Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG
Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität
Institut für Informatik

Drawing Graphs with Circular Arcs and Right-Angle Crossings

Steven Chaplick[1], Henry Förster[2], **Myroslav Kryven**[3], Alexander Wolff[3]

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SWAT 2020

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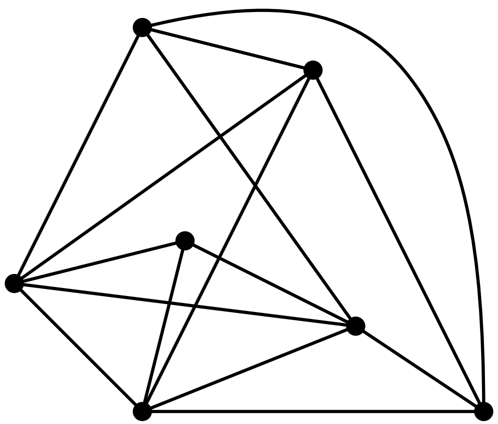

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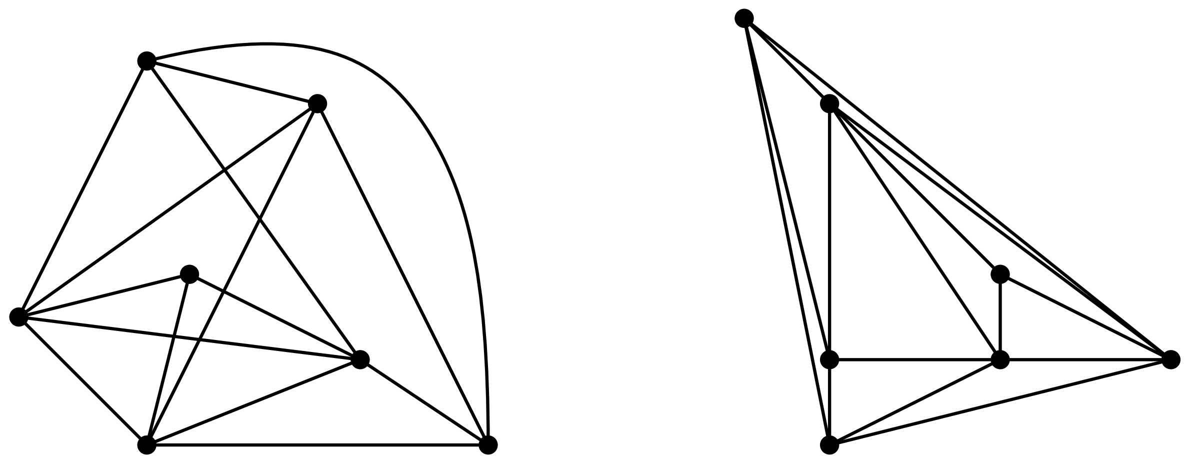

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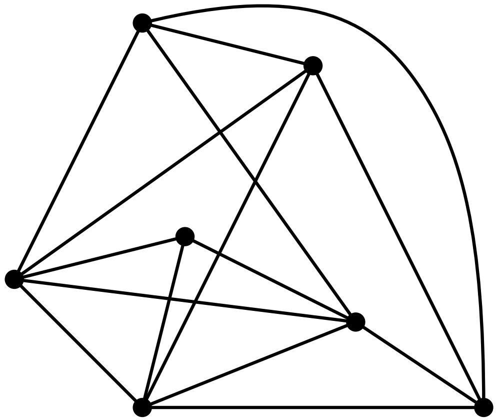

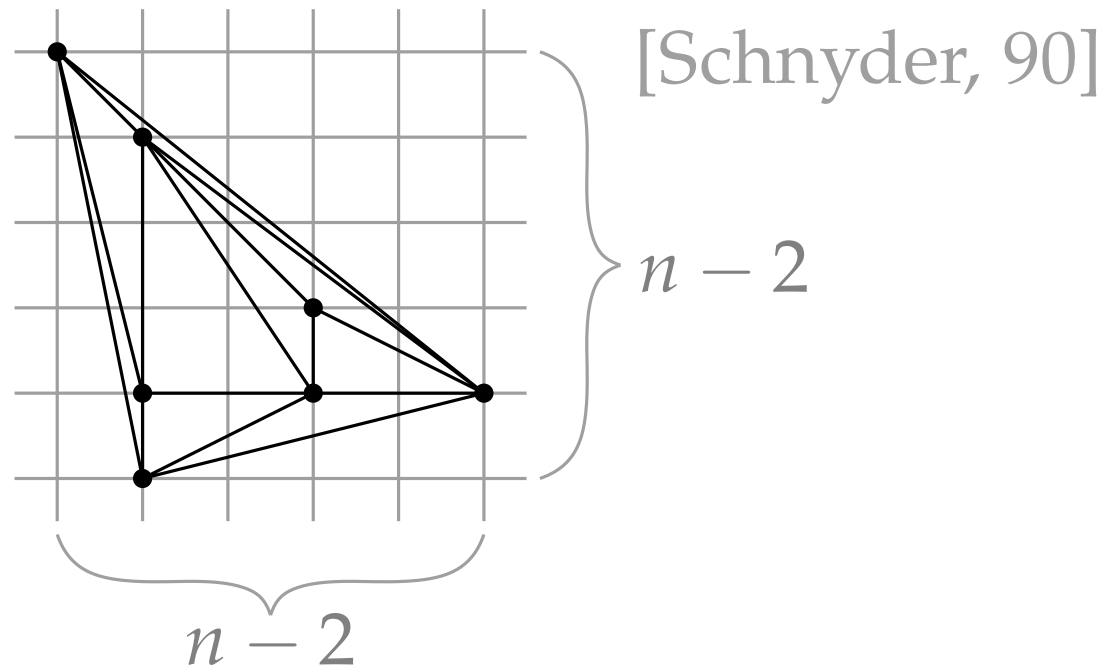

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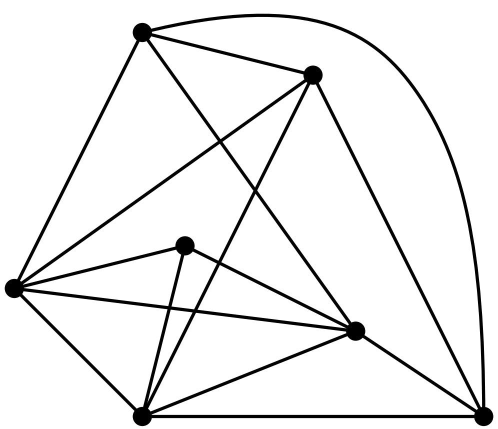

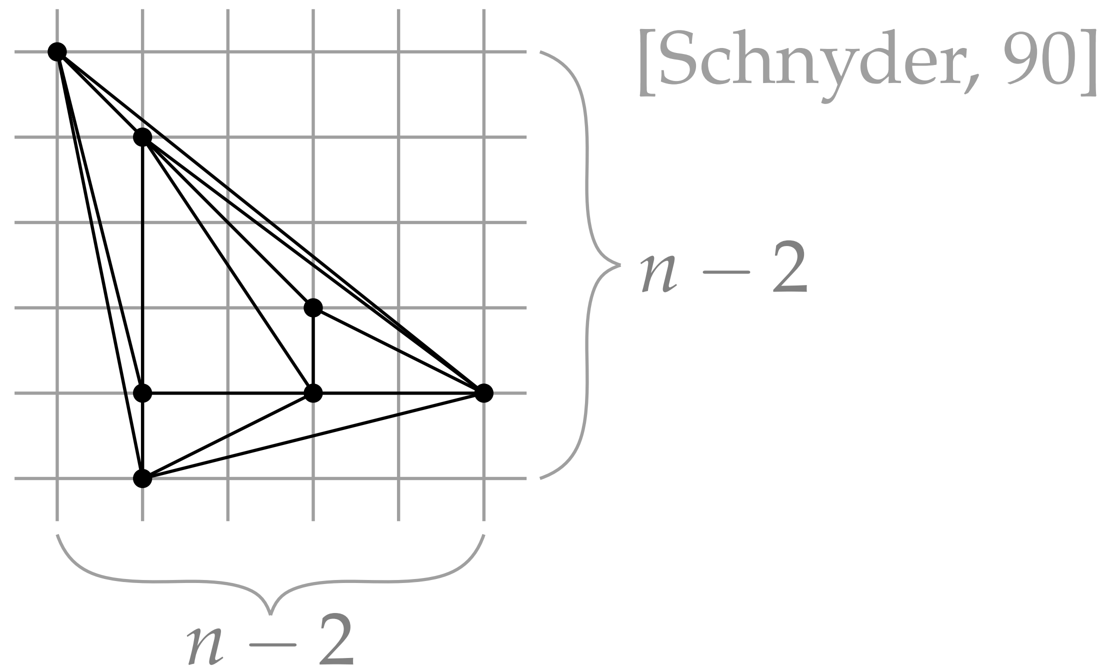

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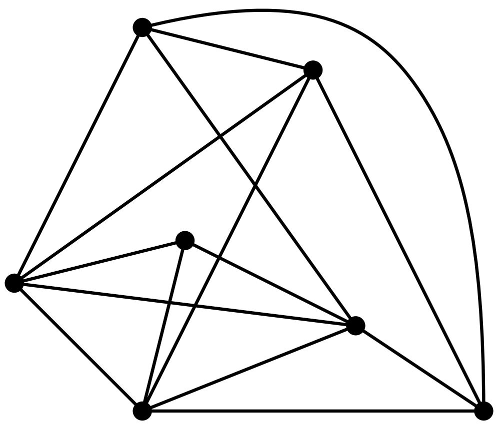

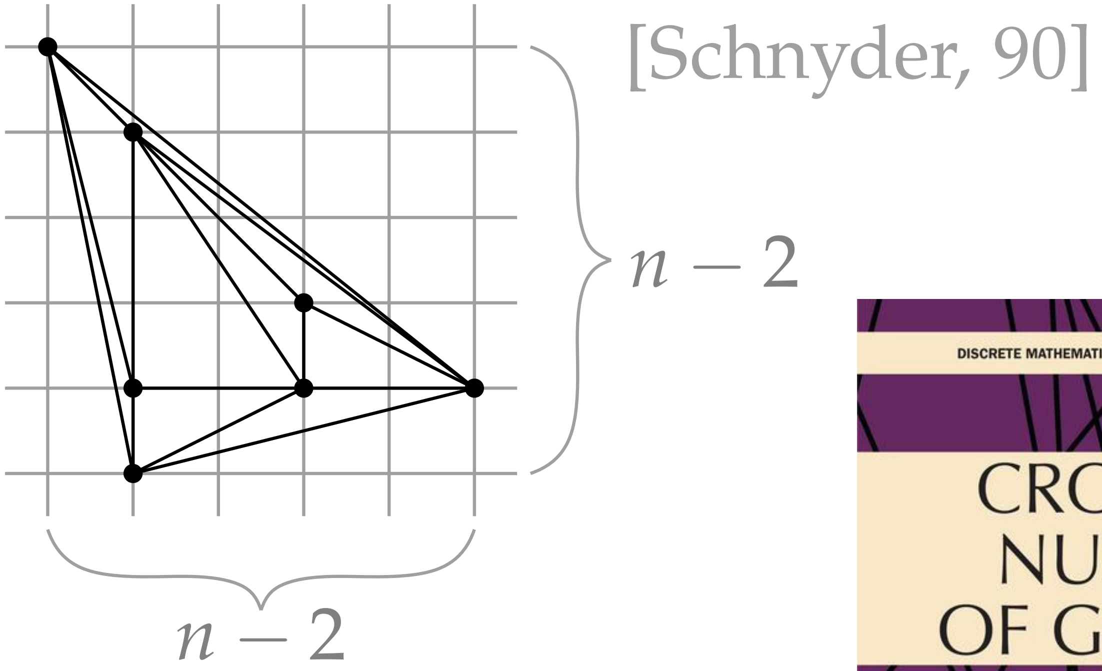

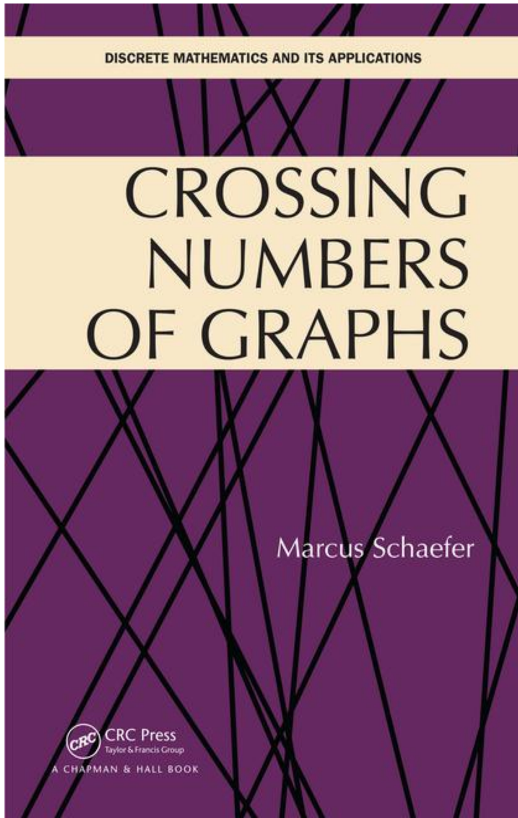

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That's a fundamental problem in GD.

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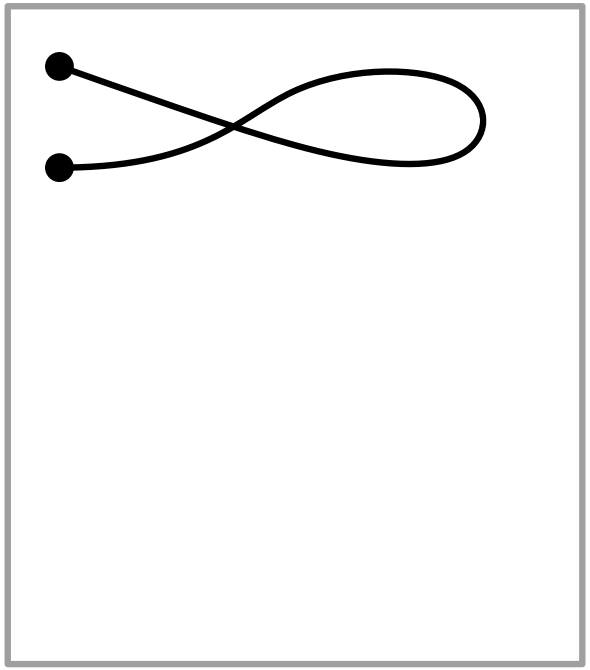

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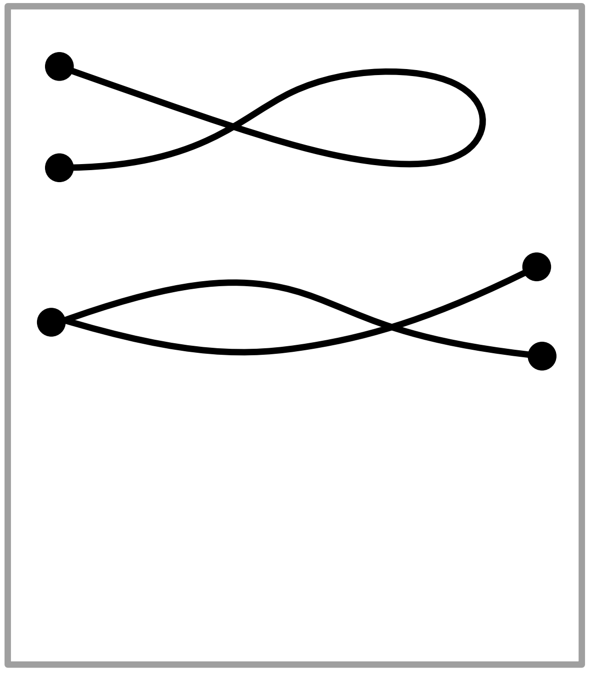

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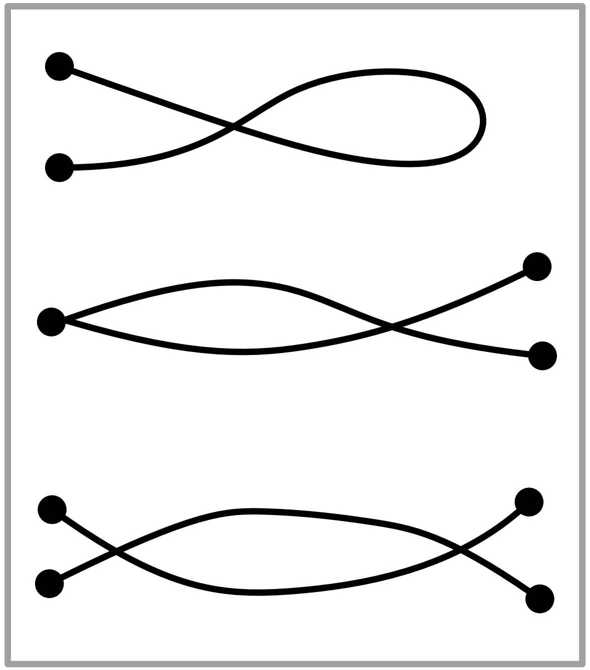

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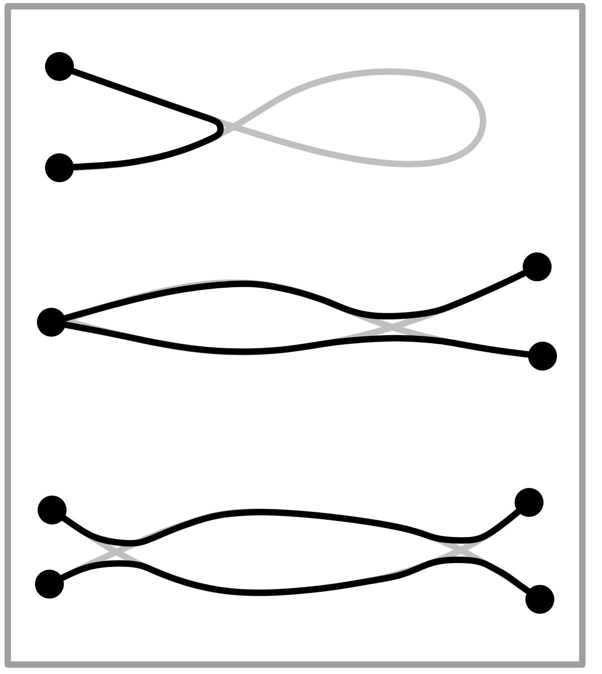

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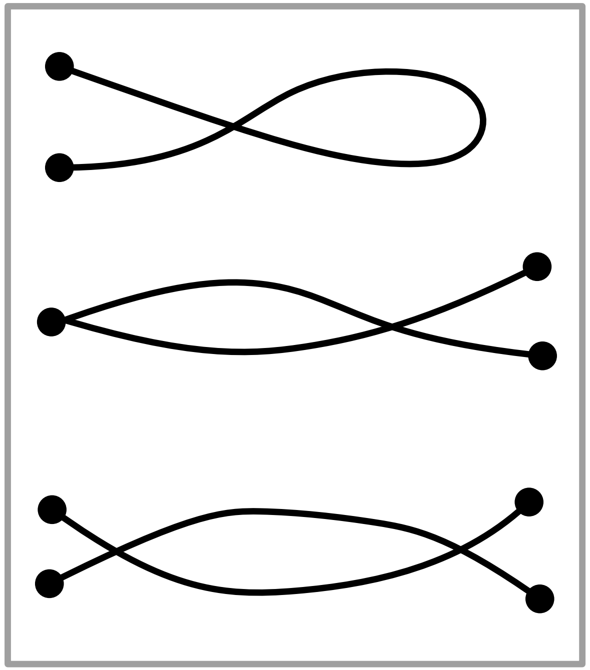

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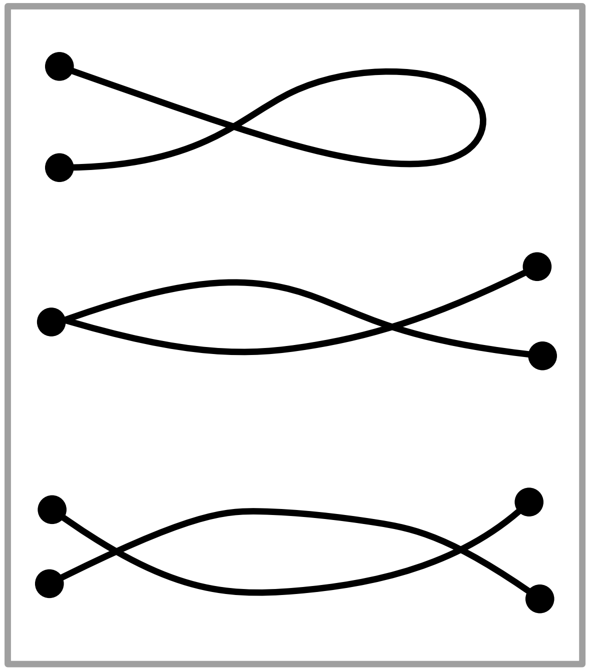

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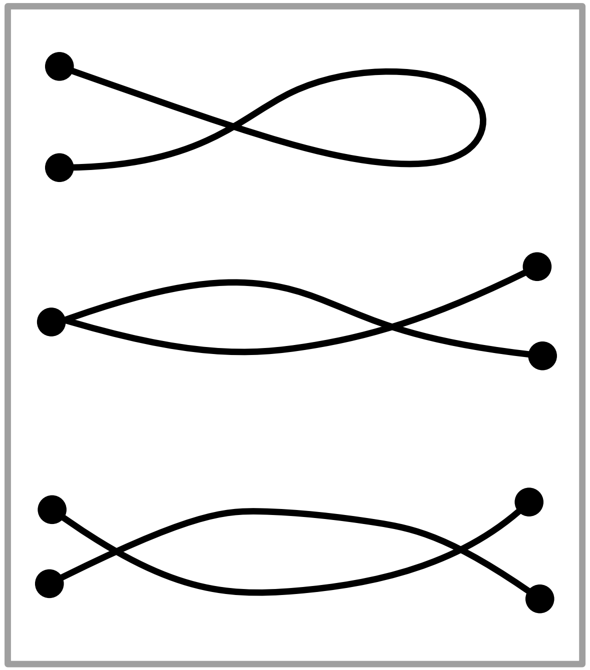

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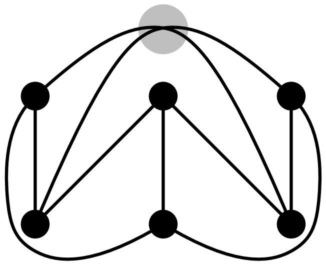

Crossing Lemma

A dense n-vertex m-edge graph G requires many crossings.

Thm. [Ajtai, Chvátal, Newborn, Szemerédi '82, Leighton '84]

$$m \geq 4n \;\Rightarrow \mathrm{cr}(G) \geq \frac{1}{64} \cdot \frac{m^3}{n^2}.$$

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For a complete graph K_n: [Guy, '60]

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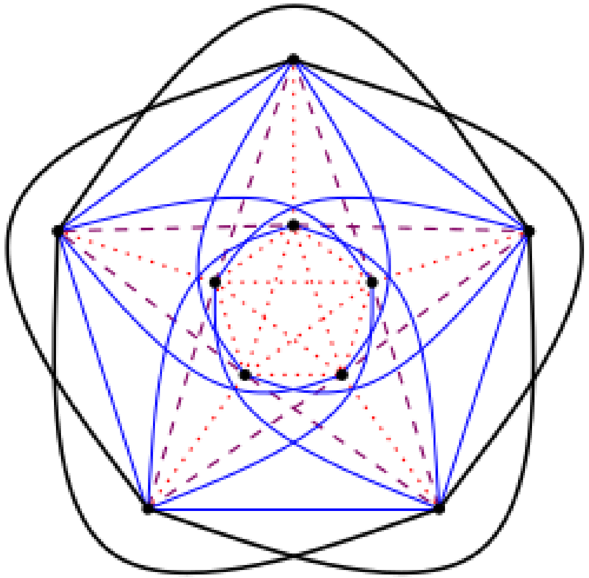

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[Balko et al., '14]

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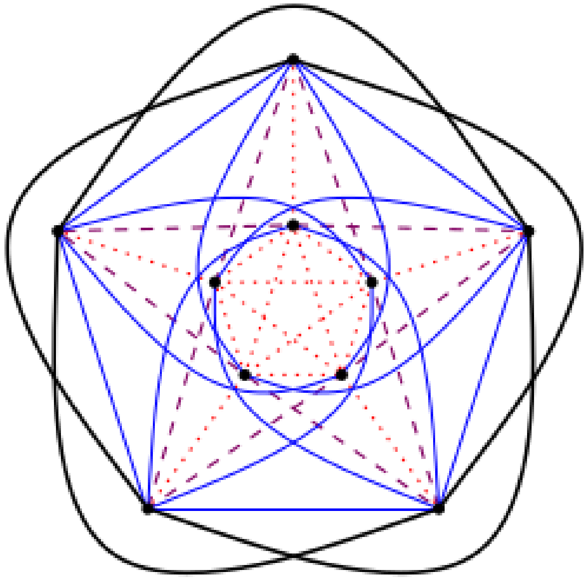

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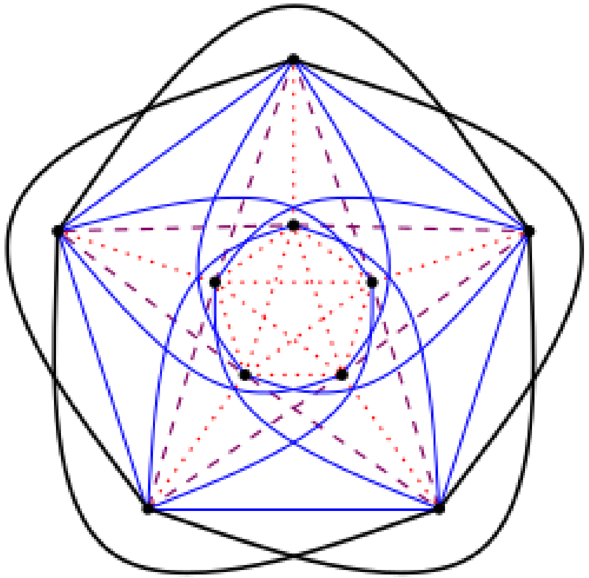
$\Omega(n^4)$ is a lot of crossings:(



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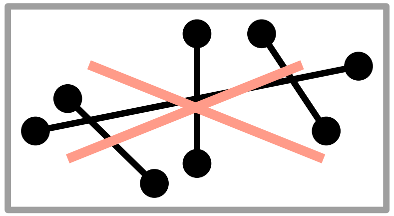

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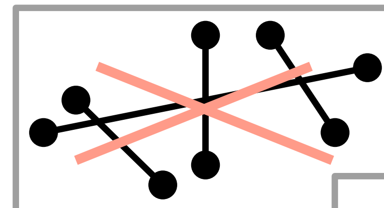

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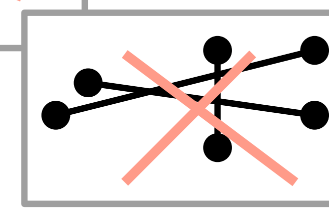

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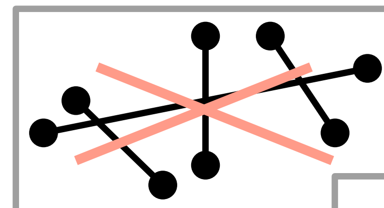

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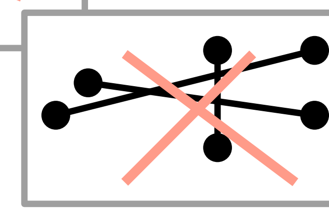

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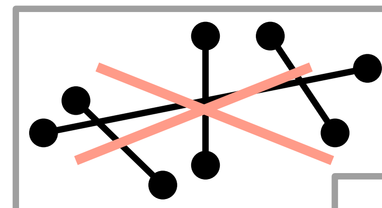

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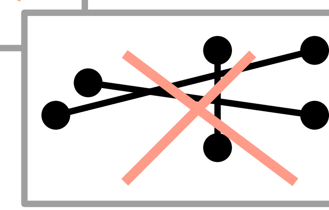

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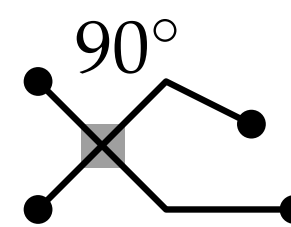

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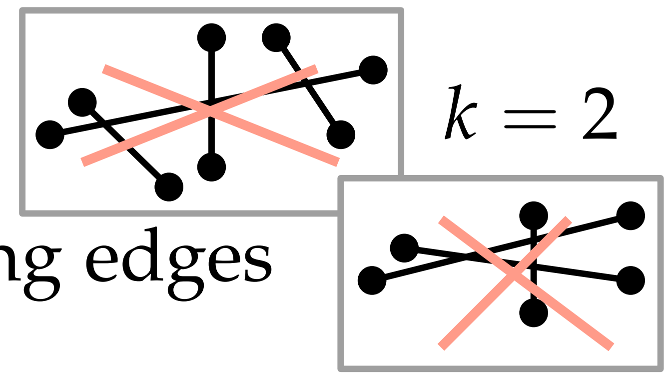

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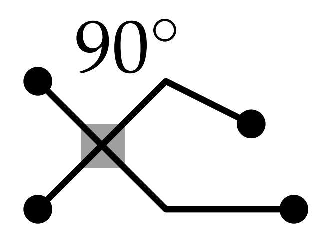

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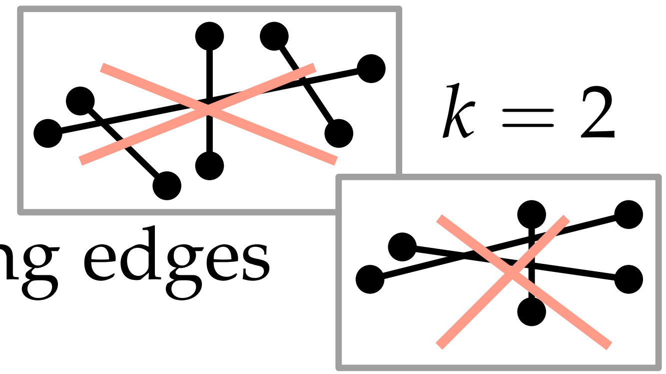

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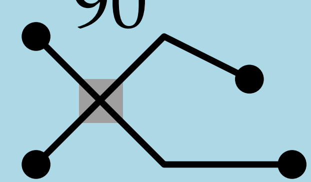

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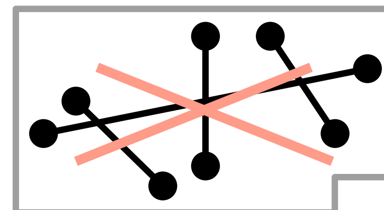

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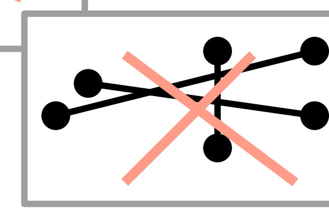

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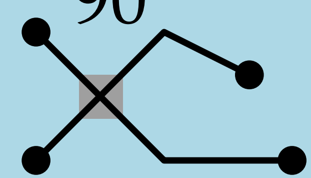

new!

- edges are circular-arc and cross at 90° angle only (*arc-RAC graphs*).

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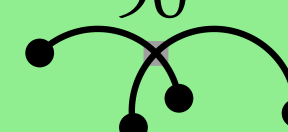

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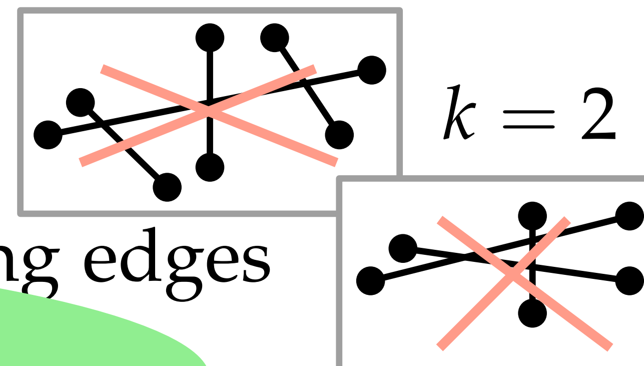

But why arcs?

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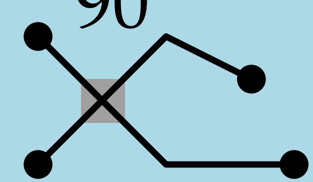

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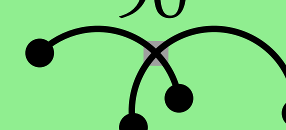
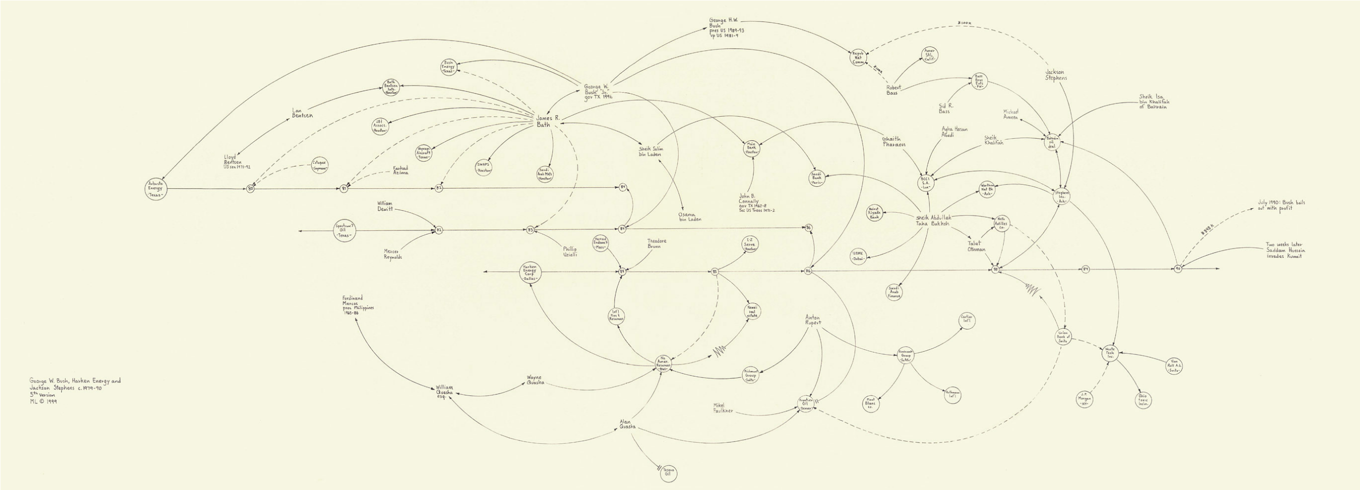


Circular Arcs in Graph Drawing



[Lombardi, ’99]

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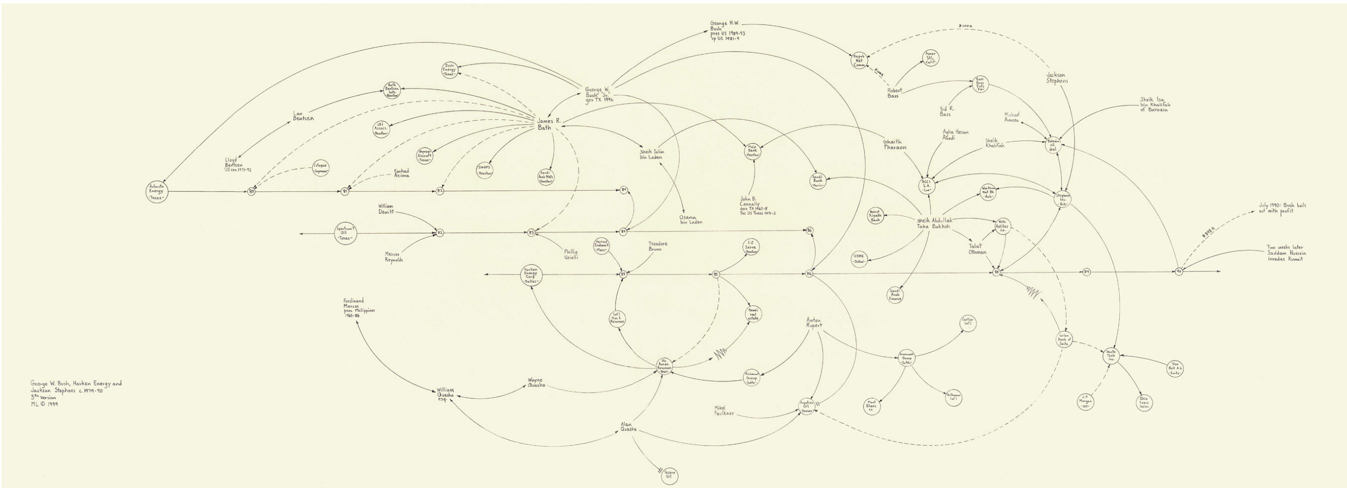

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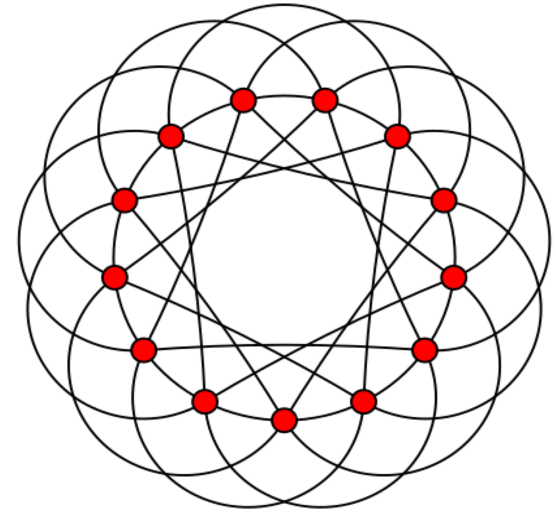

[Duncan et al., ’10]

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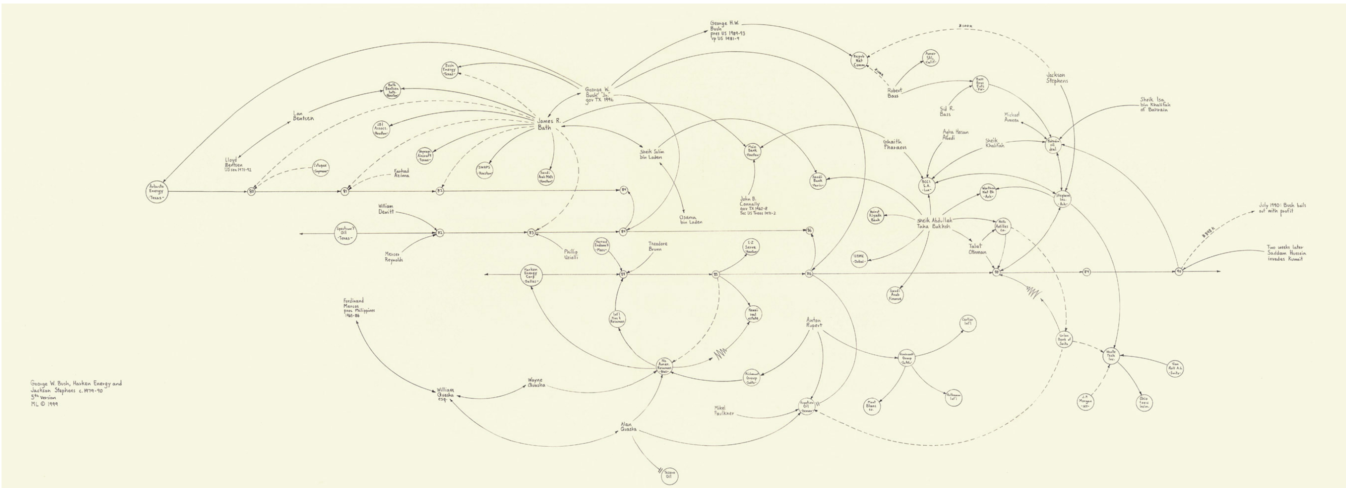

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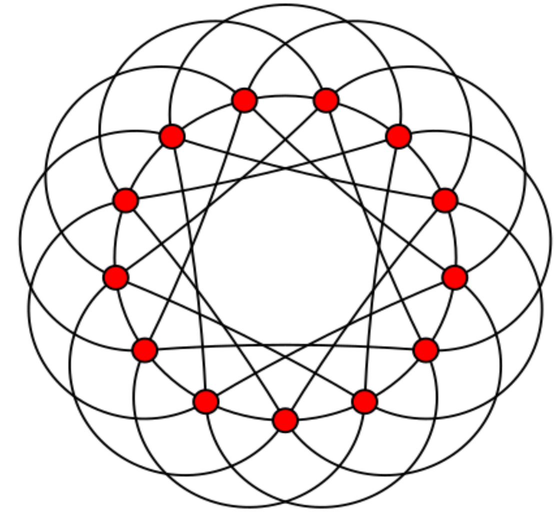

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[Xu et al., ’12] [Purchase et al., ’13]

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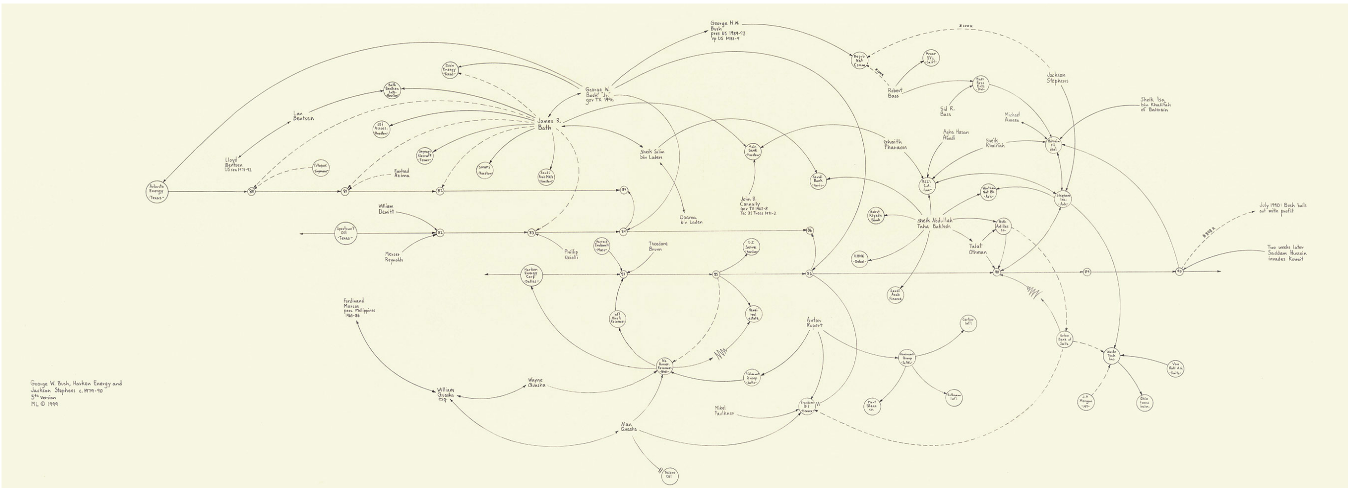

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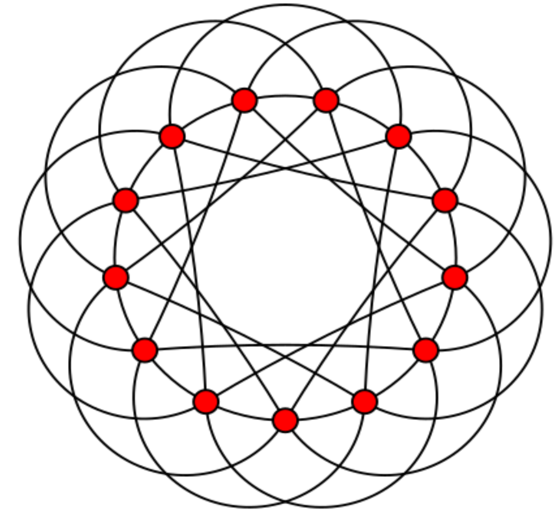

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Improve other drawing criteria:

– area of the drawing [Schulz, ’15]
– angular resolution. [Cheng et al., ’01] [Duncan et al., ’10]

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Upper bound on MED				
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(RAC_3 = all graphs)

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Upper bound on MED	$4n - 10$	$O(n^{4/3})$	$O(n^{7/4})$	
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[Didimo+ '11]

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Upper bound on MED	$4n - 10$	$O(n^{4/3})$ $6.5n - 13$	$O(n^{7/4})$ $74.2n$	
Lower bound on MED	$4n - 10$	$4.5n - O(\sqrt{n})$	$7.83n - O(\sqrt{n})$	

[Didimo+ ’11] [Arikushi+ ’12]

RAC graphs

Maximum edge density (MED) is a classical property used to characterize graph classes.

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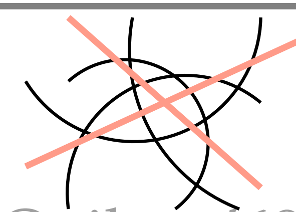

[Ogilvy, '69]

Graph class	RAC_0	RAC_1	RAC_2	4-quasi-planar arc-RAC
Upper bound on MED	$4n - 10$	$O(n^{4/3})$ $6.5n - 13$ $5.5n - O(1)$	$O(n^{7/4})$ $74.2n$	$72(n-2)$
Lower bound on MED	$4n - 10$	$4.5n - O(\sqrt{n})$ $5.5n - O(1)$	$7.83n - O(\sqrt{n})$	

[Didimo+ '11] [Arikushi+ '12] [Angelini+ '20] [Ackerman '09]

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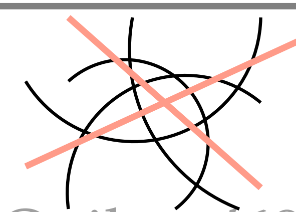

[Ogilvy, '69]

4-quasi-planar

Graph class	RAC_0	RAC_1	RAC_2	arc-RAC
Upper bound on MED	$4n-10$	$O(n^{4/3})$ $6.5n-13$ $5.5n-O(1)$	$O(n^{7/4})$ $74.2n$	$72(n-2)$ $14n-12$
Lower bound on MED	$4n-10$	$4.5n-O(\sqrt{n})$ $5.5n-O(1)$	$7.83n-O(\sqrt{n})$	

[Didimo+ '11] [Arikushi+ '12] [Angelini+ '20] [Ackerman '09]

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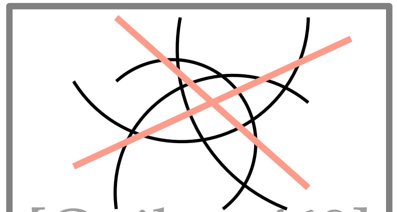

Charging method (RAC_3 = all graphs)

Graph class	RAC_0	RAC_1	RAC_2	4-quasi-planar / arc-RAC
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Charging method (RAC$_3$ = all graphs)

[Ogilvy, '69]

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Upper bound on MED	$4n - 10$ [Dujmović et al. '10]	$O(n^{4/3})$ $6.5n - 13$ $5.5n - O(1)$	$O(n^{7/4})$ $74.2n$	$72(n-2)$ $14n - 12$
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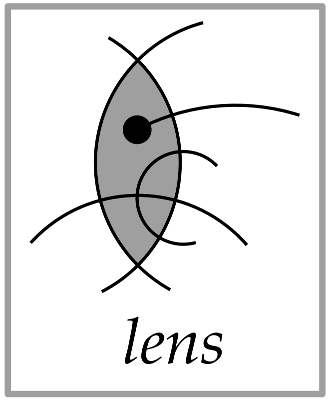
Simplification

Consider an arc-RAC graph G and its drawing D.



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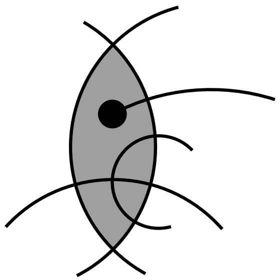
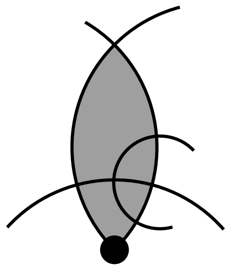


lens



empty 1-lens

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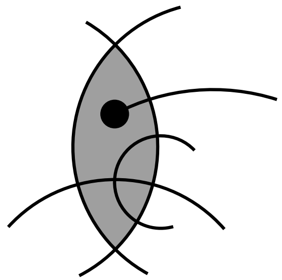
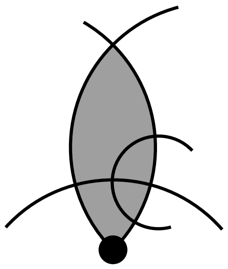
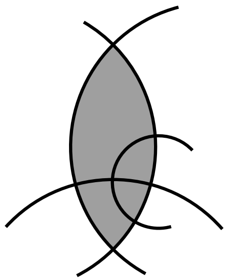


lens



empty 1-lens



empty 0-lens

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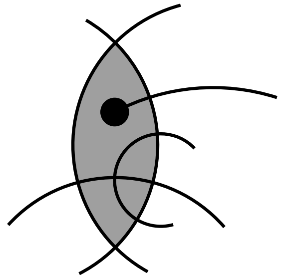

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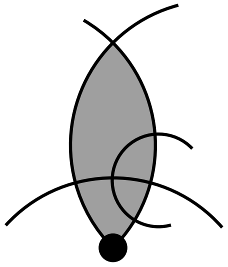
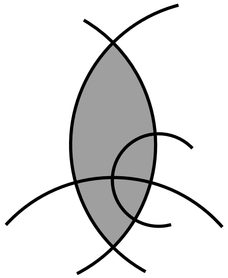
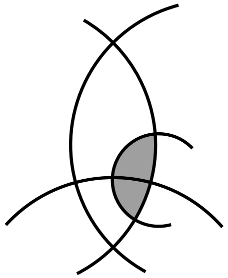


empty 1-lens



empty 0-lens



smallest empty 0-lens

Simplification

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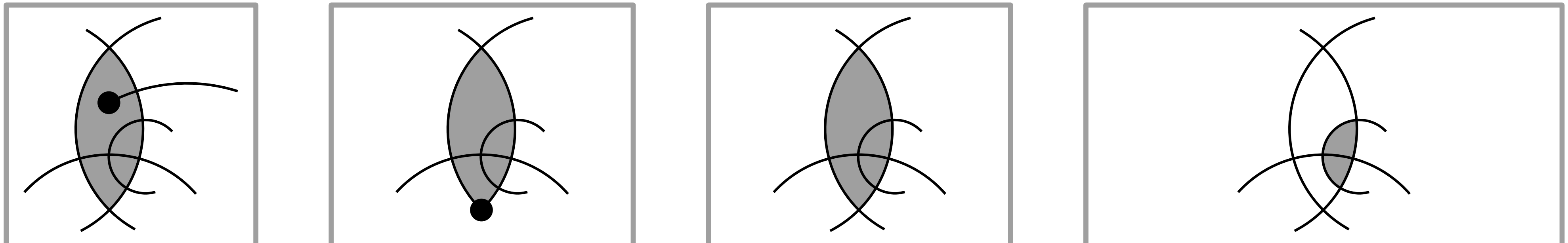
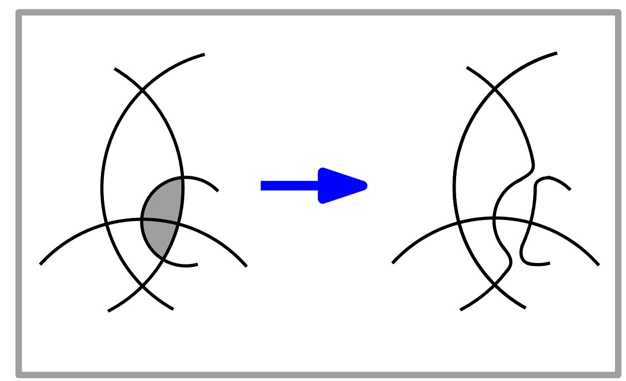


lens | *empty 1-lens* | *empty 0-lens* | *smallest empty 0-lens*

Simplification of a smallest empty 0-lens:



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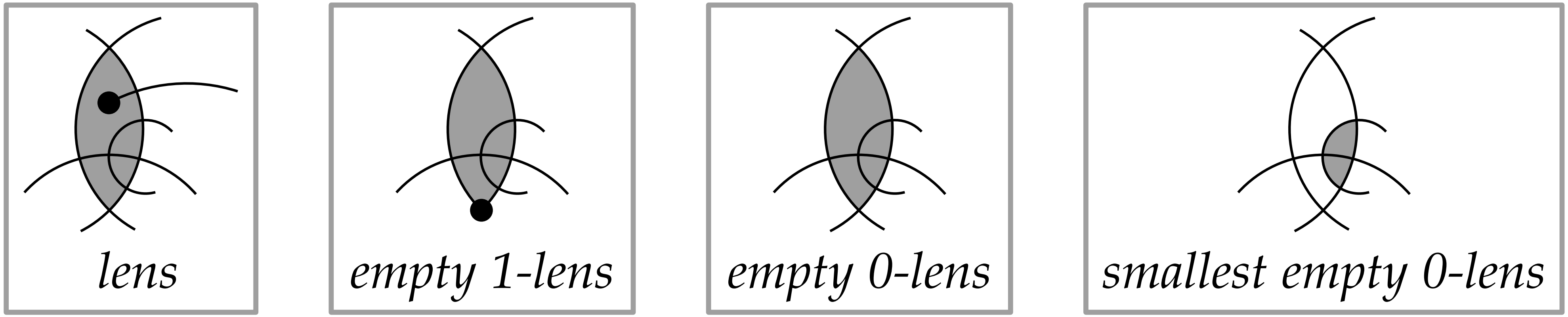

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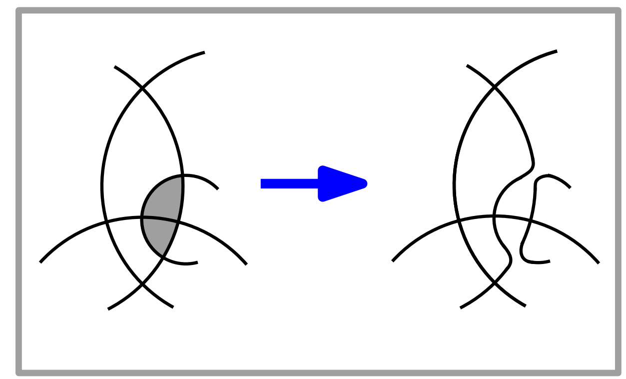

During the simplification process:

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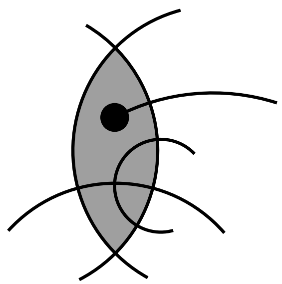
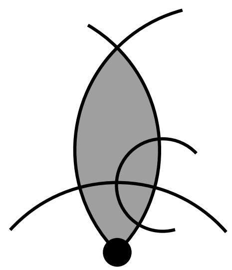
Simplification

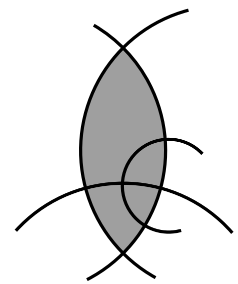
Consider an arc-RAC graph G and its drawing D.



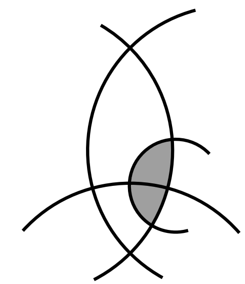
lens



empty 1-lens

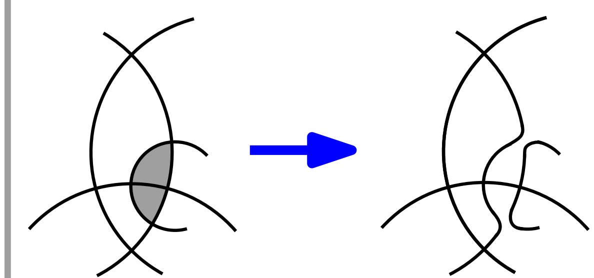


empty 0-lens



smallest empty 0-lens

Simplification of a smallest empty 0-lens:



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We *simplify* smallest empty 0-lenses until none are left.

Simplification

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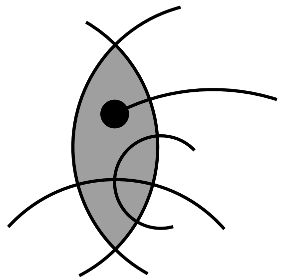
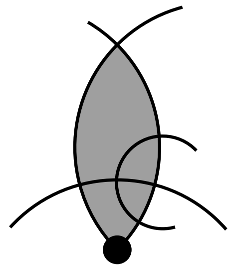
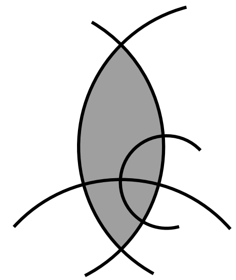


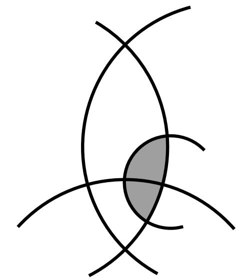
lens



empty 1-lens



empty 0-lens



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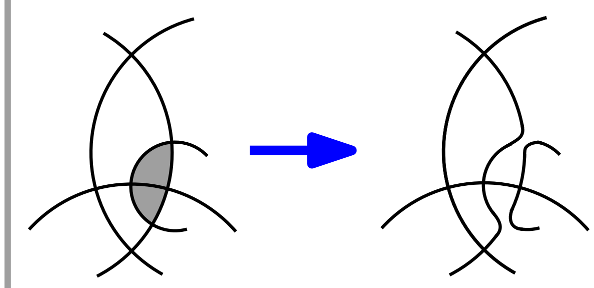

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We call the resulting drawing D' the *simplification* of D.

Main theorem

Thm. An arc-RAC graph with n vertices can have at most $14n - 12$ edges.

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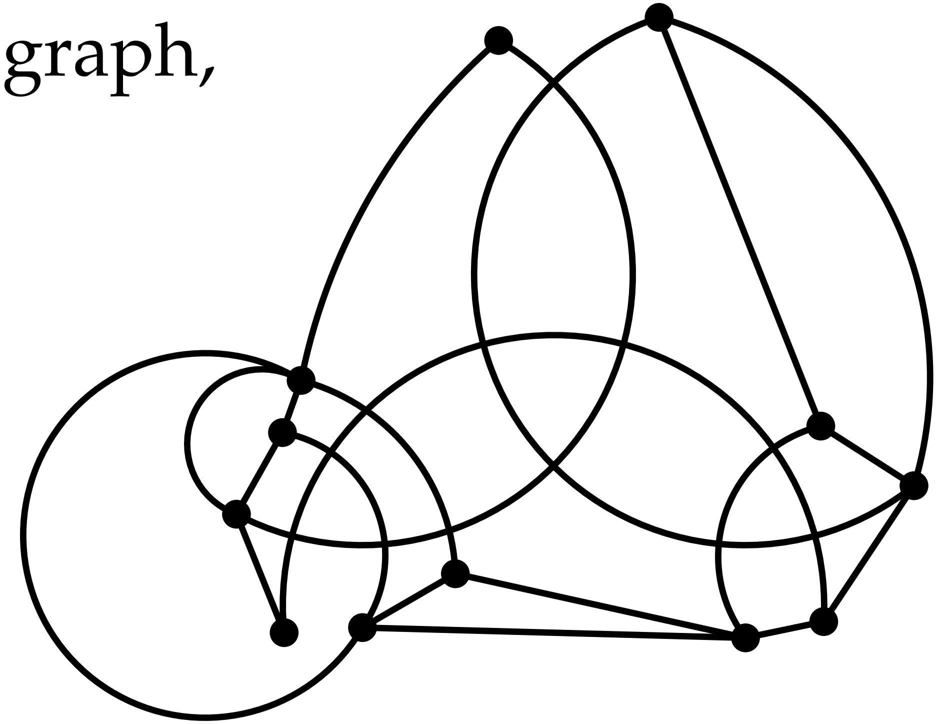

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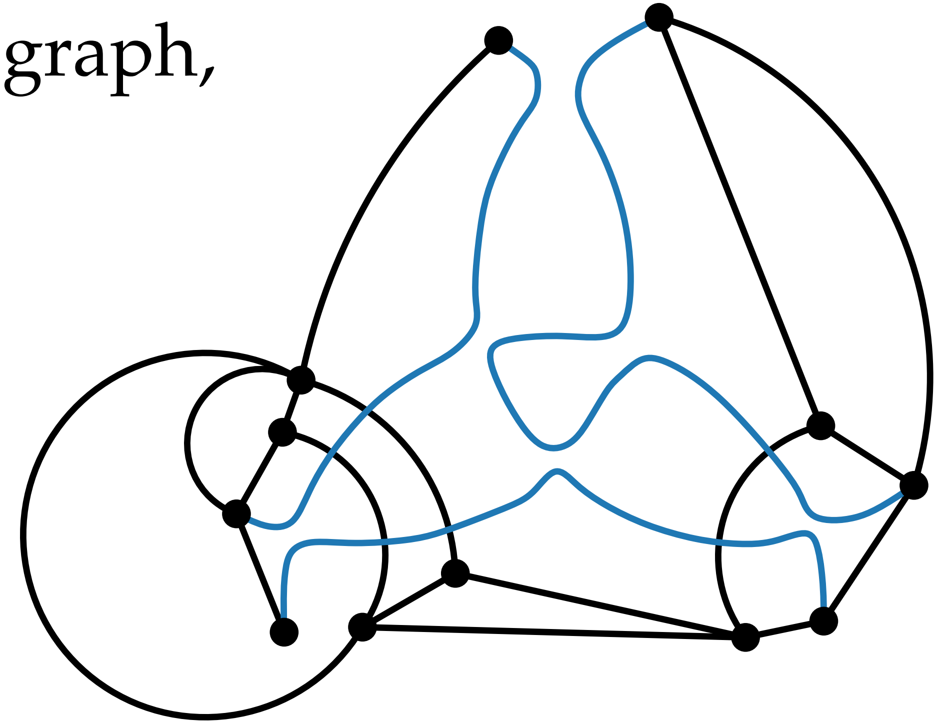

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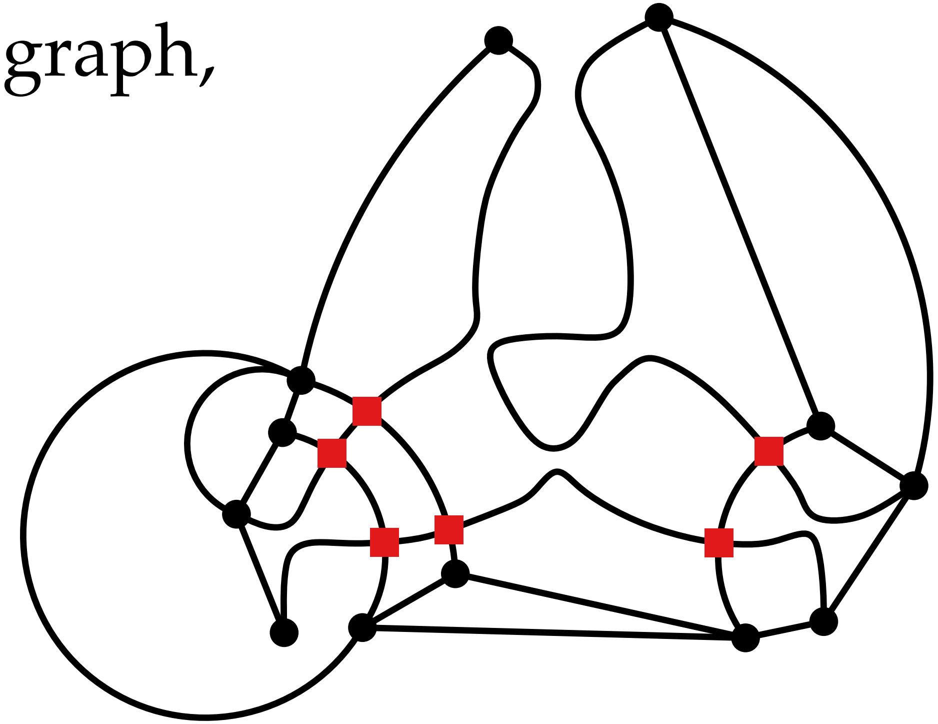

Main theorem proof overview

1. Assigning each face f of G' a charge of:
   $ch(f) = |f| + v(f) - 4$, where
   $|f|$ is the degree of f in the planarization G' and
   $v(f)$ is the number of vertices of G on the boundary of f.
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2. Redistributing charge among faces and vertices so that:
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[Dujmović et al., '10]

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And the bound follows: $28n/3 - 8 \geq \sum_{f \in G'} ch(f) \geq$

$$\sum_{f \in G'} v(f)/3 = \sum_{v \in G} \deg(v)/3 = 2|E|/3.$$

Step 1 (initial charge)

Initial charge:

- for each vertex v of G:
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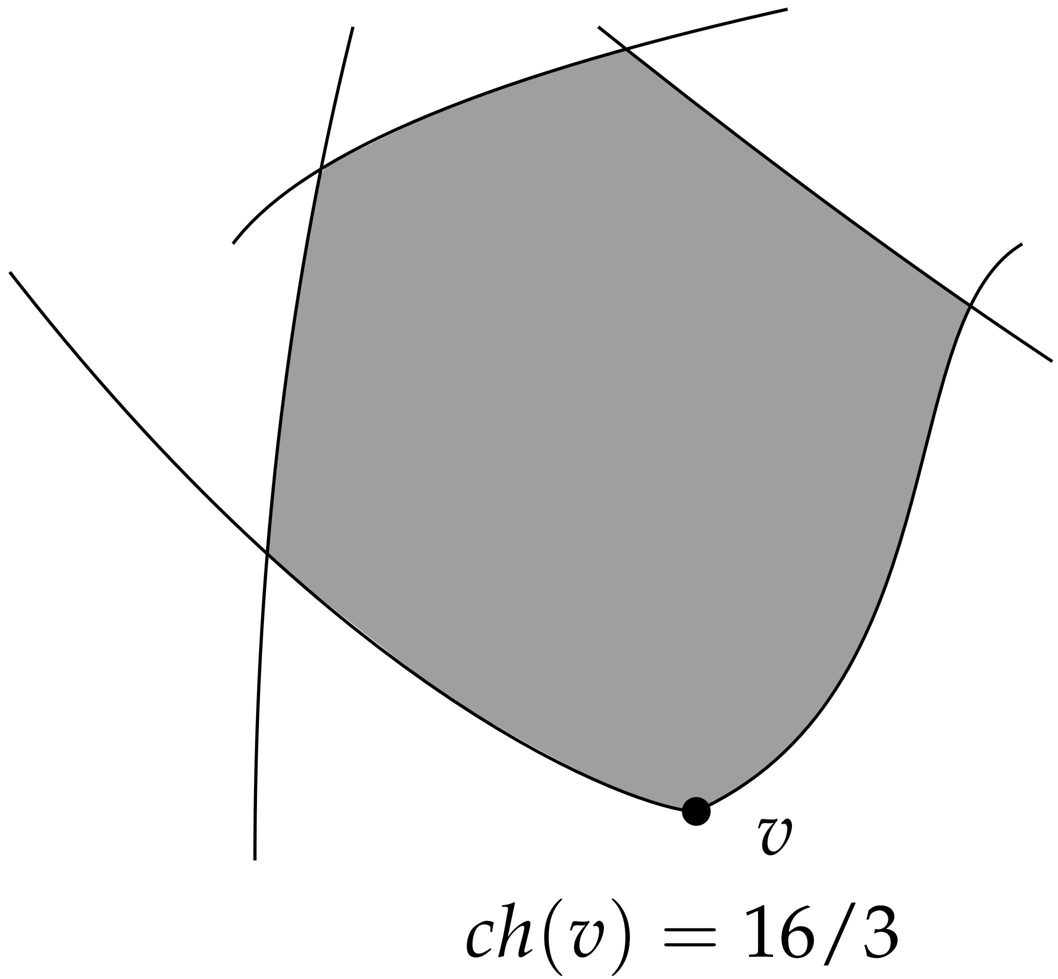

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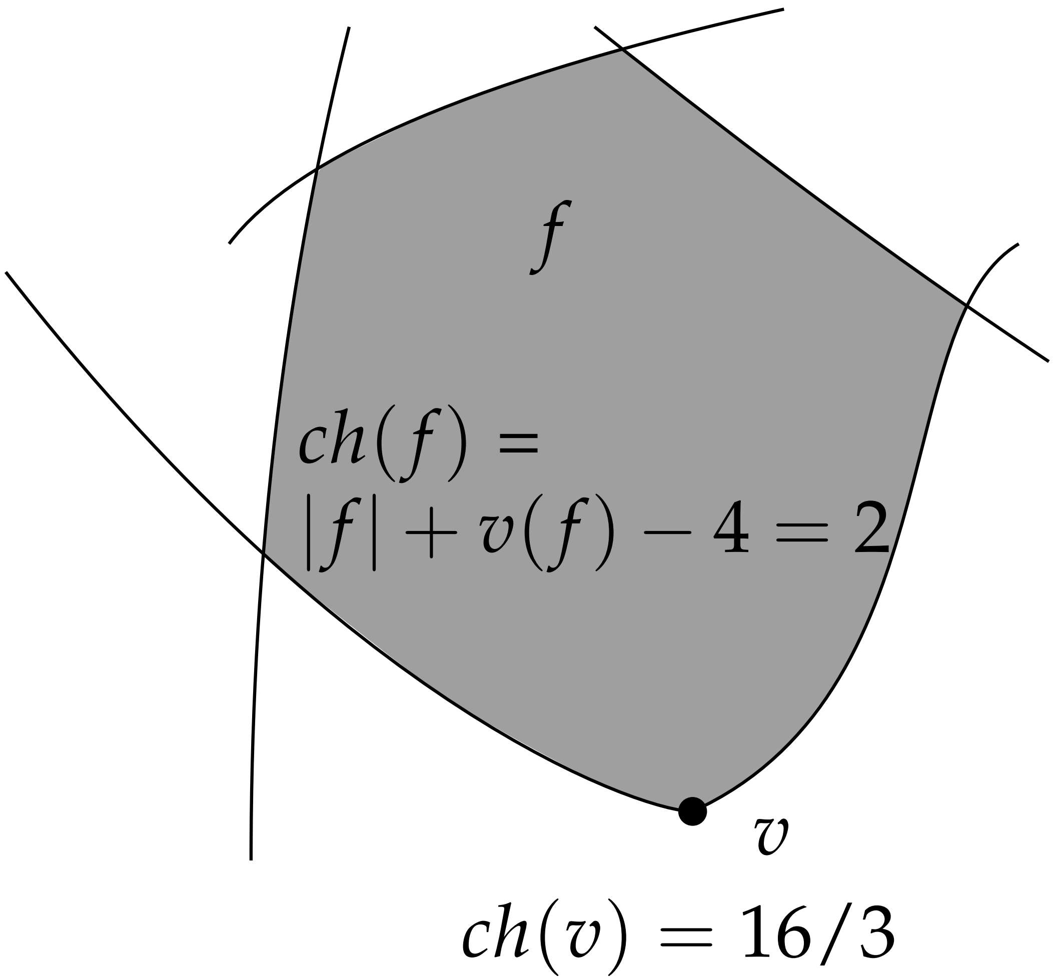

Types of faces

We call a face f of G' a k-triangle, k-quadrilateral, or k-pentagon if f has the corresponding shape and $v(f) = k$.

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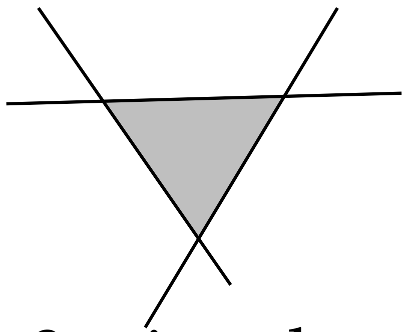
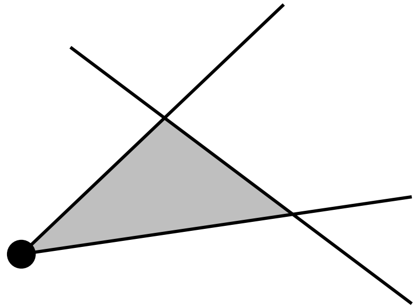
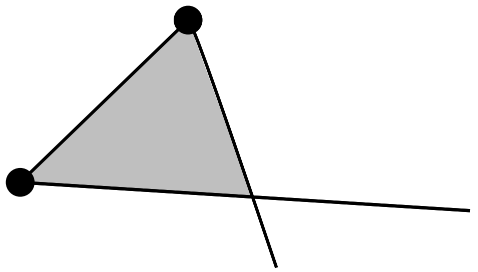


0-triangle



1-triangle



2-triangle

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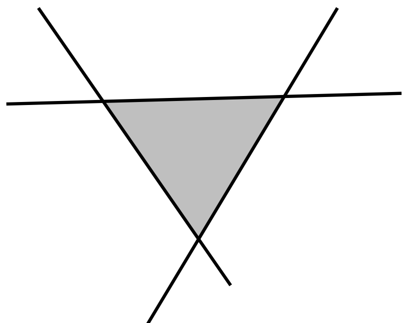
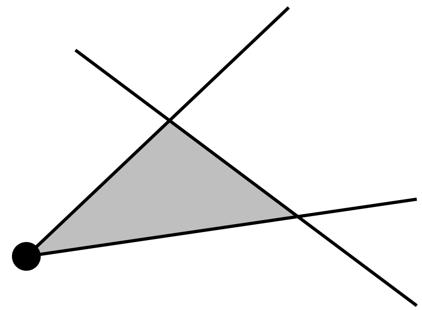


0-triangle



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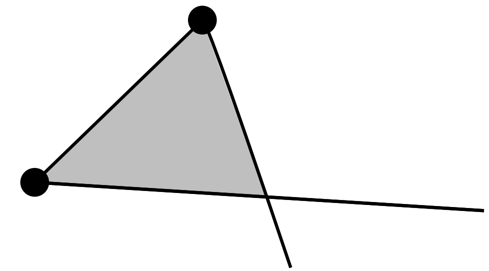
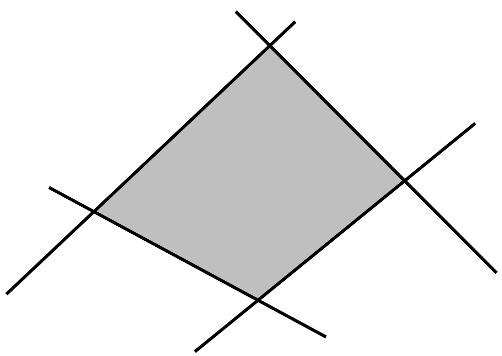
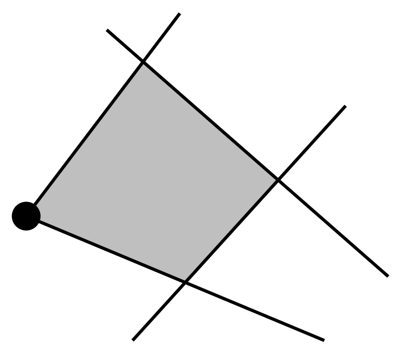


2-triangle



0-quadrangle



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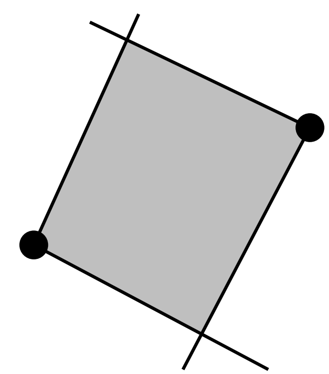

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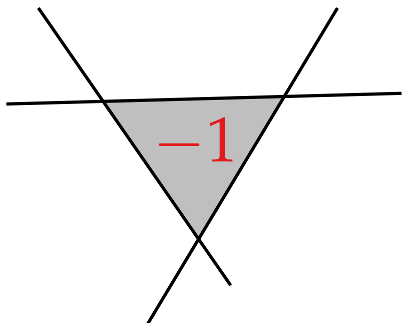

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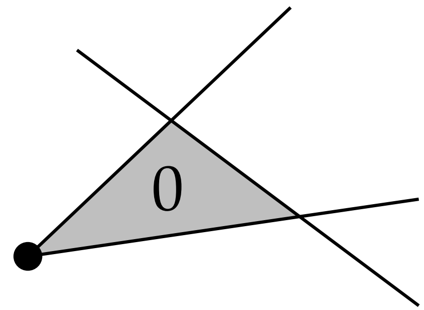
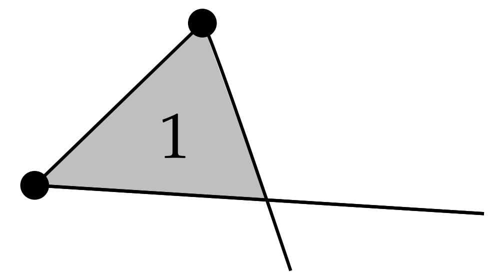
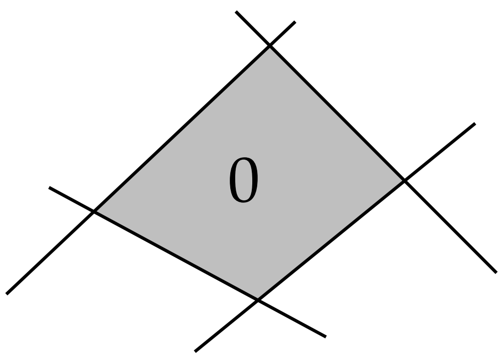


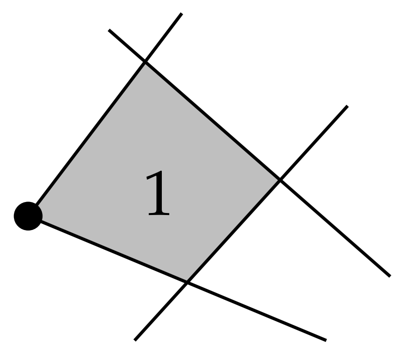
1-triangle



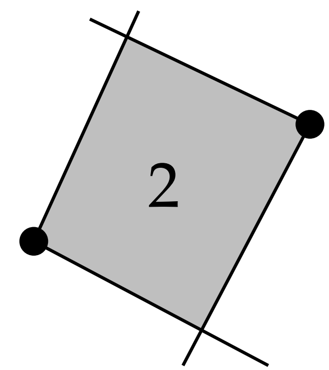
2-triangle



0-quadrangle



1-quadrangle

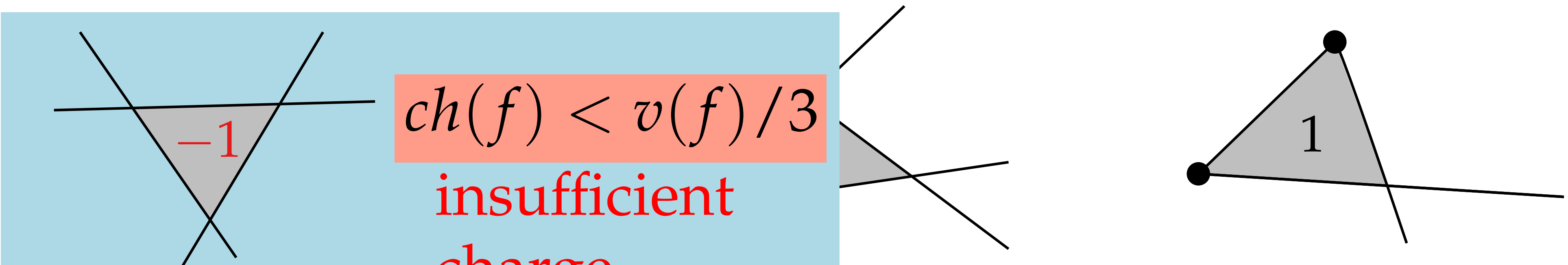


2-quadrangle

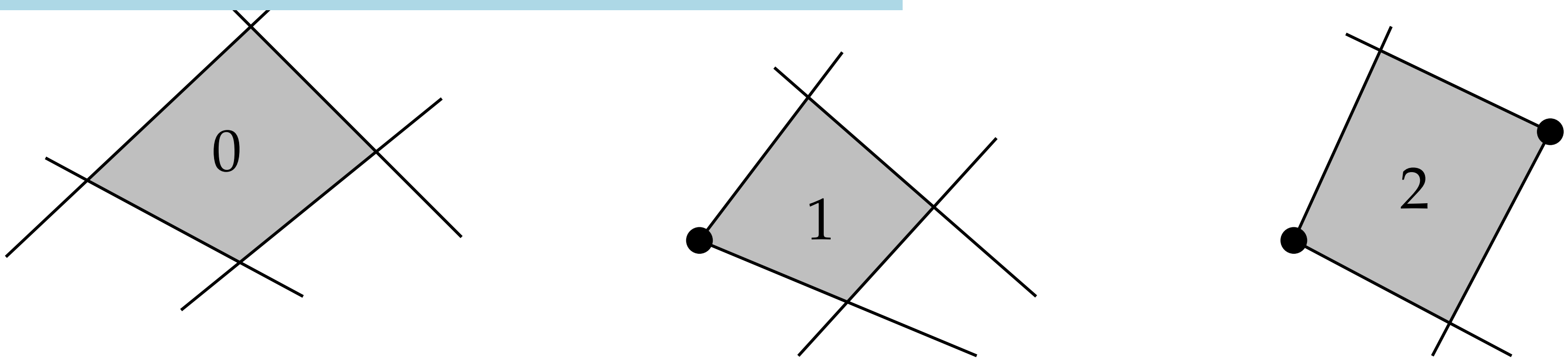
Initial charge at Step 1 is $ch(f) = |f| + v(f) - 4.$

Types of faces

We call a face f of G' a k-triangle, k-quadrilateral, or k-pentagon if f has the corresponding shape and $v(f) = k$.



0-triangle ngle 2-triangle



0-quadrangle 1-quadrangle 2-quadrangle

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Step 2 (charging small faces)

After Step 1 $ch(f) = |f| + v(f) - 4$,
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- $ch(d) \quad = -1$ for each digon d,

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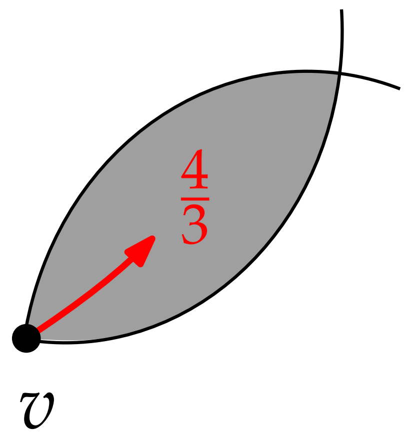
Each digon d is a 1-digon incident to some vertex v.



We say v *contributes* charge to d.

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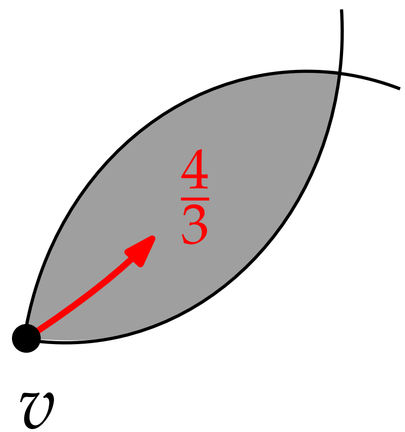

We say v *contributes* charge to d.

After the distribution $ch(d) = 1/3$, i.e., $ch(d) \geq v(d)/3$.

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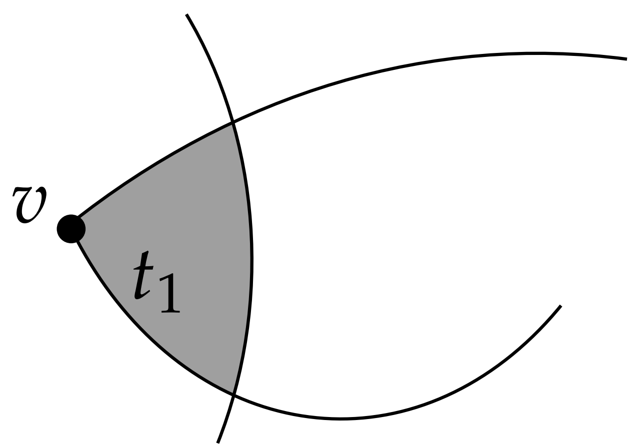
Consider the 1-triangle t_1.



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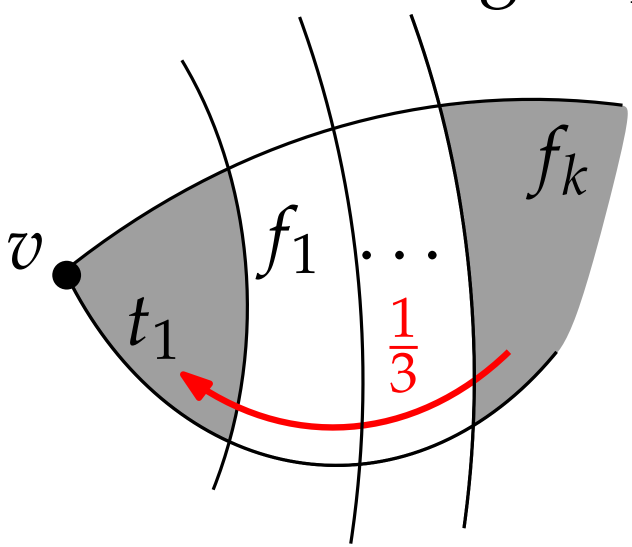

f_k is not a 0-quadrangle

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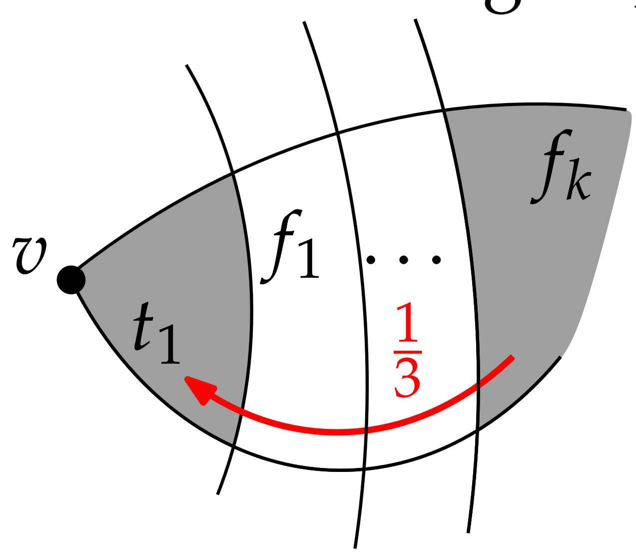

f_k is not a 0-quadrangle

$|f_k| \geq 4$

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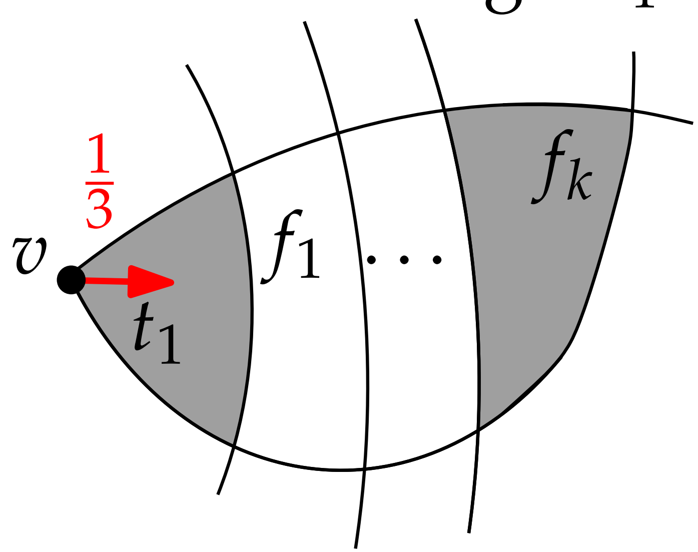

f_k is not a 0-quadrangle

$|f_k| = 3$

Step 2 (charging small faces)

After Step 1 $ch(f) = |f| + v(f) - 4$,
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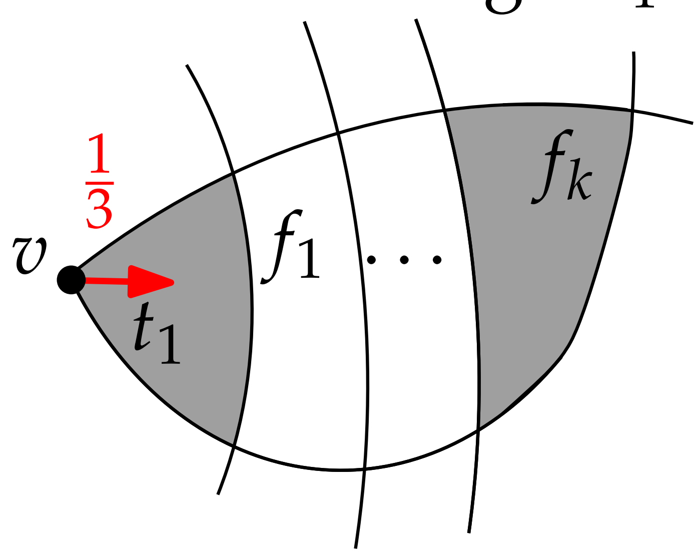

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After the distribution $ch(t_1) = 1/3$, i.e., $ch(t_1) \geq v(t_1)/3$.

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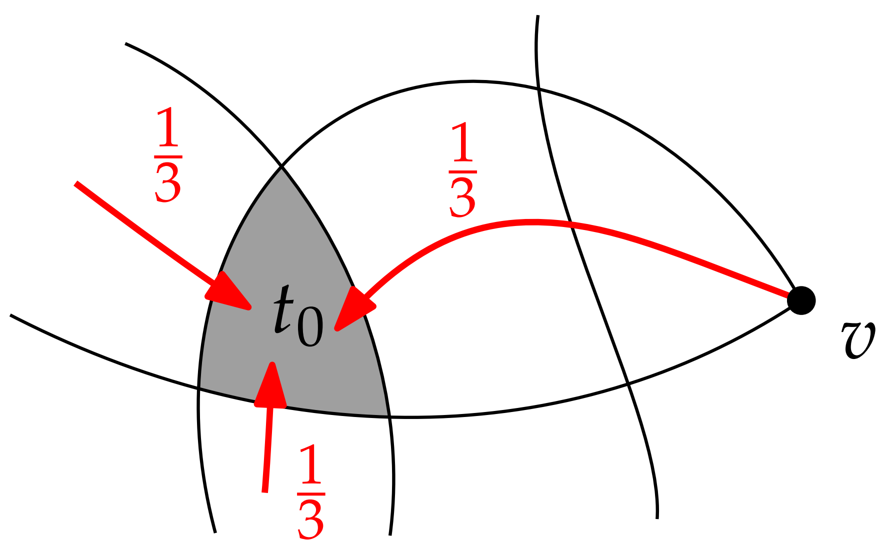

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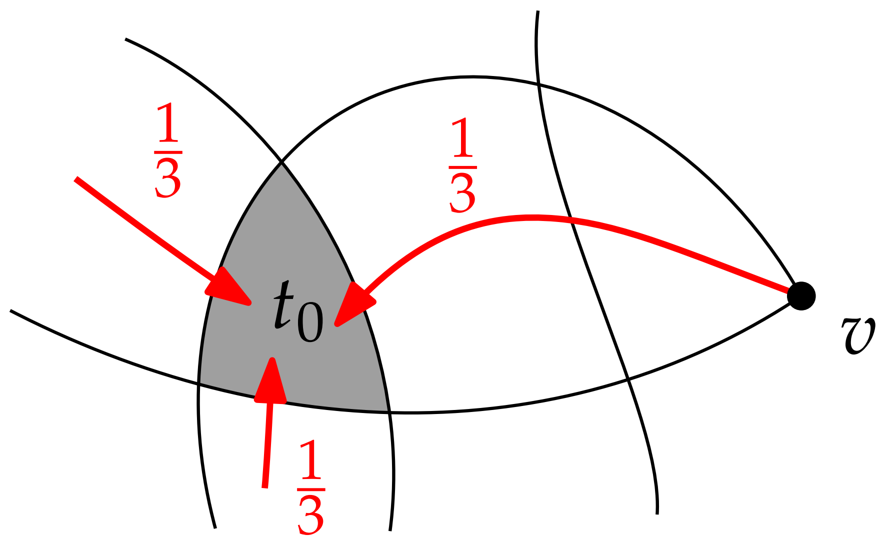
Consider the 0-triangle t_0.



After the distribution $ch(t_0) = 0$, i.e., $ch(t_0) \geq v(t_0)/3$.

Step 3 (verification)

We need to show that:

a) $ch(v) \geq 0$ for all v in G and

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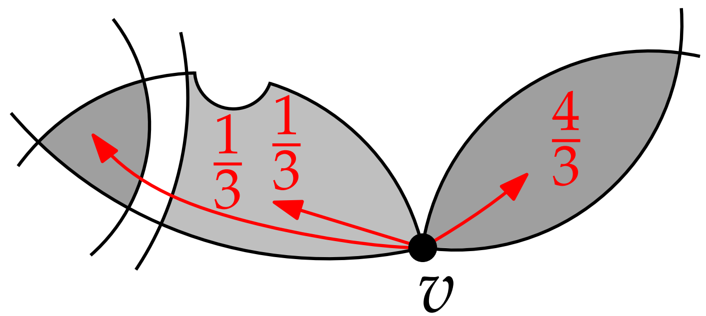

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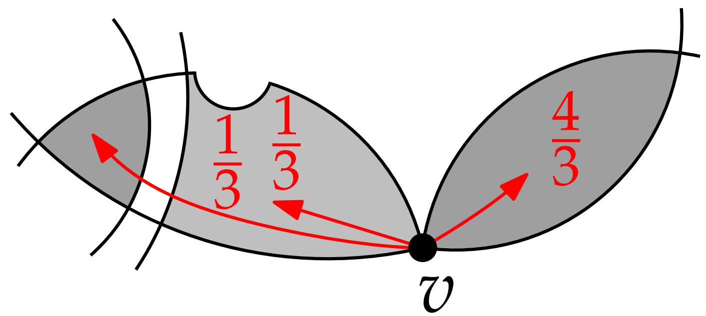

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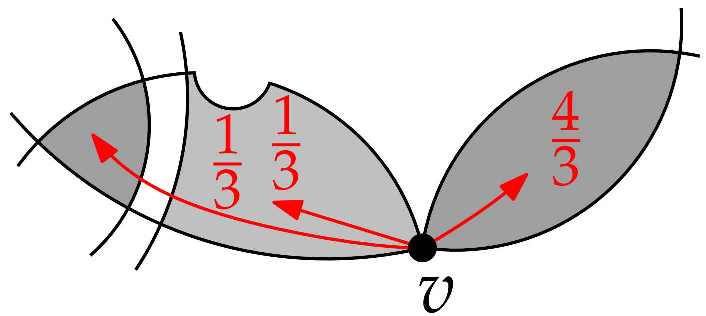

Lem Each vertex is incident to at most four non-overlapping empty 1-lenses.

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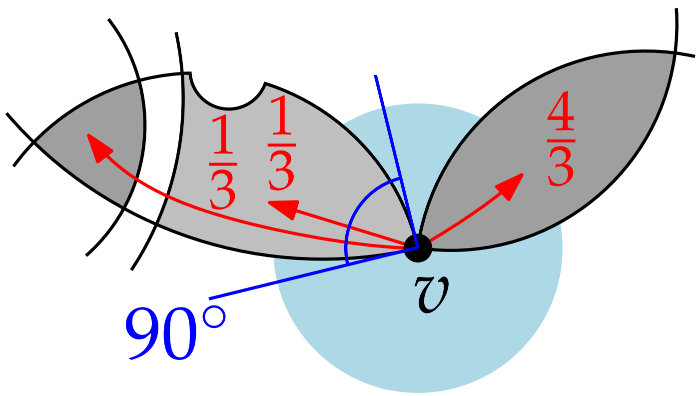

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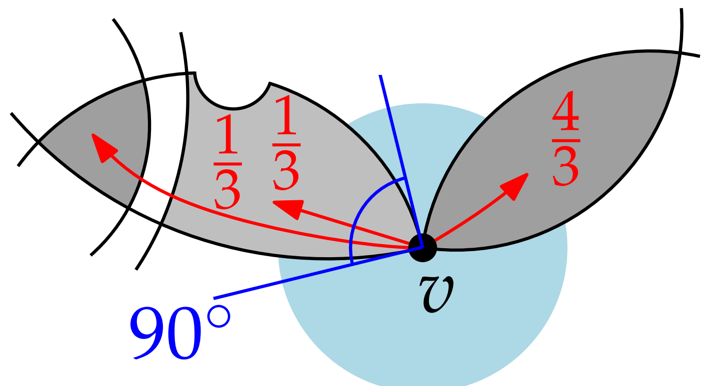

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Worst case: v contributes charge to at most 4 digons.

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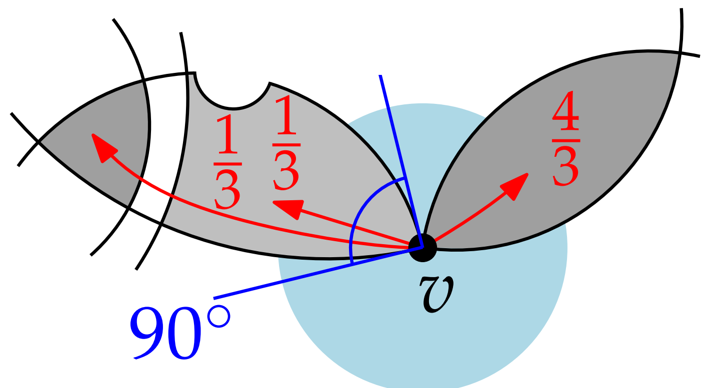

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Worst case: v contributes charge to at most 4 digons.

But aster Step 1 $ch(v) = 16/3$, thus,
after Step 2 $ch(v) \geq 0$.

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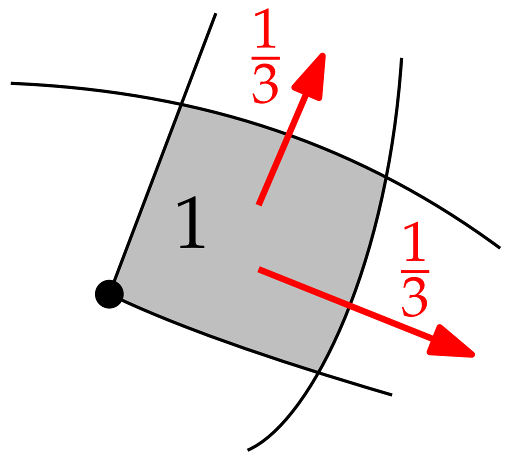

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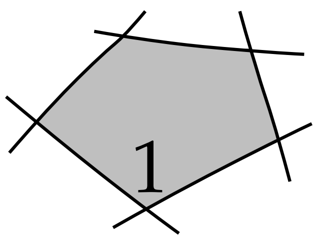

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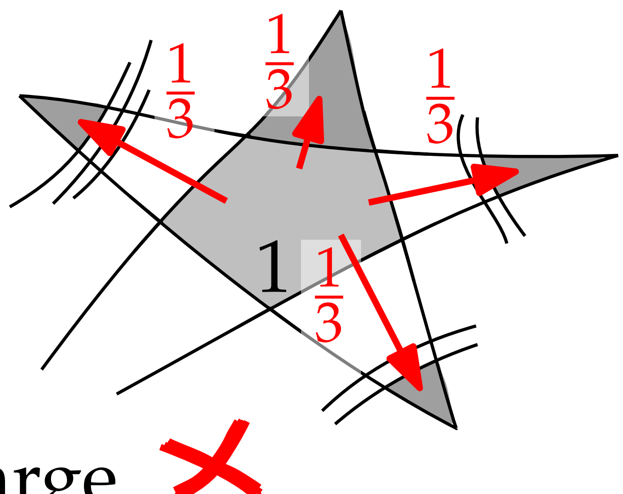

Lem.* A 0-pentagon cannot contribute charge to four or more triangles.

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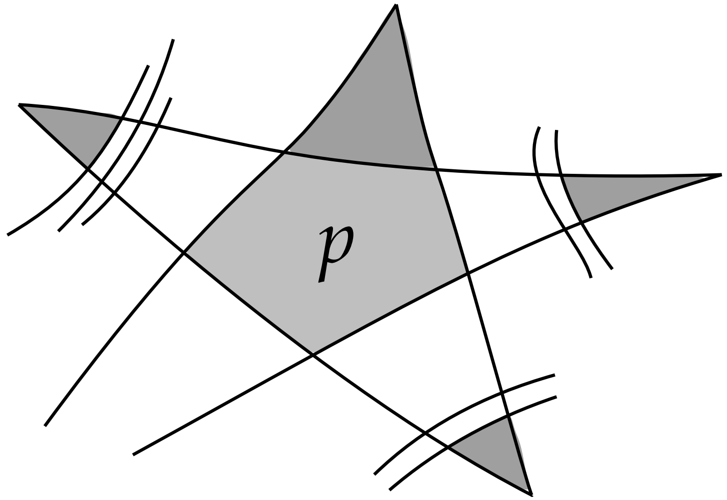
Proof (idea).

in D':



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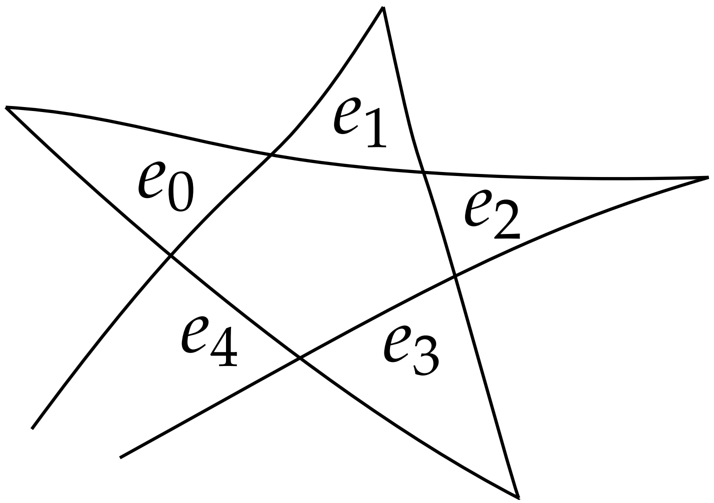

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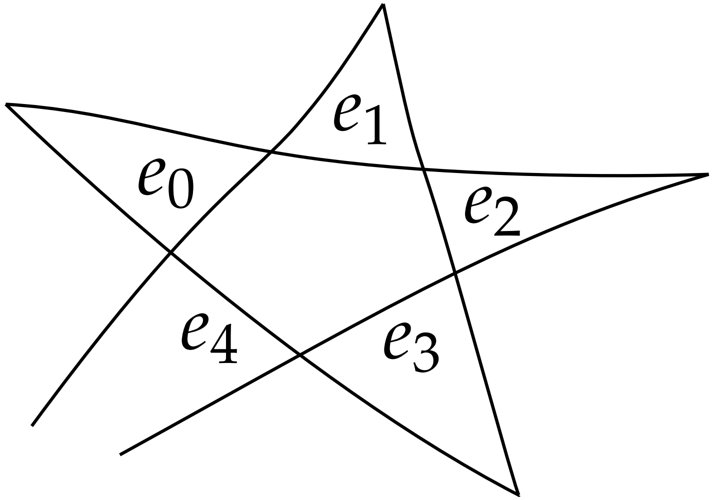
in D': 

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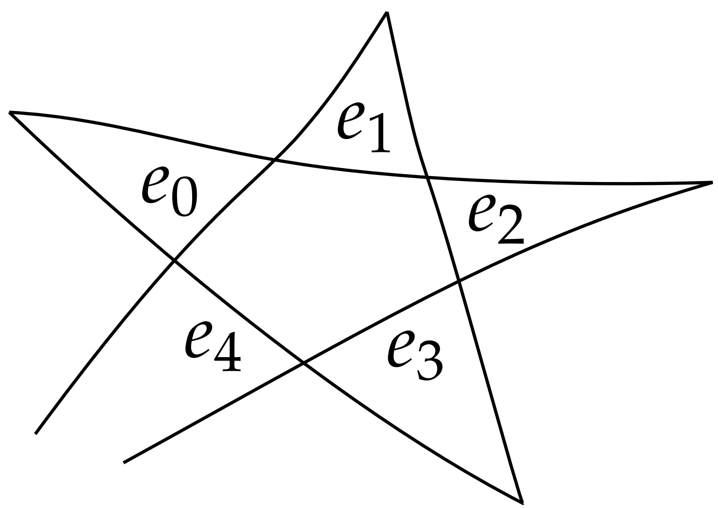
in D': 

in D: ?

1. Property of simplification:
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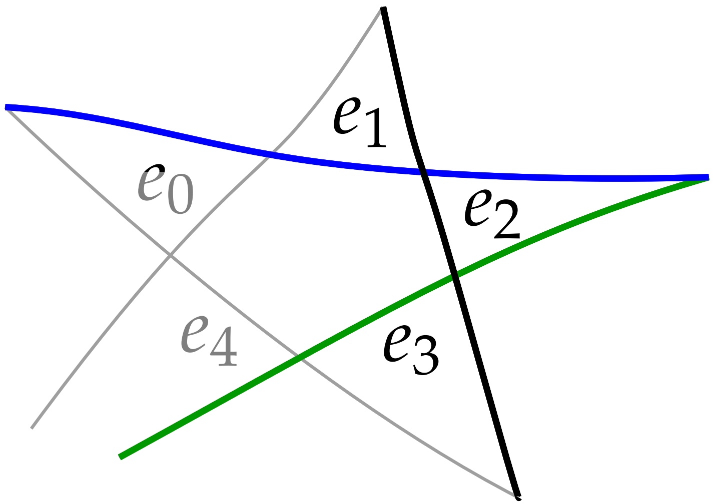

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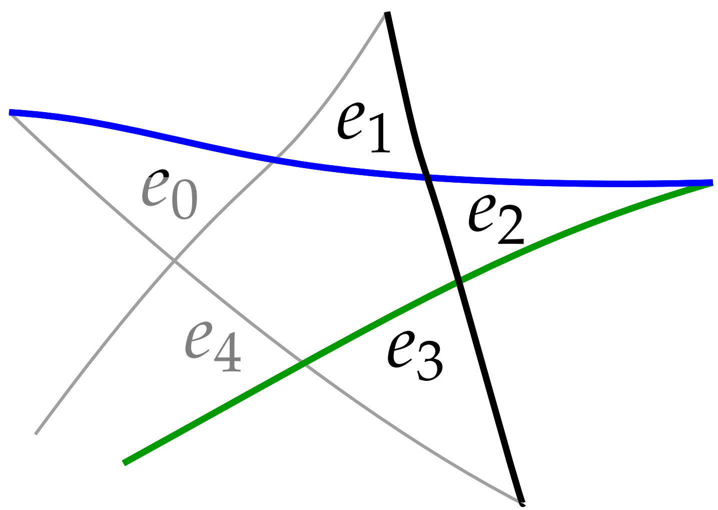

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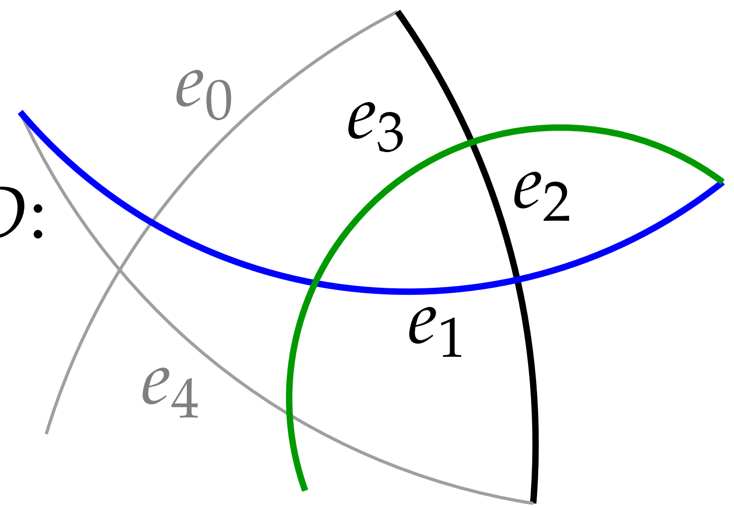

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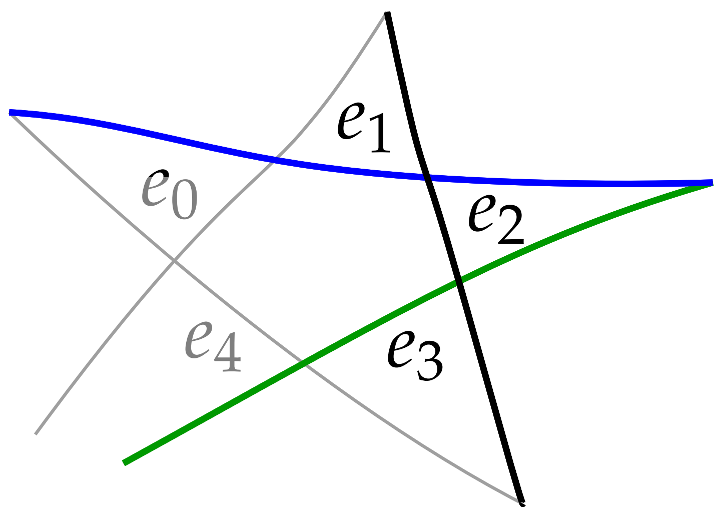

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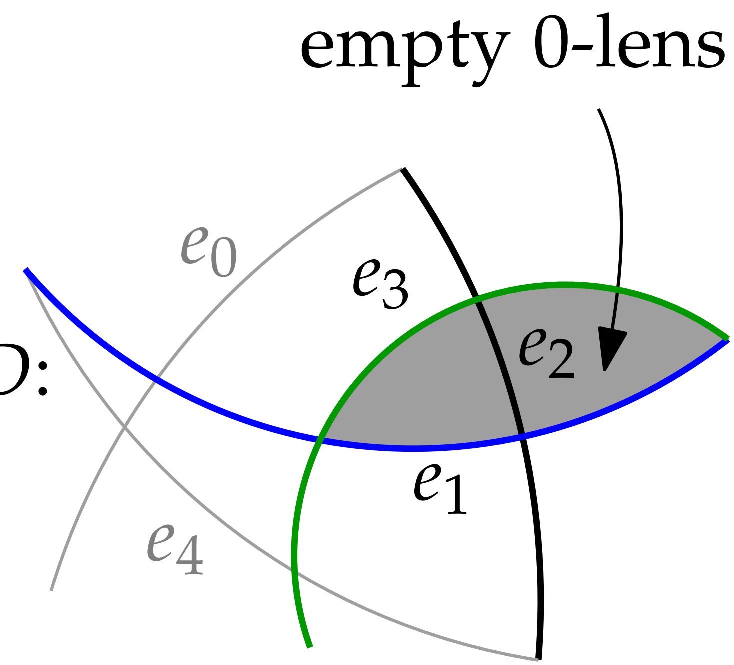

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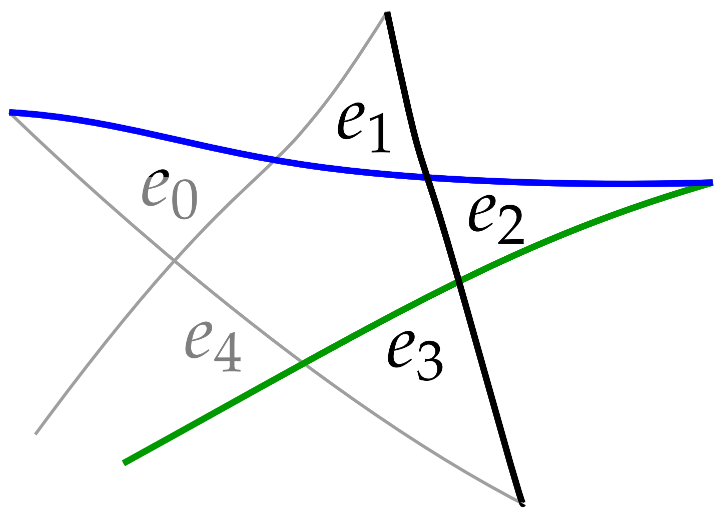

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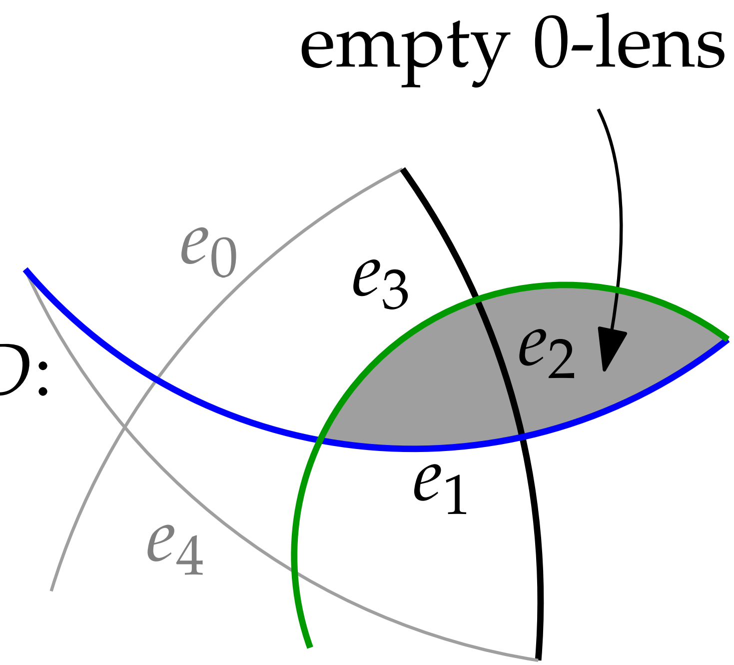

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2. No two edges e_i and e_j can form an empty 0-lens in D.

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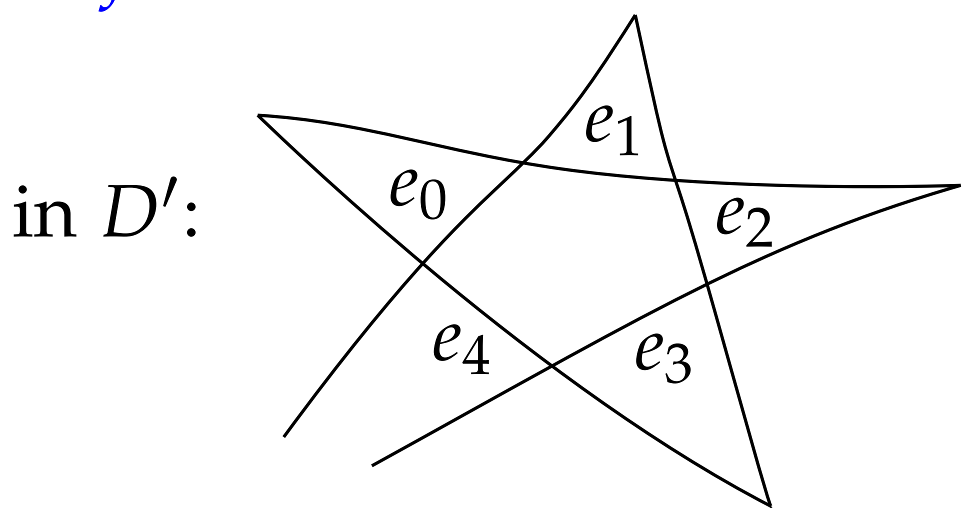

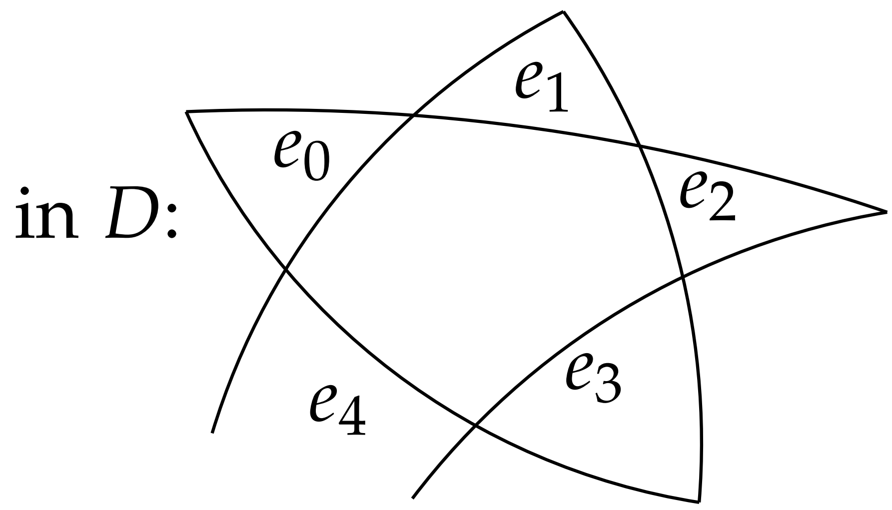

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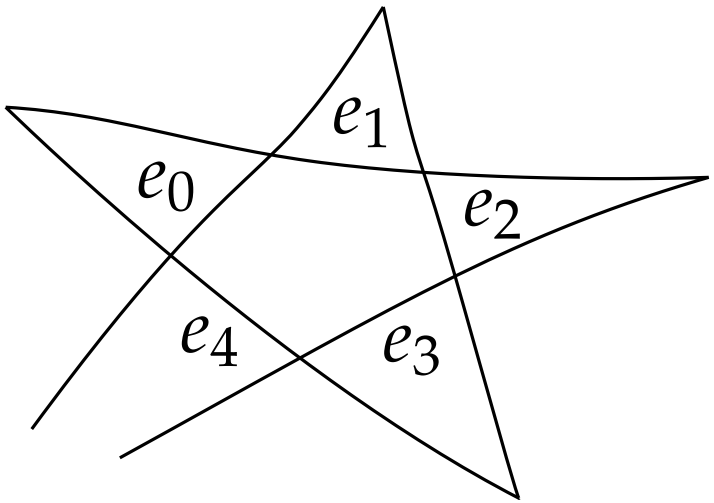

circular arcs & right-angle crossings

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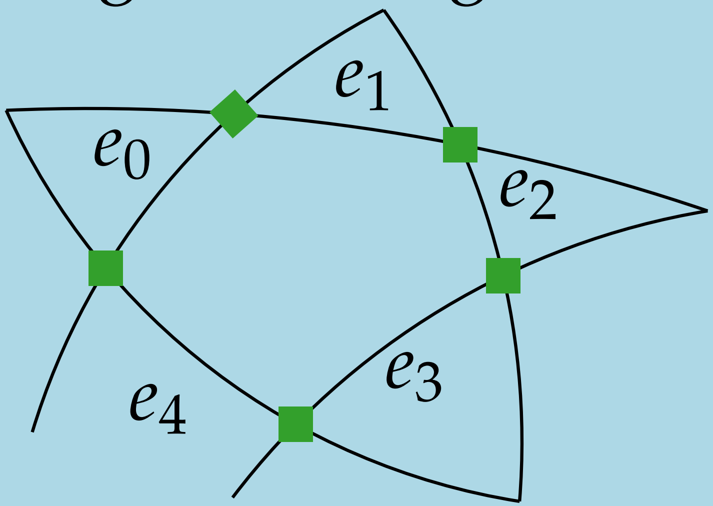

1. Property of simplification:
   if e_i and e_j cross e_k with different order in D than in D', then e_i and e_j form an empty 0-lens in D.

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Lemma*

Lem.* A 0-pentagon cannot contribute charge to four or more triangles.

Proof (idea).

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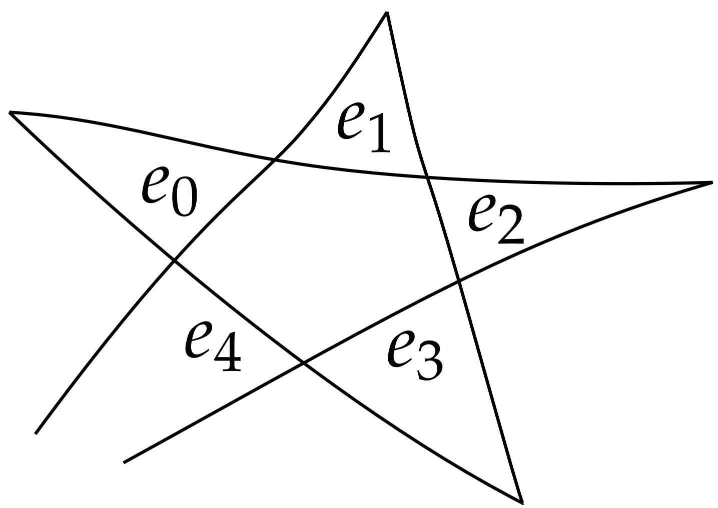

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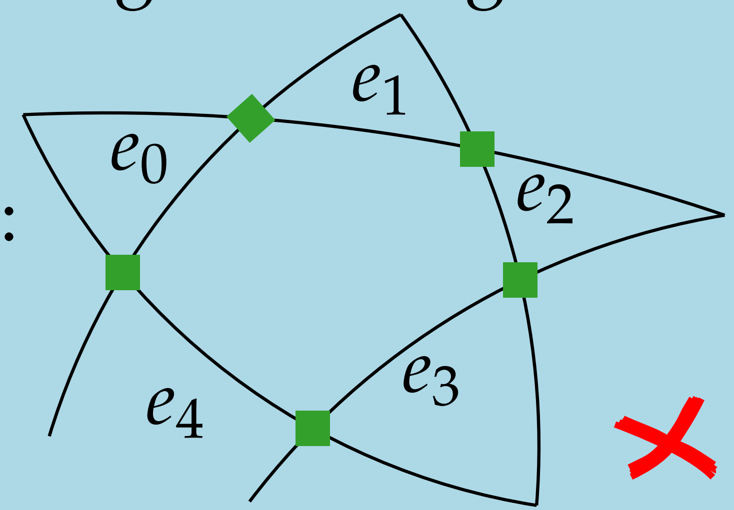

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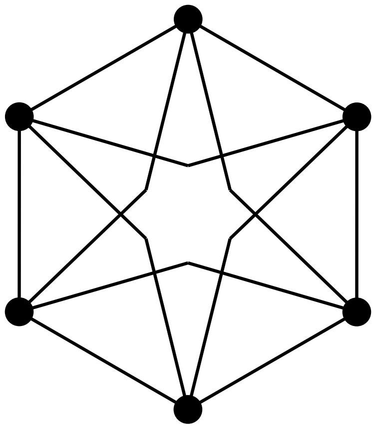

[Arikushi+ '12]

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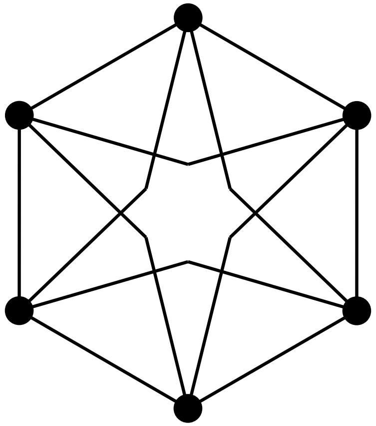

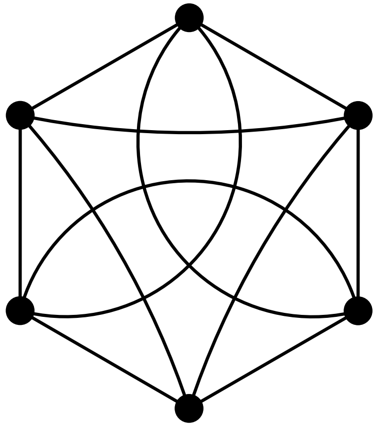

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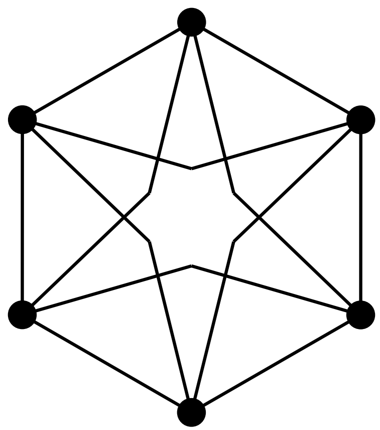

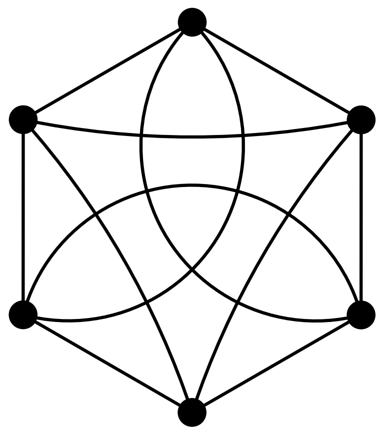

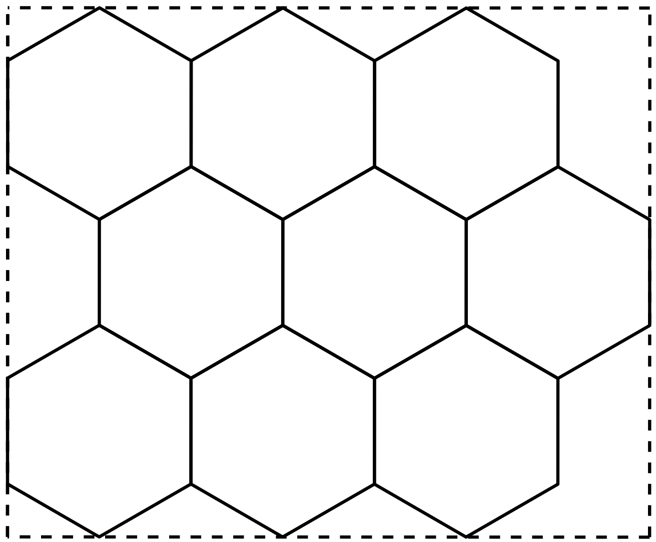

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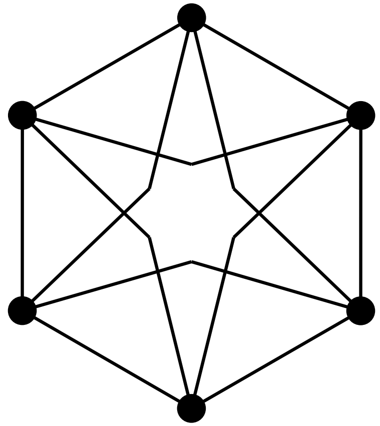

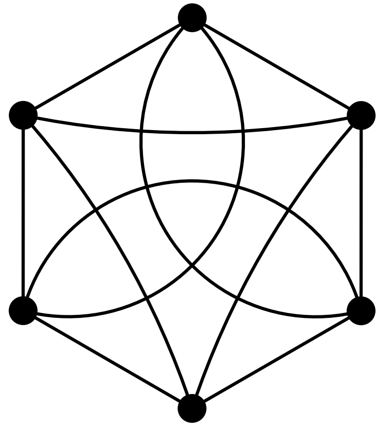

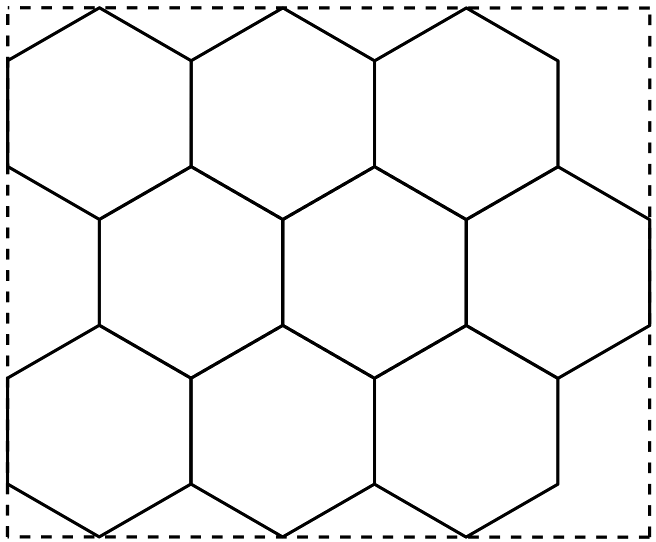

[Arikushi+ '12]

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$\Rightarrow$ G has $4.5n - O(\sqrt{n})$ edges.

Open Questions

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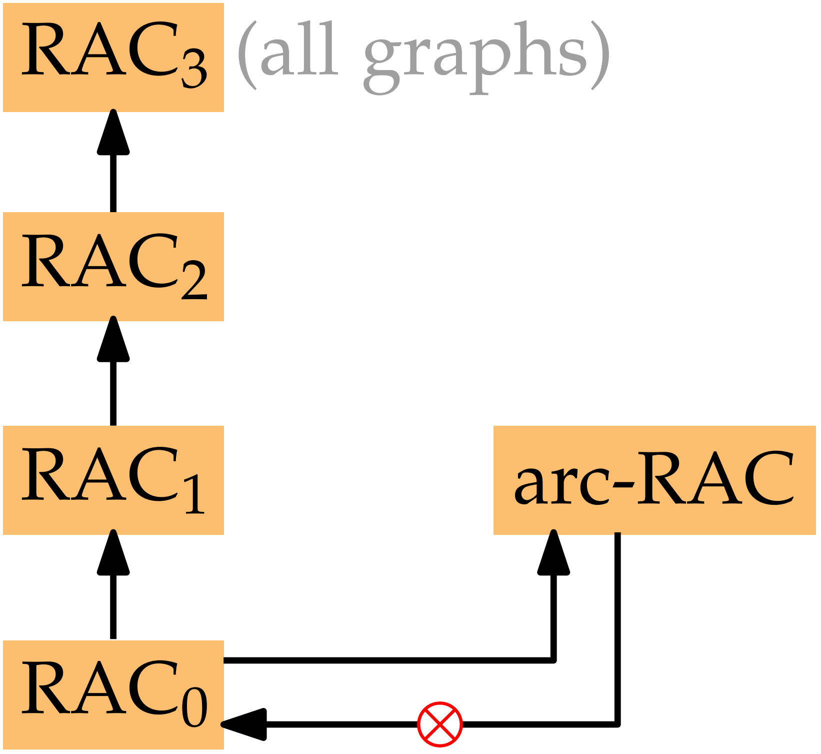

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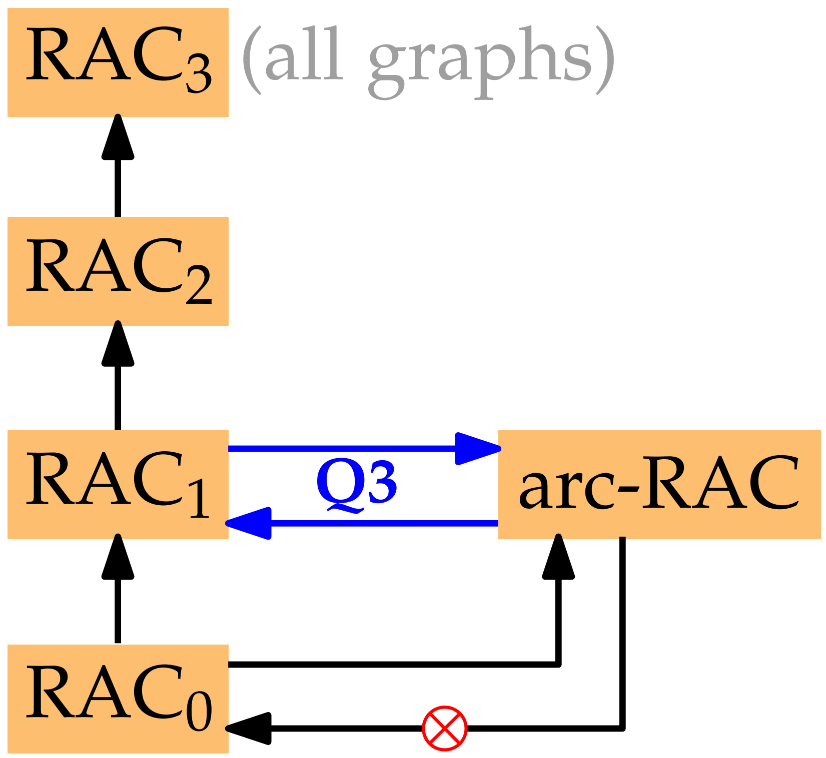

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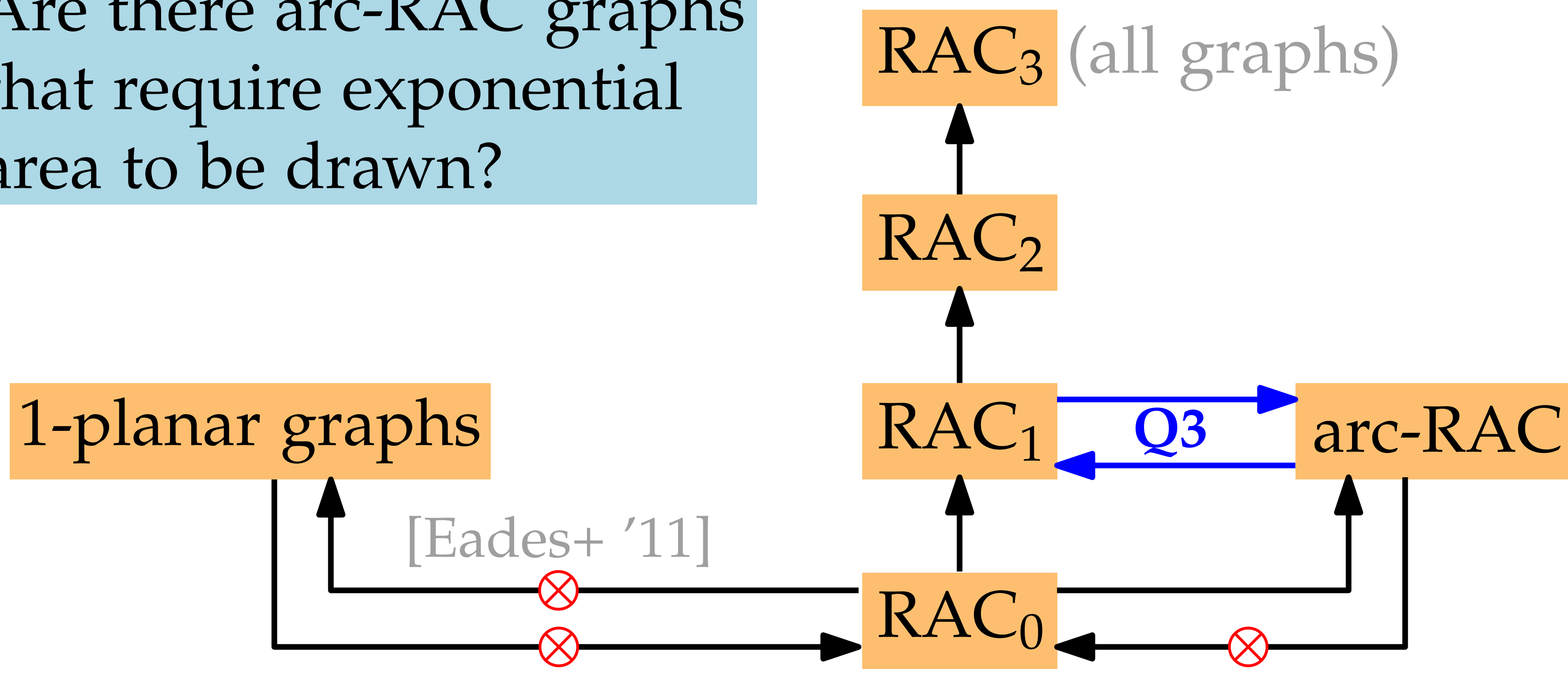

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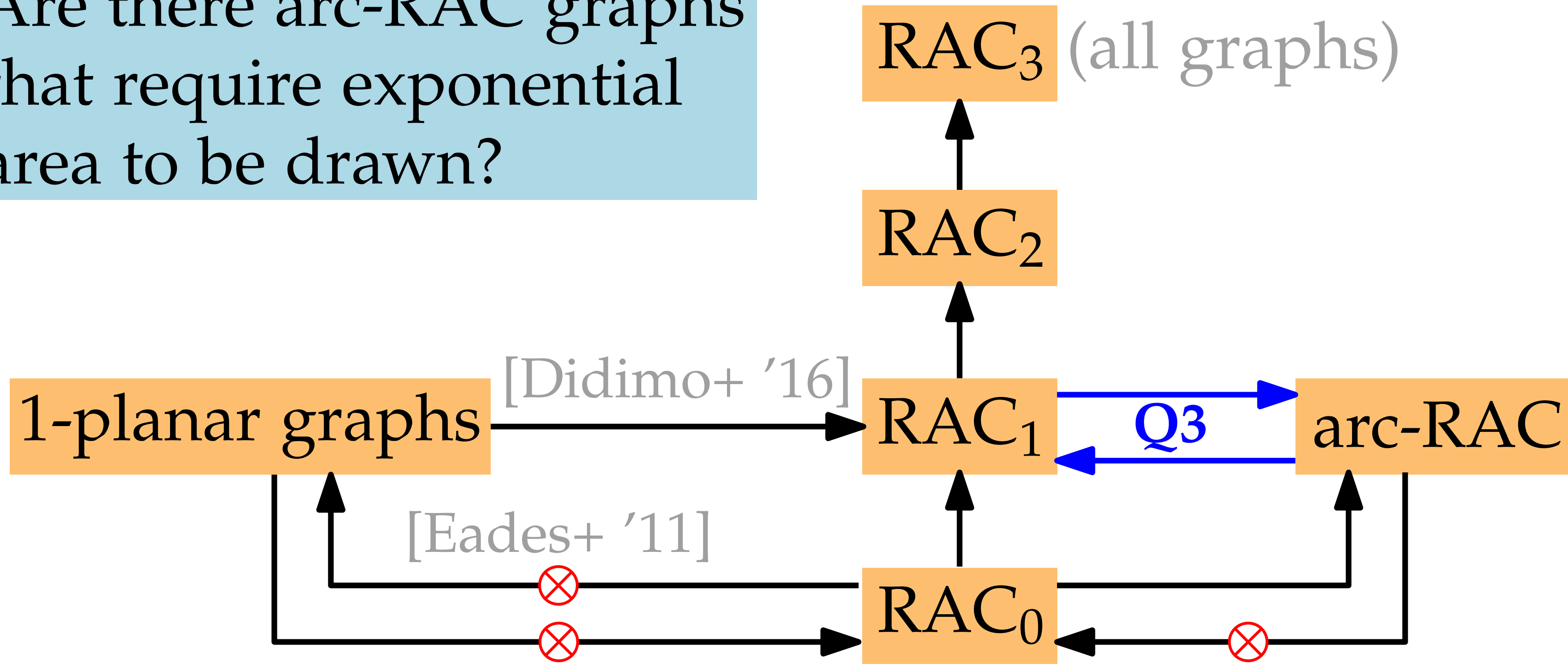

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