Minimum Monotone Spanning Trees

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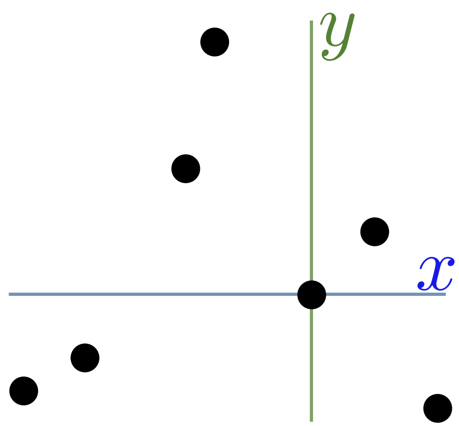
Eleni Katsanou, Lena Schlipf,
Antonios Symvonis, Alexander Wolff

SOFSEM 2025

Introduction

Minimum Spanning Trees

- Input:
  - A set S of points in the plane



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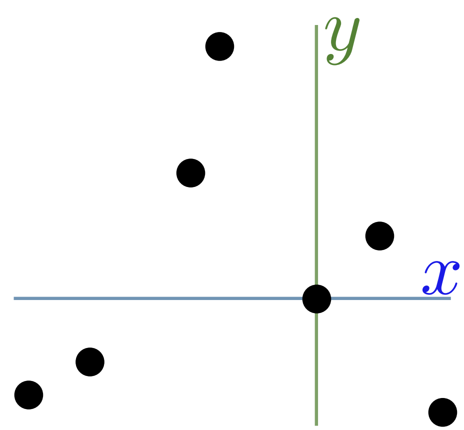
- Input:
  - A set S of points in the plane
- Output:
  - A geometric tree T that spans all points in S and has the minimum total length



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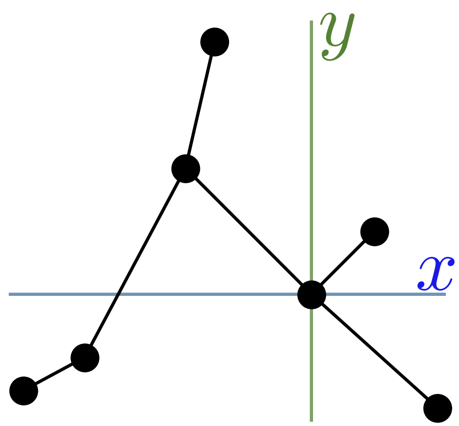

Minimum Spanning Trees

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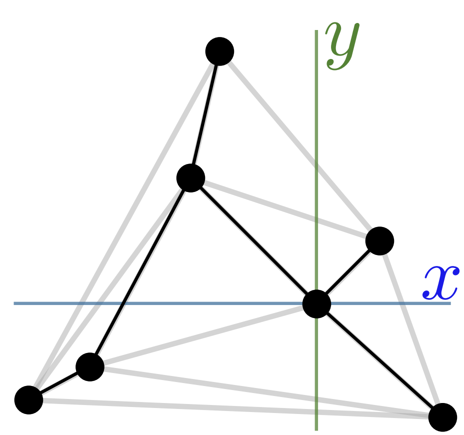
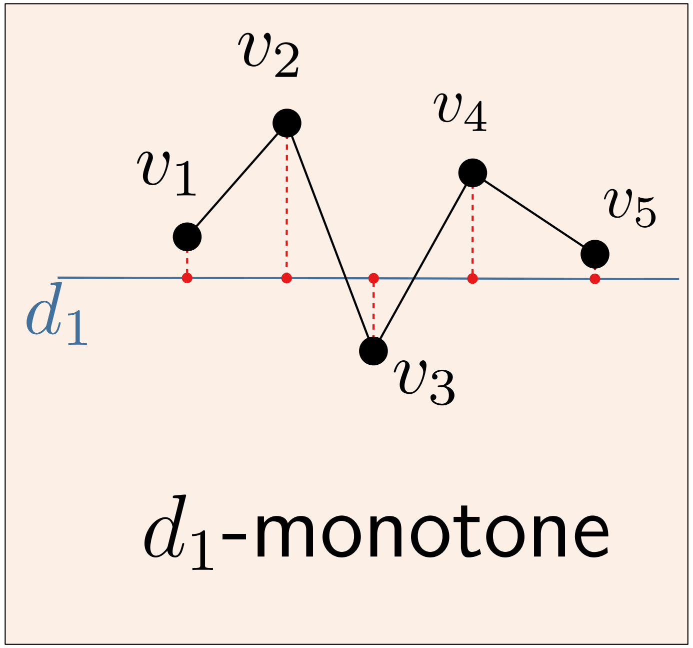
✔In $O(n \log n)$ time using the Delaunay Triangulation



Monotonicity

- Given a direction d:
  - A geometric path $\langle v_1, v_2, \ldots, v_n \rangle$ is d-monotone if the order of the vertices coincides with the order of their projections on a line parallel to d



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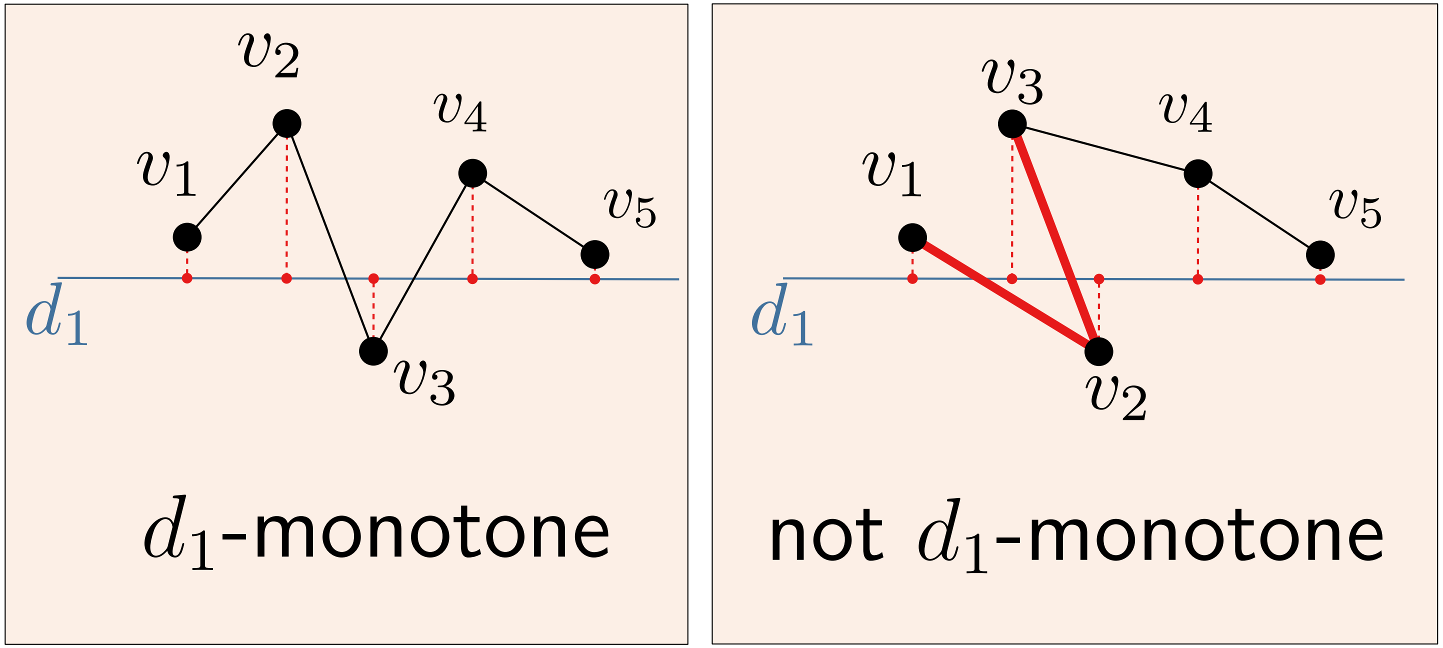

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- Given a direction d:

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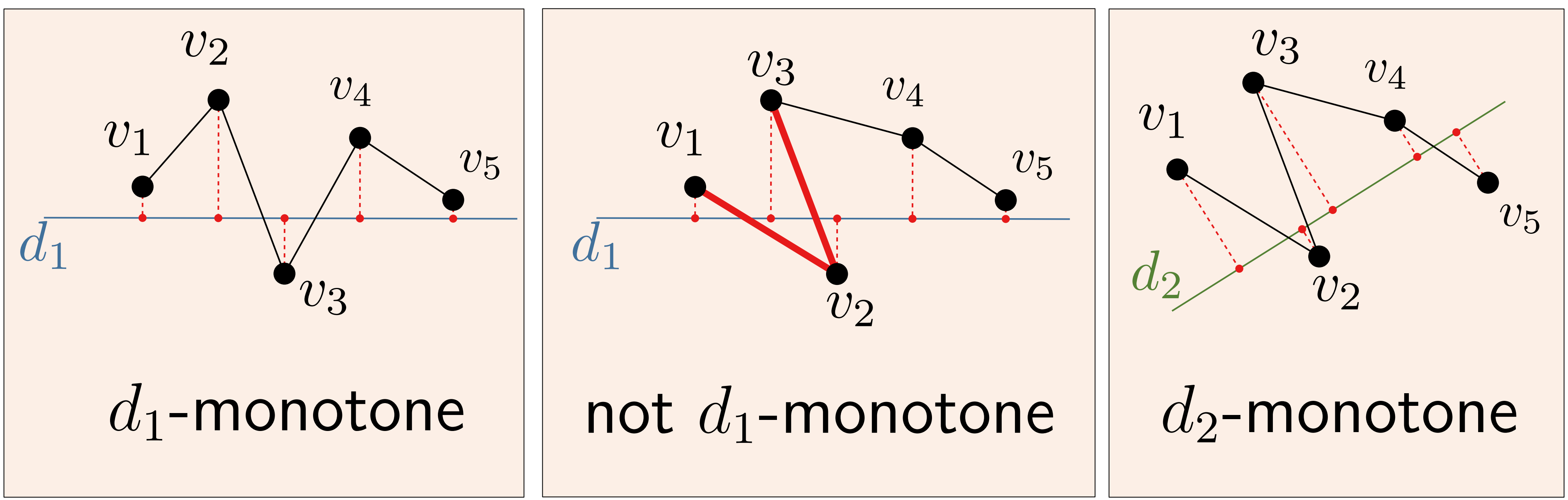
- It is monotone if it is d-monotone with respect to some direction d.



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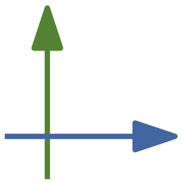
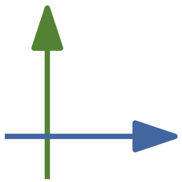


- Given a set $\mathcal{D}$ of (non-opposite) directions:
  - A tree T is $\mathcal{D}$-monotone if the path between any two vertices in T is d-monotone, for $d \in \mathcal{D}$

Monotonicity



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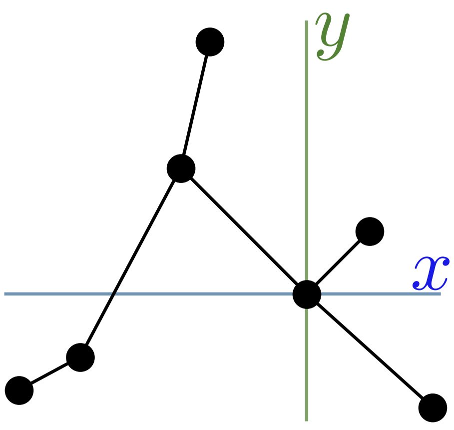

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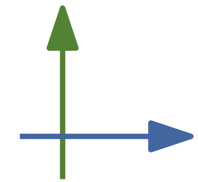

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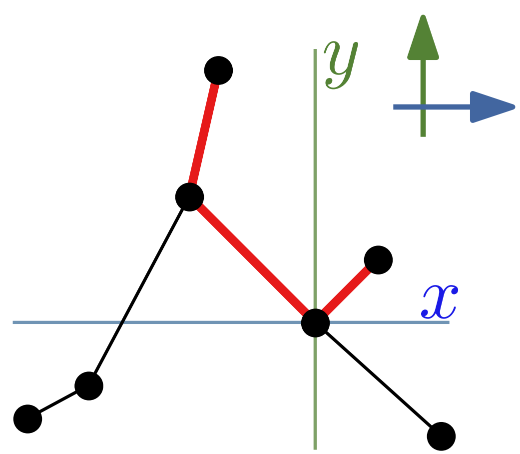

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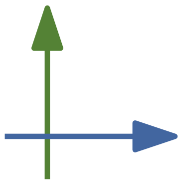

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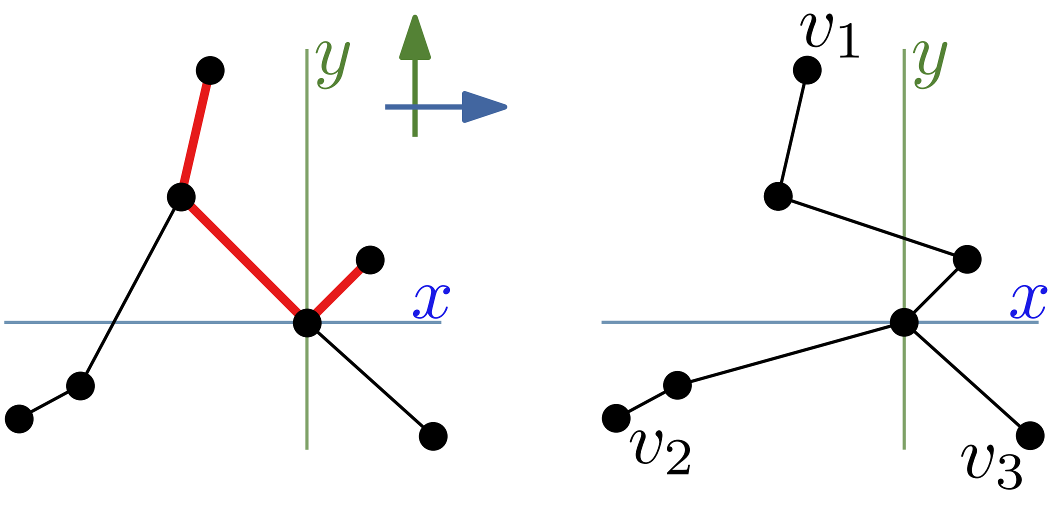

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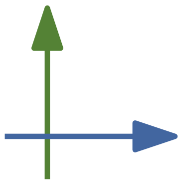

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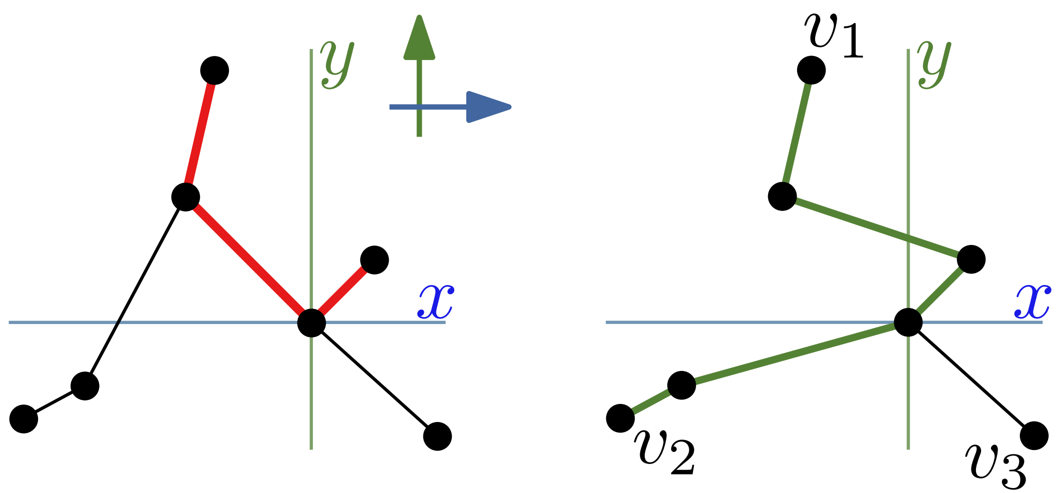

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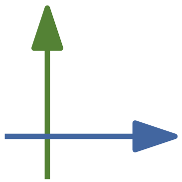

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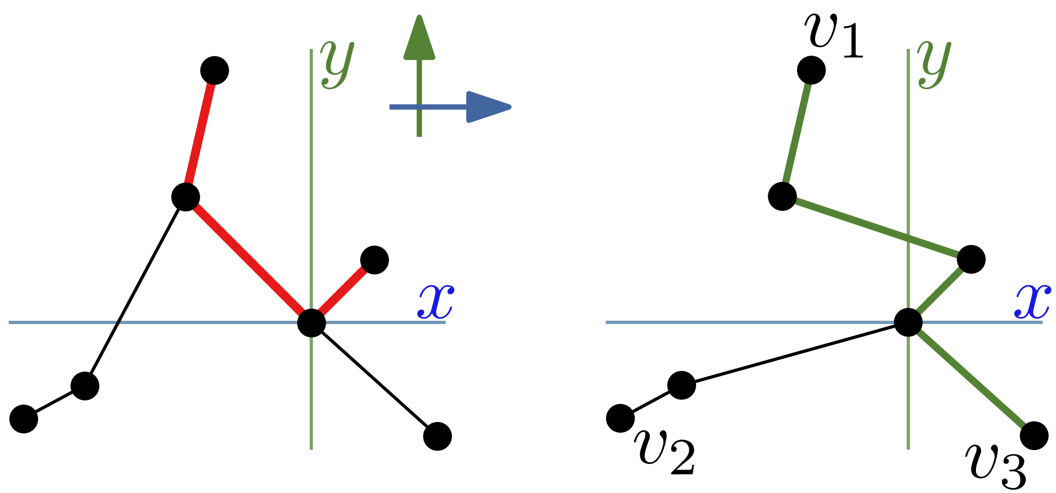

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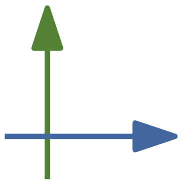

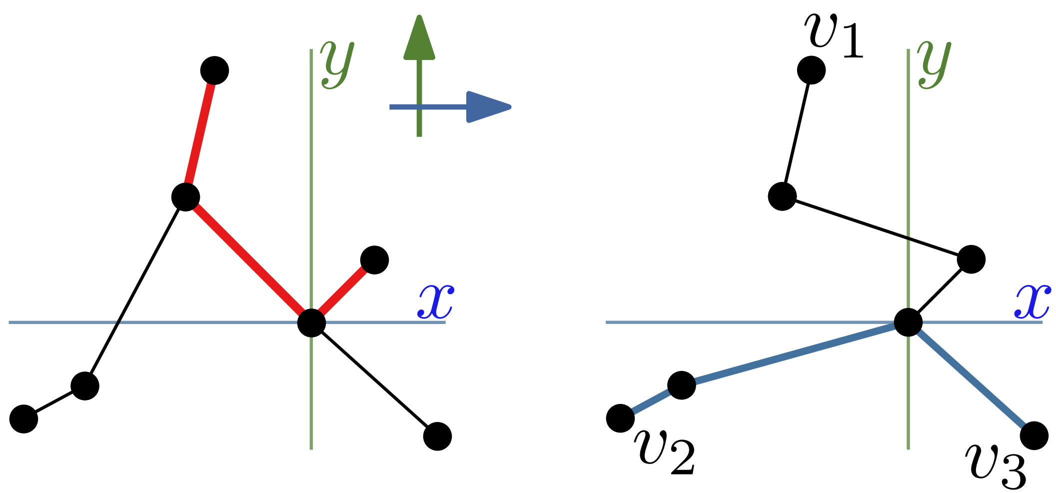

Minimum **Monotone** Spanning Trees

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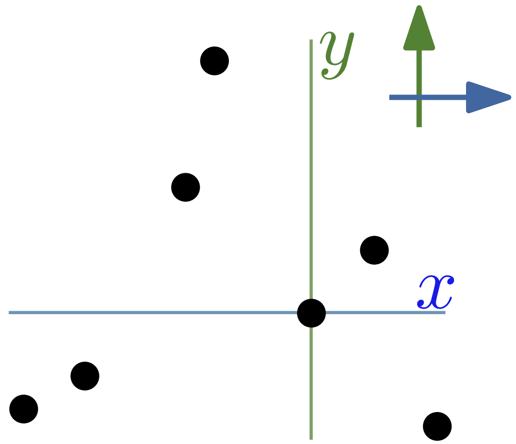

Minimum Monotone Spanning Trees

- Input:
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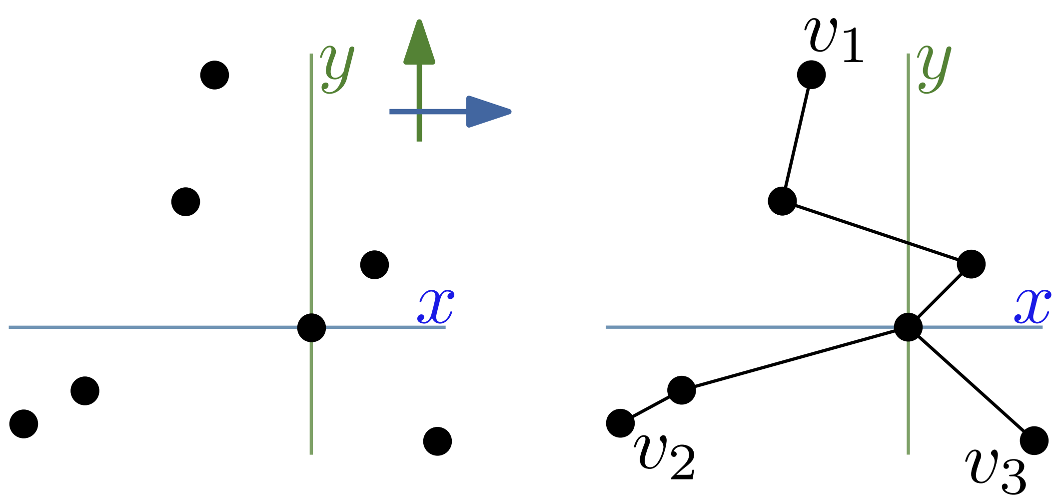
- Output:
  - A $\mathcal{D}$-monotone tree T that spans all points in S and has minimum total length among all such trees



Known Results

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- Angelini, Colasante, Di Battista, Frati, and Patrignani. Monotone drawings of graphs. (JGAA 2012)

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Area requirements of monotone drawings of trees

- Angelini et al. (2012): grid of size $O(n^{1.6}) \times O(n^{1.6})$ (BFS-based algorithm)
- Angelini et al. (2012): grid of size $O(n) \times O(n^2)$ (DFS-based algorithm)
- Kindermann et al. (2014): grid of size $O(n^{1.5}) \times O(n^{1.5})$
- He and He (2015): grid of size $O(n^{1.205}) \times O(n^{1.205})$
- He and He (2016): grid of size $O(n \log n) \times O(n \log n)$
- He and He (2016): grid of size $12n \times 12n$
- Oikonomou and Symvonis (2017): grid of size $n \times n$

Known Results

- Mastakas and Symvonis. Rooted uniform monotone minimum spanning trees. (2017)
  Input:
  – A set S of points in the plane and a designated root $r \in S$
  Output:
  – An MST such that the path from r to any other point of S is monotone with respect to (i) one direction or (ii) two orthogonal directions.
- Mastakas. Uniform 2d-monotone minimum spanning graphs. (2018)
- Mastakas. Drawing a rooted tree as a rooted y-monotone minimum spanning tree. (2021)

Our Contribution

Minimum $\mathcal{D}$-Monotone Spanning Tree, MMST$(S, \mathcal{D})$

- Input:
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- MMST$(S, \mathcal{D})$ can be solved in $O(f(|\mathcal{D}|)n^{2|\mathcal{D}|-1} \log n)$ time, i.e. is in XP when parameterized by $|\mathcal{D}|$

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OBVIOUS

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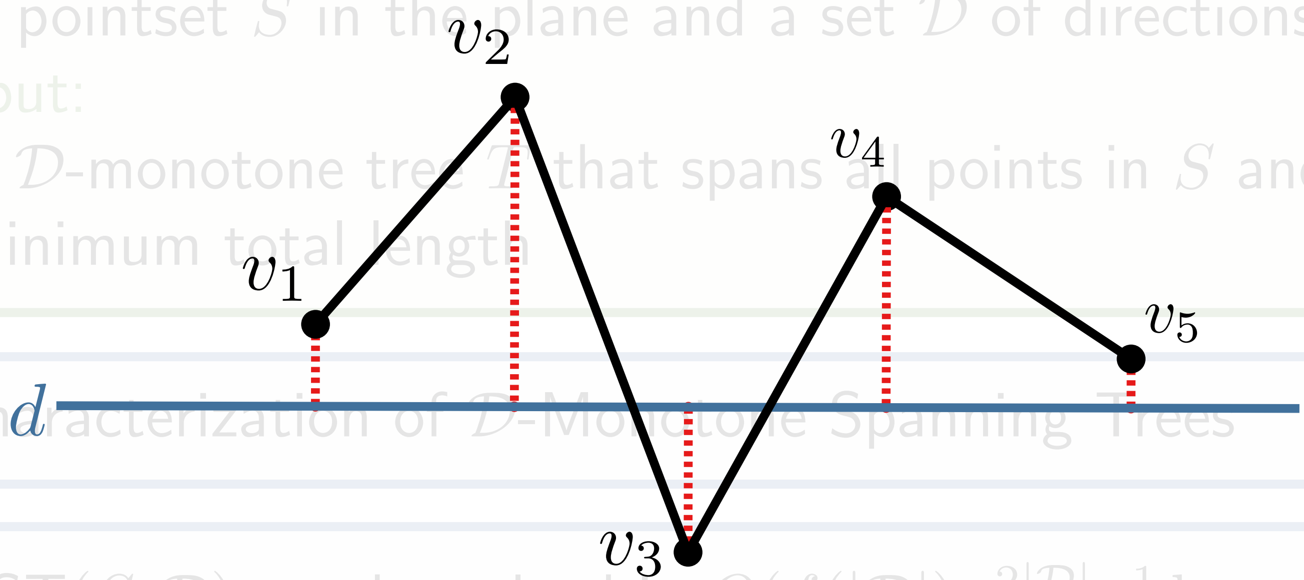

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- For $|\mathcal{D}| = 2$, MMST$(S, \mathcal{D})$ can be solved in $O(n^2)$ time

- For $|\mathcal{D}| = 1$, MMST$(S, \mathcal{D})$ [illegible] $O(n \log n)$ time

OBVIOUS

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Theorem

For every even integer $k \geq 2$, there exists a point set S and a set $\mathcal{D}$ of k directions such that any minimum-length $\mathcal{D}$-monotone spanning tree of S has maximum vertex degree $2k$.

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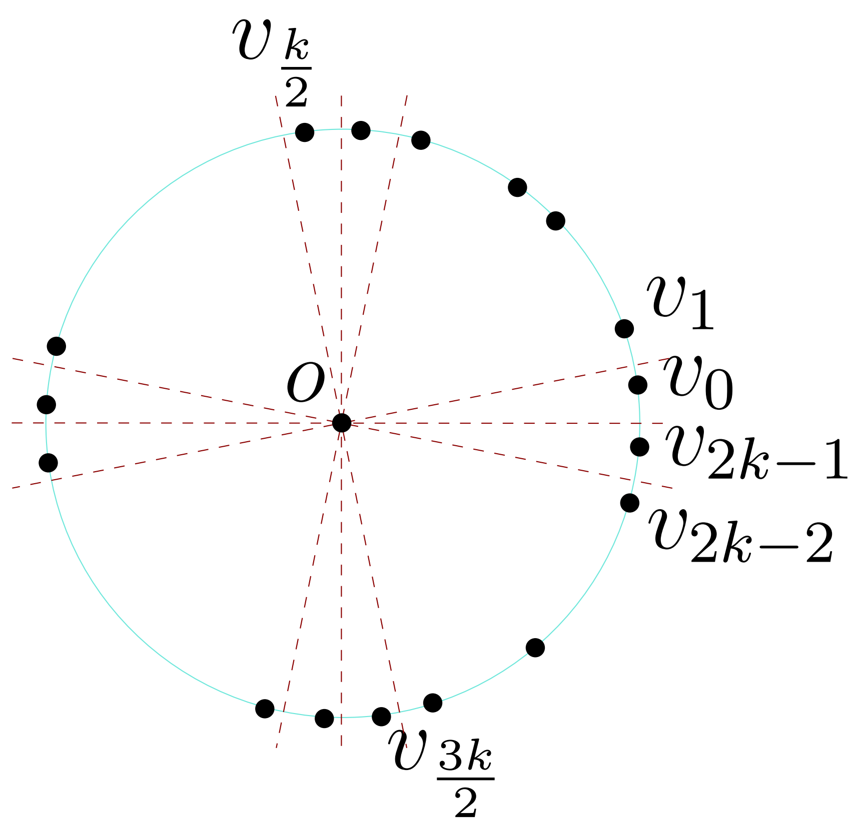

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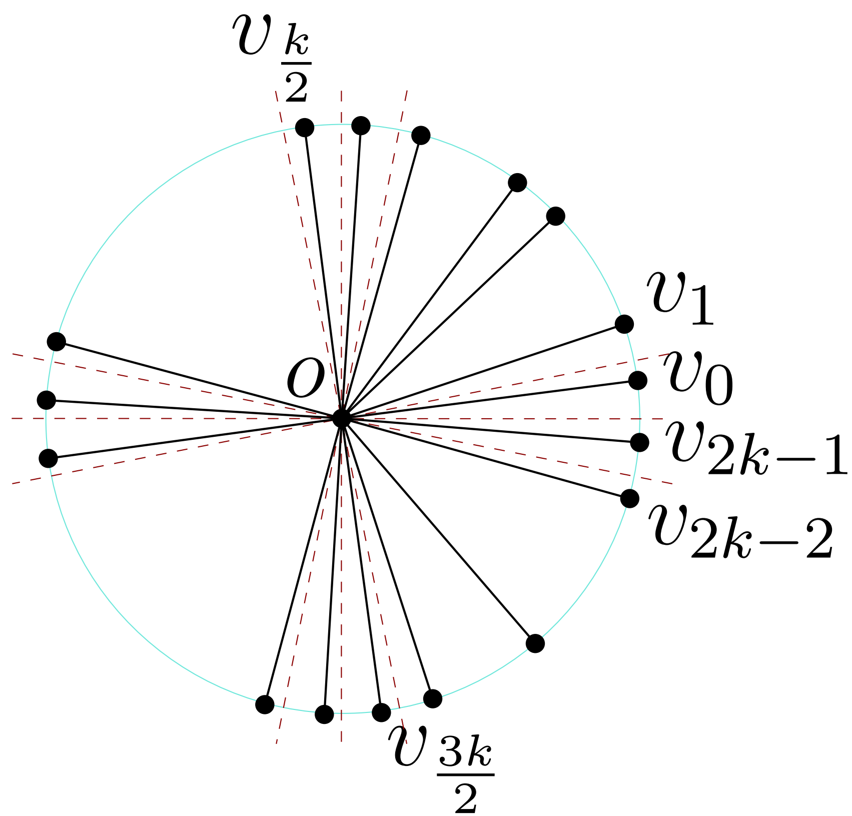

Basic Properties

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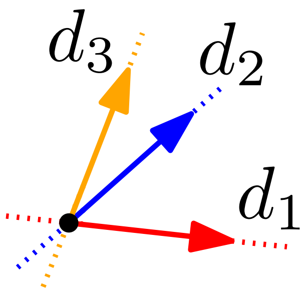
set $\mathcal{D}$ of k directions



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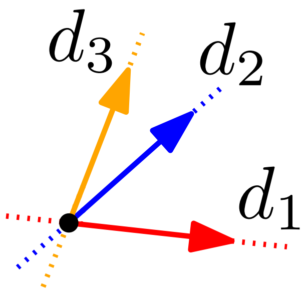
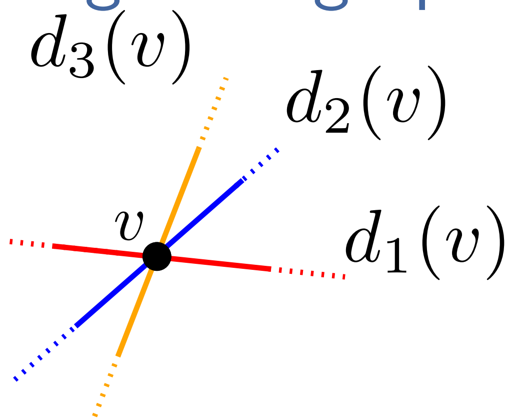


lines passing through point v



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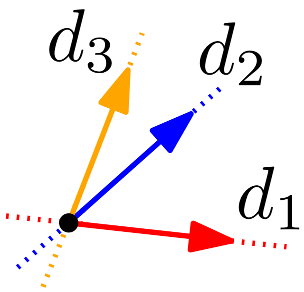
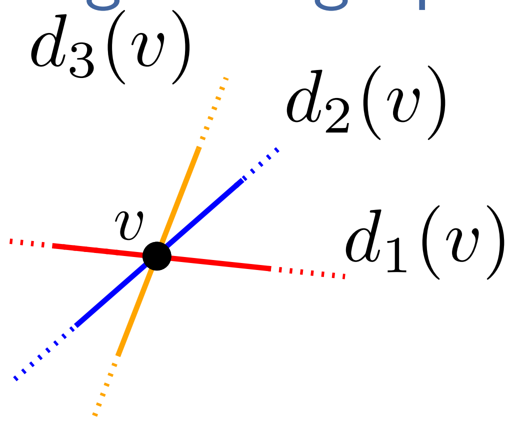
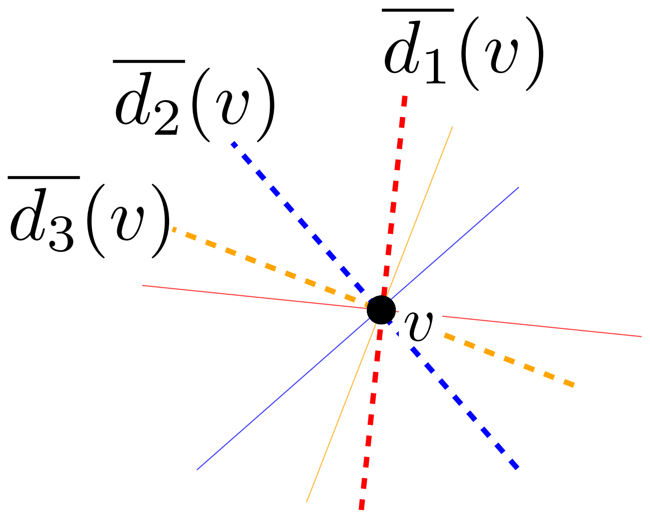


lines passing through point v



orthogonals



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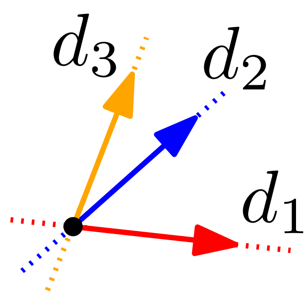

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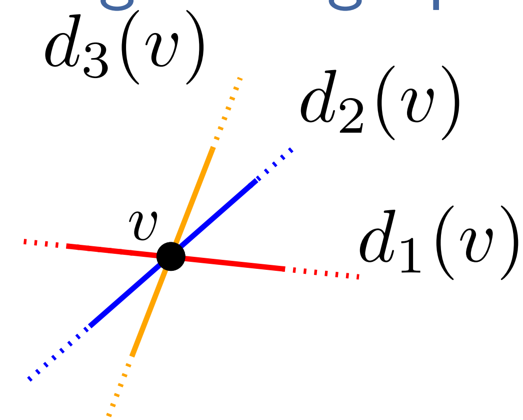

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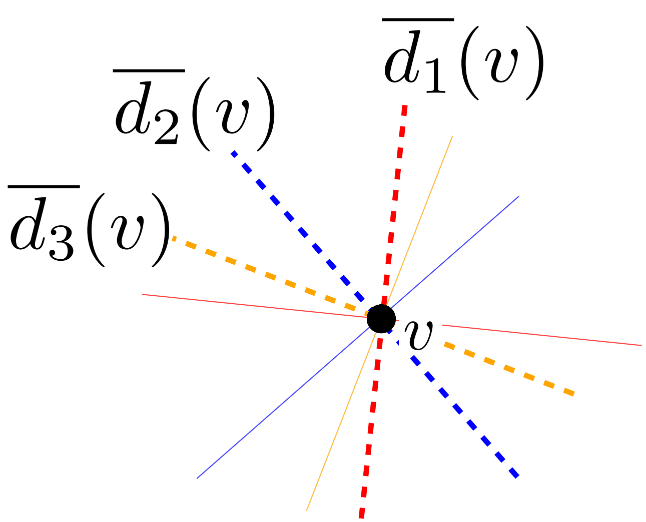
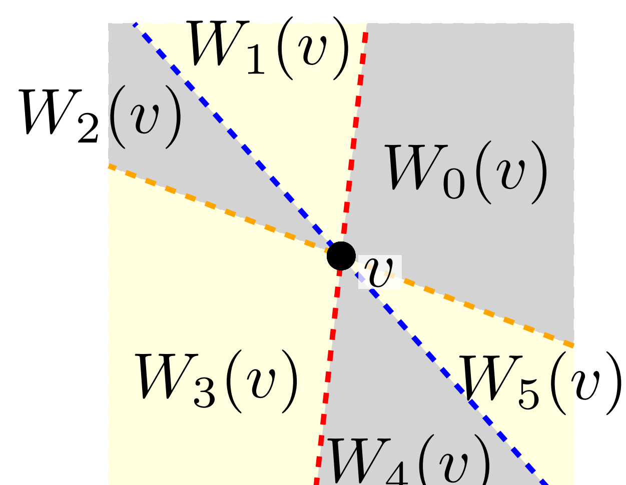


wedge set

$$\mathcal{W}_{\mathcal{D}}(v) = \{W_0(v), W_1(v), \ldots, W_{2k-1}(v)\}$$

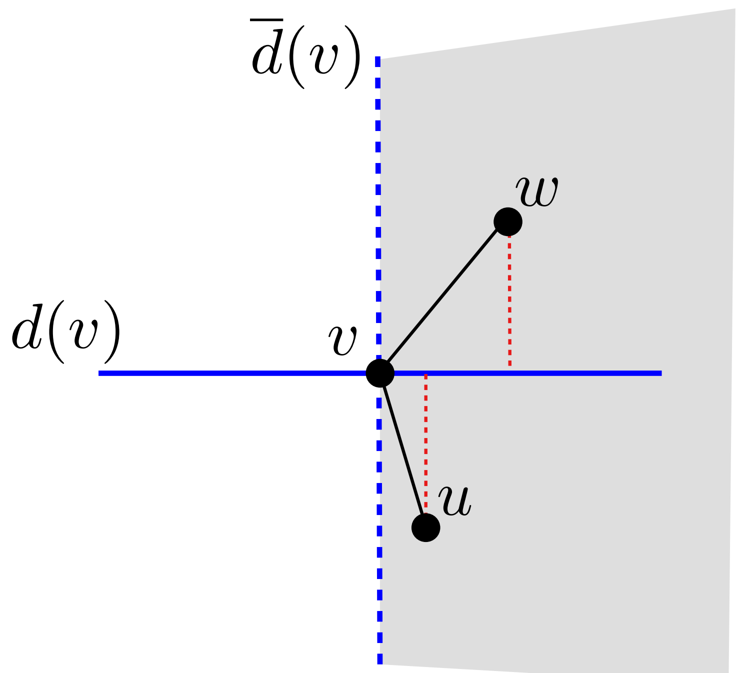


wedges based on orthogonal lines

Basic Properties

Lemma 1

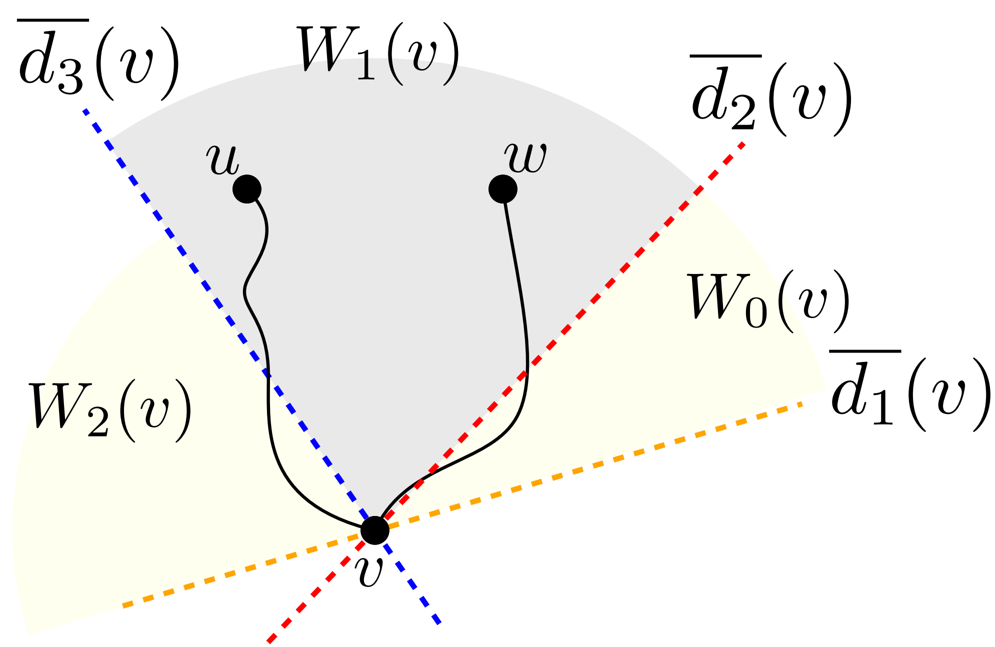
Let $\langle u, v, w \rangle$ be a geometric path. If u and w lie in the same half-plane determined by $\overline{d}(v)$, then the path between u and w is not d-monotone.



Basic Properties

Lemma 2

Let $P = \langle u, \ldots, v, \ldots, w \rangle$ be a geometric path. If u and w lie in the same wedge in $\mathcal{W}_{\mathcal{D}}(v)$, then the path P is not $\mathcal{D}$-monotone.



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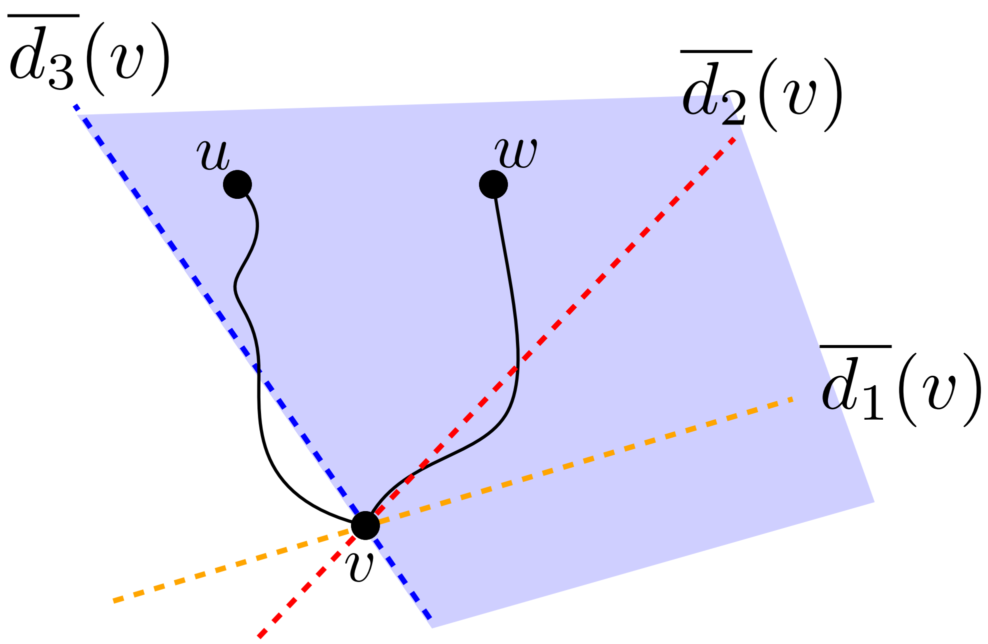

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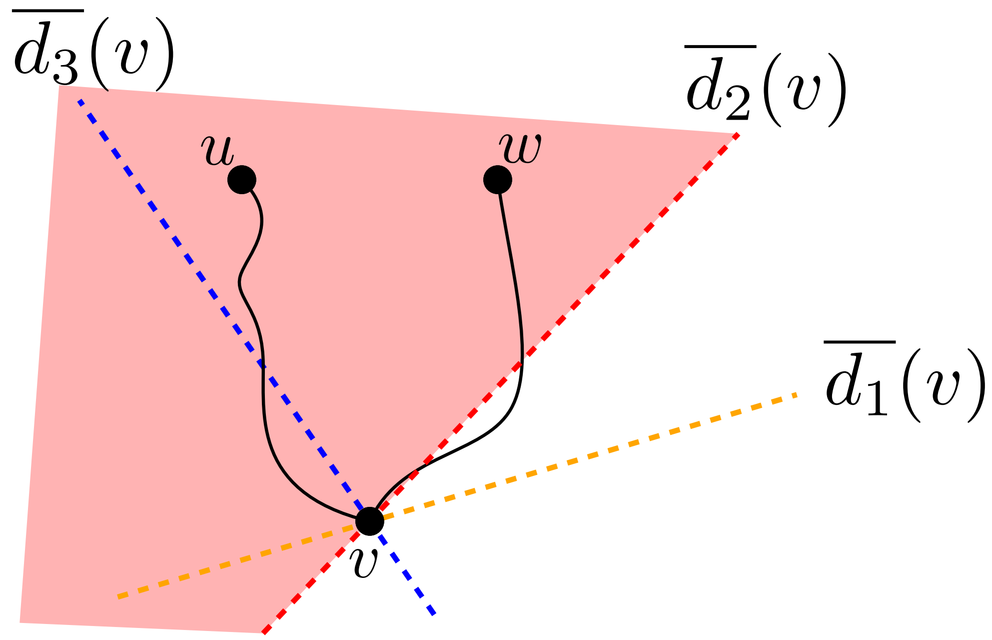

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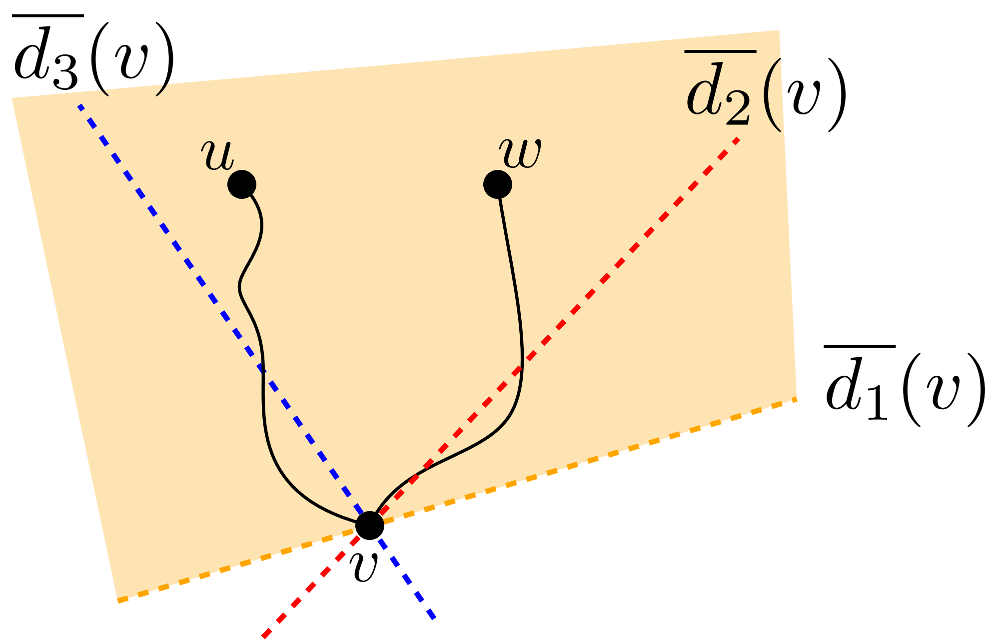

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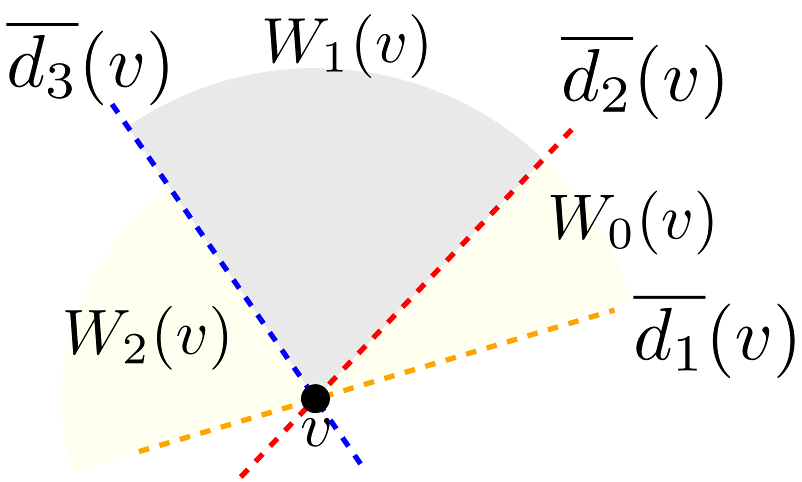
Lemma 3

Let T be a $\mathcal{D}$-monotone spanning tree of S. Then, $\Delta(T) \leq 2k$.



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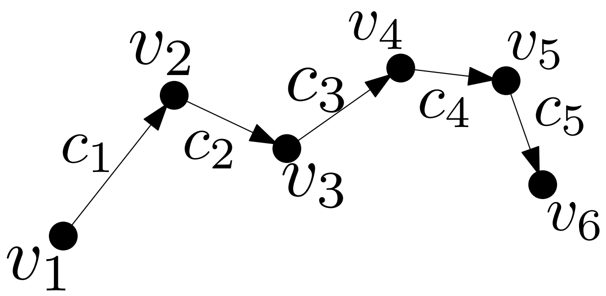
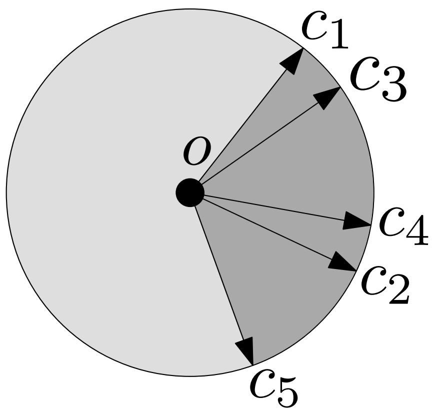


sector of directions of P, sec(P)



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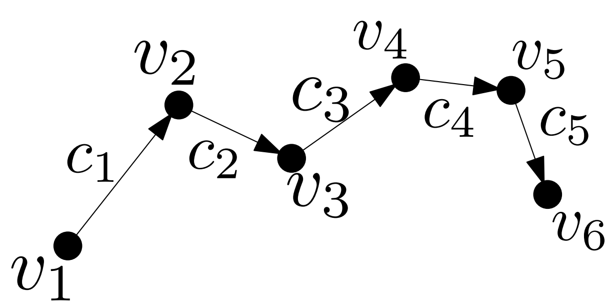
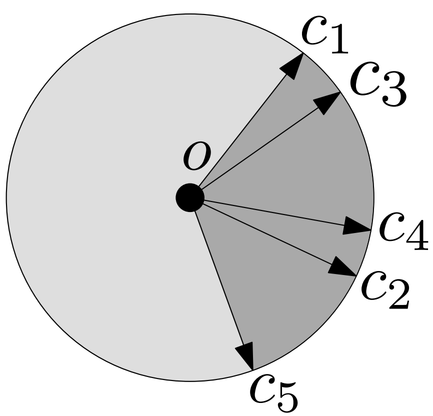
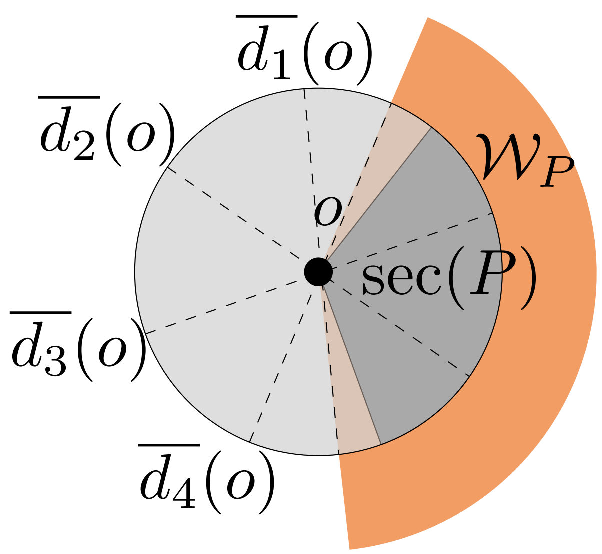


sector of directions of P, sec(P)



wedge set $\mathcal{W}_P$ of the directed path P



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Lemma 4 [Angelini et al. (2012)]

Given a directed geometric path P,
P is monotone $\Leftrightarrow$ the angle of sec(P) is smaller than π.

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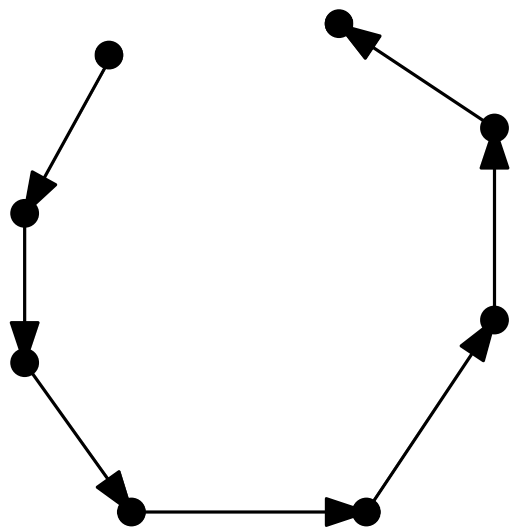

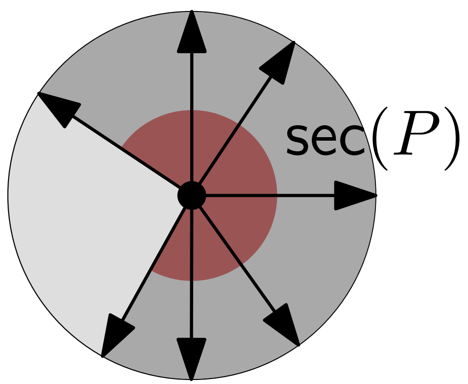

A Characterization of $\mathcal{D}$-Monotone Spanning Trees

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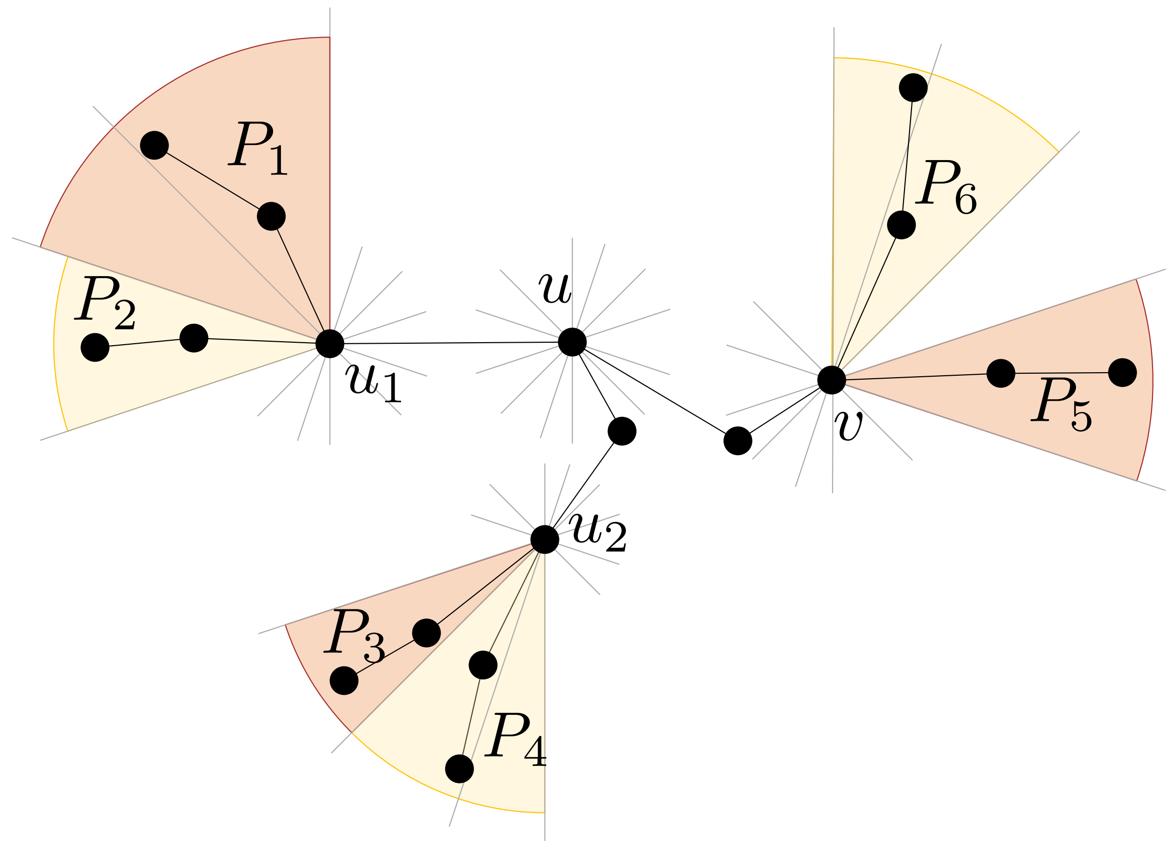
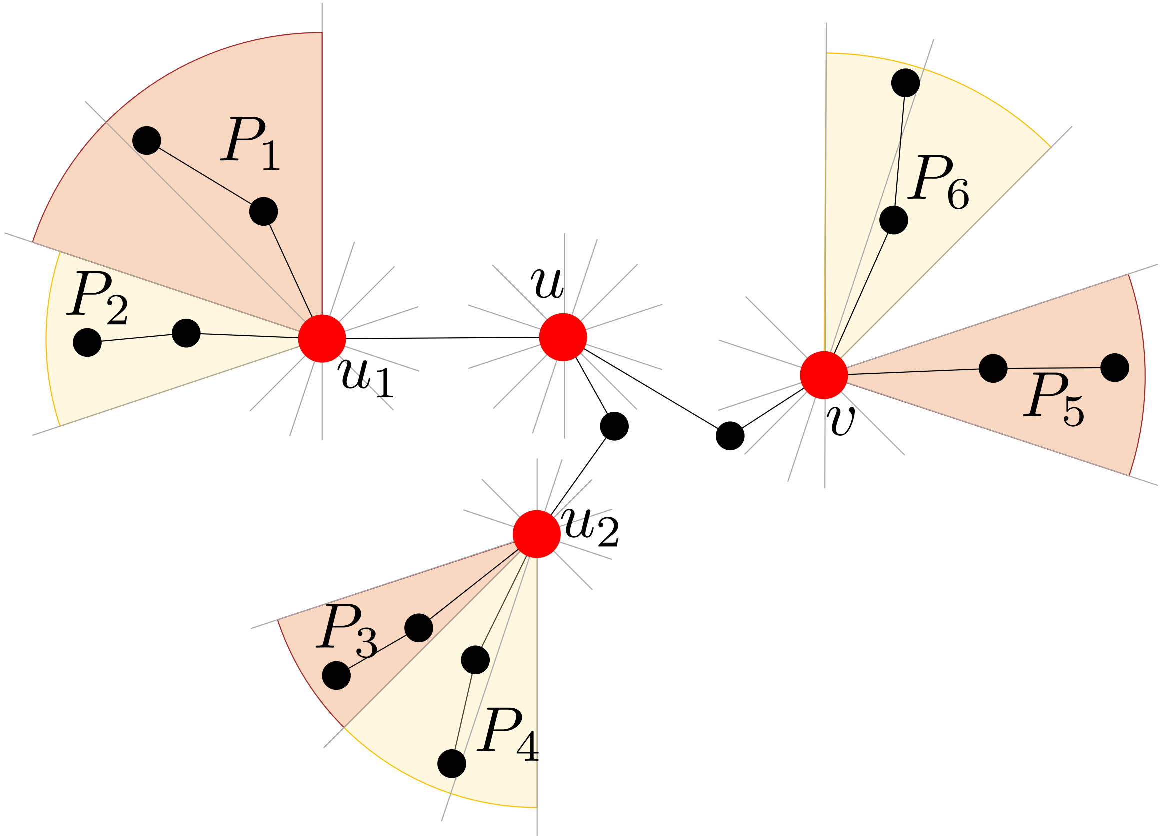


A Characterization of $\mathcal{D}$-Monotone Spanning Trees

● **Branching vertex**

A vertex of degree at least 3



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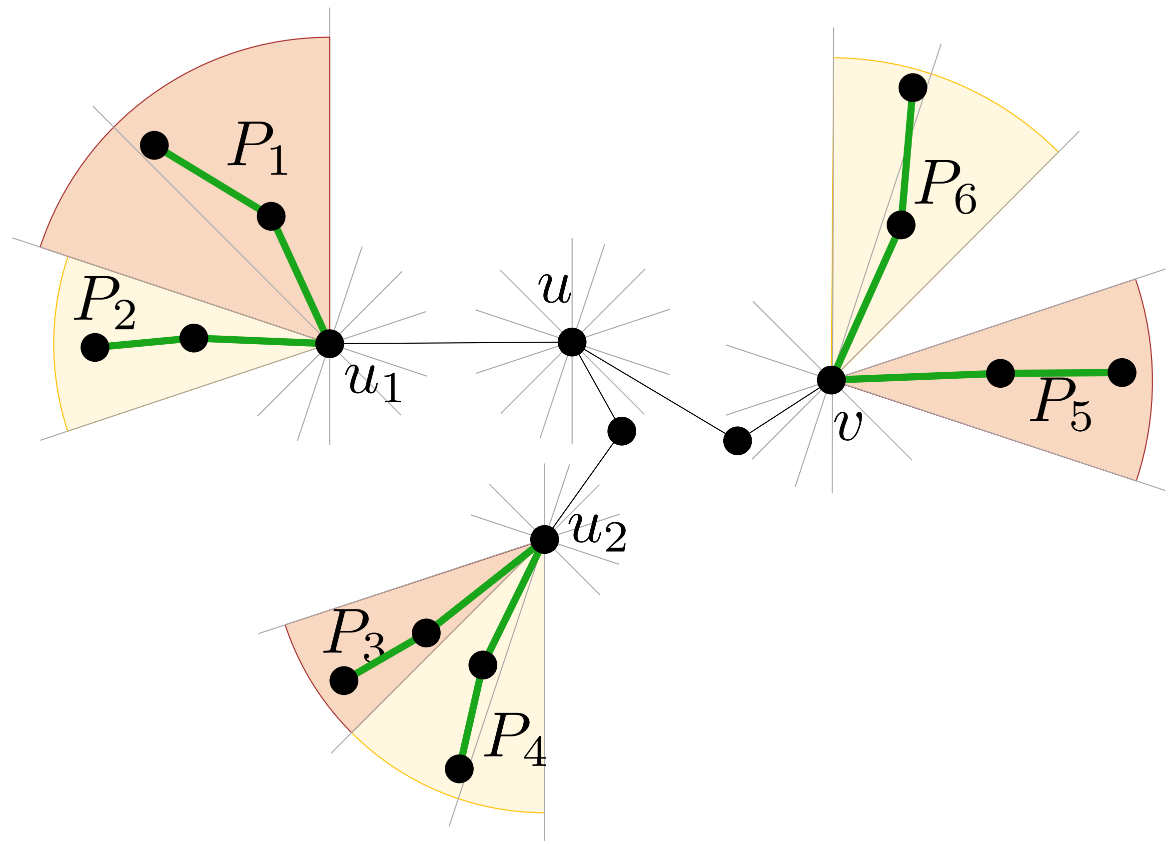

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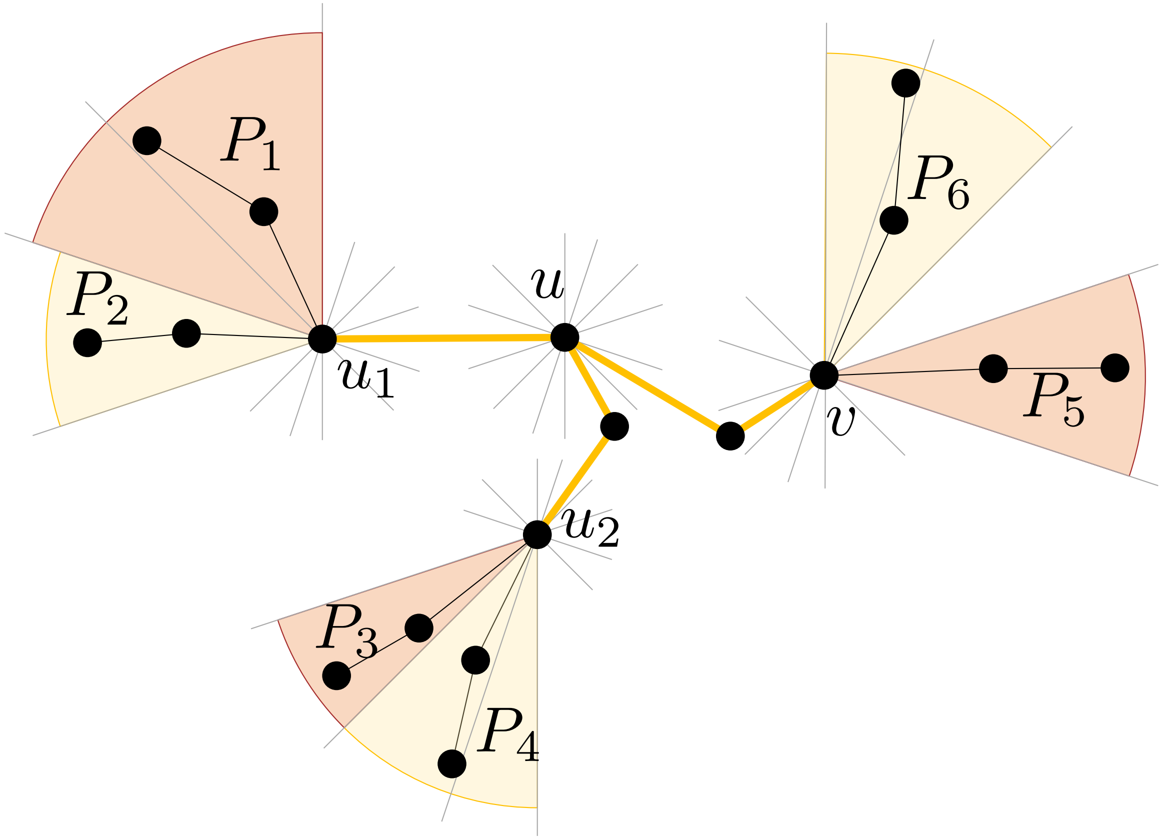
A path connecting "adjacent" branching vertices u, v



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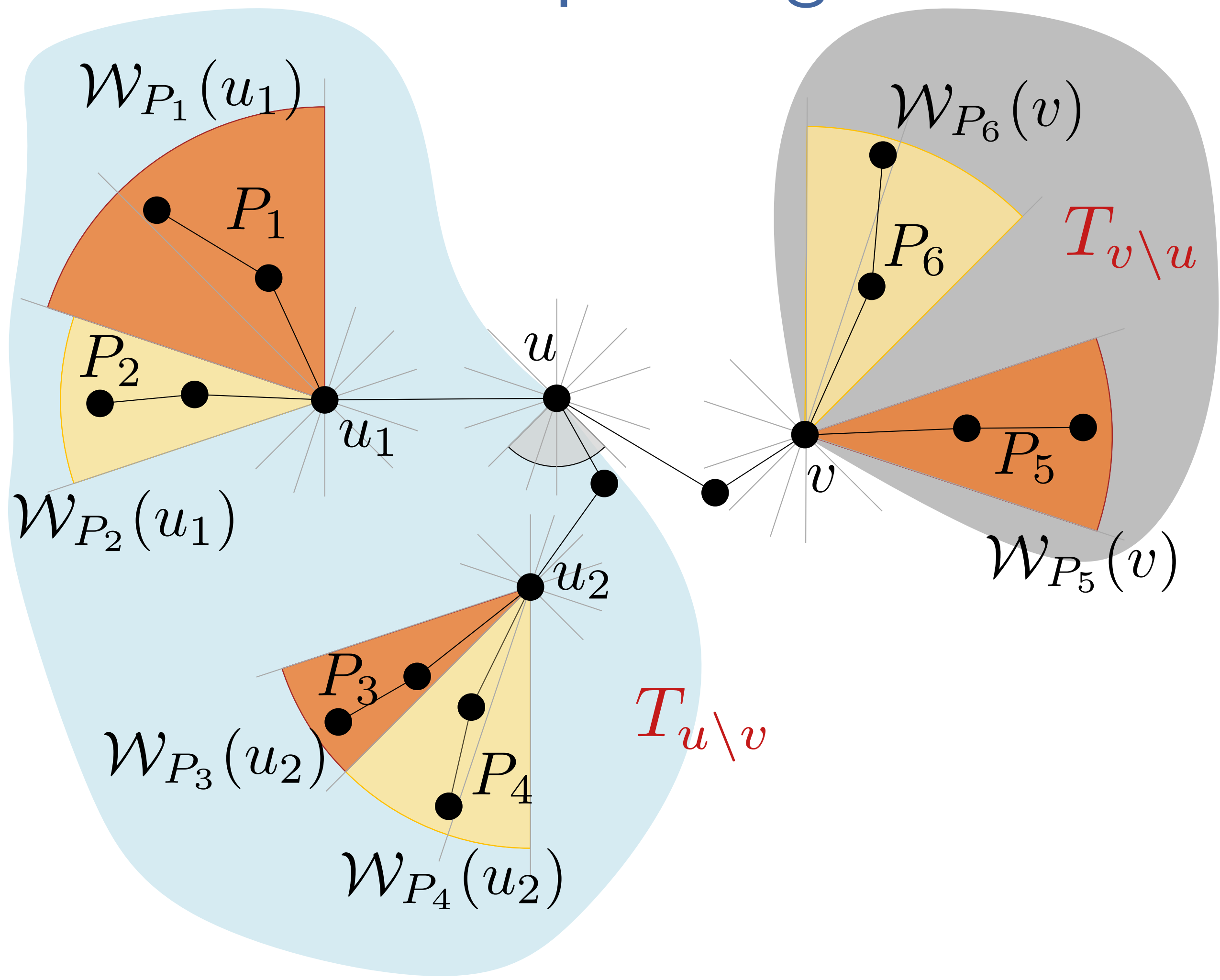

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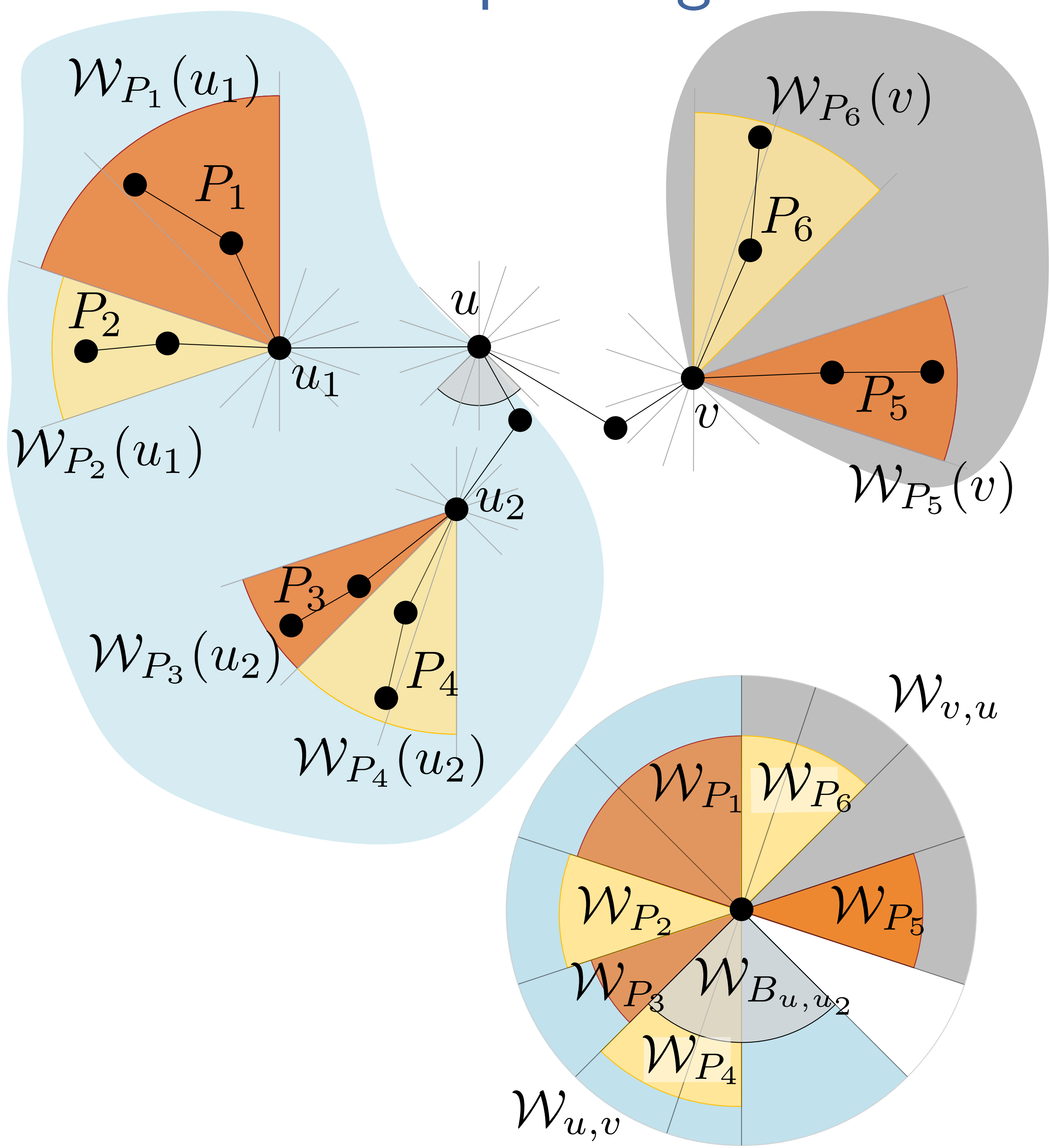

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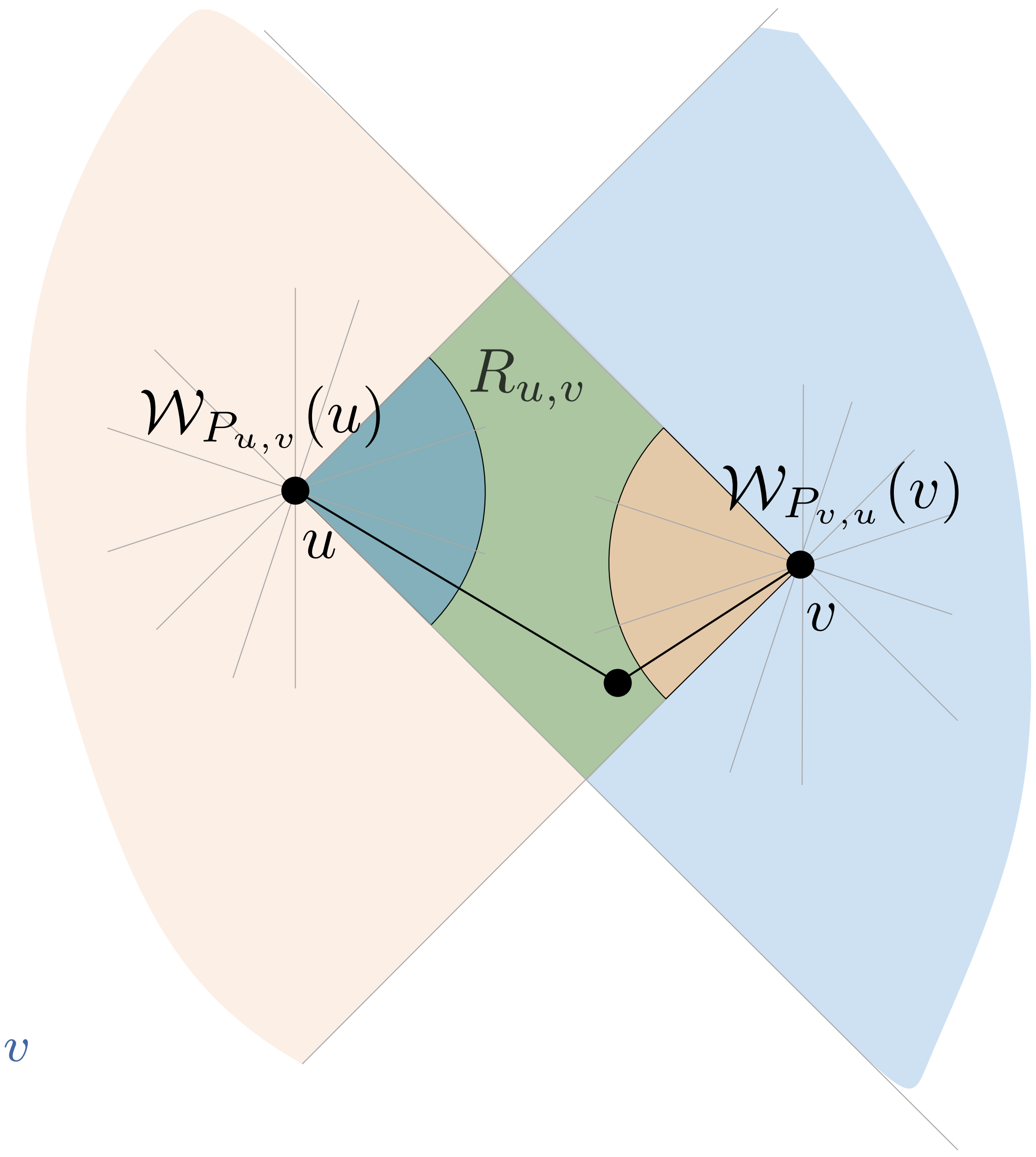
Region $R_{u,v}$ **of path** $P_{u,v}$
$R_{u,v} = \mathcal{W}_{P_{u,v}}(u) \cap \mathcal{W}_{P_{v,u}}(v)$



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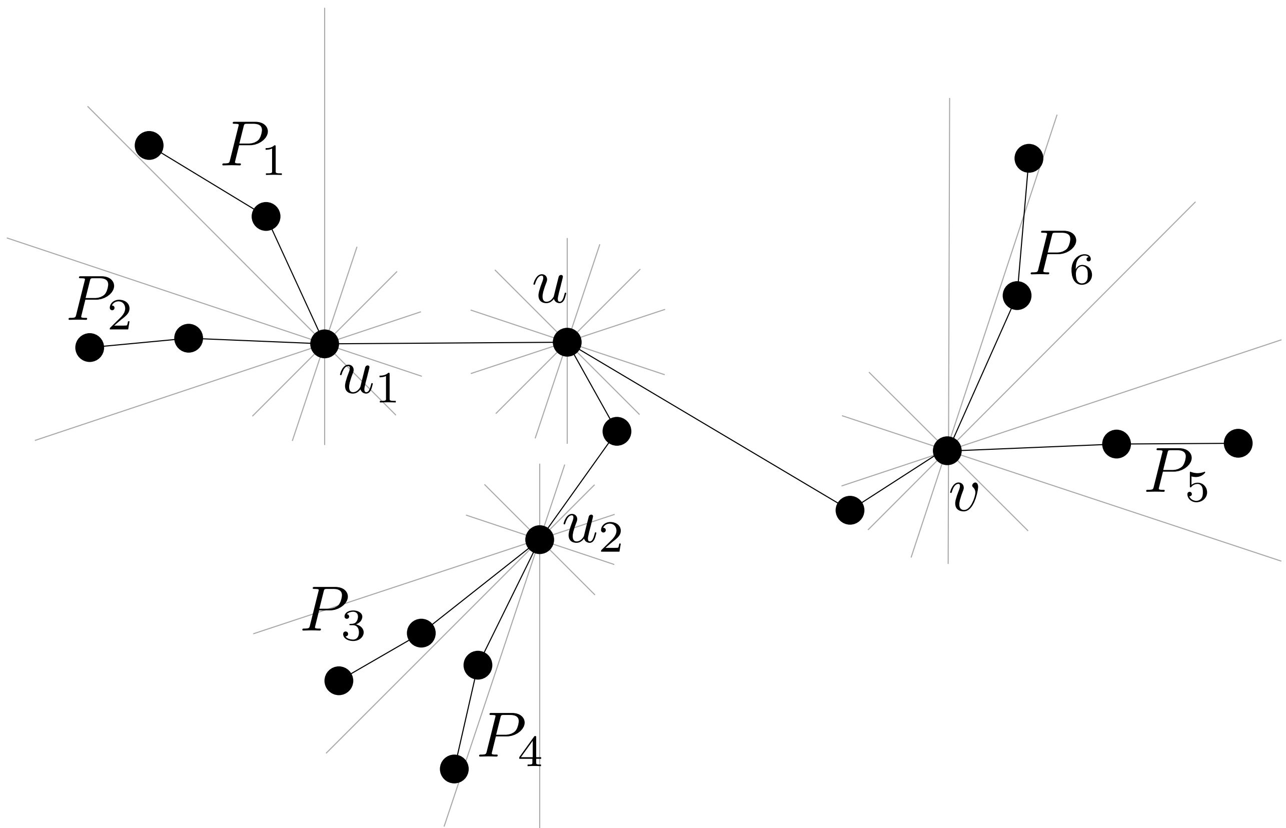

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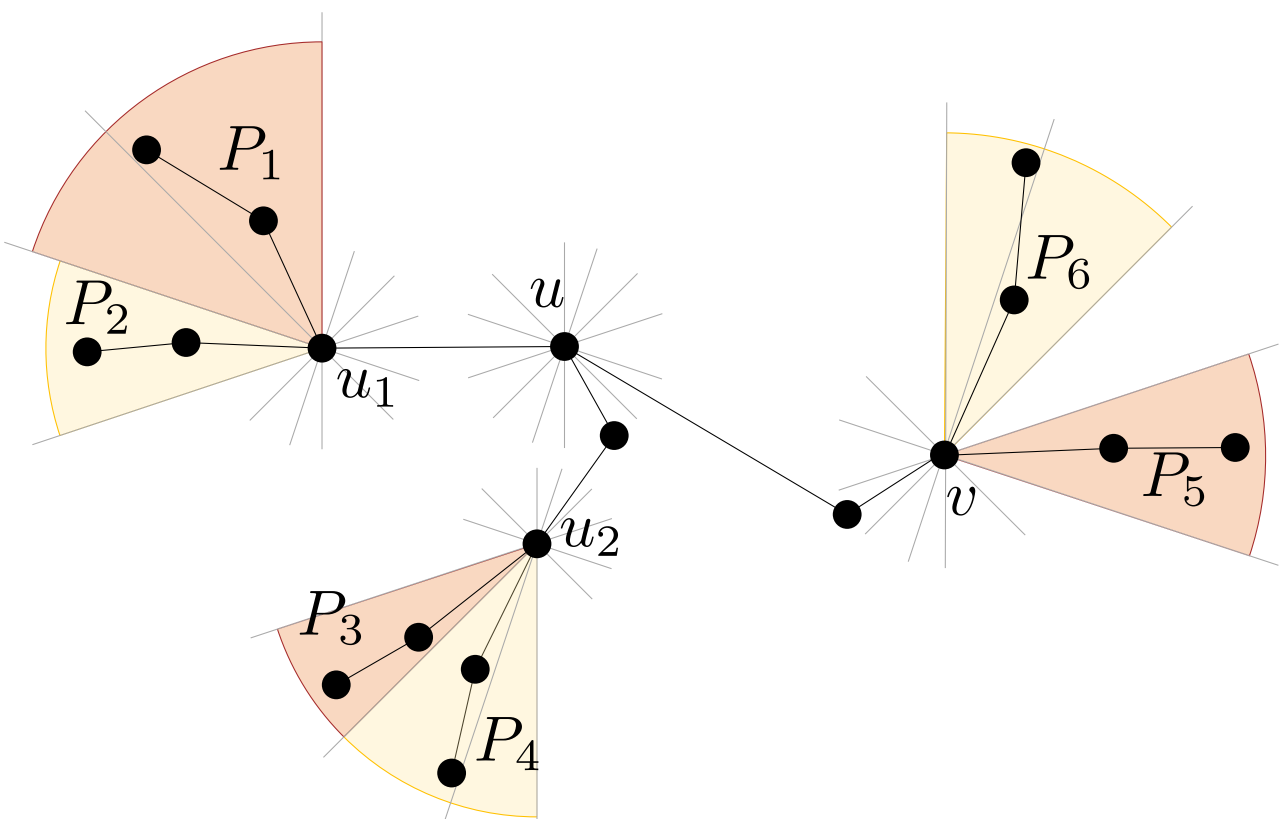

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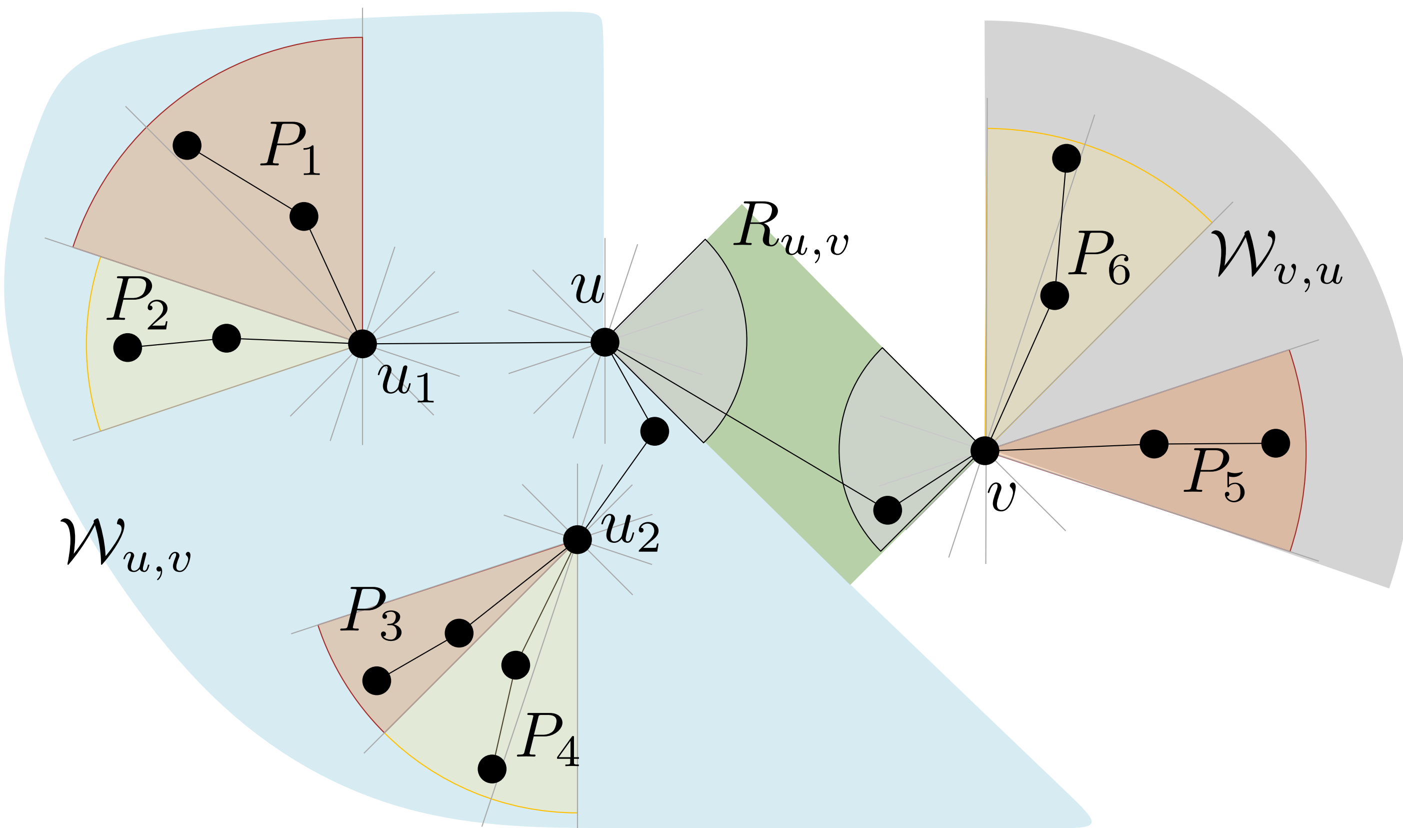

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Lemma

If T is a $\mathcal{D}$-monotone spanning tree of S, then T has at most $2k$ leaves.

An algorithm for MMST$(S, \mathcal{D})$

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Homeomorphically Irreducible Tree (HIT)

- An embedded tree without vertices of degree two

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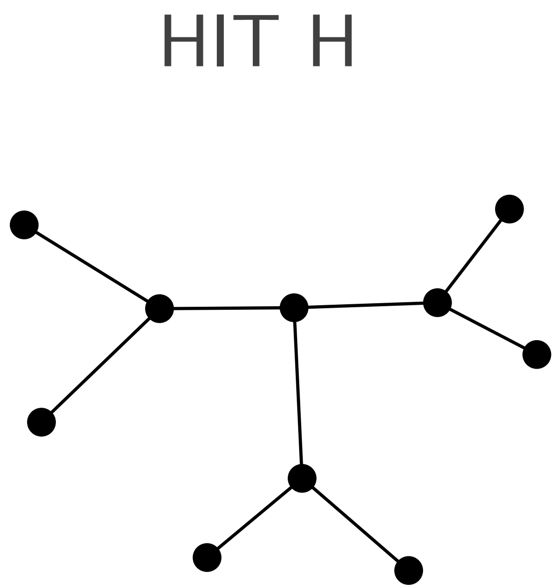
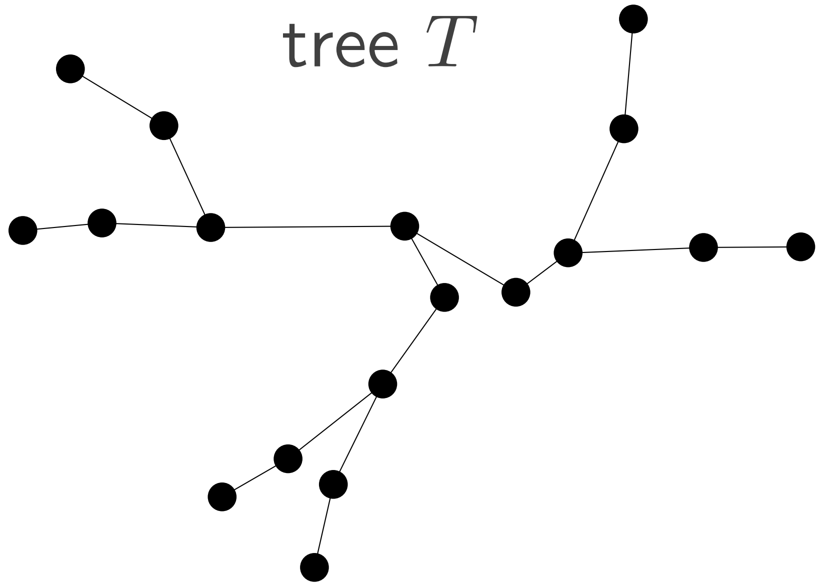




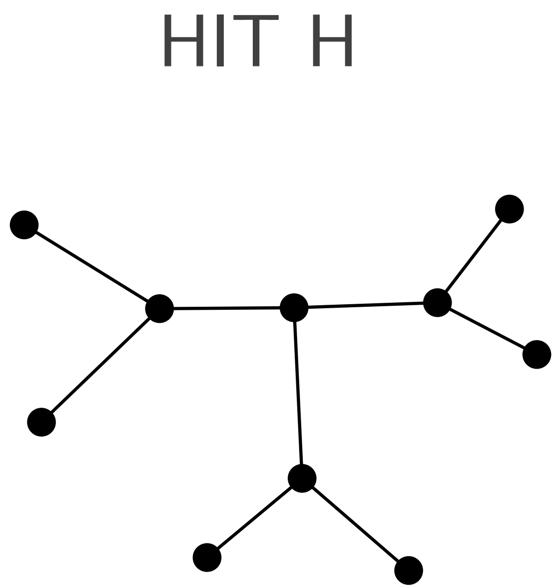
An algorithm for MMST$(S, \mathcal{D})$

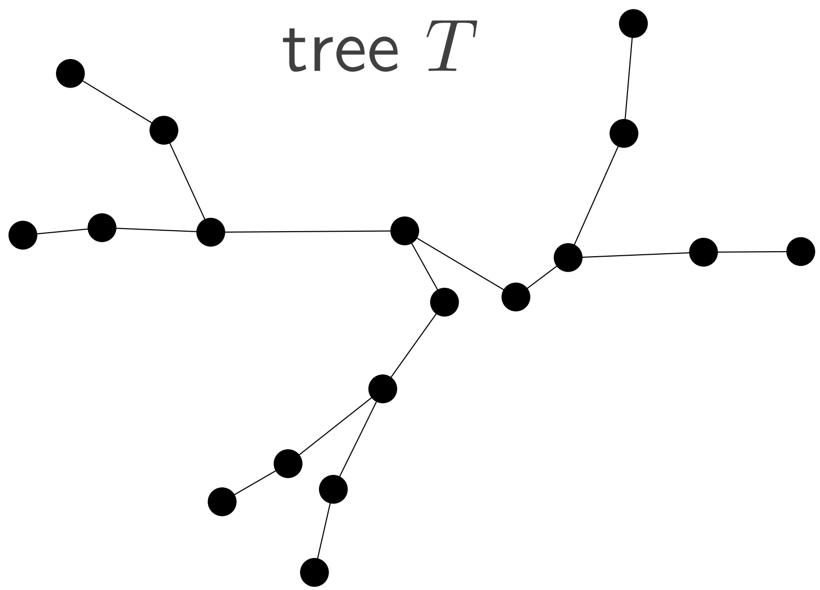
Homeomorphically Irreducible Tree (HIT)

- An embedded tree without vertices of degree two

Lemma

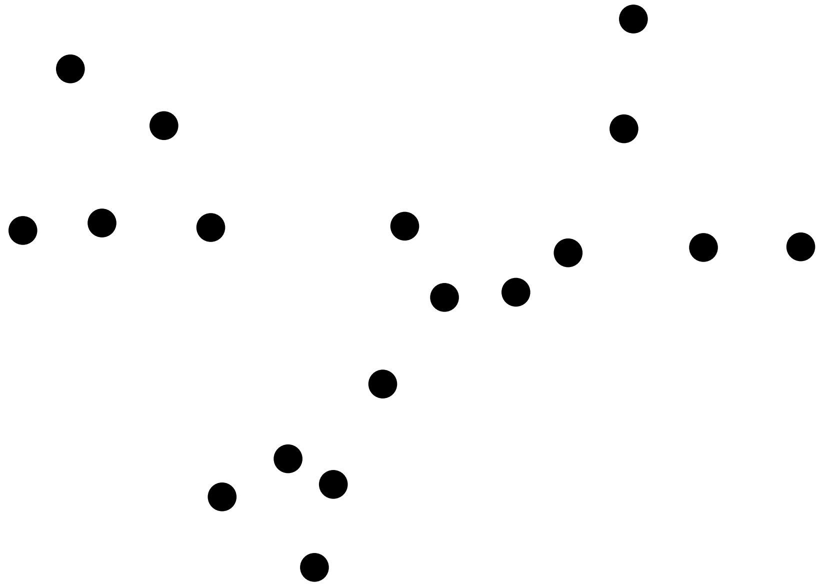
The number of different HITs with at most $2k$ leaves is $O(7^{2k} \cdot 2k!)$, and these HITs can be enumerated in $O(7^{2k} \cdot 2k!)$ time.





An algorithm for MMST$(S, \mathcal{D})$

Algorithm



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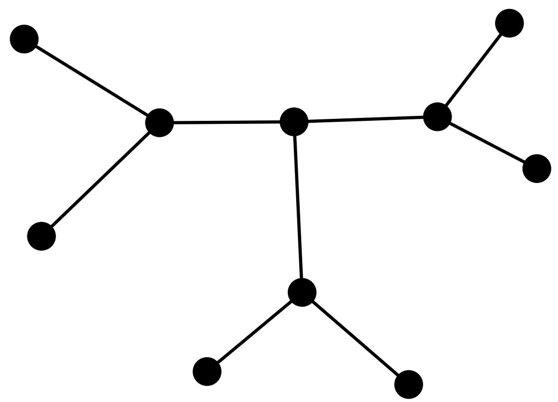
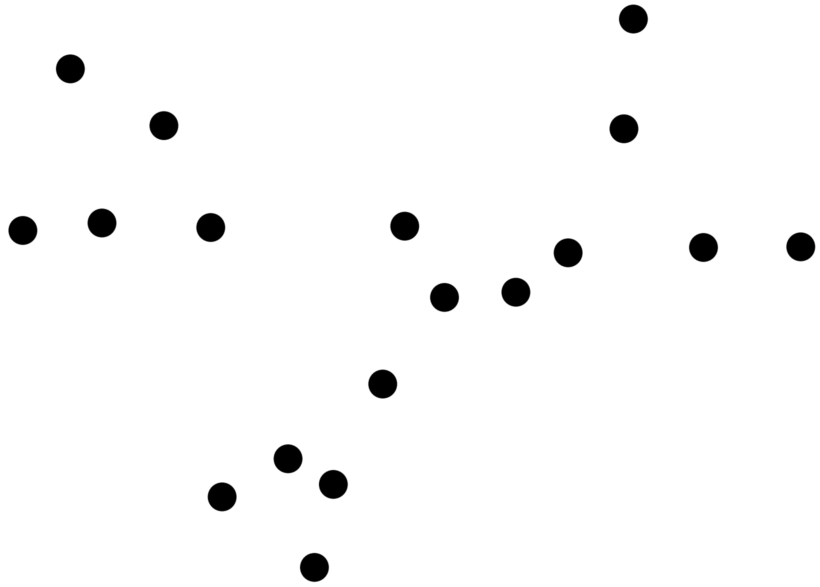
for every HIT H $O(7^{2k} \cdot 2k!)$





An algorithm for MMST$(S, \mathcal{D})$

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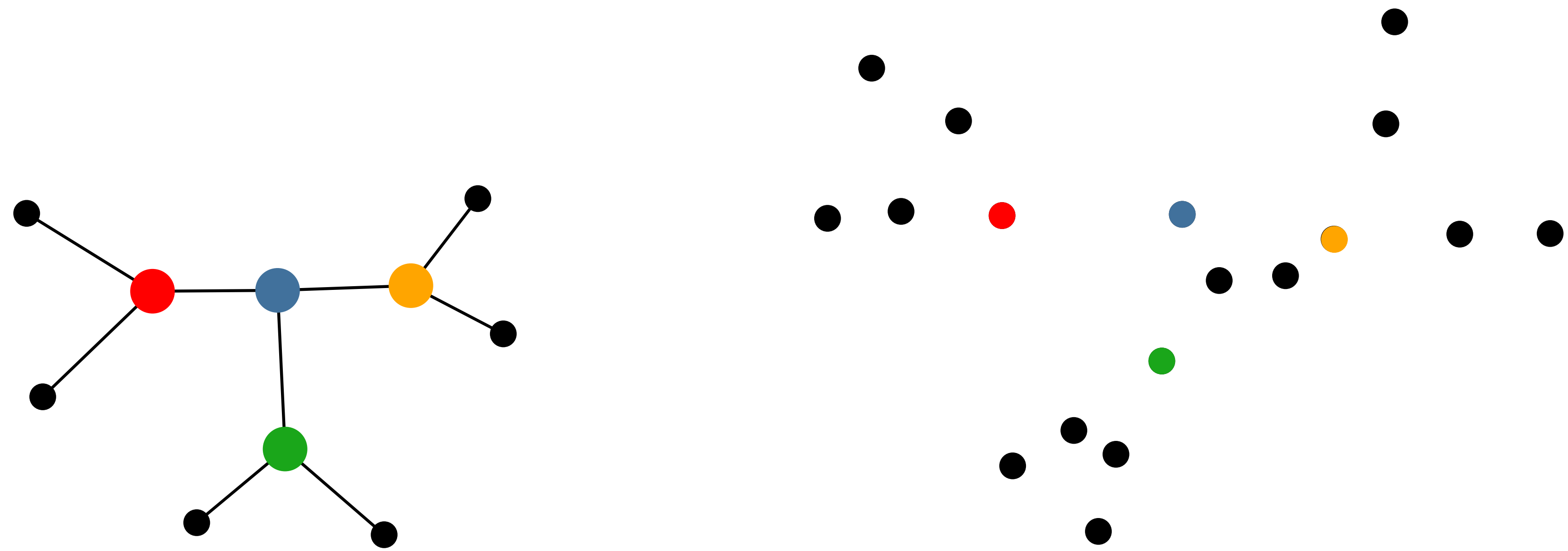
for every HIT H — $O(7^{2k} \cdot 2k!)$

for every mapping M of internal vertices H to points in S — $O(n^{2k-2})$



An algorithm for MMST$(S, \mathcal{D})$

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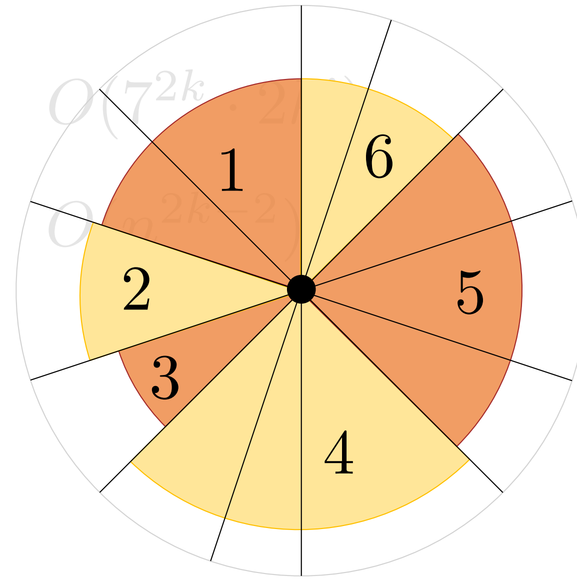

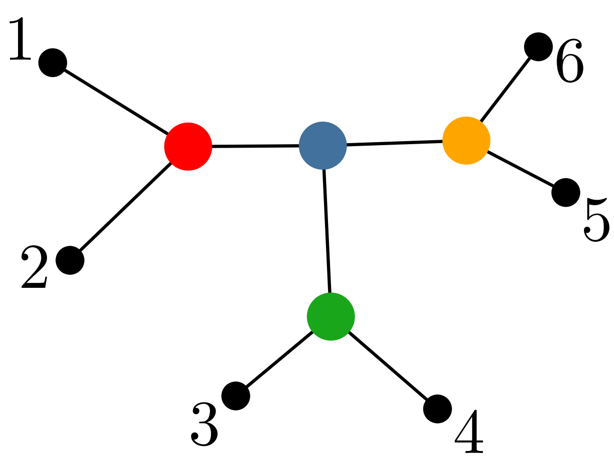

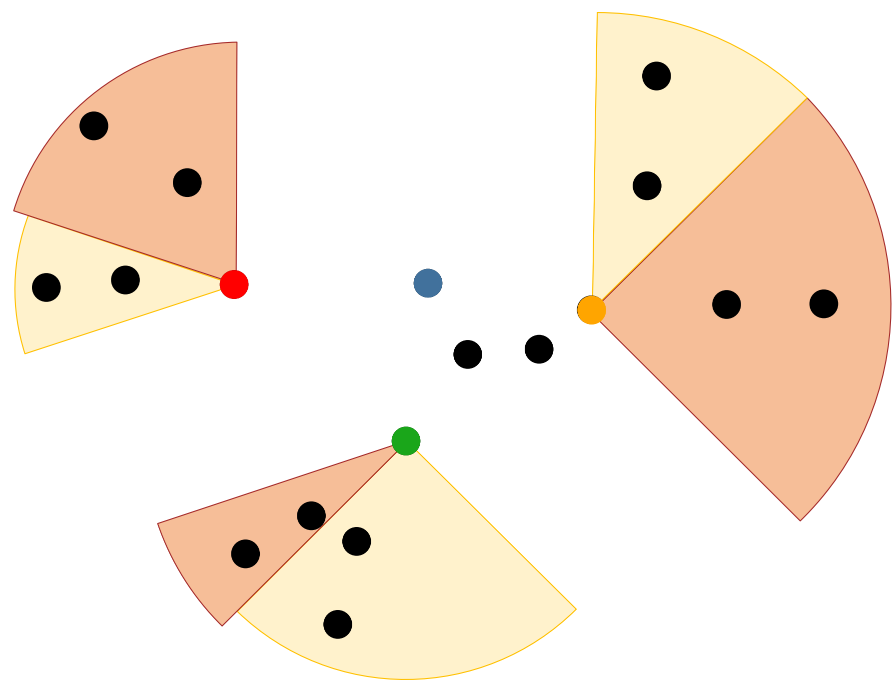

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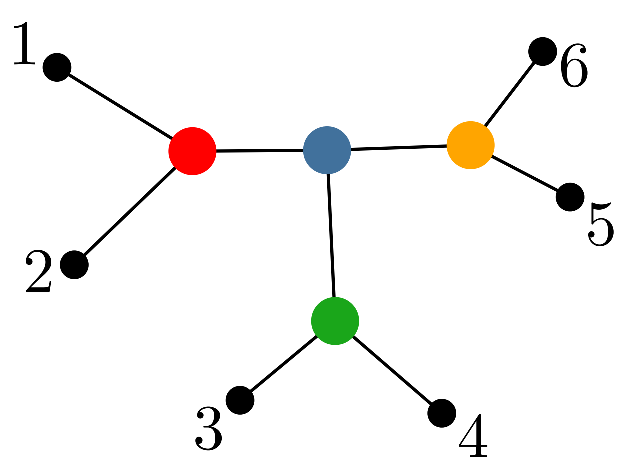

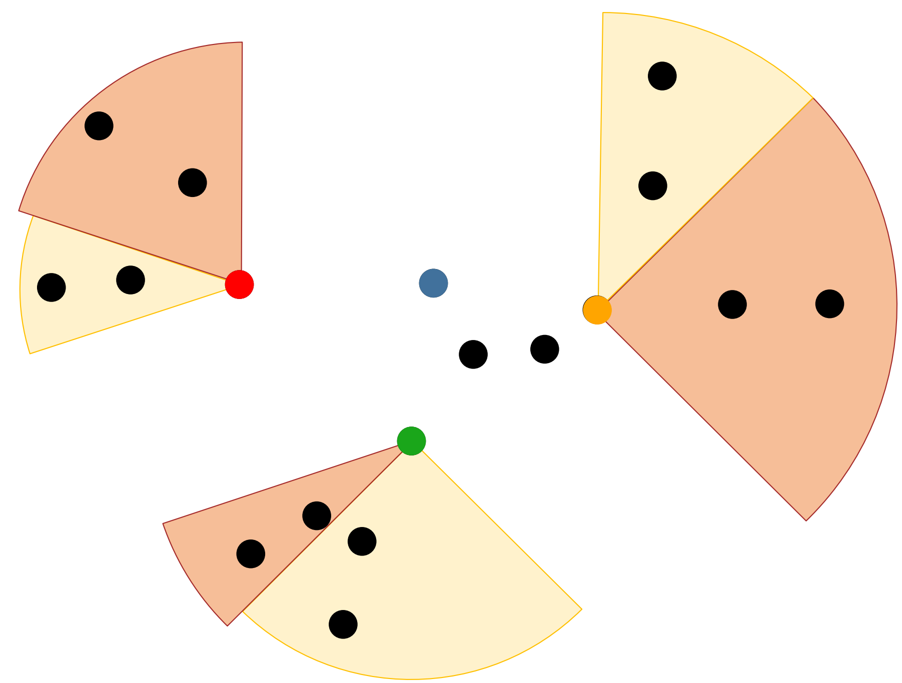

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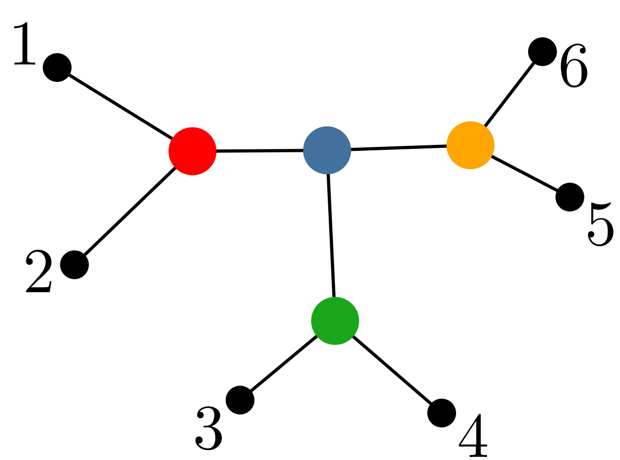

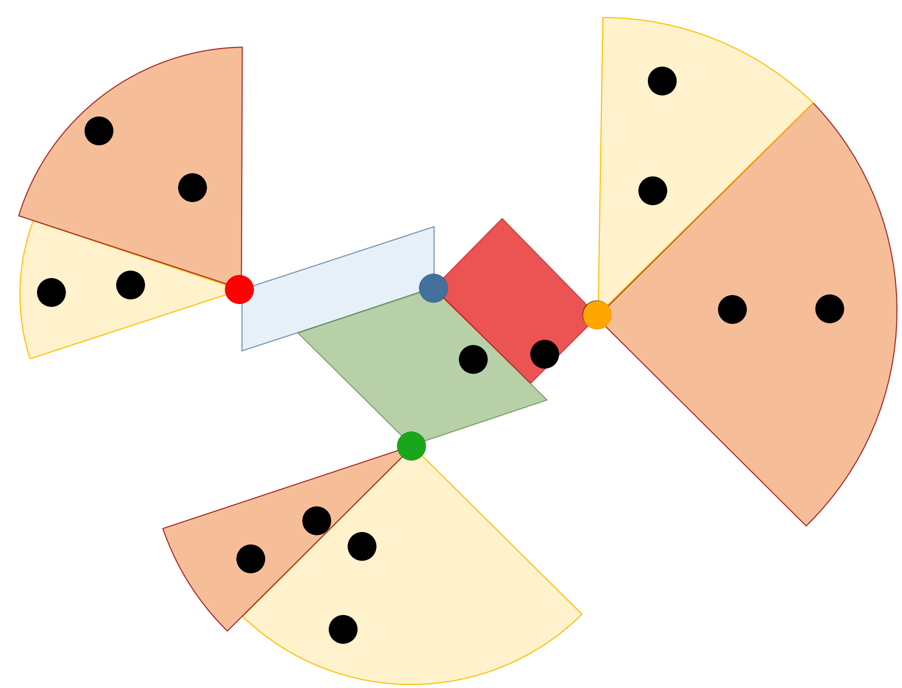

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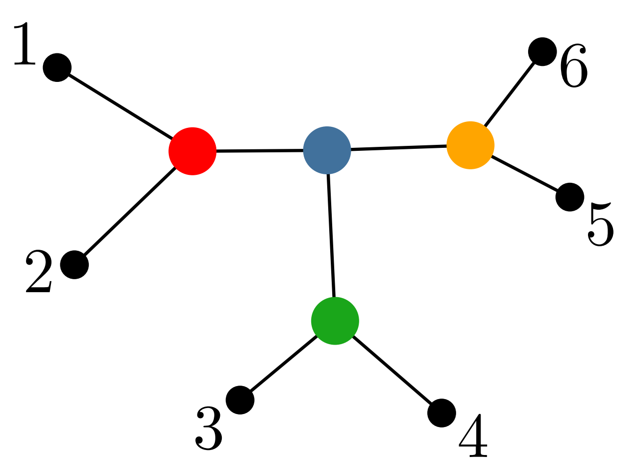

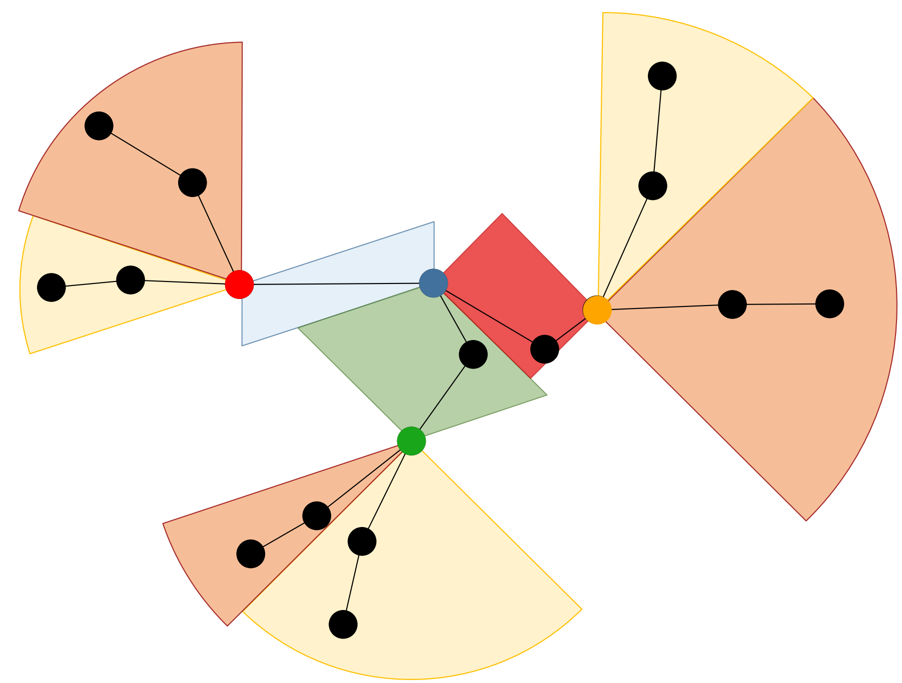

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MMST$(S, \mathcal{D})$ can be solved in $O(f(k) \cdot n^{2k-1} \log n)$

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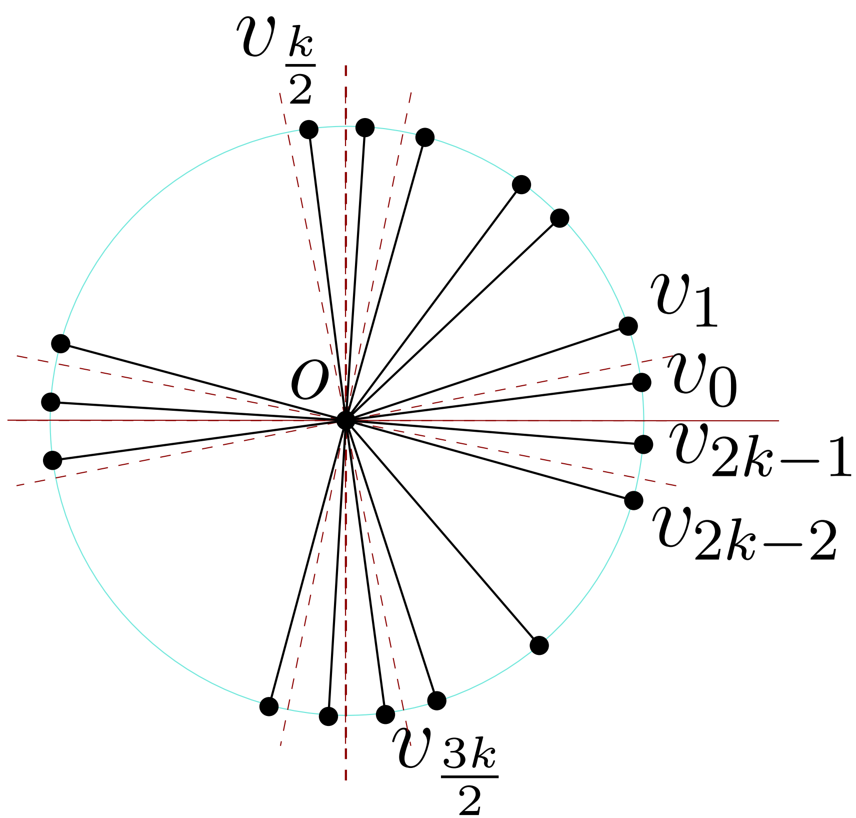

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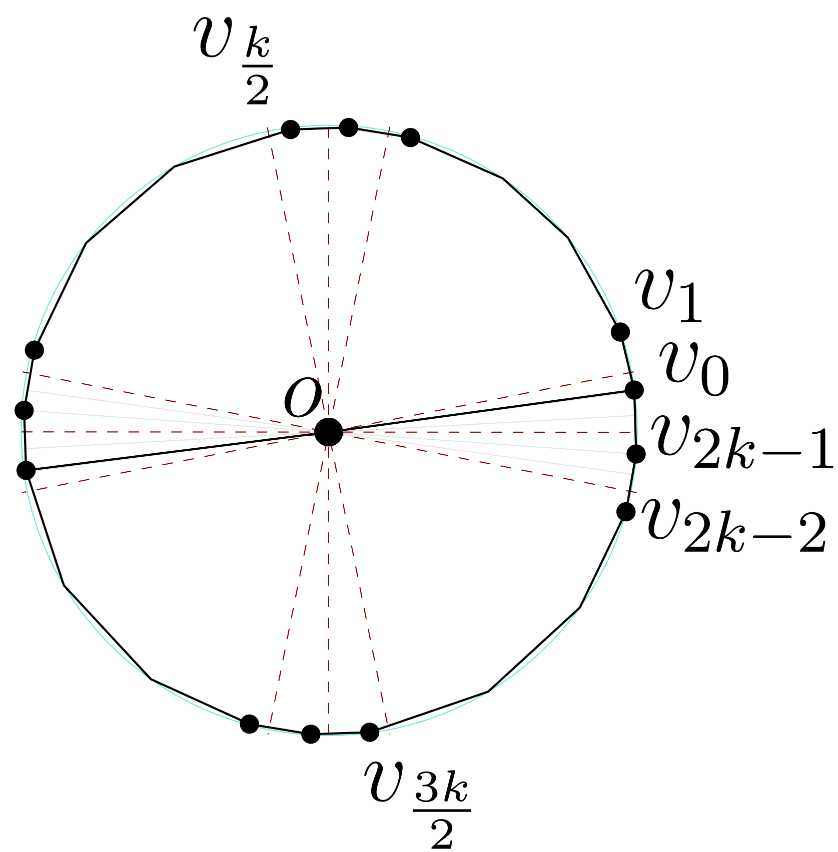

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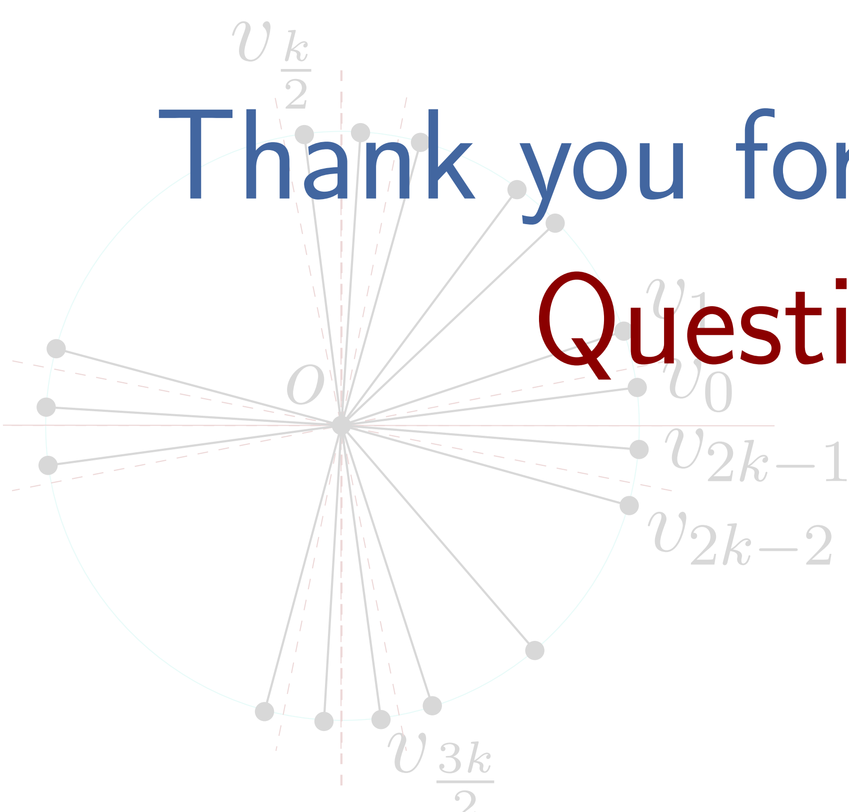

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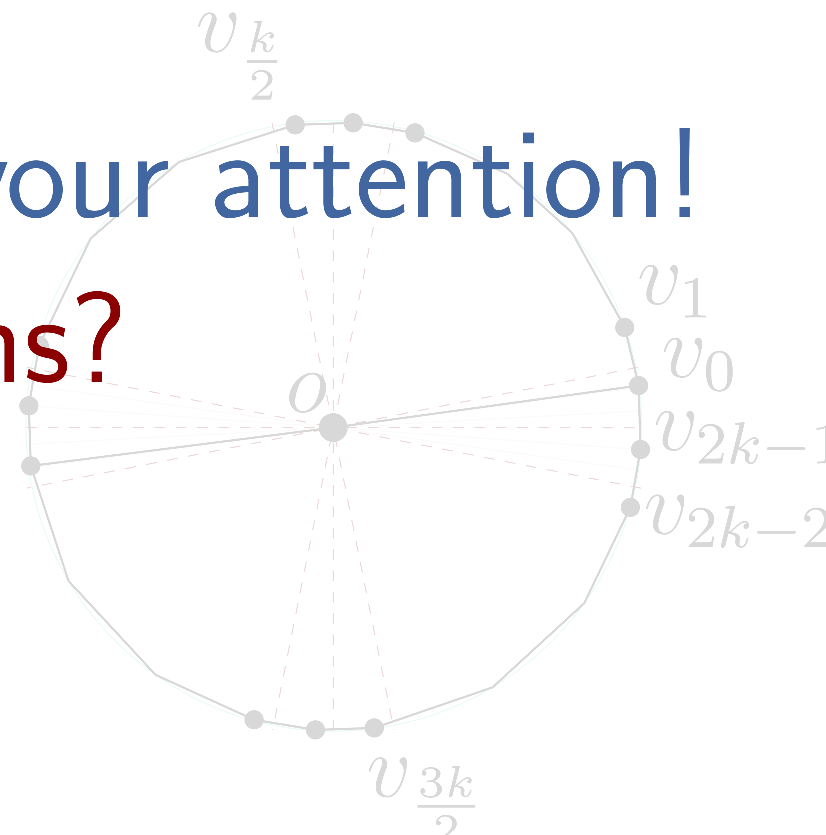

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Thank you for your attention!

Questions?

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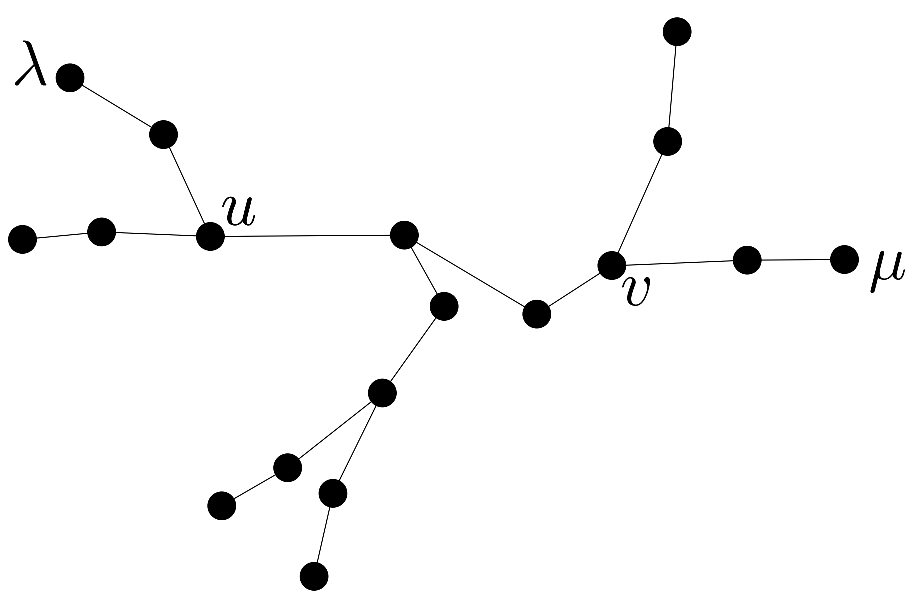

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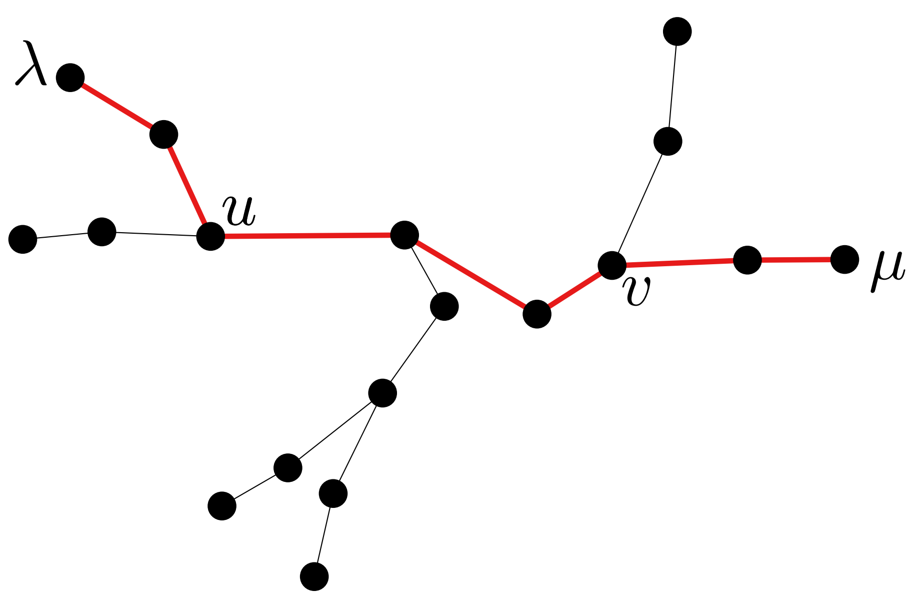

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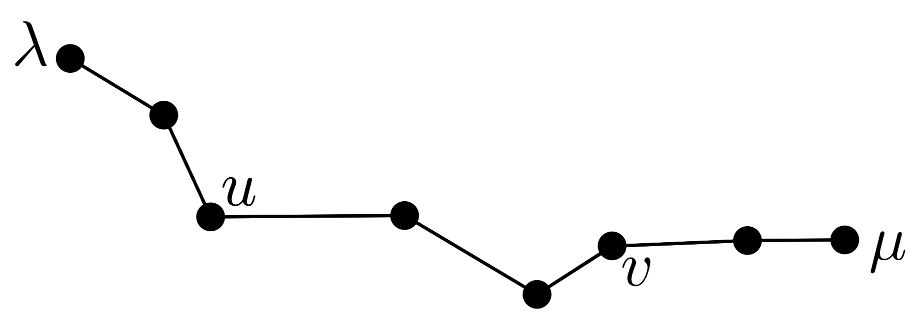

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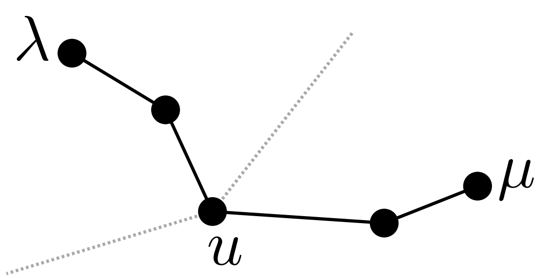

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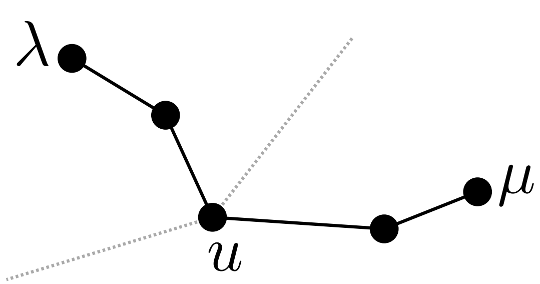

(a) $\Rightarrow$ both leaf paths $P_{u,\lambda}$ and $P_{u,\mu}$ are $\mathcal{D}$-monotone.

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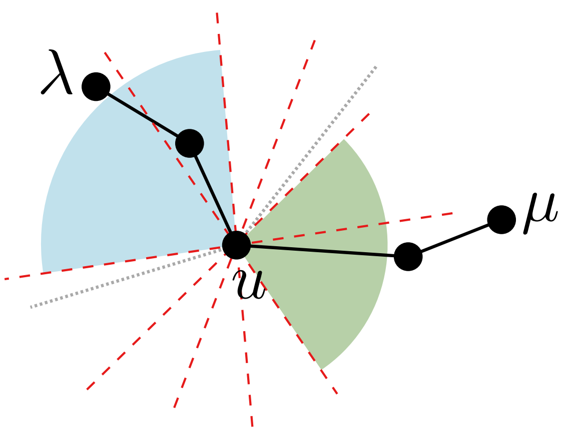

(a) $\Rightarrow$ both leaf paths $P_{u,\lambda}$ and $P_{u,\mu}$ are $\mathcal{D}$-monotone.

$\Rightarrow |\mathcal{W}_{P_{u,\lambda}}| \leq k$ and $|\mathcal{W}_{P_{u,\mu}}| \leq k$.

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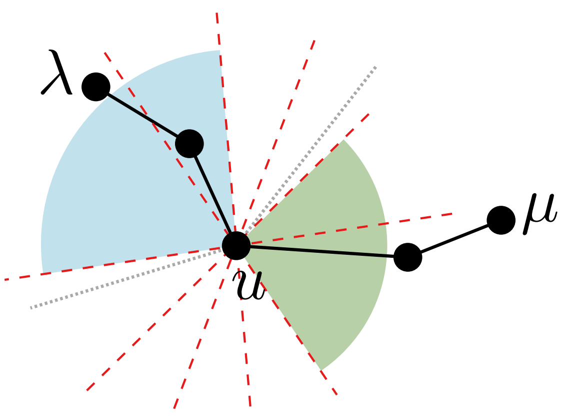

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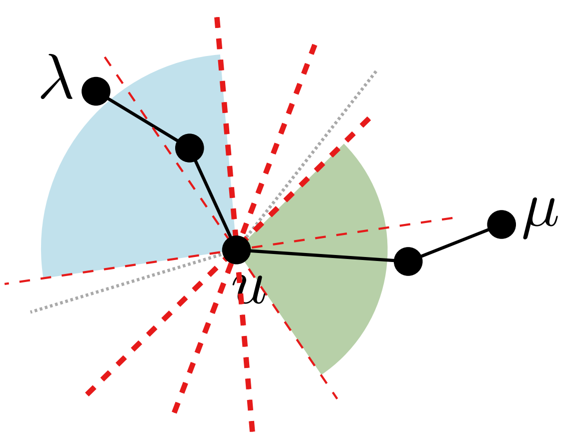

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$\Rightarrow \exists\, d \in \mathcal{D}$ s.t. $\overline{d}(u)$ separates $\mathcal{W}_{P_{u,\lambda}}(u)$ and $\mathcal{W}_{P_{u,\mu}}(u)$ and does not intersect the interior of either of them.

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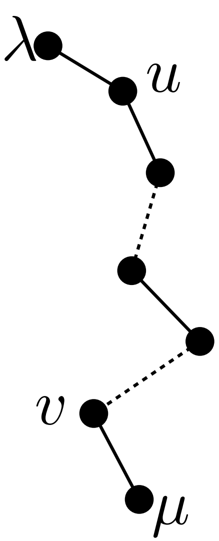

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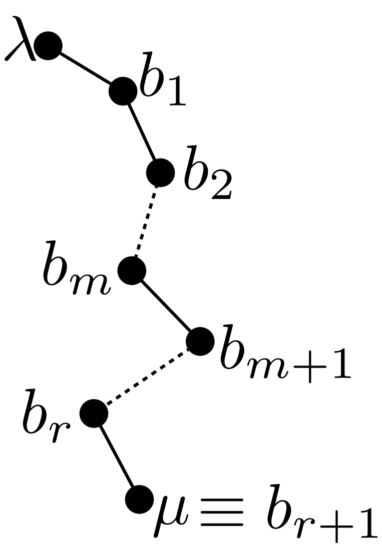

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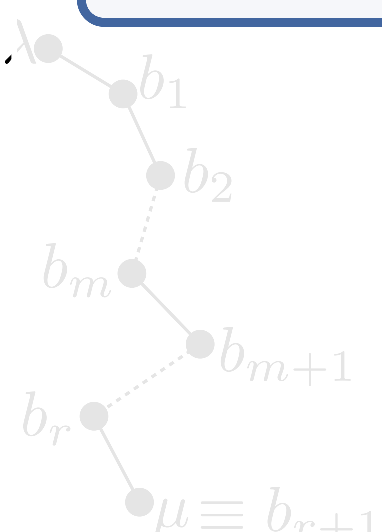
Corollary 6

Let $\mathcal{D}$ be a set of k (pairwise non-opposite) directions, and let P be a directed geometric path. Given a direction $d \in \mathcal{D}$, P is d-monotone if and only if $\bar{d}(o)$ does not intersect the interior of $\mathcal{W}_P$, where o is the origin.



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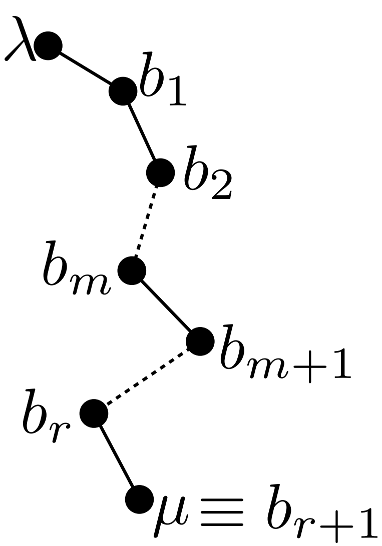

It suffices to show that there is a direction d such that line $\overline{d}(\mu)$ does not intersect the interior of $\mathcal{W}_{P_{\mu,\lambda}}(\mu) \Rightarrow |\mathcal{W}_{P_{\mu,\lambda}}| \leq k$.

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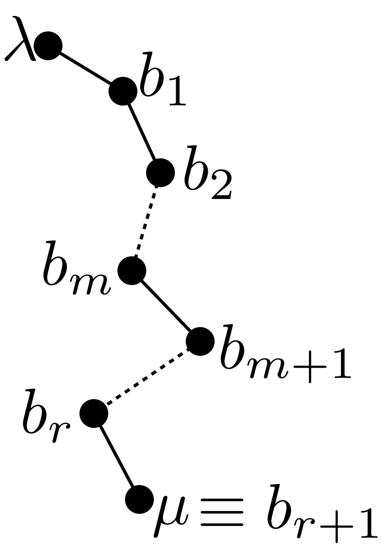

It suffices to show that there is a direction d such that line $\overline{d}(\mu)$ does not intersect the interior of $\mathcal{W}_{P_{\mu,\lambda}}(\mu) \Rightarrow |\mathcal{W}_{P_{\mu,\lambda}}| \leq k$.

- Let $\mathcal{P}_i$ be the path from b_i to λ
- We show by induction on the number of the branching vertices that $|\mathcal{W}_{\mathcal{P}_i}| \leq k$
- $\mathcal{P}_{r+1}$ is by definition the oriented path from μ to λ

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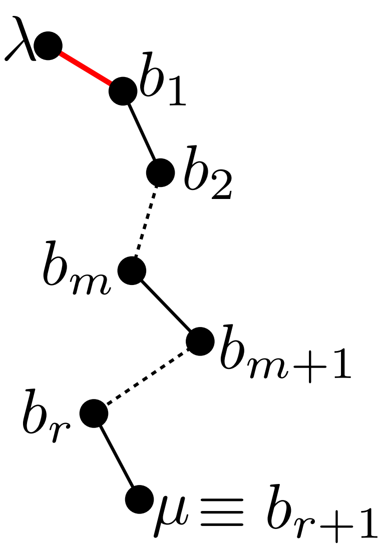

Base case: $\mathcal{P}_1$ is the path from $b_1 \equiv u$ to λ

$\Rightarrow \mathcal{D}$-monotone from (a)

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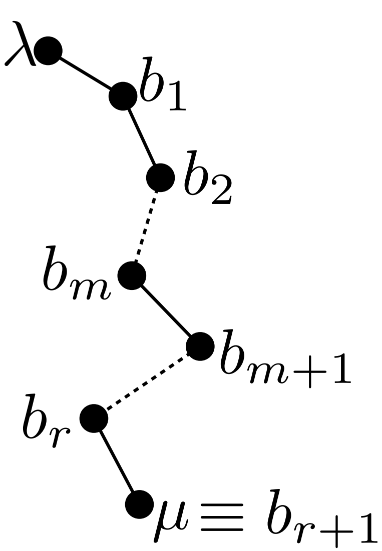

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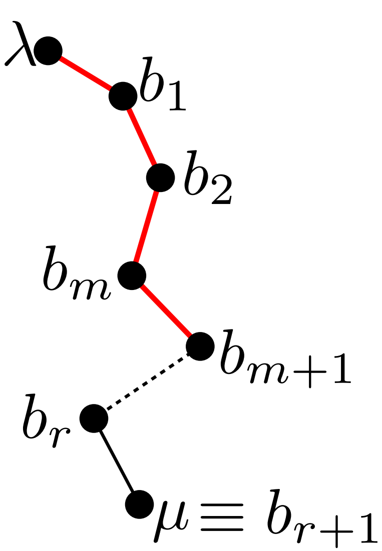

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We show that $|\mathcal{W}_{\mathcal{P}_{m+1}}| \leq k$.

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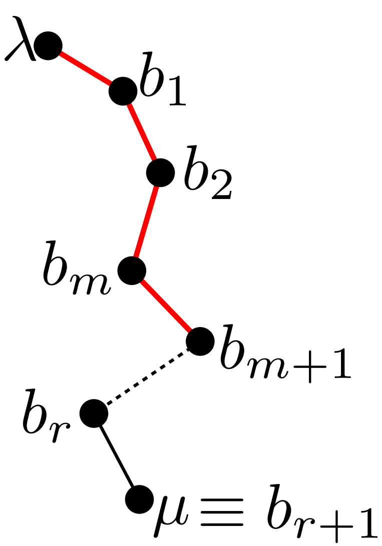
Assume, for a contradiction, that $|\mathcal{W}_{\mathcal{P}_{m+1}}| > k$.



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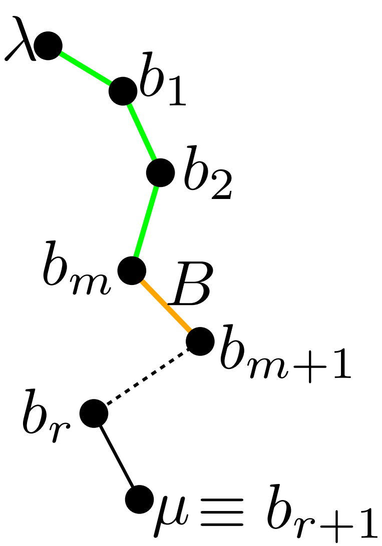

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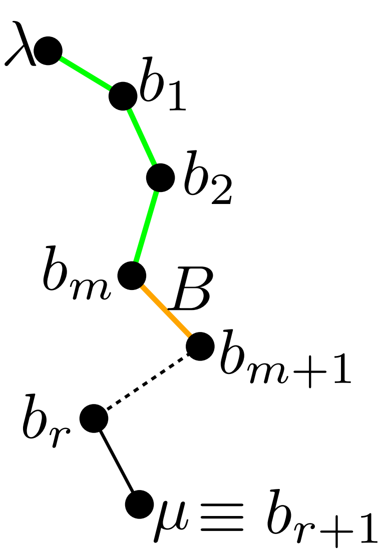

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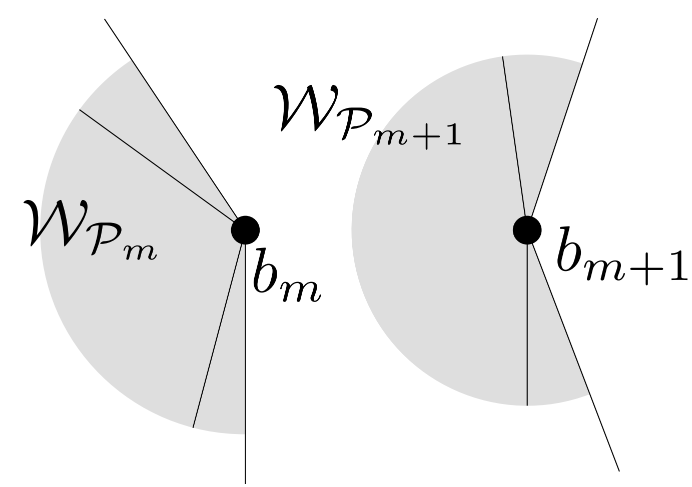

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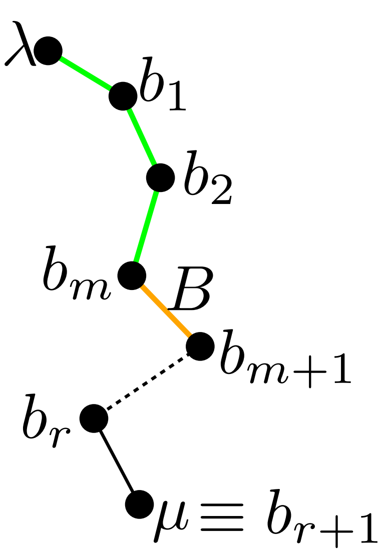

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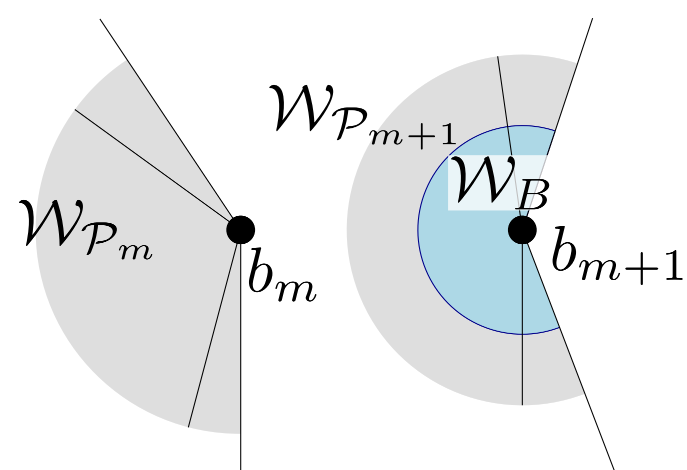

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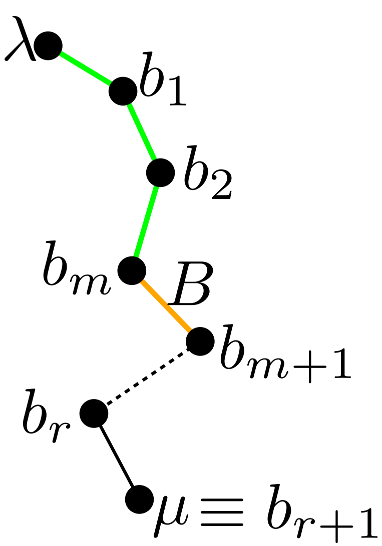

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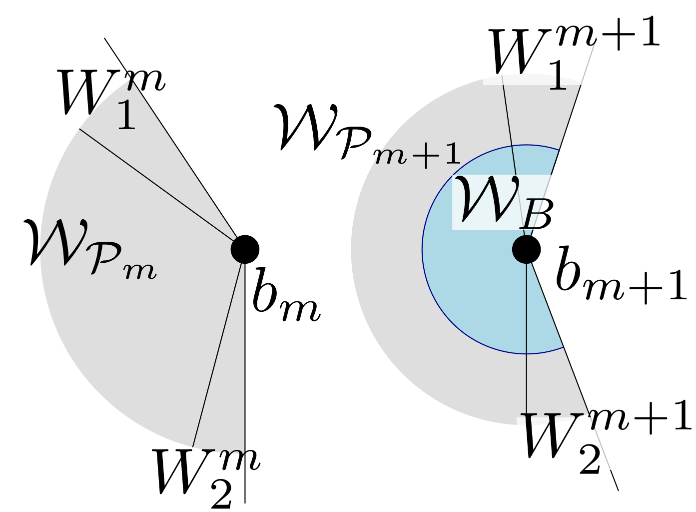

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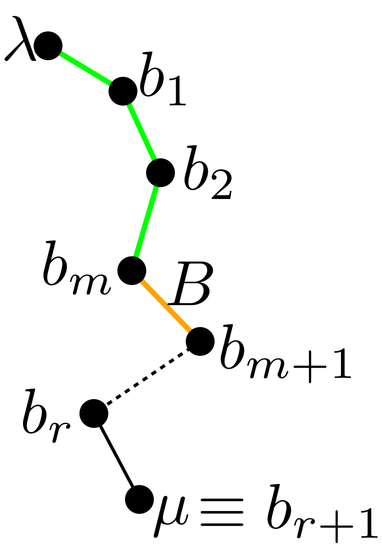

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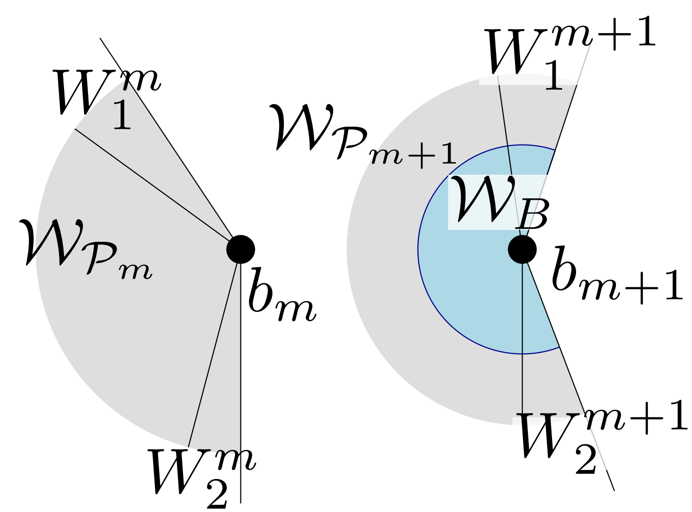
$|\mathcal{W}_B| > k$

A contradiction due to (a)!



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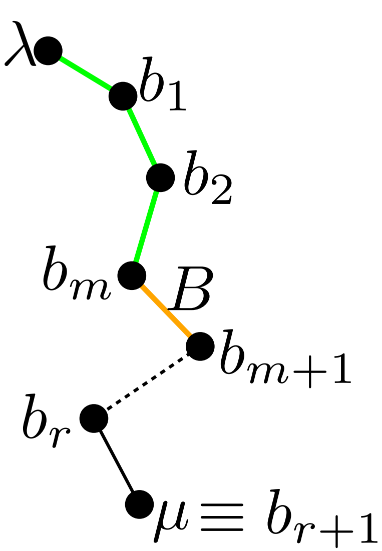

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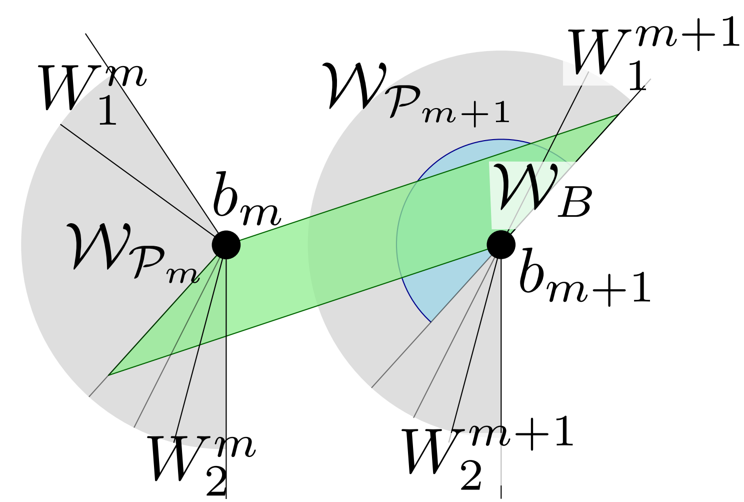

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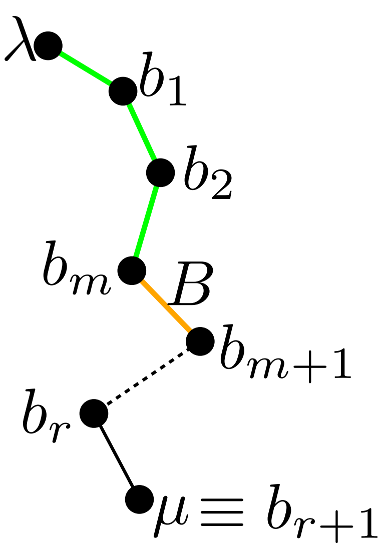

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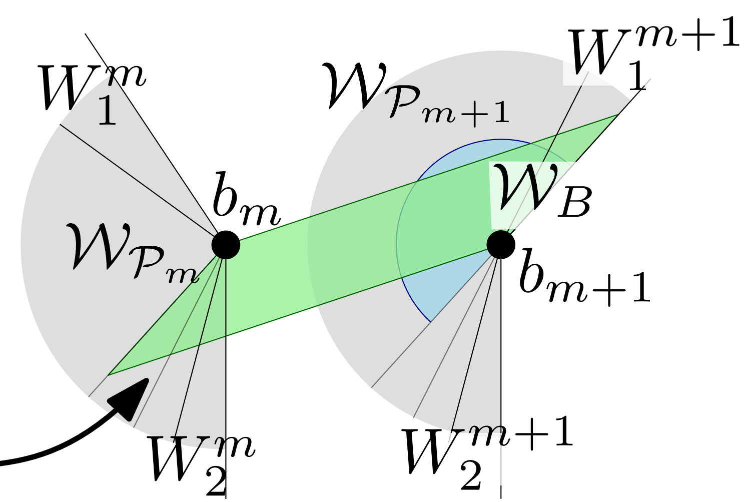
The region of the branch intersects $\mathcal{W}_{\mathcal{P}_m}$

A contradiction due to (c)!



Maximum degree of $\mathcal{D}$-Monotone Spanning Trees

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Theorem

In contrast to the MST, whose vertex degree is at most six, for every even integer $k \geq 2$, there exists a point set S_k and a set $\mathcal{D}$ of k directions such that any minimum-length $\mathcal{D}$-monotone spanning tree of S_k has maximum vertex degree $2k$.

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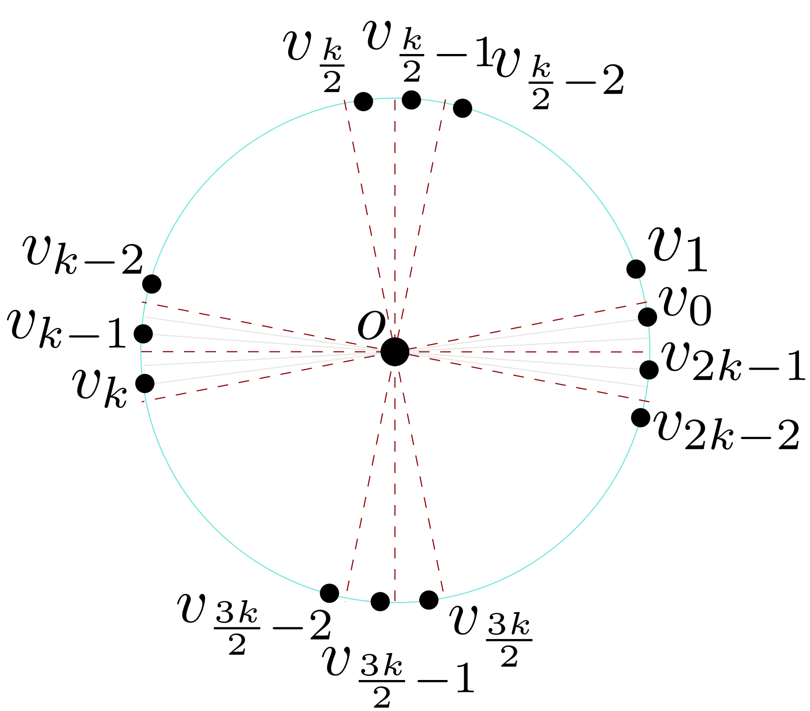

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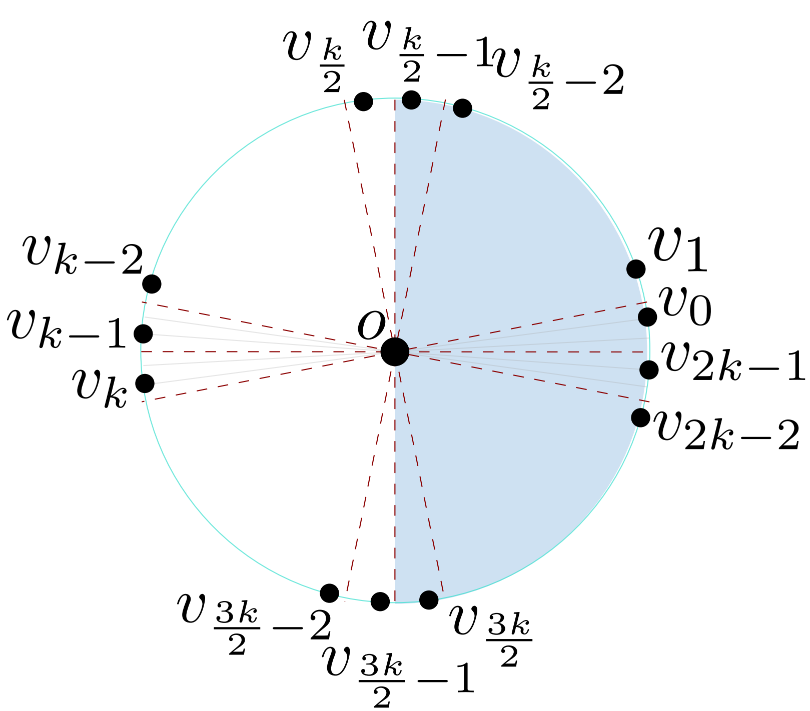

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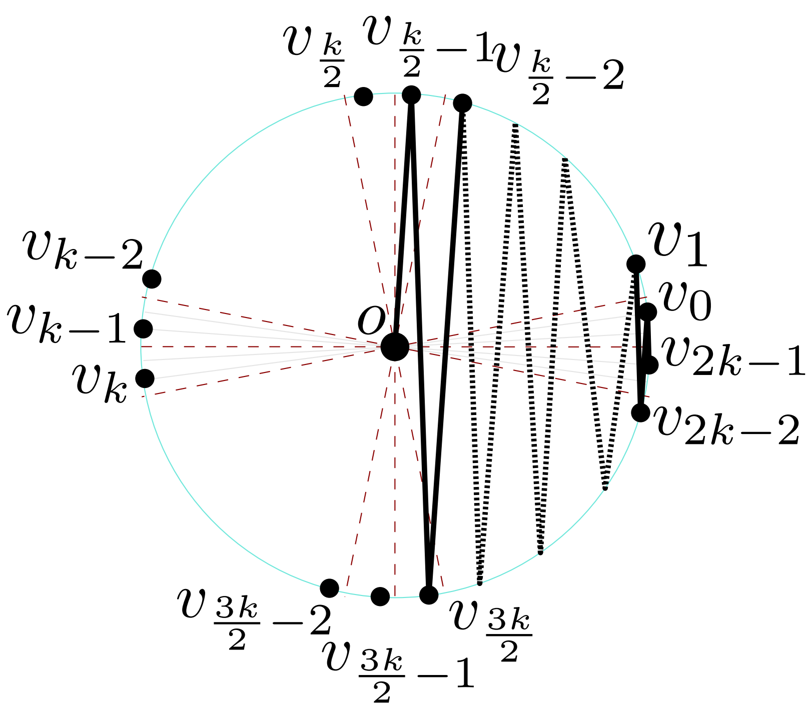

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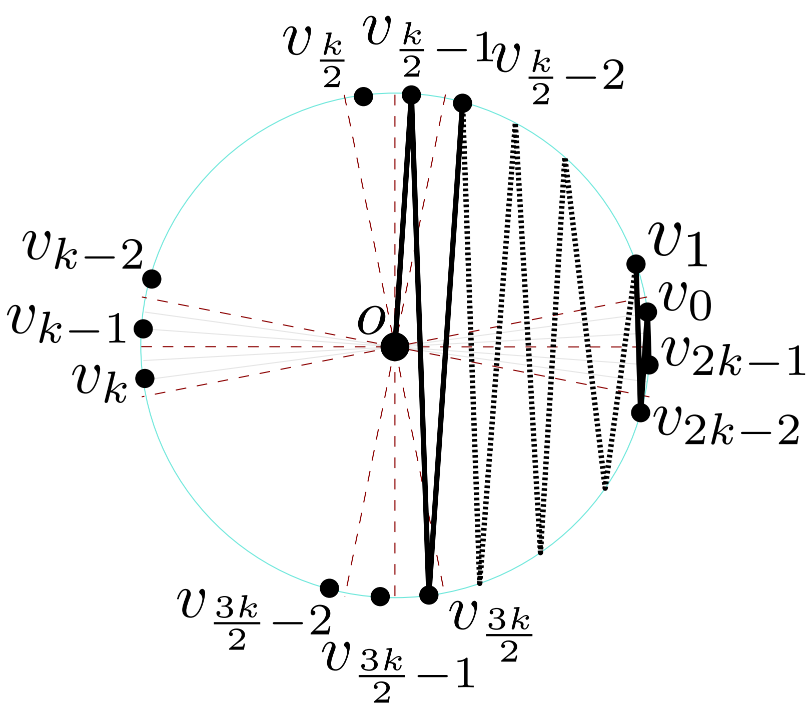

length: $\cot(\frac{\pi}{4k})$

Maximum degree of $\mathcal{D}$-Monotone Spanning Trees

Theorem

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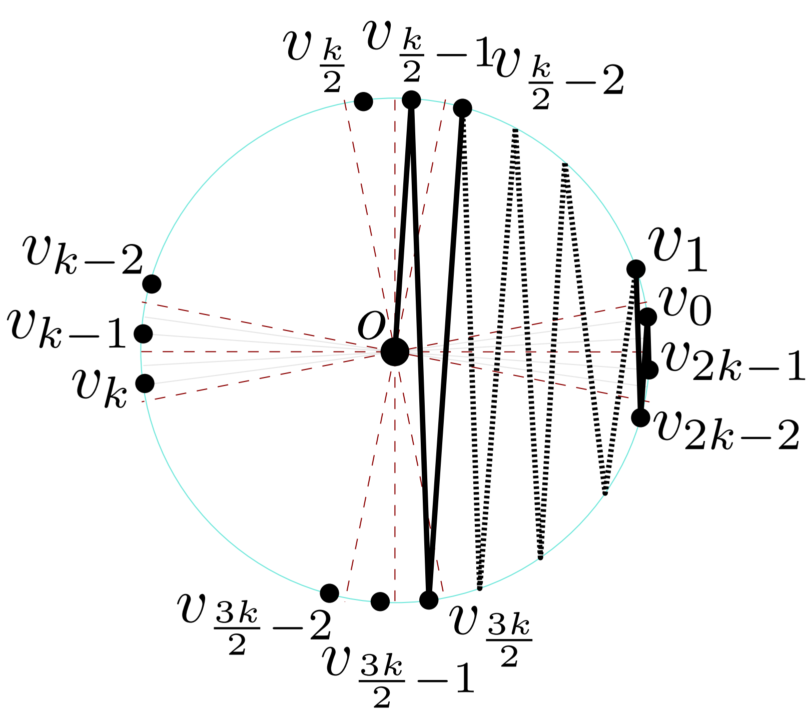
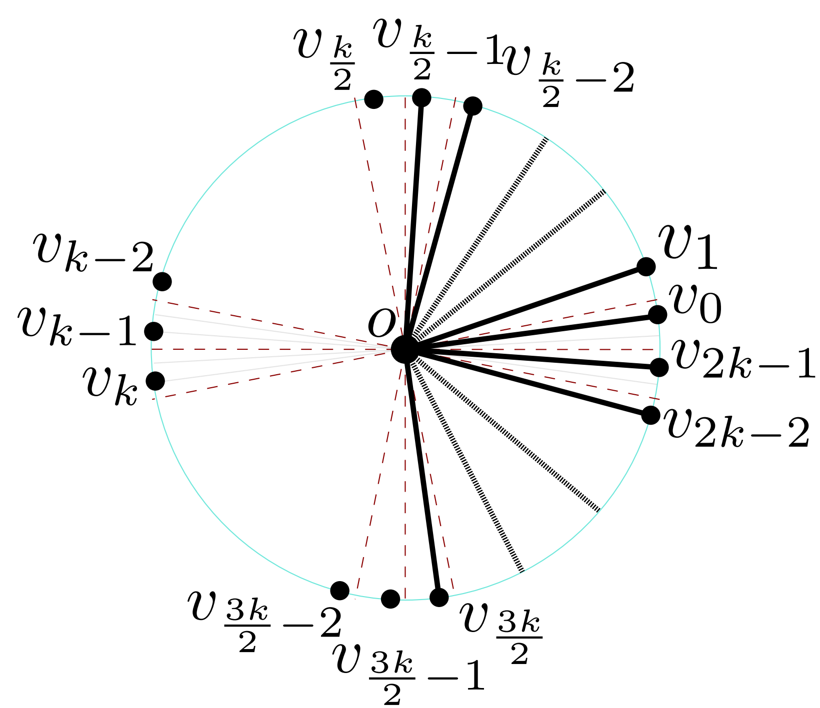


length: $\cot(\frac{\pi}{4k})$



length: k

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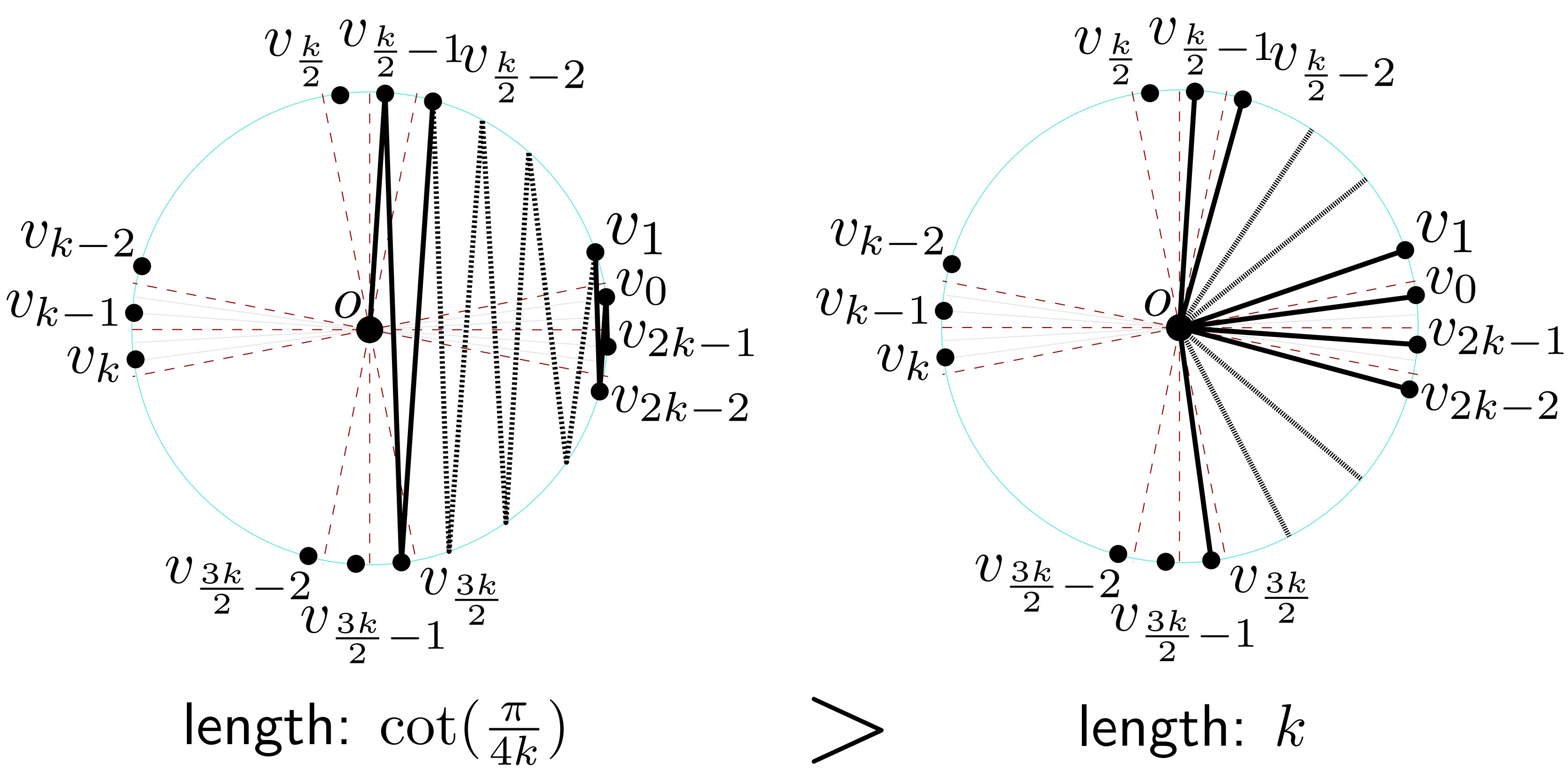

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