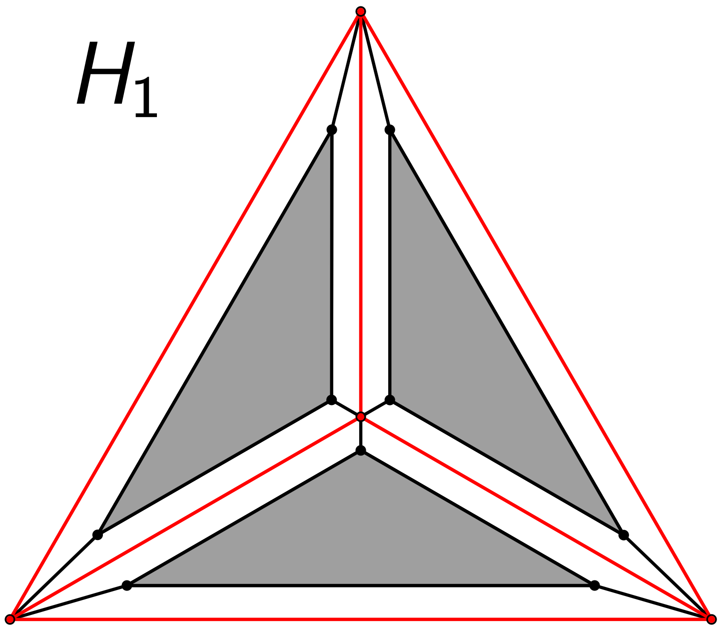

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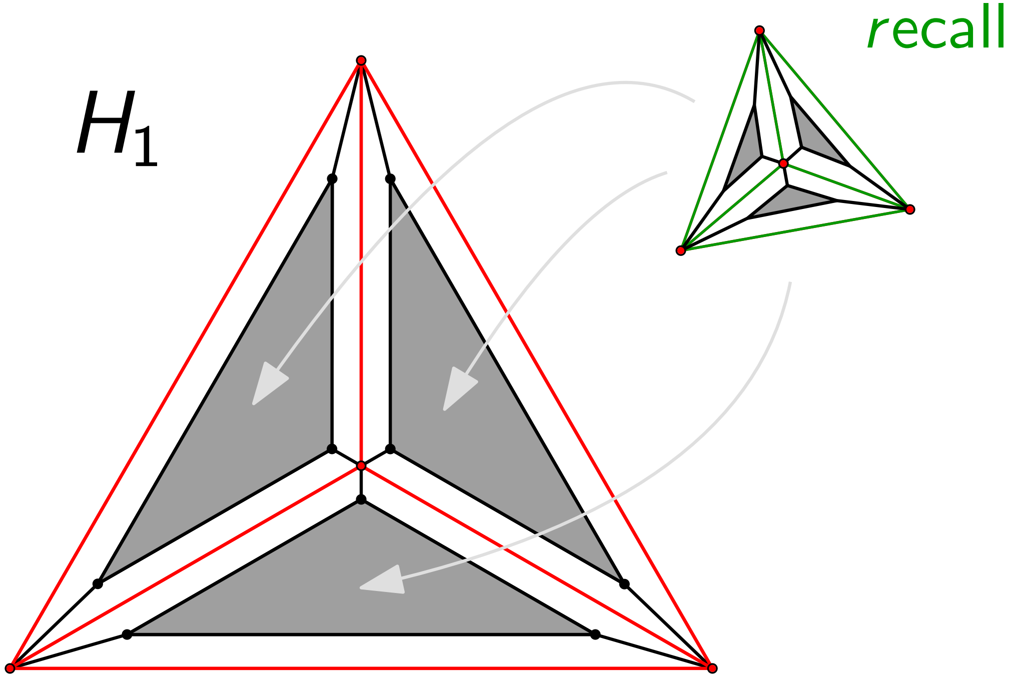

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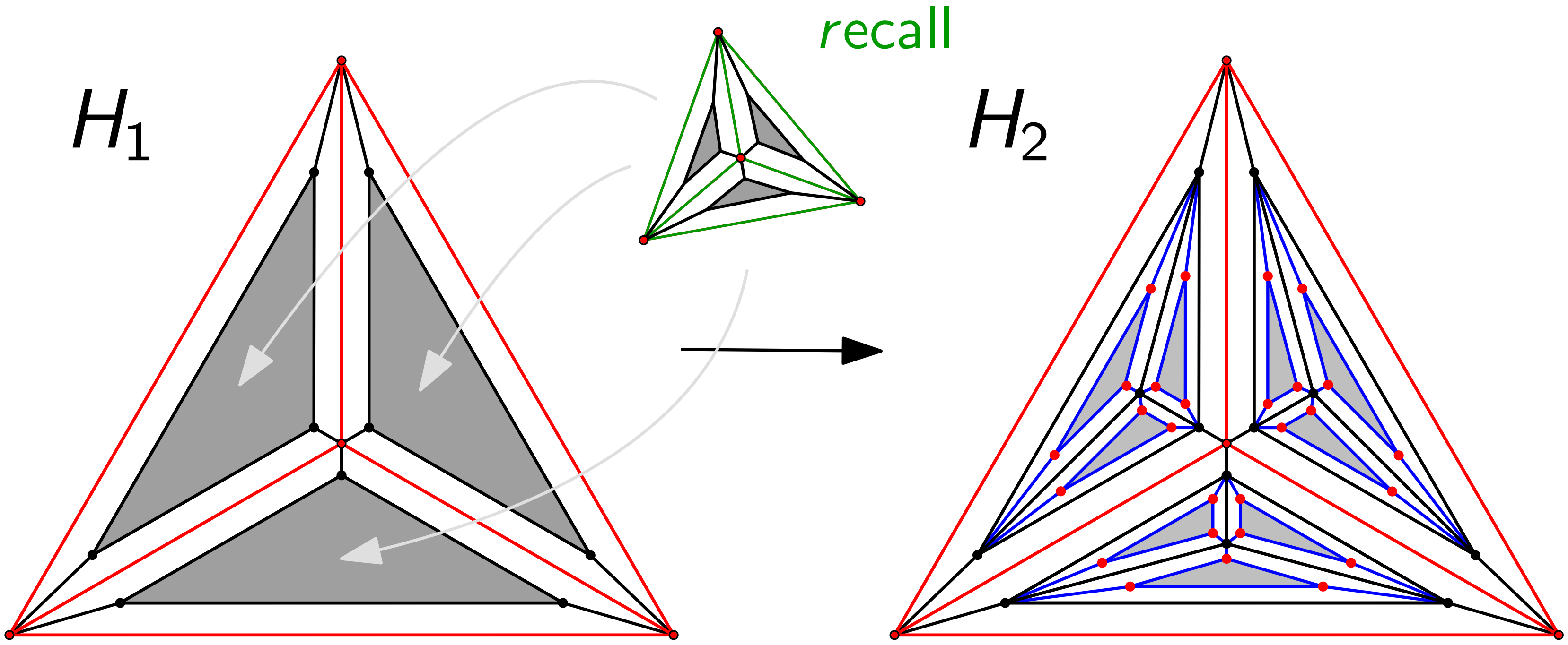

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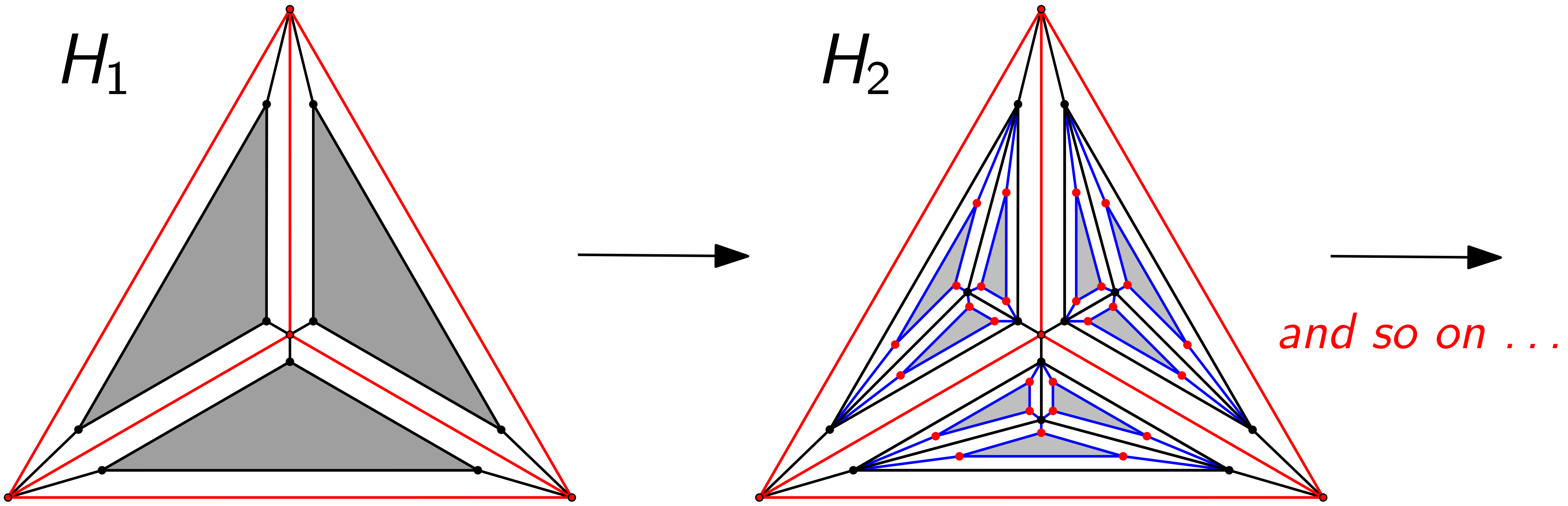

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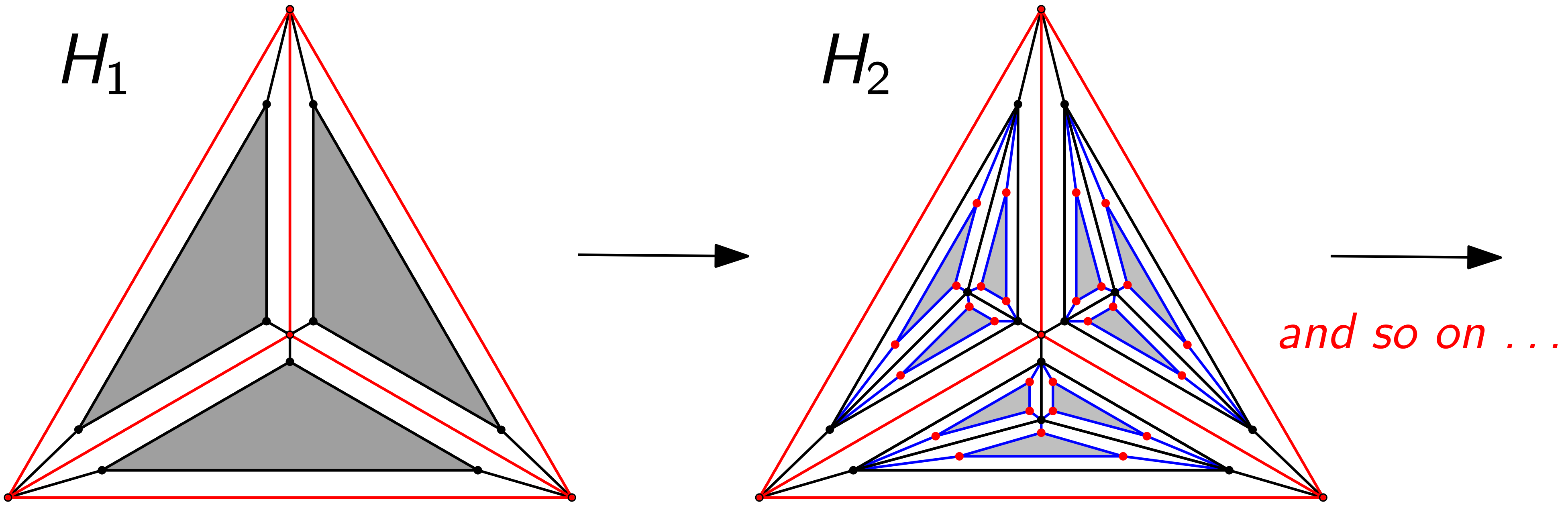


maximum degree

*6*

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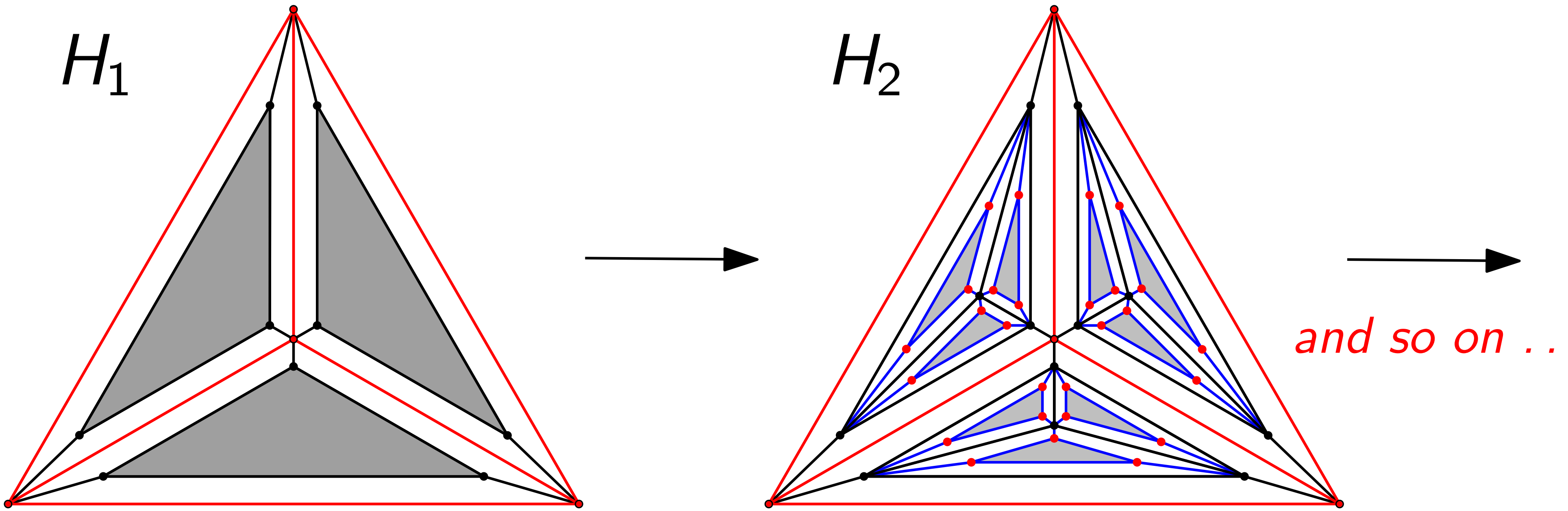


maximum degree

*6*

treewidth

*3*

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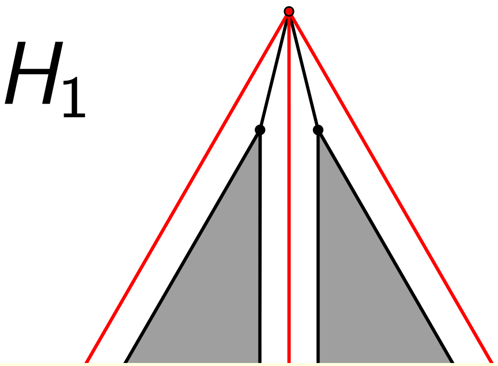


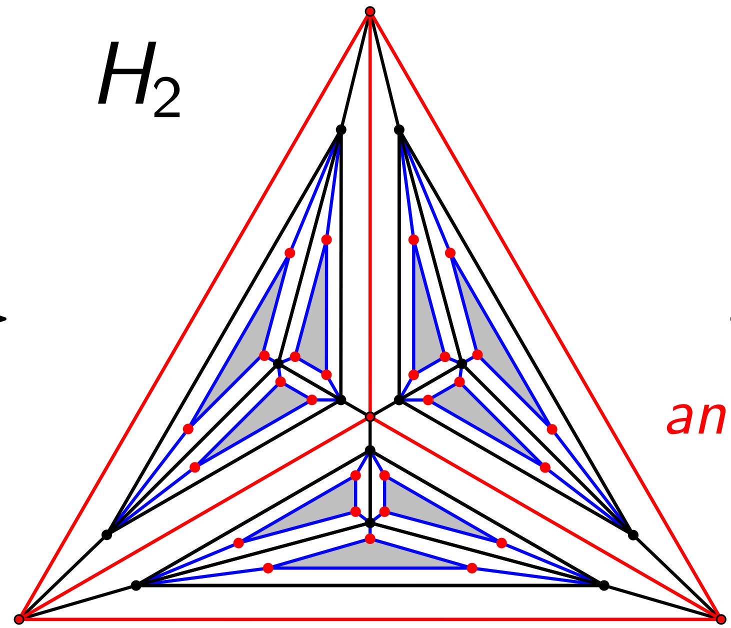


*and so on . . .*

Why?

treewidth
*3*

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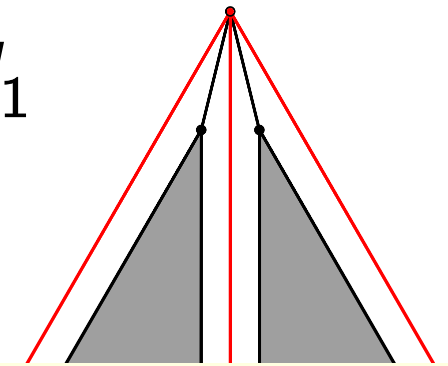


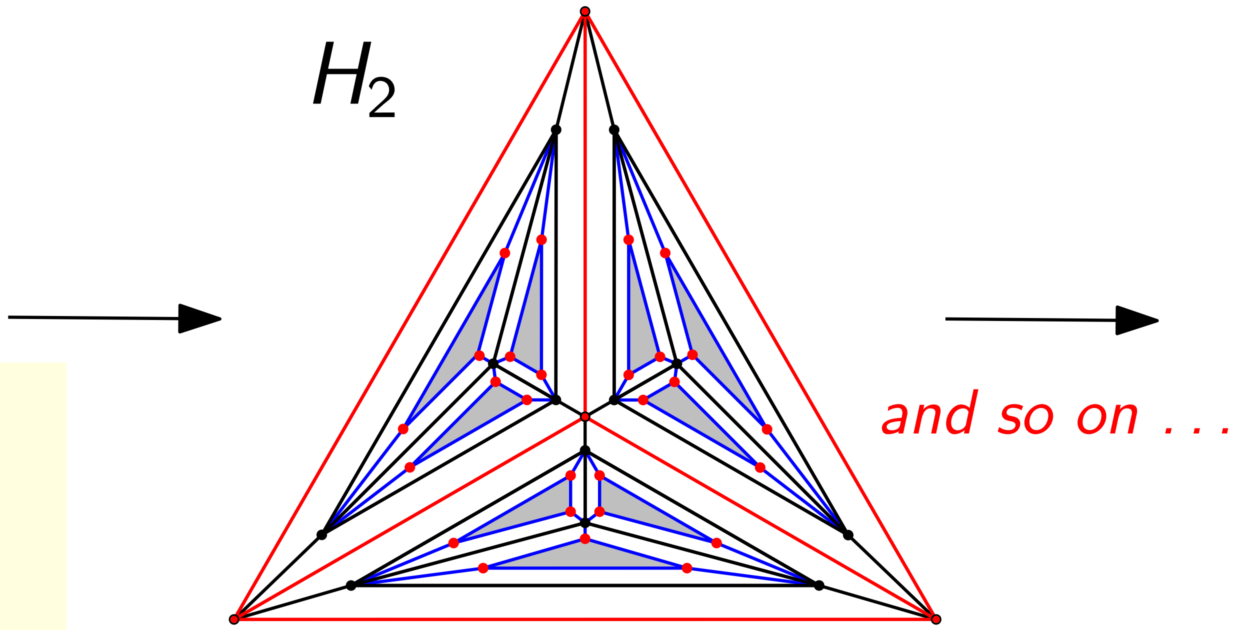


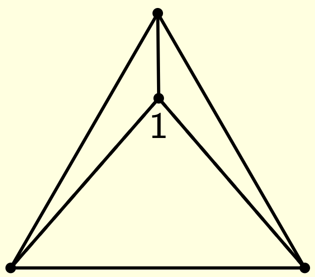


3-tree

Why?

treewidth
*3*

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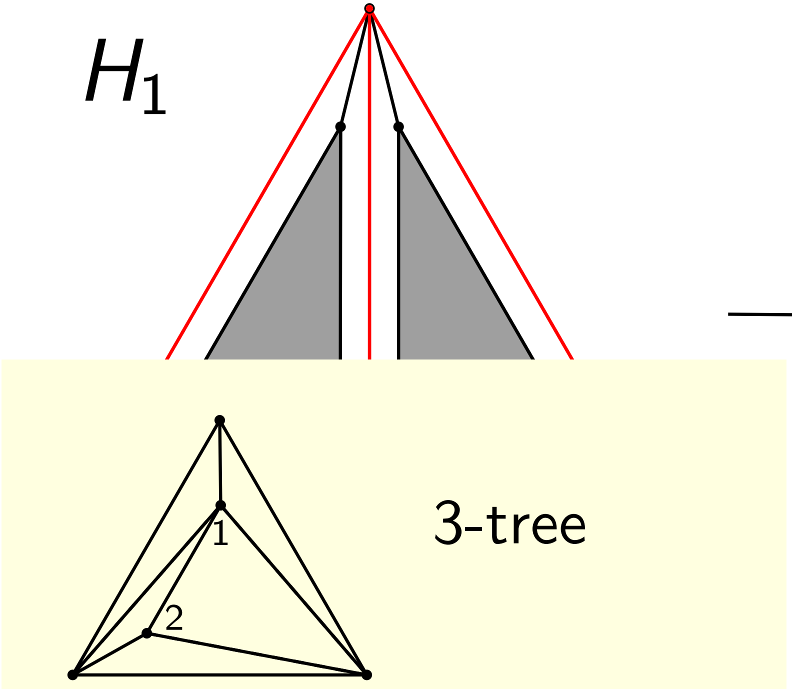


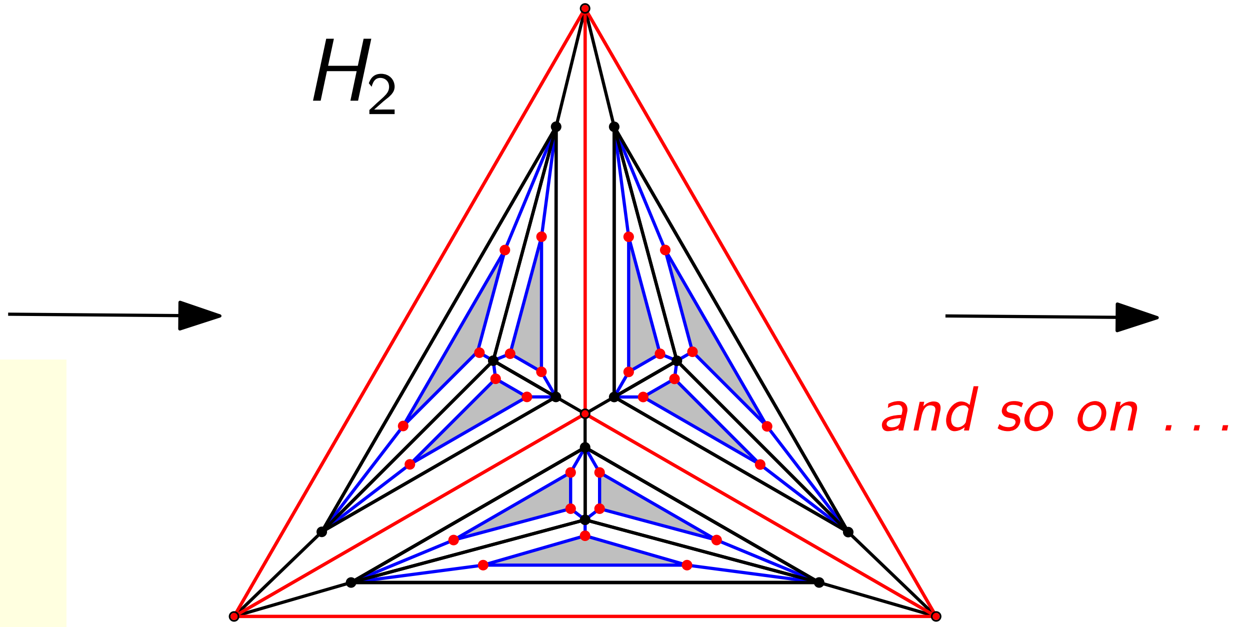


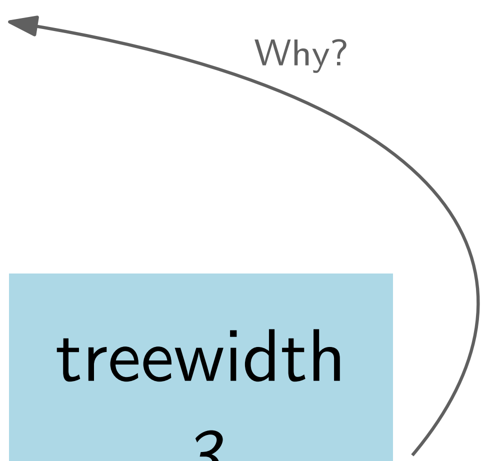


treewidth

*3*

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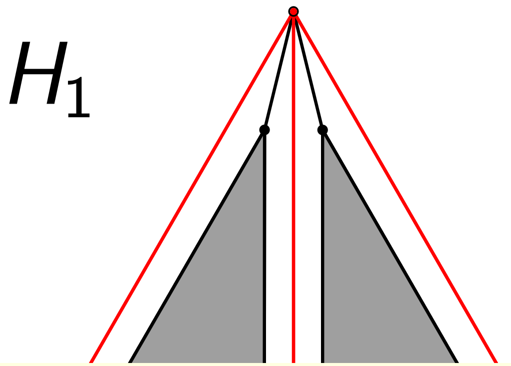


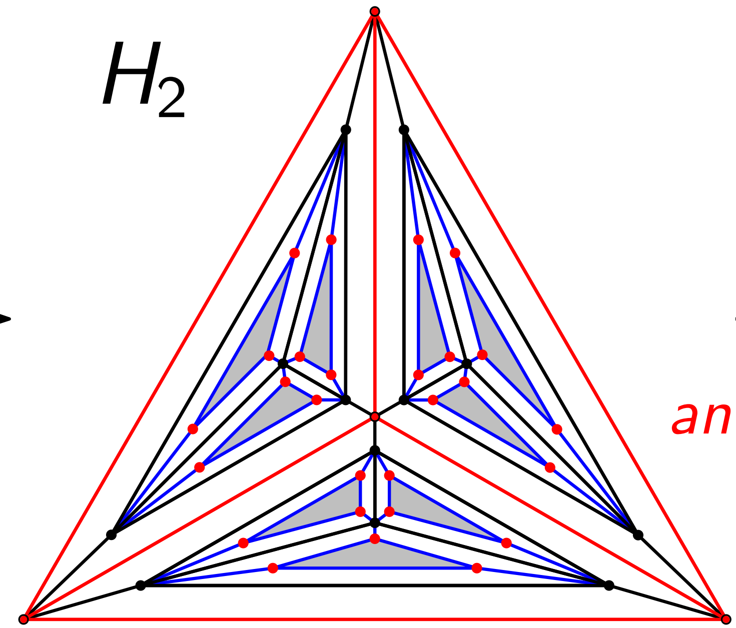


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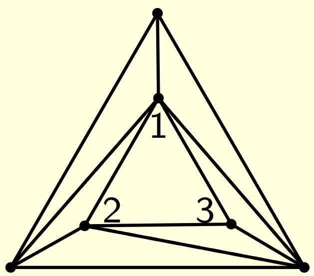


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Why?

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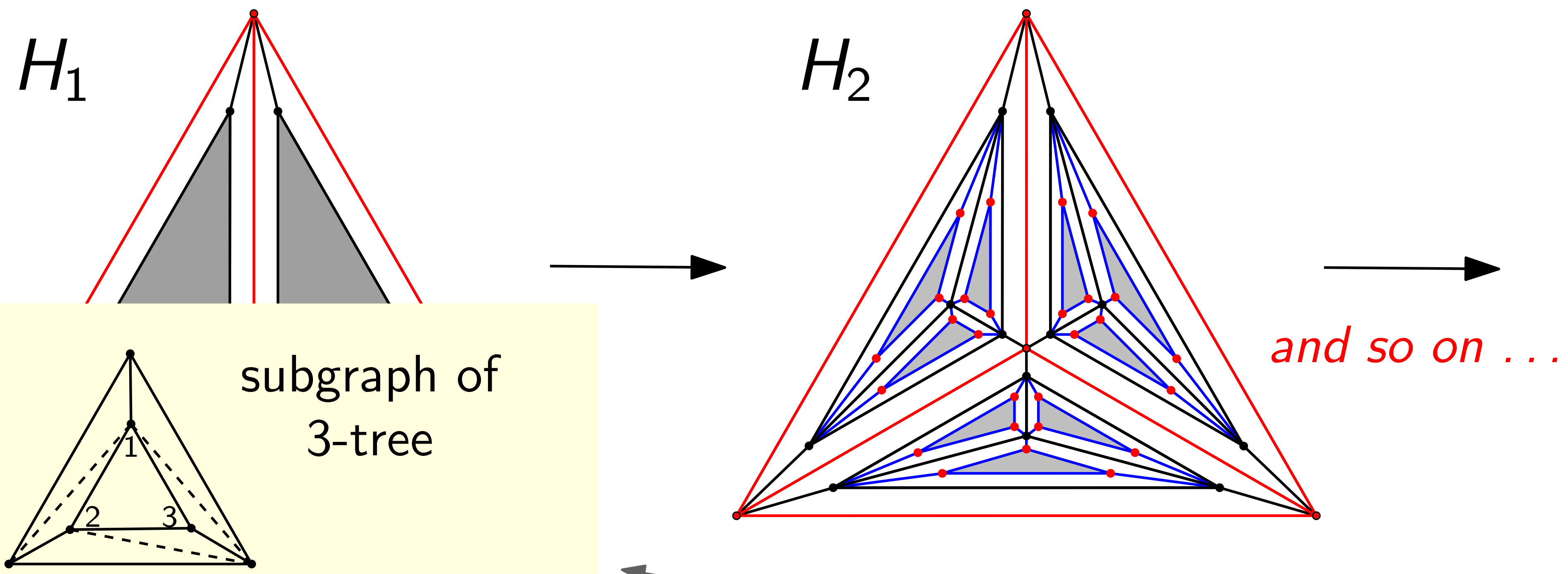


subgraph of 3-tree

Why?

treewidth *3*

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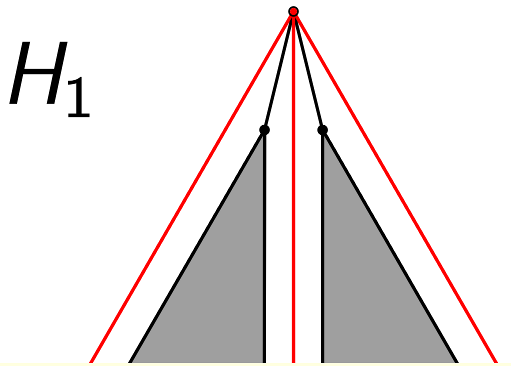


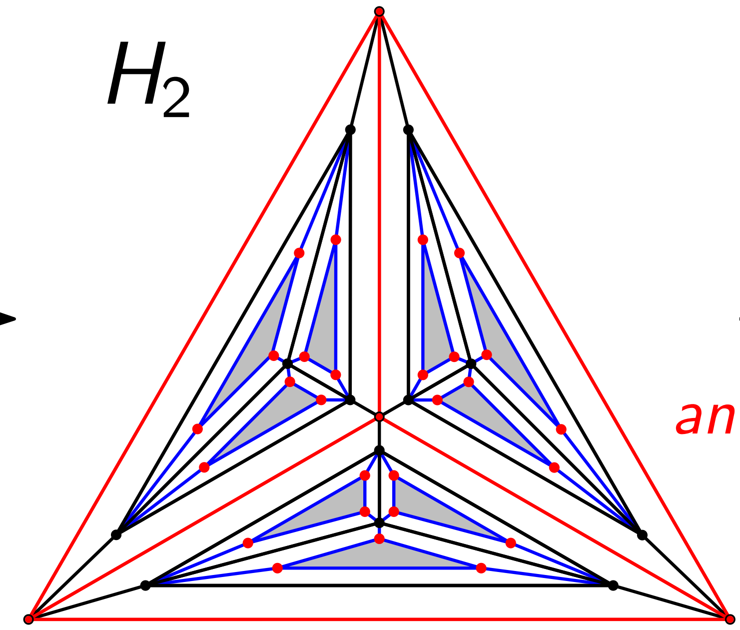


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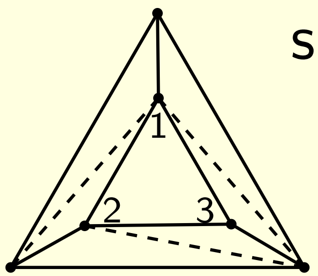


subgraph of 3-tree

$\text{tw}(G_i) \leq 3$

Why?

treewidth

*3*

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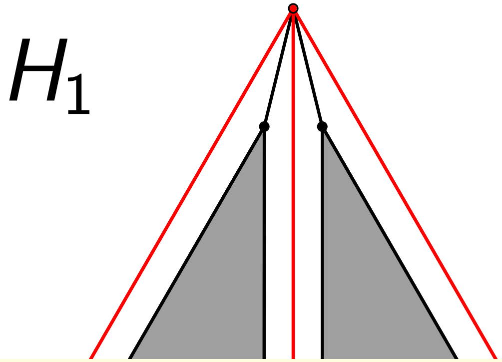


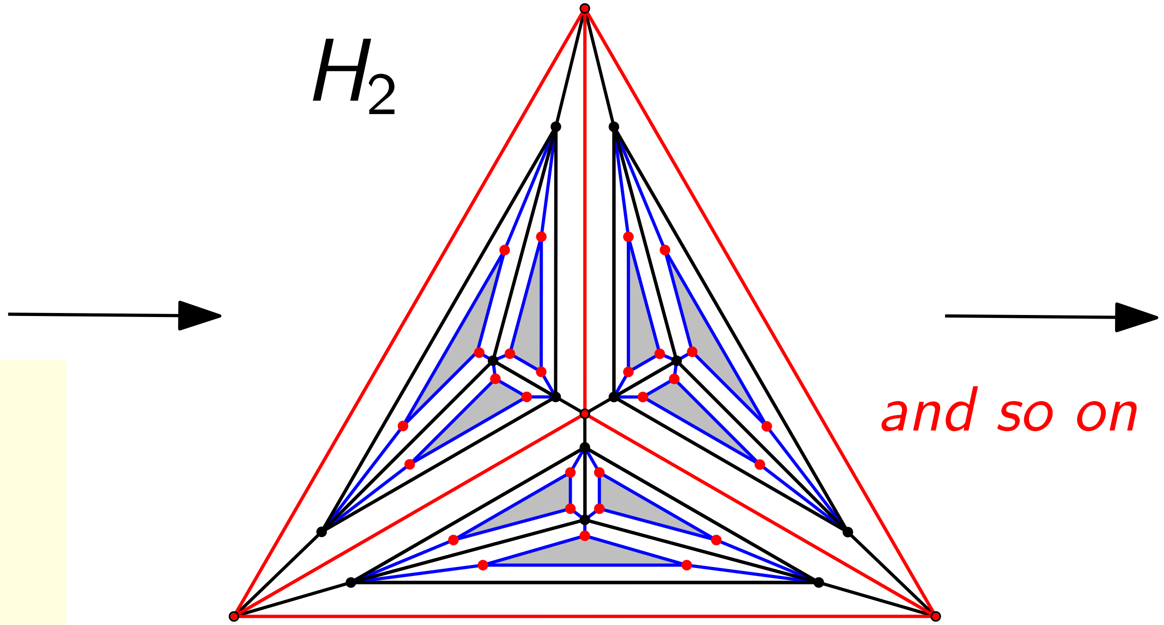


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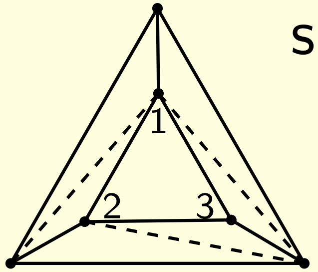


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treewidth
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Why?

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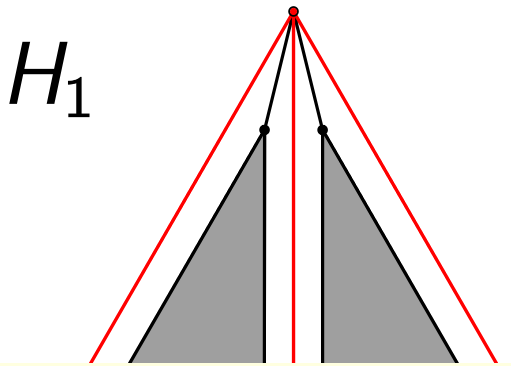


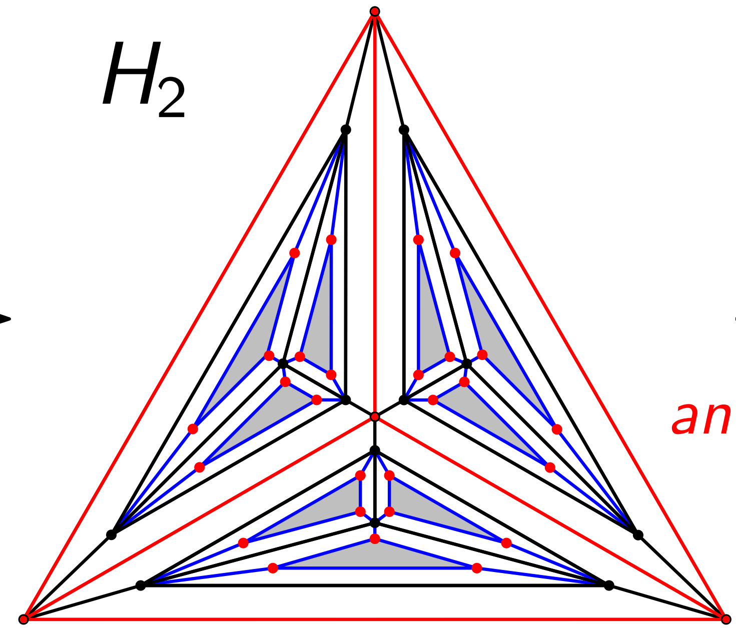


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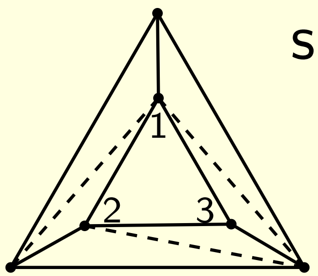


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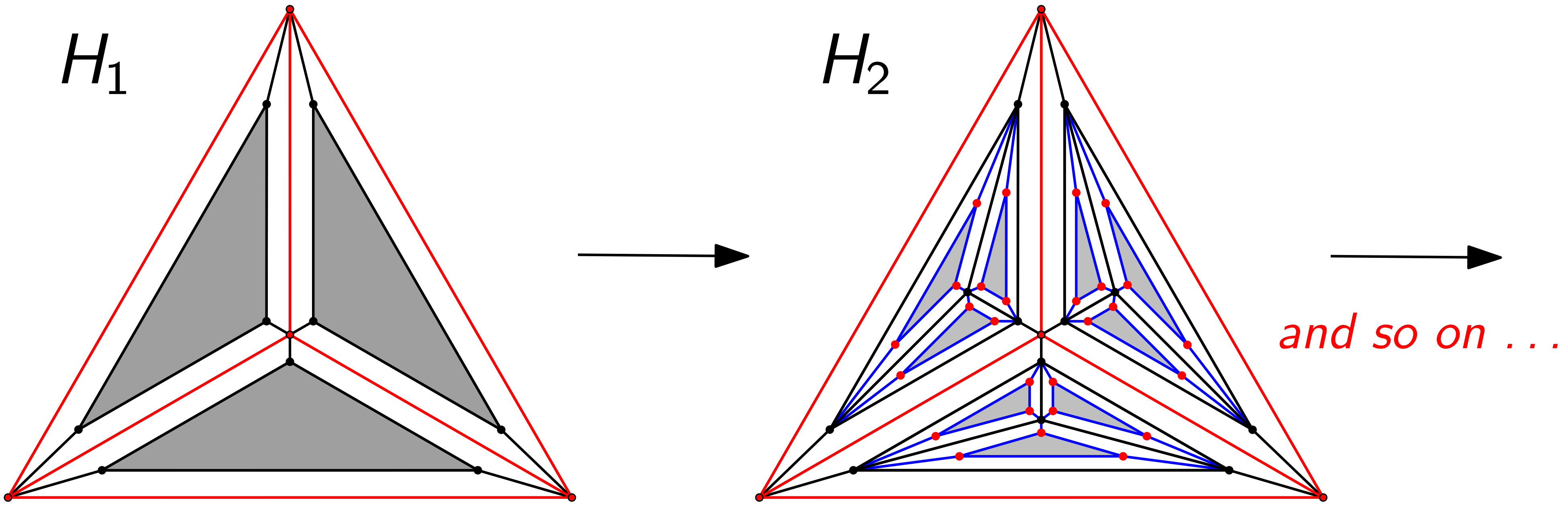


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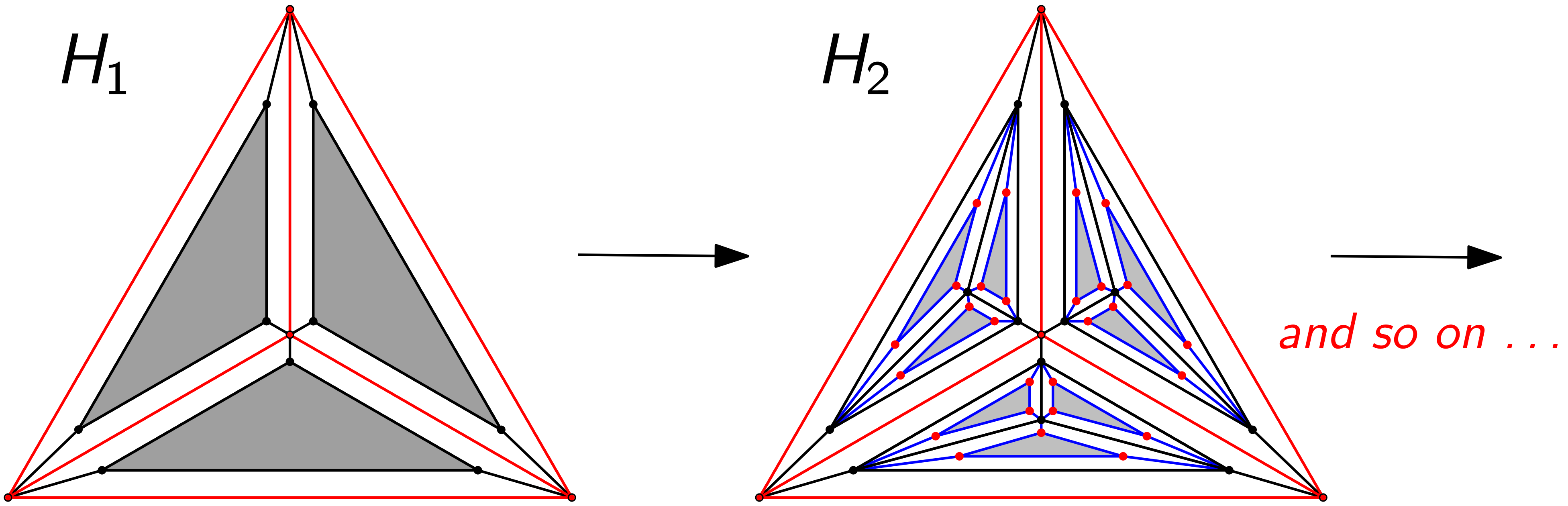


maximum degree
*6*

treewidth
*3*

2D weak line cover number
*unbounded*

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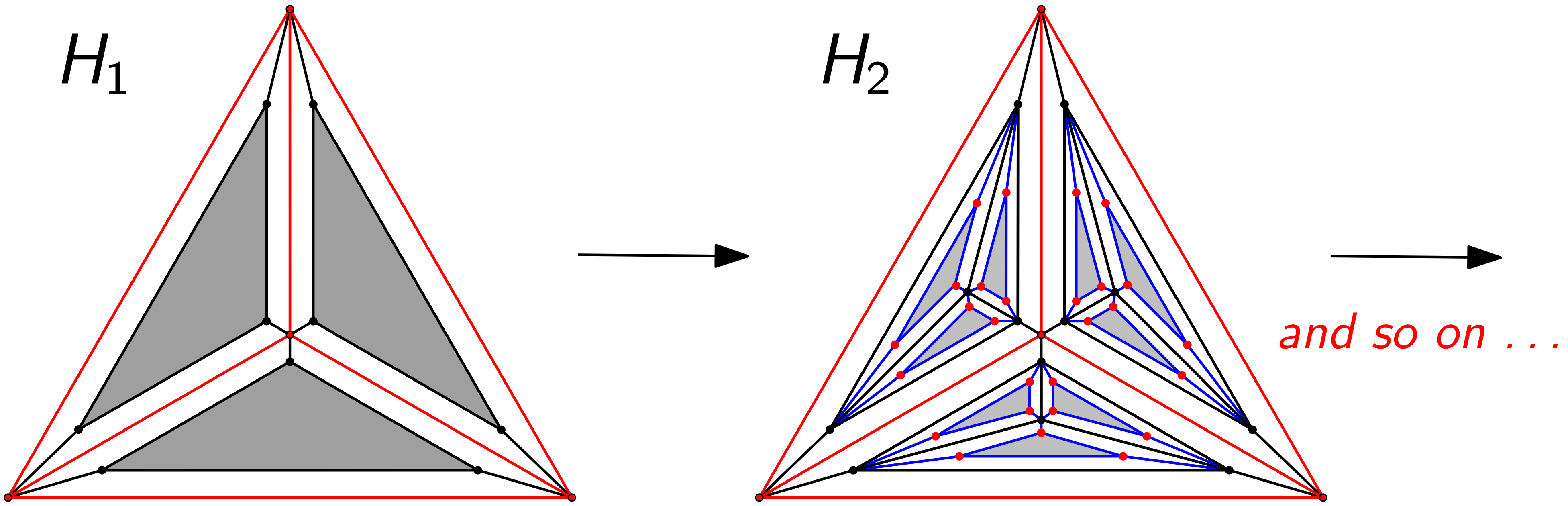


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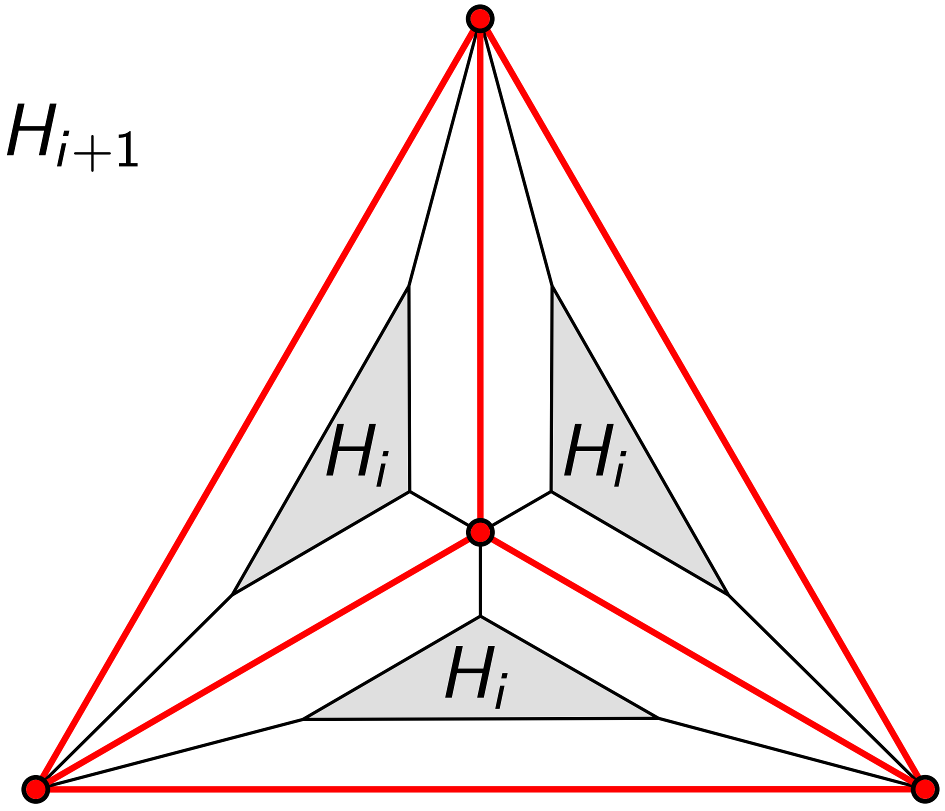
Consider the graph $H_{i+1}, i = 1, 2, 3, \ldots$.



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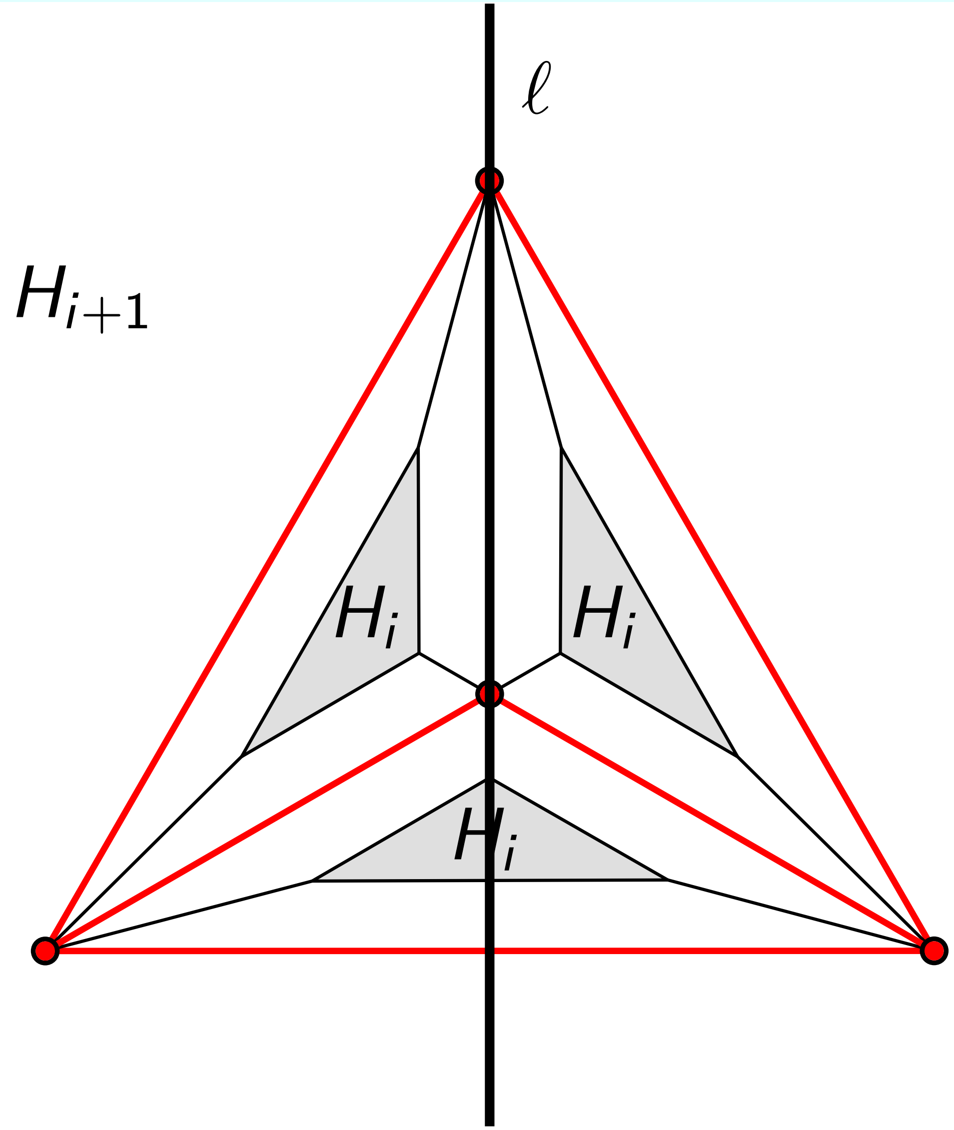

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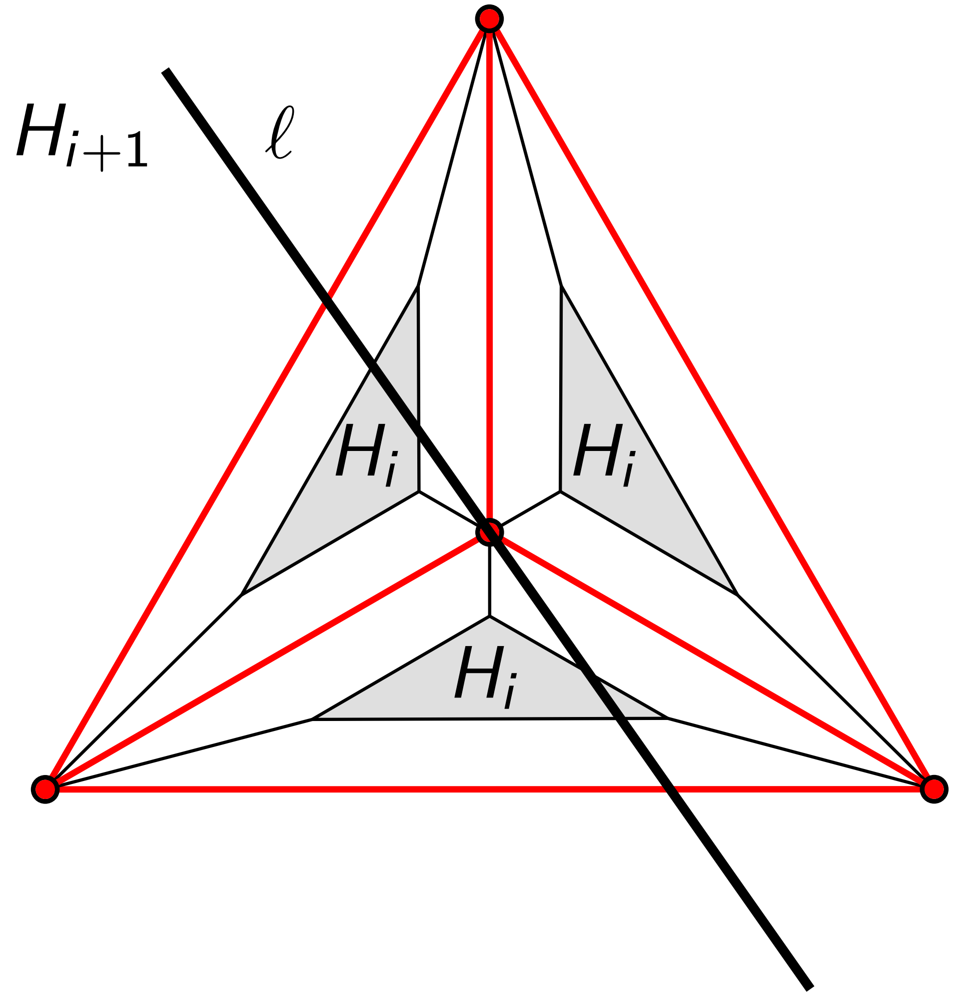

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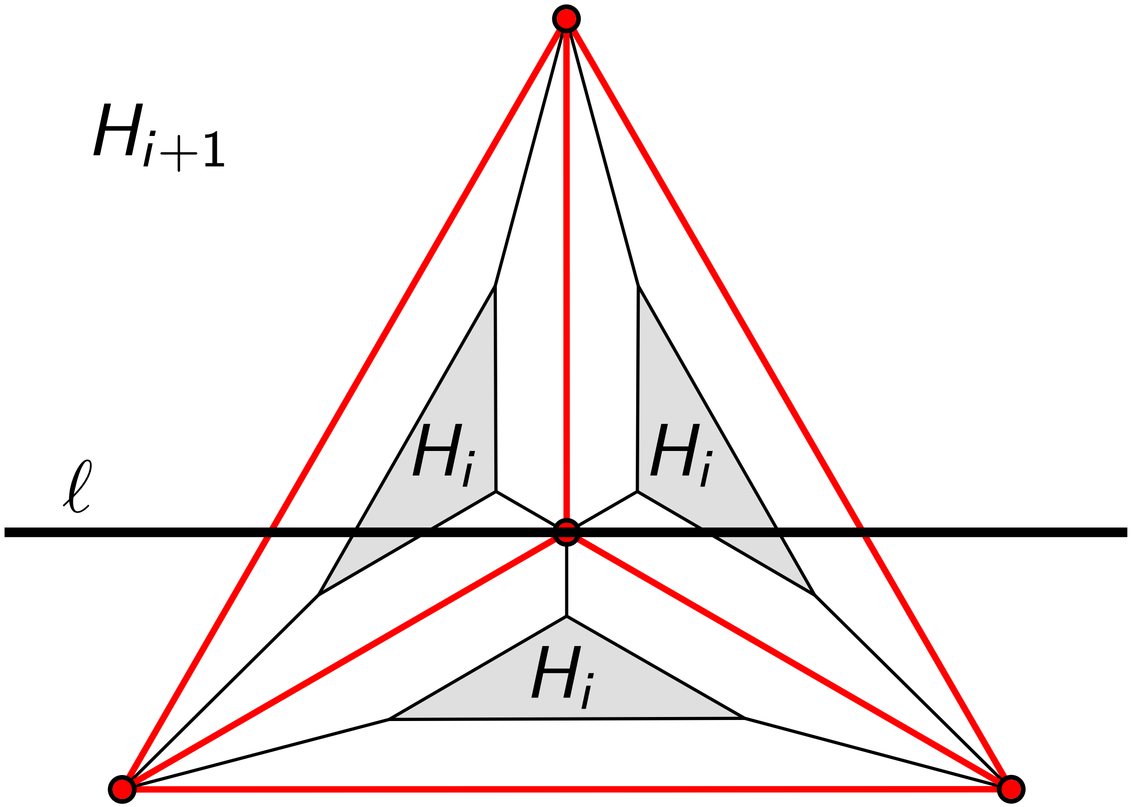

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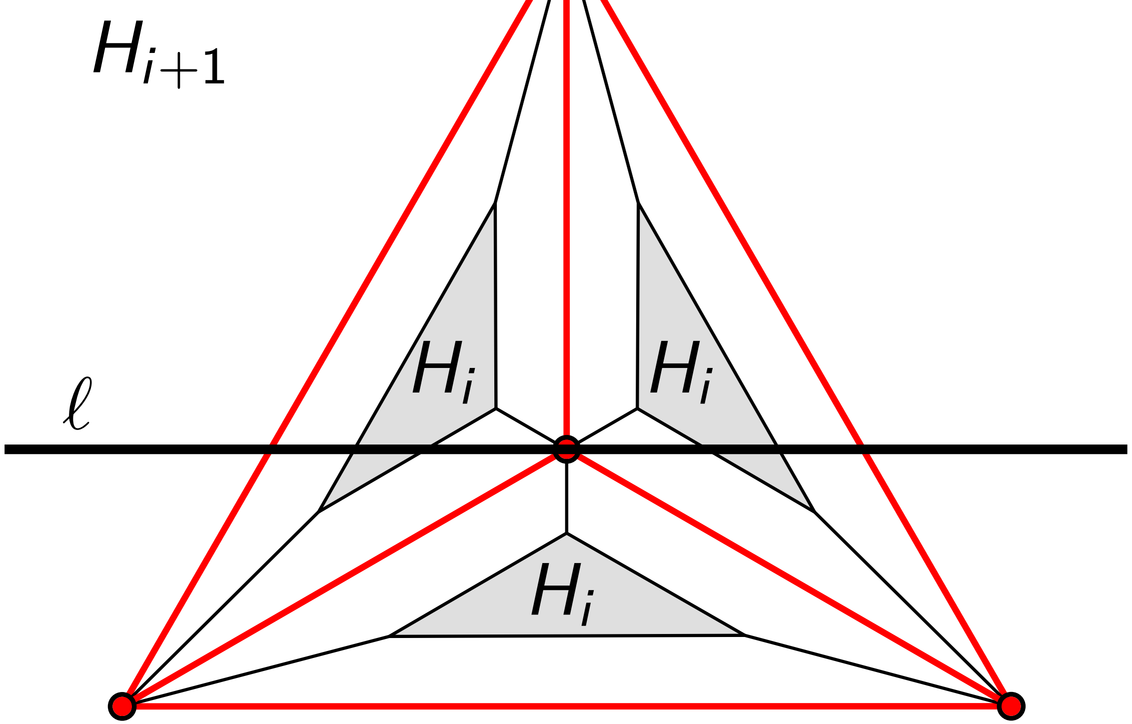
in each step
**+1** line



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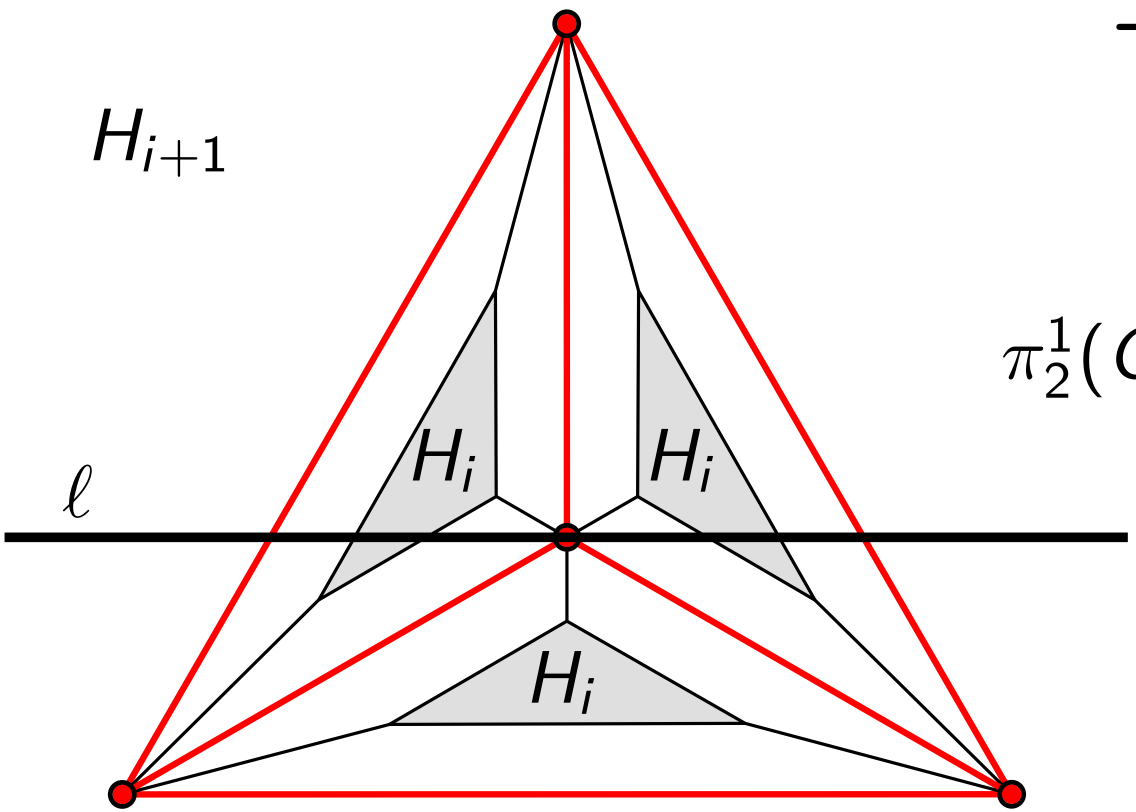


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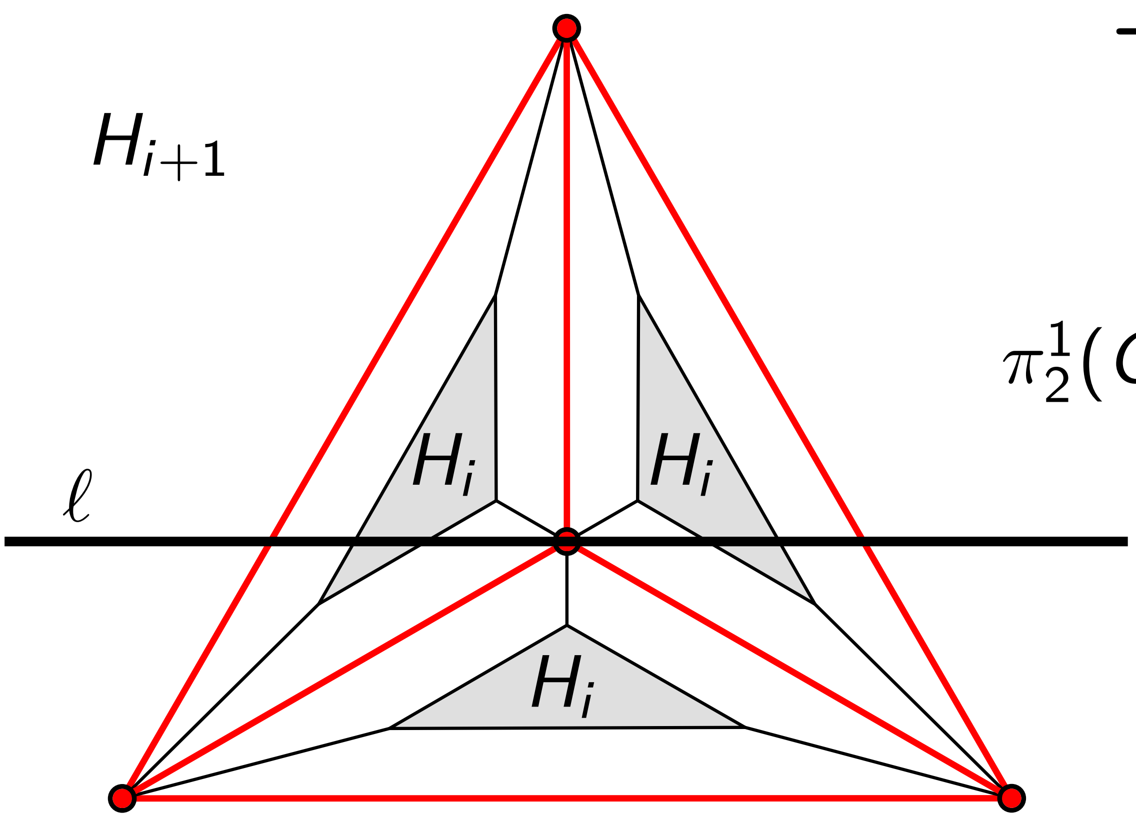


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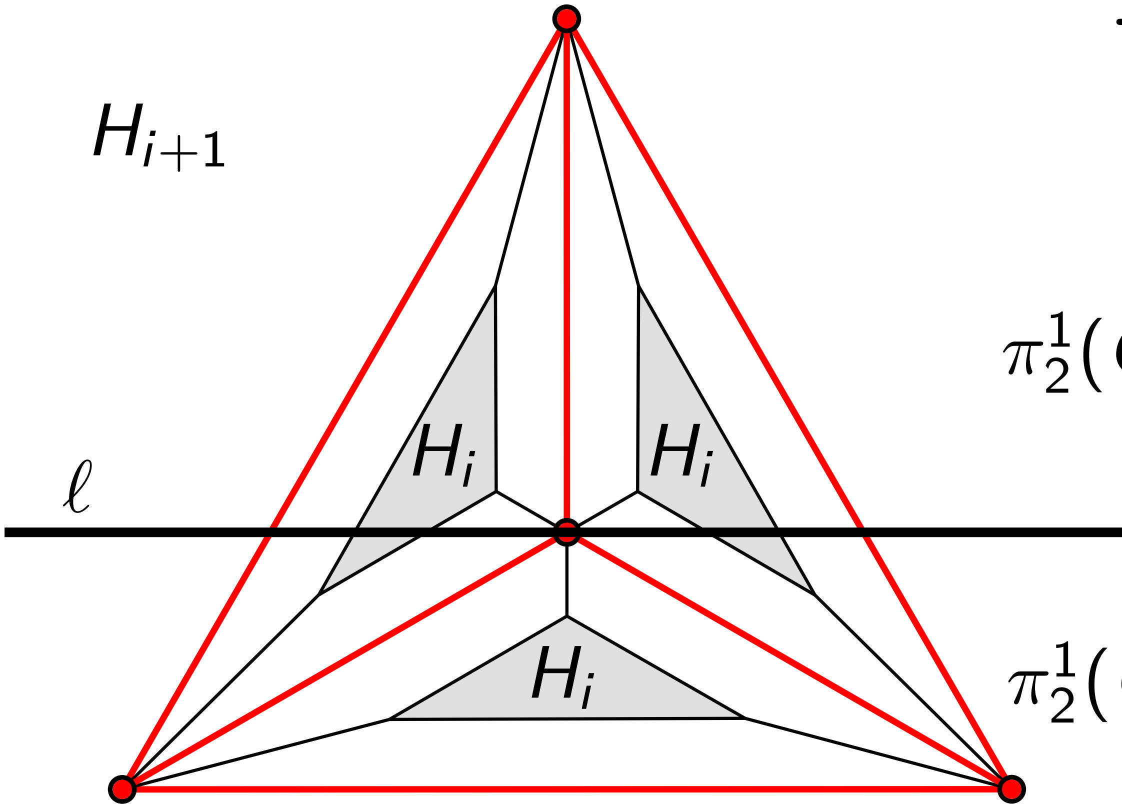


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$\pi^1_2(G_i) \in \Omega(\log n_i)$

# Open problems

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How small can we make the maximum degree in a family of planar graphs such that their $\pi^1_2$-value is still unbounded?

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Is it NP-hard to compute $\pi^1_2(G)$ for a given graph $G$?

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Yes, by reduction from (a restricted version of) Level Planarity.

[Biedl, Evans, Felsner, Lazard, Meijer, Valtr, Whitesides, Wismath, Wolff 2018]