Angle Covers

Algorithms and Complexity
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Angle Covers Definition

Given a graph G (not necessarily planar) and an embedding (ordered adjacency lists), does there exist an angle cover?

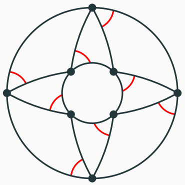


Figure 1: A plane graph with an angle cover using each cover exactly once.

In an angle cover, each vertex chooses two adjacent edges to cover, such that they must also be adjacent in the circular ordering around the vertex, i.e. there is an "angle" between them.

What we Mean by Embedding

Instead of discussing embeddings, we use the term "rotation system". This is a circular ordering of the edges around each vertex, so we do not care if edges intersect.

In fact, we will soon simplify to the case of a plane graph.

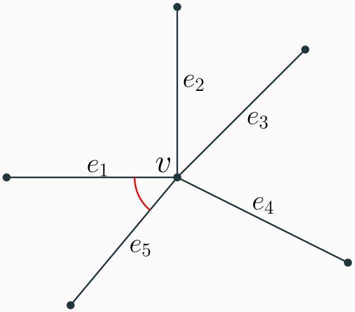


Figure 2: A vertex v with the circular ordering e_1, e_2, e_3, e_4, e_5, and the marked angle (e_5, e_1).

The "Easy" Version (1)

There is a simple relaxation of this problem that takes a general graph as input: For each vertex, choose any two adjacent edges to cover.

A graph G has a "relaxed angle cover" if and only if: For every subgraph $G' \subset G$, the number of edges $||G'|| \leq 2|G'|$.

If we got to choose our embedding for the angle cover problem, we can choose one so that the "relaxed angle cover" is also an angle cover. As such, we assume the embedding/rotation system is given in the problem.

The "Easy" Version (2)

The density constraint also simplifies the original problem:
If it is not met, we get an "easy no", i.e. if there is some subset with a dense set of induced edges, then there are insufficiently many angles to form an angle cover for this subset (and therefore the whole graph).

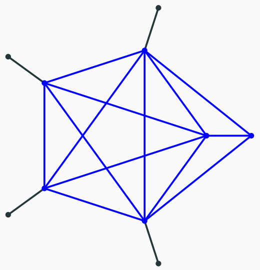


Figure 3: A graph with 10 vertices and 17 edges, and a subgraph highlighted in blue with 6 vertices and 13 edges, violating the density constraint.

A Non-Trivial "No" Instance

Lemma: The following graph does not admit an angle cover:

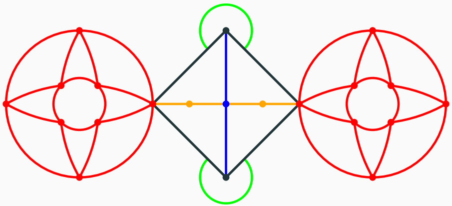


Figure 4: A low-density graph and embedding that does not have an angle cover.

Proof: This graph has a total of 21 vertices and 42 edges, so no edge can be covered twice. The two orange angles are the only angles available for the orange vertices to choose. There are 16 red vertices, and 32 red edges. Every red edge is adjacent to two red vertices, and so all the angles of red vertices must cover red edges. This induces the two green angles. Finally, the blue vertex is unable to cover both of the blue edges. ■

A Reduction to the Planar Case

Lemma: Suppose we have an embedding of a topological graph G. In particular, assume that no three edges cross at a single point. Then, we can create a plane graph G' that has an angle cover if and only if G and its embedding has an angle cover.

Replace each intersection with a vertex ("planarization"):

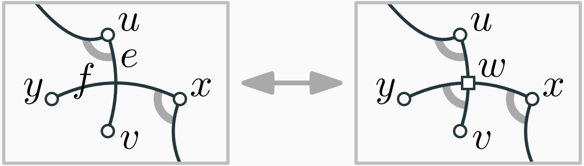


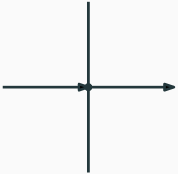
Figure 5: Any topological graph (left) admits an angle cover if and only if its planarization (right) admits an angle cover.

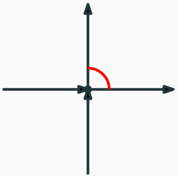
Euler Tour for Maximum Degree 4

Theorem: For any graph with maximum degree 4 and a rotation system, we can find an angle cover in linear time.

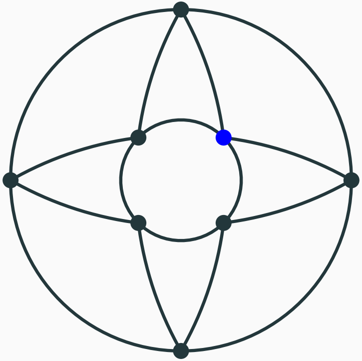
Augment to Degree 4:
For any vertices with degree < 4, add "imaginary" edges between them until there is at most one such vertex, which must have degree 2 (and can be dealt with a number of ways).

Modified Euler Tour:
When the maximum degree is 4, we can modify the Euler tour algorithm to find an angle cover. Whenever you enter a node, simply exit the node using the edge opposite to the entry-edge. The two "out"-oriented edges form the angle at the vertex.



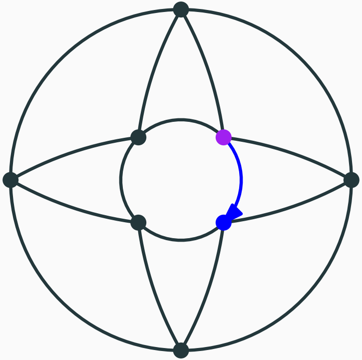


Euler Tour for Maximum Degree 4 - Example

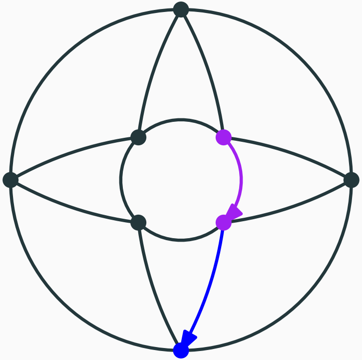


We will quickly look at an example of finding an angle cover using Euler tours, starting from this blue vertex (chosen arbitrarily).

Euler Tour for Maximum Degree 4 - Example

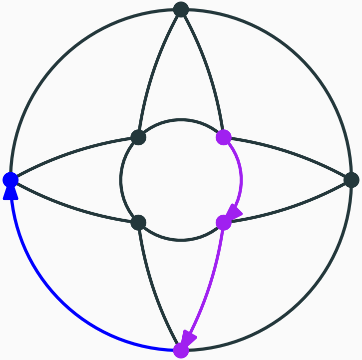


Euler Tour for Maximum Degree 4 - Example



Whenever we visit a vertex, we exit on the side opposite.

Euler Tour for Maximum Degree 4 - Example



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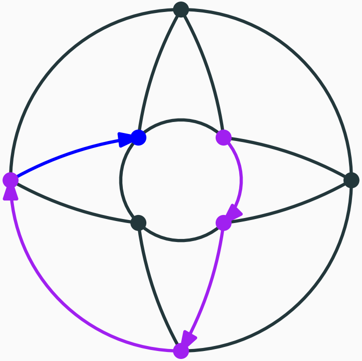

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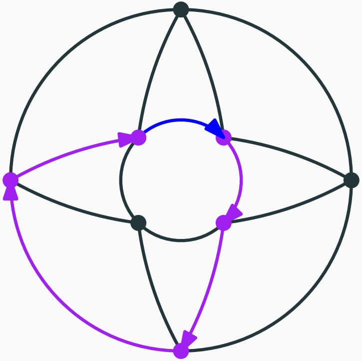

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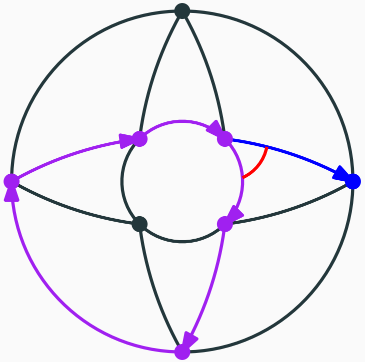
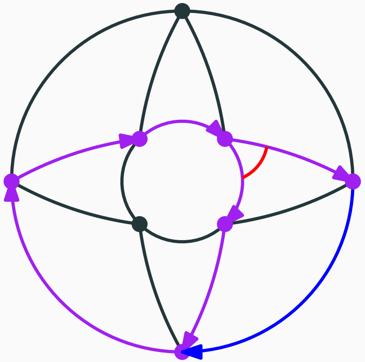


When we've found our two out-oriented edges for a vertex, we can mark the angle.

Euler Tour for Maximum Degree 4 - Example



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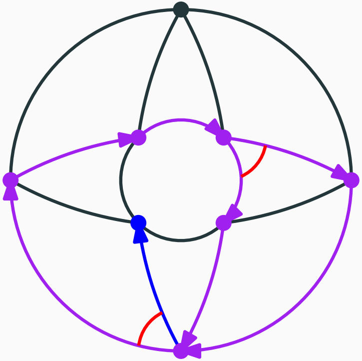

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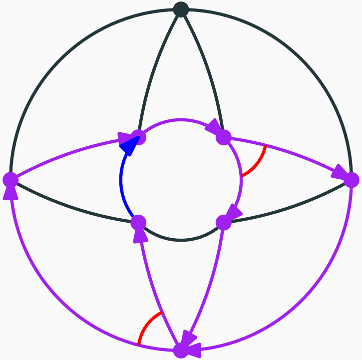

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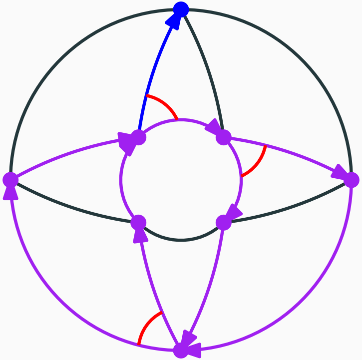

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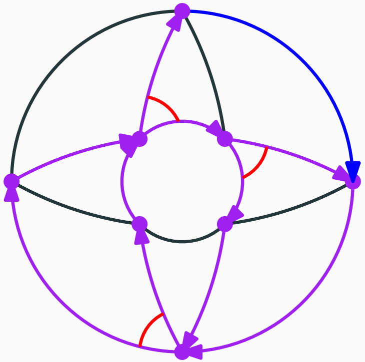

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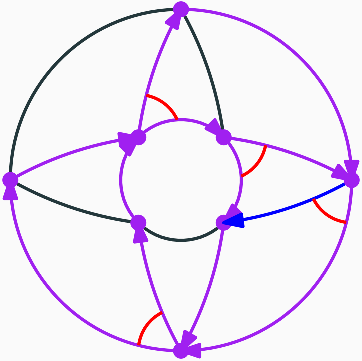

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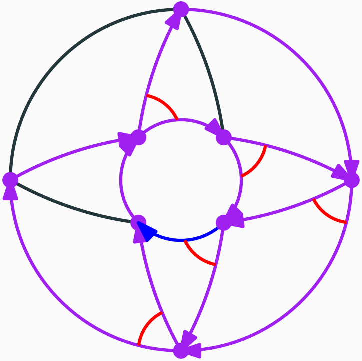

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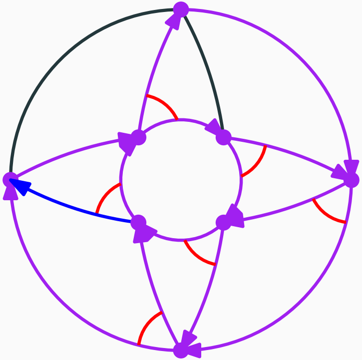

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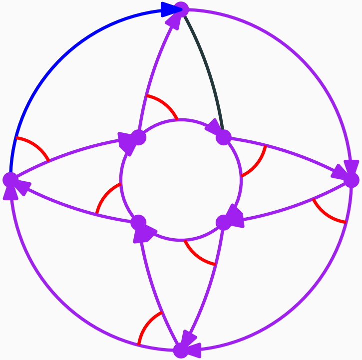

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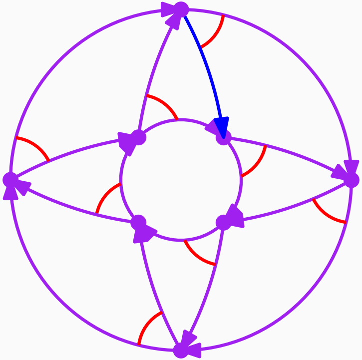

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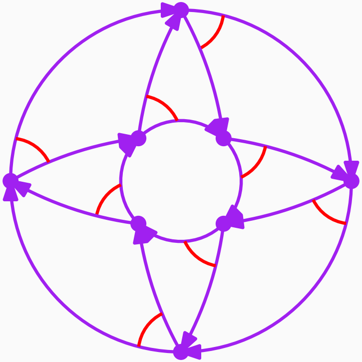


When we arrive at a vertex with no more edges to orient, we stop. If need be, we start again elsewhere.

Hardness (1)

Theorem: It is NP-Hard to determine if a graph with maximum degree 5 and a rotation system has an angle cover.

Proof: **Reduction from 3-colouring.** Start with a graph G for the 3-colouring problem. We will create a graph H and rotation system for the angle cover problem.

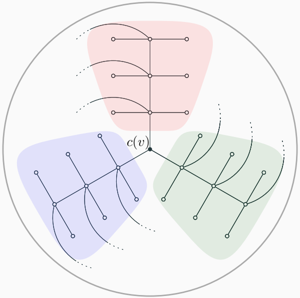
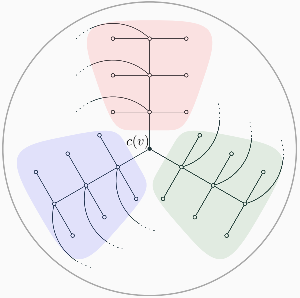


Figure 7: The construction at each vertex.

Hardness (2)

For each vertex v in G of degree d:

- Create the "colour selector vertex" $c(v)$ with degree 3.
  - Edges connected to $c(v)$ correspond to colours. The uncovered edge is the colour of the vertex.
- Turn each of the edges coming out of $c(v)$ into a "chain" of length d. Each vertex in the chain will have degree 1 vertices connected to it to "split" the angle choices.



- For the each of the three edges coming out of $c(v)$:
  - Create a chain of length d, where each vertex in the chain has two connected degree 1 edges ("splitters"), splitting up the angle choices.

Between each pair of connected vertices u and v in G:

- Add edges between each of their 3 corresponding colour chains, on the side of the splitters that can be covered when at least one vertex is *not* that colour.

Hardness (3)

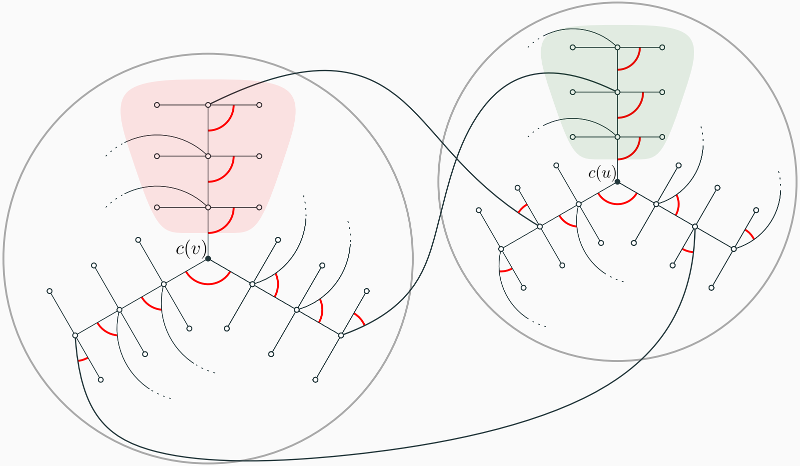


Figure 8: The resulting constructions for two vertices *v* and *u* that were connected. Here *v* is coloured red, and *u* is coloured green.

Another Tractable Case: No Vertices of Degree 3

Theorem: For graphs with no vertices of degree exactly 3 and any rotation system, there is a quadratic time algorithm to decide if there is an angle cover.

Proof: We reduce to 2-SAT. Create a variable for each half-edge, representing whether or not that half-edge is covered by an angle.

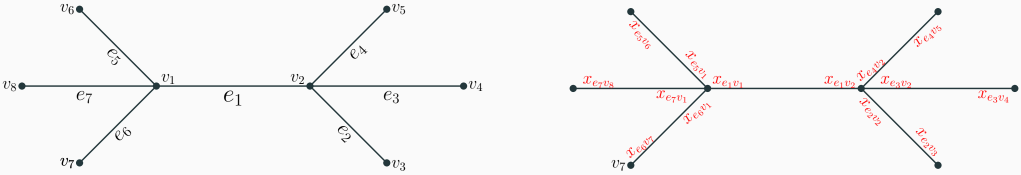


Figure 9: The variables for the 2-SAT encoding of a small plane graph.

Another Tractable Case: 2-SAT Encoding Continued

The kinds of constraints are:

- **All edges covered**: For each edge $e = v_i v_j$, $(x_{ev_i} \lor x_{ev_j})$
- **Non-adjacent edges not both covered**: For each vertex v with degree ≥ 4, and for every pair of edges e_i and e_j that are not adjacent in the circular ordering around v, $(\neg x_{e_i v} \lor \neg x_{e_j v})$

Any maximal satisfying assignment corresponds to an angle cover.

This encoding is quadratic in size.

Generalizations and Open Problems

What if every vertex could select two angles? We call this a 2-angle cover.
In our paper, we give upper and lower bounds for the complexity of this problem, as well as generalizations with even more angles. The complexity of the following problem remains open:

> *For a graph with maximum degree* 7 *with a rotation system, does it admit a* 2*-angle cover?*

It is known to be NP-hard for maximum degree 8, and all graphs with rotation systems with maximum degree ≤ 6 always admit angle covers. One of these cases also applies to maximum degree 7.

We also do not know the complexity of our problem on **Laman graphs**, which have a similar density constraint (all subgraphs G' have $\leq 2|G'| - 3$ edges), but the full graph also has *exactly* $2|G| - 3$ edges.

More in Our Paper

In our paper, we consider several generalizations and relaxations of the angle cover problem, for which we provide both algorithms and hardness results.

We also give an application of angle covers to isomorphic thickness.

Thank you!

Thanks for listening!

Now: Q&A.

Later: I'm always happy to answer more questions over email. I can be reached at `jacketsj@alumni.ubc.ca`