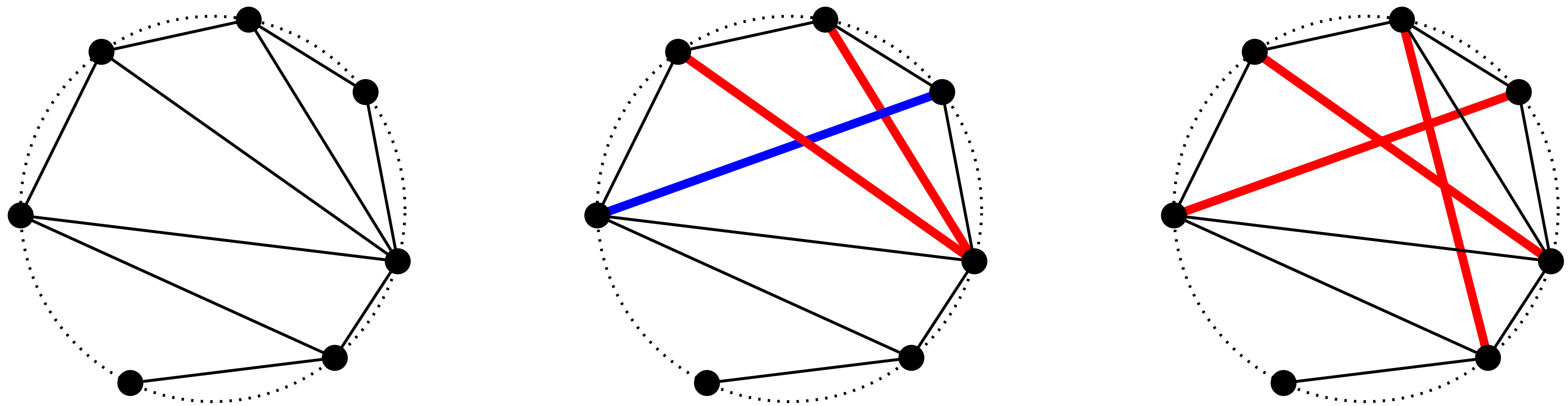
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Chair for INFORMATICS I Efficient Algorithms and Knowledge-Based Systems

Institute for Informatics

Beyond Outerplanarity



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Generalizing Planarity – “nice” crossings

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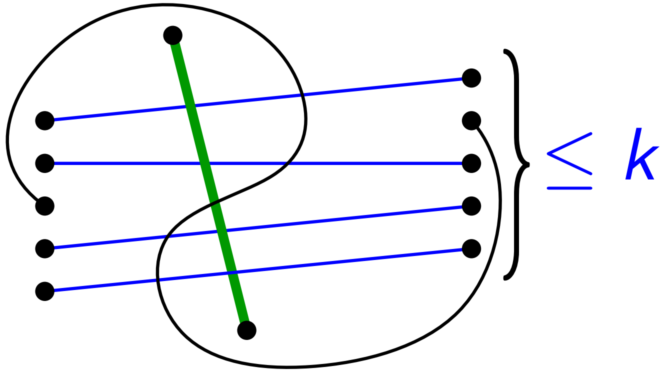

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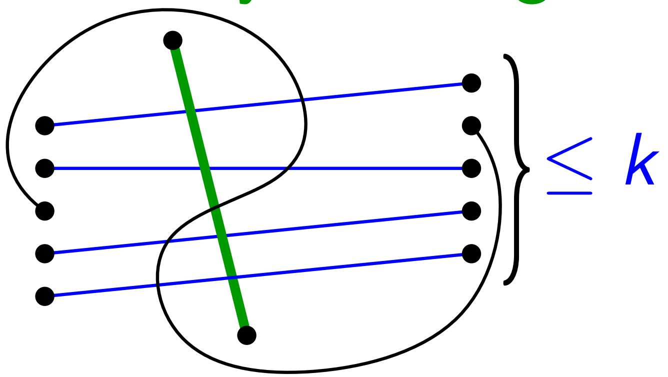

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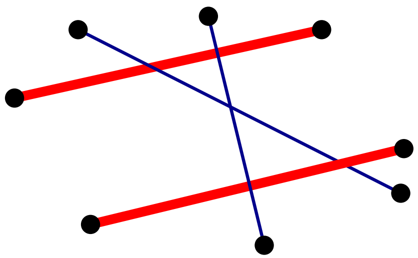

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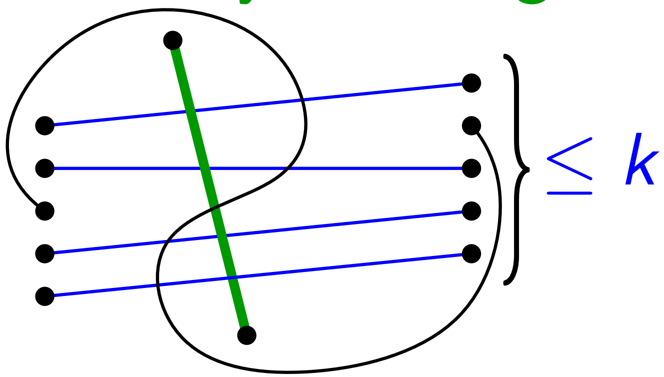

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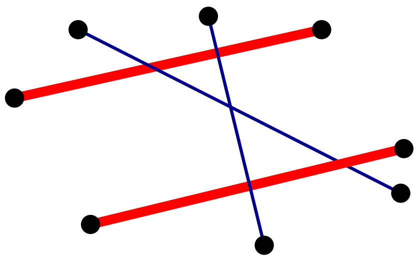

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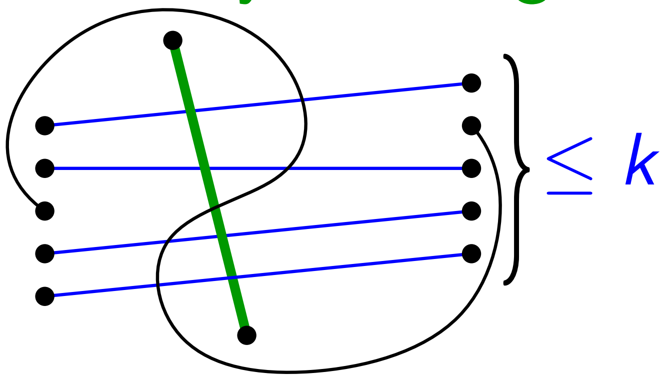

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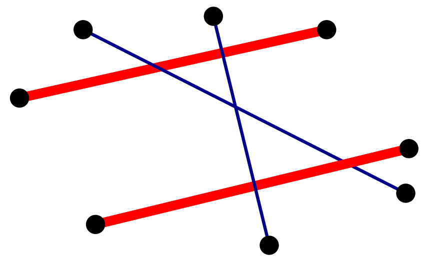

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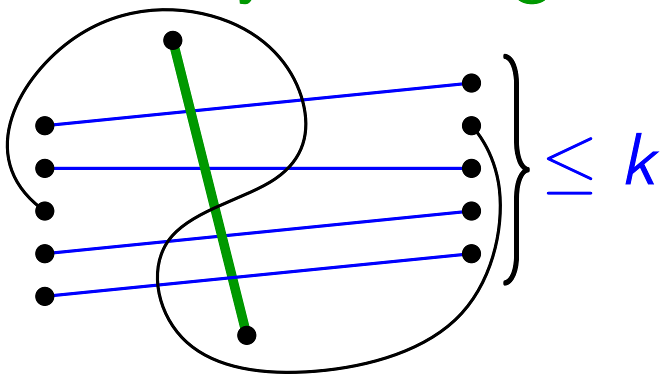

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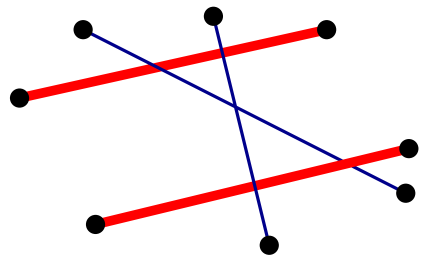

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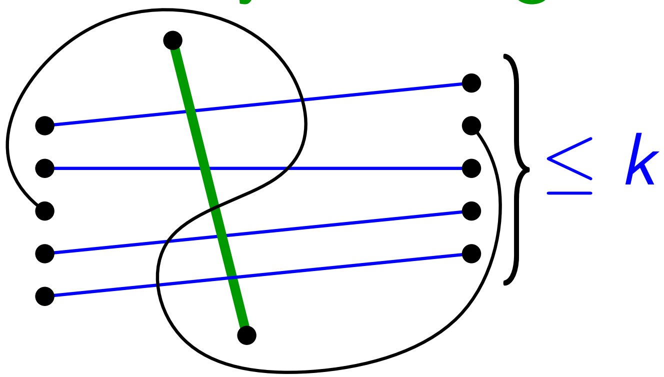

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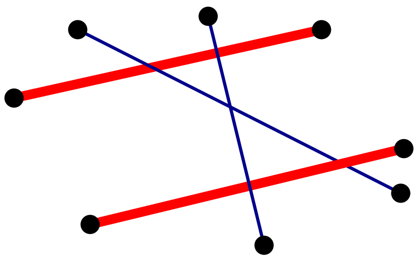

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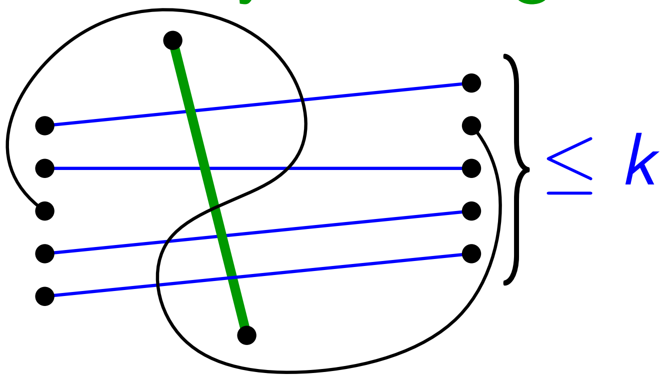

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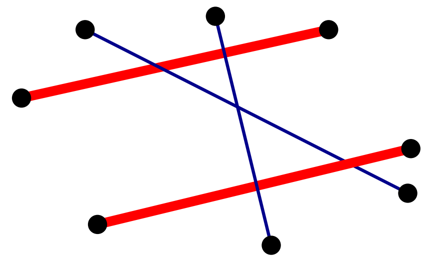

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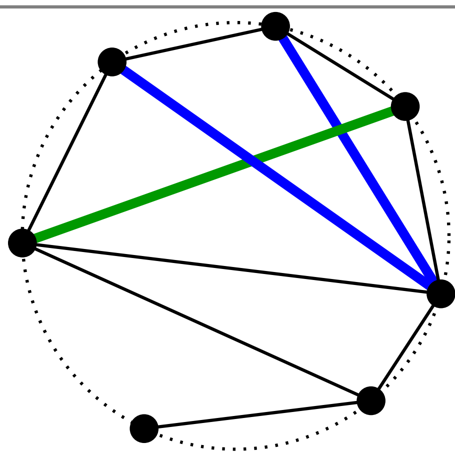
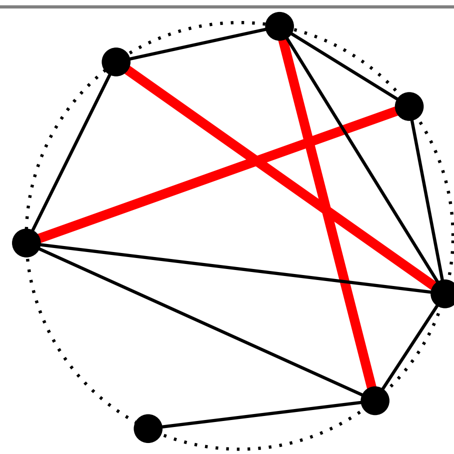
outer k-planarity





outer k-quasi-planarity

Concepts/Problems

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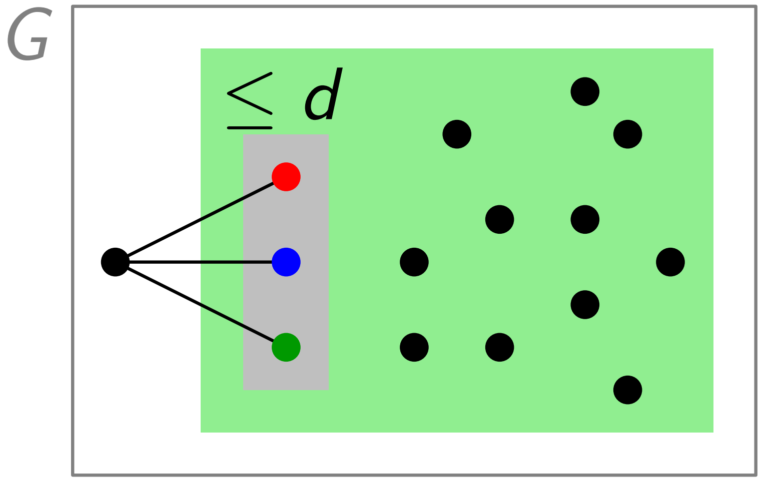

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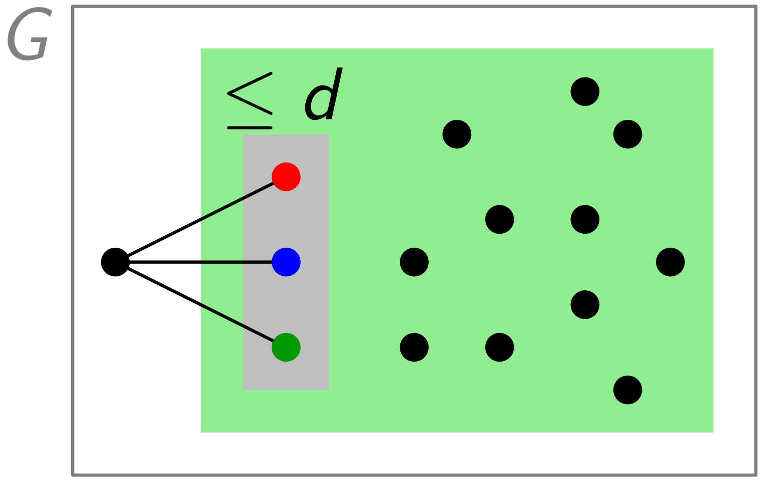

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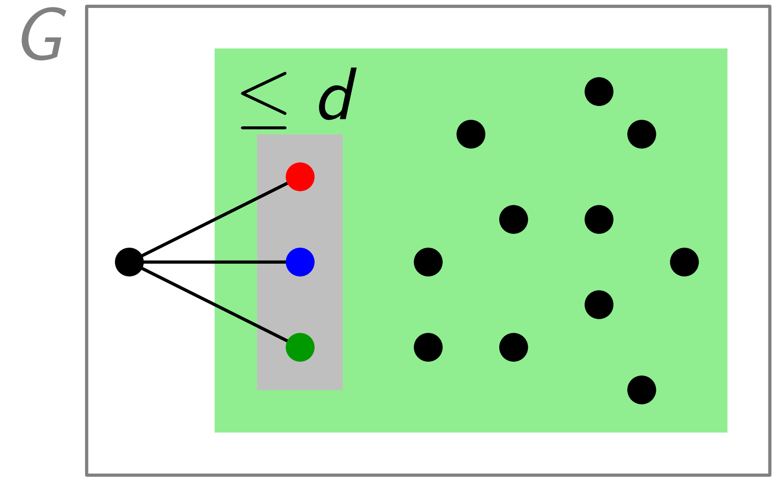

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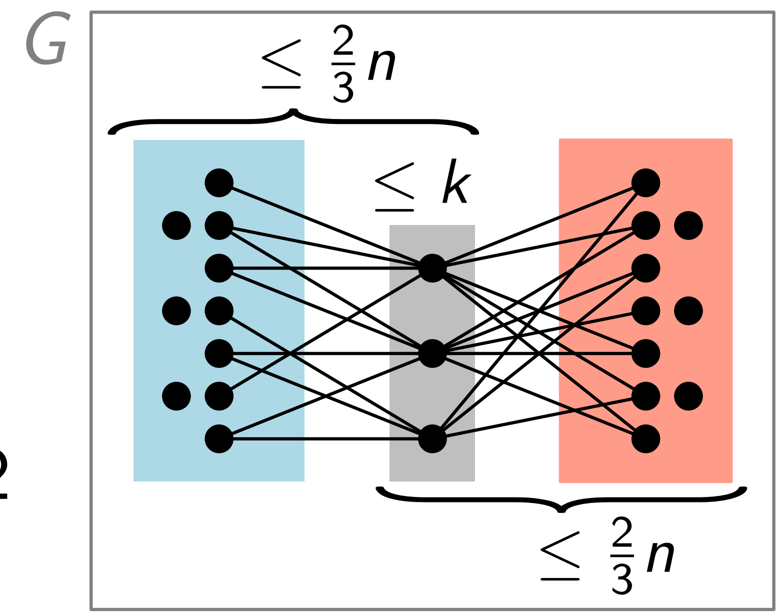

Separation Number (sn): a graph class has sn $\leq k$ when every graph G in it has a balanced separator of size $\leq k$.

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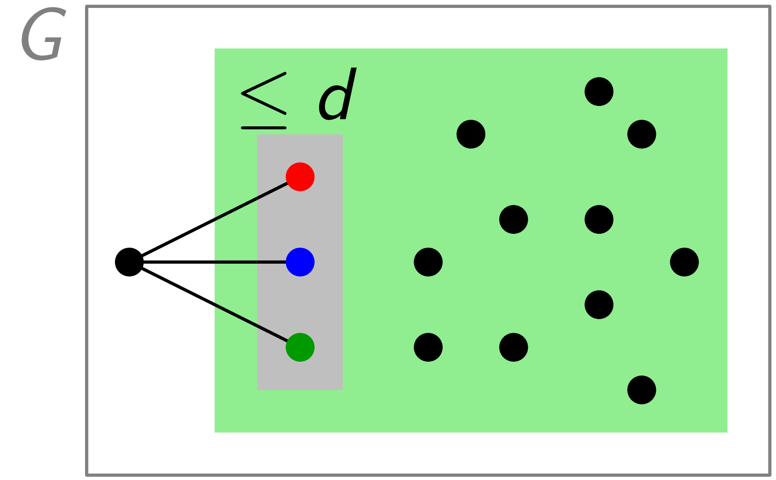

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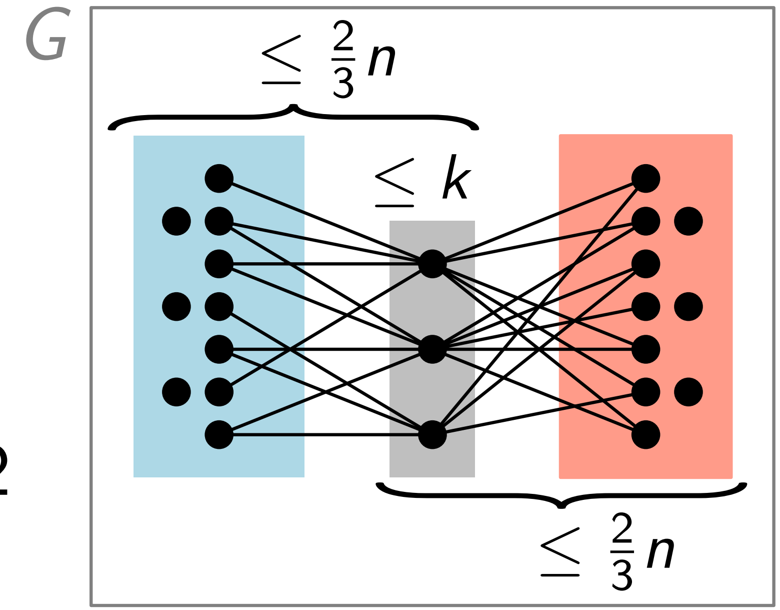

Recognition: Testing for membership in a graph class.
both planarity and outerplanarity can be tested in linear time.

Background : General Drawings

k-planar graphs – introduced by Ringel '65.

- Edge density: $4.108n\sqrt{k}$ [Pach, Tóth '97]
  $\rightarrow 8.216\sqrt{k}$-degenerate (via avg. degree)
- $O(\sqrt{kn})$ treewidth [Dujmović, Eppstein, Wood '17]
  $\rightarrow$ sn $\in O(\sqrt{kn})$
- 1-planarity testing is NP-hard [Grigoriev, Bodlaender '07]

k-quasi-planar graphs

- Edge density: $(n \log n)2^{\alpha(n)^{c_k}}$ [Fox, Pach, Suk '13]
  Conjectured to be $c_k n$ [Pach et al '96]

Comparing Classes:

- k-planar $\subset$ $(k+1)$-quasi-planar:
  $k > 2$ [Angelini et al '17], $k = 2$ [Hoffmann, Tóth '17]

Background : Outer Drawings

Outer k-crossing ($\leq k$ crossings in the whole drawing)

- $O(\sqrt{k})$ treewidth $\rightarrow$ sn $\in O(\sqrt{k})$
- Ext. Monadic Second Order Logic (MSO_2) formula for outer k-crossing

  $\rightarrow$ testing outer k-crossing in time $O(f(k)(n+m))$

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- treewidth $\leq 3k+11 \rightarrow$ sn $\leq 3k+12$ [Wood, Telle '07]
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  outer 1-planar in linear time [Auer et al '16, Hong et al '15]
  full outer 2-planar in linear time [Hong, Nagamochi '16]

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Outer k-quasi-planarity

- Edge density: $\leq 2(k-1)n - \binom{2k-1}{2}$ [Capoyleas, Pach '92]
  $\rightarrow$ $(4k-5)$-degenerate

Results

Outer k-planar graphs

- $(\lfloor\sqrt{4k+1}\rfloor+1)$-degenerate $\rightarrow$ $(\lfloor\sqrt{4k+1}\rfloor+2)$-colorable
- separation number $\leq 2k+3$ $\rightarrow$ quasi-poly time recognition

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Closed Drawings in MSO_2

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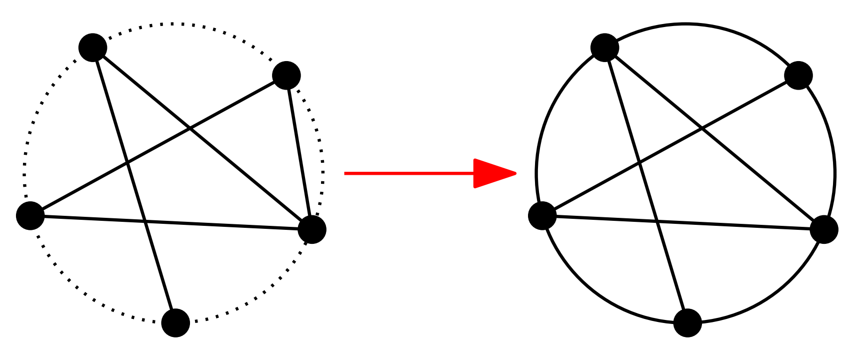

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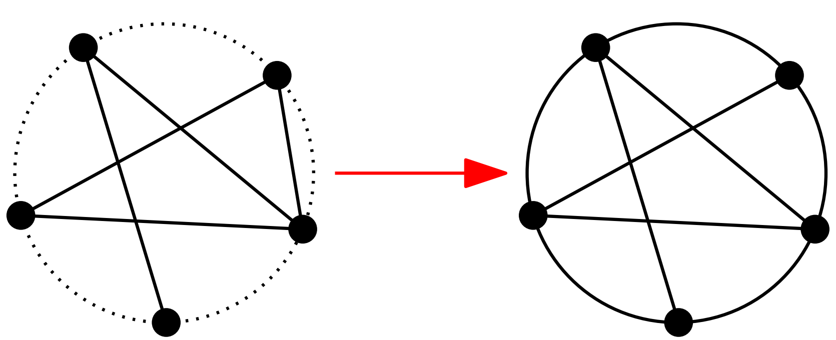

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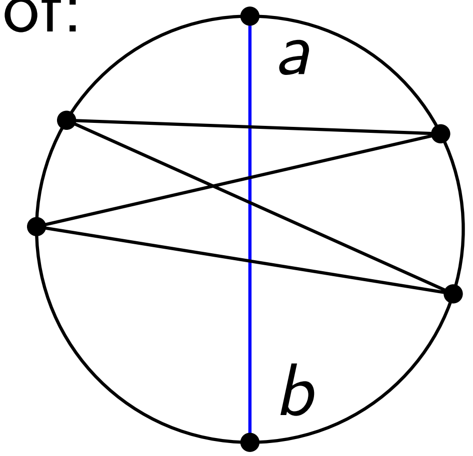

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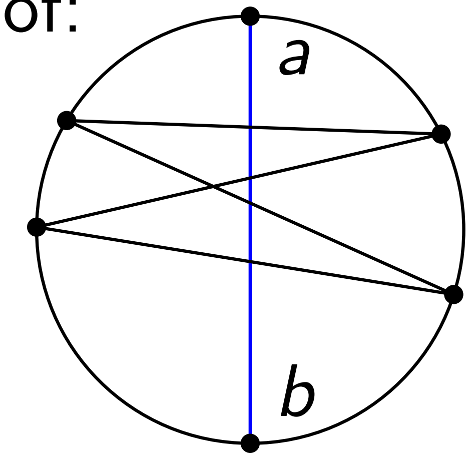

- a complete bipartite graph crosses ab.
- thus, for even n, $k \geq (\frac{n-2}{2})^2$, and
  for odd n, $k \geq \frac{1}{4}(n-3)(n-1)$

$\rightarrow n \leq \lfloor \sqrt{4k+1} \rfloor + 2$

∎

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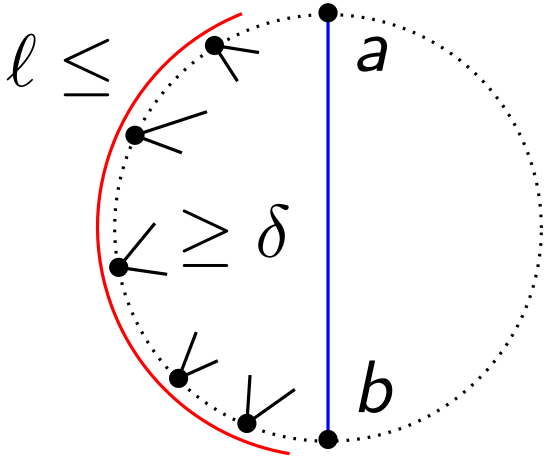

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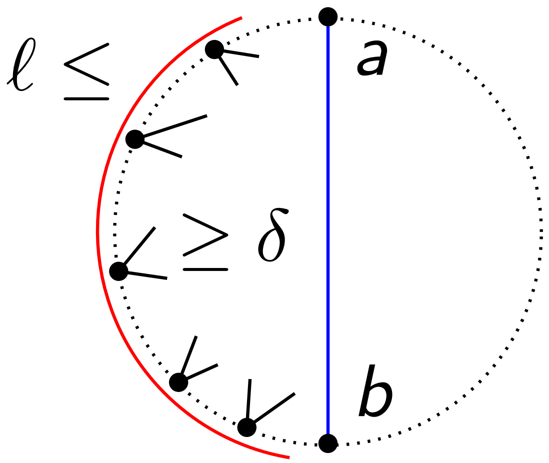

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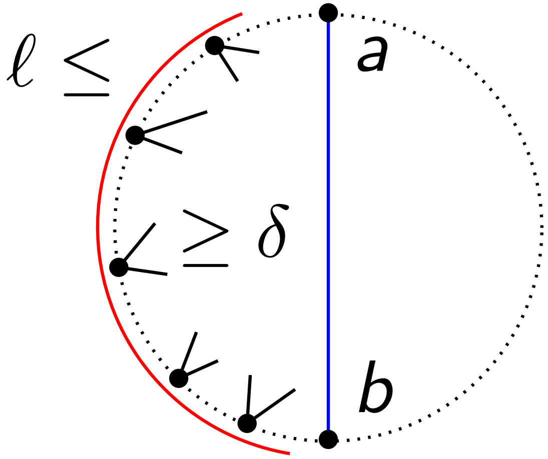

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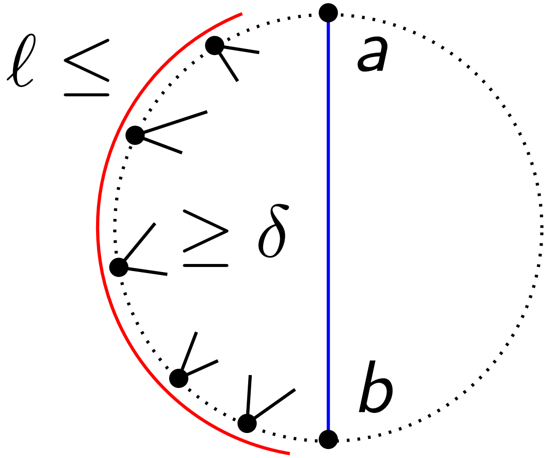

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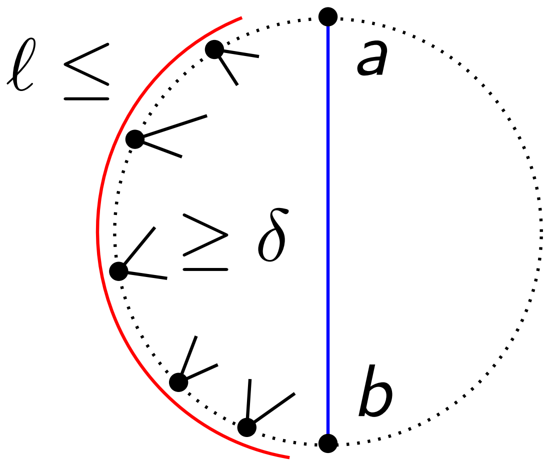

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Cor: Outer k-planarity $\rightarrow$ $(\lfloor \sqrt{4k+1} \rfloor + 2)$-colorable (tight).

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Thm: Outer k-planar graphs have sn $\leq 2k+3$, and such separators imply quasi-polynomial time ($2^{\text{polylog}(n)}$) recognition. i.e., assuming ETH, recognition is not NP-hard.

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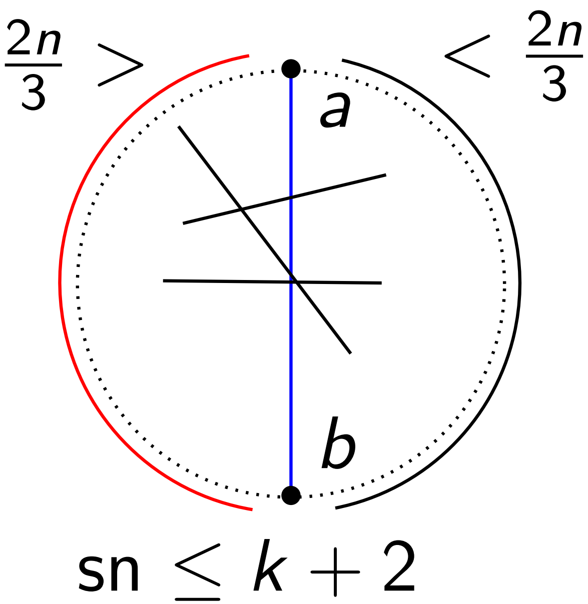

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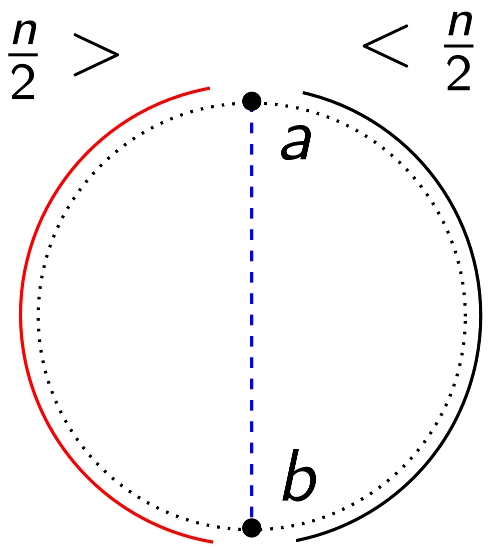

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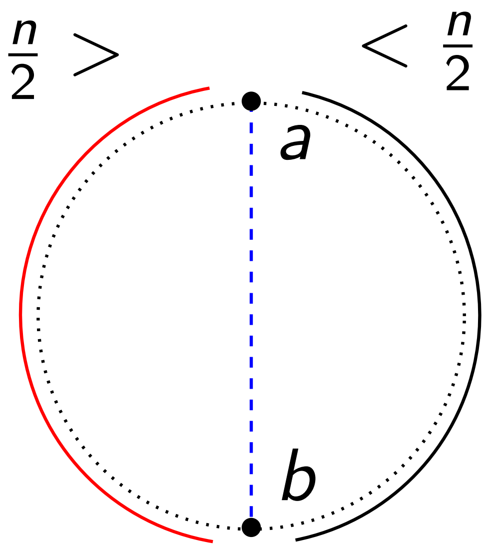

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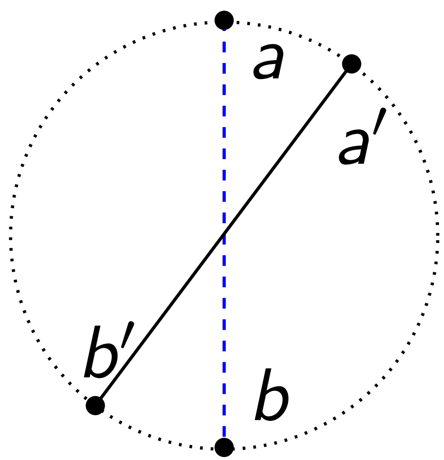

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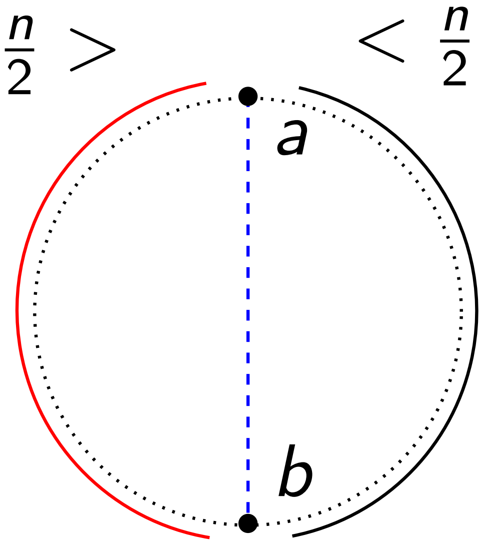

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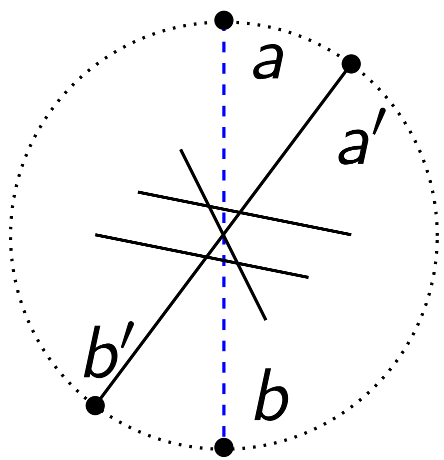

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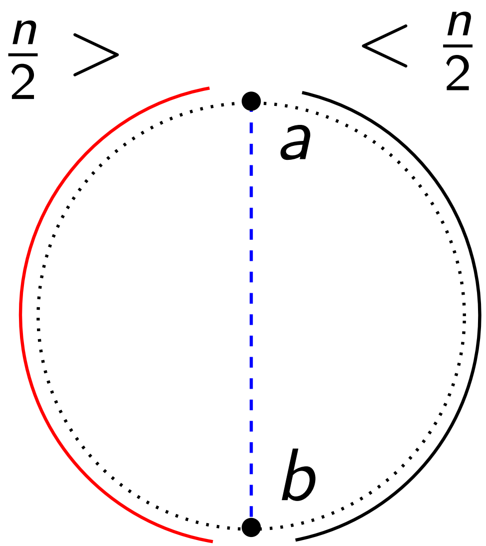

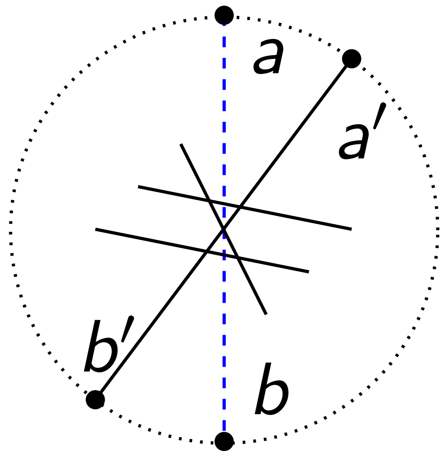

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Outer k-planarity

Thm: Outer k-planar graphs have $\mathrm{sn} \leq 2k+3$, and such separators imply quasi-polynomial time ($2^{\mathrm{polylog}(n)}$) recognition. i.e., assuming ETH, recognition is not NP-hard.

Proof (sketch):

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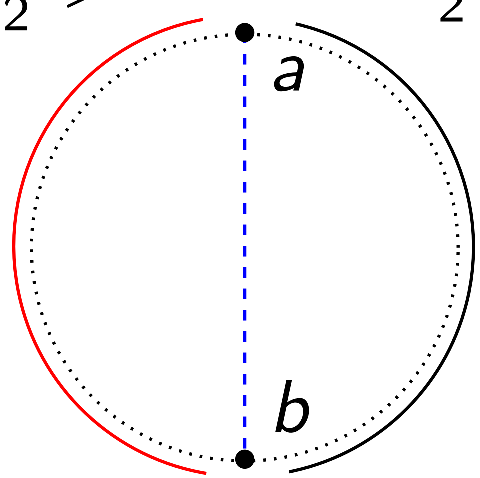

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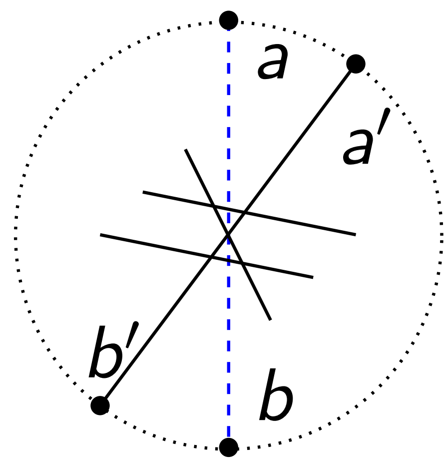

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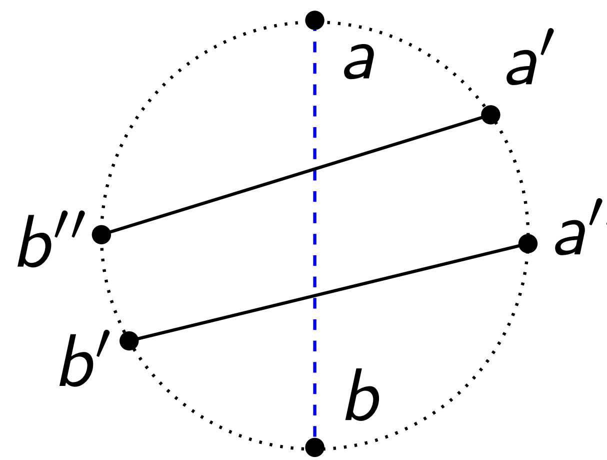

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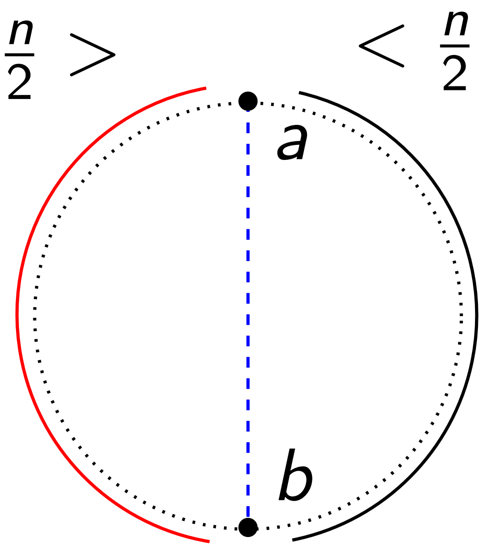

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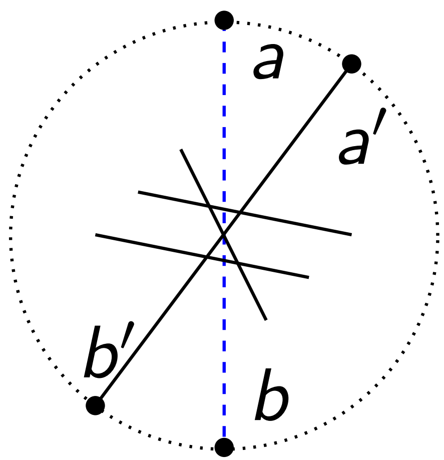

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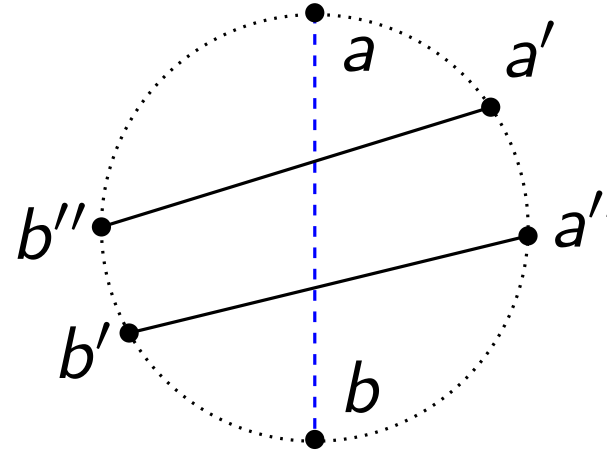

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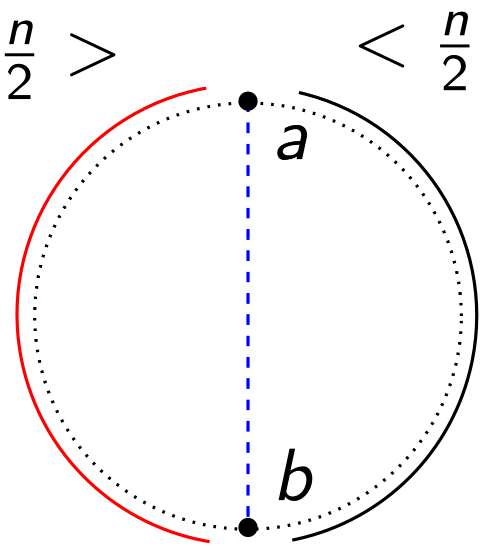

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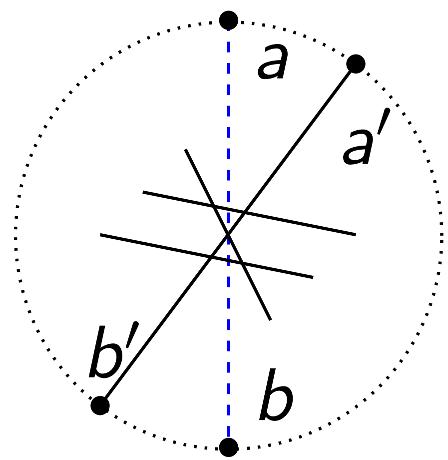

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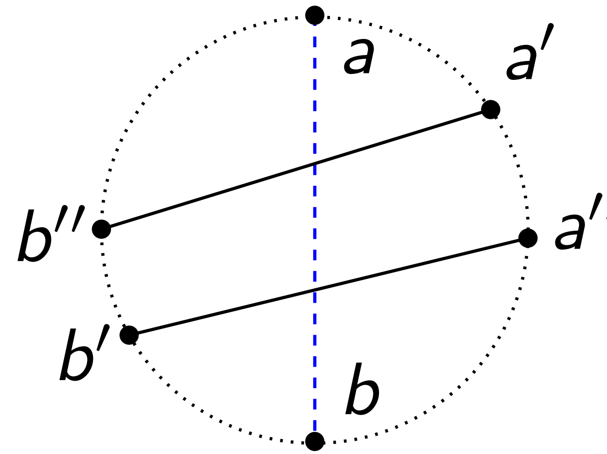

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Outline

Outer k-planar graphs

- $(\lfloor\sqrt{4k+1}\rfloor + 1)$-degenerate $\rightarrow$ $(\lfloor\sqrt{4k+1}\rfloor + 2)$-colorable
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Outer k-quasi-planar graphs

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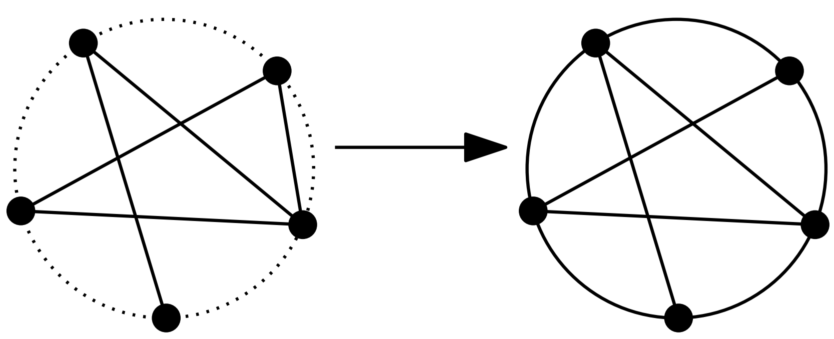

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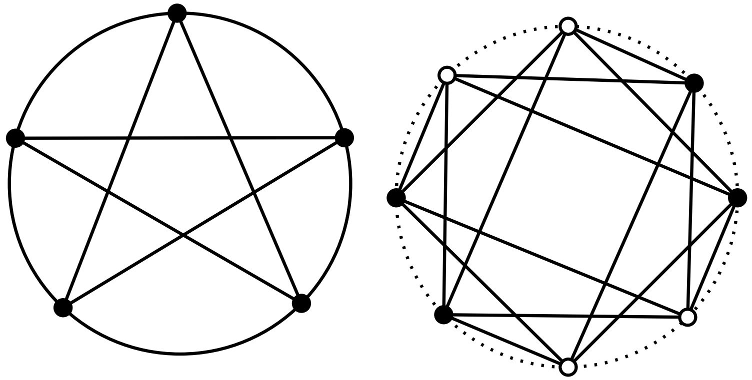

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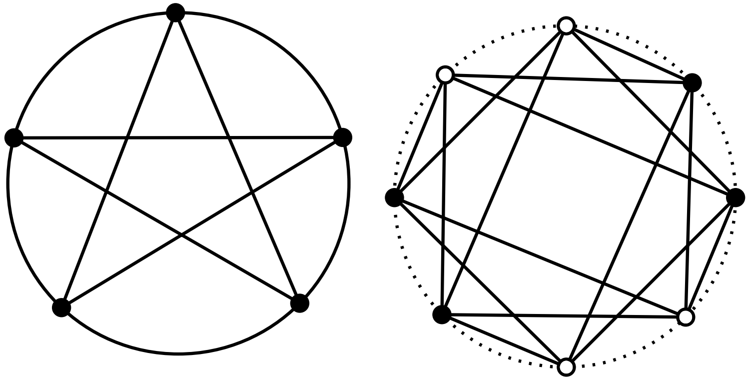

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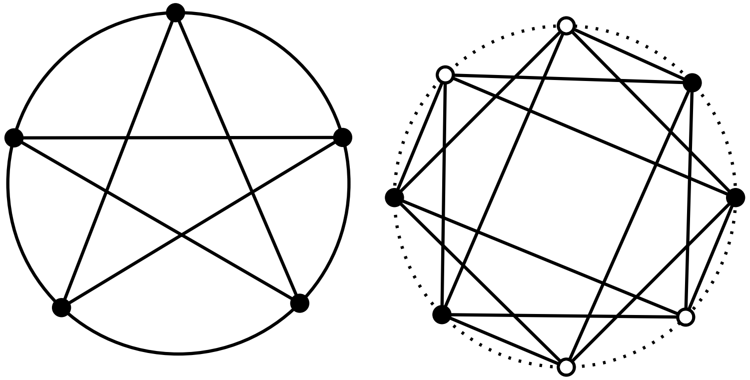

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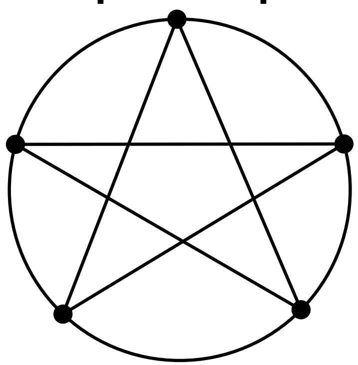

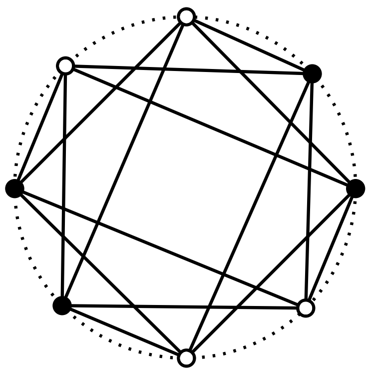

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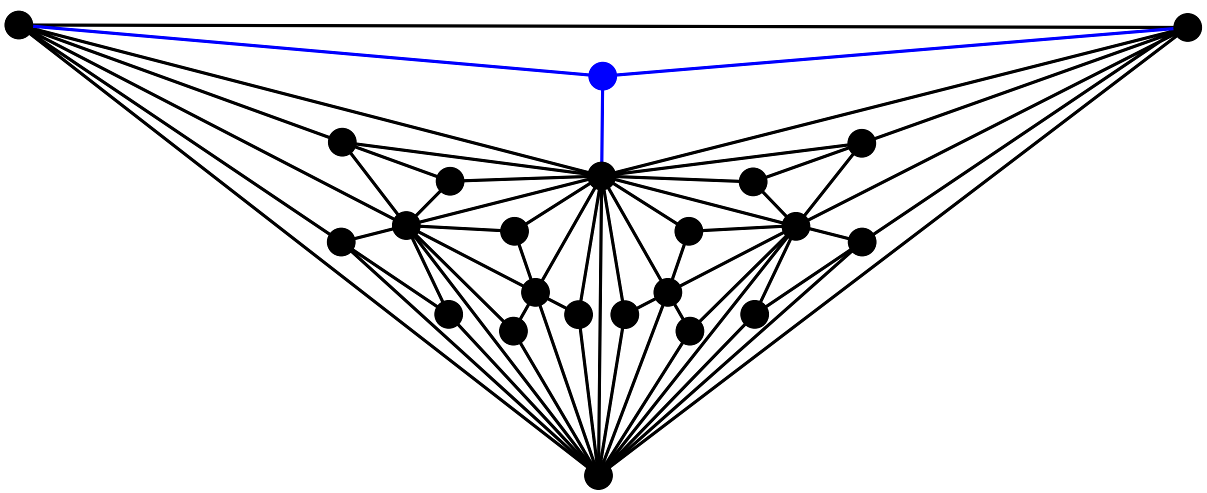

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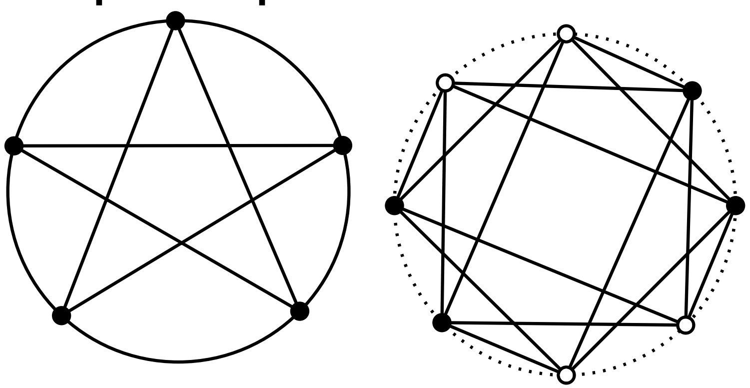

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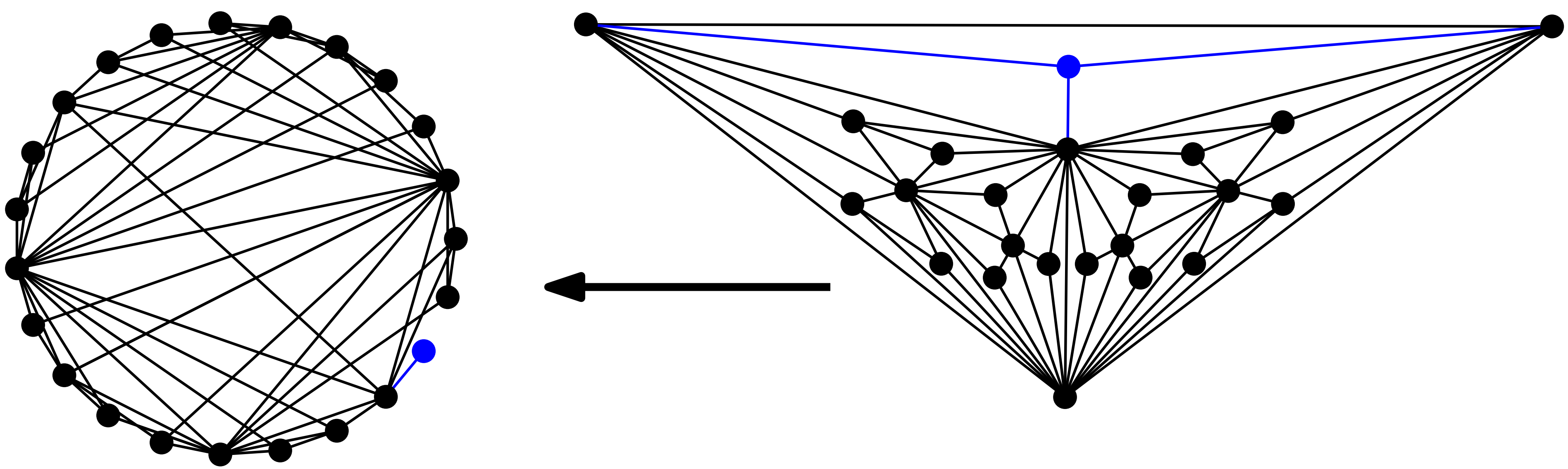

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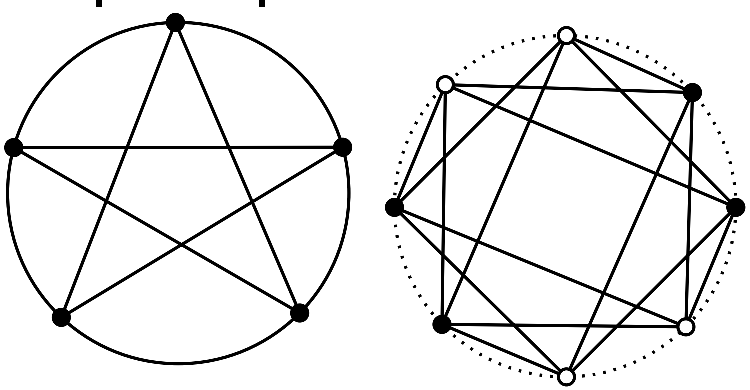

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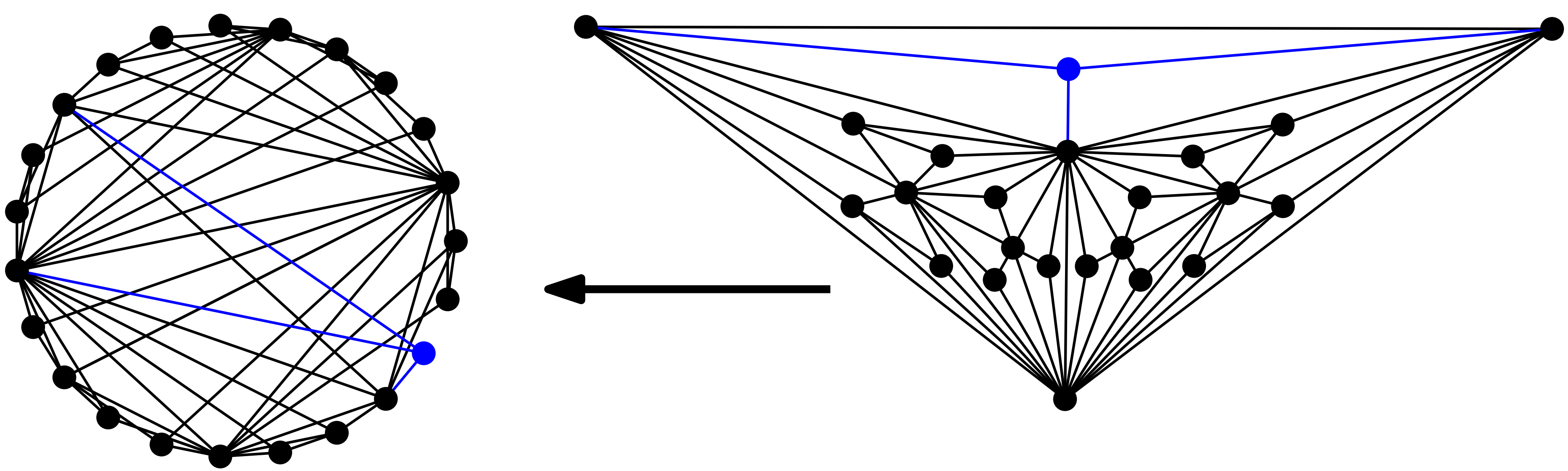

Edge maximal outer k-quasi-planar drawings

Thm: Each edge maximal outer k-quasi-planar drawing of $G = (V, E)$ has

$$|E| = \begin{cases} \binom{|V|}{2} & \text{if } |V| \leq 2k-1, \\ 2(k-1)|V| - \binom{2k-1}{2} & \text{if } |V| \geq 2k-1. \end{cases}$$

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What is the biggest line arrangment in the hyperbolic plane with $\leq n$ points at ∞ and without k mutually crossing lines (*Karzanov number* $\leq k-1$) ? [Dress et al. 2002]

Results

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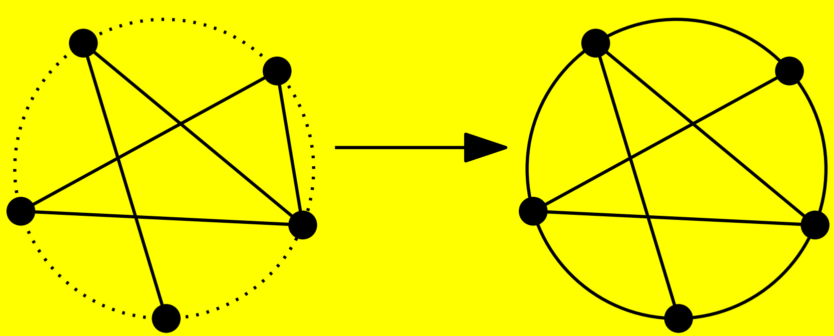

Monadic Second Order Logic (MSO_2)

Thm (Courcelle): If a property P is expressed as $\varphi \in MSO_2$, then for every graph G with treewidth at most t, P can be tested in time $O(f(t, |\varphi|)(n + m))$ for a computable function f.

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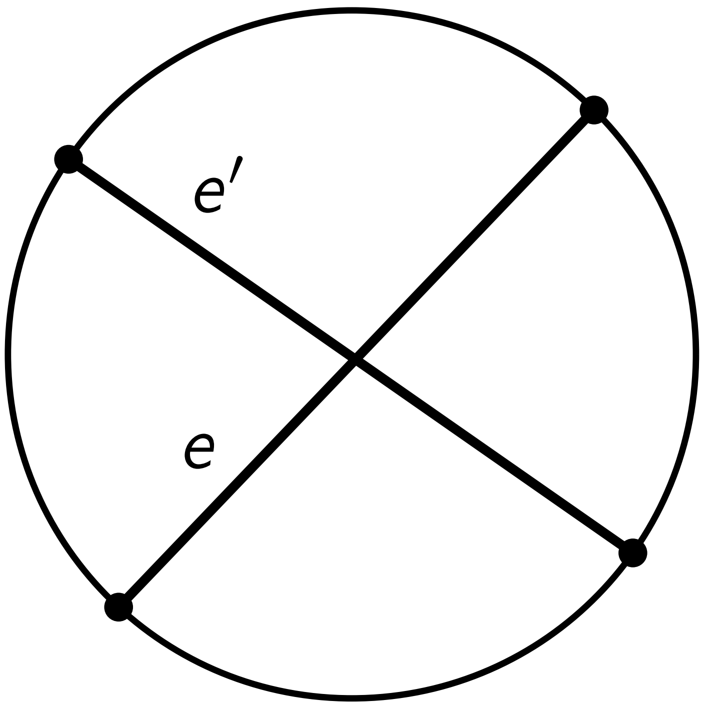
Formally:

- variables: vertices, edges, sets of vertices, and sets of edges;
- binary relations: equality ($=$), set membership ($\in$), subset of a set ($\subseteq$), and edge–vertex incidence (I);
- standard propositional logic operators: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$.
- standard quantifiers ($\forall, \exists$).

Encoding a Crossing in MSO_2

$\textsc{V-Partition}(A, B, C) \equiv$
(A, B, C) is a partition of the vertex set.



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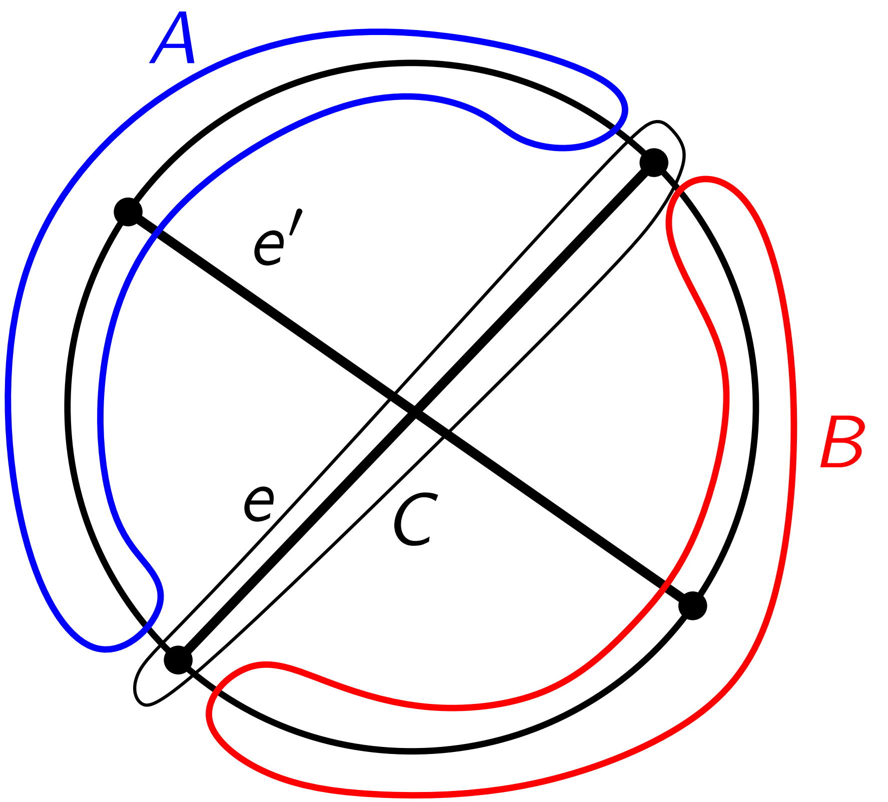

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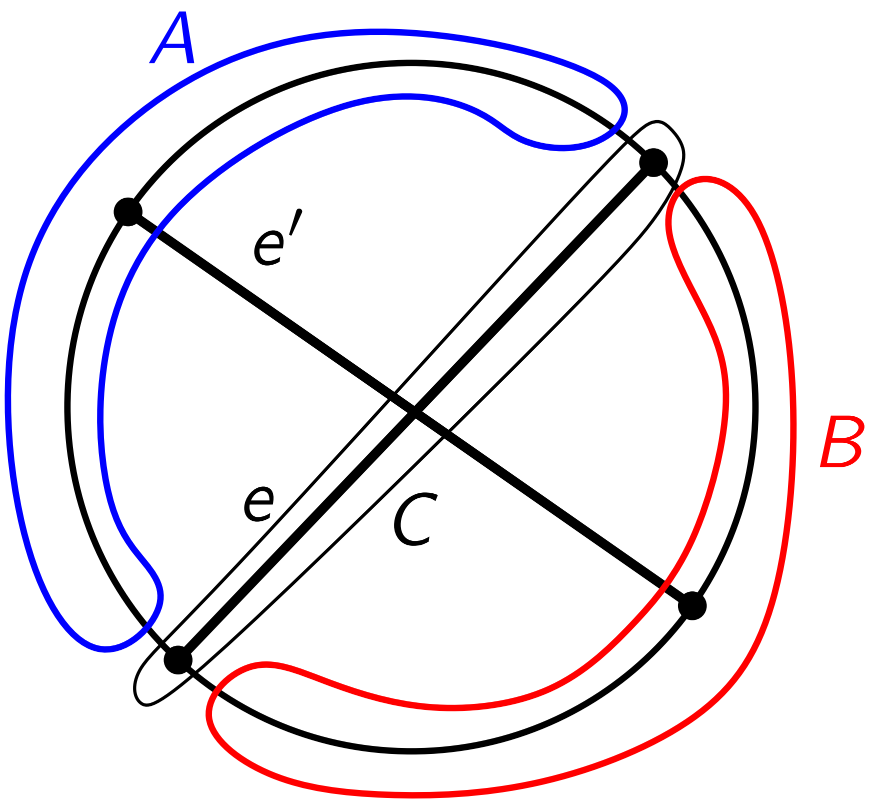

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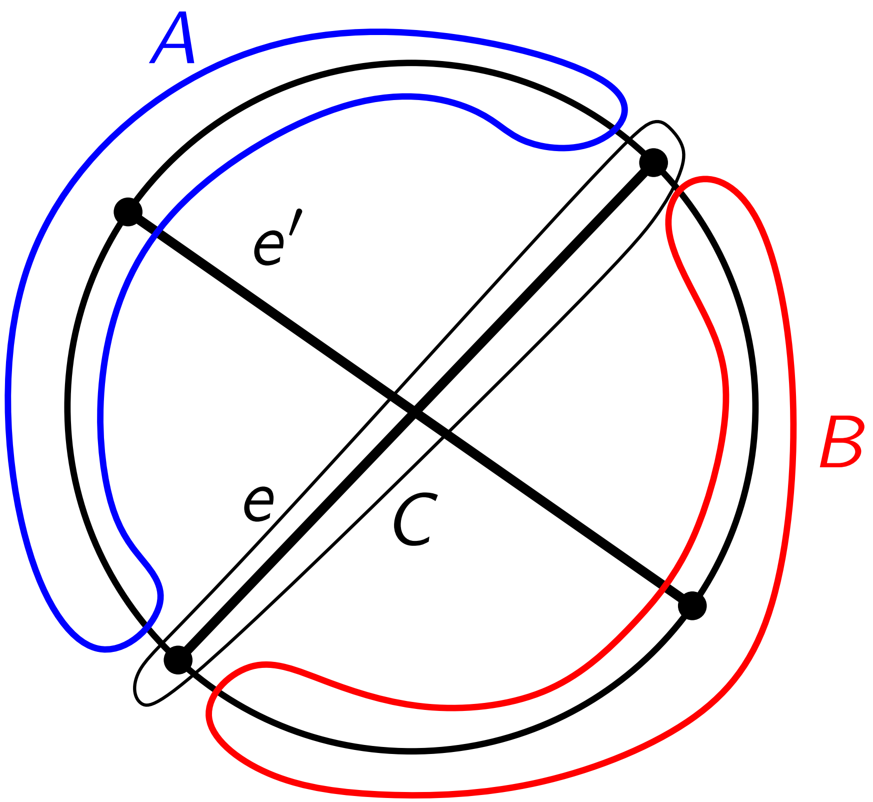

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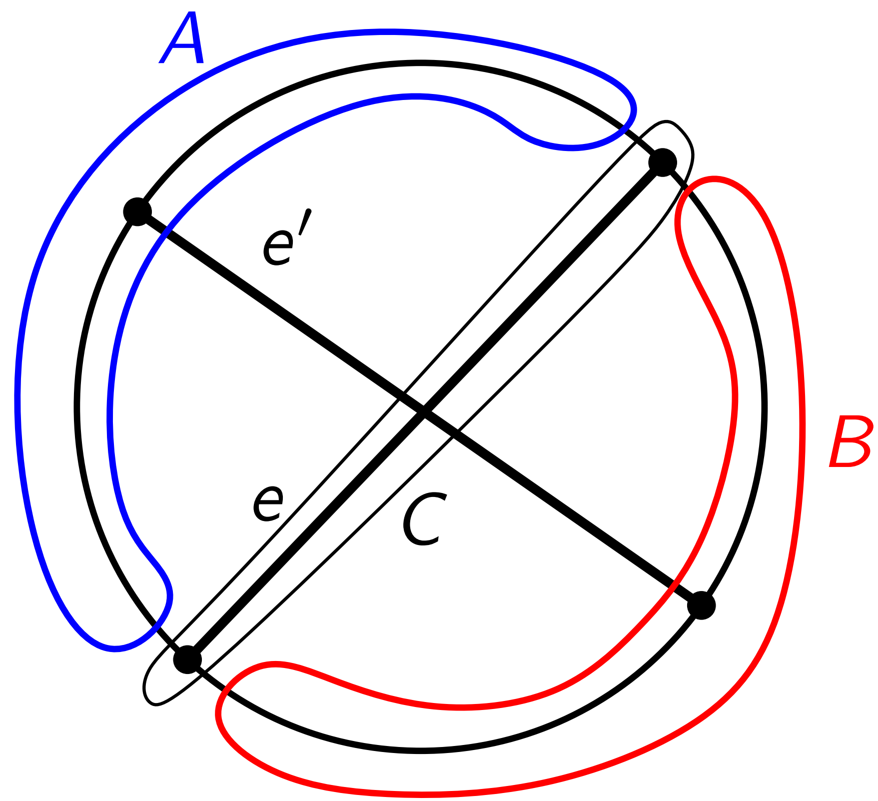

$$\textsc{Crossing}(E^*, e, e') \equiv (\forall A, B, C)\Big[\Big(\textsc{V-Partition}(A, B, C) \wedge (x \in C \leftrightarrow I(e, x)) \wedge \textsc{Conn}(A, E^*) \wedge \textsc{Conn}(B, E^*)\Big) \rightarrow (\exists a \in A)(\exists b \in B)[I(e', a) \wedge I(e', b)]\Big]$$

Implications of our MSO_2 formulae

- *closed* drawings which are k-planar or k-quasi planar can be expressed in MSO_2.
- closed k-planarity can be tested in linear FPT-time (parameterized by k).
- closed k-quasi-planarity can be tested in linear FPT-time (parameterized by both k and treewidth).
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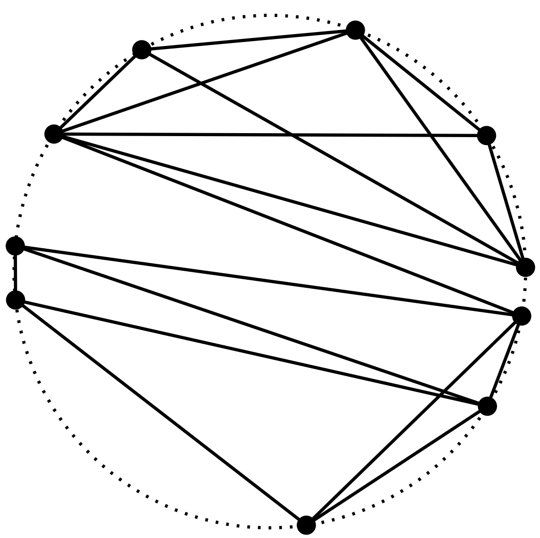

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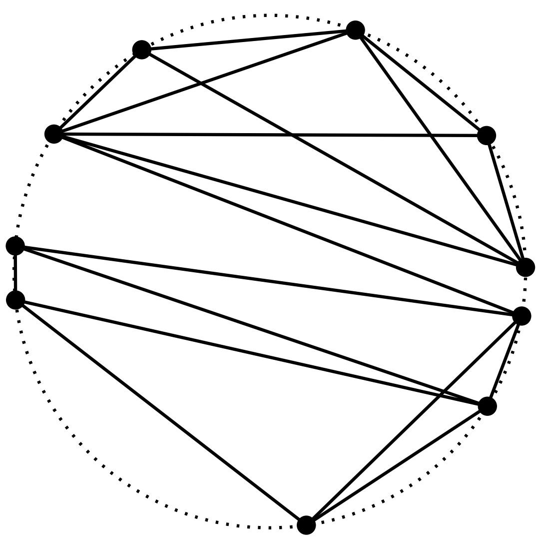

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full outer 2-planarity testing in linear time [Hong, Nagamochi '16]

Conclusion

Outer k-planar graphs:

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  Quasi-polynomial time recognition via balanced separators,
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Thank you for your attention :-)