The Complexity of Drawing Graphs on Few Lines and Few Planes

Steven Chaplick Krzysztof Fleszar
Fabian Lipp Alexander Wolff

Julius-Maximilians-Universität Würzburg, Germany

Alexander Ravsky

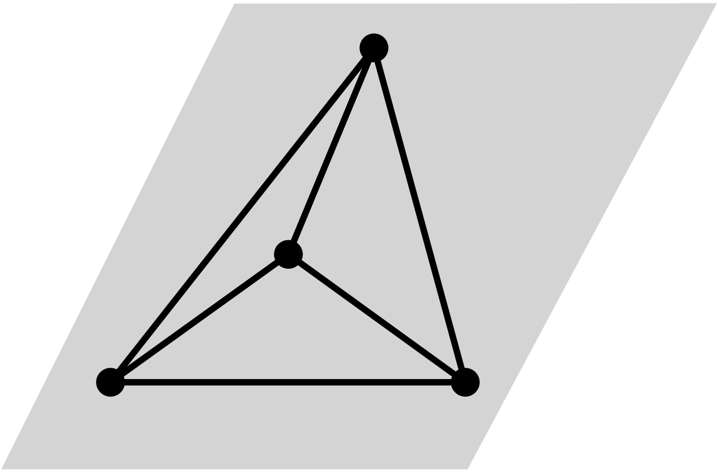
National Academy of Sciences of Ukraine, Lviv, Ukraine

Oleg Verbitsky

Humboldt-Universität zu Berlin, Germany

Our Task

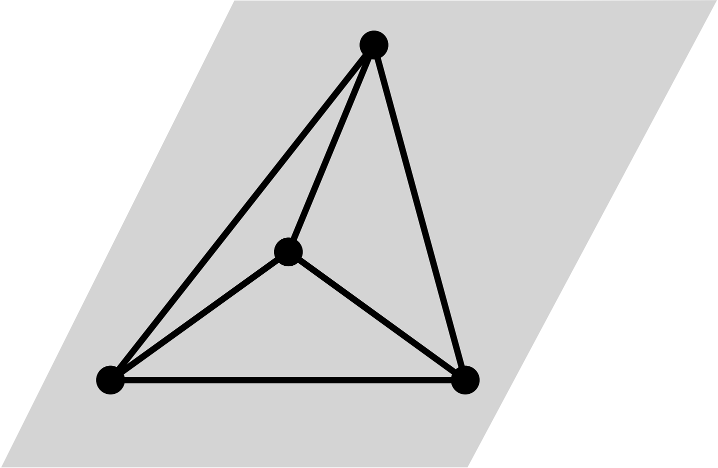
For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.



Our Task

For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

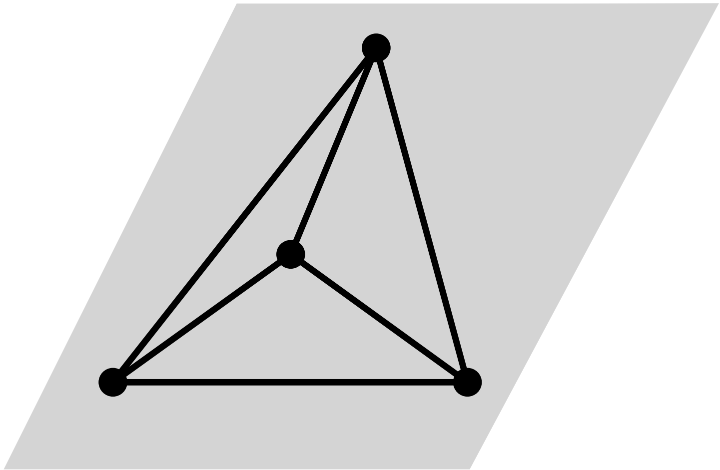


Our Task

For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

Use more than one plane to draw it.



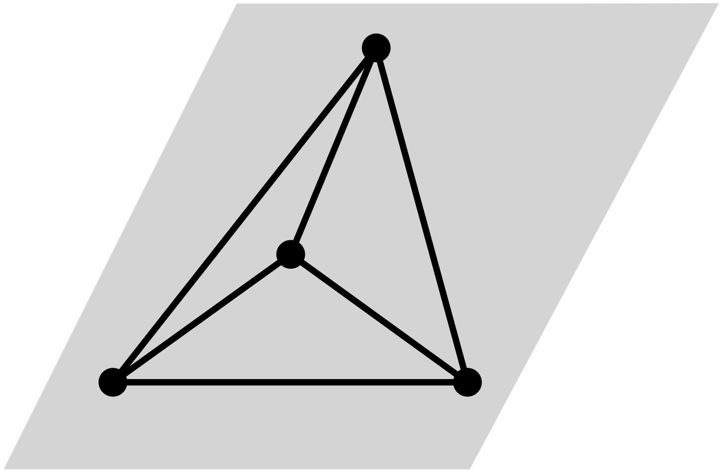
Our Task

For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

Use more than one plane to draw it.

How many planes in 3-space do we need to cover all vertices and edges of a crossing-free straight-line drawing of K_5?



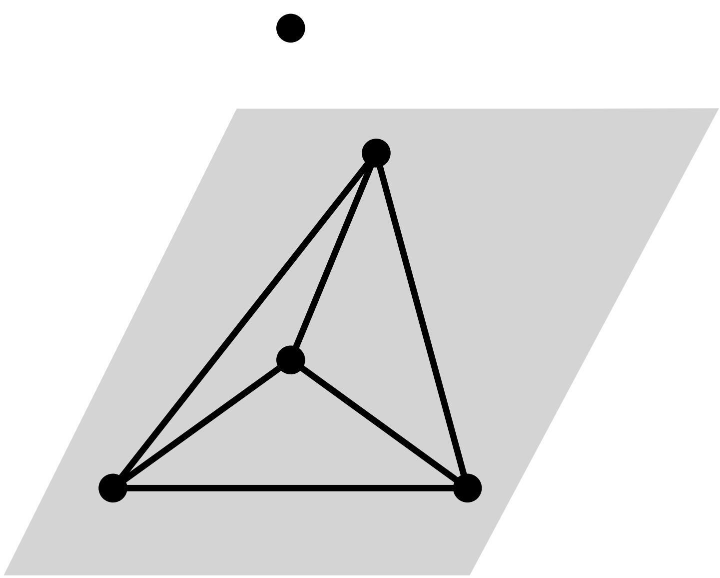
Our Task

For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

Use more than one plane to draw it.

How many planes in 3-space do we need to cover all vertices and edges of a crossing-free straight-line drawing of K_5?



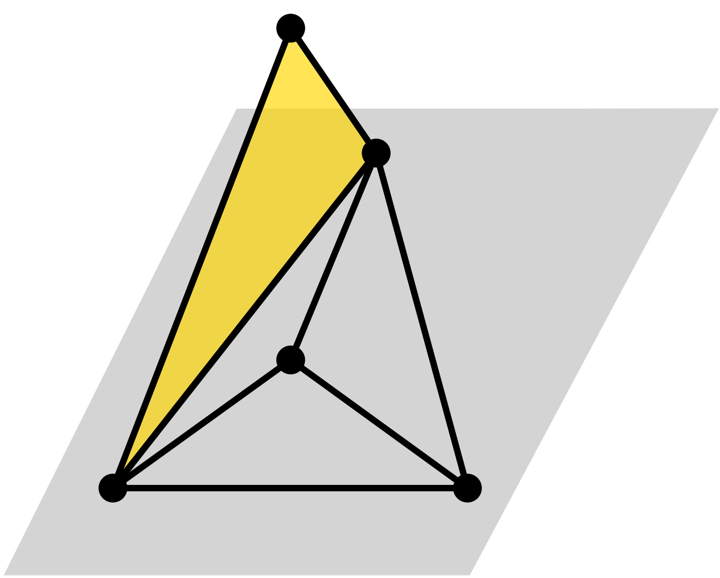
Our Task

For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

Use more than one plane to draw it.

How many planes in 3-space do we need to cover all vertices and edges of a crossing-free straight-line drawing of K_5?



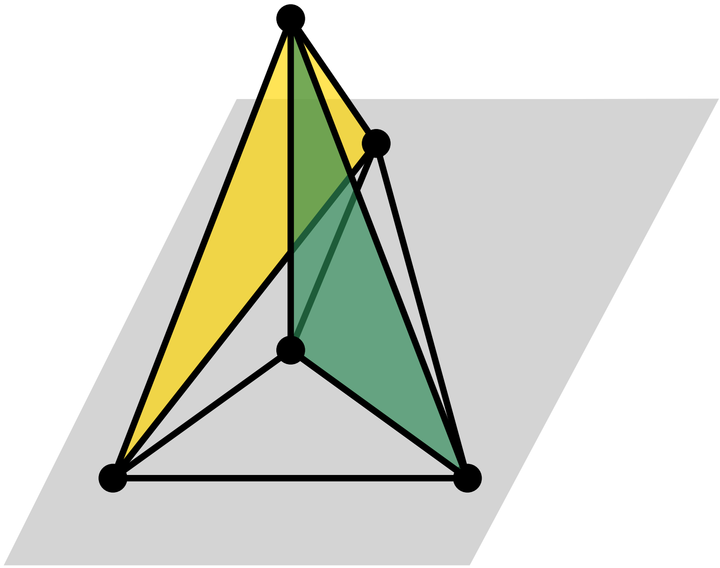
Our Task

For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

Use more than one plane to draw it.

How many planes in 3-space do we need to cover all vertices and edges of a crossing-free straight-line drawing of K_5? **3**



Our Task

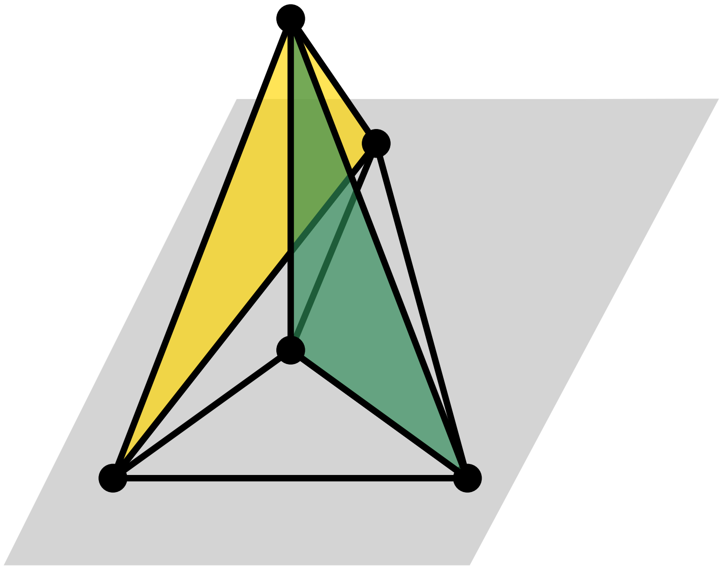
For any planar graph (like K_4) there is a *crossing-free straight-line drawing* in the plane.

But what about beyond-planar graphs?

Use more than one plane to draw it.

How many planes in 3-space do we need to cover all vertices and edges of a crossing-free straight-line drawing of K_5? **3**

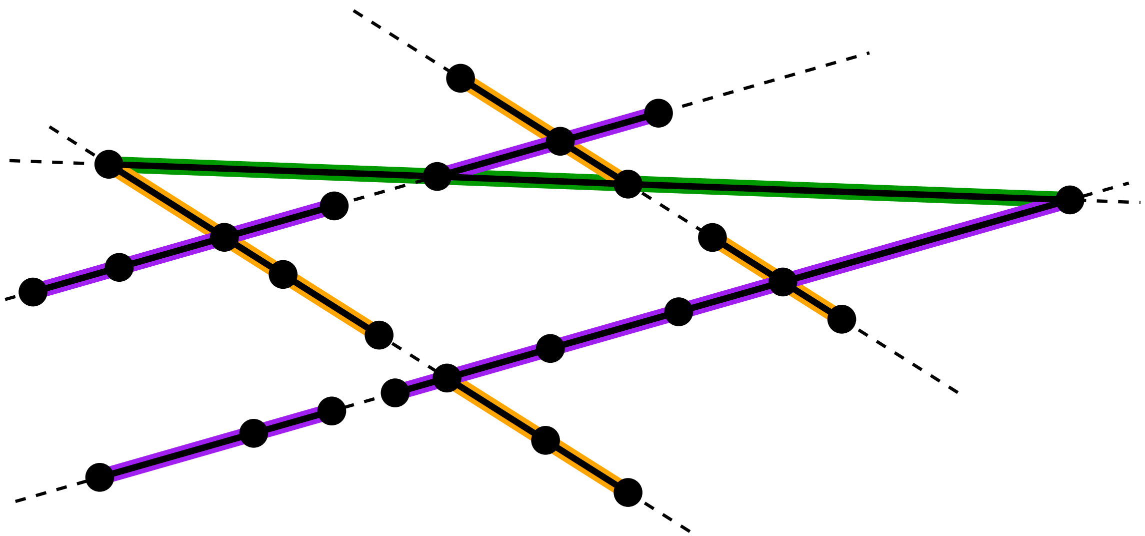
We propose the number of planes needed as a parameter for classifying beyond-planar graphs.



Related Work: Low Visual Complexity

Minimum number of segments

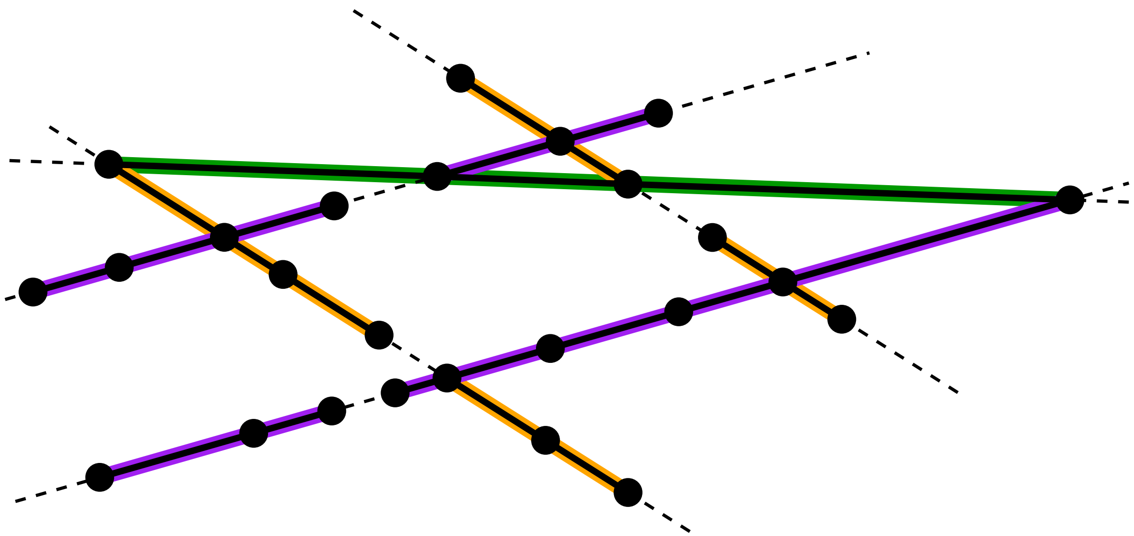
[Durocher et al., JGAA 2013]



Related Work: Low Visual Complexity

Minimum number of segments

[Durocher et al., JGAA 2013]



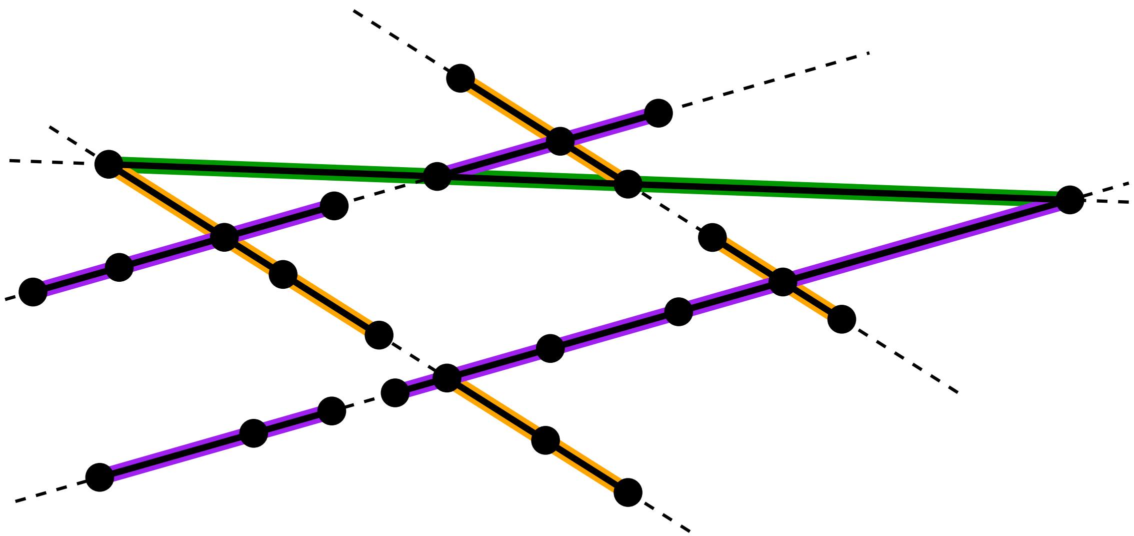
9 segments

5 lines

Related Work: Low Visual Complexity

Minimum number of segments

[Durocher et al., JGAA 2013]



9 segments
5 lines

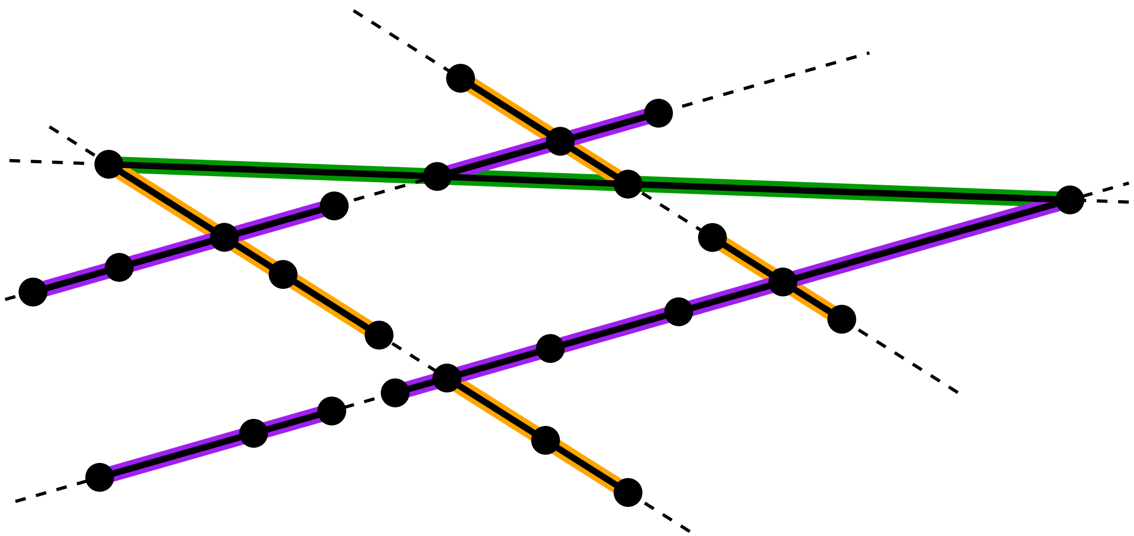
Low number of arcs

[Schulz, JGAA 2015]

Related Work: Low Visual Complexity

Minimum number of segments

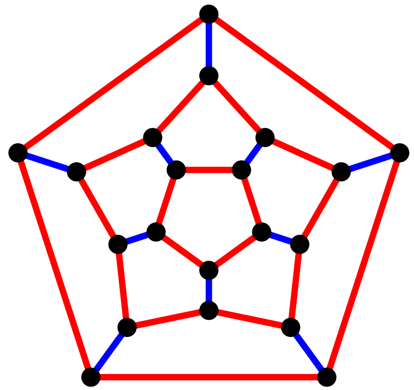
[Durocher et al., JGAA 2013]



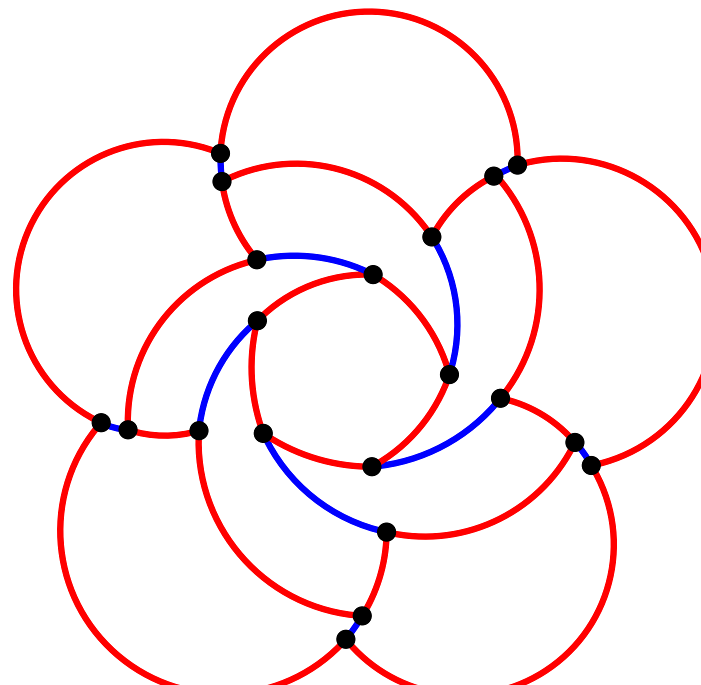
9 segments
5 lines

Low number of arcs

[Schulz, JGAA 2015]



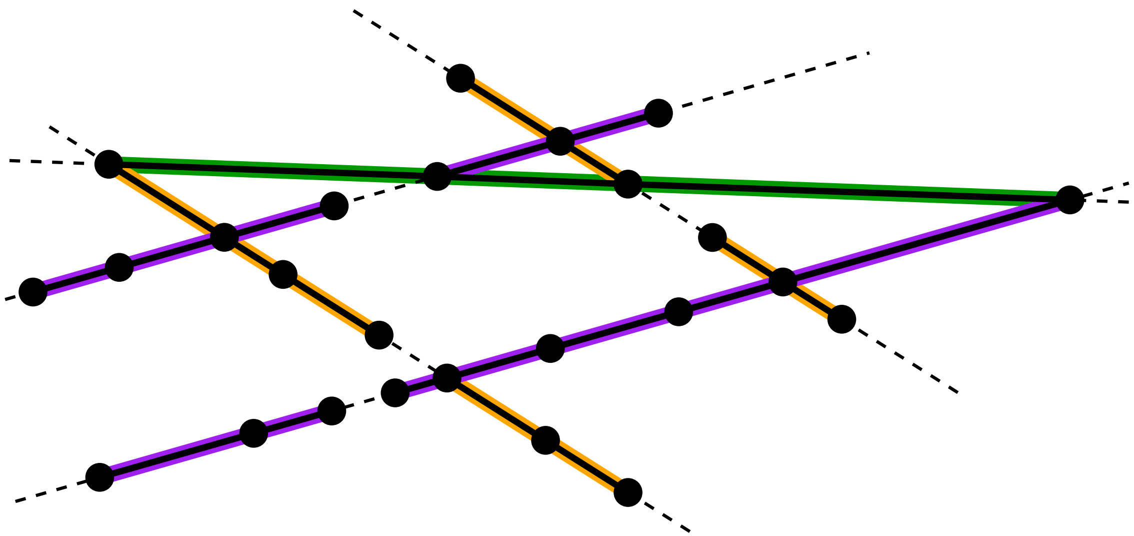
dodecaedron



Related Work: Low Visual Complexity

Minimum number of segments

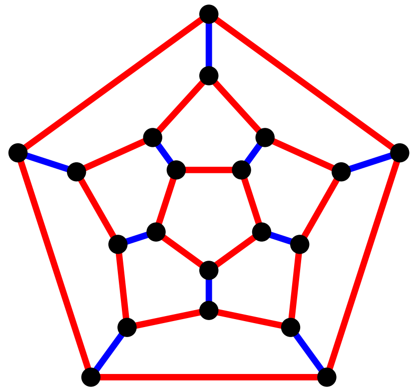
[Durocher et al., JGAA 2013]



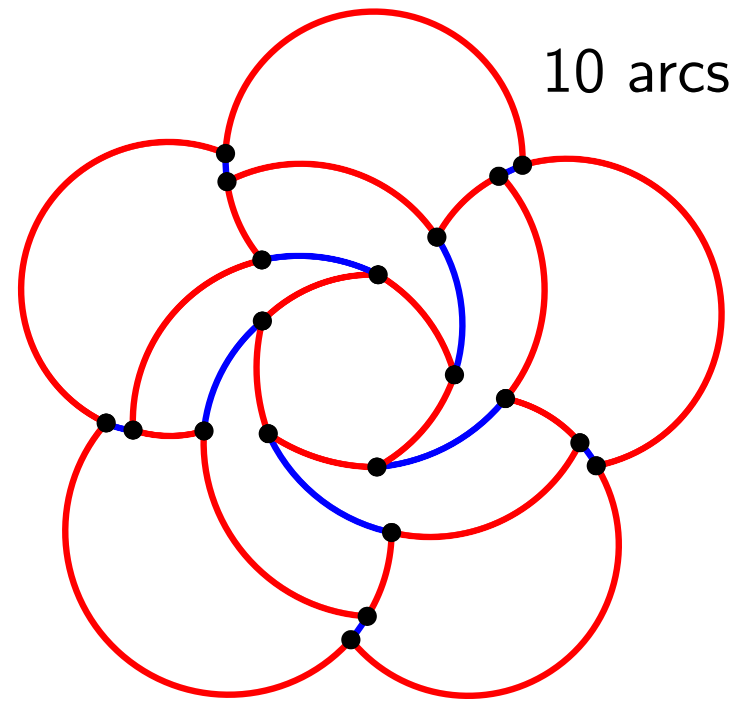
9 segments
5 lines

Low number of arcs

[Schulz, JGAA 2015]



dodecaedron



10 arcs

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho^m_d(G)$:

minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ s.t. G has a crossing-free straight-line drawing that is contained in these planes

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$:

minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ s.t. G has a crossing-free straight-line drawing that is contained in these planes

Interesting cases:

- Line cover numbers in 2D and 3D: ρ_2^1, ρ_3^1
- Plane cover numbers: ρ_3^2

Definitions

Let G be a graph and $1 \leq m < d$.

Affine cover number $\rho_d^m(G)$:

minimum number of m-dimensional hyperplanes in $\mathbb{R}^d$ s.t. G has a crossing-free straight-line drawing that is contained in these planes

Interesting cases:

- Line cover numbers in 2D and 3D: ρ_2^1, ρ_3^1
- Plane cover numbers: ρ_3^2

Known results:

- Combinatorial bounds for various classes of graphs
- Relations to other graph characteristics

[Chaplick et al., GD 2016]

Our Results

- Computing line cover numbers is $\exists\mathbb{R}$-hard
  (ρ_2^1 and ρ_3^1)

- Computing line cover numbers is fixed-parameter tractable
  (ρ_2^1 and ρ_3^1)

- Computing plane cover numbers is NP-hard
  (ρ_3^2)

Our Results

- Computing line cover numbers is $\exists\mathbb{R}$-hard (ρ_2^1 and ρ_3^1)

- Computing line cover numbers is fixed-parameter tractable (ρ_2^1 and ρ_3^1)

- Computing plane cover numbers is NP-hard (ρ_3^2)

$\exists\mathbb{R}$-hardness

Decision problem for the existential theory of the reals: decide if first-order formula about the reals of the form $\exists x_1 \ldots \exists x_m \Phi(x_1, \ldots, x_m)$ is true where formula Φ uses:

- no quantifiers
- constants 0, 1
- basic arithmetic operations
- order and equality relations

$\exists\mathbb{R}$-hardness

Decision problem for the existential theory of the reals:
decide if first-order formula about the reals of the form
$\exists x_1 \ldots \exists x_m \Phi(x_1, \ldots, x_m)$ is true where formula Φ uses:

- no quantifiers
- constants 0, 1
- basic arithmetic operations
- order and equality relations

$\exists\mathbb{R}$: problems that can be reduced to that problem

[Schaefer, GD 2009]

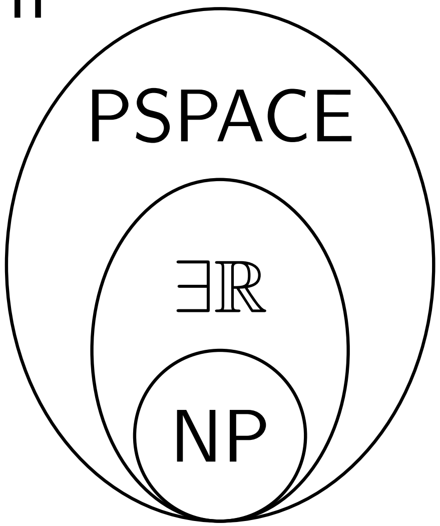
$\exists\mathbb{R}$-hardness

Decision problem for the existential theory of the reals:
decide if first-order formula about the reals of the form
$\exists x_1 \ldots \exists x_m \Phi(x_1, \ldots, x_m)$ is true where formula Φ uses:

- no quantifiers
- constants 0, 1
- basic arithmetic operations
- order and equality relations

$\exists\mathbb{R}$: problems that can be reduced to that problem

[Schaefer, GD 2009]



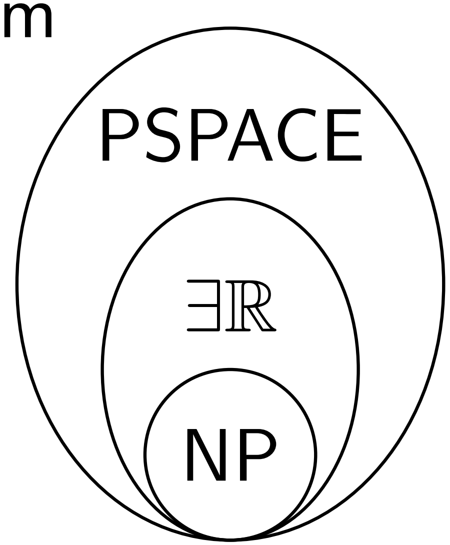
$\exists\mathbb{R}$-hardness

Decision problem for the existential theory of the reals:
decide if first-order formula about the reals of the form
$\exists x_1 \ldots \exists x_m \Phi(x_1, \ldots, x_m)$ is true where formula Φ uses:

- no quantifiers
- constants 0, 1
- basic arithmetic operations
- order and equality relations

$\exists\mathbb{R}$: problems that can be reduced to that problem

[Schaefer, GD 2009]



Natural $\exists\mathbb{R}$-complete problems:

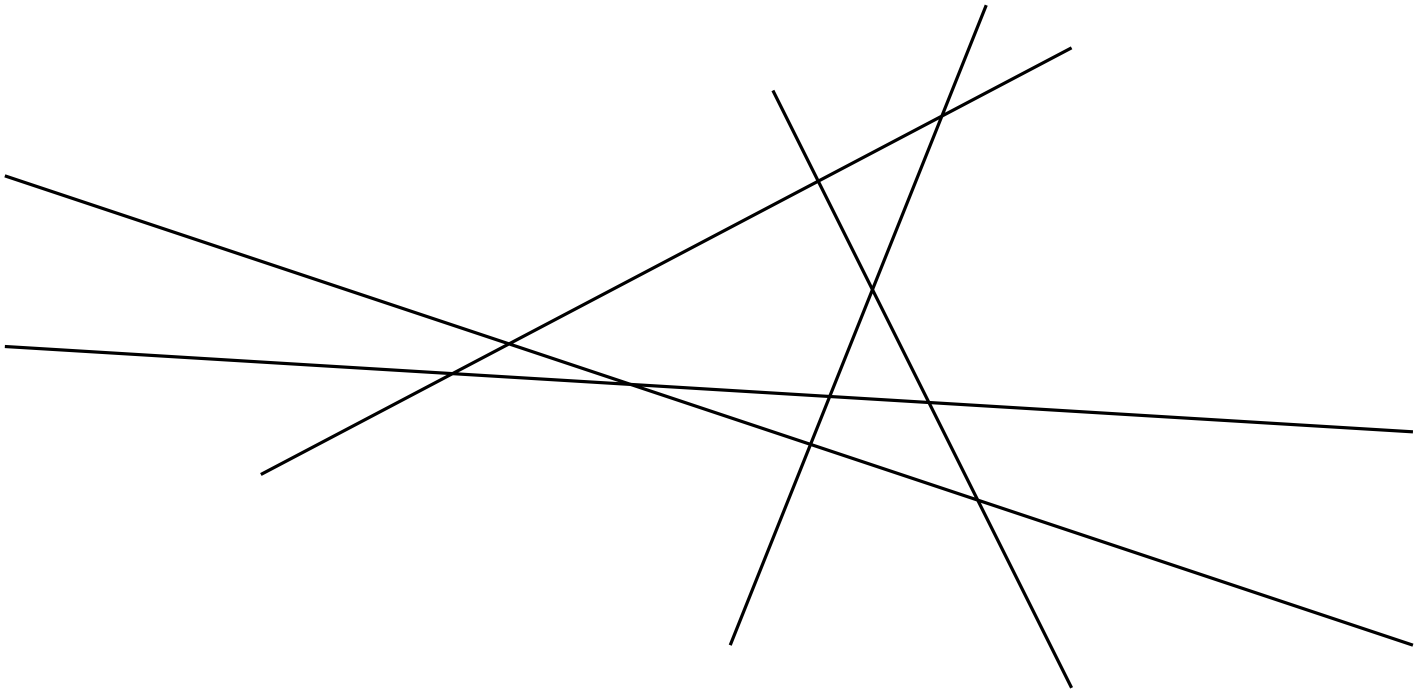
- rectilinear crossing number
- recognition of segment intersection graphs
- recognition of unit disk graphs

Arrangement Graph Recognition (AGR)

Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$

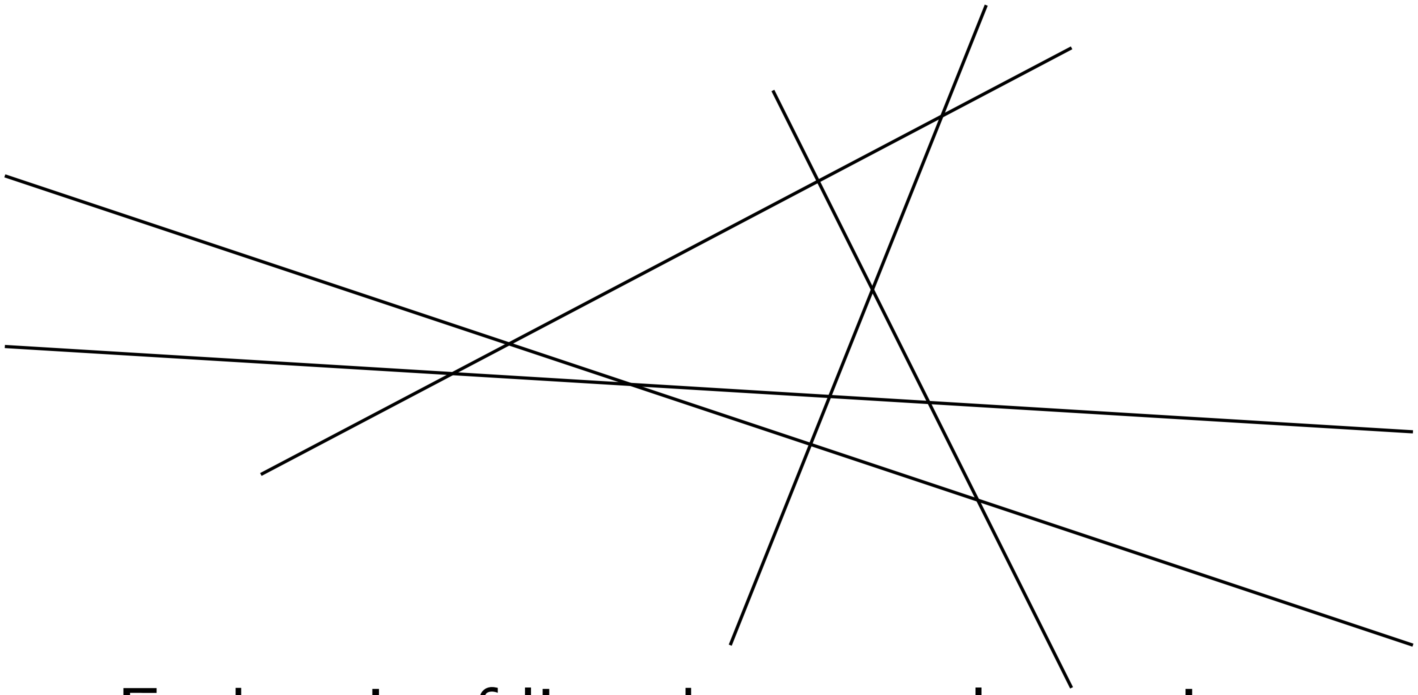
Arrangement Graph Recognition (AGR)

Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$



Arrangement Graph Recognition (AGR)

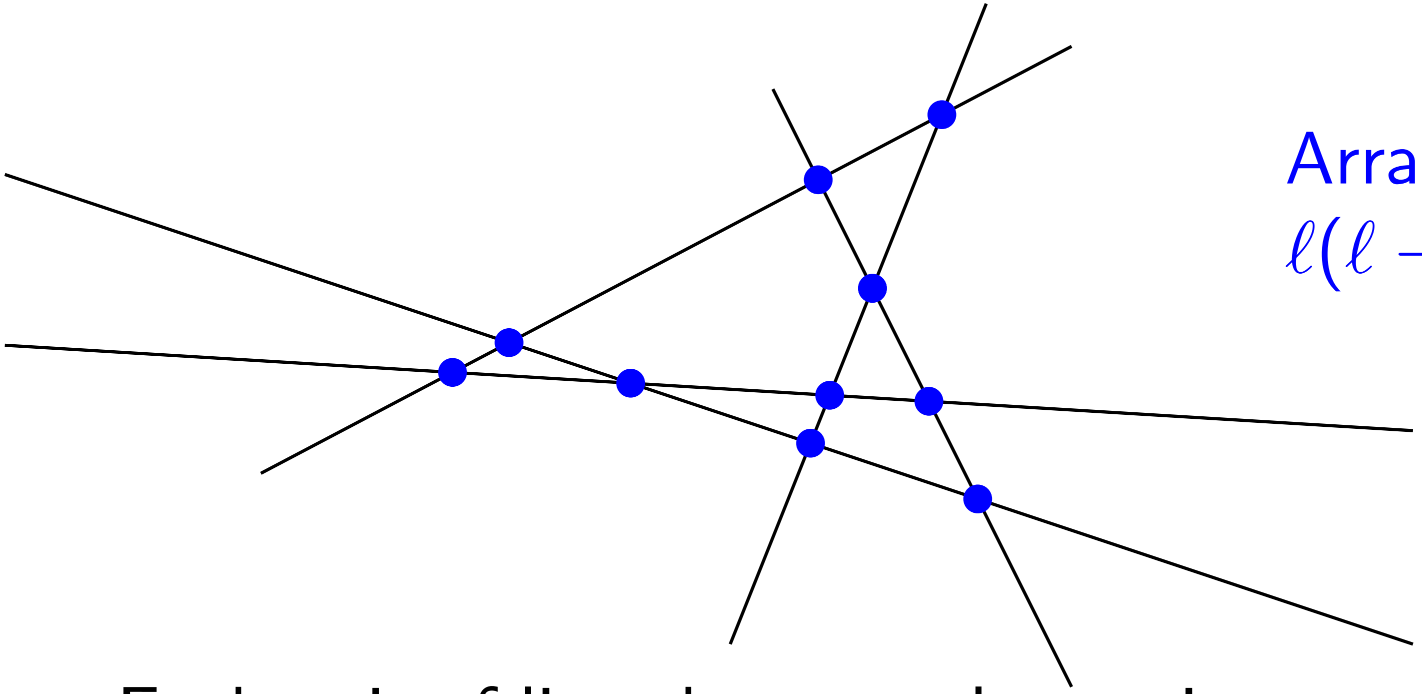
Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$



- Each pair of lines has exactly one intersection
- No three lines share a common point

Arrangement Graph Recognition (AGR)

Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$

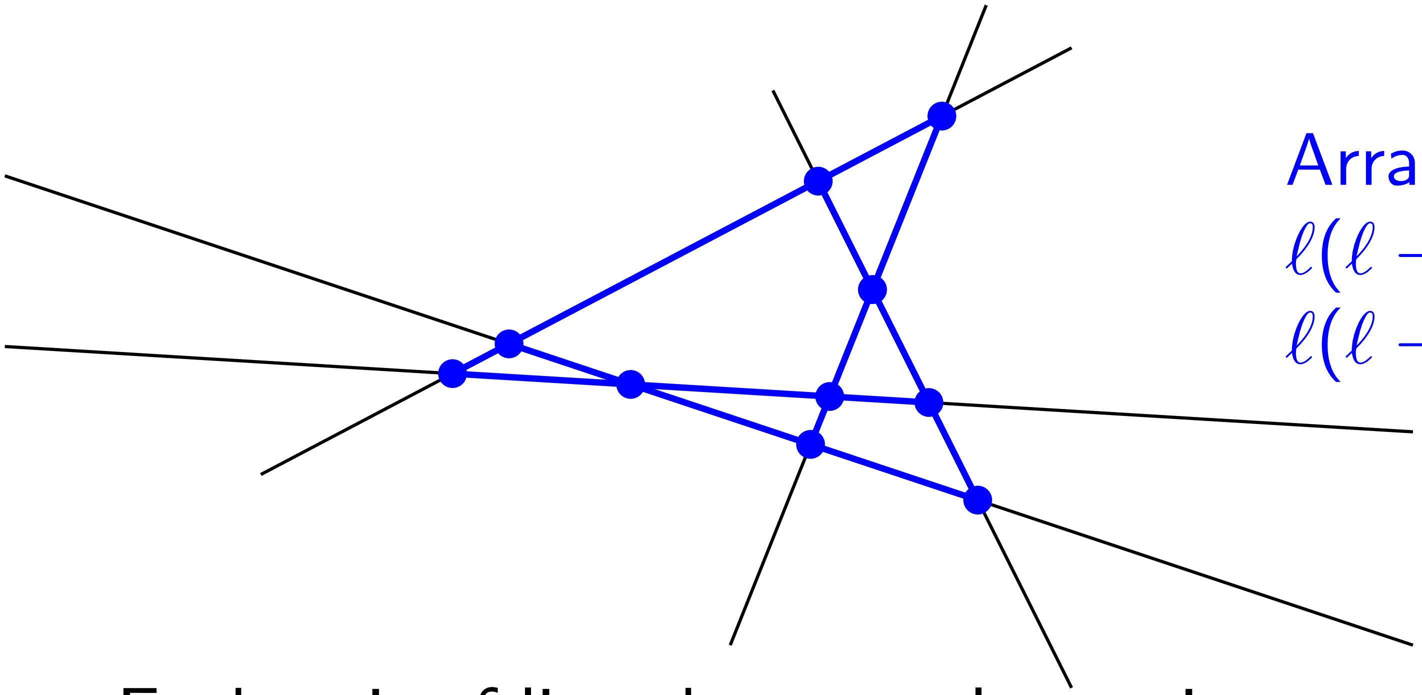


Arrangement graph:
$\ell(\ell - 1)/2$ vertices

- Each pair of lines has exactly one intersection
- No three lines share a common point

Arrangement Graph Recognition (AGR)

Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$

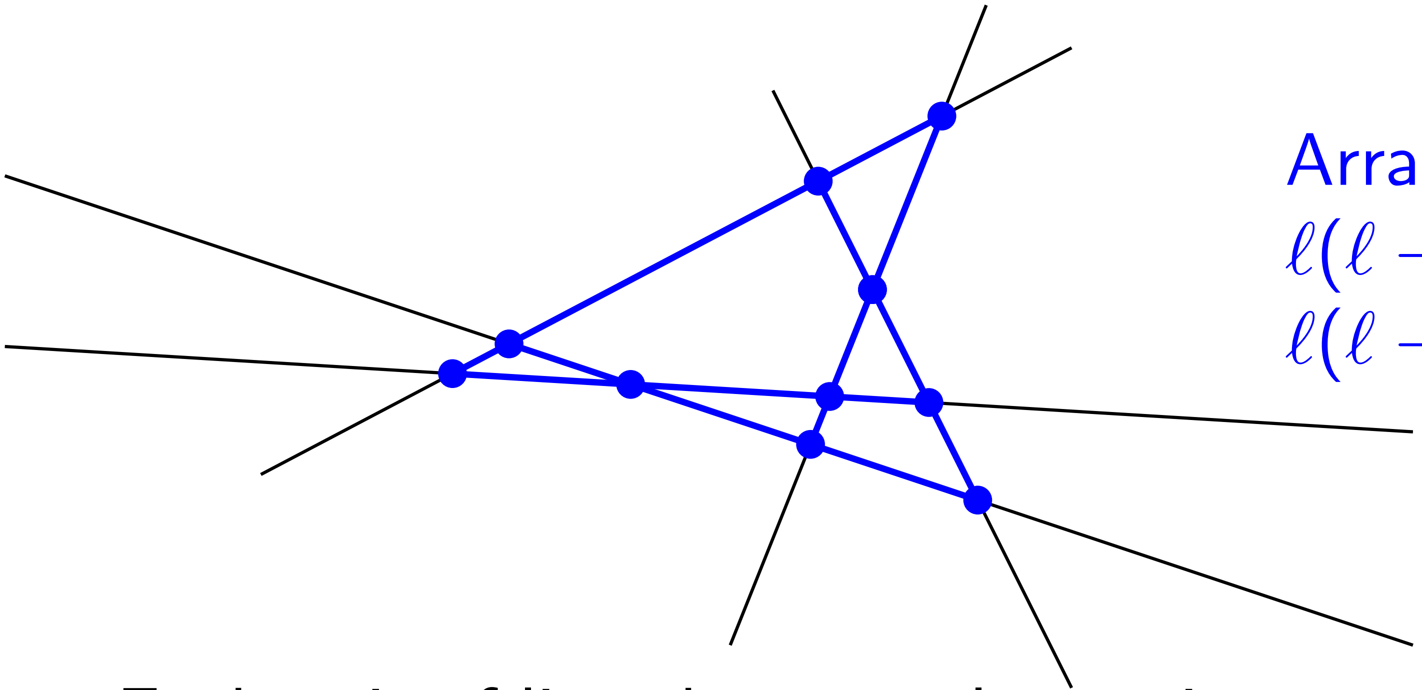


Arrangement graph:
$\ell(\ell-1)/2$ vertices
$\ell(\ell-2)$ edges

- Each pair of lines has exactly one intersection
- No three lines share a common point

Arrangement Graph Recognition (AGR)

Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$



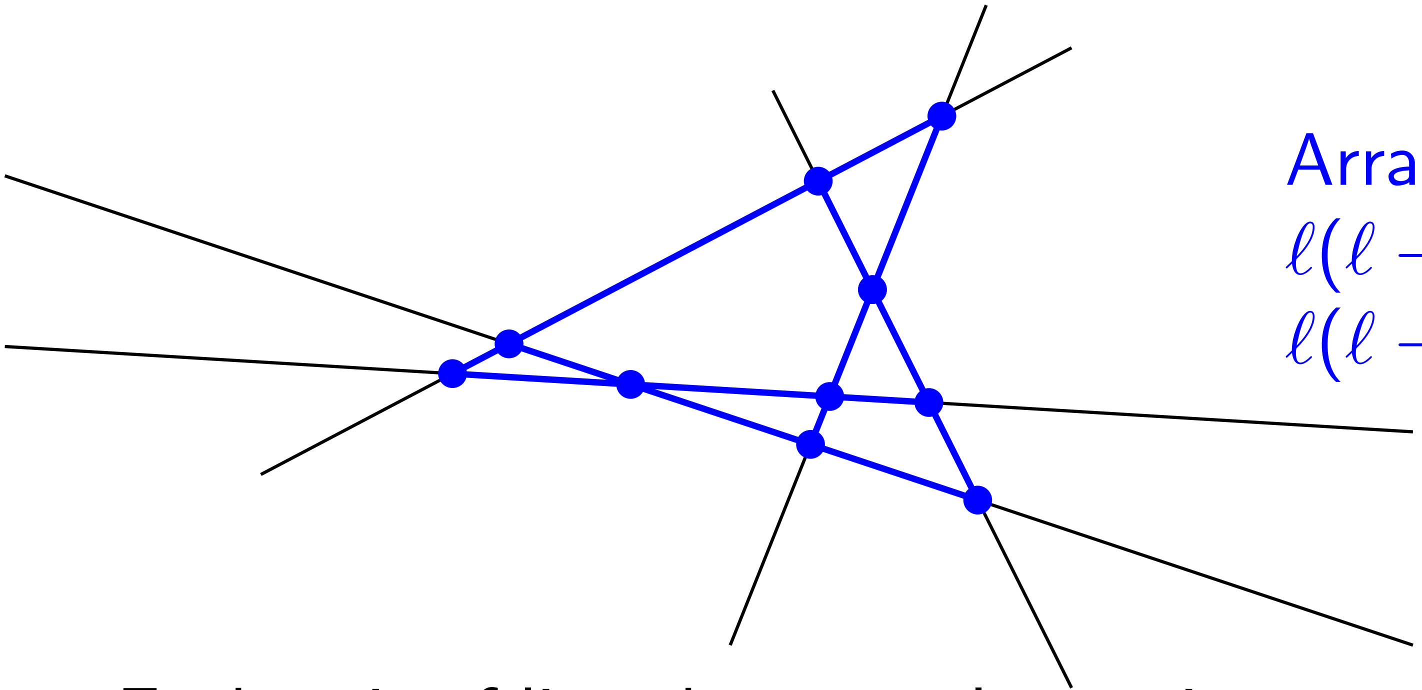
Arrangement graph:
$\ell(\ell-1)/2$ vertices
$\ell(\ell-2)$ edges

- Each pair of lines has exactly one intersection
- No three lines share a common point

Problem AGR: Given a graph $G = (V, E)$, decide if it is an arrangement graph of some set of lines.

Arrangement Graph Recognition (AGR)

Simple line arrangement: set of ℓ lines in $\mathbb{R}^2$



Arrangement graph:
$\ell(\ell-1)/2$ vertices
$\ell(\ell-2)$ edges

- Each pair of lines has exactly one intersection
- No three lines share a common point

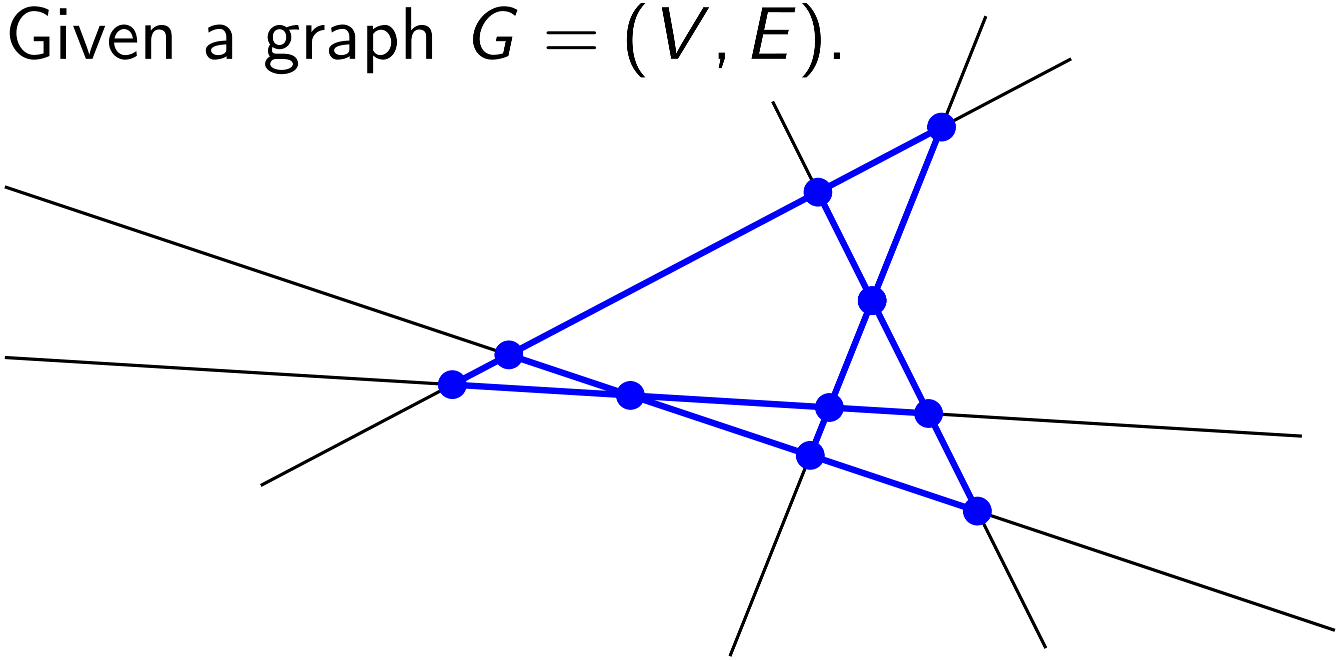
Problem AGR: Given a graph $G = (V, E)$, decide if it is an arrangement graph of some set of lines.

AGR is known to be $\exists\mathbb{R}$-hard

Line Cover Number is $\exists\mathbb{R}$-hard

Reduction from AGR

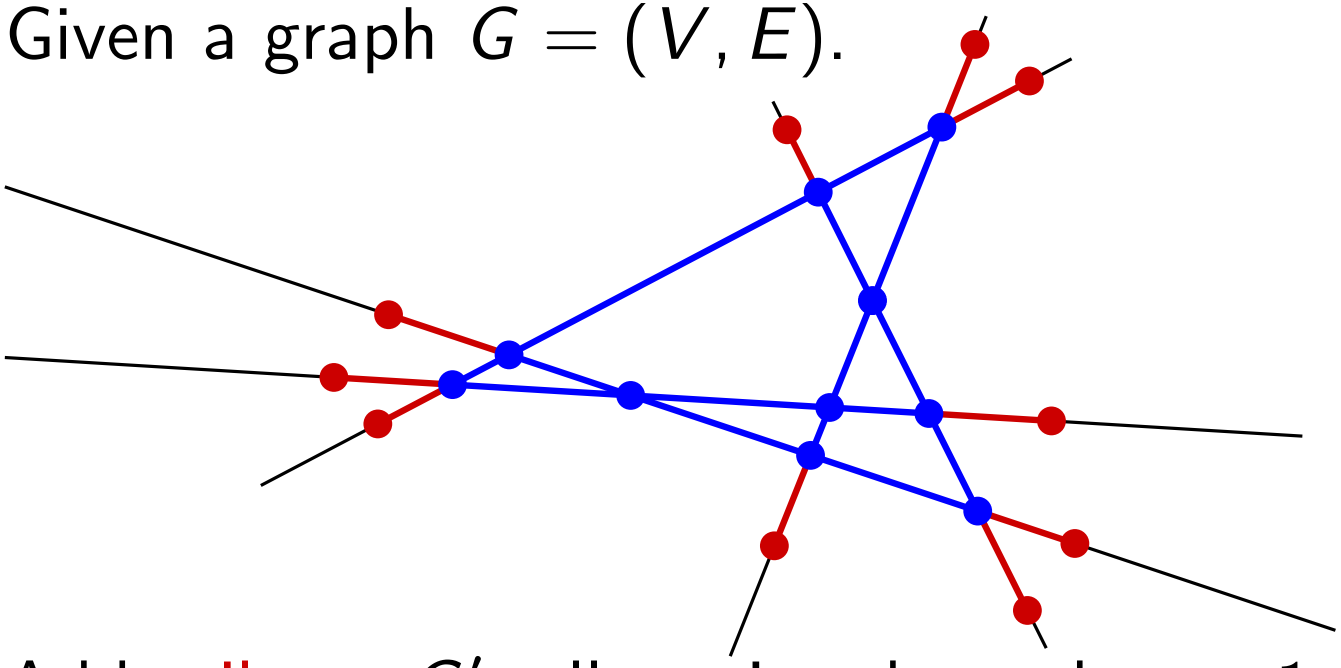
Given a graph $G = (V, E)$.



Line Cover Number is $\exists\mathbb{R}$-hard

Reduction from AGR

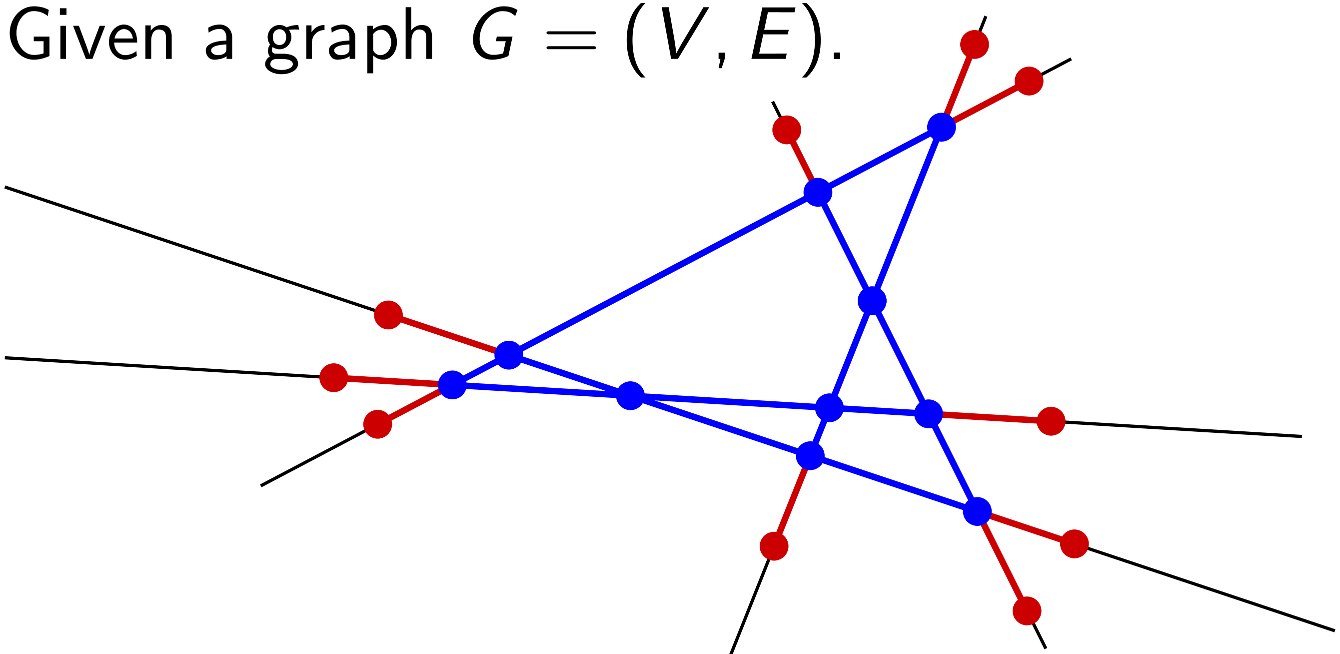
Given a graph $G = (V, E)$.



Add tails $\Rightarrow$ G': all vertices have degree 1 or 4.

Line Cover Number is $\exists\mathbb{R}$-hard

Reduction from AGR
Given a graph $G = (V, E)$.



Add tails $\Rightarrow$ G': all vertices have degree 1 or 4.

Claim: G is an arrangement graph iff $\rho_2^1(G') \leq \ell$
Proof similar to hardness proof for segment number.
[Durocher, Mondal, Nishat, Whitesides, JGAA 2013]

Theorem: Deciding whether $\rho_2^1(G) \leq k$ and $\rho_3^1(G) \leq k$ is $\exists\mathbb{R}$-complete.

Our Results

- Computing line cover numbers is $\exists\mathbb{R}$-hard (ρ_2^1 and ρ_3^1)

- Computing line cover numbers is fixed-parameter tractable (ρ_2^1 and ρ_3^1)

- Computing plane cover numbers is NP-hard (ρ_3^2)

Fixed-parameter Tractability

Kernelization: Reduce G to an instance G' with size bounded by $f(k)$

Assume that G has a k-line cover.

Fixed-parameter Tractability

Kernelization: Reduce G to an instance G' with size bounded by $f(k)$

Assume that G has a k-line cover. at most $\binom{k}{2}$ intersections

Fixed-parameter Tractability

Kernelization: Reduce G to an instance G' with size bounded by $f(k)$

Assume that G has a k-line cover. at most $\binom{k}{2}$ intersections

Remove connected components that are paths.

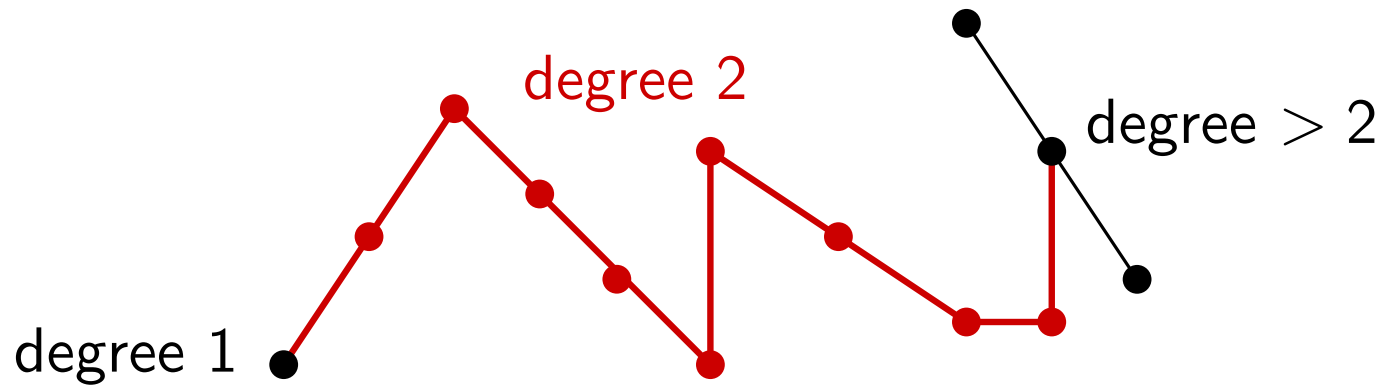
Fixed-parameter Tractability

Kernelization: Reduce G to an instance G' with size bounded by $f(k)$

Assume that G has a k-line cover. at most $\binom{k}{2}$ intersections

Remove connected components that are paths.

Contract long **paths** to at most $\binom{k}{2}$ vertices.



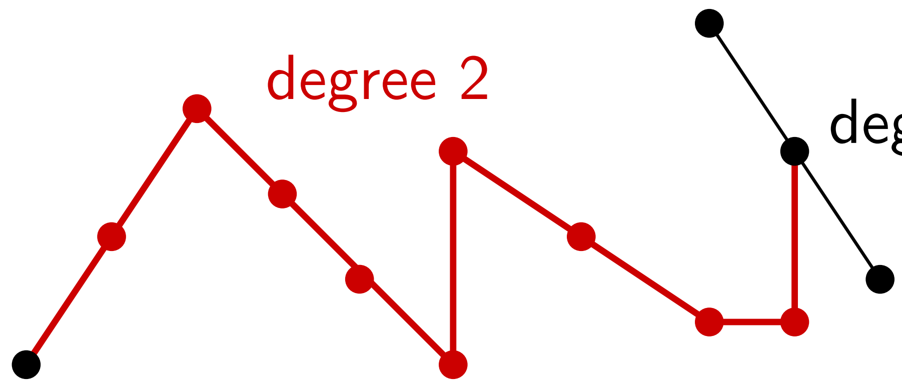
Fixed-parameter Tractability

Kernelization: Reduce G to an instance G' with size bounded by $f(k)$

Assume that G has a k-line cover. at most $\binom{k}{2}$ intersections

Remove connected components that are paths.

Contract long **paths** to at most $\binom{k}{2}$ vertices.



Number of paths: $O(k^2)$

Vertices per path: at most $\binom{k}{2}$

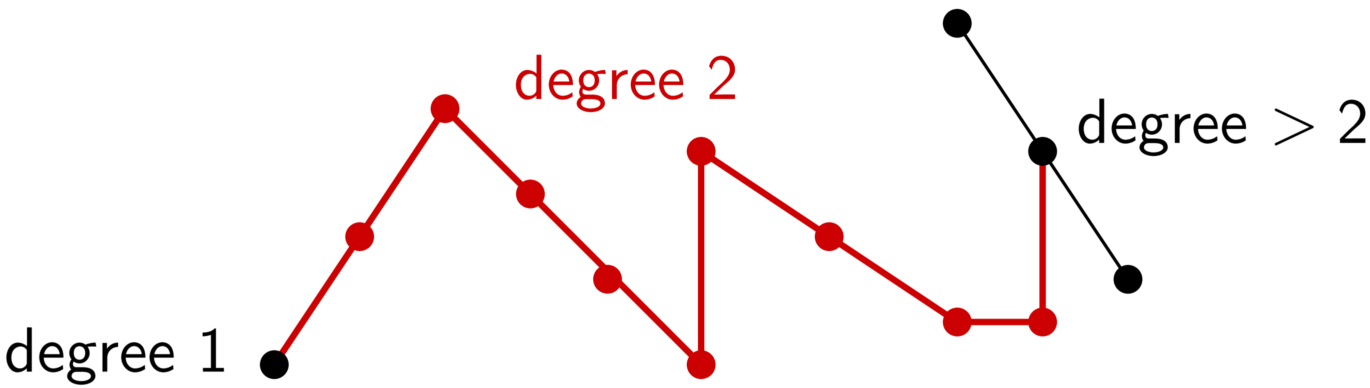
Fixed-parameter Tractability

Kernelization: Reduce G to an instance G' with size bounded by $f(k)$

Assume that G has a k-line cover. at most $\binom{k}{2}$ intersections

Remove connected components that are paths.

Contract long **paths** to at most $\binom{k}{2}$ vertices.



Number of paths: $O(k^2)$

Vertices per path: at most $\binom{k}{2}$

$\Big\}$ $O(k^4)$ vertices, $O(k^4)$ edges

Fixed-parameter Tractability

Test line cover number for reduced graph G' of size $O(k^4)$. Describe as a formula Φ in the existential theory of the reals and use Renegar's decision algorithm.

Fixed-parameter Tractability

Test line cover number for reduced graph G' of size $O(k^4)$. Describe as a formula Φ in the existential theory of the reals and use Renegar's decision algorithm.

Theorem: For graph G and integer k, in time $2^{O(k^3)} + O(n+m)$ we can decide

- whether $\rho^1_d(G) \leq k$ and, if so,
- give a combinatorial description of the drawing.

Our Results

- Computing line cover numbers is $\exists\mathbb{R}$-hard (ρ_2^1 and ρ_3^1)
- Computing line cover numbers is fixed-parameter tractable (ρ_2^1 and ρ_3^1)
- Computing plane cover numbers is NP-hard (ρ_3^2)

Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT

(x_1, x_2, x_6) (x_1, x_4, x_5) (x_1, x_3, x_4) (x_3, x_5, x_6) (x_2, x_3, x_6)

Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT

(x_1, x_2, x_6) (x_1, x_4, x_5) (x_1, x_3, x_4) (x_3, x_5, x_6) (x_2, x_3, x_6)

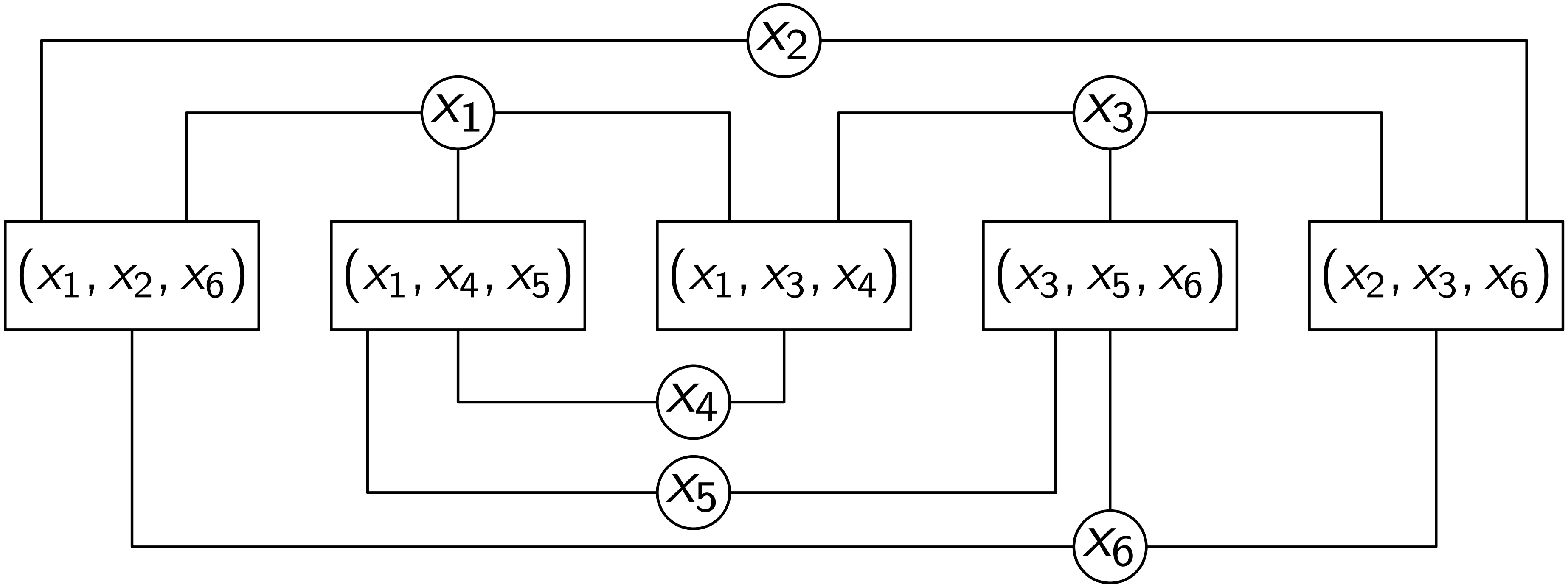
Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT

(x_1, x_2, x_6) (x_1, x_4, x_5) (x_1, x_3, x_4) (x_3, x_5, x_6) (x_2, x_3, x_6)

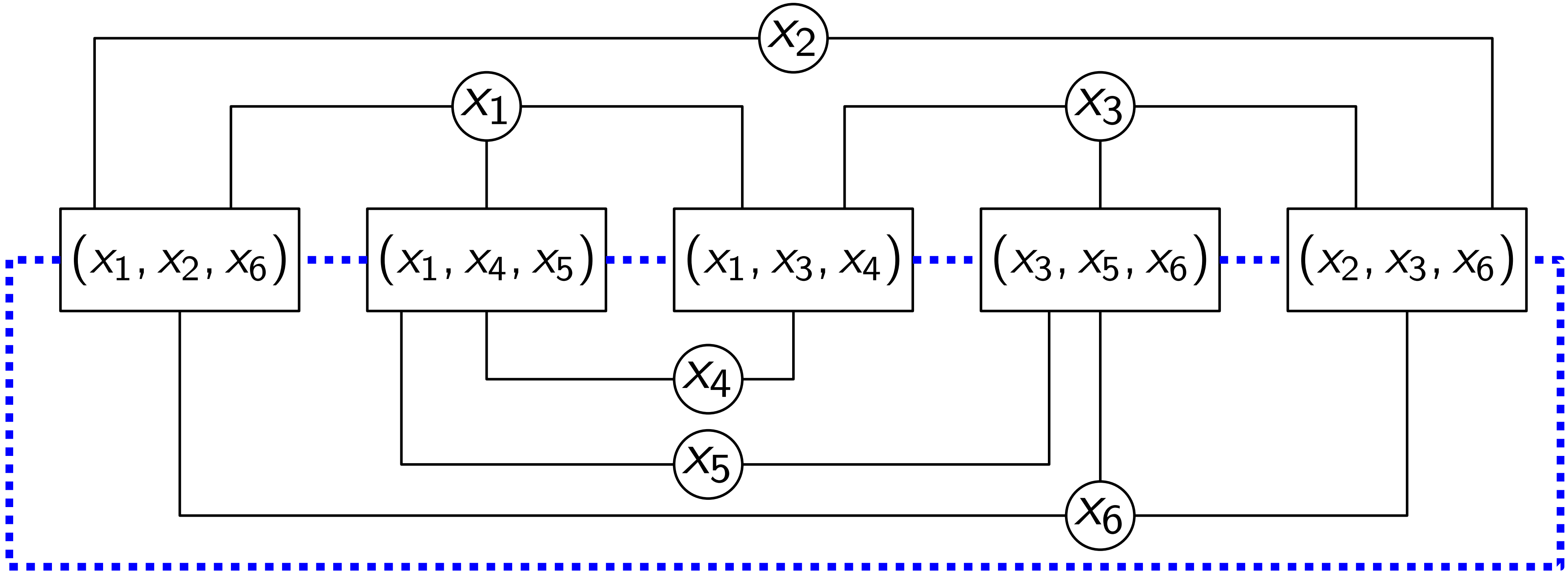
Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT



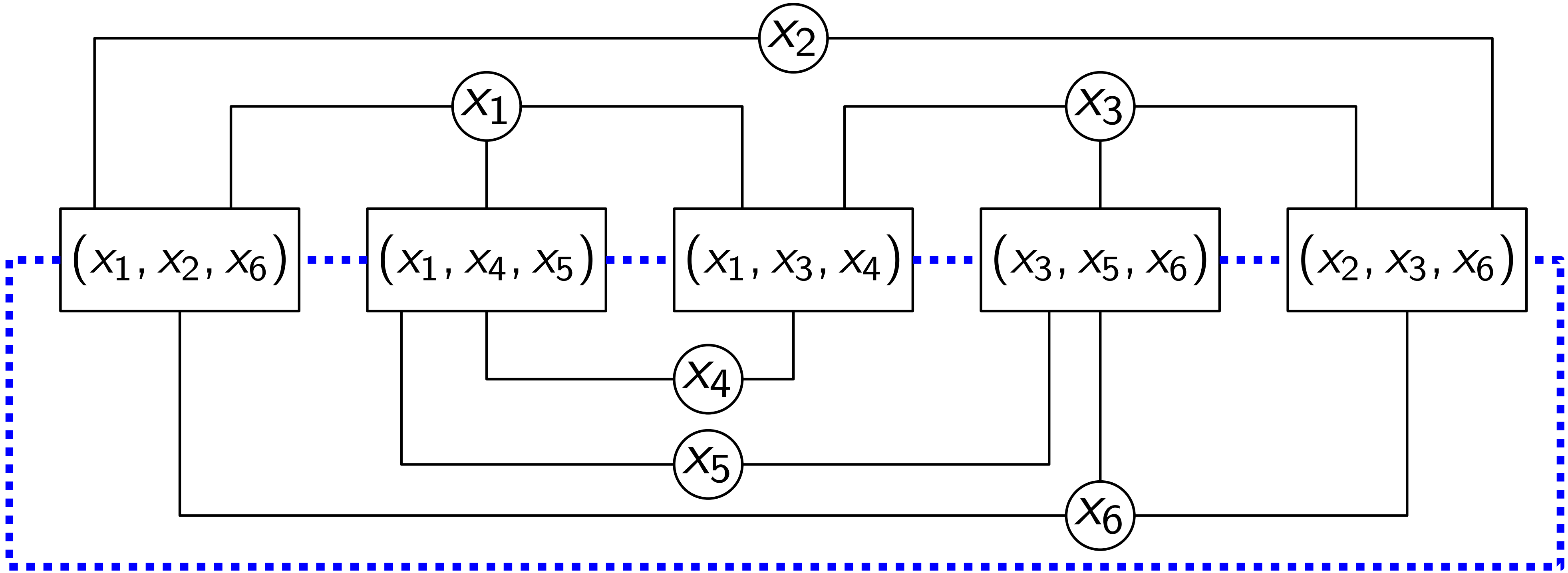
Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT



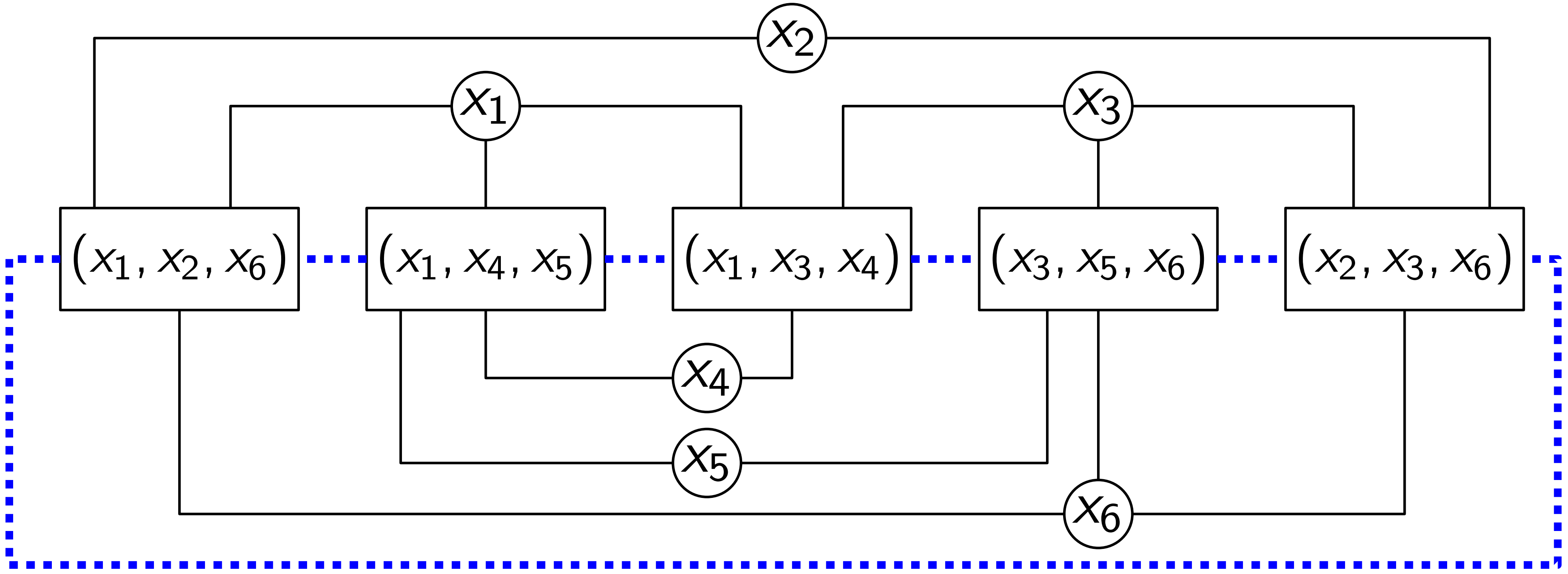
Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT



Detour: 3-SAT Variant

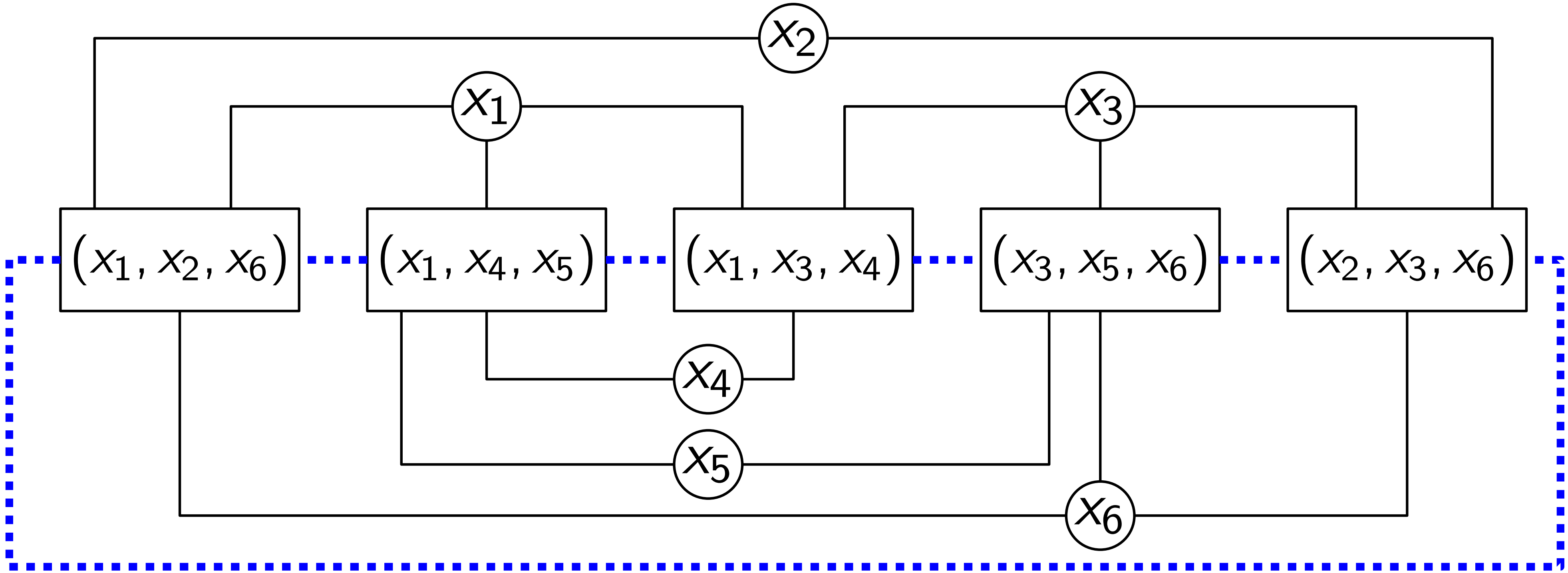
We use a reduction from Positive Planar Cycle 1-in-3-SAT



NP-hardness proof:

Detour: 3-SAT Variant

We use a reduction from Positive Planar Cycle 1-in-3-SAT



NP-hardness proof:

Planar Cycle 3-SAT
[Kratochvíl, Lubiw, Nešetřil, 1991]

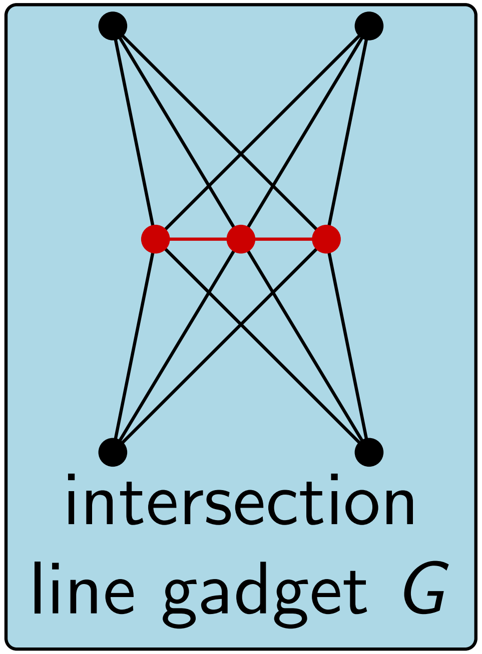
+

Positive Planar 1-in-3-SAT
[Mulzer, Rote, 2008]

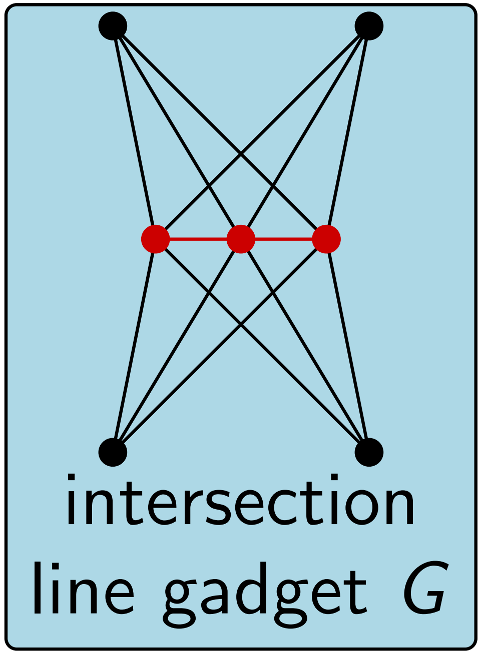
⇓ combined and modified

Positive Planar Cycle 1-in-3-SAT is NP-hard

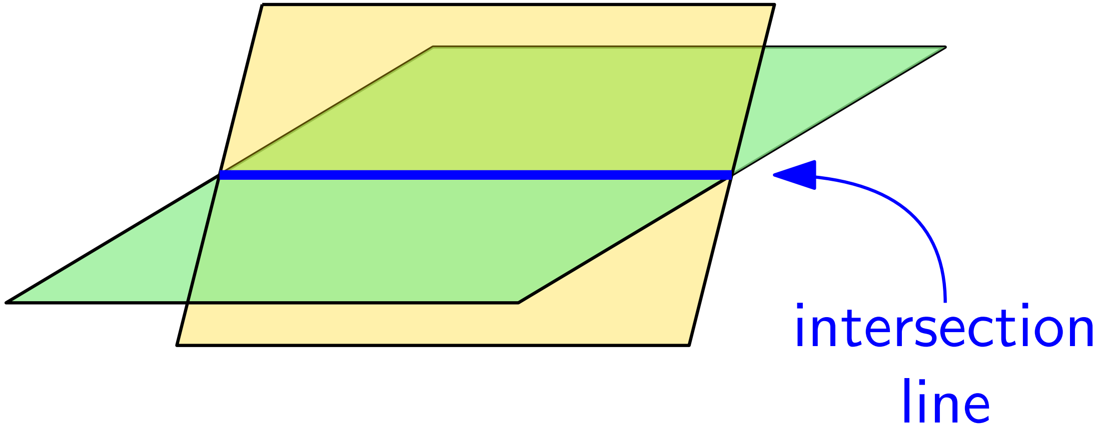
Intersection Line Gadget



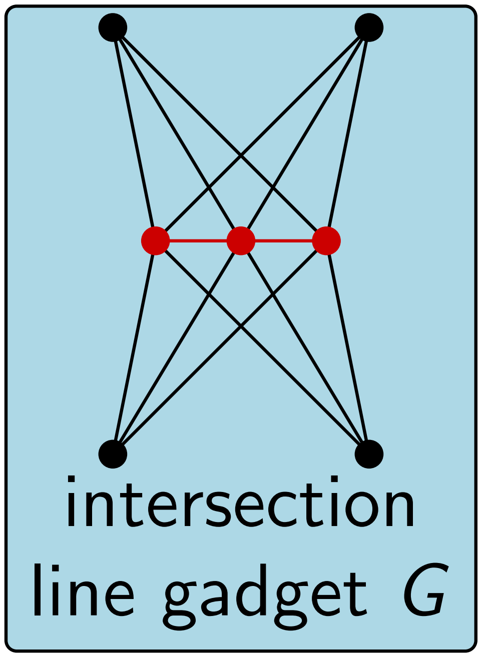
Intersection Line Gadget



Assume that G is embedded on two planes

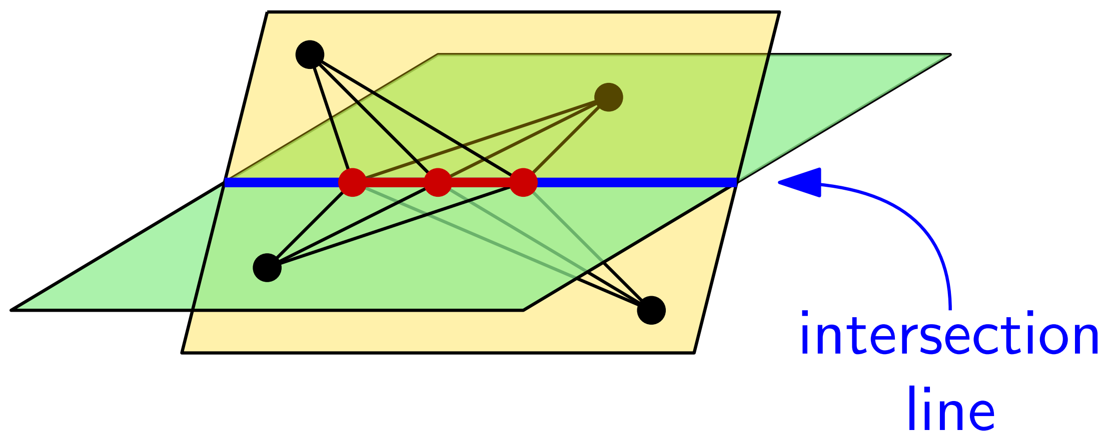


Intersection Line Gadget



Assume that G is embedded on two planes

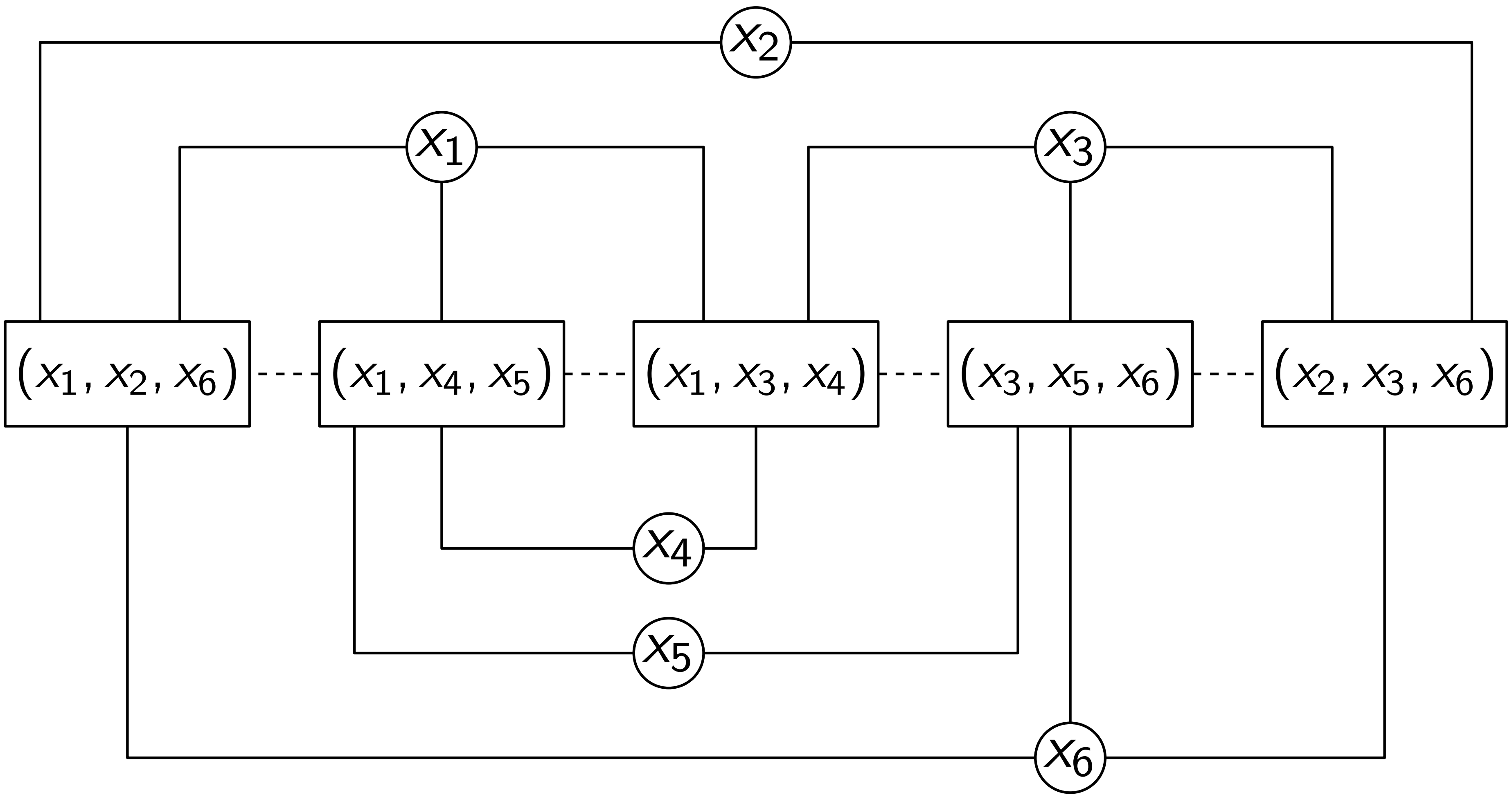
$\Rightarrow$ Red vertices are placed on the intersection line



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

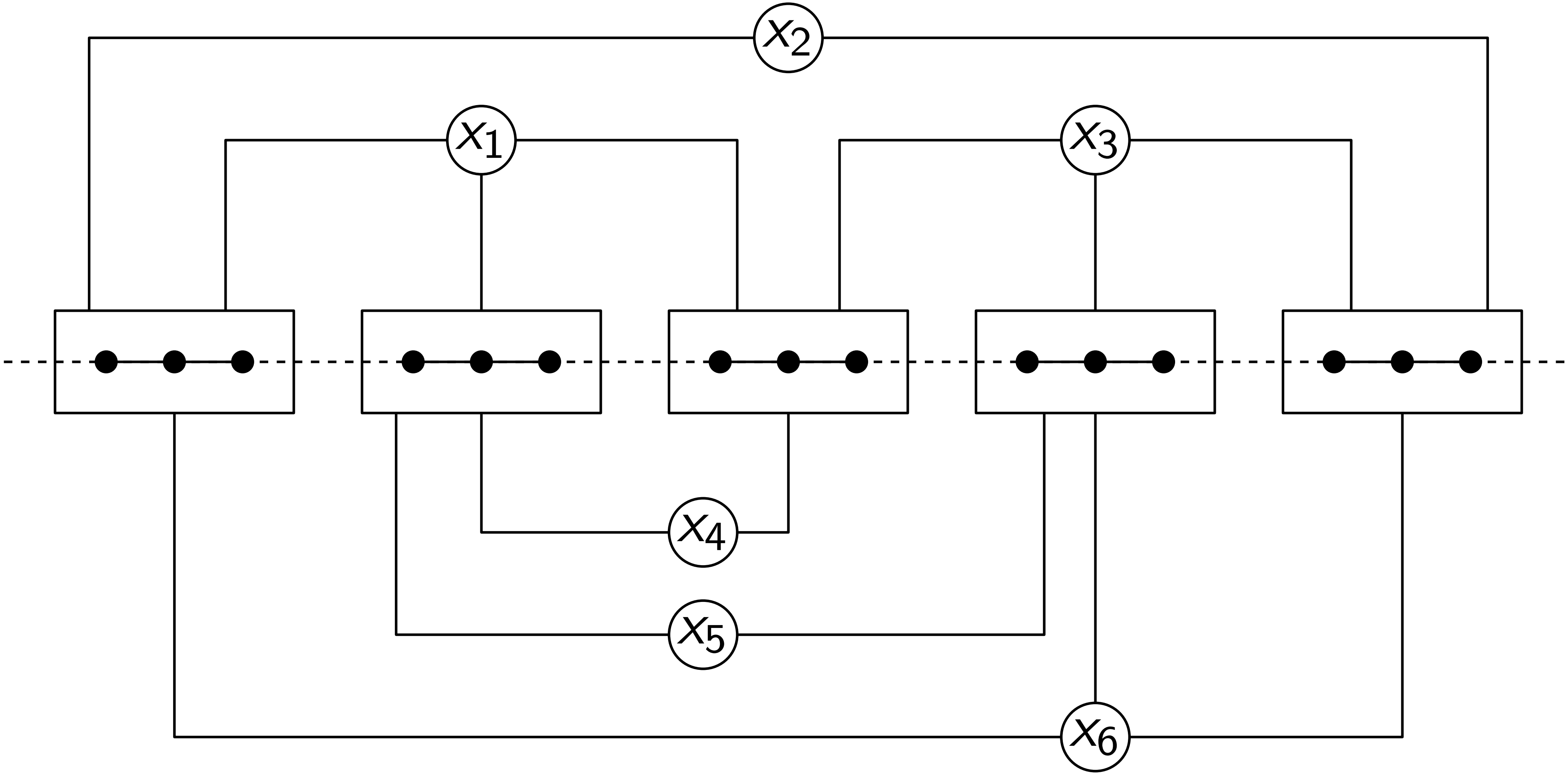
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

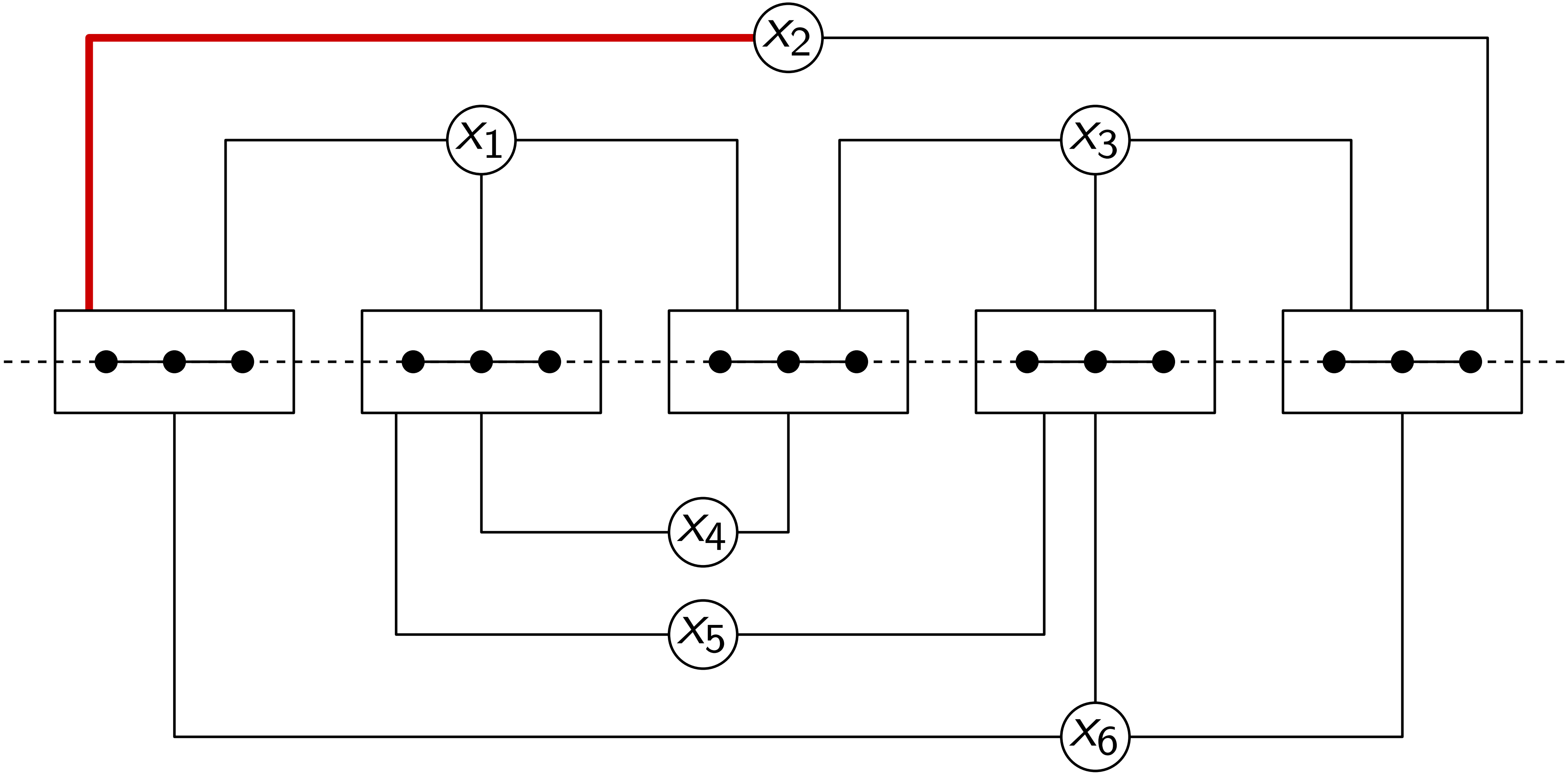
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

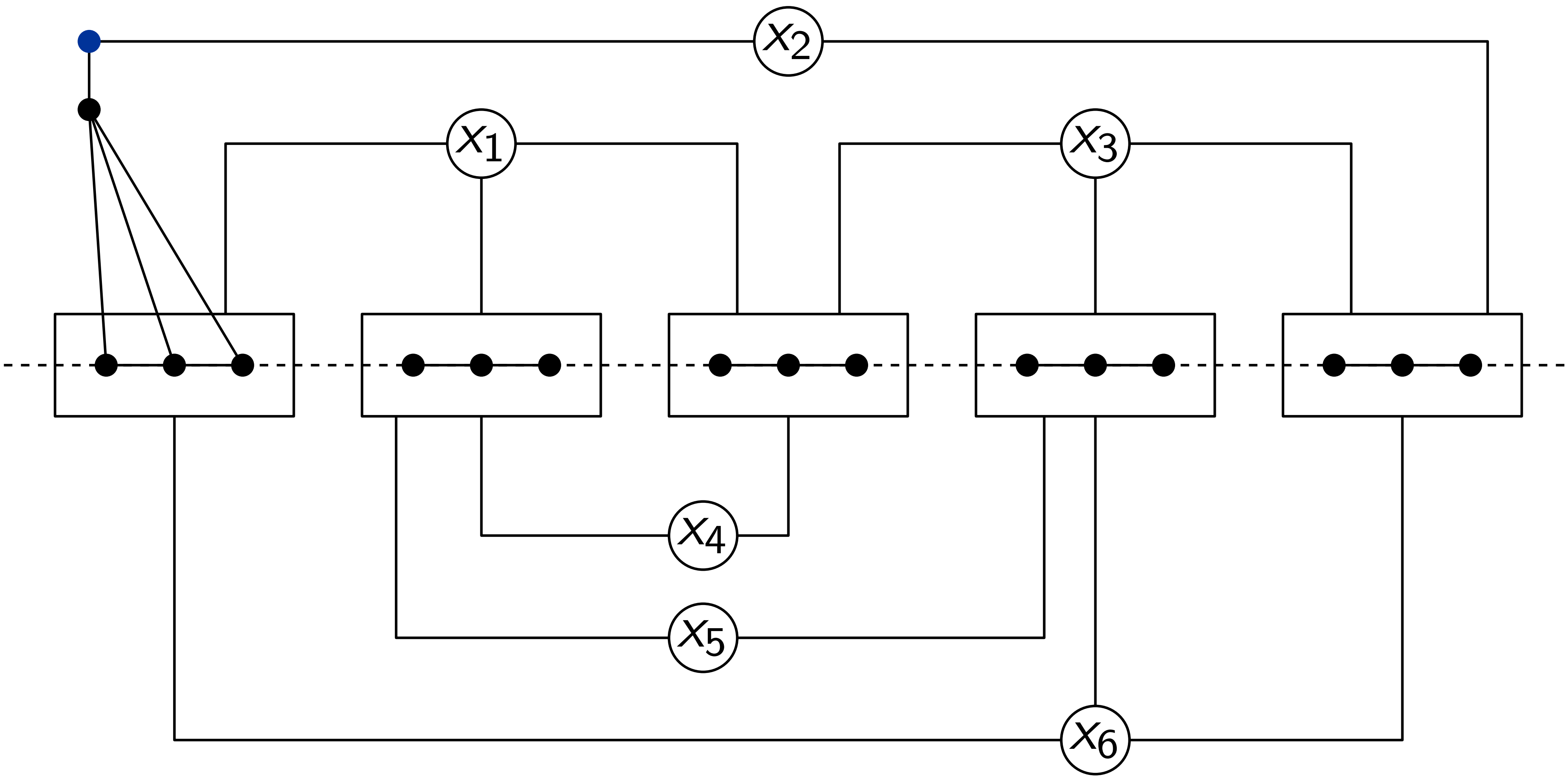
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

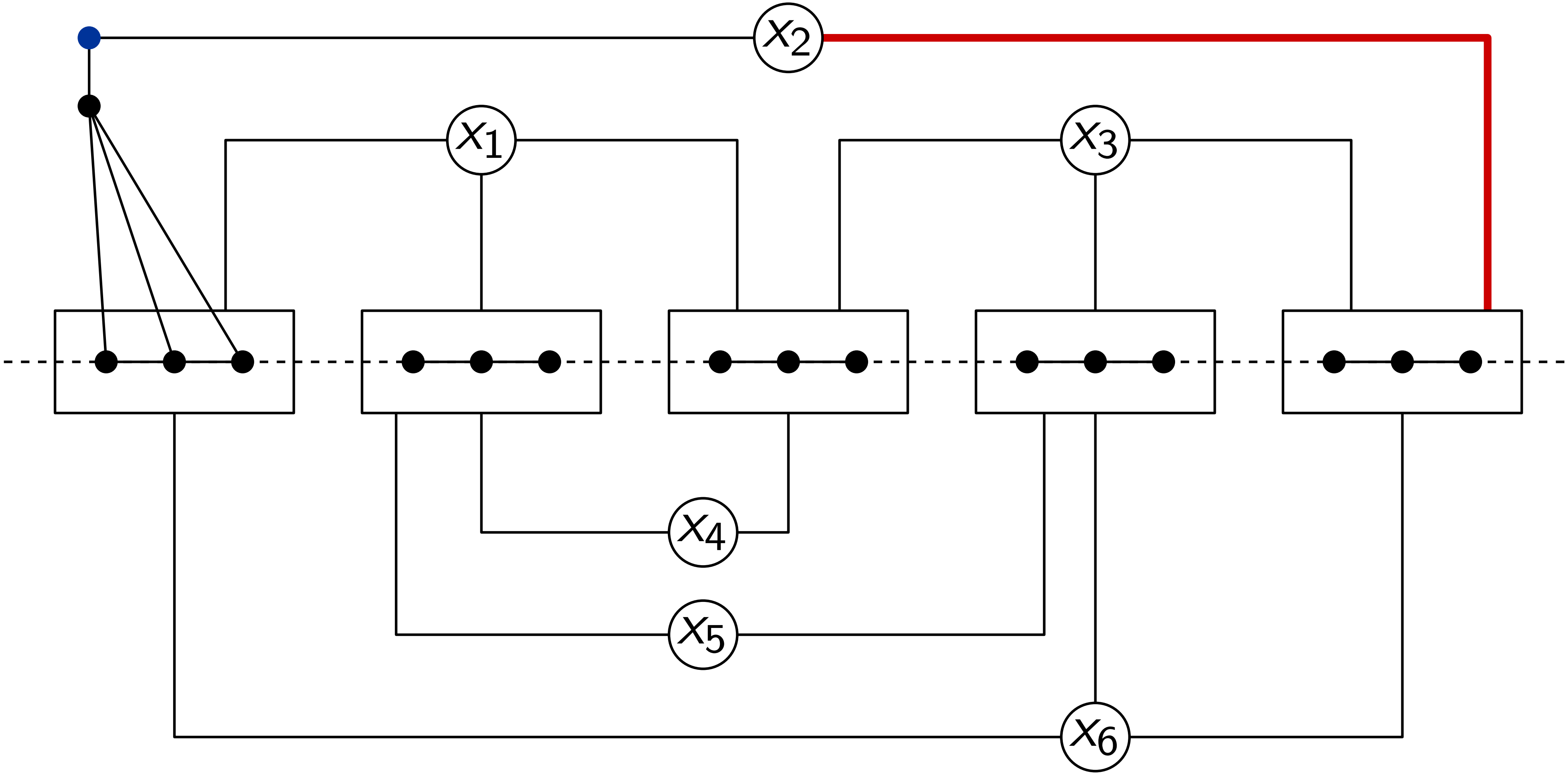
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

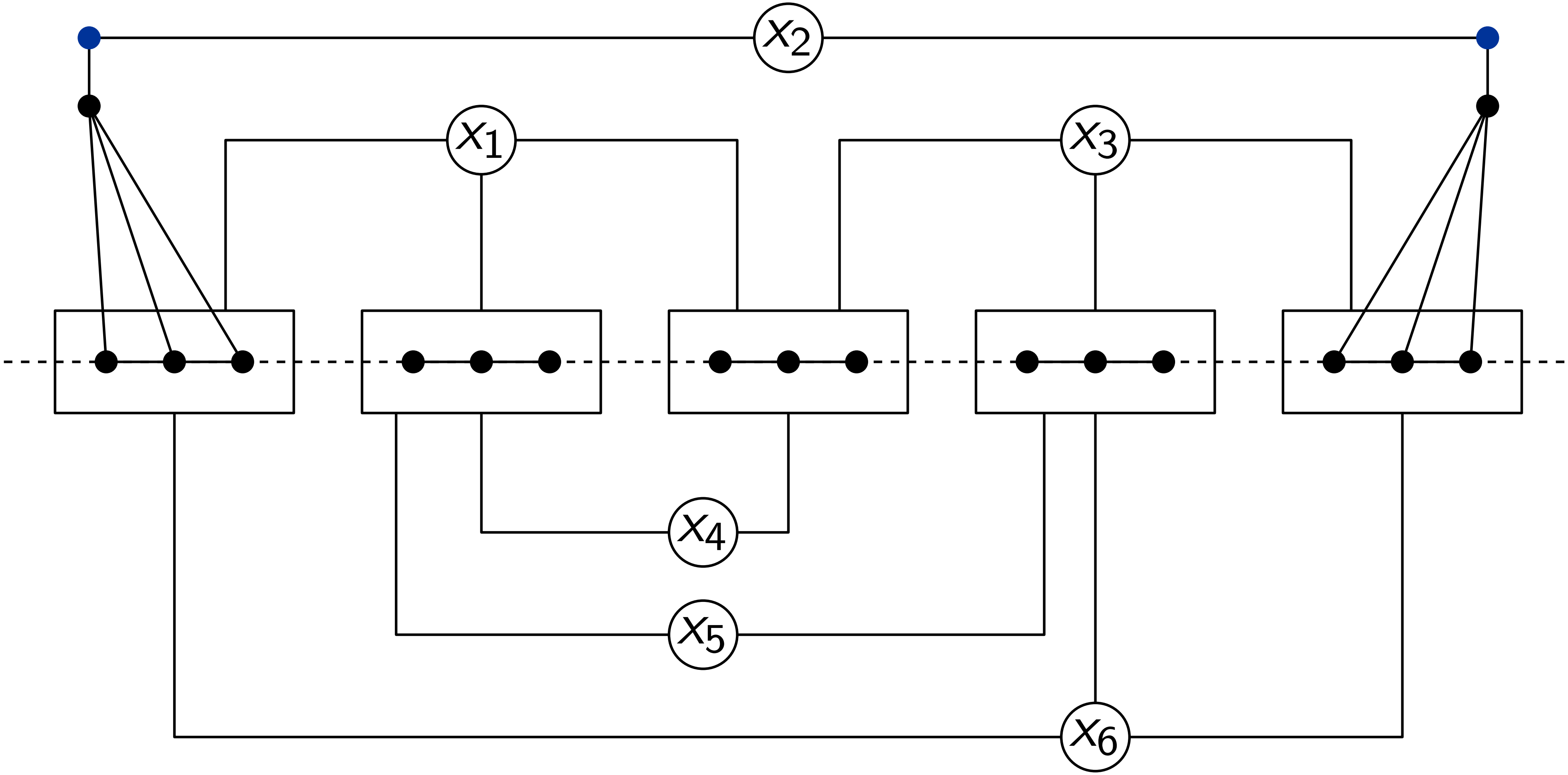
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

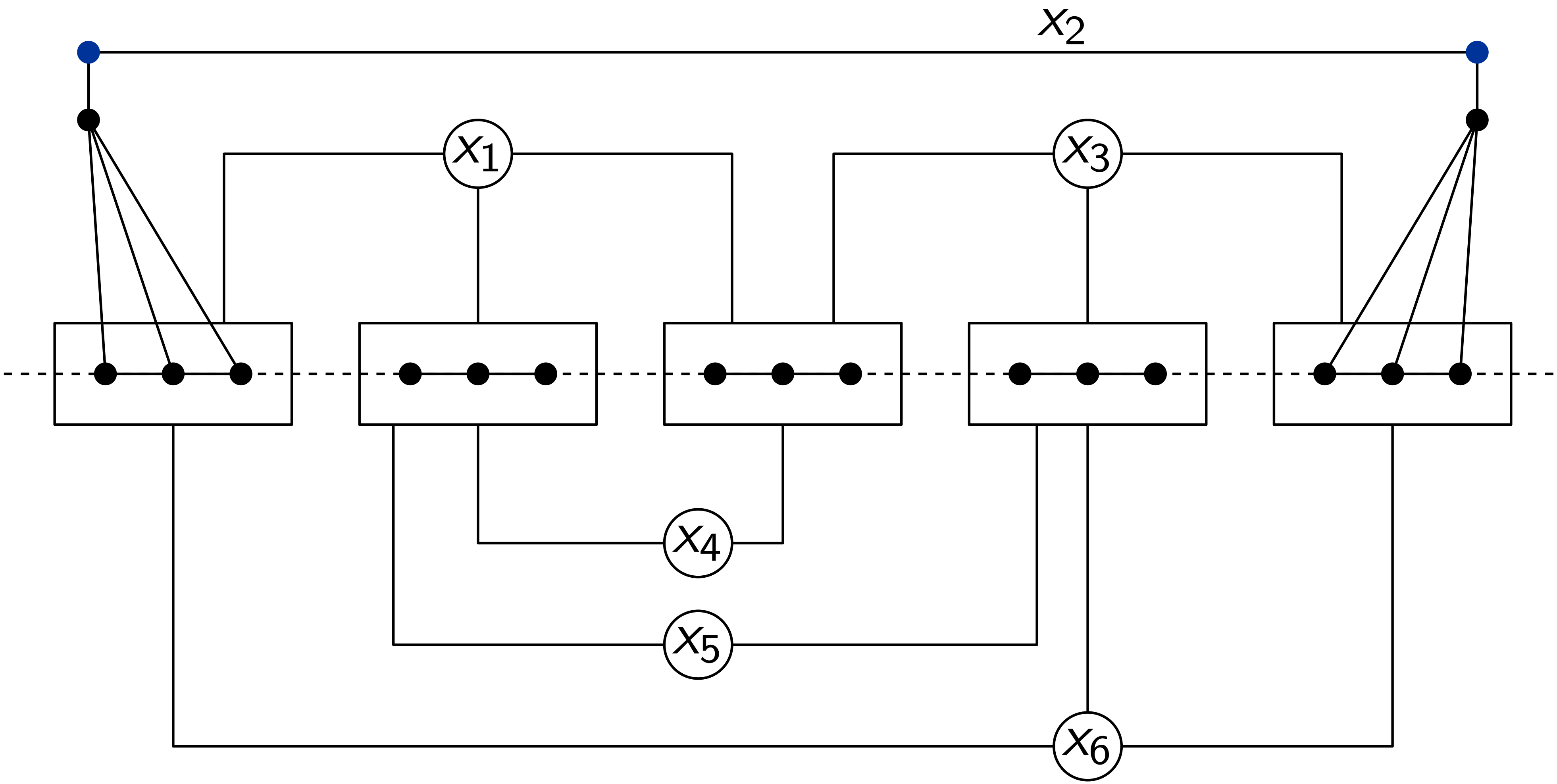
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

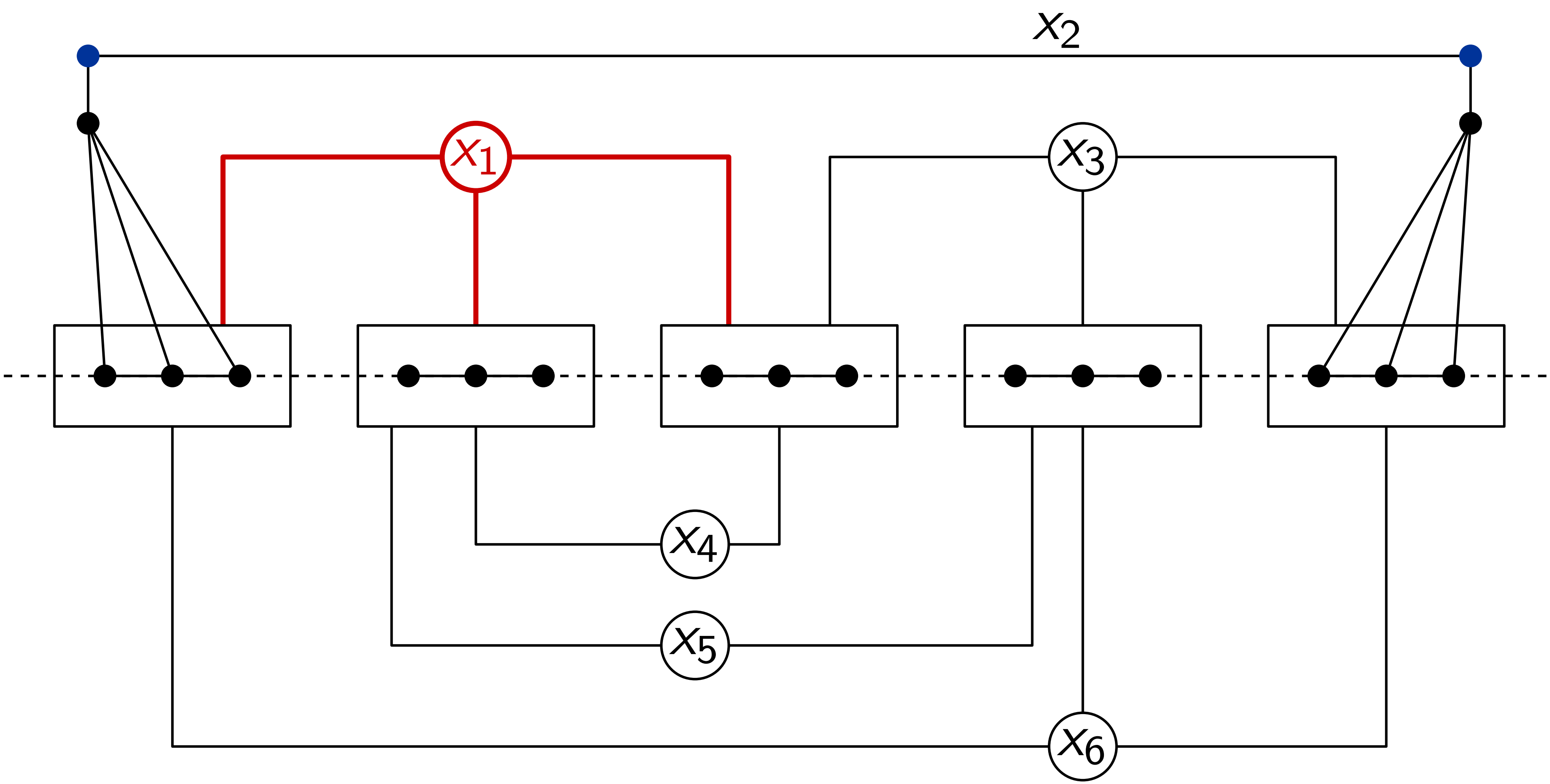
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

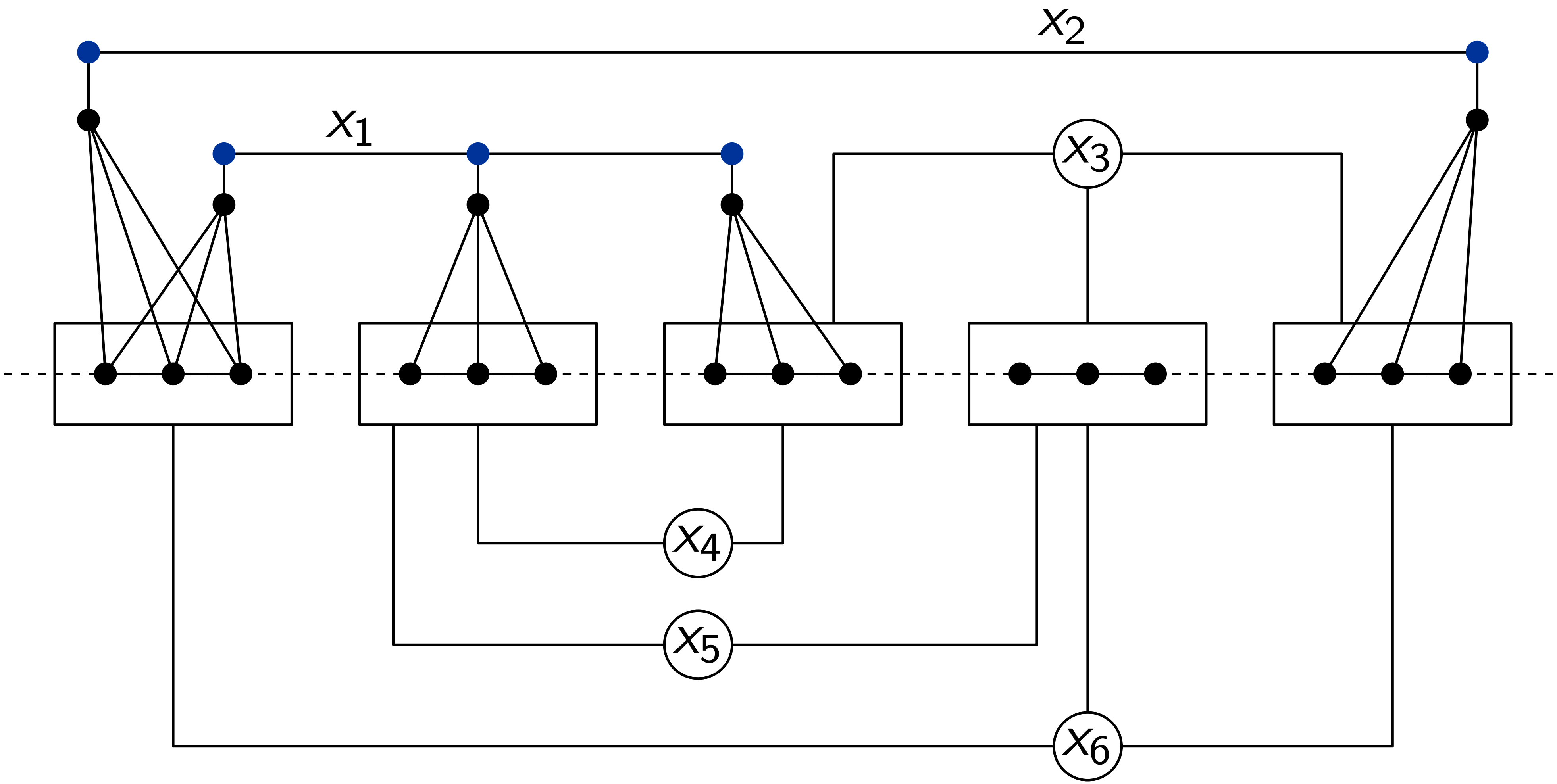
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

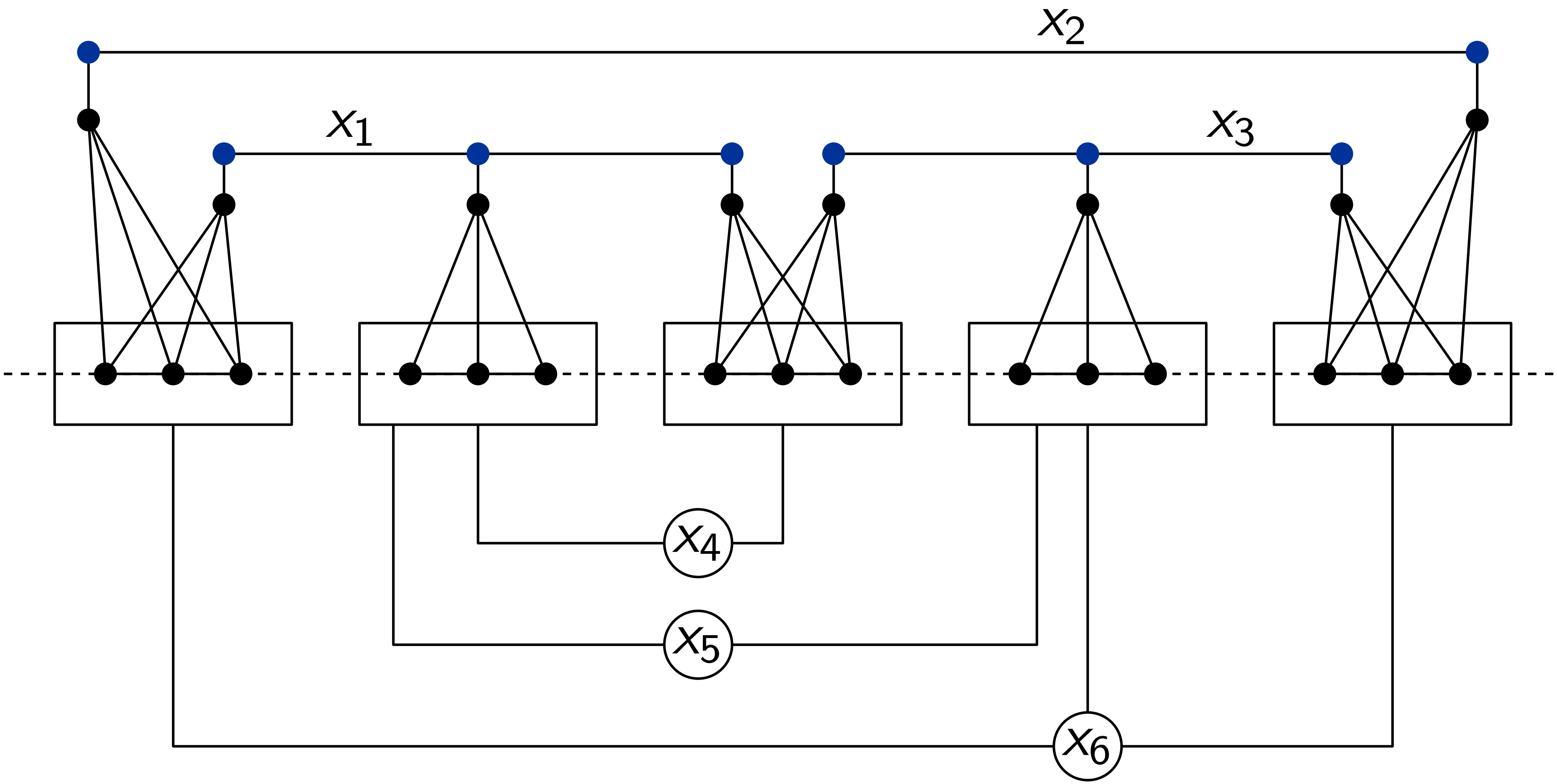
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

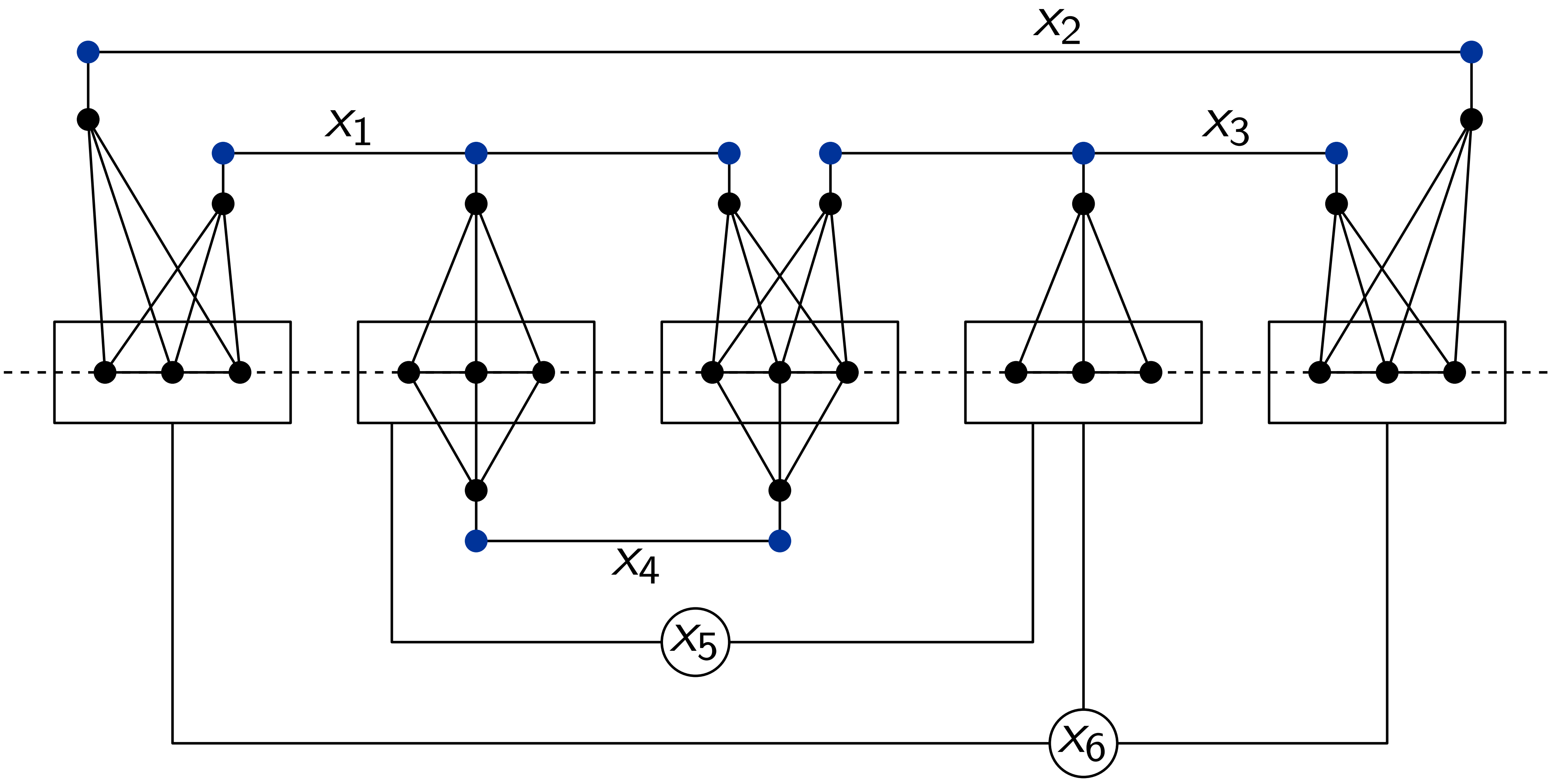
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

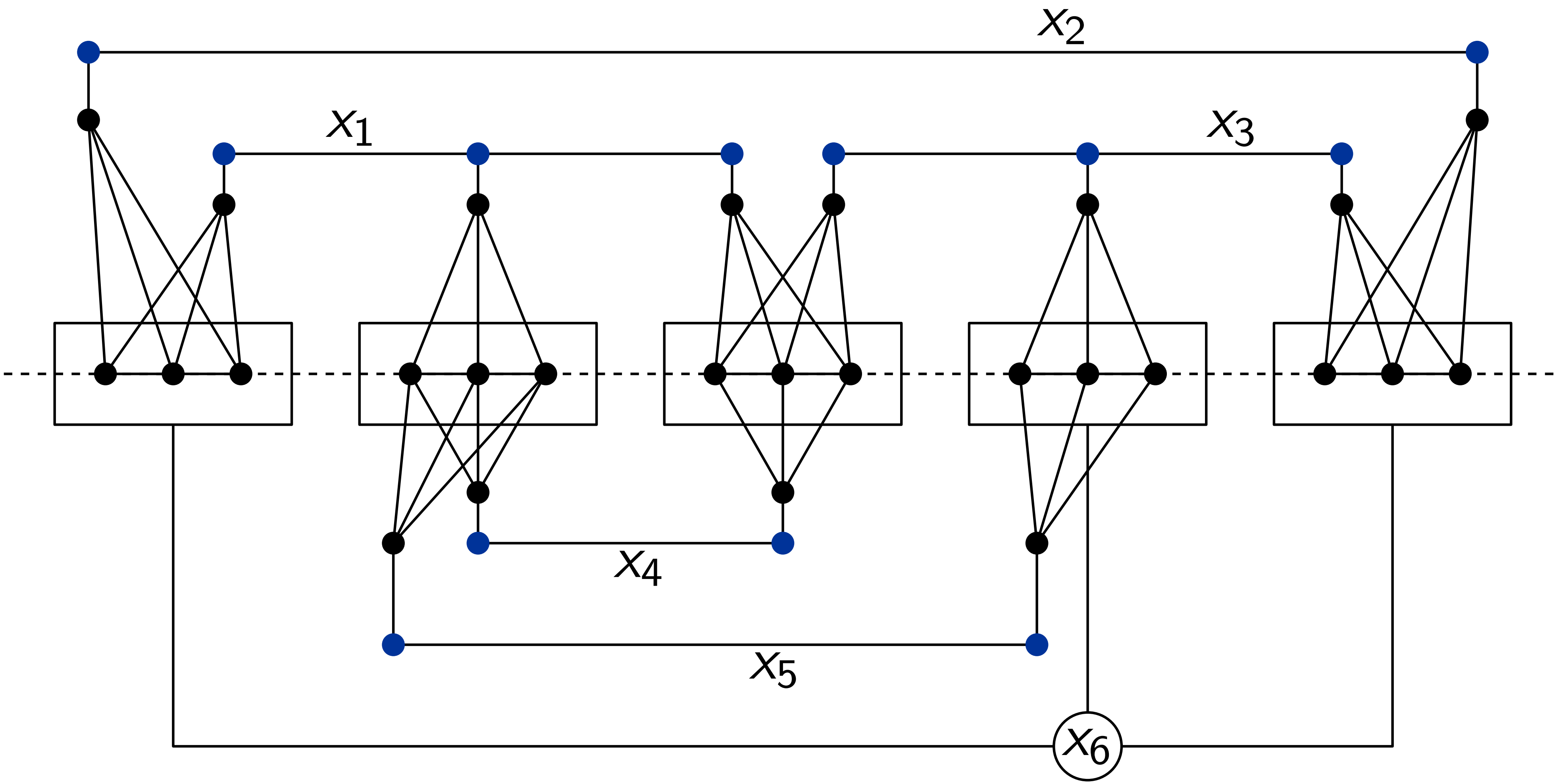
Proof:



Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

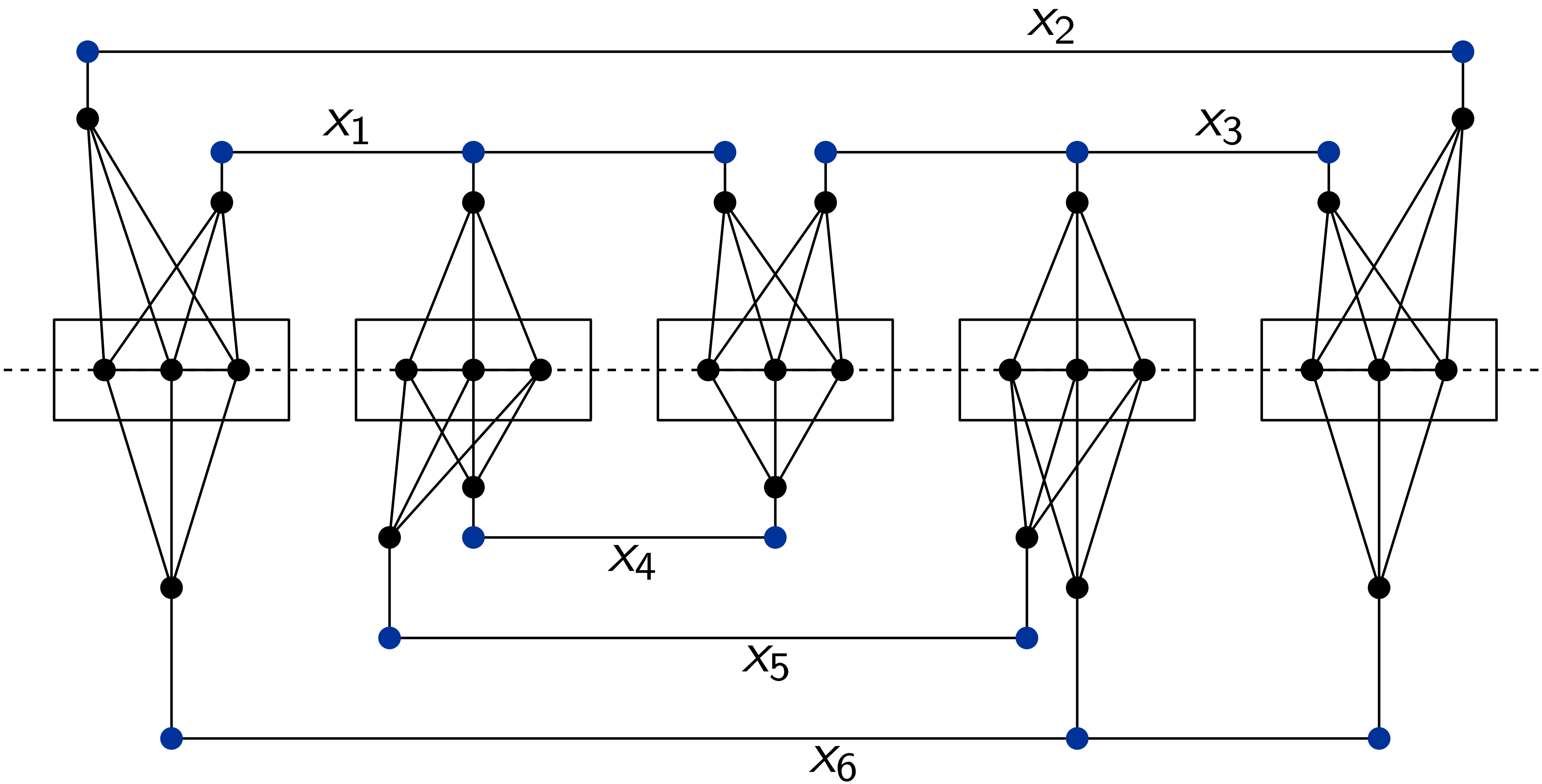
Proof:



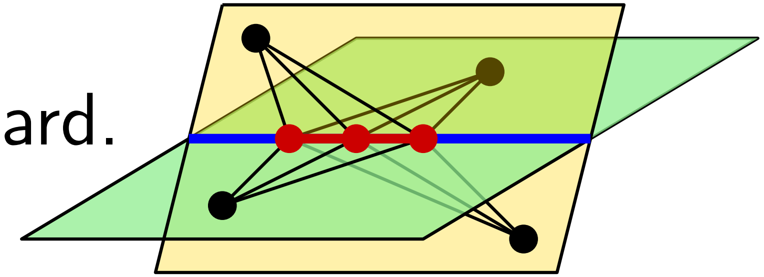
Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

Proof:

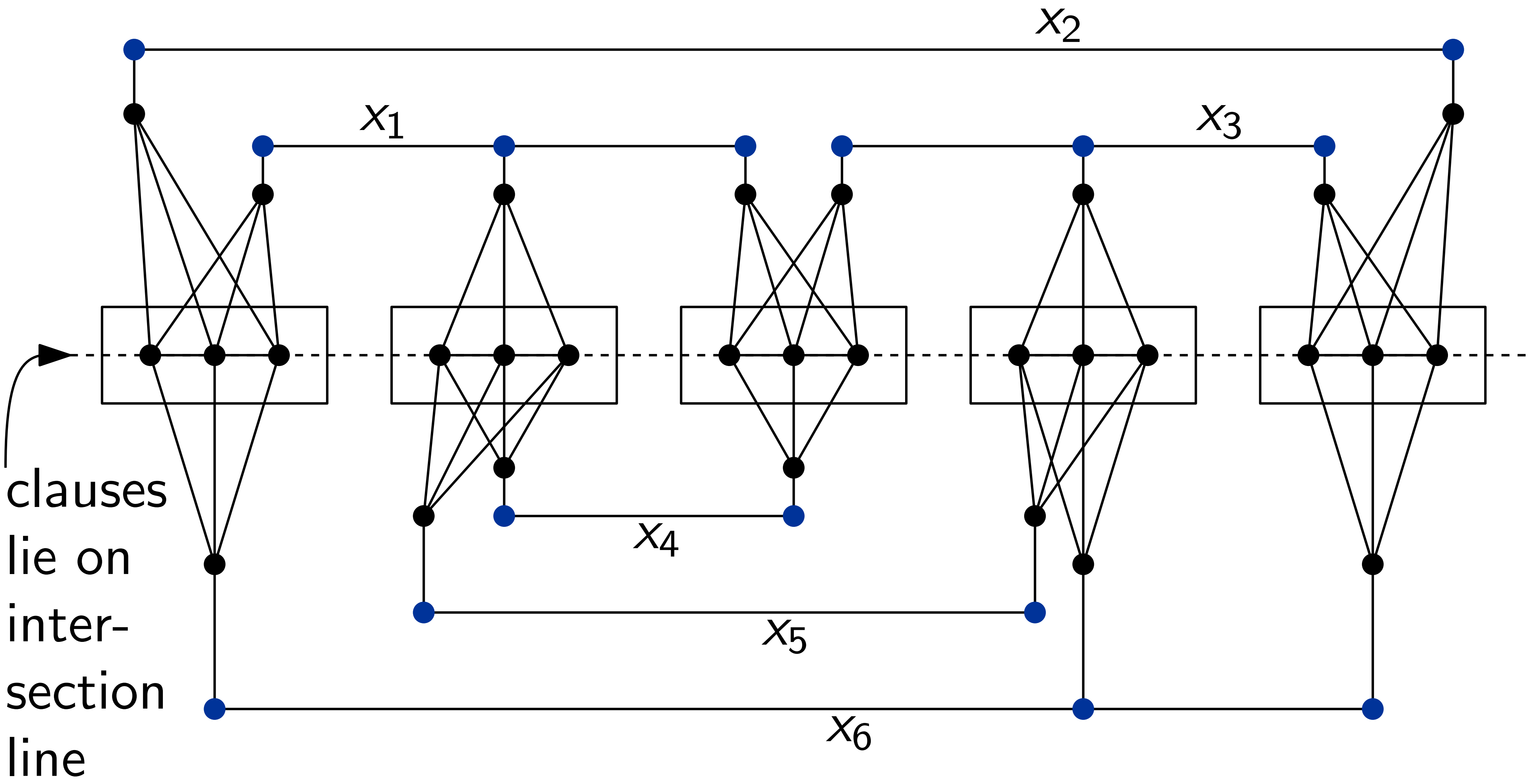


Reduction to Plane Cover Number



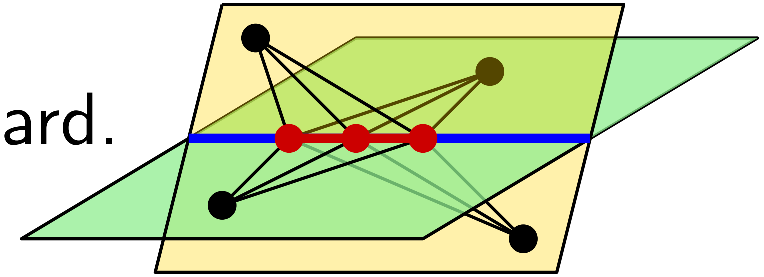
Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

Proof:

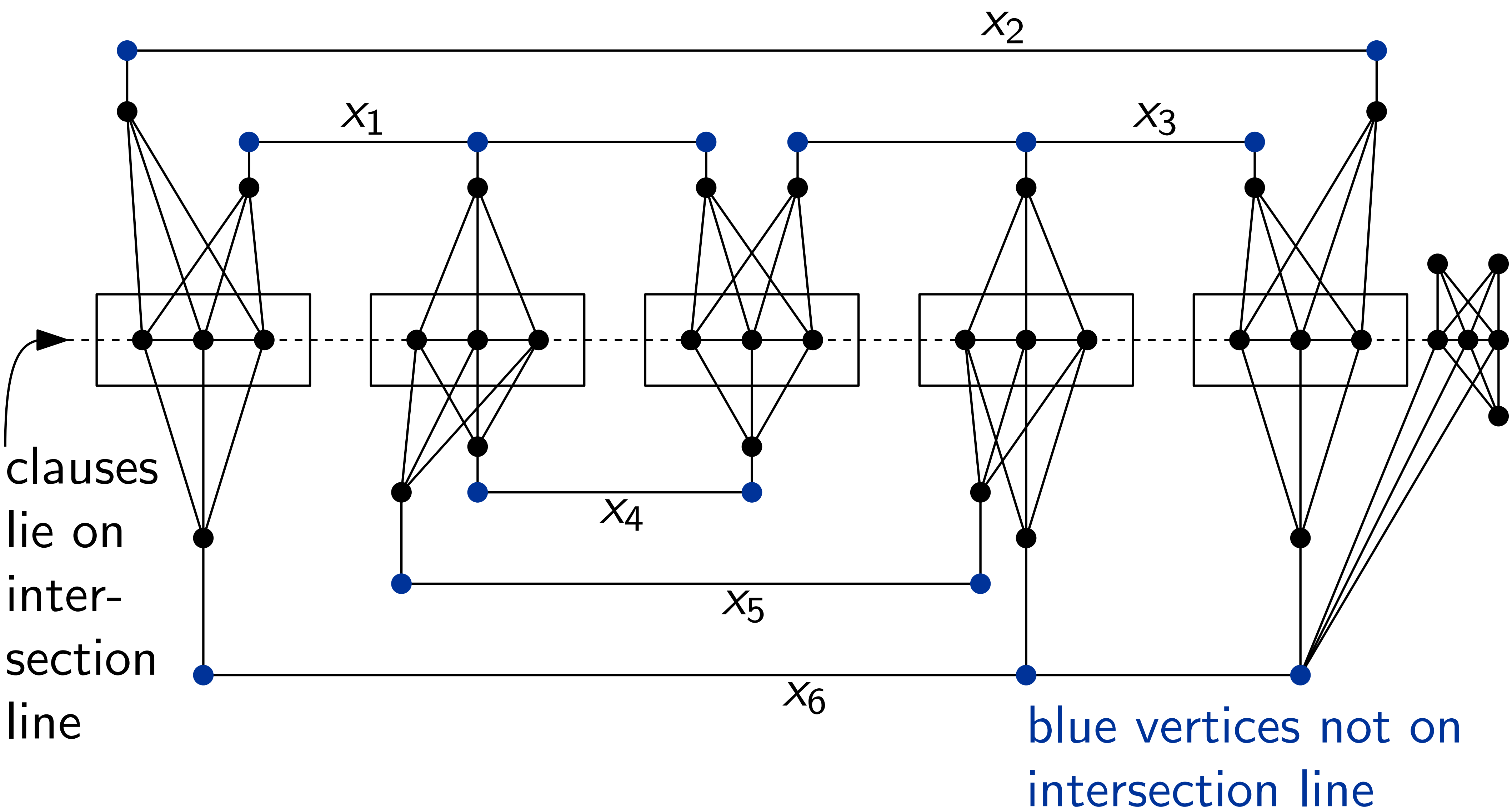


Reduction to Plane Cover Number

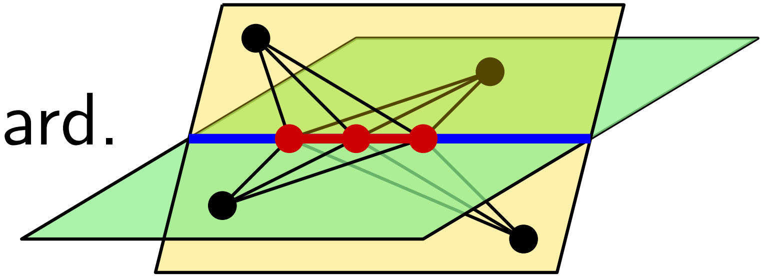
Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.



Proof:

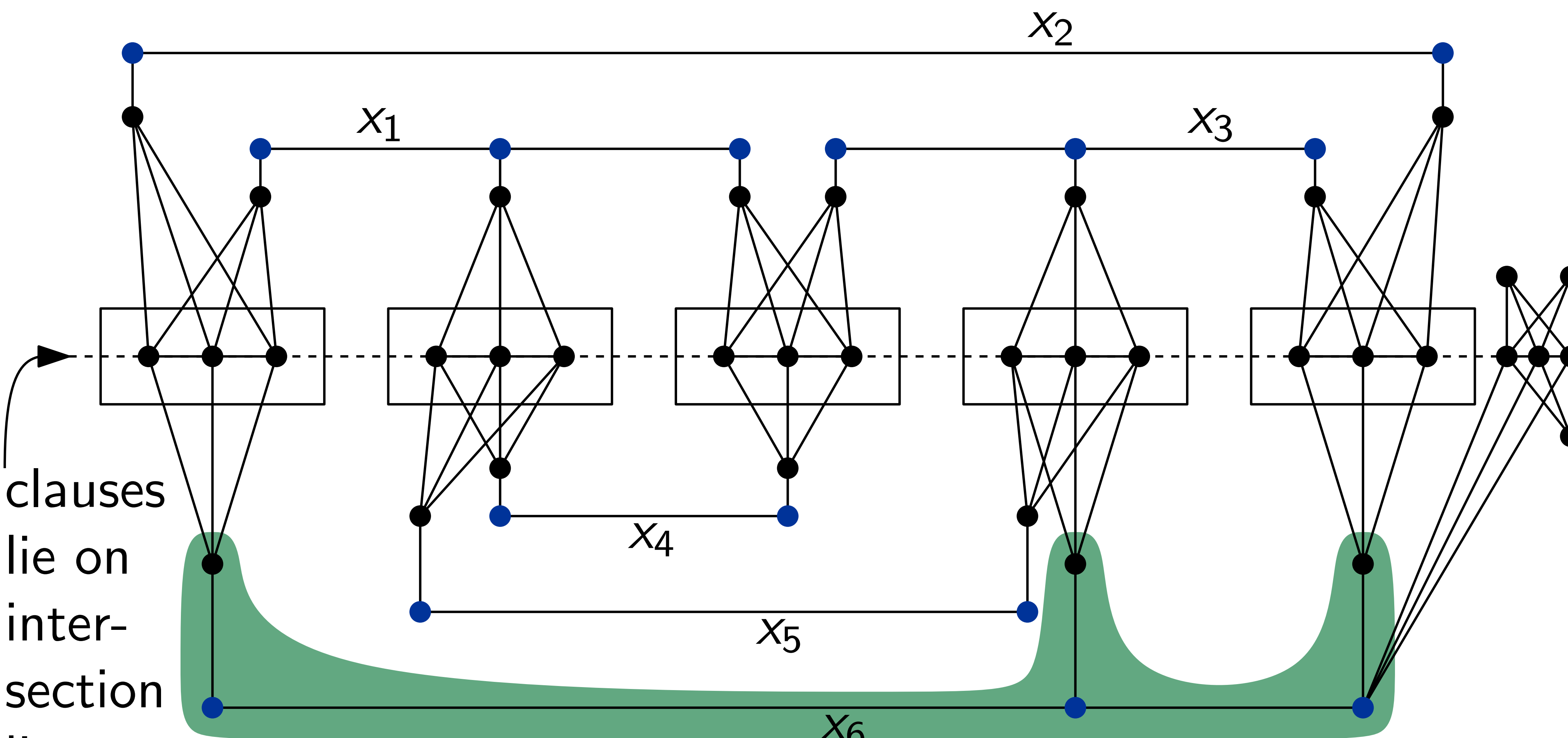


Reduction to Plane Cover Number



Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

Proof:



clauses lie on inter-section line

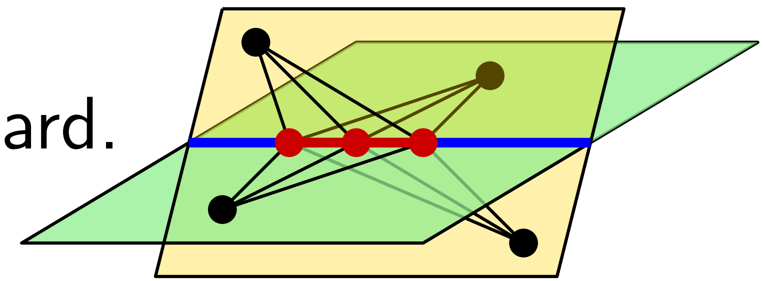
each variable only on one plane

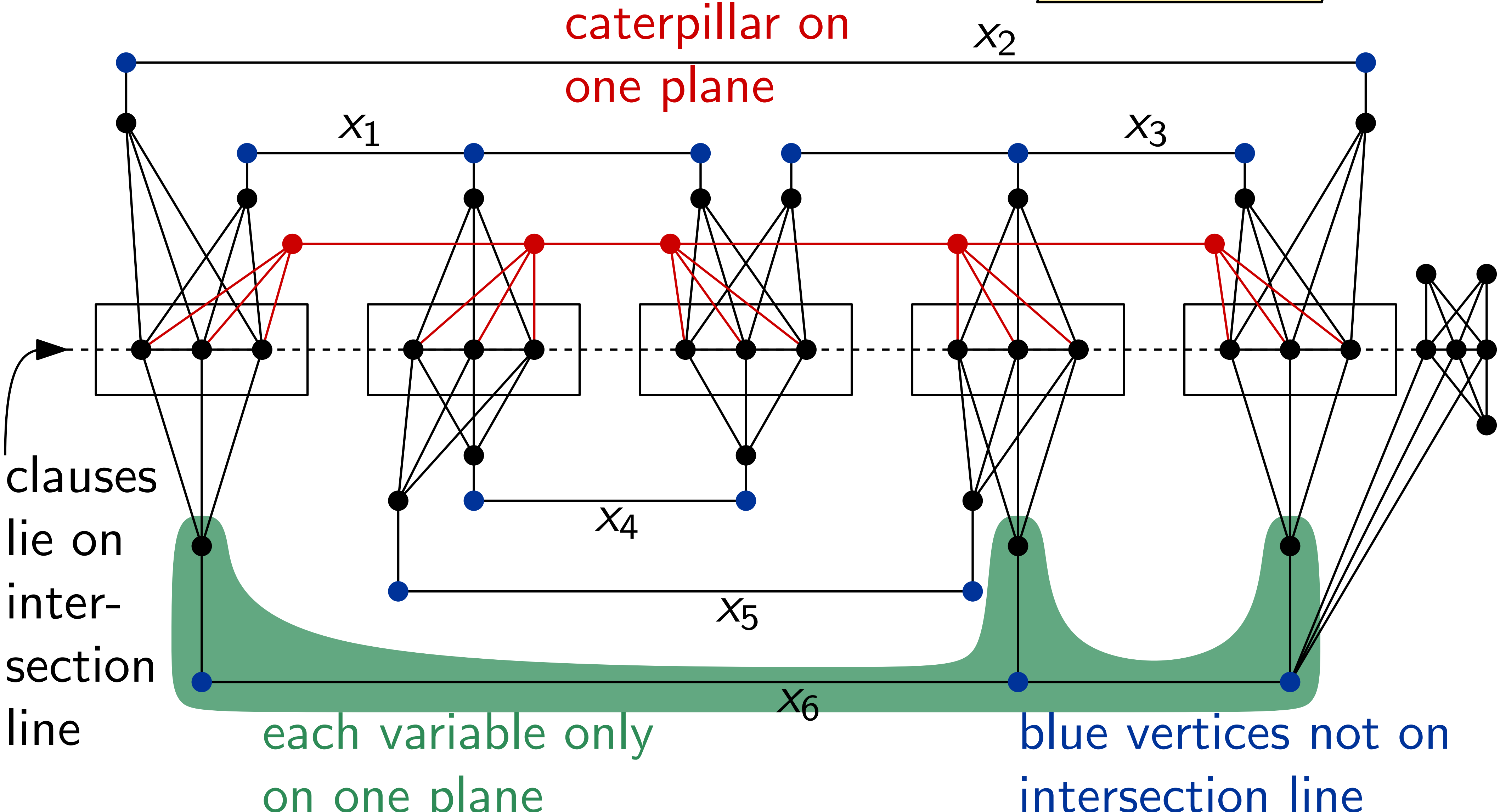
blue vertices not on intersection line

Reduction to Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

Proof:

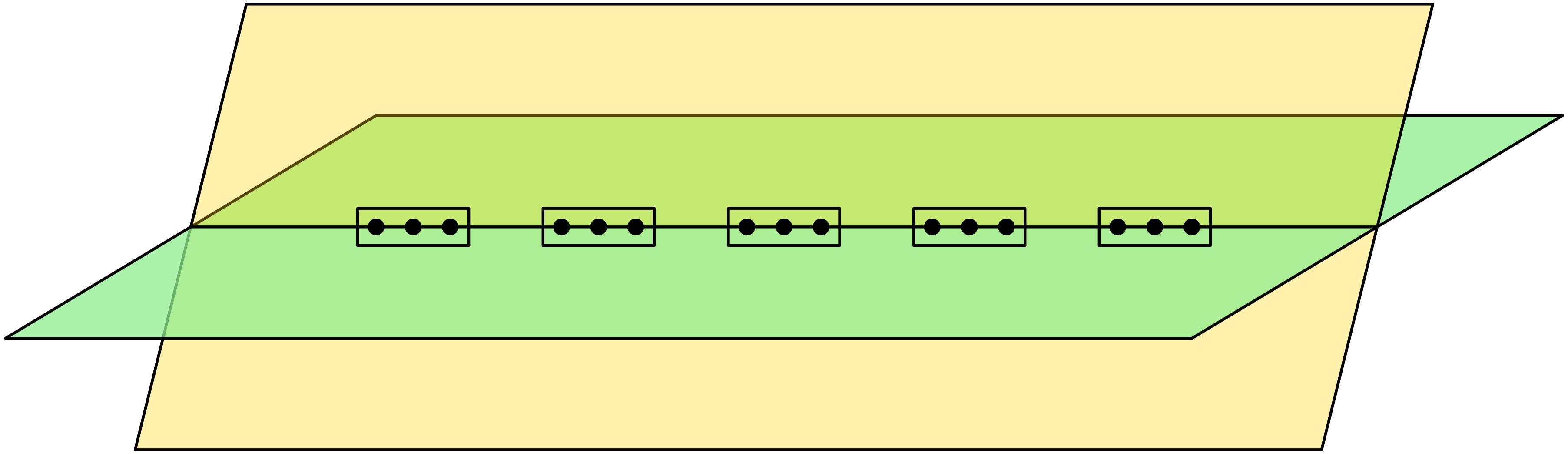




Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

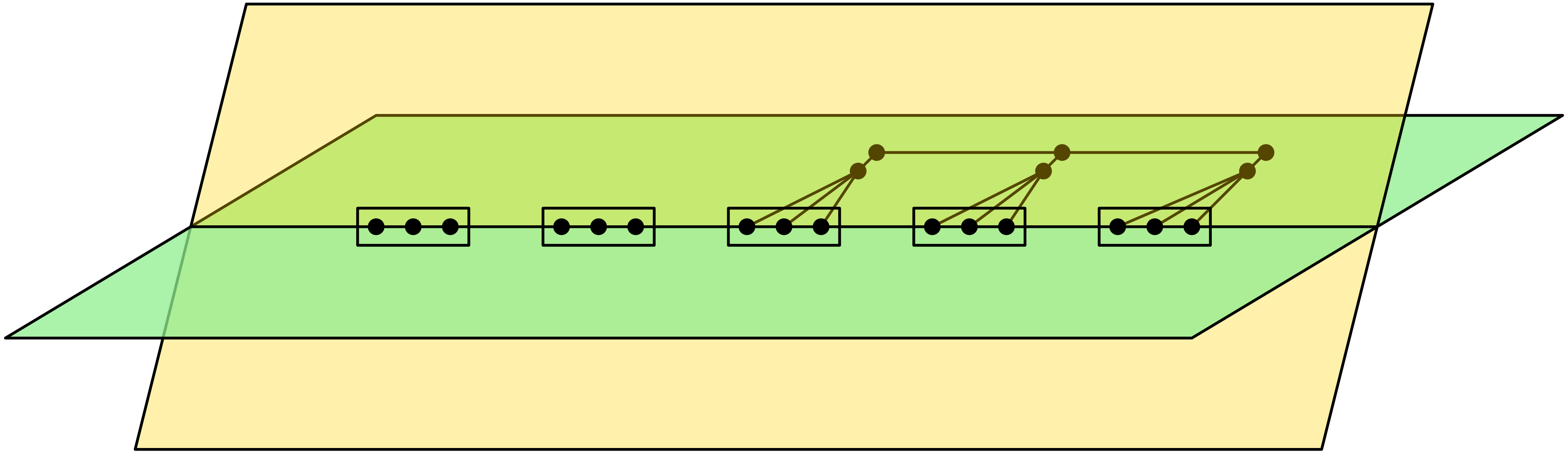
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho^2_3(G) = 2$ is NP-hard.

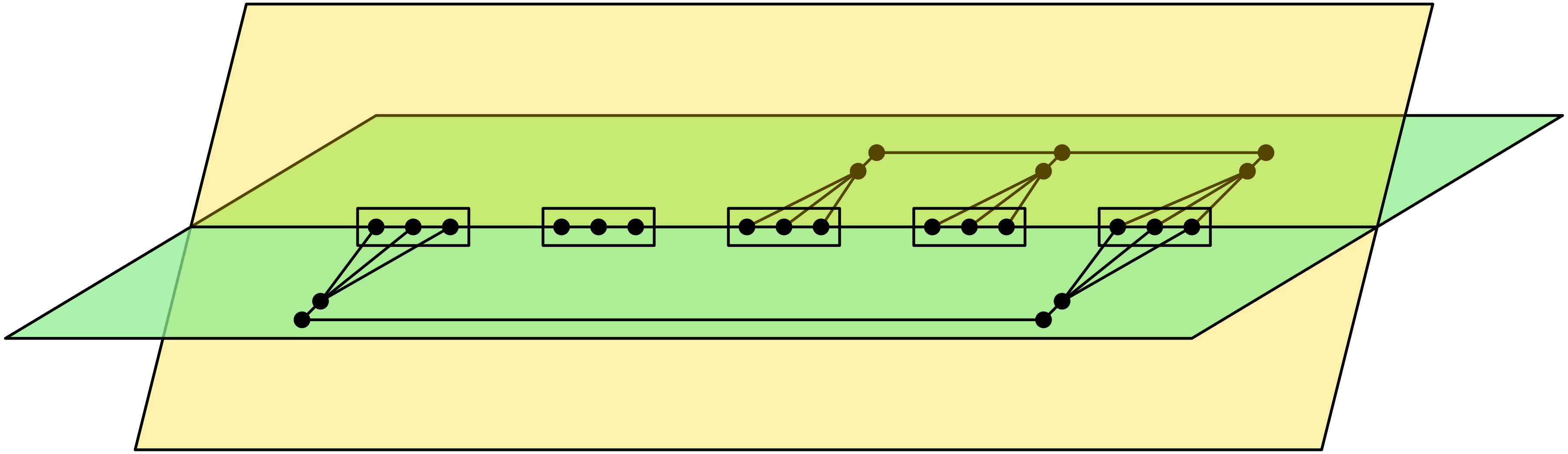
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

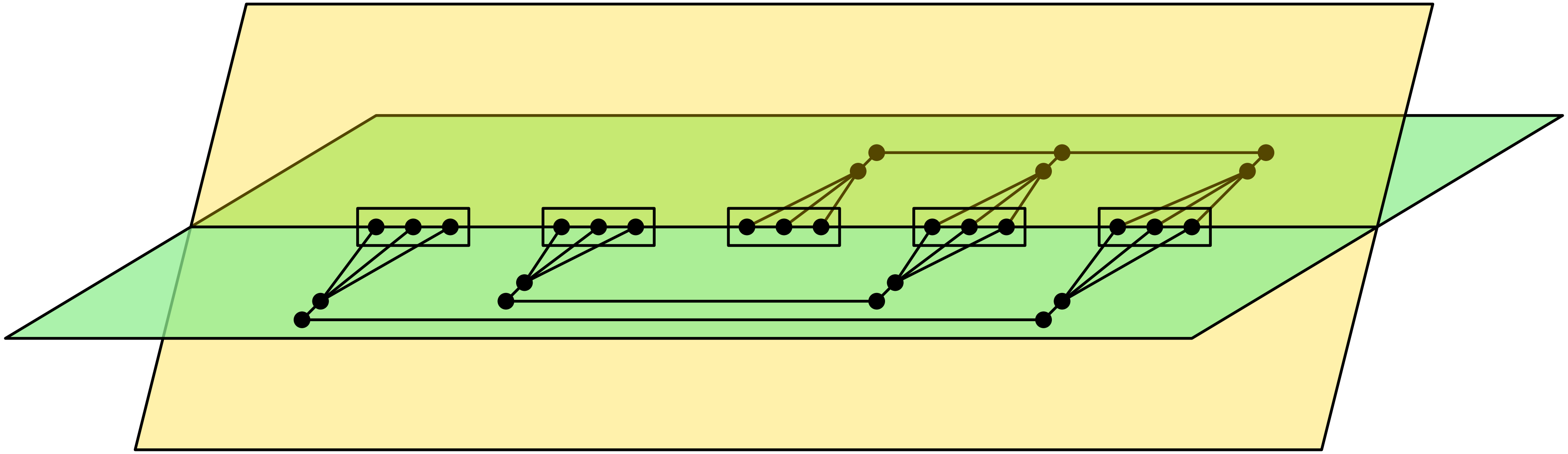
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

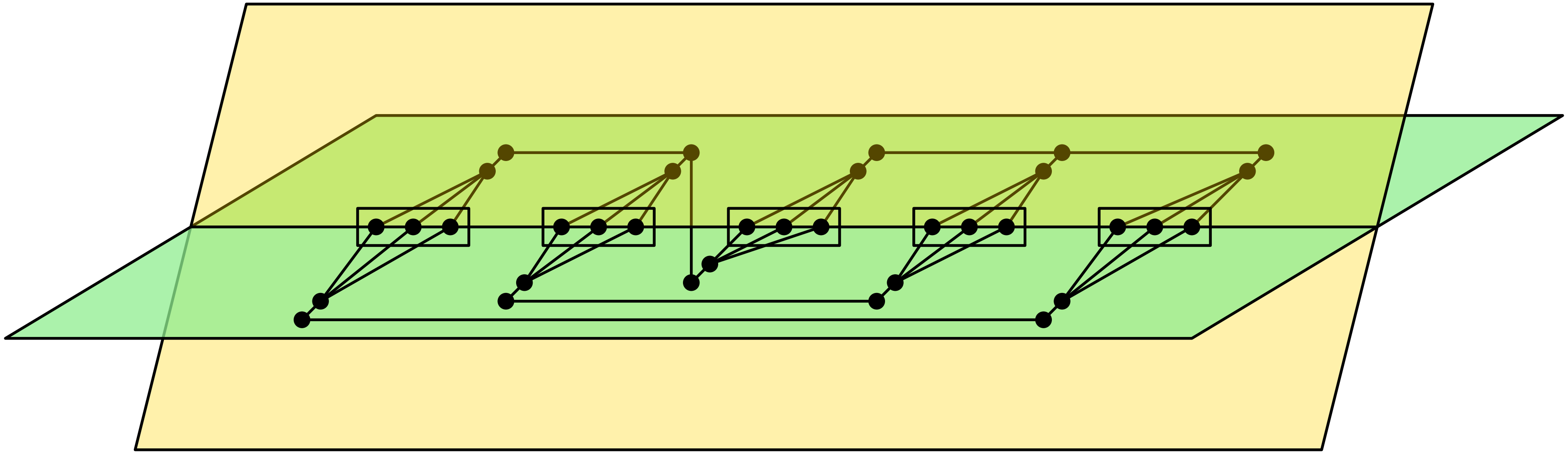
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

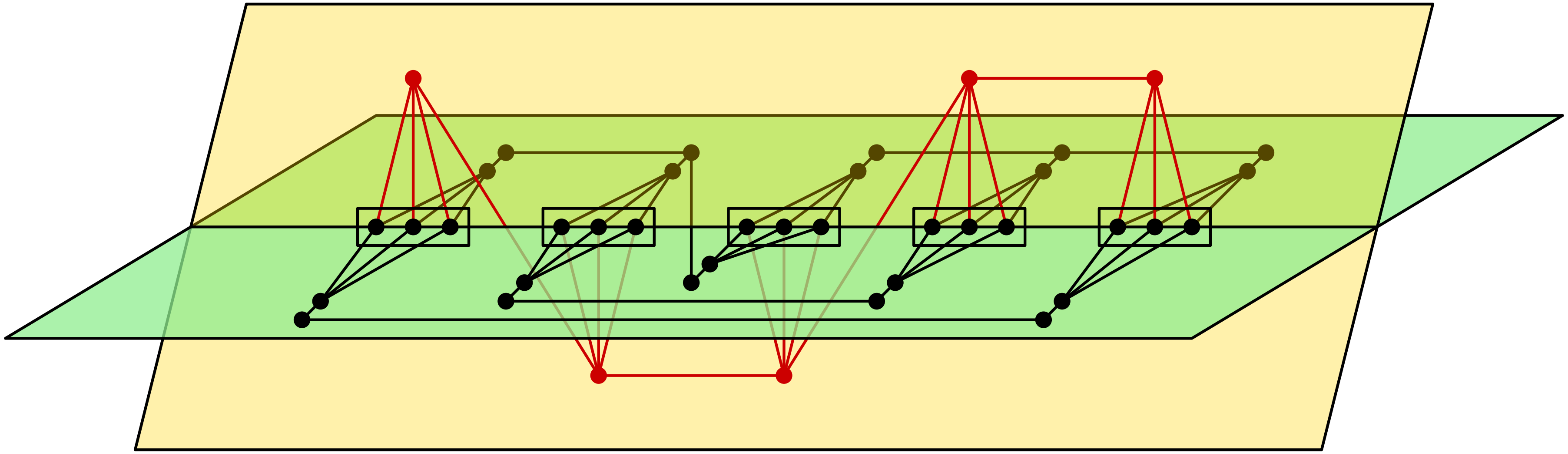
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho^2_3(G) = 2$ is NP-hard.

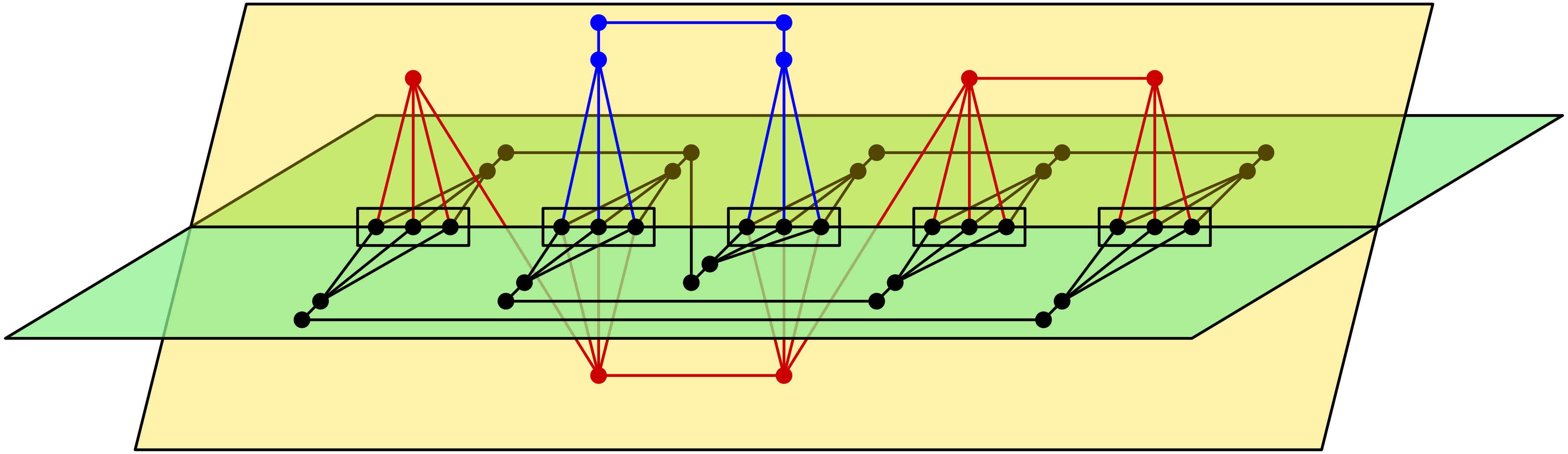
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

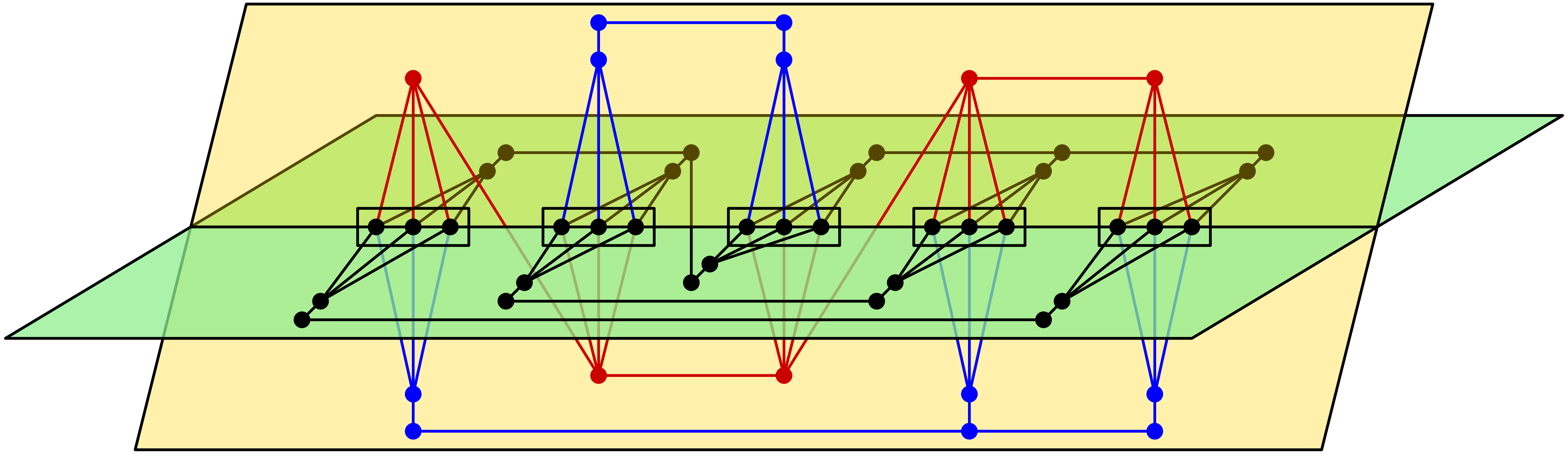
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

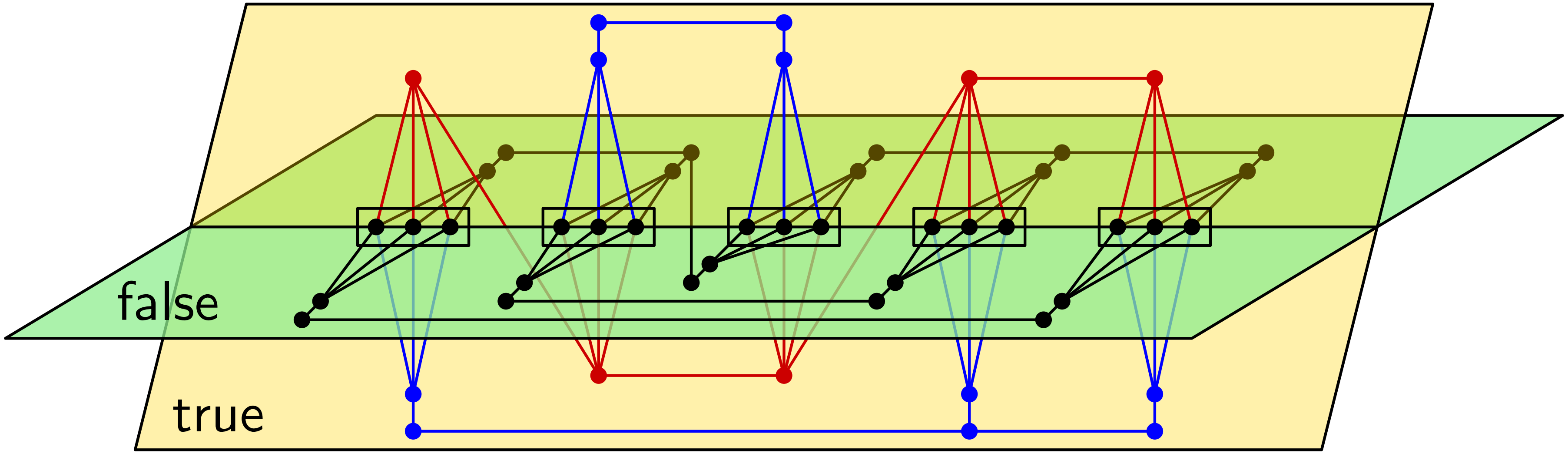
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

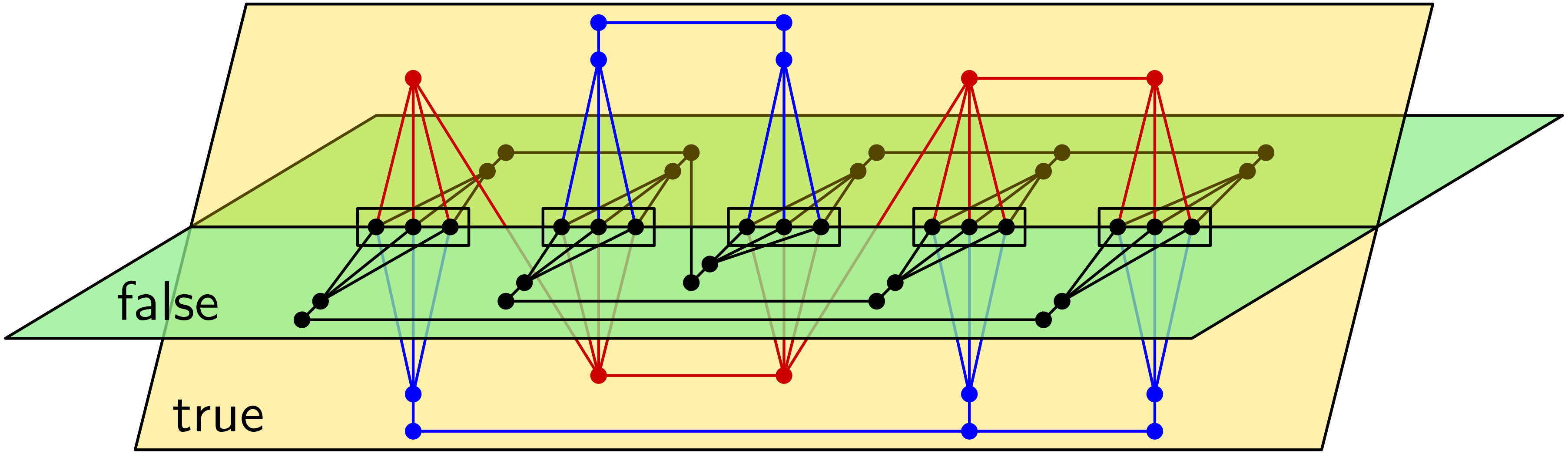
Proof:



Hardness of the Plane Cover Number

Theorem: Deciding $\rho_3^2(G) = 2$ is NP-hard.

Proof:



Corollary: Deciding $\rho_3^2(G) = k$ is NP-hard for any $k \geq 2$.
(add more blocking gadgets)

Plane cover number not FPT in k.

Conclusion

- Computing the line cover numbers is hard, but fixed-parameter tractable.
  Can it be computed efficiently for trees?

Conclusion

- Computing the line cover numbers is hard, but fixed-parameter tractable.
  Can it be computed efficiently for trees?

- Positive Planar Cycle 1-in-3-SAT is NP-hard.

Conclusion

- Computing the line cover numbers is hard, but fixed-parameter tractable.
  Can it be computed efficiently for trees?

- Positive Planar Cycle 1-in-3-SAT is NP-hard.

- Computing the plane cover number is NP-hard and contained in $\exists\mathbb{R}$.
  Is it also $\exists\mathbb{R}$-hard?

Conclusion

- Computing the line cover numbers is hard, but fixed-parameter tractable.
  Can it be computed efficiently for trees?

- Positive Planar Cycle 1-in-3-SAT is NP-hard.

- Computing the plane cover number is NP-hard and contained in $\exists\mathbb{R}$.
  Is it also $\exists\mathbb{R}$-hard?

- Approximation Algorithms?

Conclusion

- Computing the line cover numbers is hard, but fixed-parameter tractable.
  Can it be computed efficiently for trees?

- Positive Planar Cycle 1-in-3-SAT is NP-hard.

- Computing the plane cover number is NP-hard and contained in $\exists\mathbb{R}$.
  Is it also $\exists\mathbb{R}$-hard?

- Approximation Algorithms?