Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Approximating Minimum Manhattan Networks in Higher Dimensions

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ESA'11

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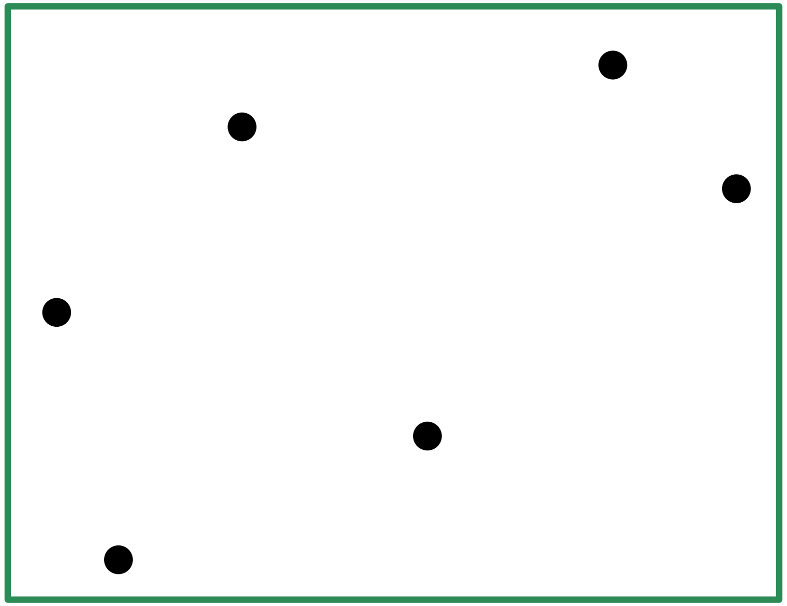
Minimum Manhattan Networks

Given a set of points called **terminals** in $\mathbb{R}^d$,
find a minimum-length network such that each pair of terminals is connected by a **Manhattan path**.



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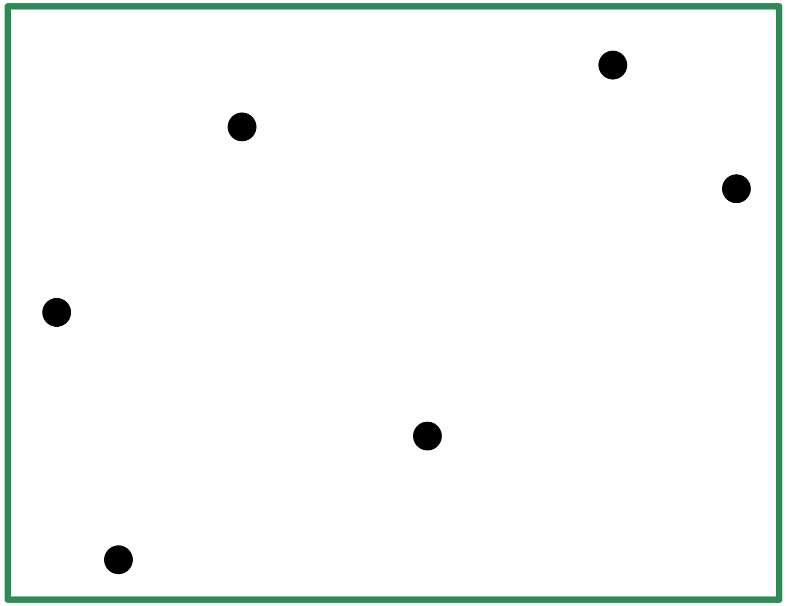

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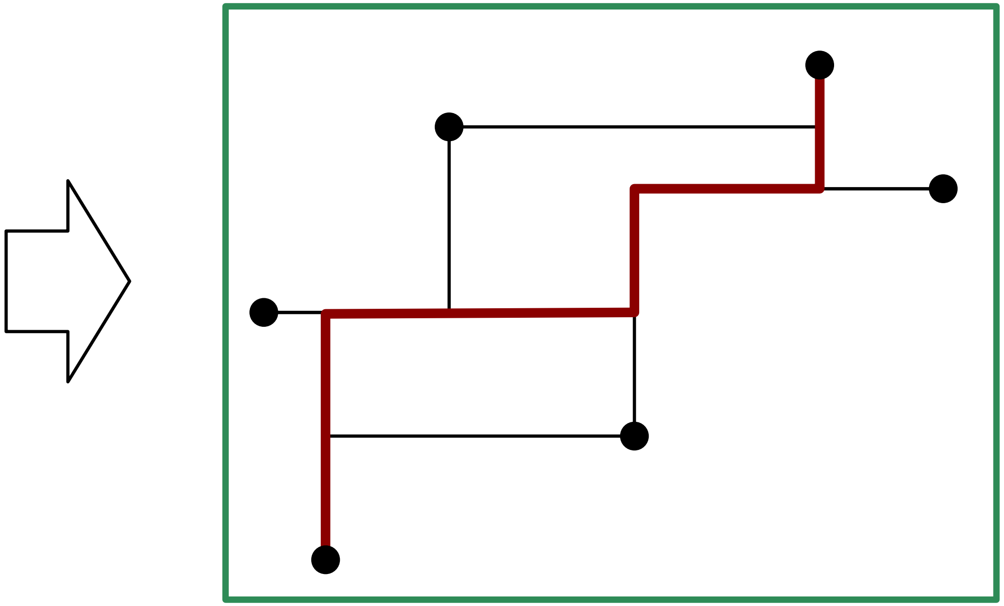

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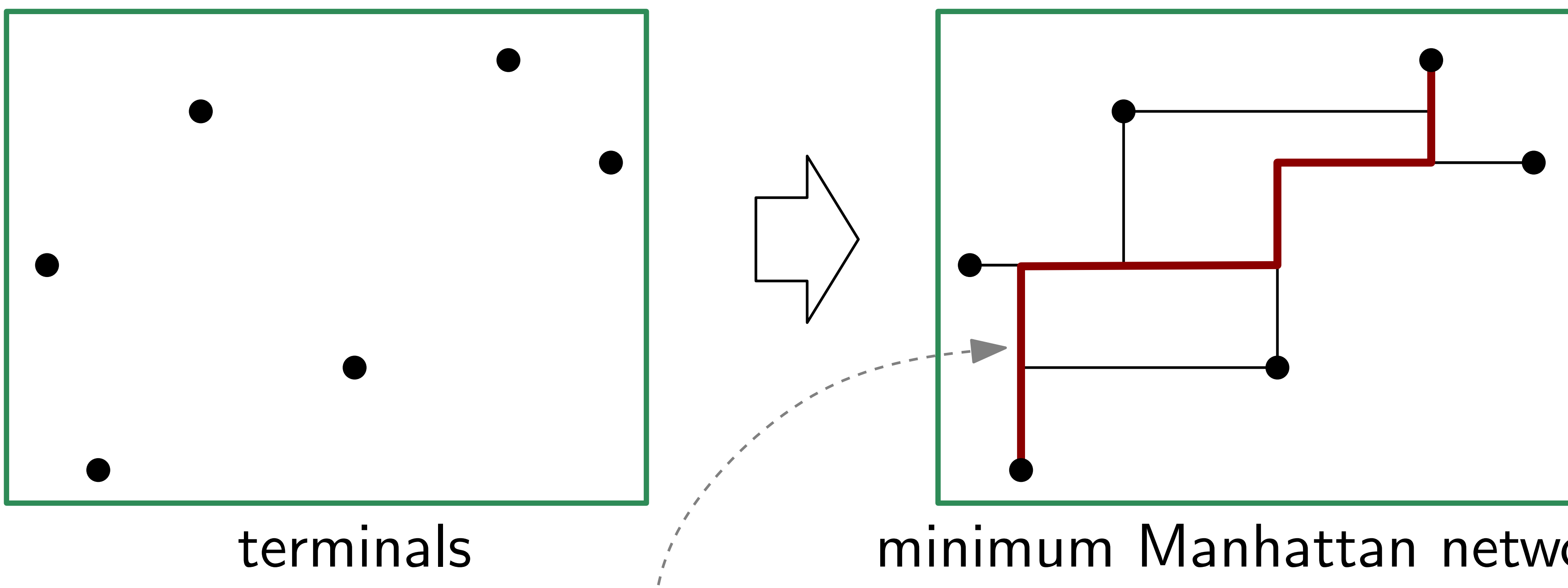

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A Manhattan path is a chain of axis-parallel line segments whose total length is the Manhattan distance of the chain's endpoints.

Previous Results

Results for 2D

- introduced by Gudmundsson et al. (NJC'01)

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Results for 3D (or higher dimensions)

- constant factor approximation for very restricted 3D case
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- Non-trivial approximations for unrestricted version?

Our Results

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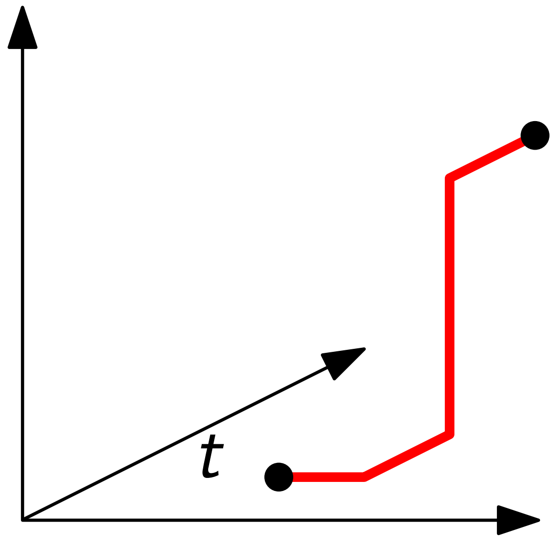
Decomposition into Directional Subproblems

Directional Subproblem: M-connect all pairs of terminals $t = (x, y, z)$ and $t' = (x', y', z')$ with $x \leq x'$, $y \leq y'$, $z \leq z'$.



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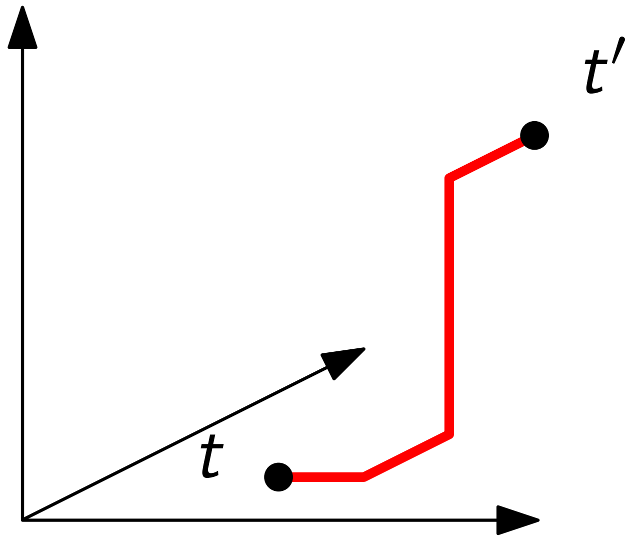

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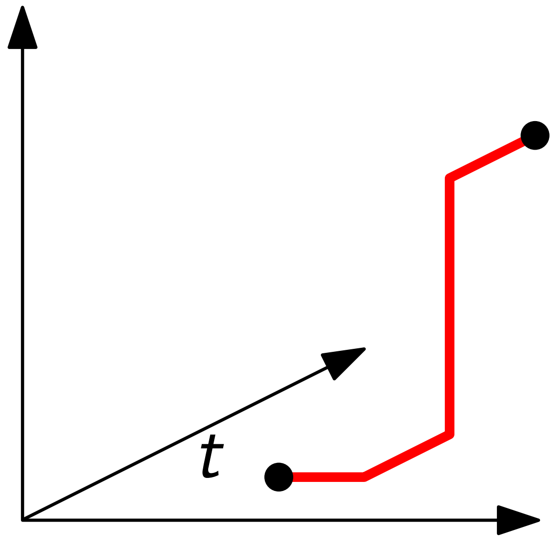
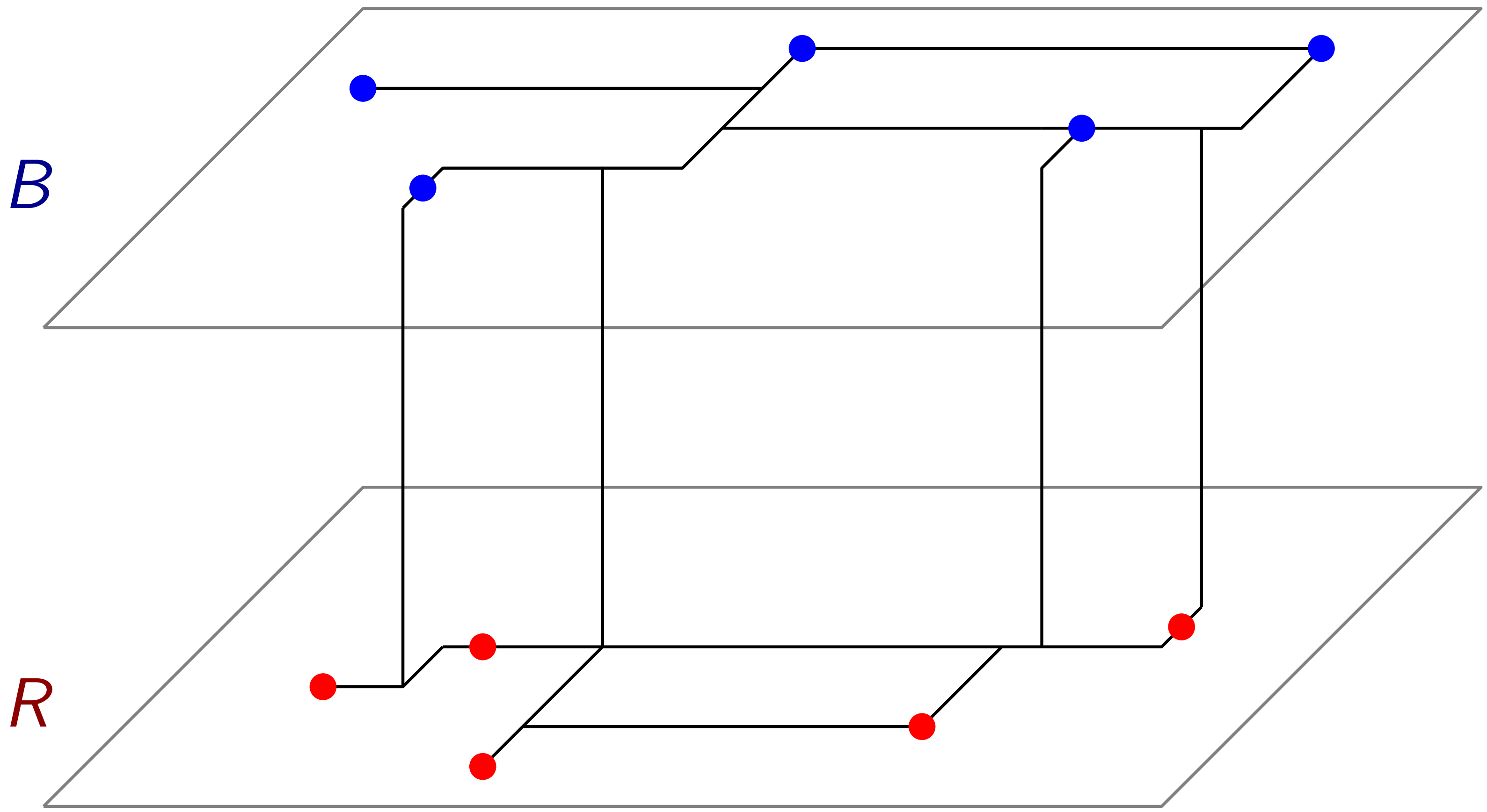


General problem can be decomposed into four directional subproblems

Two Horizontal Planes

Let N be some directional Manhattan network.



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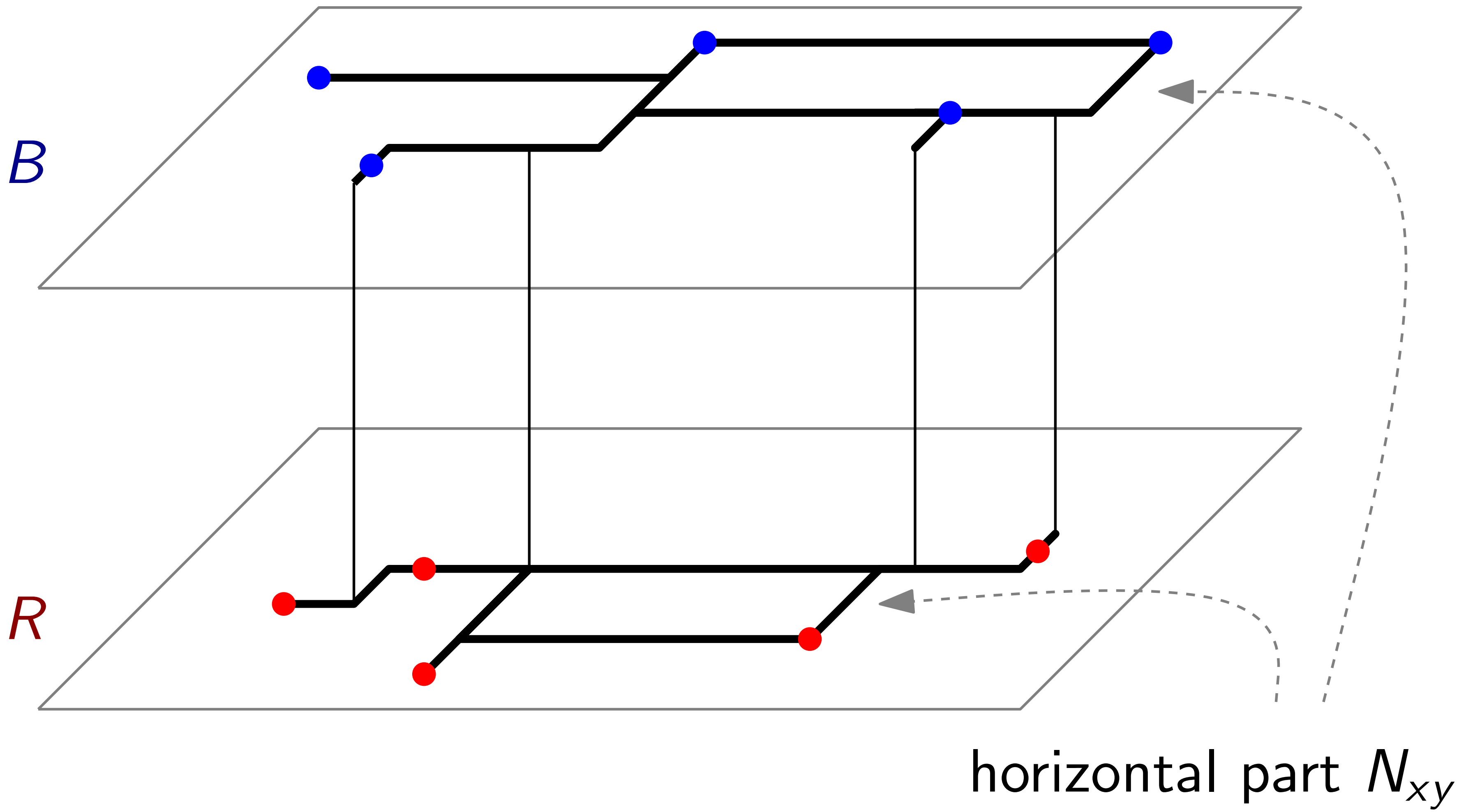

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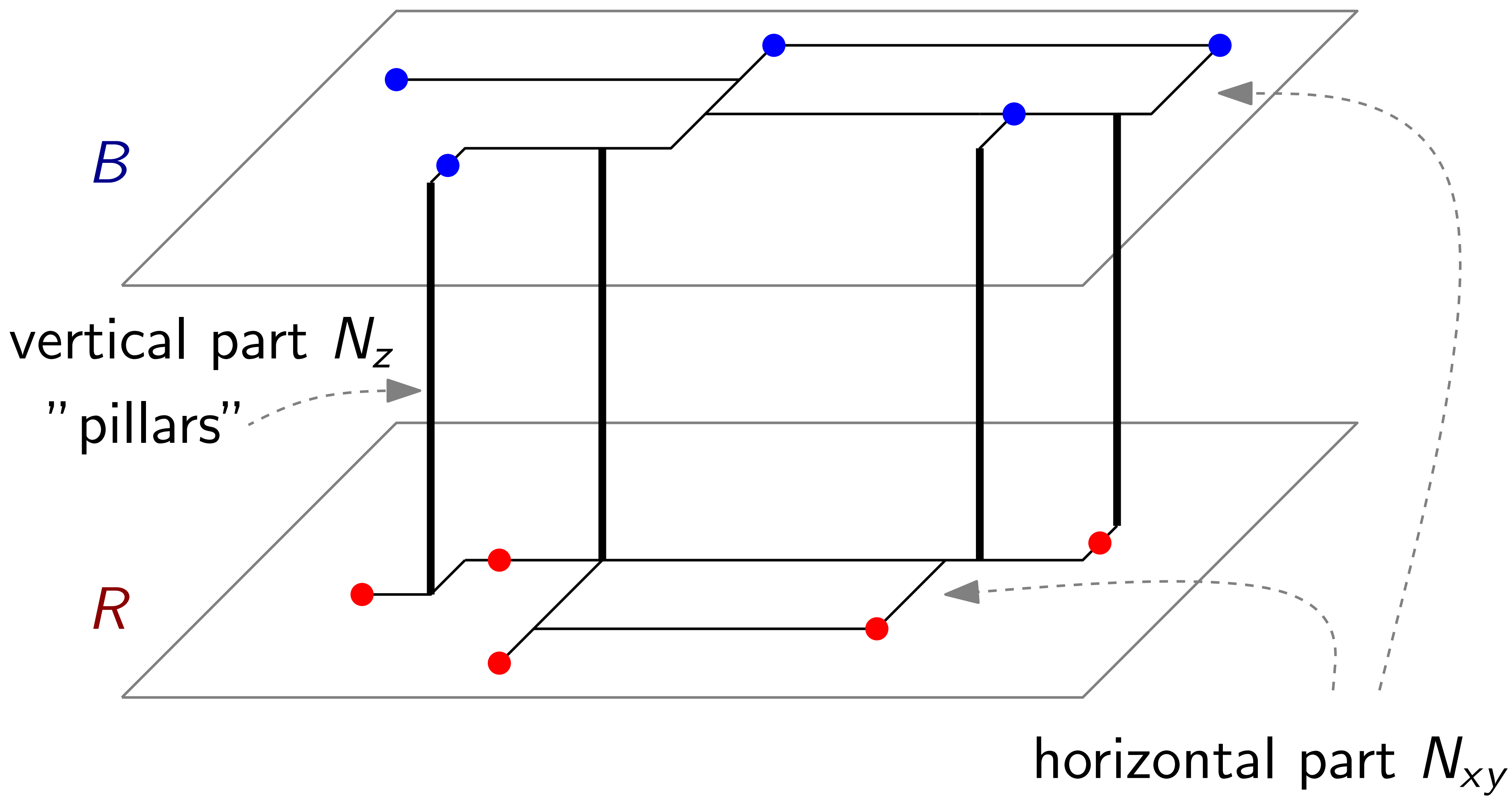

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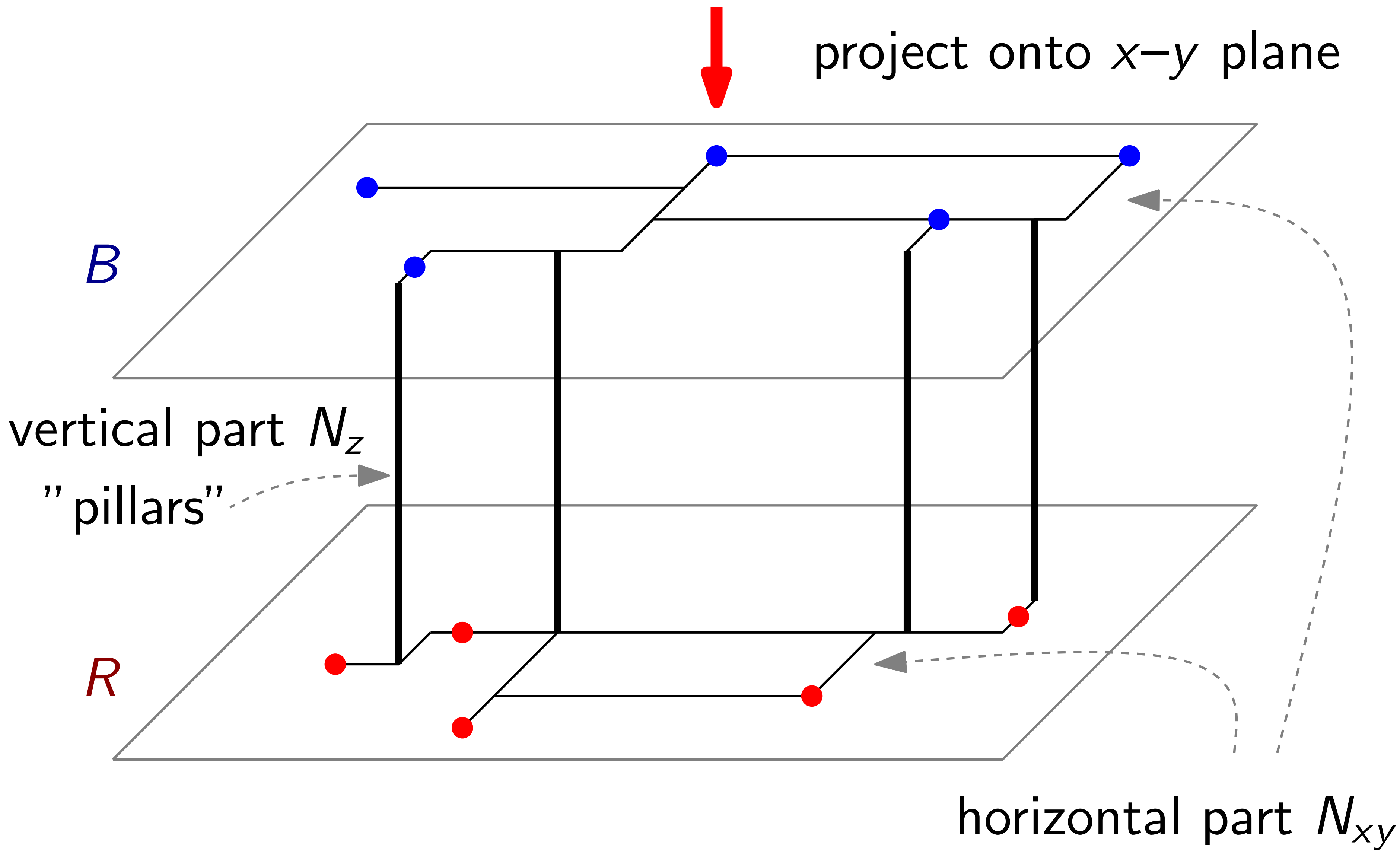
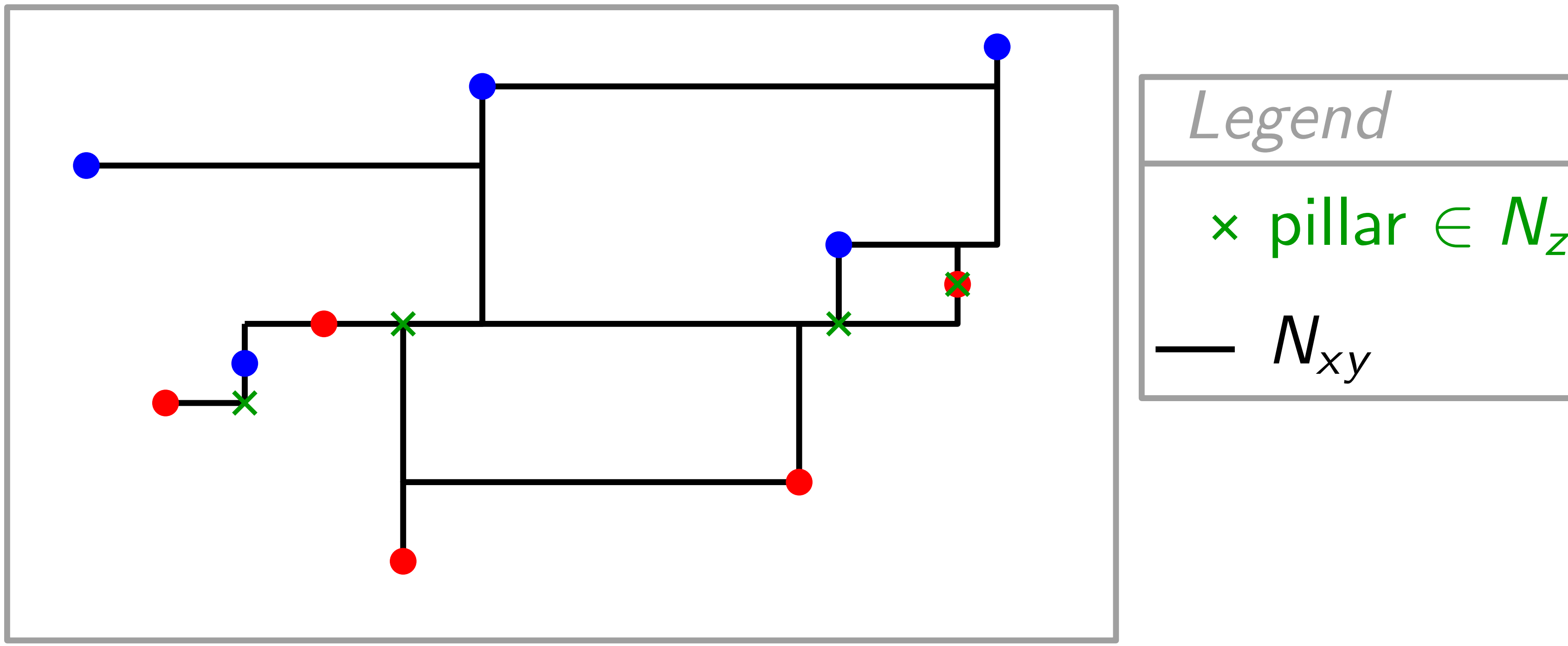


2D Projection



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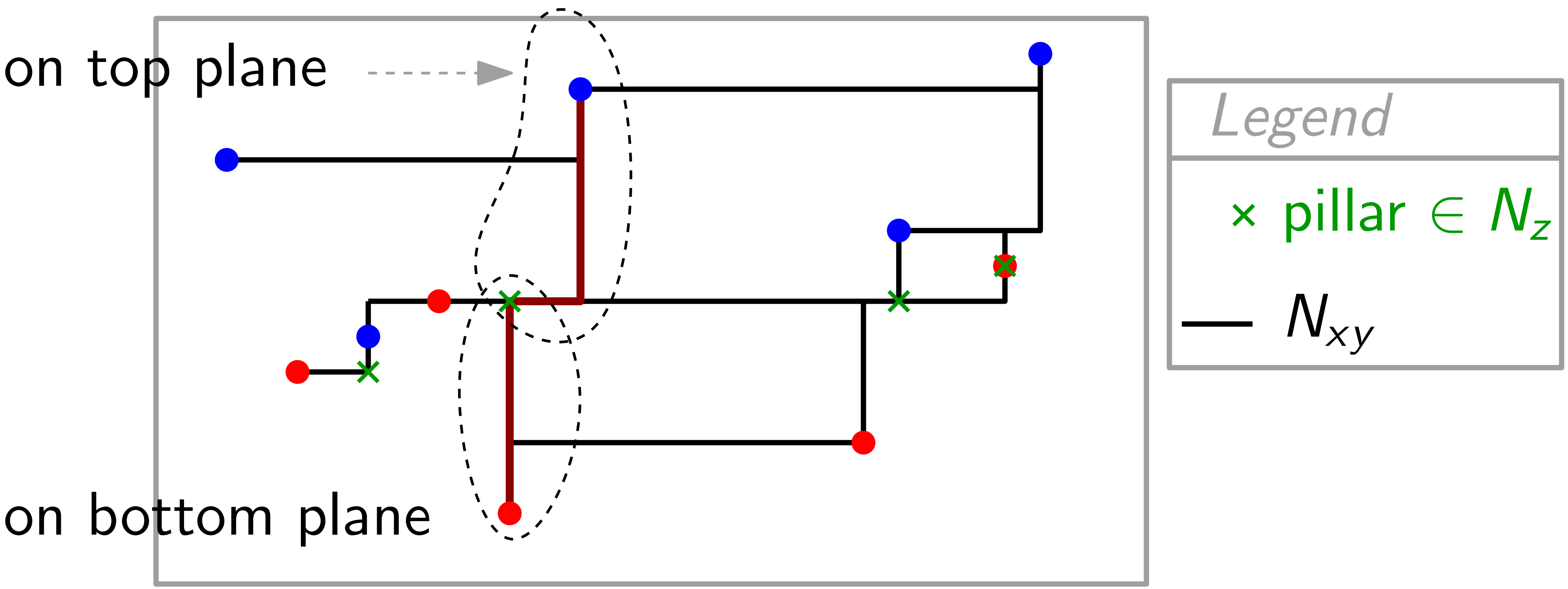
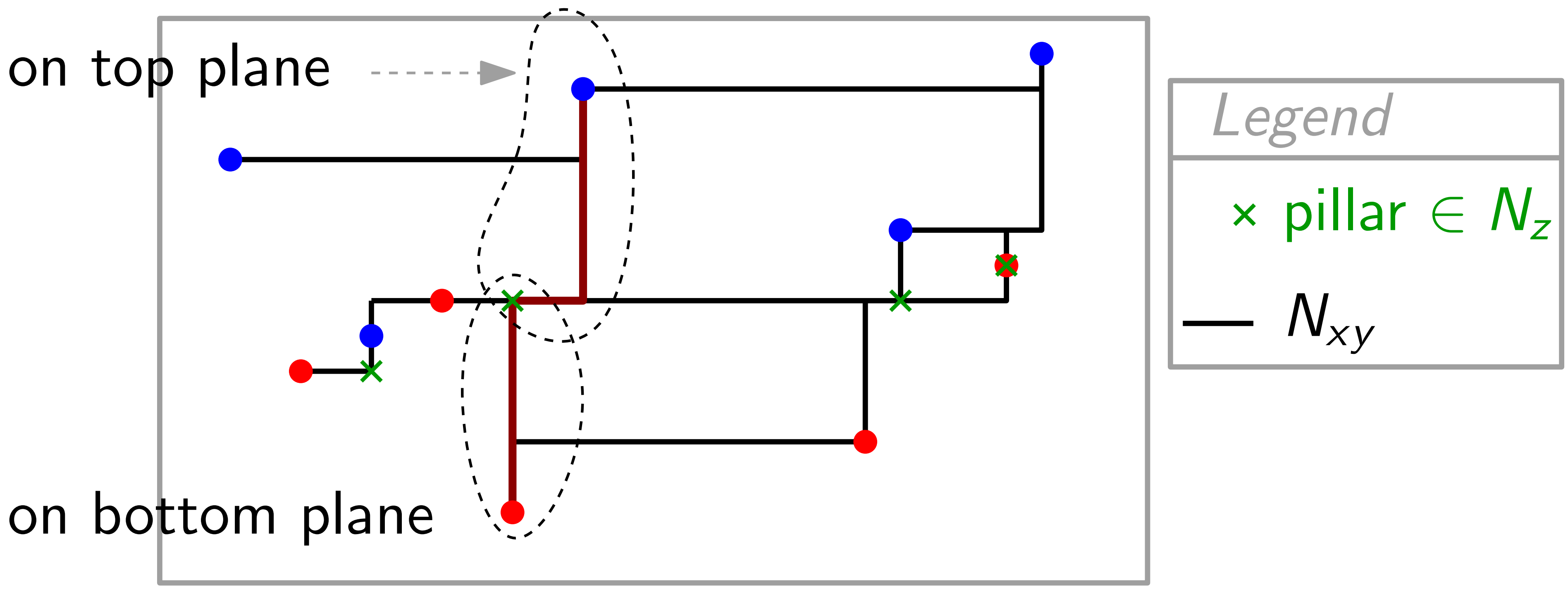


2D Projection

N_{xy} is a directional **2D Manhattan network** for $R \cup B$



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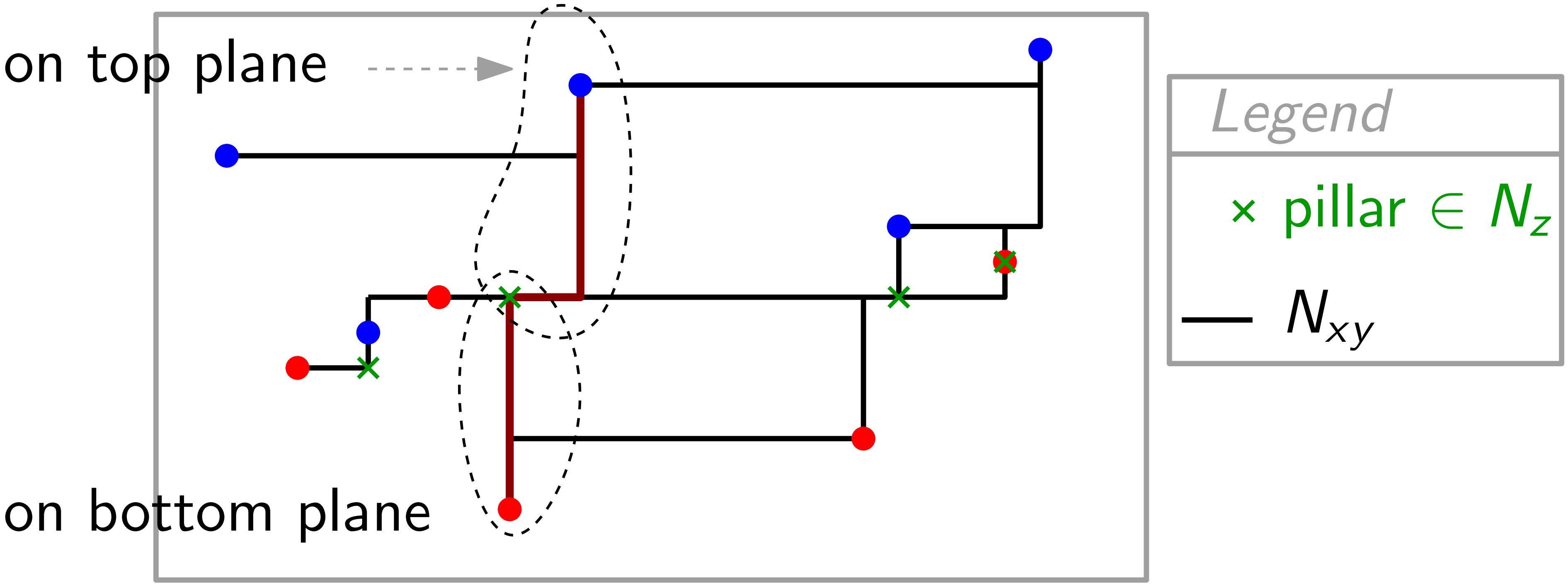
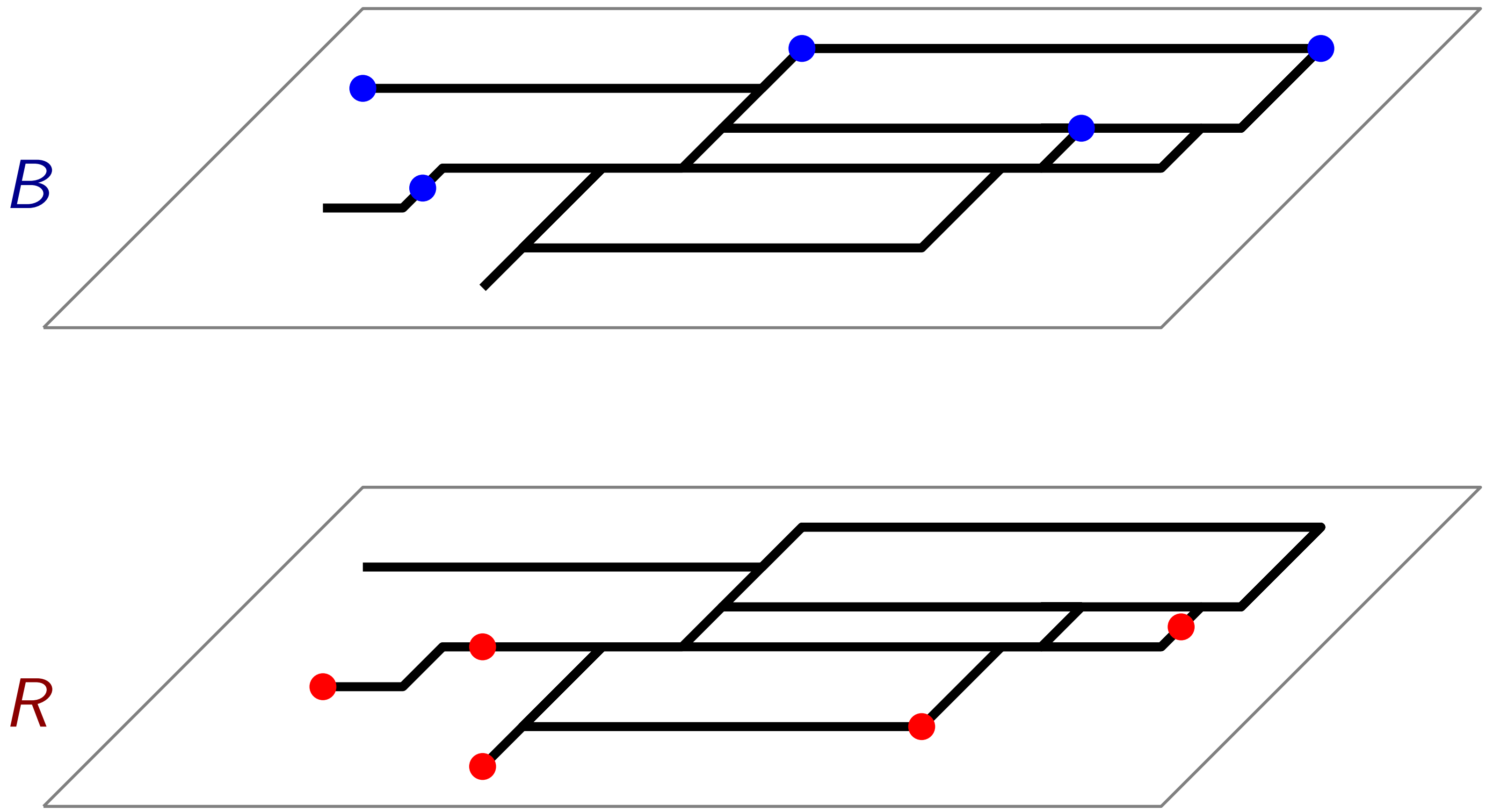
Use 2D approximation on both planes



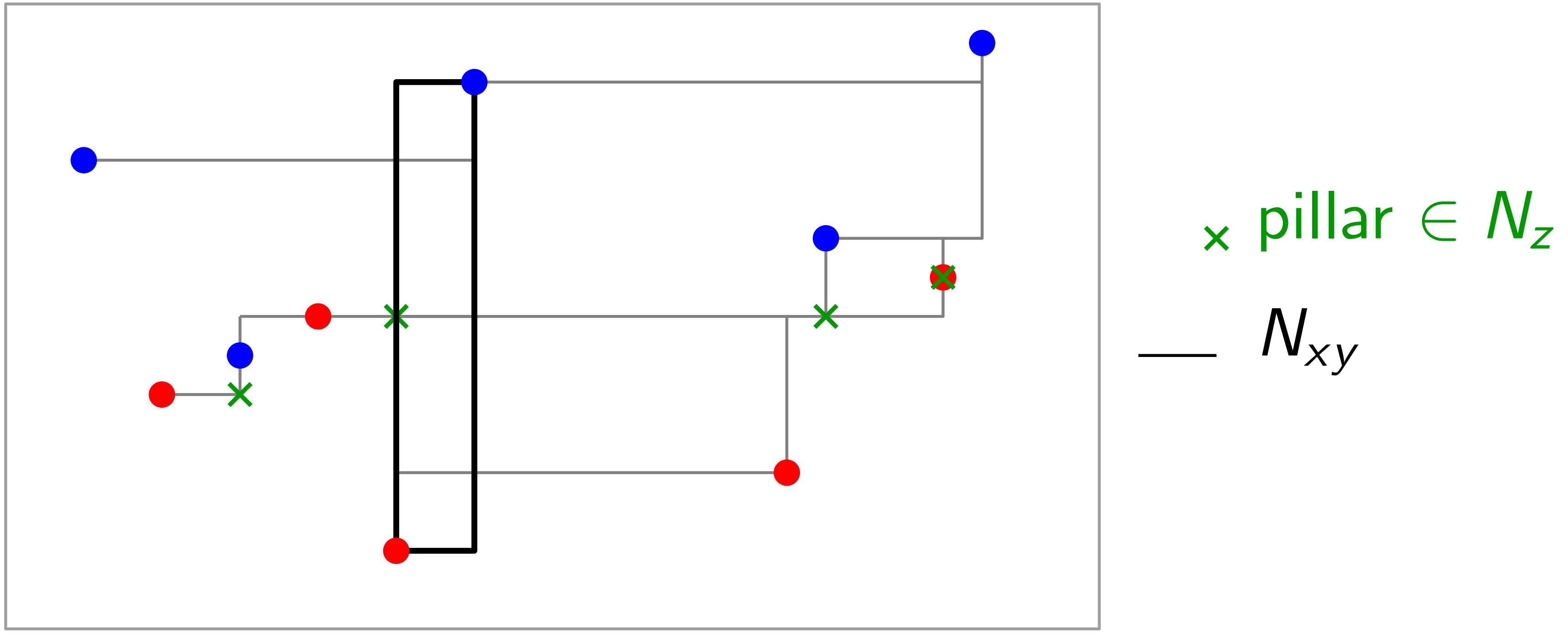
Approximating the Horizontal Part is Easy

Copy 2-approximate 2D network for $R \cup B$ onto both planes



But How to Find the Pillars?

Each rectangle spanned by a relevant red-blue terminal pair is **pierced** by some pillar in N.



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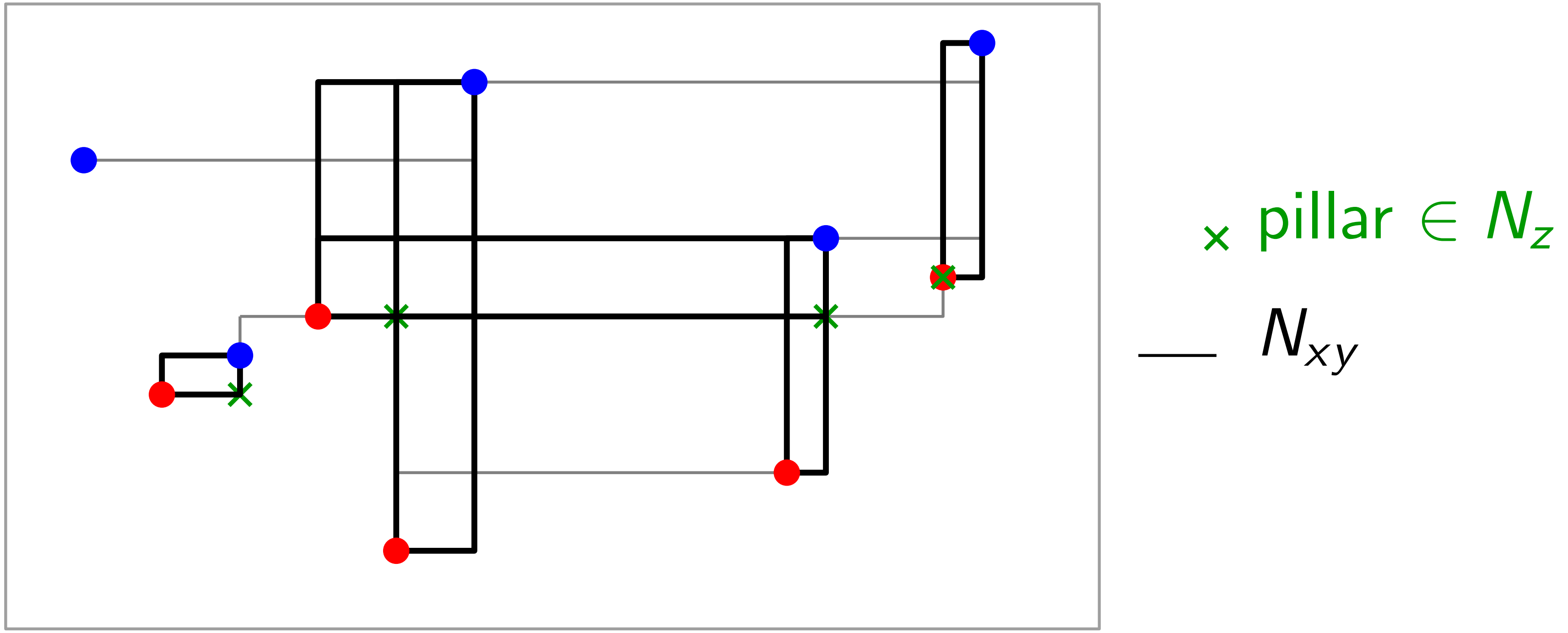

Lower Bounding by Red-Blue Piercings

Subproblem RBP:

Given a set of red and blue points in the plane,

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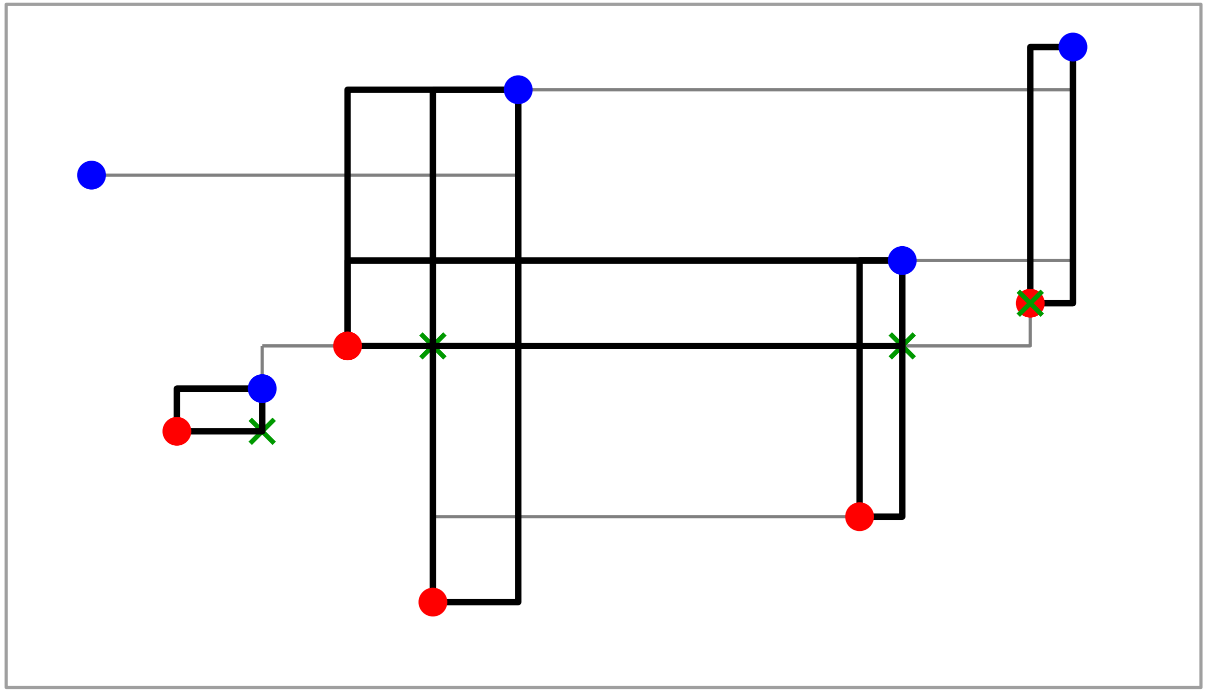

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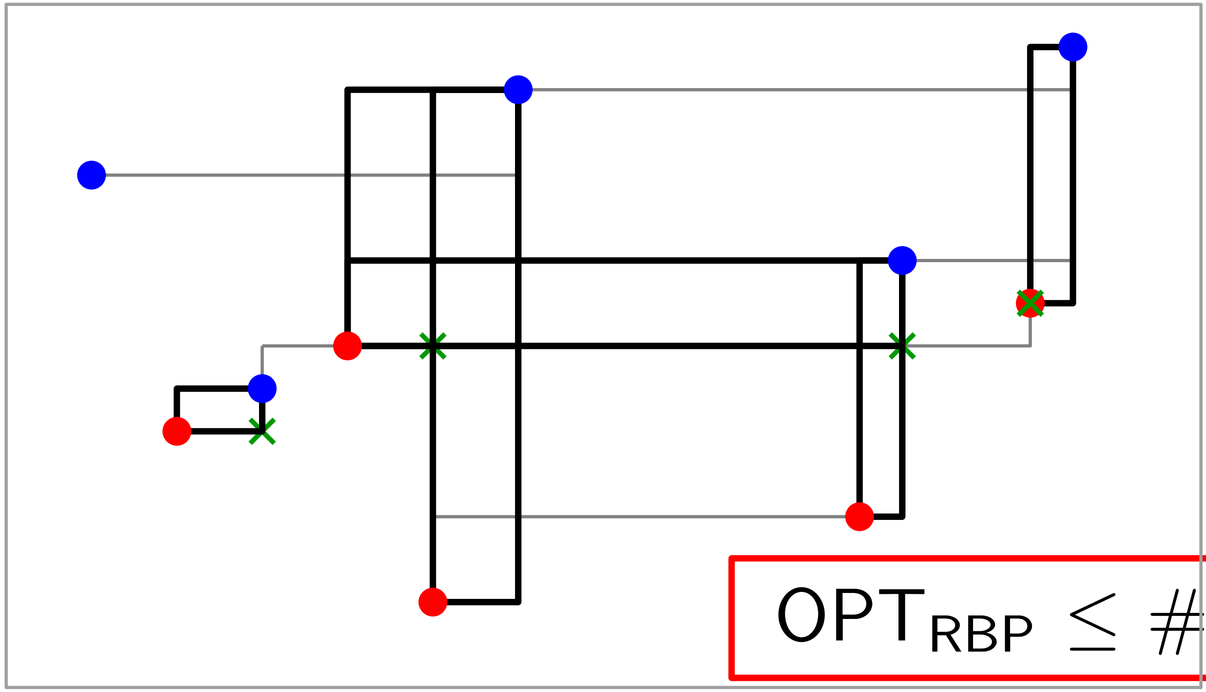

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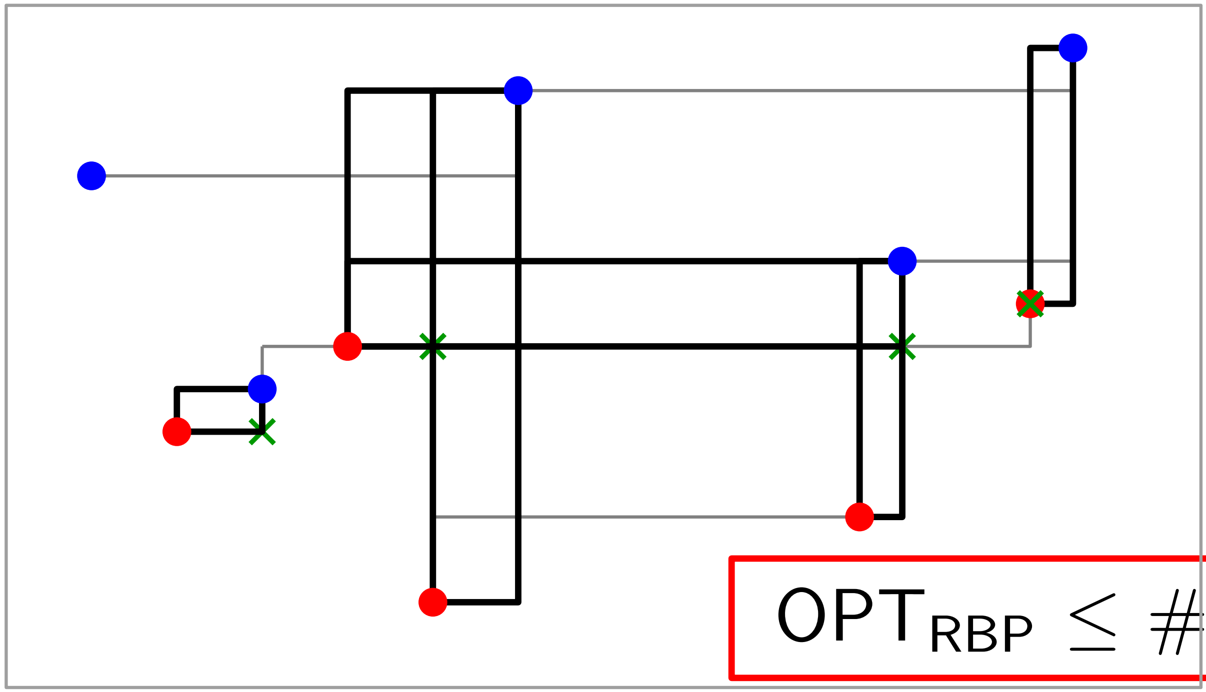

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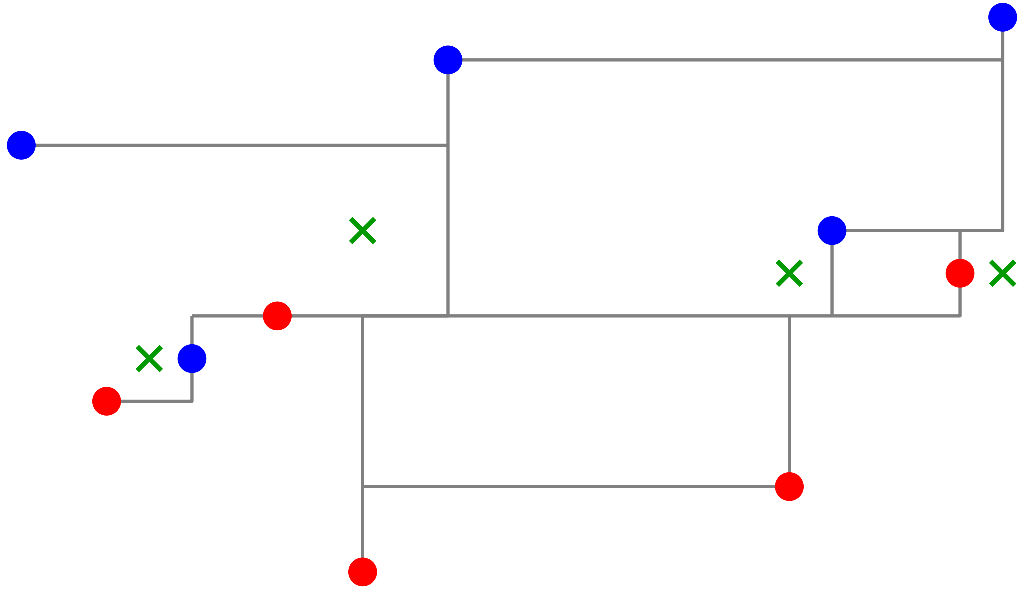
Theorem (Soto & Telha, IPCO'11)

Red-blue piercing can be solved in polynomial time.

Converting Piercings to Pillars (I)

Lemma

Given red-blue piercing S and Manhattan network for $R \cup B$, we can *move* the needles (pts) in S so that
for each relevant pair (r, b)
there is an M-path that contains a needle of S.



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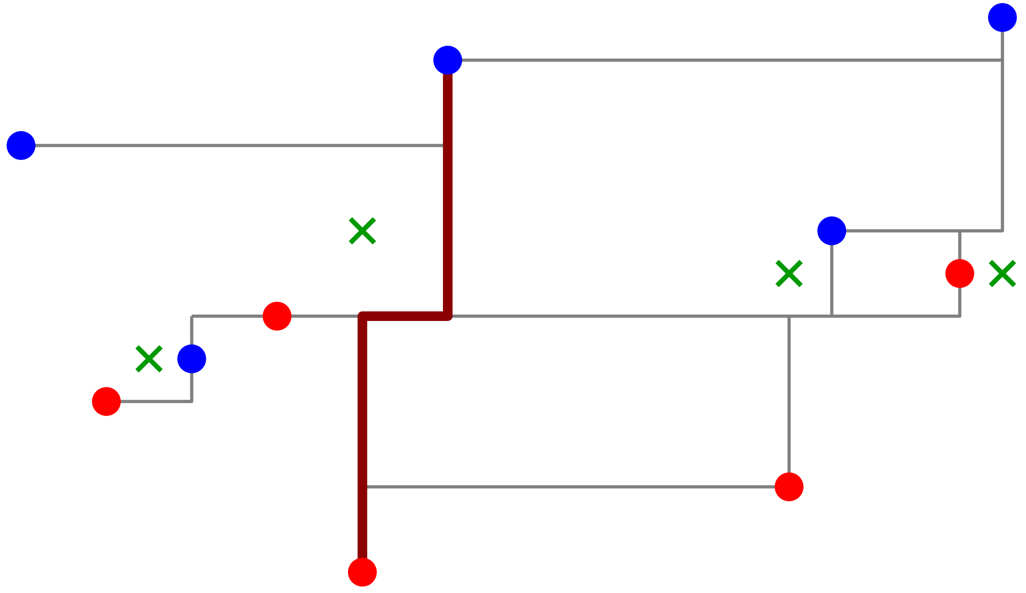

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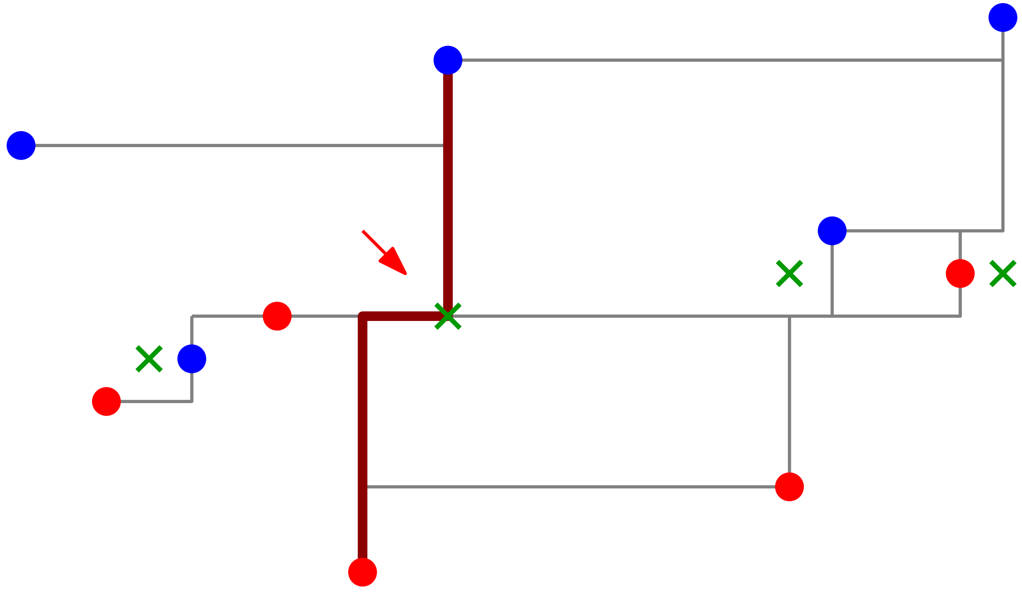

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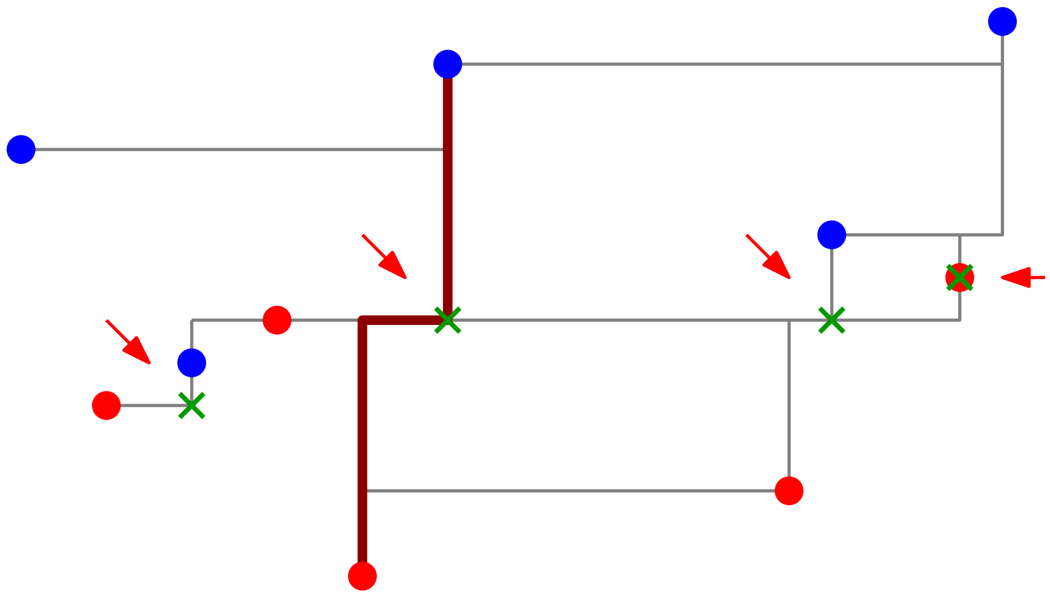

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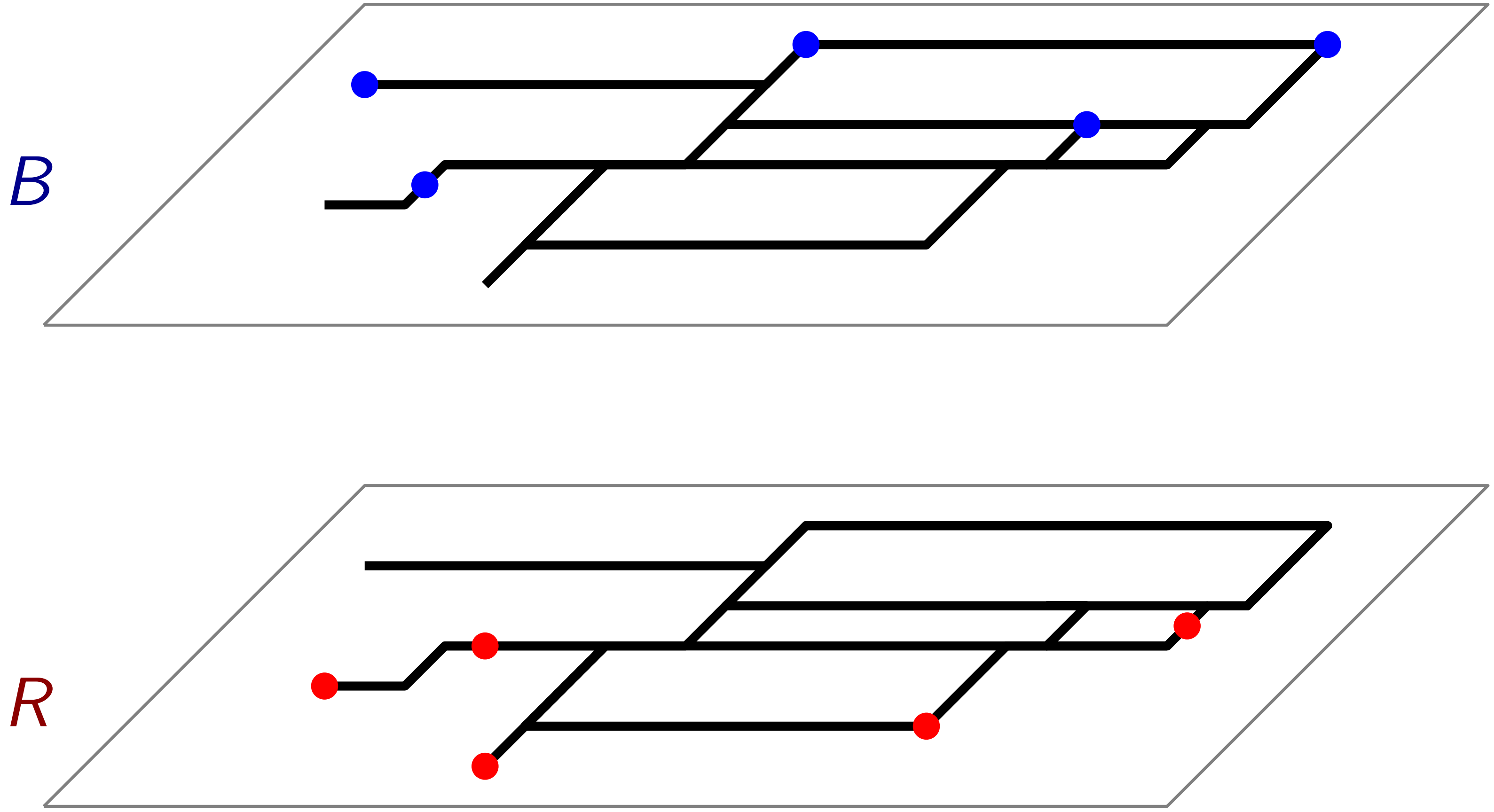

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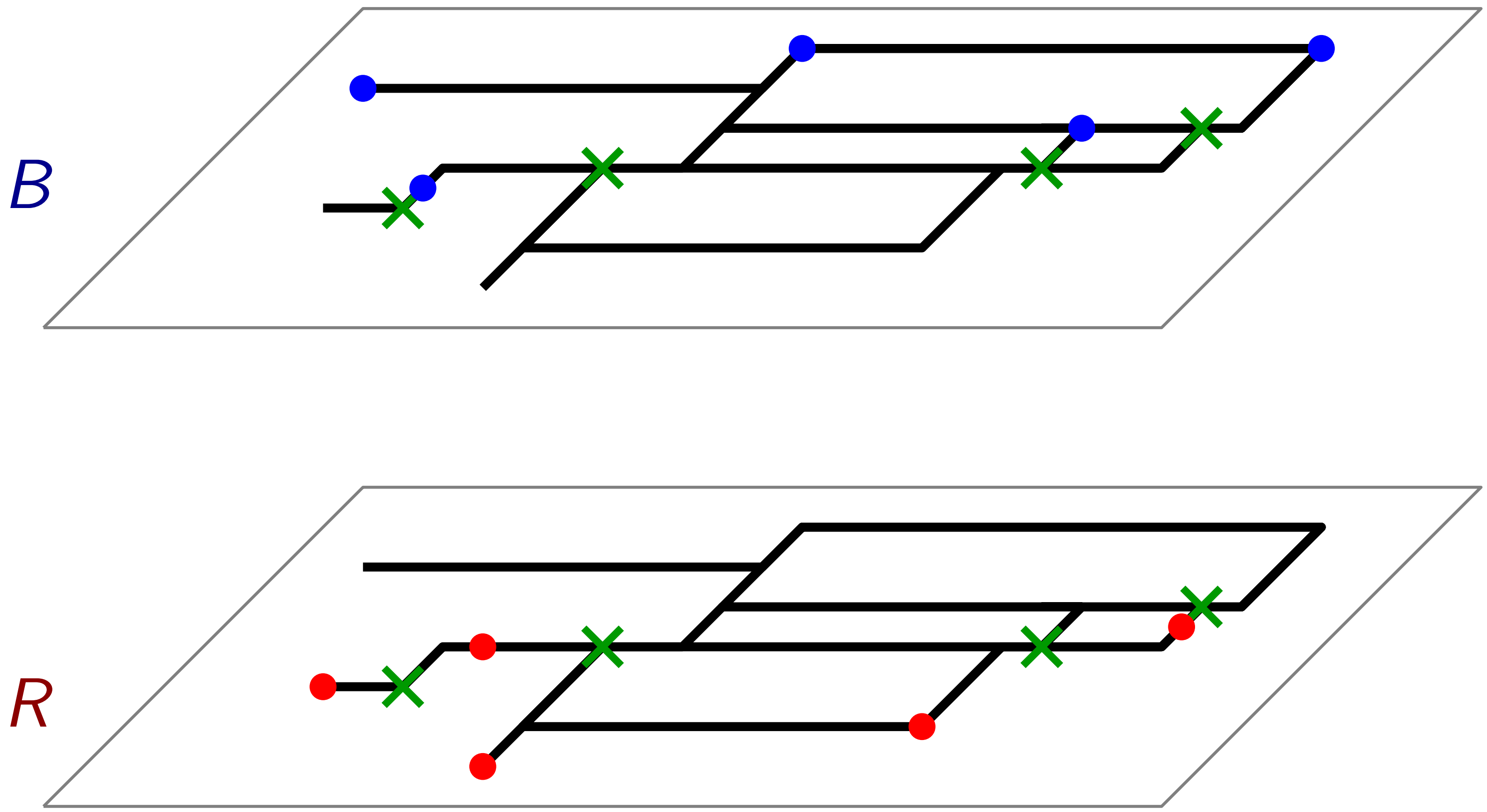
Extend piercing pts to pillars



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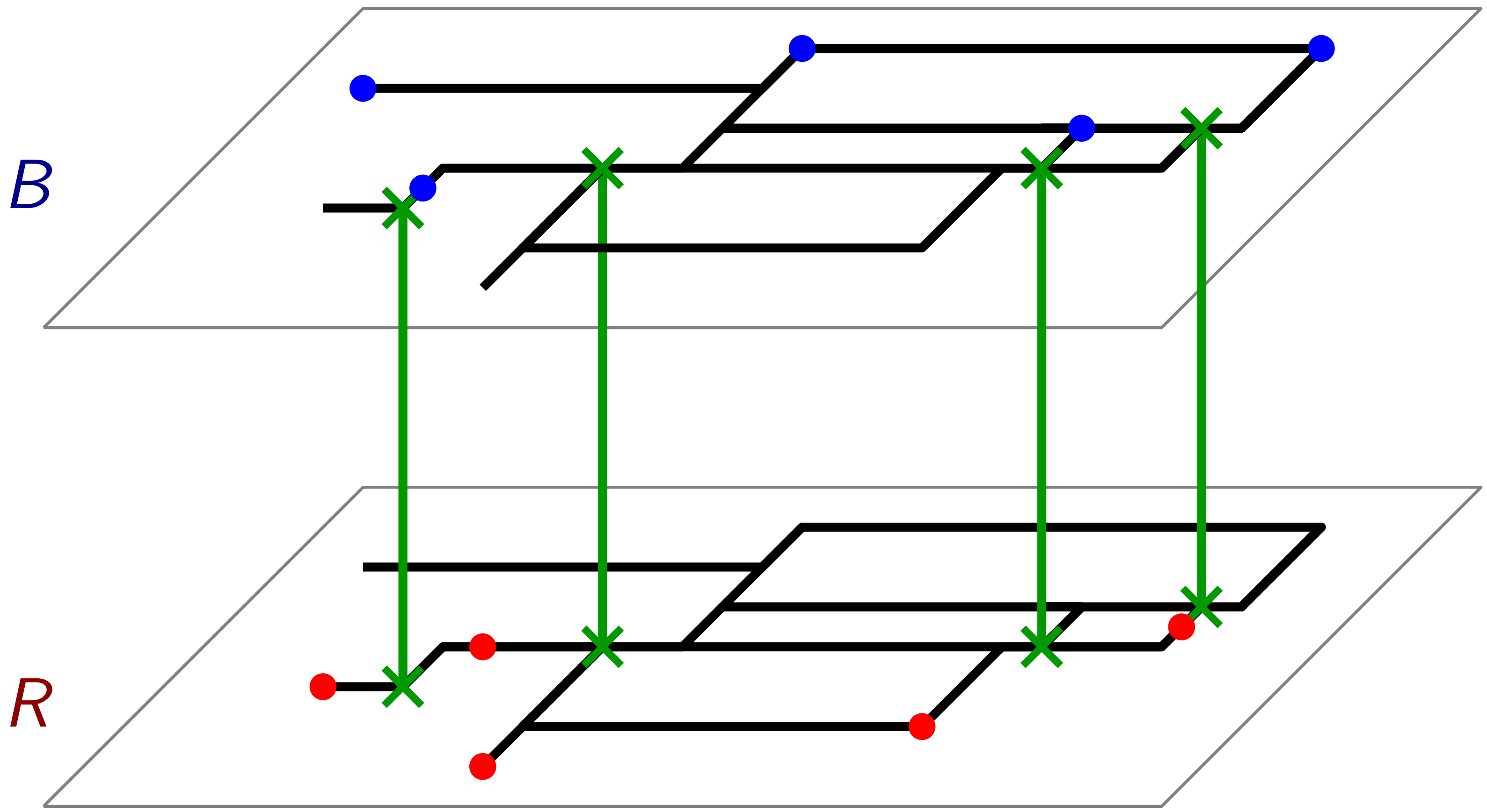
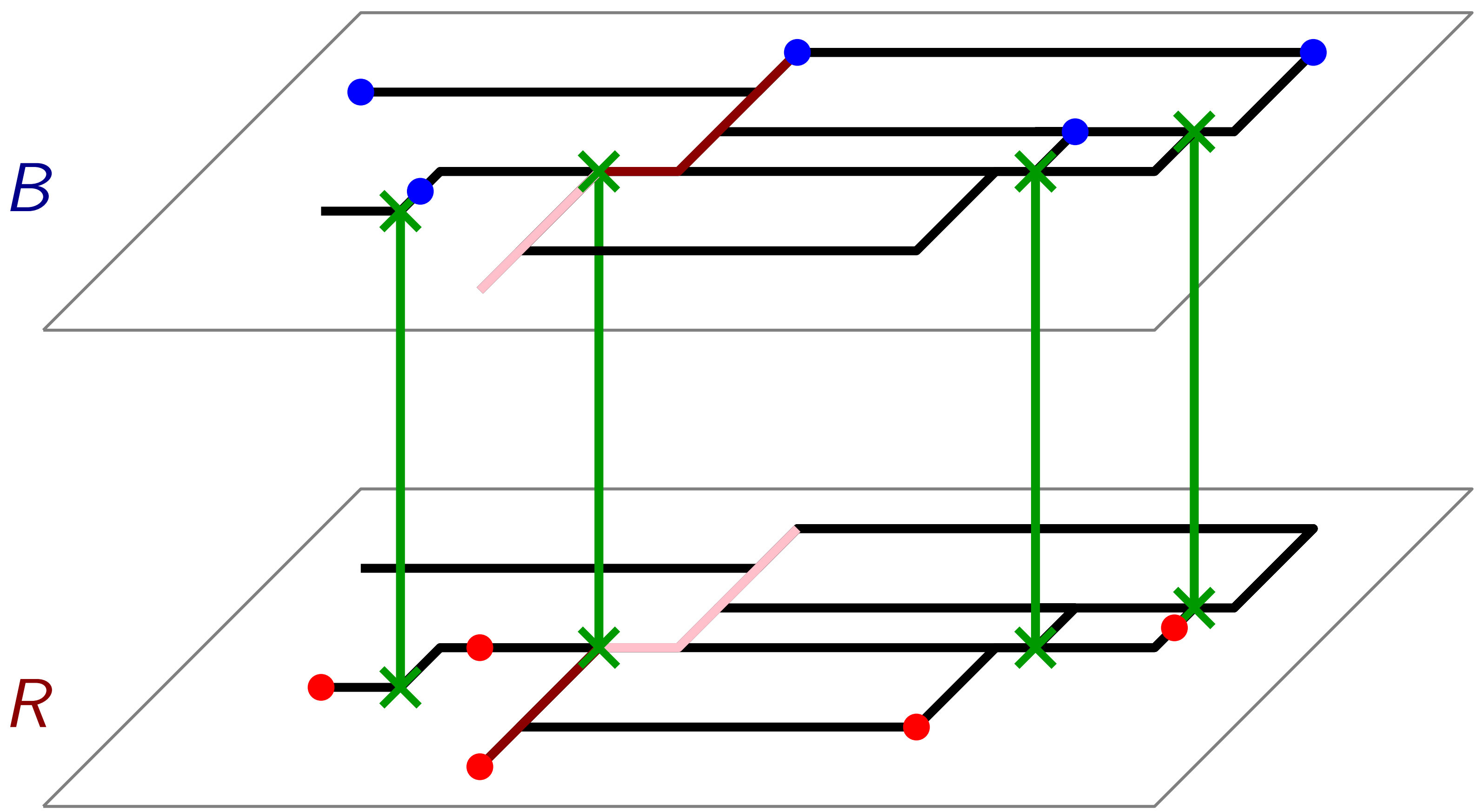


Converting Piercings to Pillars (II)

Extend piercing pts to pillars $\rightsquigarrow$ feasible 3D Manhattan network!



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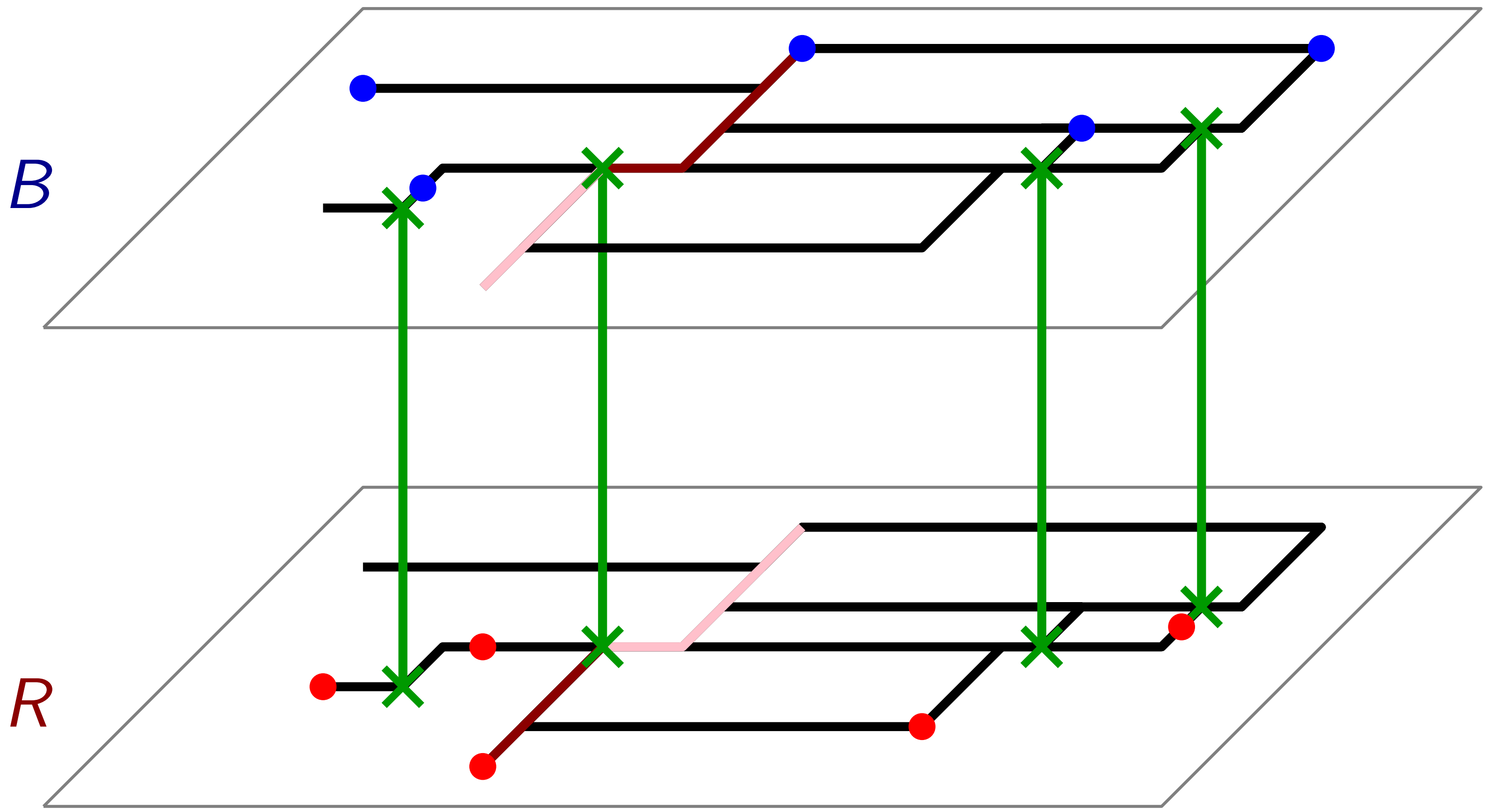
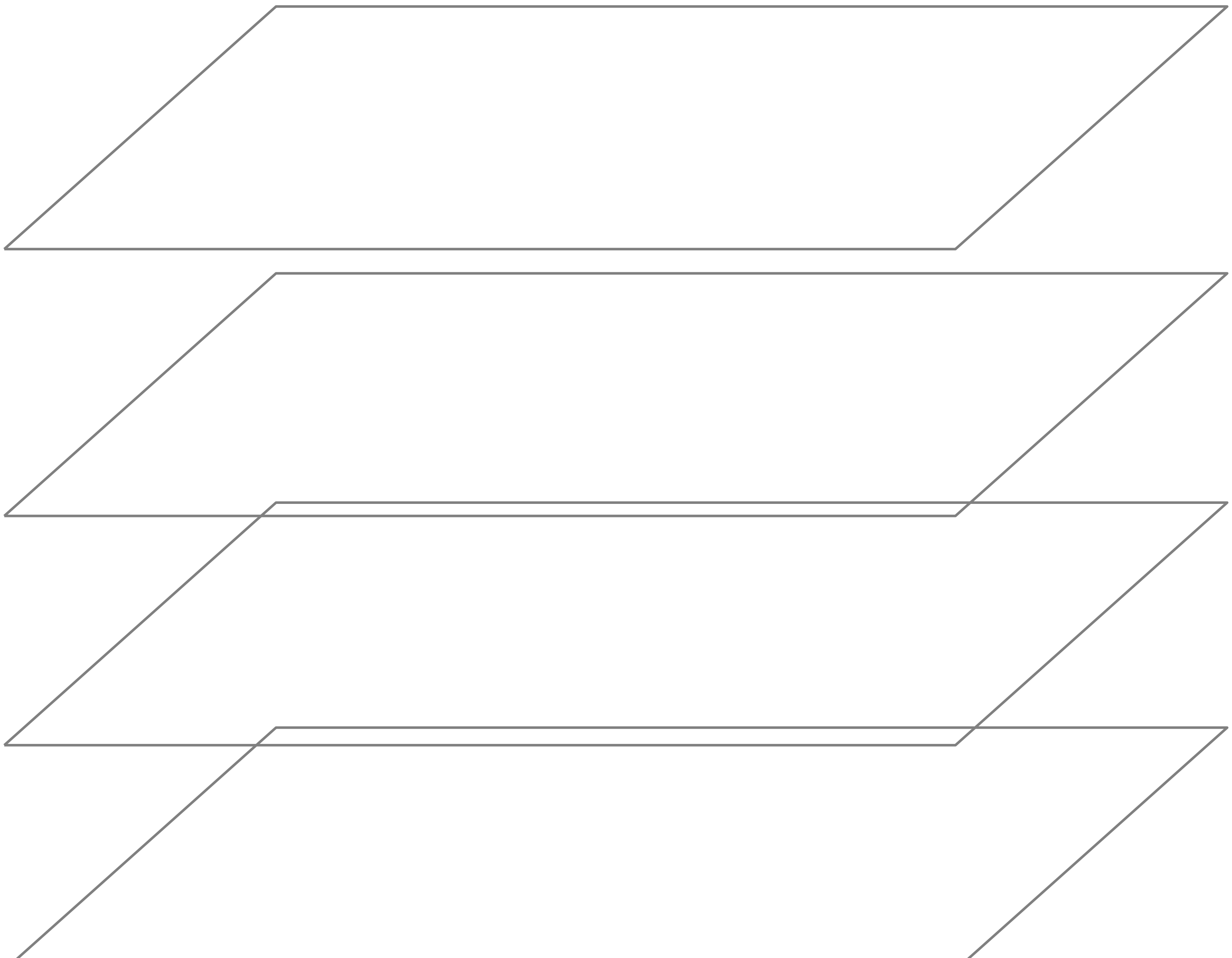


cost $\leq 4 \cdot \mathrm{OPT}$ (due to the four directions)

k Planes – Horizontal Part

copy 2D Manhattan network onto each plane



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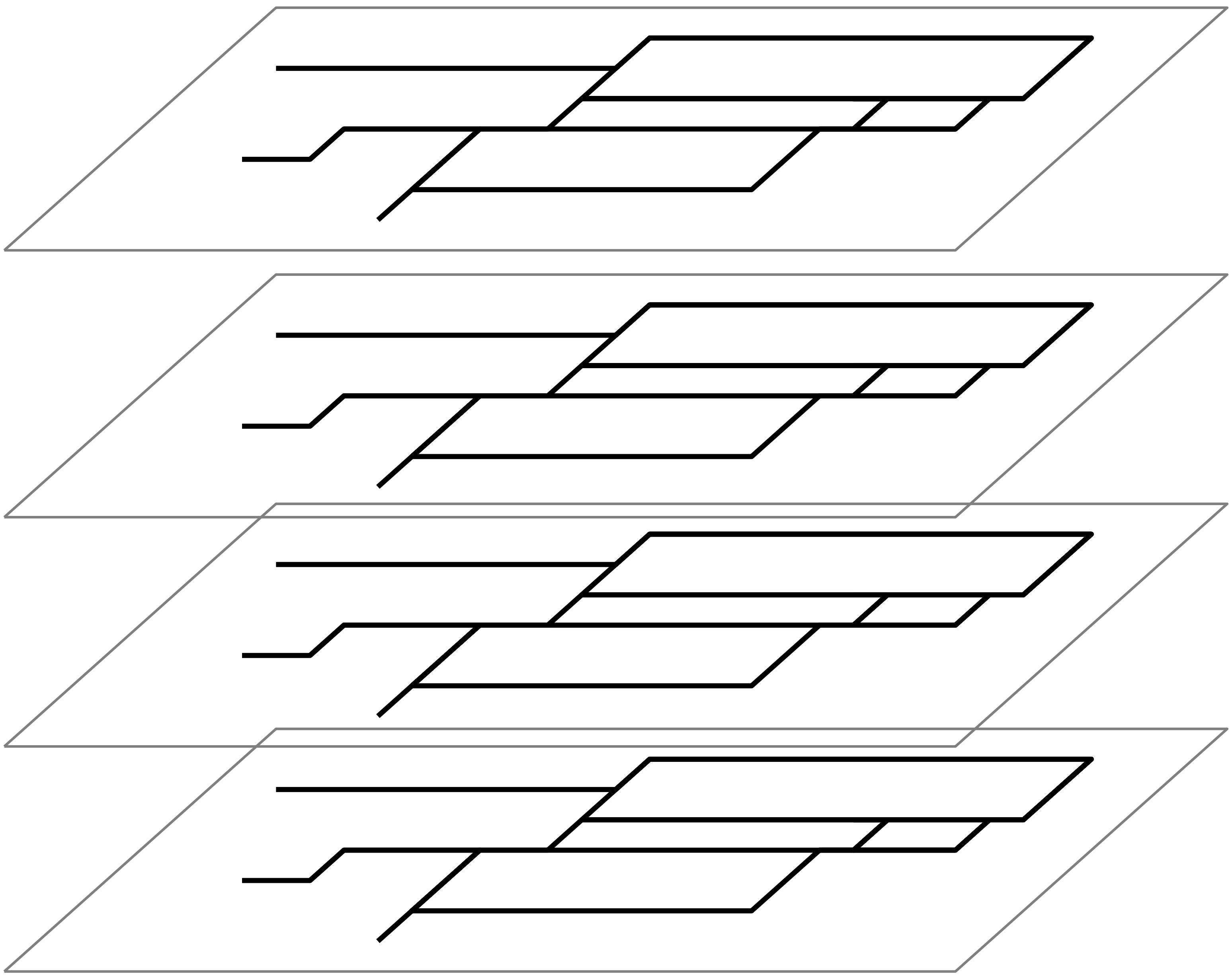

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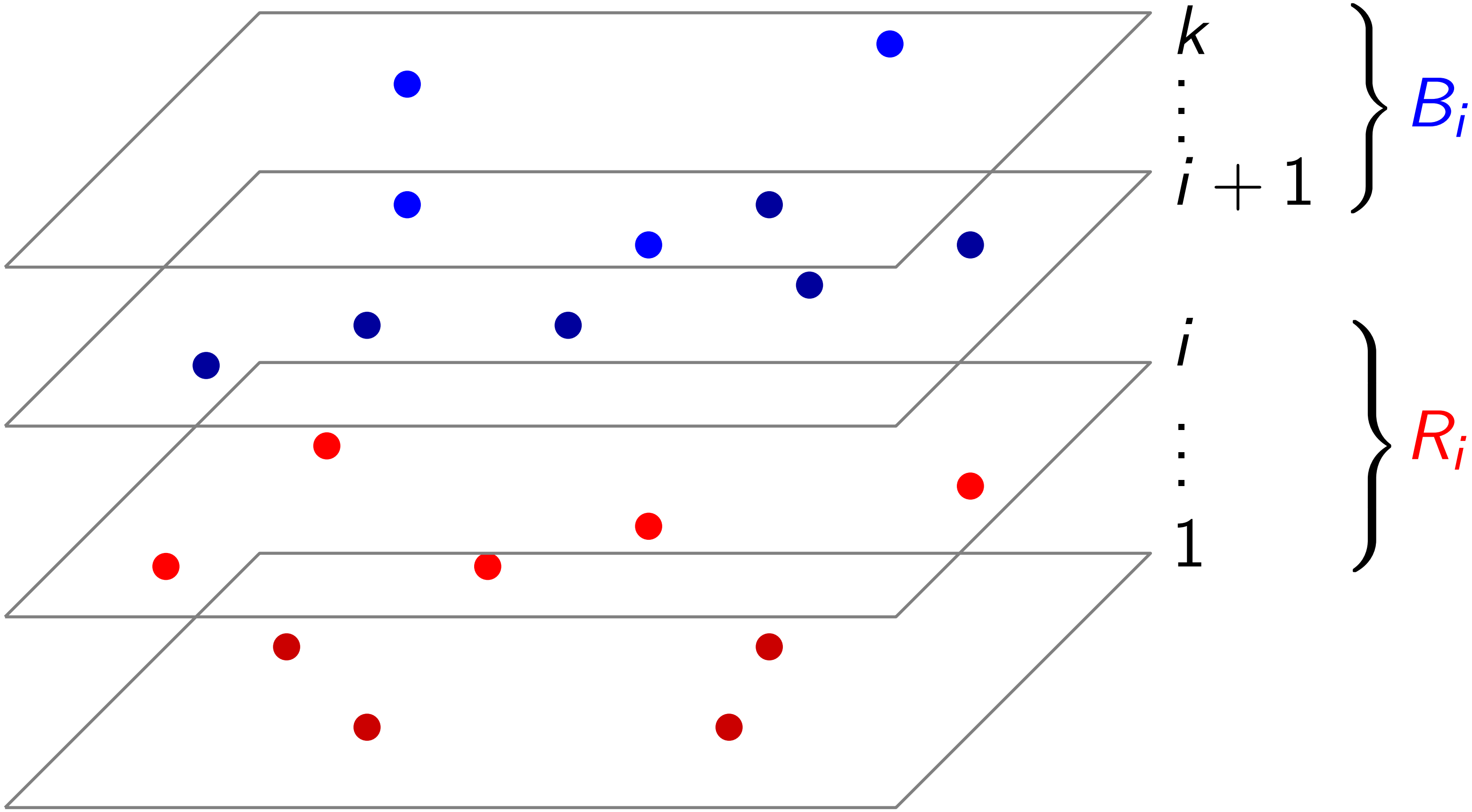

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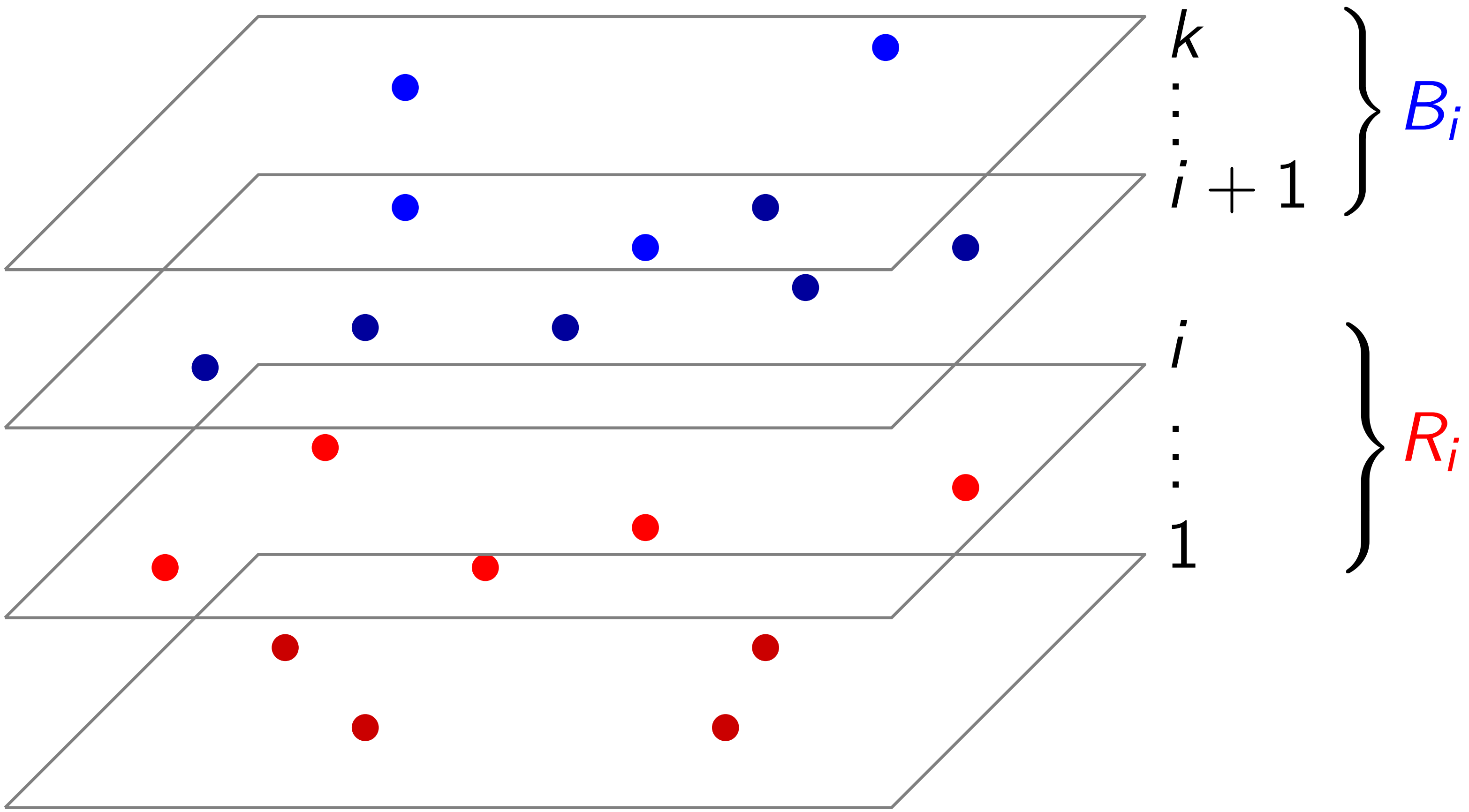

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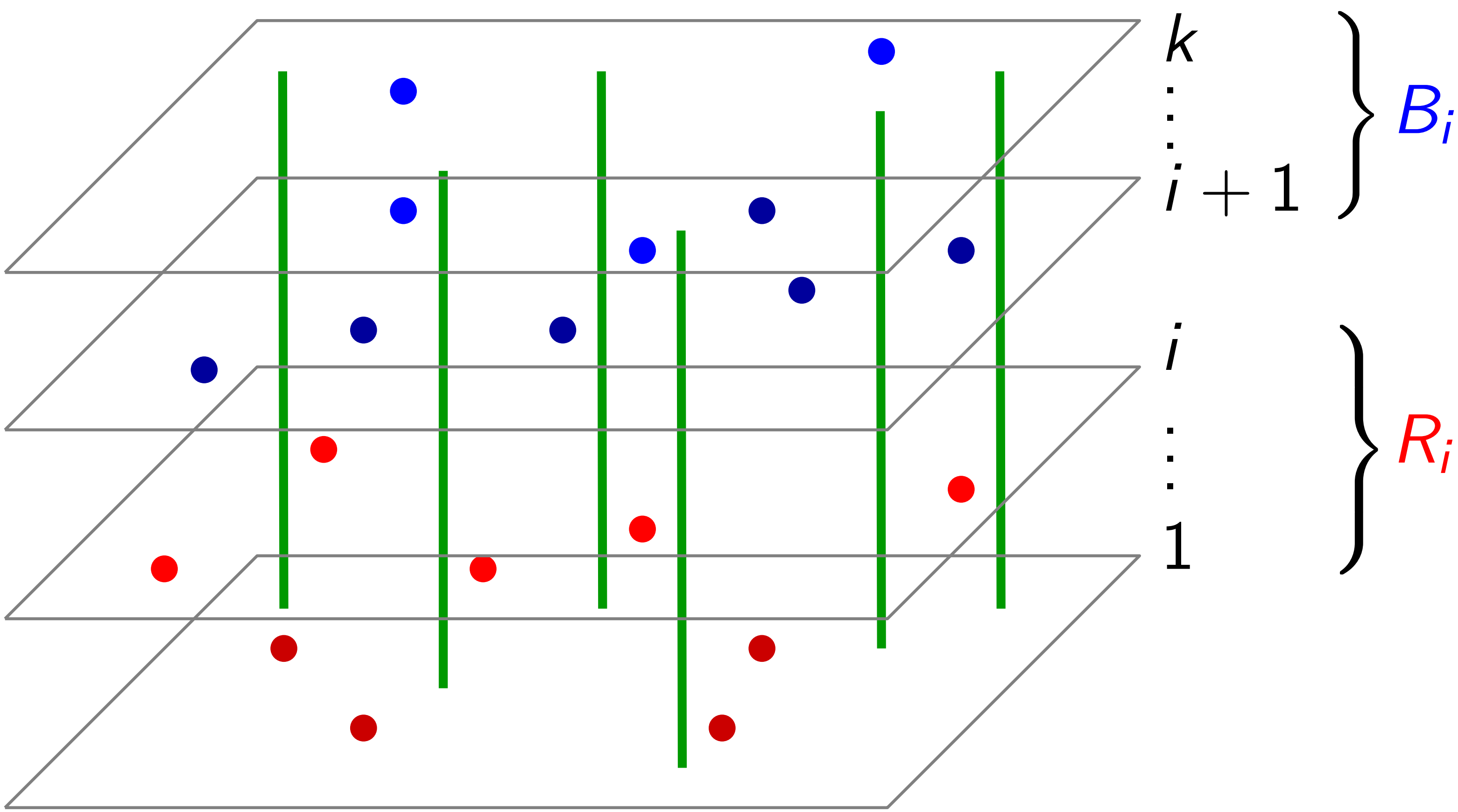

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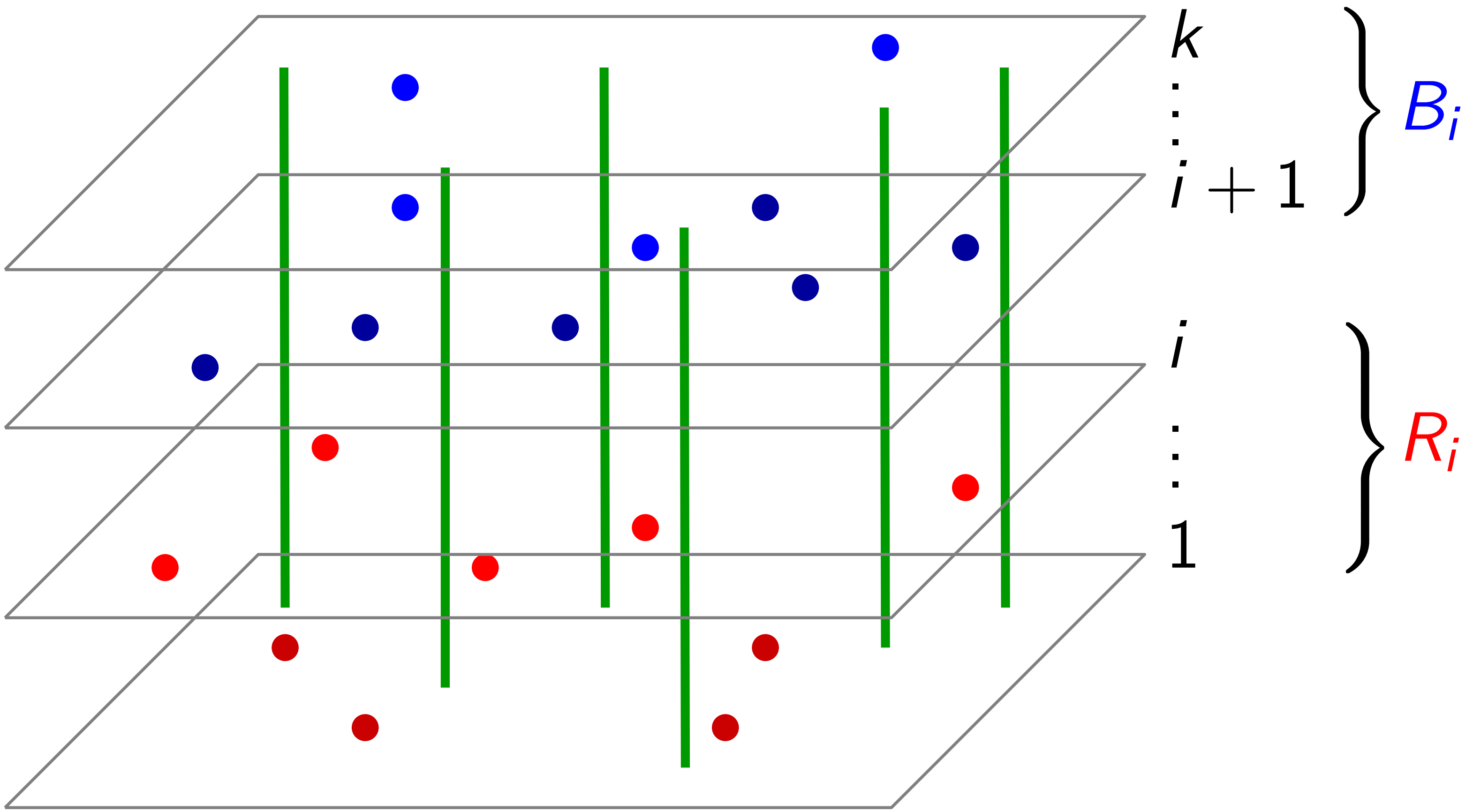

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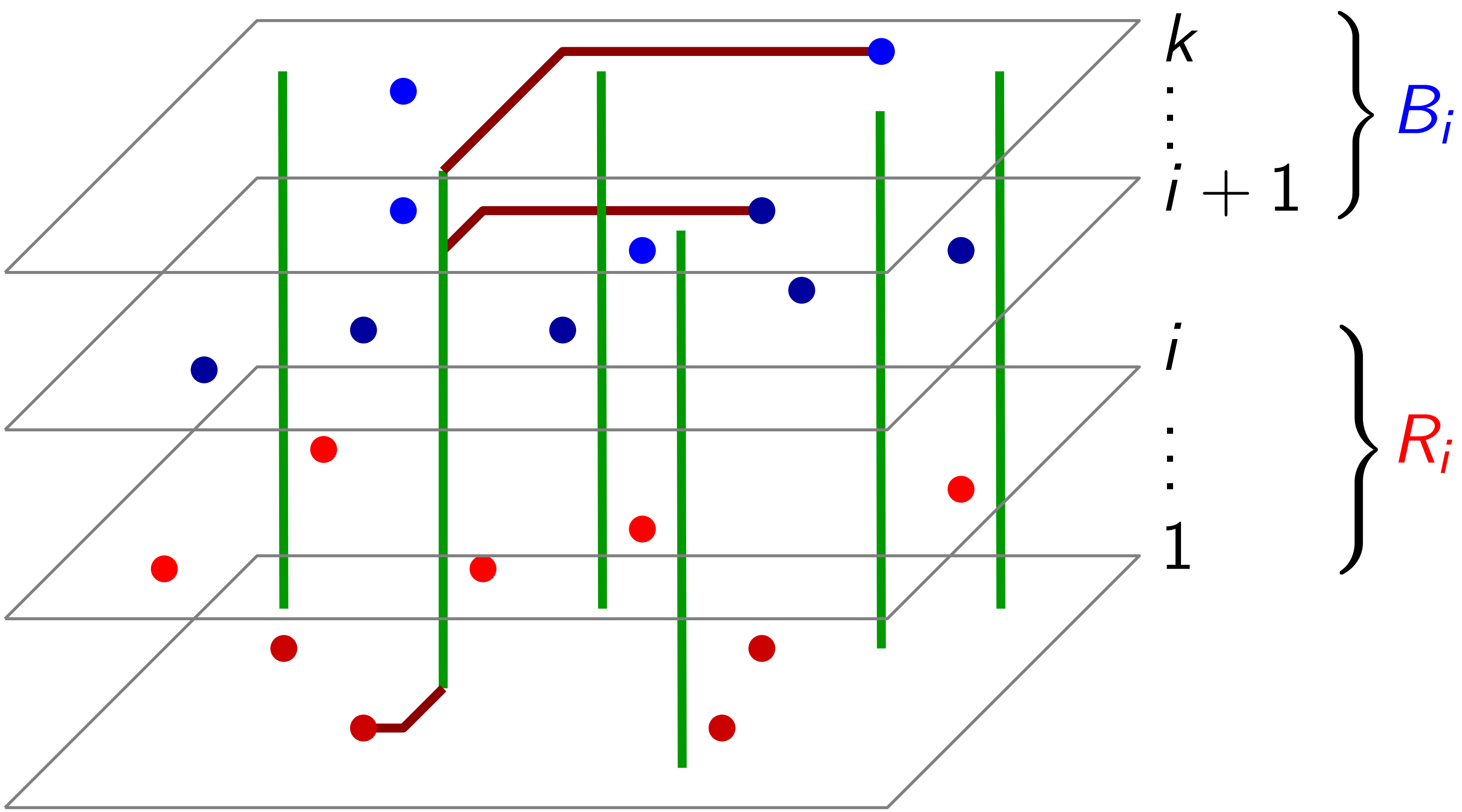

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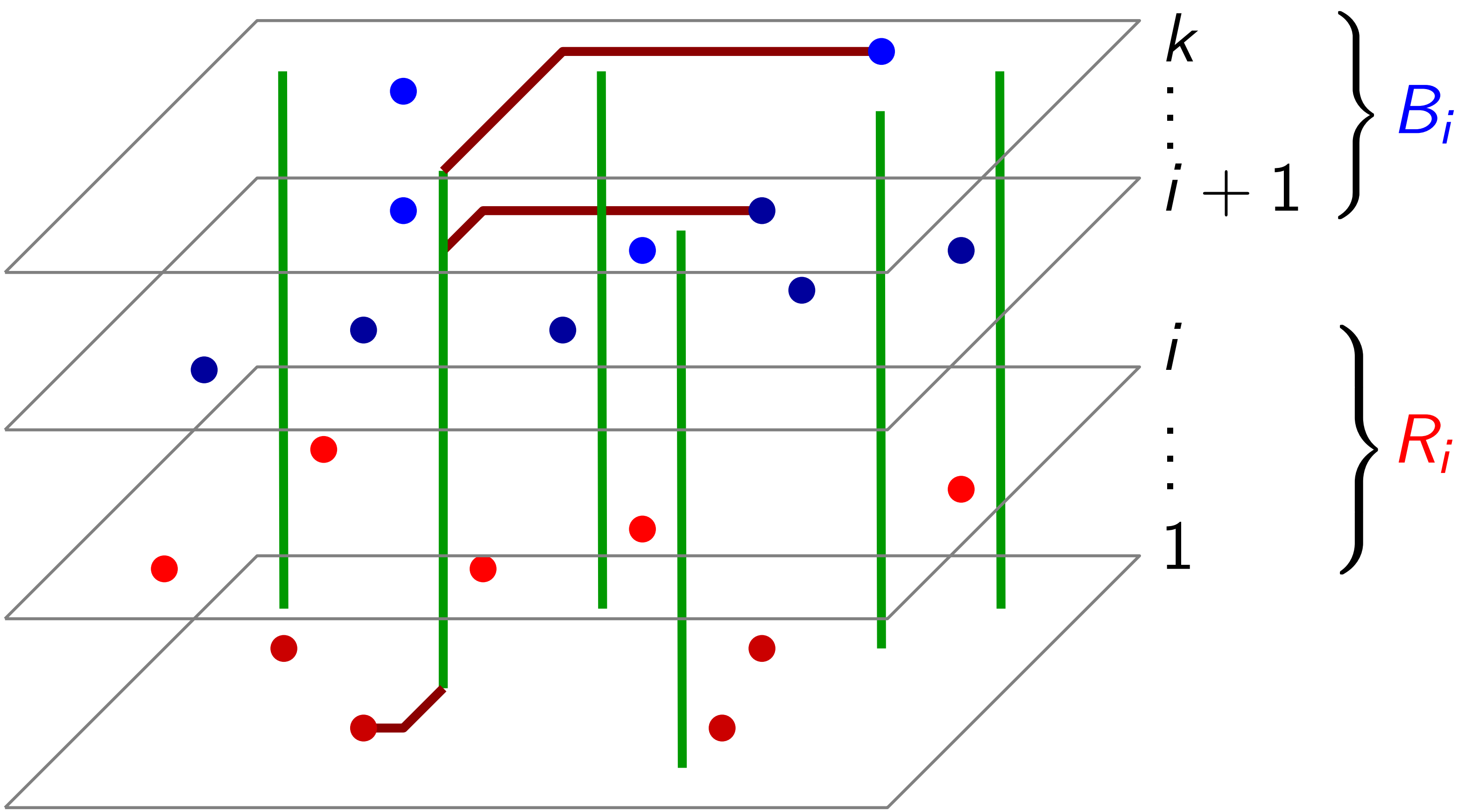

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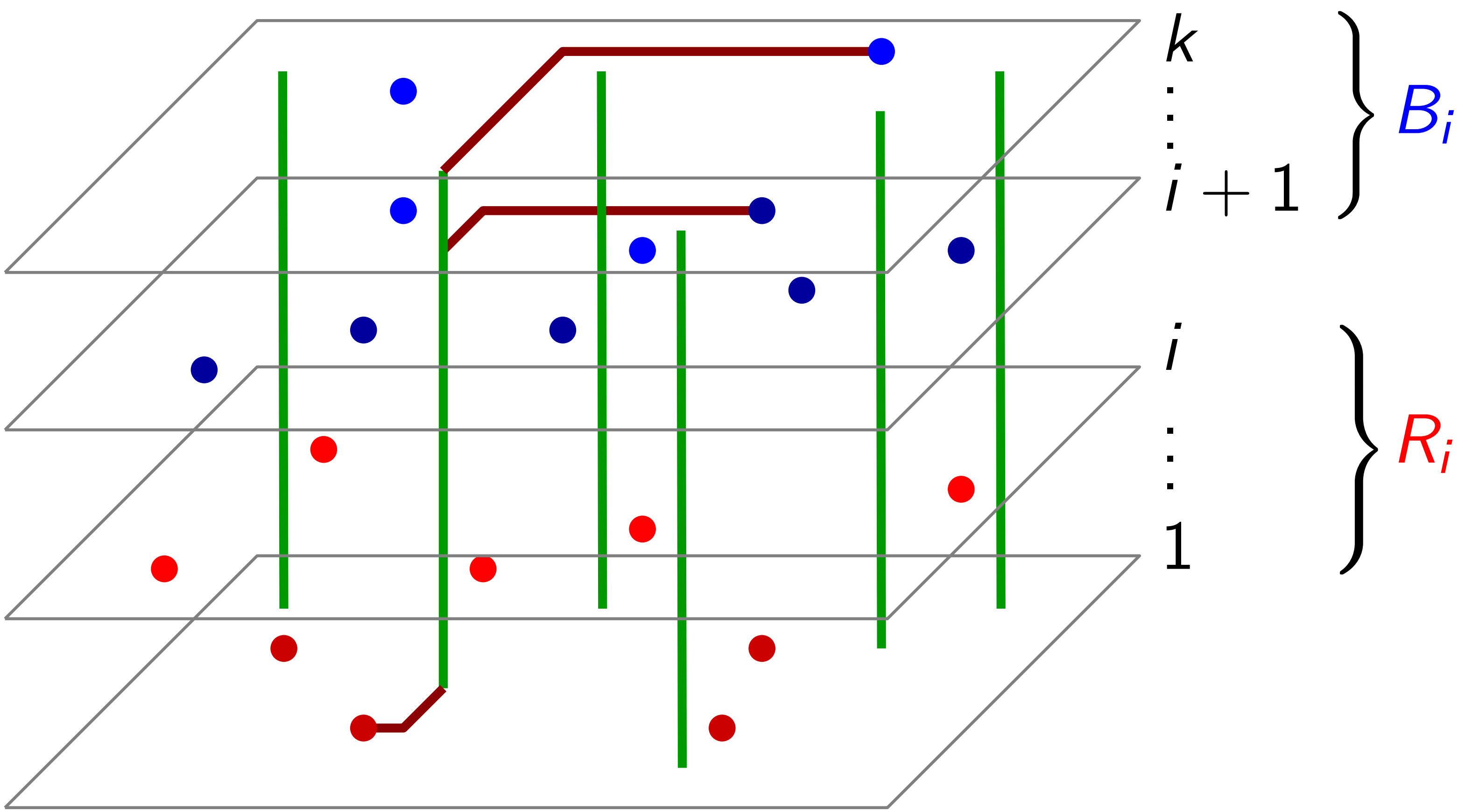

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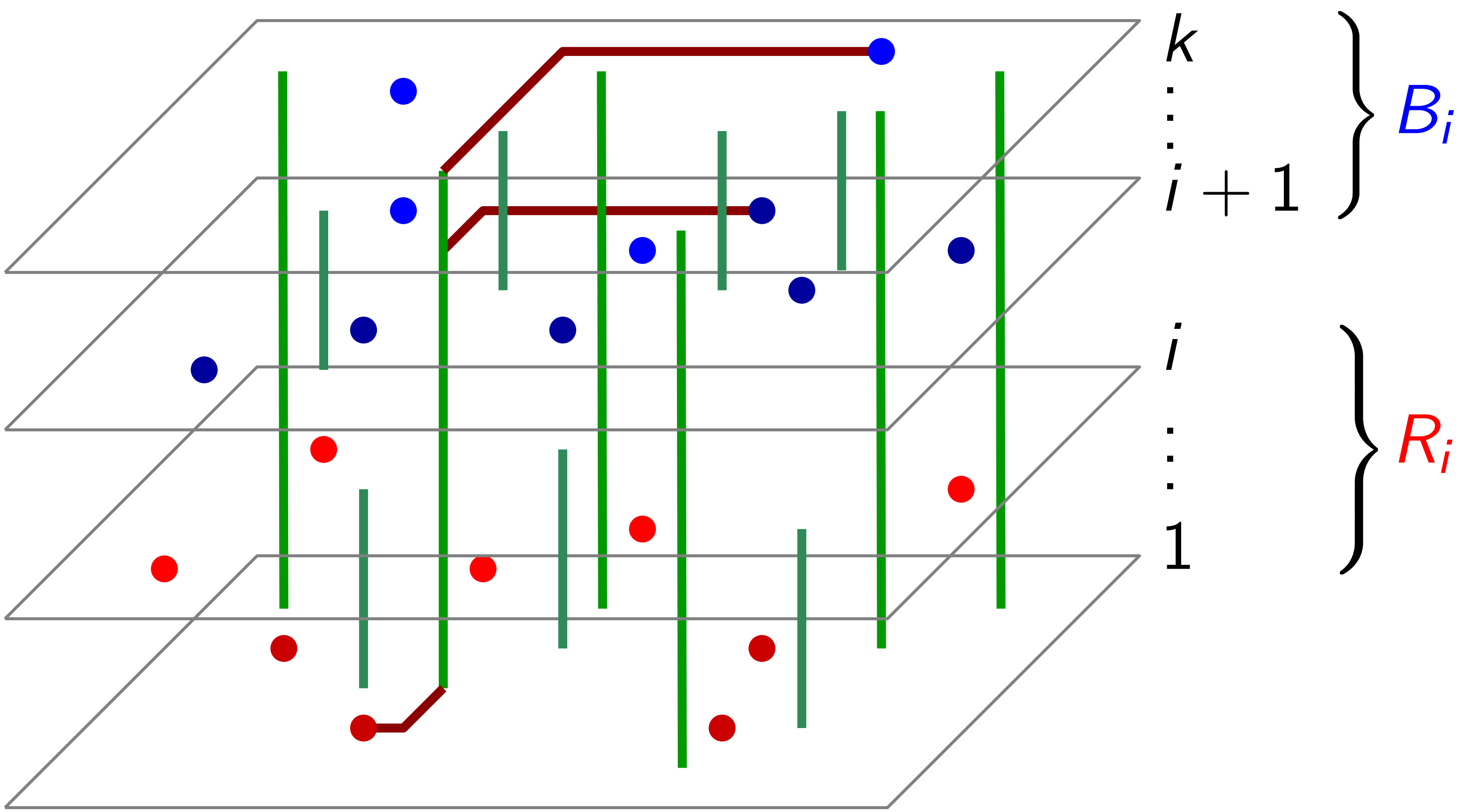

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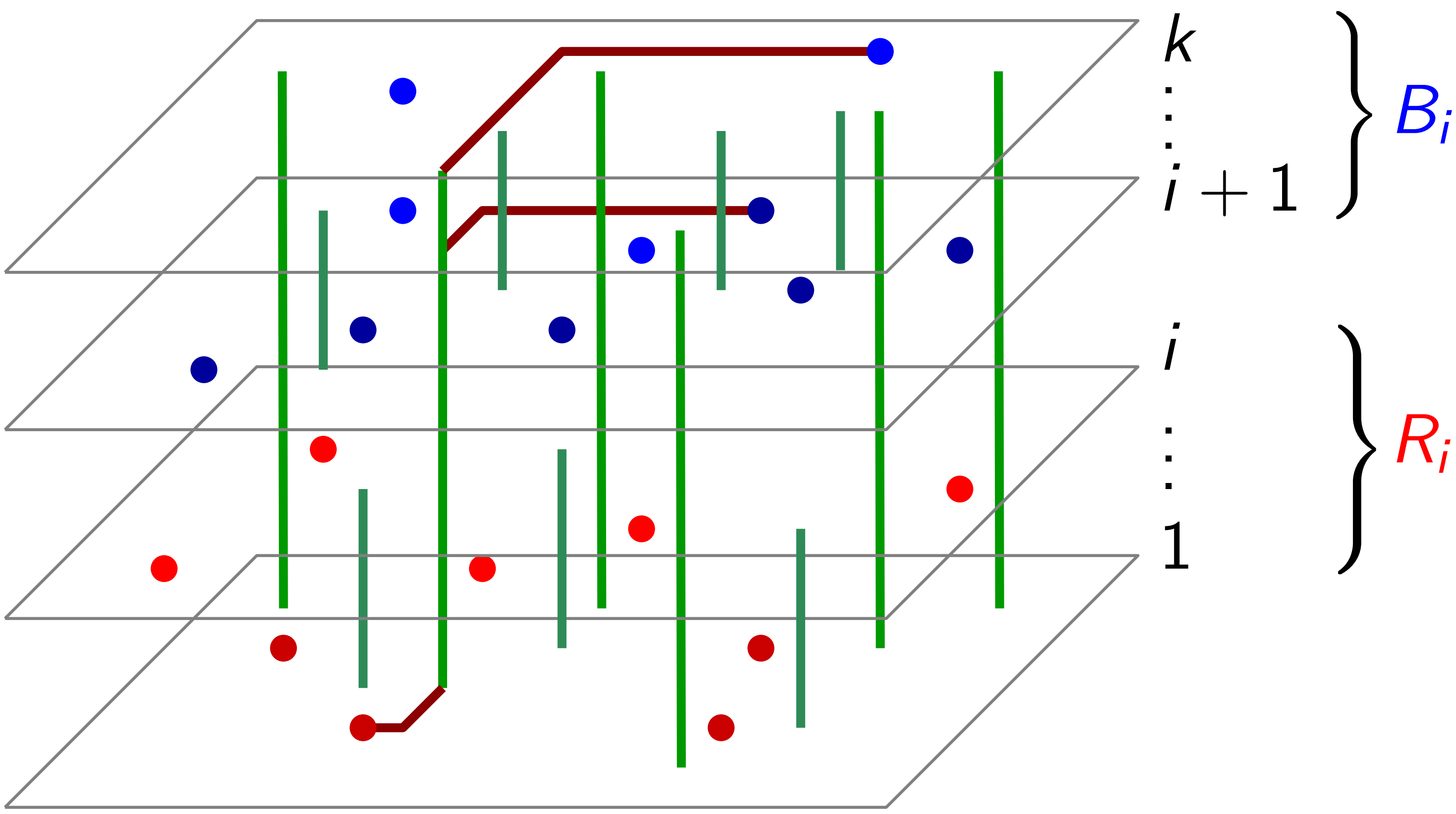

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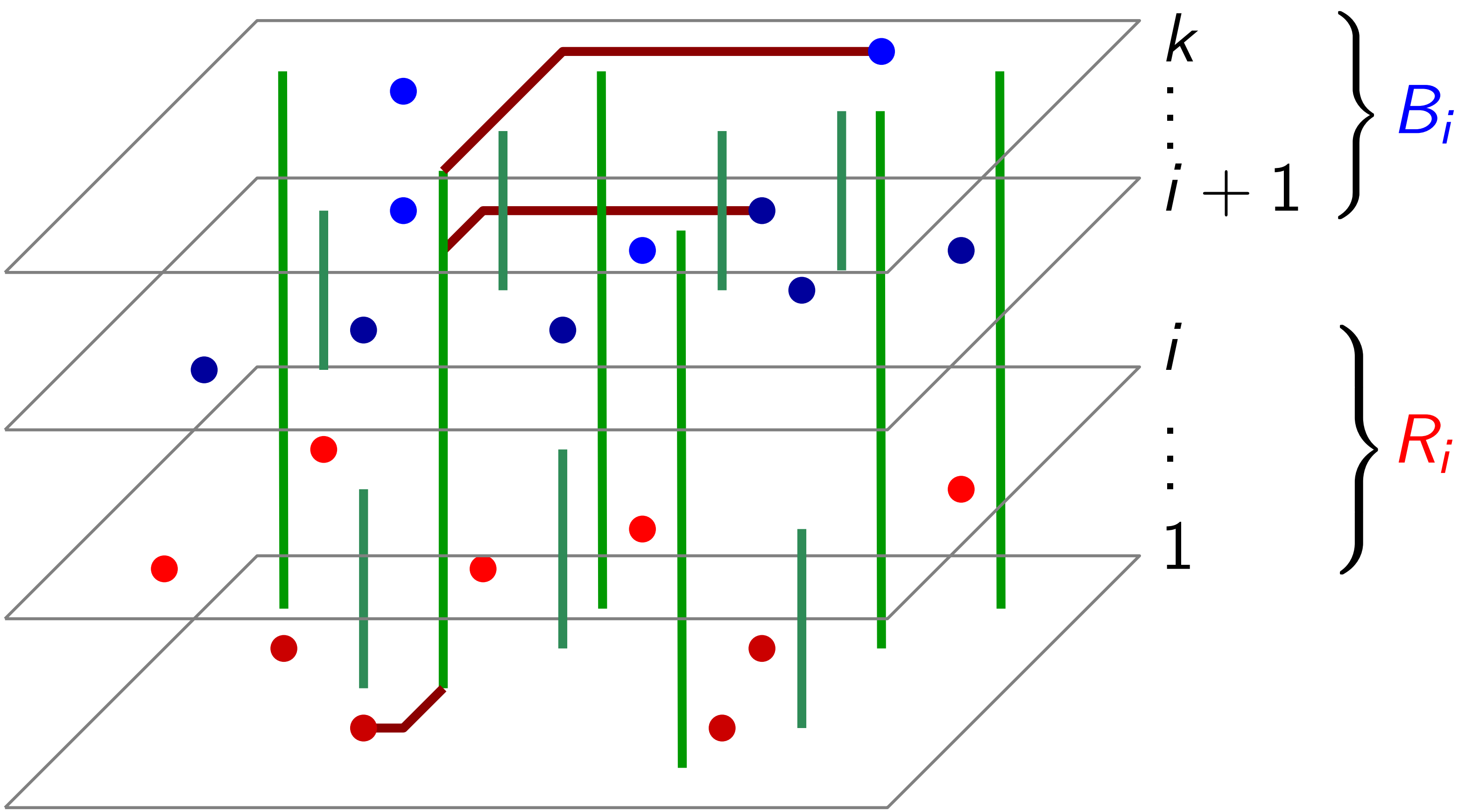

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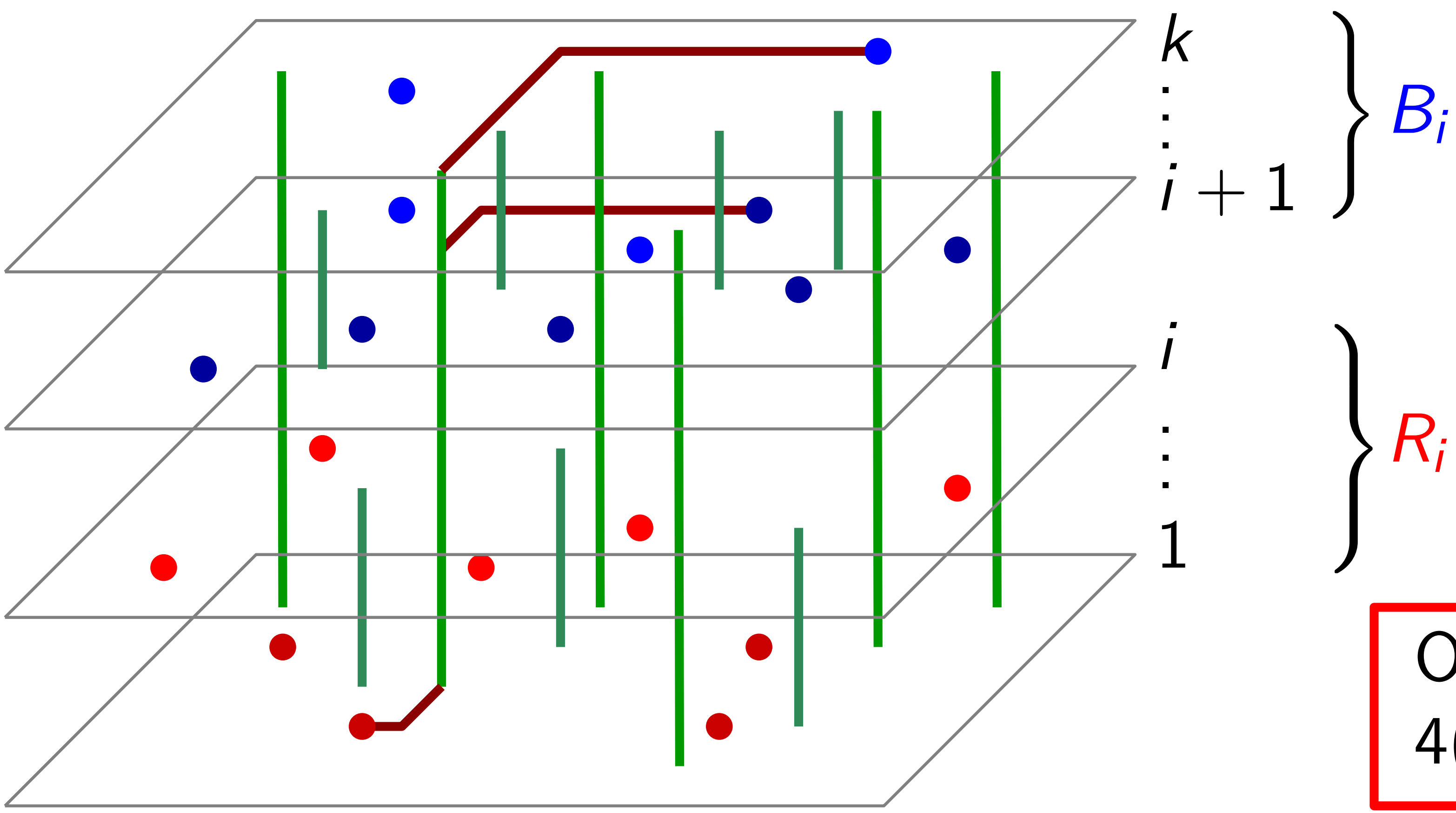

Overall ratio $4(k-1)$

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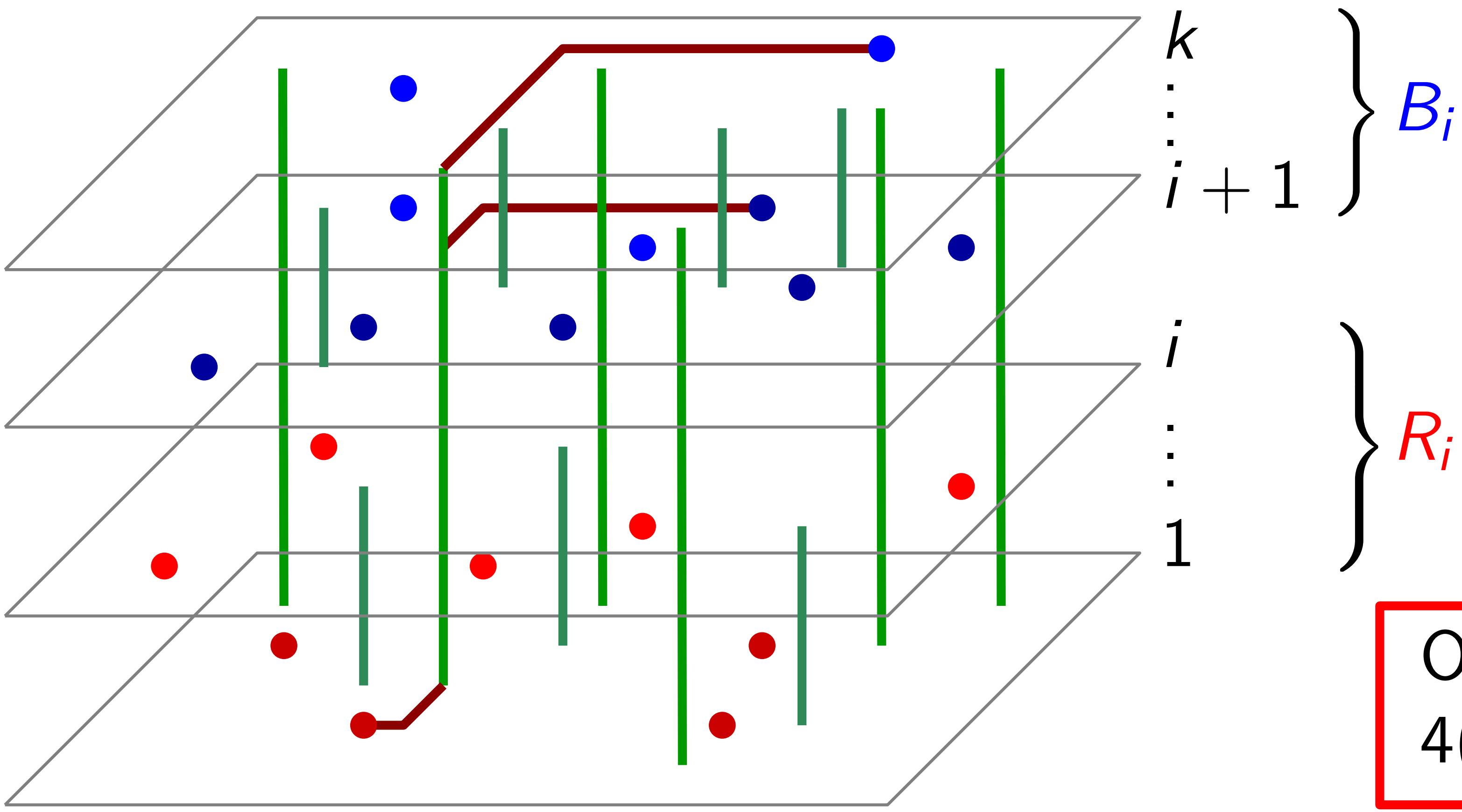

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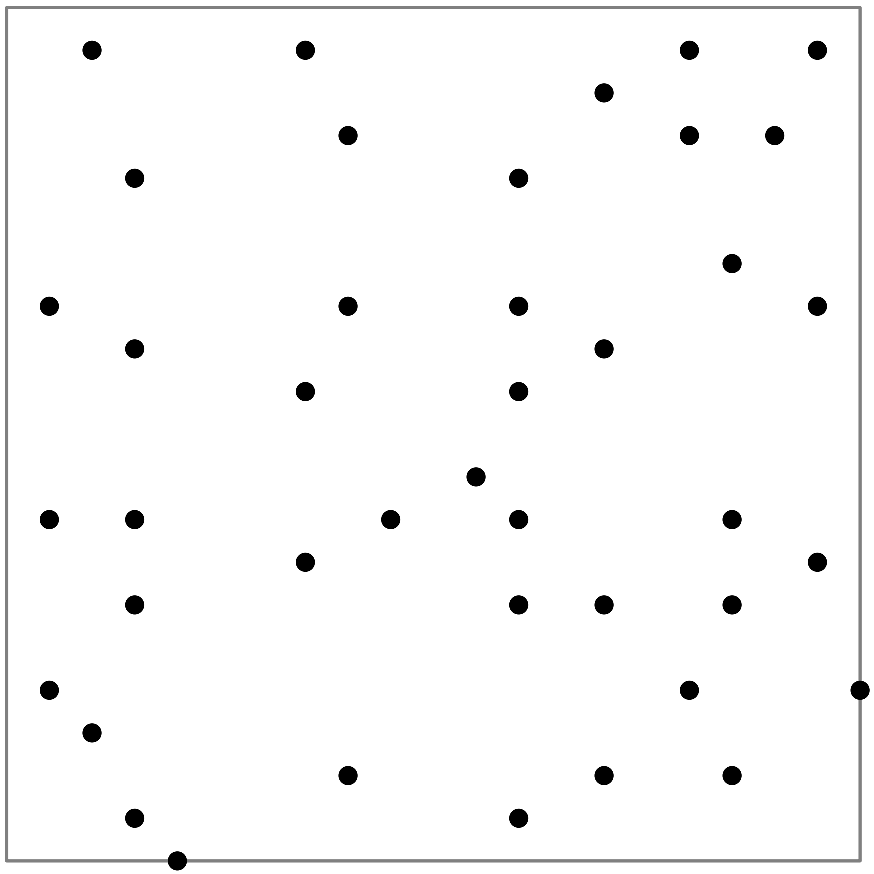
Our Results for Higher Dimensions

- $4(k-1)$ approximation for 3D –
  if the terminals lie in the union of k horizontal planes

- $O(n^\epsilon)$ approximation for general case
  in any fixed dimension and for any fixed $\epsilon > 0$

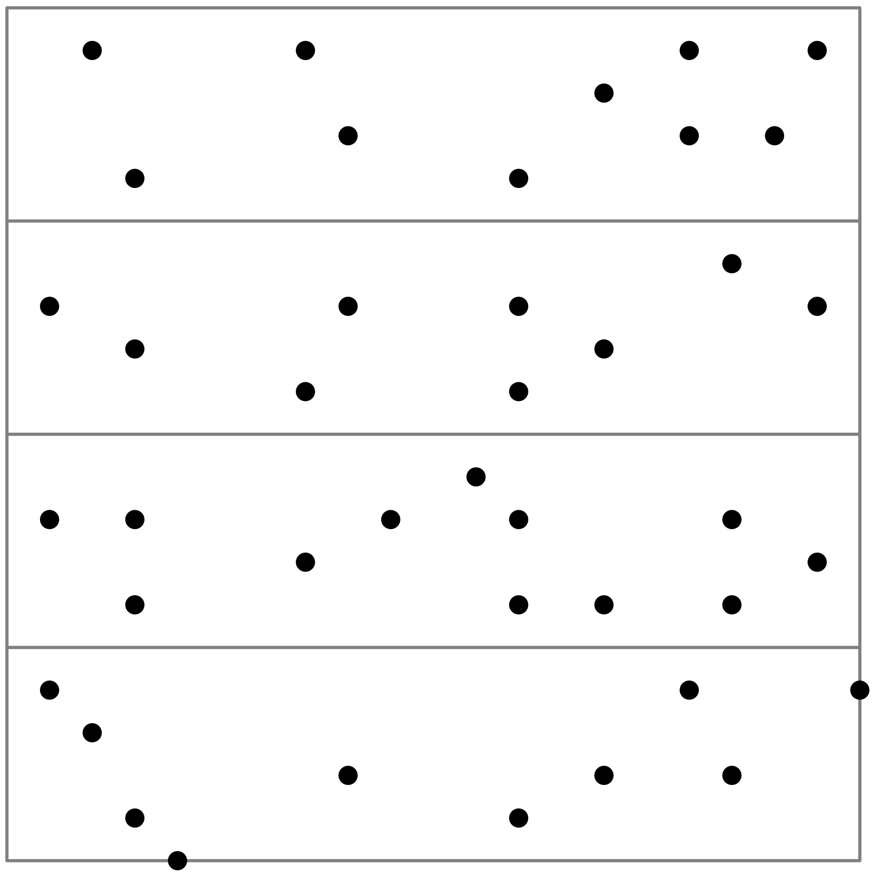
Grid Algorithm for General Case

- determine bounding cuboid



Grid Algorithm for General Case

- determine bounding cuboid
- partition into $c \times c$ **slabs** with n/c terminals each



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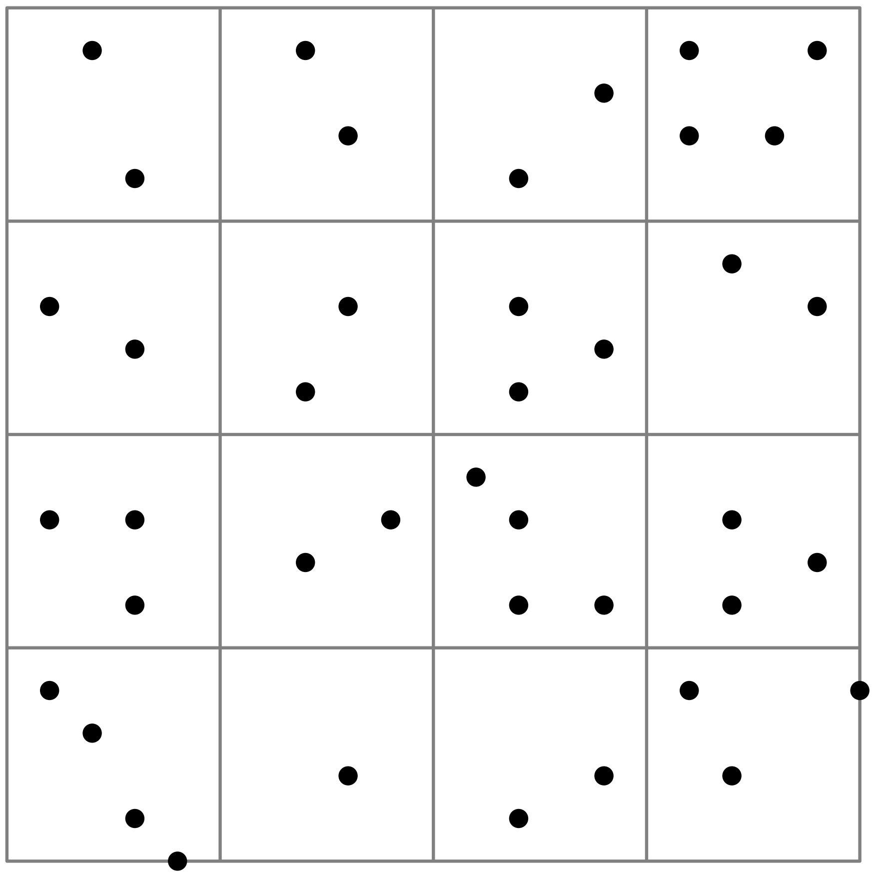

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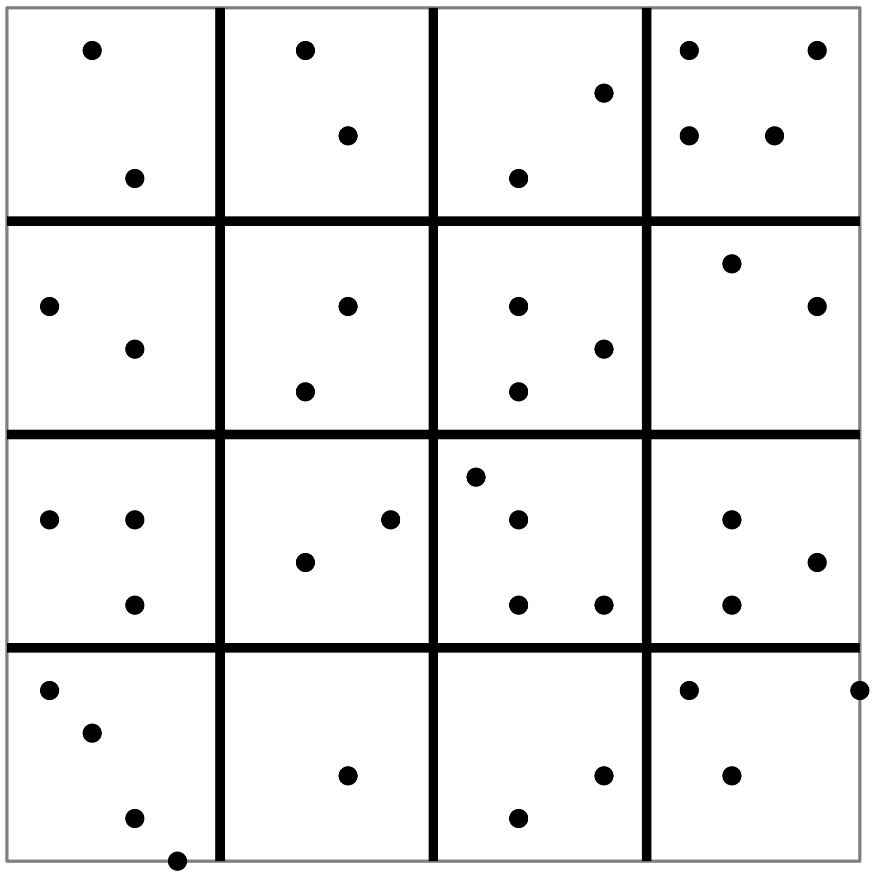

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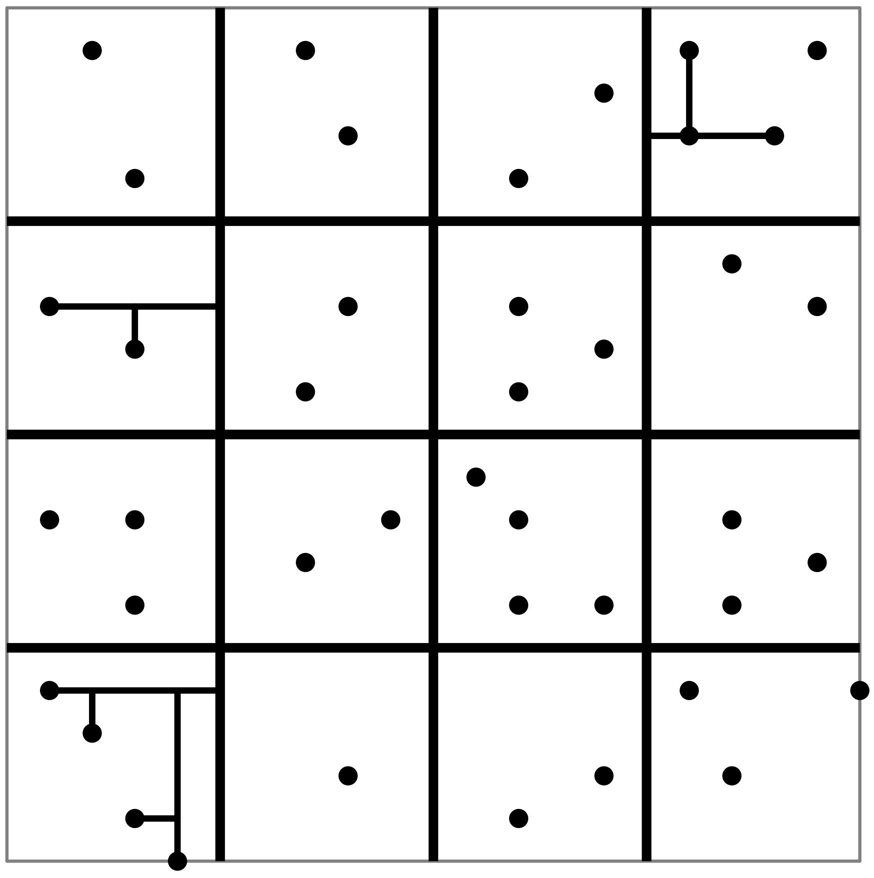

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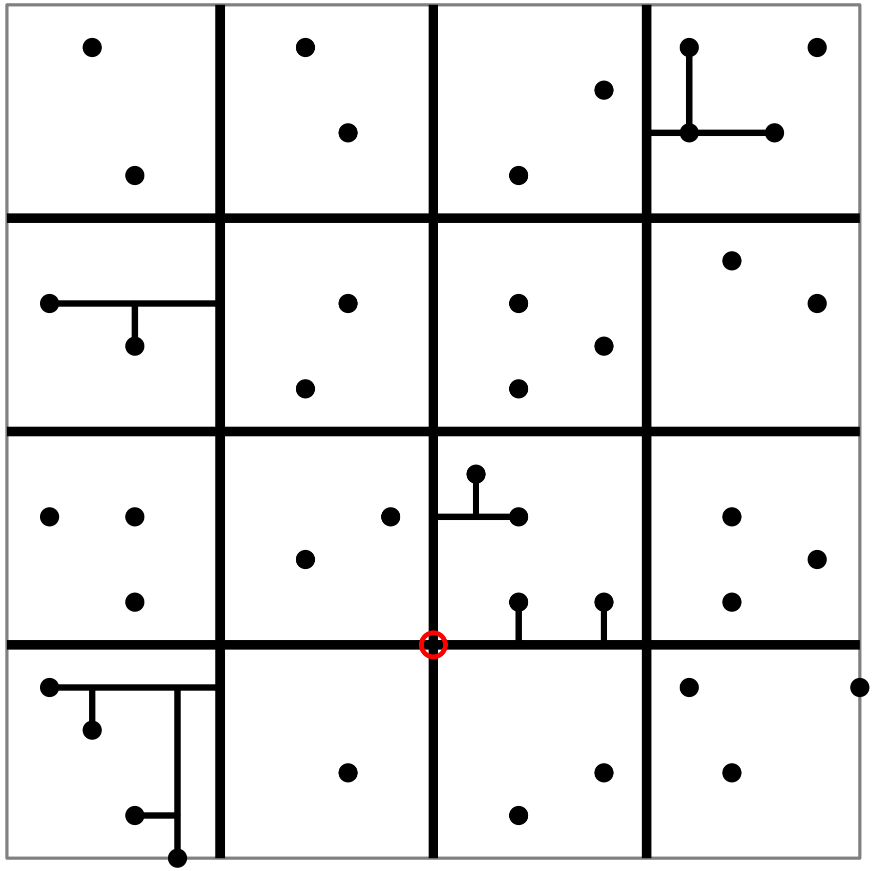

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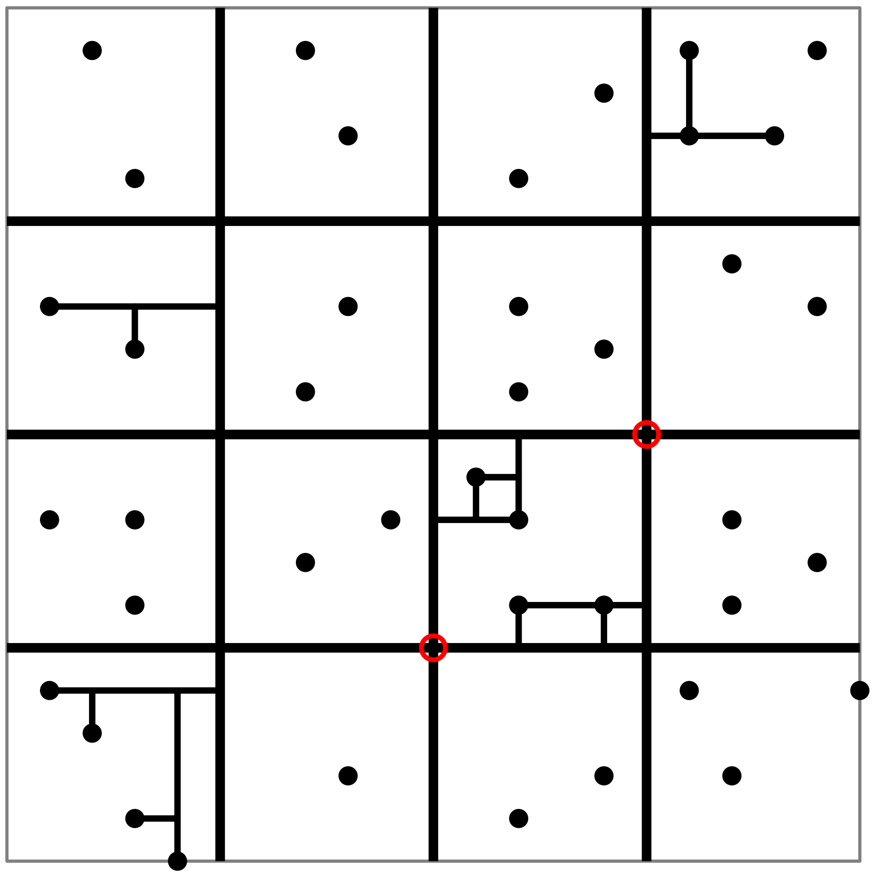

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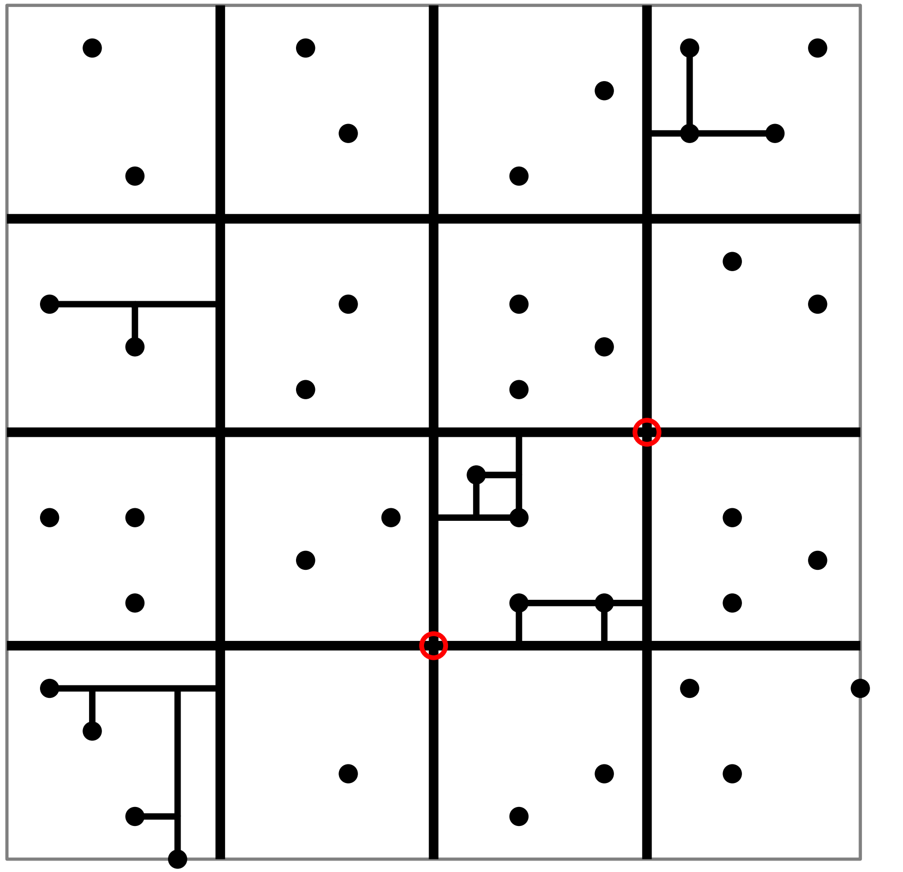

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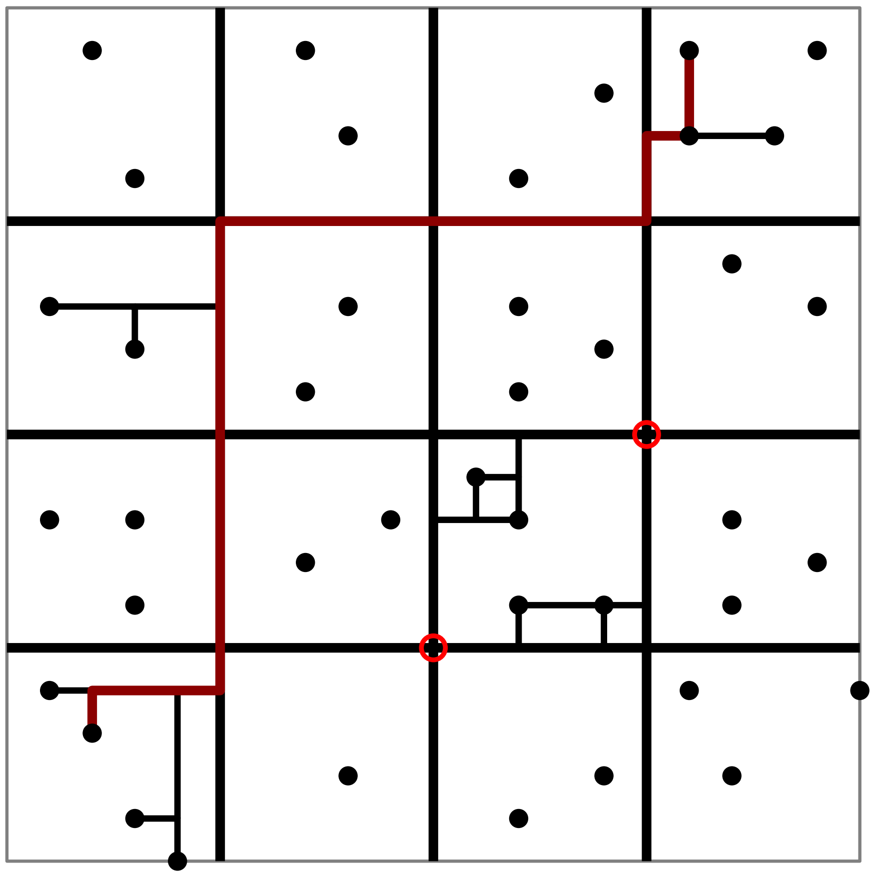

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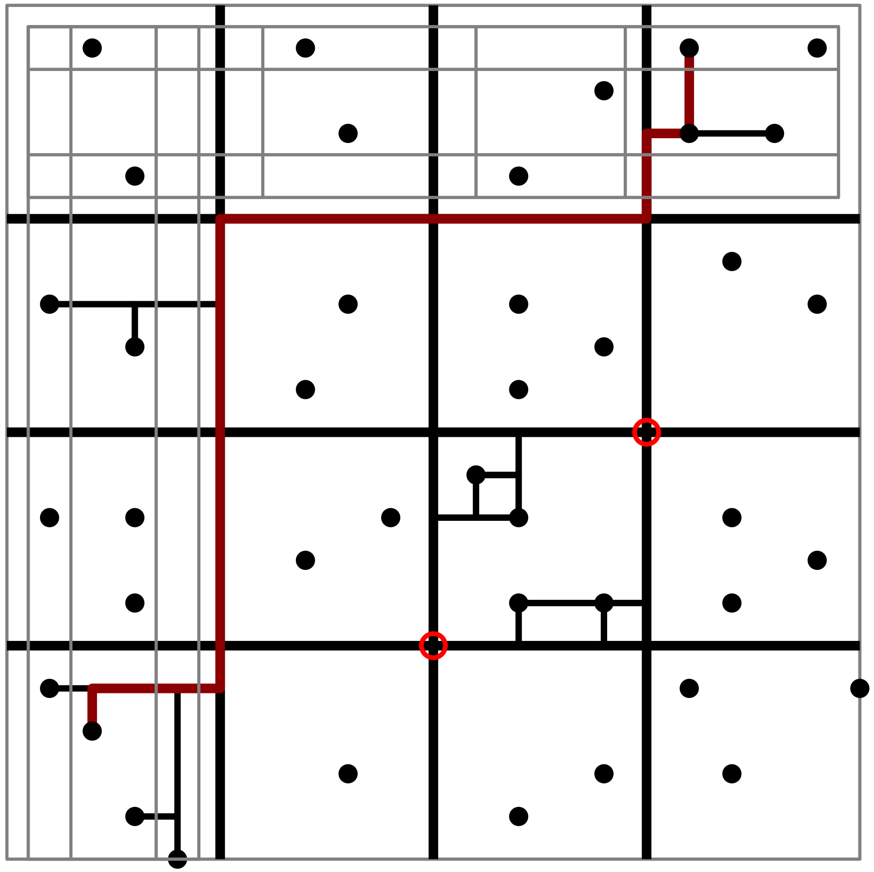

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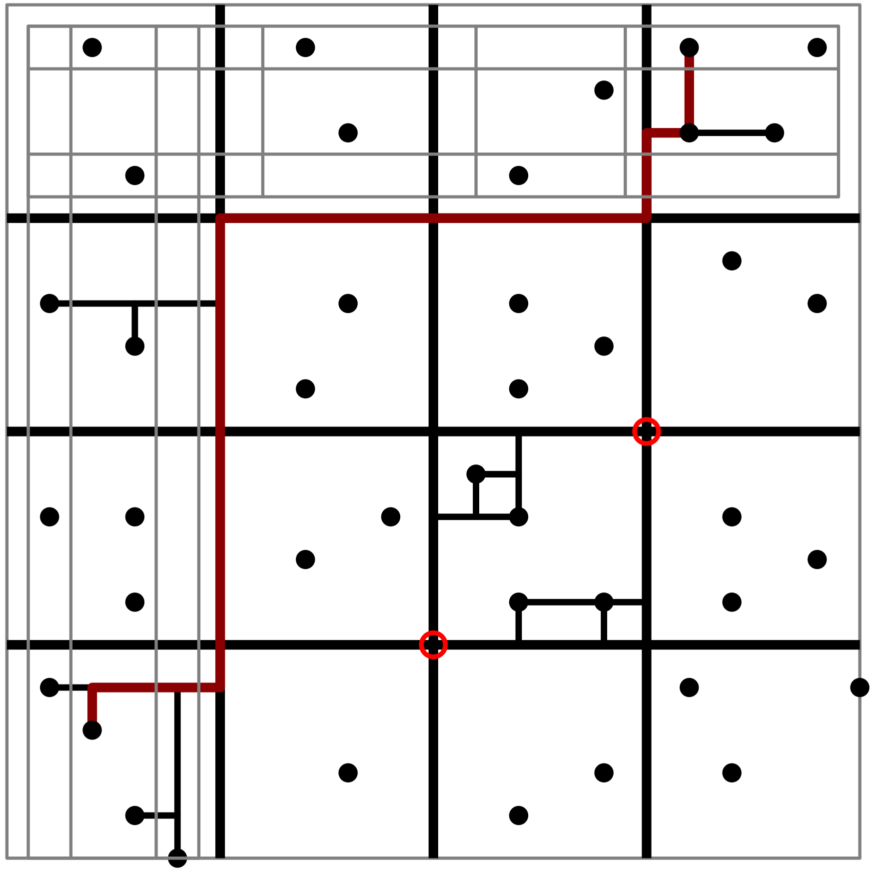
- apply recursively to slabs
- overall ratio $O(n^\epsilon)$

  (by choosing c accordingly)



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- $O(\log^{d+1} n)$-approximation algorithm for d dimensions.
- $O(\log n)$-approximation algorithm for 2D.

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- Both these results hold for GMMN as well.