# Representing Graphs and Hypergraphs by Touching Polygons in 3D

Paweł Rzążewski, Noushin Saeedi

joint work with William Evans, Chan-Su Shin, and Alexander Wolff

# How to draw a graph? (in 2d)

- non-crossing drawings

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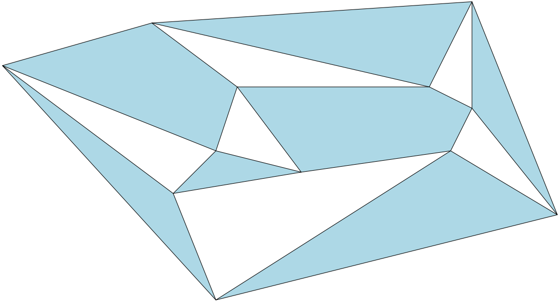

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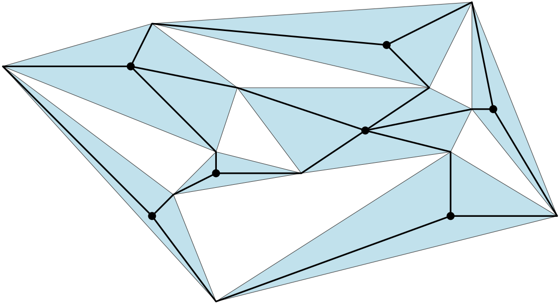

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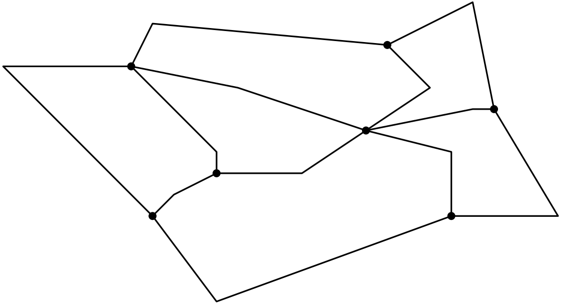

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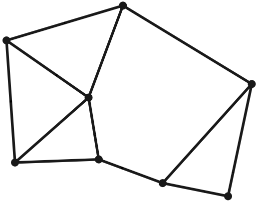

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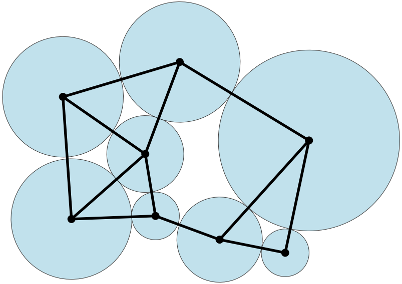

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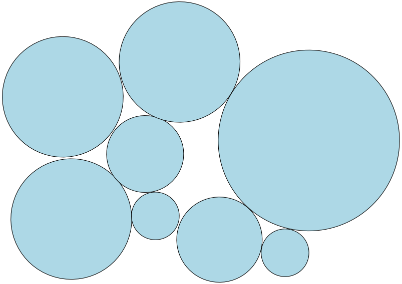

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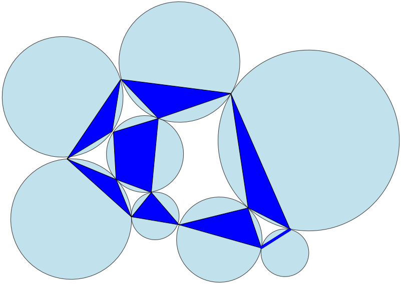

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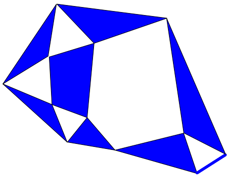

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# Contact representations by touching polygons

Theorem. Every graph can be represented by touching convex polygons in 3d.

- in particular, this is an intersection representation by convex sets

# Key lemma

Lemma. For every $n \geq 3$ there is an arrangement of lines $\ell_1, \ell_2, \ldots, \ell_n$, such that:

a) $\ell_i$ intersects $\ell_1, \ell_2, \ldots, \ell_n$ in this ordering ($p_{i,j} := \ell_i \cap \ell_j$),

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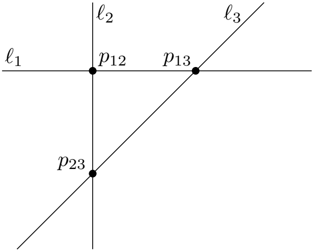

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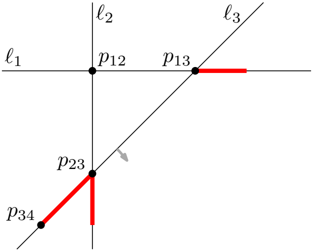

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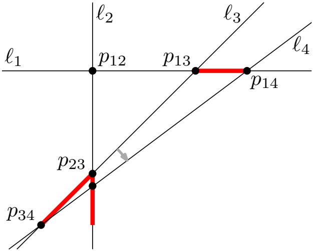

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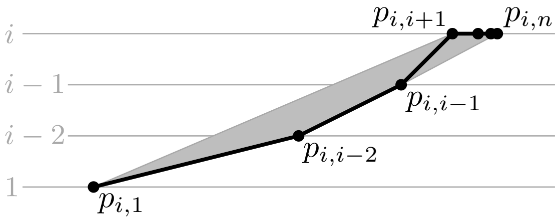

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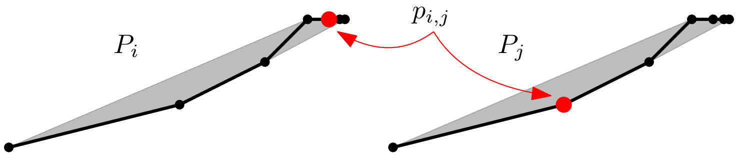

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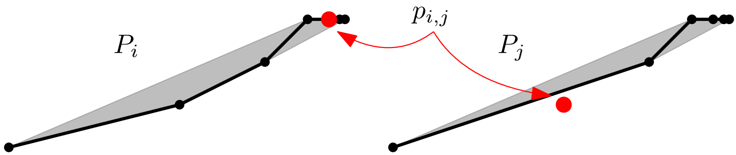
- for arbitrary graphs: if $v_i v_j$ is a non-edge, remove $p_{i,j}$ from $P_i$ and $P_j$



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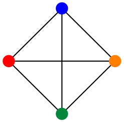
# Grid size

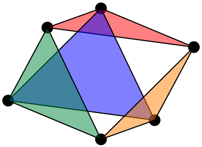
- our representation requires exponential-sized grid
- we consider also special classes of graphs

| Graph class | general | bipartite | 1-plane cubic | subcubic |
|---|---|---|---|---|
| Grid volume | super-poly | $O(n^4)$ | $O(n^2)$ | $O(n^3)$ |
| Running time | $O(n^2)$ | linear | linear | $O(n \log^2 n)$ |

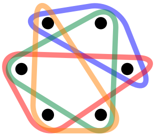
# Drawing Hypergraphs

Graph $G = (V, E)$





Hypergraph $H = (V, E)$



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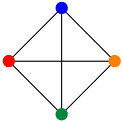

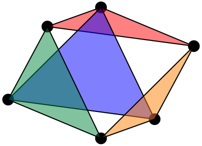

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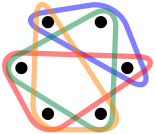

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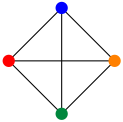

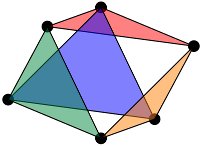

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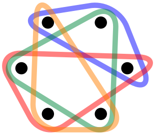

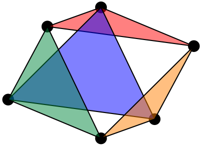

# Complete 3-uniform Hypergraphs

A hypergraph is 3-uniform if all its hyperedges are of cardinality 3.

### Theorem (Carmesin [ArXiv'19])

*Complete 3-uniform hypergraphs with* $n \geq 6$ *vertices cannot be realized by non-crossing triangles in 3d.*

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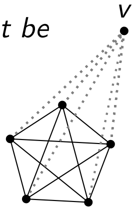


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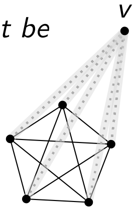


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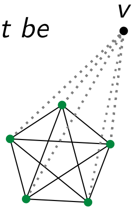


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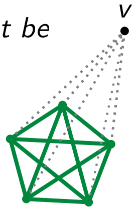


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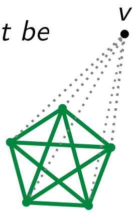


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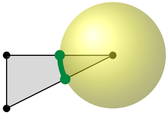
- If there is a non-crossing drawing, the link graph at any vertex must be planar.



# Steiner Systems

A Steiner system $S(t, k, n)$ is an $n$-element set $S$ together with a set of $k$-element subsets of $S$, called blocks, such that each $t$-element subset of $S$ is contained in exactly one block.

Steiner Triple Systems[1]

| $S(2,3,7)$ |
|---|
| 1 2 3 |
| 1 4 7 |
| 1 5 6 |
| 2 4 6 |
| 2 5 7 |
| 3 4 5 |
| 3 6 7 |

| $S(2,3,9)$ | |
|---|---|
| 1 2 3 | 1 5 9 |
| 4 5 6 | 2 6 7 |
| 7 8 9 | 3 4 8 |
| 1 4 7 | 1 6 8 |
| 2 5 8 | 2 4 9 |
| 3 6 9 | 3 5 7 |

Steiner Quadruple System

| $S(3,4,8)$ | |
|---|---|
| 1 2 4 8 | 3 5 6 7 |
| 2 3 5 8 | 1 4 6 7 |
| 3 4 6 8 | 1 2 5 7 |
| 4 5 7 8 | 1 2 3 6 |
| 1 5 6 8 | 2 3 4 7 |
| 2 6 7 8 | 1 3 4 5 |
| 1 3 7 8 | 2 4 5 6 |

[1] Ossona de Mendez [JGAA'02] shows that any 3-uniform hypergraph with incidence poset dimension 4 has a non-crossing drawing with triangles. This implies the existence of 3d representations (with exponential coordinates) for the two smallest Steiner triple systems.

# Steiner Triple Systems

**Theorem**

*The Fano plane $S(2,3,7)$ has a non-crossing drawing.*

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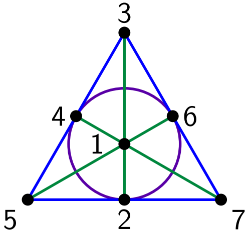
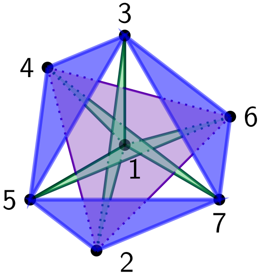
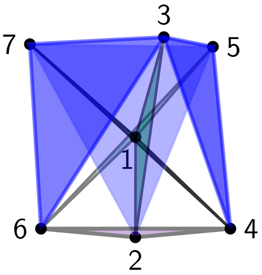
| 3 6 7 |




2d drawing

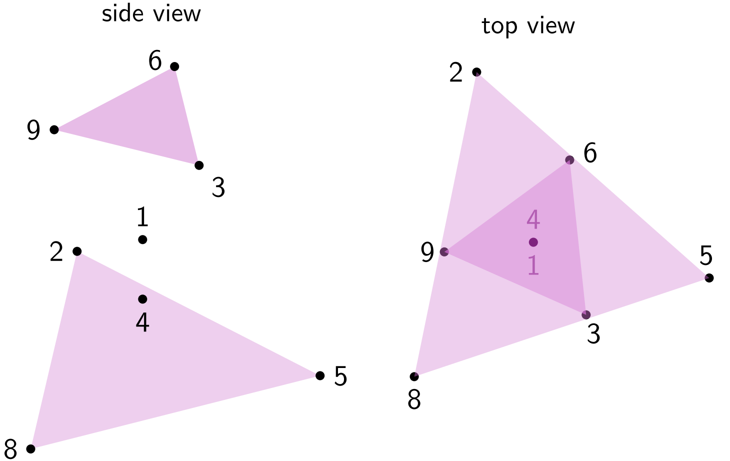



top 3d view




side 3d view

# Steiner Triple Systems (cont.)

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| 2 6 7 |
| 3 4 8 |
| 1 6 8 |
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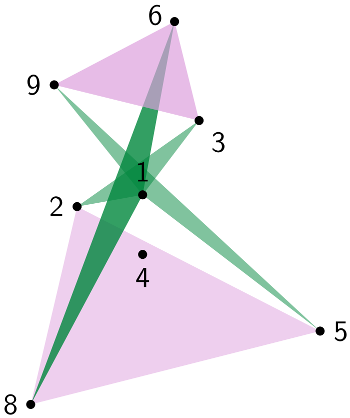
| 3 5 7 |

## Theorem

*The Steiner triple system $S(2,3,9)$ has a non-crossing drawing.*

# Steiner Triple Systems (cont.)

side view

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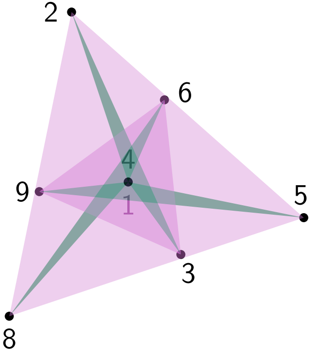


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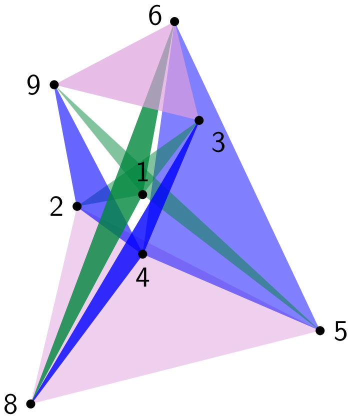


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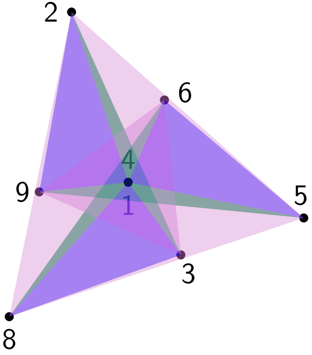


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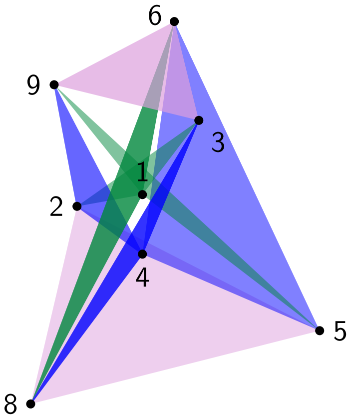
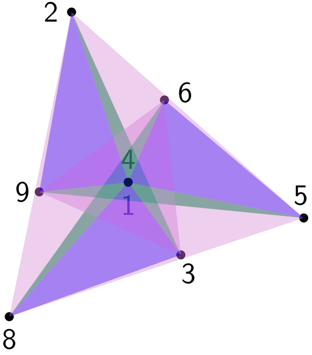
# Steiner Triple Systems (cont.)

side view




top view

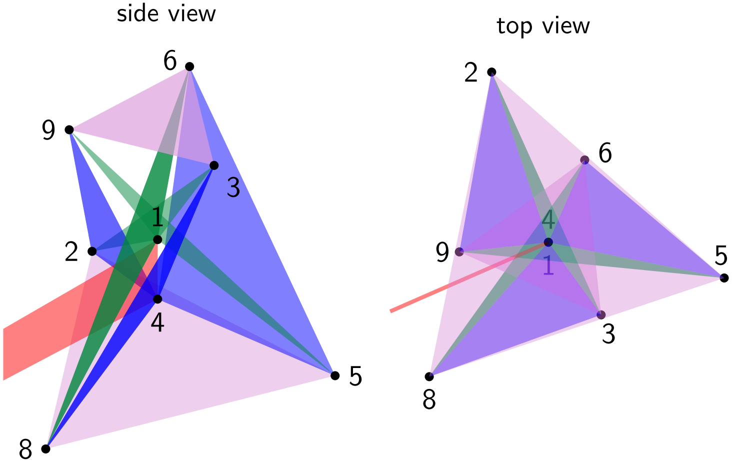



| $S(2,3,9)$ |
|---|
| 1 2 3 |
| 4 5 6 |
| 7 8 9 |
| 1 4 7 |
| 2 5 8 |
| 3 6 9 |
| 1 5 9 |
| 2 6 7 |
| 3 4 8 |
| 1 6 8 |
| 2 4 9 |
| 3 5 7 |

## Theorem

*The Steiner triple system $S(2,3,9)$ has a non-crossing drawing.*

# Steiner Triple Systems (cont.)




| $S(2,3,9)$ |
|---|
| 1 2 3 |
| 4 5 6 |
| 7 8 9 |
| 1 4 7 |
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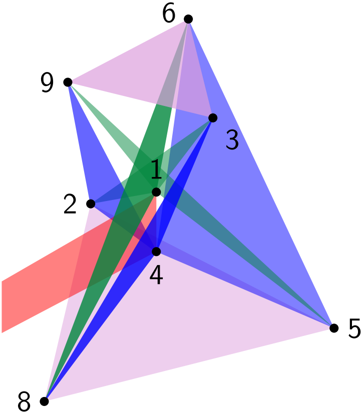


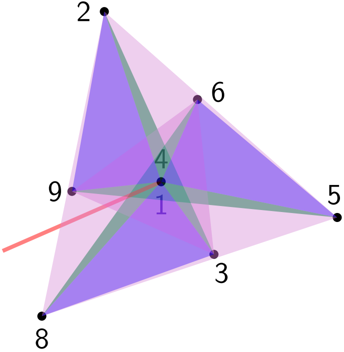


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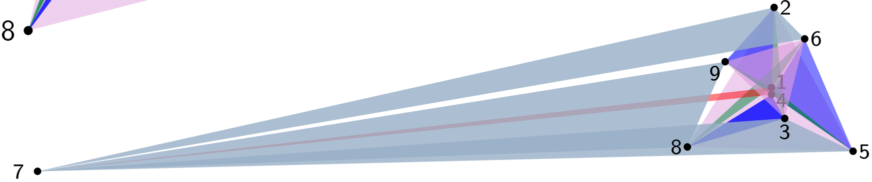

# Steiner Quadruple Systems

## Theorem

*The Steiner quadruple system $S(3,4,8)$ does not have a non-crossing drawing.*

| $S(3,4,8)$ | |
|---|---|
| 1 2 4 8 | 3 5 6 7 |
| 2 3 5 8 | 1 4 6 7 |
| 3 4 6 8 | 1 2 5 7 |
| 4 5 7 8 | 1 2 3 6 |
| 1 5 6 8 | 2 3 4 7 |
| 2 6 7 8 | 1 3 4 5 |
| 1 3 7 8 | 2 4 5 6 |

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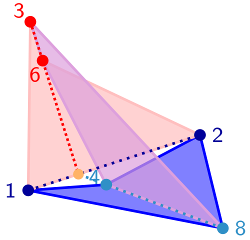
| 1248 | 36 |
|---|---|

$P_{1236} \cap P_{3468} = l_{36}$ and $l_{12} \cap l_{48} \in l_{36}$




| $S(3,4,8)$ | |
|---|---|
| 1 2 4 8 | 3 5 6 7 |
| 2 3 5 8 | 1 4 6 7 |
| 3 4 6 8 | 1 2 5 7 |
| 4 5 7 8 | 1 2 3 6 |
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| 1248 | 37 | $P_{1378} \cap P_{2347} = l_{37}$ and $l_{18} \cap l_{24} \in l_{37}$ |
| 1248 | 67 | $P_{1467} \cap P_{2678} = l_{67}$ and $l_{14} \cap l_{28} \in l_{67}$ |

| $S(3,4,8)$ | |
|---|---|
| 1 2 4 8 | 3 5 6 7 |
| 2 3 5 8 | 1 4 6 7 |
| 3 4 6 8 | 1 2 5 7 |
| 4 5 7 8 | 1 2 3 6 |
| 1 5 6 8 | 2 3 4 7 |
| 2 6 7 8 | 1 3 4 5 |
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If there is a drawing,

- 3,6, and 7 are all placed at the same point.
- 3567 is degenerate; a contradiction.
  (In fact, we can show that 3567 is just a point.) □

# Steiner Quadruple Systems

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If there is a drawing,

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## Theorem

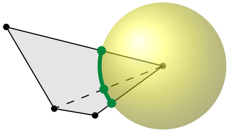
*The Steiner quadruple system $S(3,4,10)$ cannot be drawn using all convex or all non-convex non-crossing quadrilaterals.*

# Steiner Quadruple Systems (cont.)

## Theorem

*No Steiner quadruple system can be drawn using convex quadrilaterals*[2].

- Any vertex $v$ is incident to $\frac{(n-1)(n-2)}{6}$ quadrilaterals.
- Add the diagonals incident to $v$ to get a simplicial 2-complex.
- The link graph at $v$ has $\frac{(n-1)(n-2)}{3}$ edges and $n-1$ vertices.
- For $n > 8$, the link graph is not planar.



[2]We thank Arnaud de Mesmay and Eric Sedgwick for pointing us to a lemma of Dey and Edelsbrunner [DCG'94], which uses the same proof idea.

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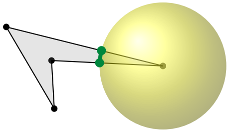
## Theorem

*No Steiner quadruple system with* 20 *or more vertices can be drawn using quadrilaterals.*



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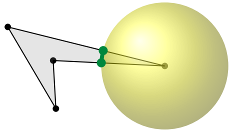

### Conjecture

*No Steiner quadruple system can be drawn using non-crossing quadrilaterals.*

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# Open problems

| | |
|---|---|
| Other hypergraphs | Larger Steiner triple systems/projective planes. |
| Hardness | Is deciding whether a 3-uniform hypergraph has a non-crossing drawing with triangles NP-hard? |
| Grid size | Can any graph be represented with convex polygons on a polynomial sized grid? |
| Nicer drawings | Small aspect ratio, large angle resolution, etc. |