Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

On Arrangements of Orthogonal Circles

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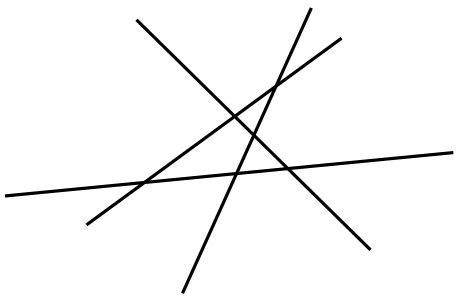
[2] Universität Tübingen, Germany

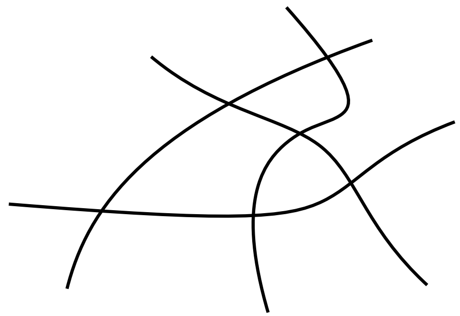
GD 2019, Prague

Arrangements of Curves

Arrangements of

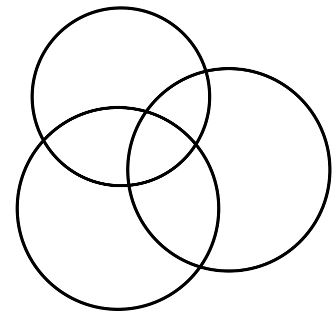


lines

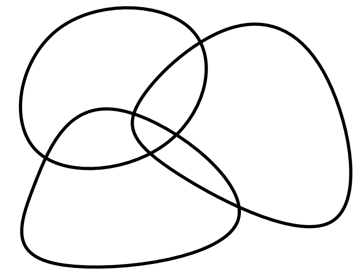


pseudolines

[Grünbaum 1972]



circles



pseudocircles

[Alon et al. 2001, Pinchasi 2002],
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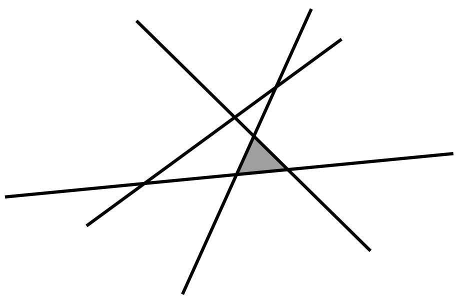

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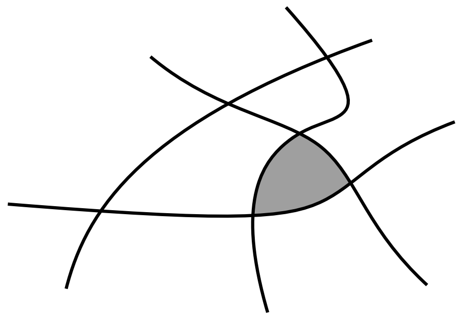

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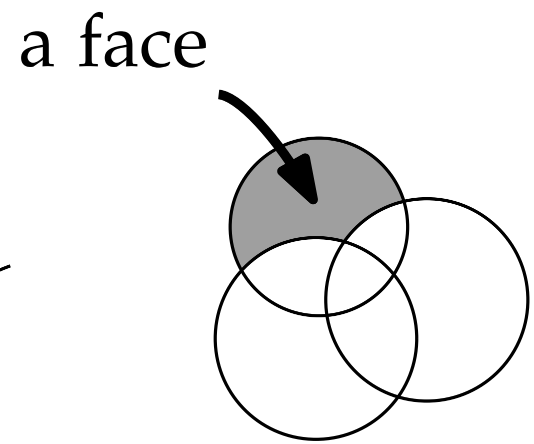

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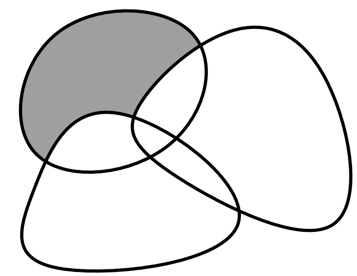

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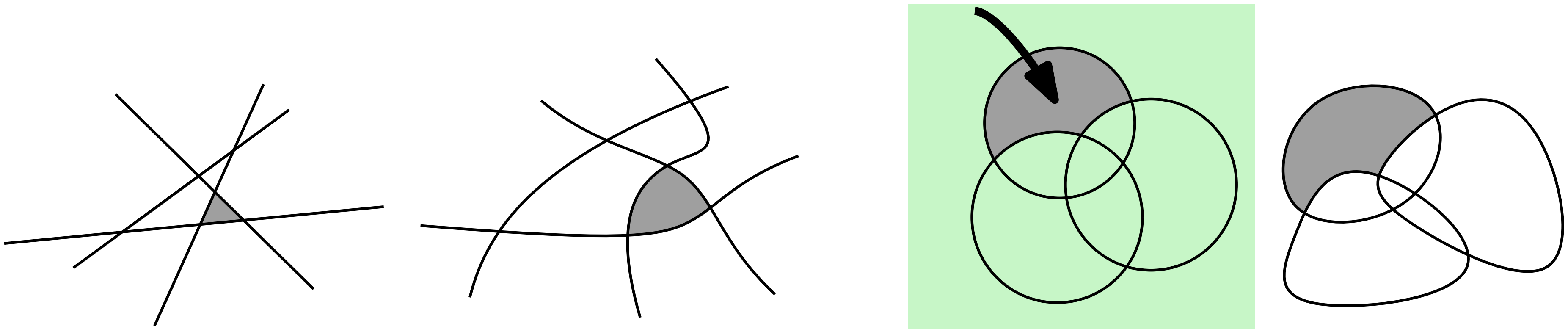
Classical question:
How many faces
can an arrangement of certain curves have?

Arrangements of Curves

Arrangements of

a face



lines pseudolines circles pseudocircles

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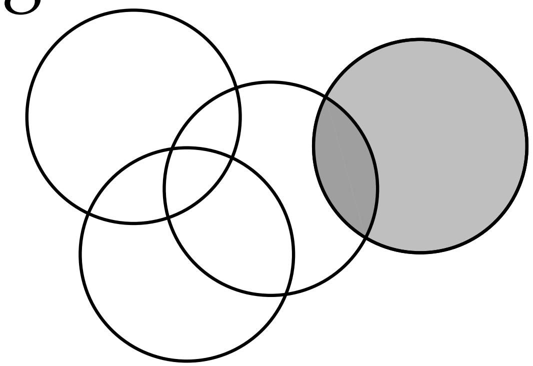
Arrangements of Circles, Digons

$p_k(\mathcal{A})$ = # of faces of degree k in an arrangement $\mathcal{A}$.



Any arrangement $\mathcal{A}$ of n unit circles has
$p_2(\mathcal{A}) = O(n^{4/3} \log n)$ digonal faces;
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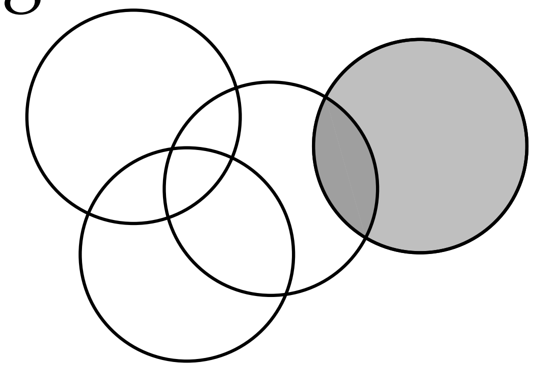

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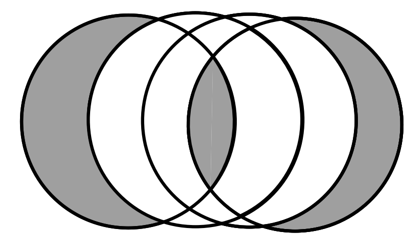

if, in addition, every pair of circles in $\mathcal{A}$ intersect, then $p_2(\mathcal{A}) \leq n + 3$.

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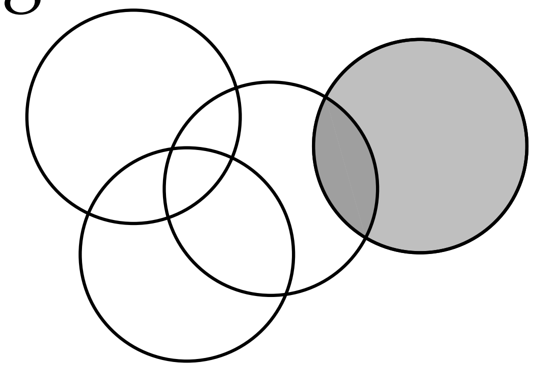

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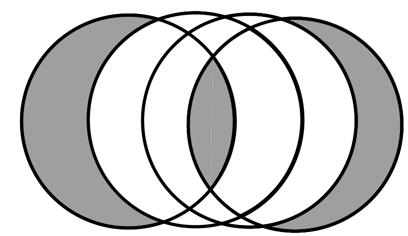
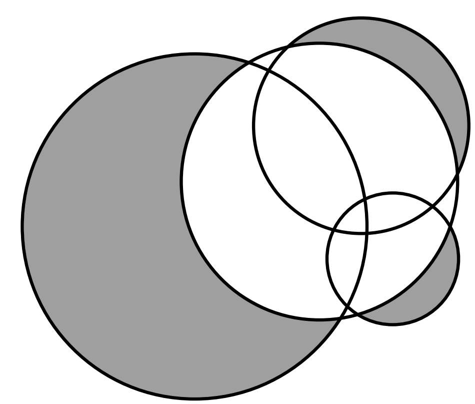


For any arrangement $\mathcal{A}$ of n circles with arbitrary radii $p_2(\mathcal{A}) \leq 20n-2$ if every pair of circles in $\mathcal{A}$ intersect.
[Alon et al. 2001]



Arrangements of Circles, Triangles

For any arrangement $\mathcal{A}$ of (pseudo)circles
$p_3(\mathcal{A}) \leq \frac{2}{3}n^2 + O(n)$. [Felsner & Scheucher 2018]

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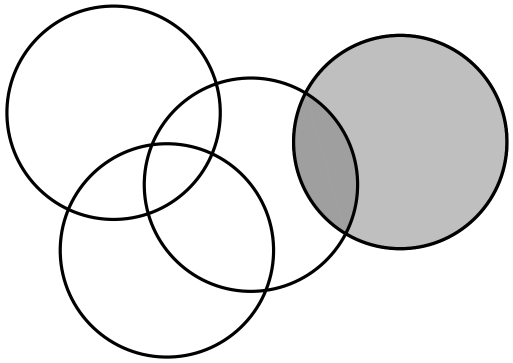
Lower bound example $\mathcal{A}$ with $p_3(\mathcal{A}) = \frac{2}{3}n^2 + O(n)$ can be constructed from a line arrangement $\mathcal{A}'$ with
$p_3(\mathcal{A}') = \frac{1}{3}n^2 + O(n)$. [Füredi & Palásti 1984]
[Felsner, S.: Geometric Graphs and Arrangements, 2004]

Arrangements of Circles, Restrictions

Types of restrictions:

Any arrangement $\mathcal{A}$ of n unit circles has $p_2^\circ(\mathcal{A}) = O(n^{4/3} \log n)$ digonal faces;



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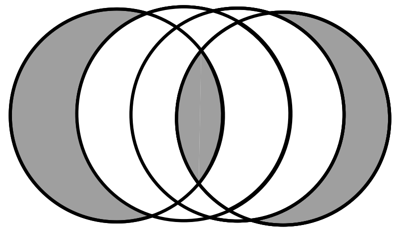

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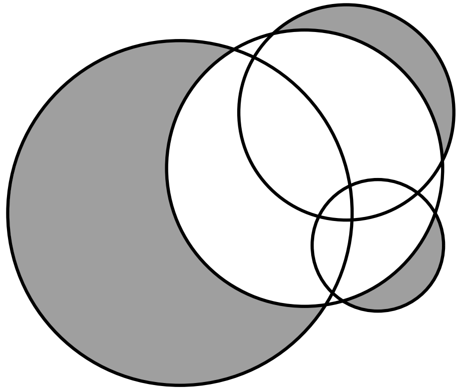
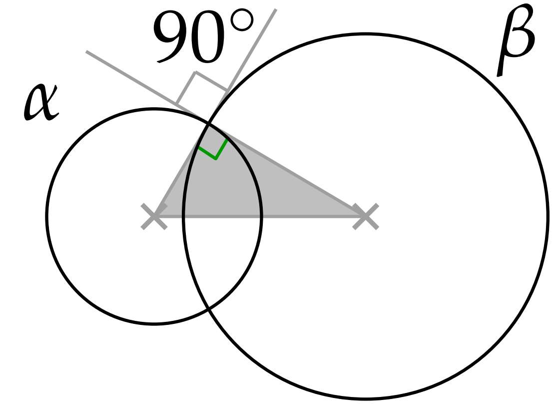


Orthogonal Circles

Two circles α and β are *orthogonal* if their tangents at one of their intersection points are orthogonal.



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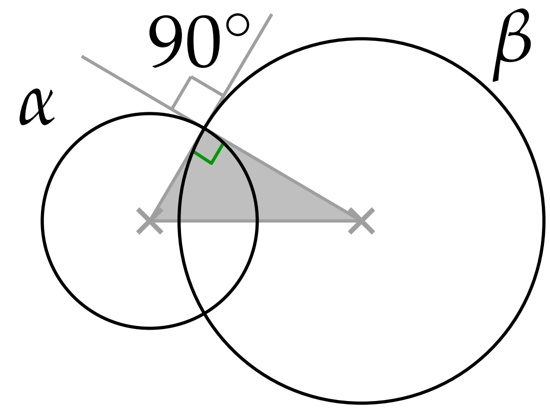
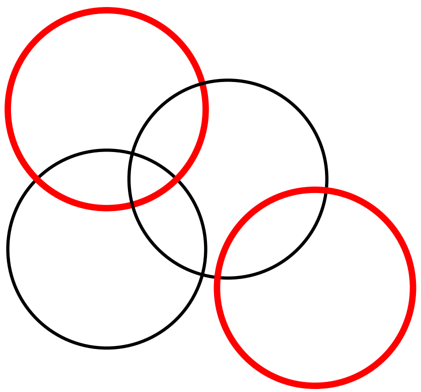


In an *arrangement of orthogonal circles* every two circles either are disjoint or orthogonal.



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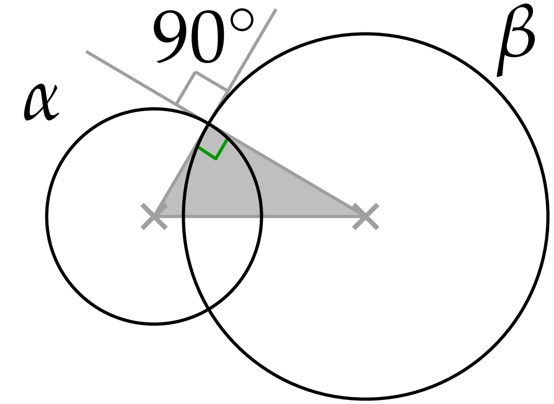

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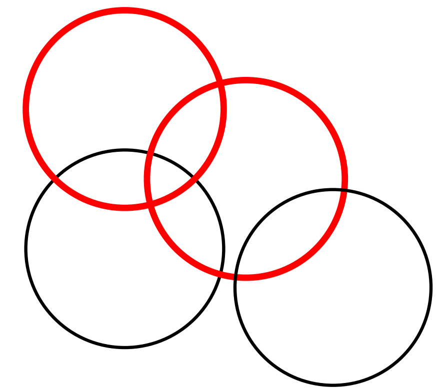

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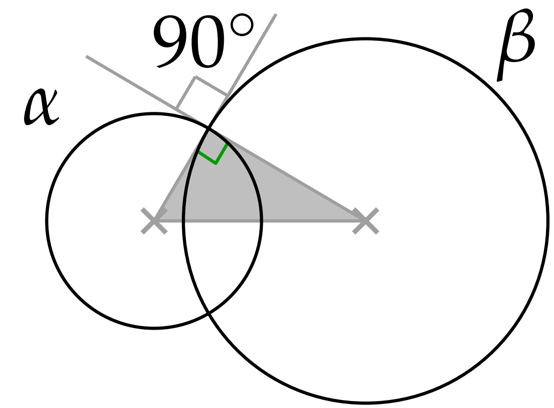

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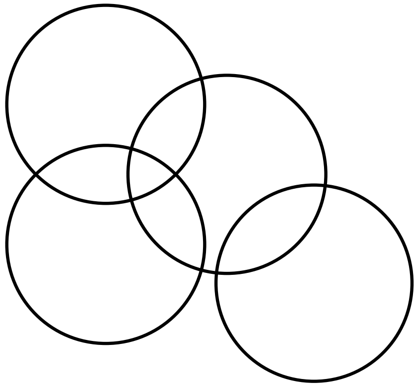
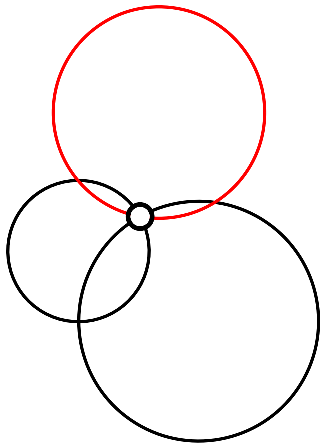


No three pairwise orthogonal circles can share the same point.



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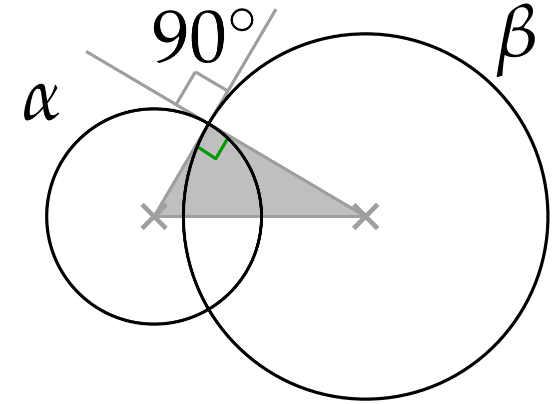

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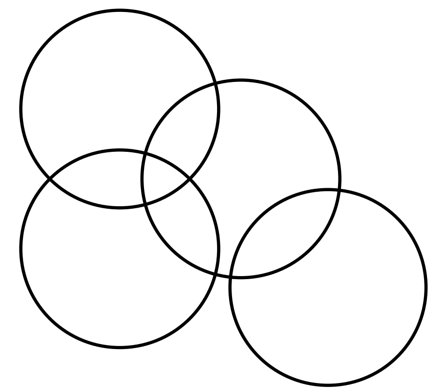

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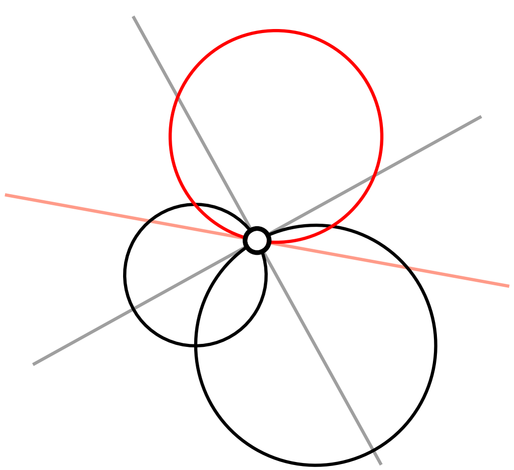
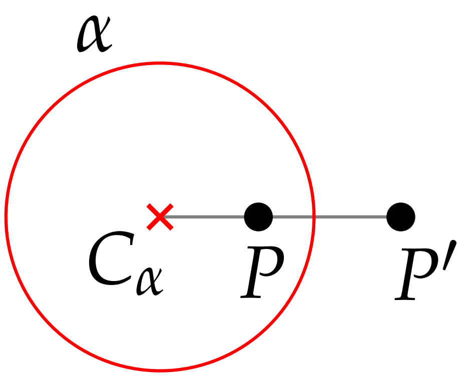


Inversion

Inversion of a point P with respect to α is a point P' on the ray $C_\alpha P$ so that $|C_\alpha P'| \cdot |C_\alpha P| = r_\alpha^2$.



Properties:

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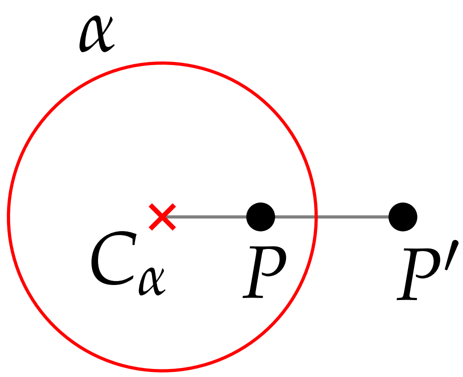
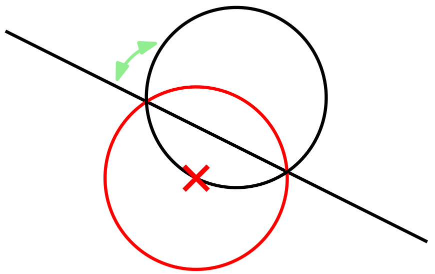
Properties:

- each circle passing through C_α is mapped to a line;





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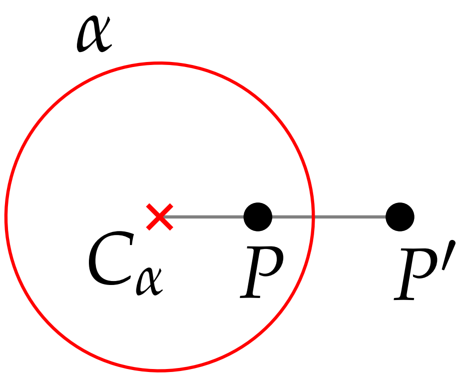

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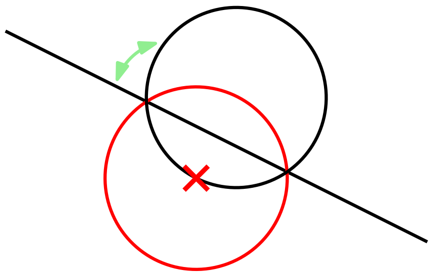
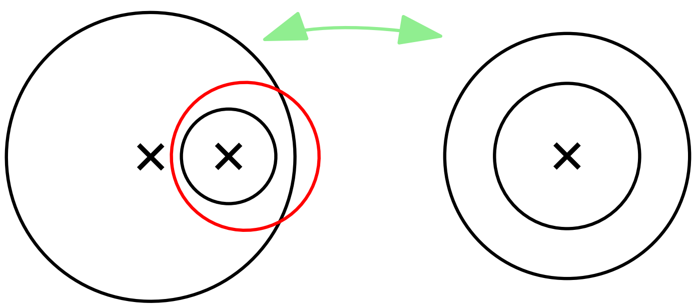


- $\exists$ an inversion that maps 2 disjoint circles into 2 concentric circles;



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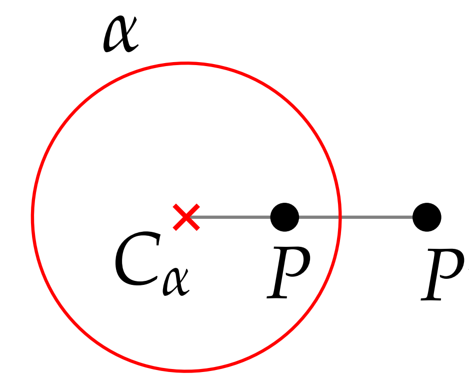

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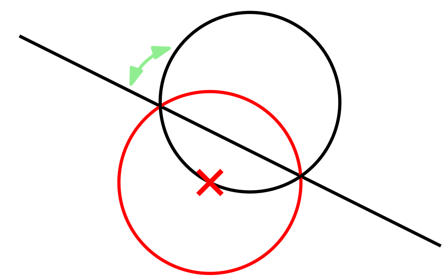

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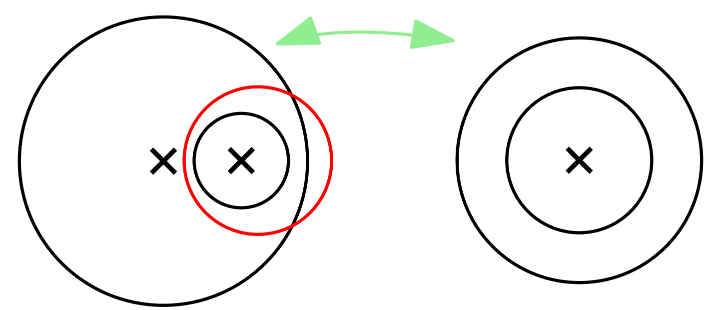
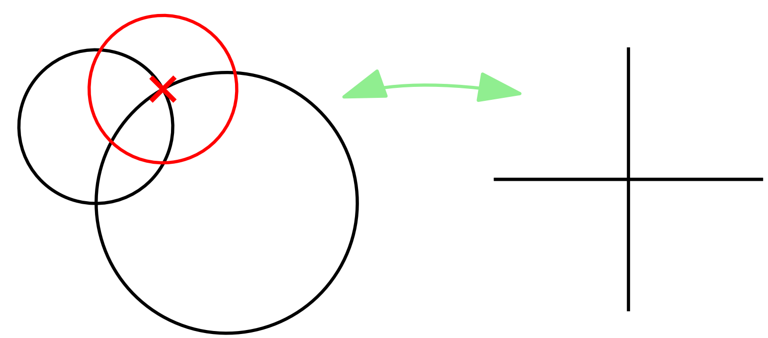


- inversion preserves angles.



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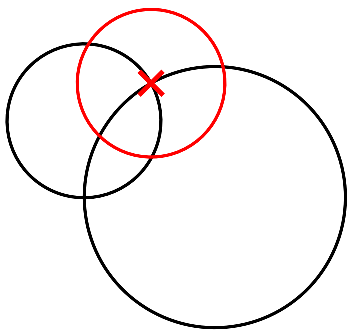
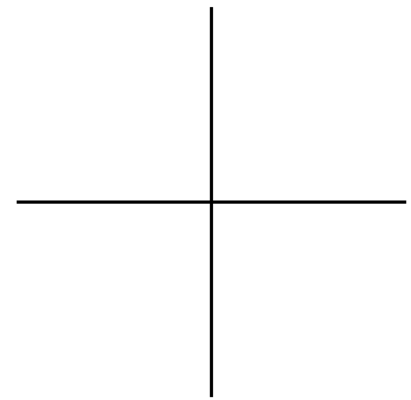
Proof: Assume for contradiction there exist such four circles.





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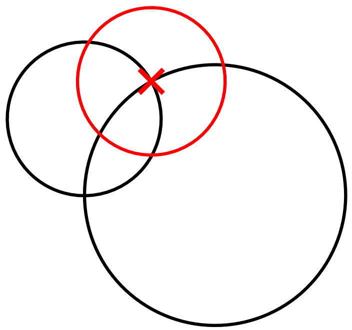

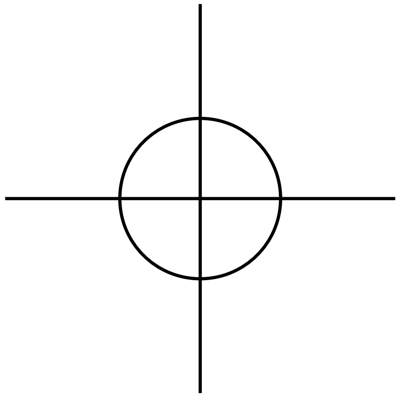

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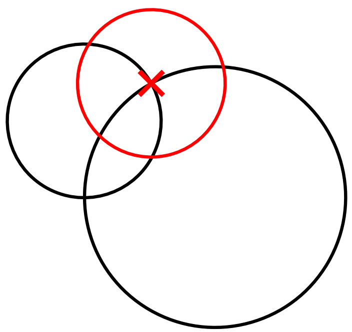

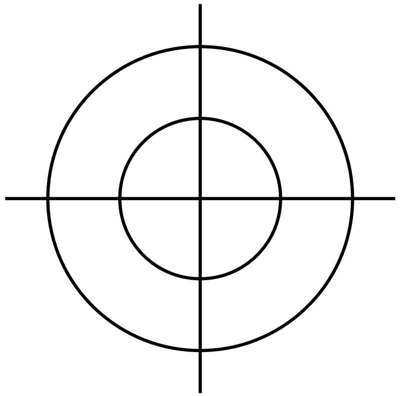

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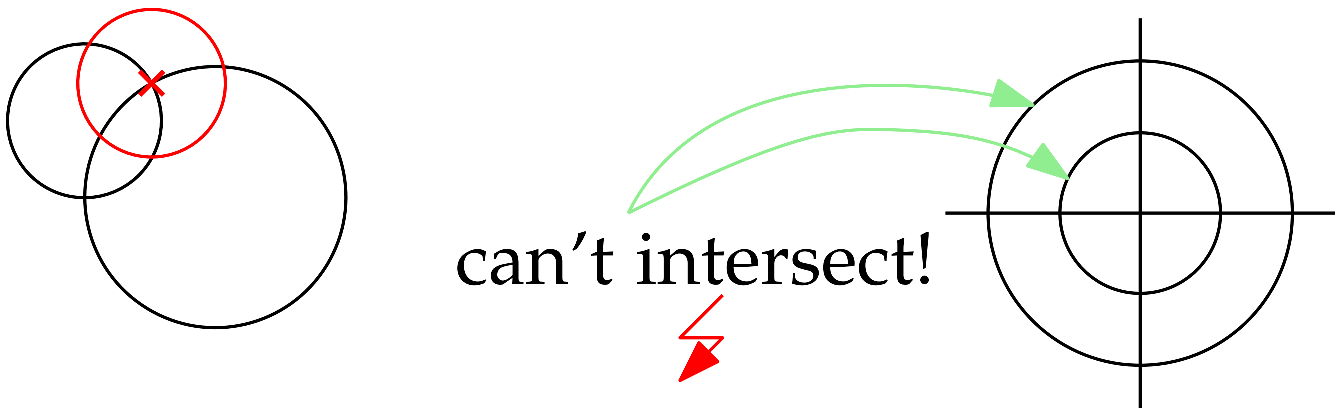

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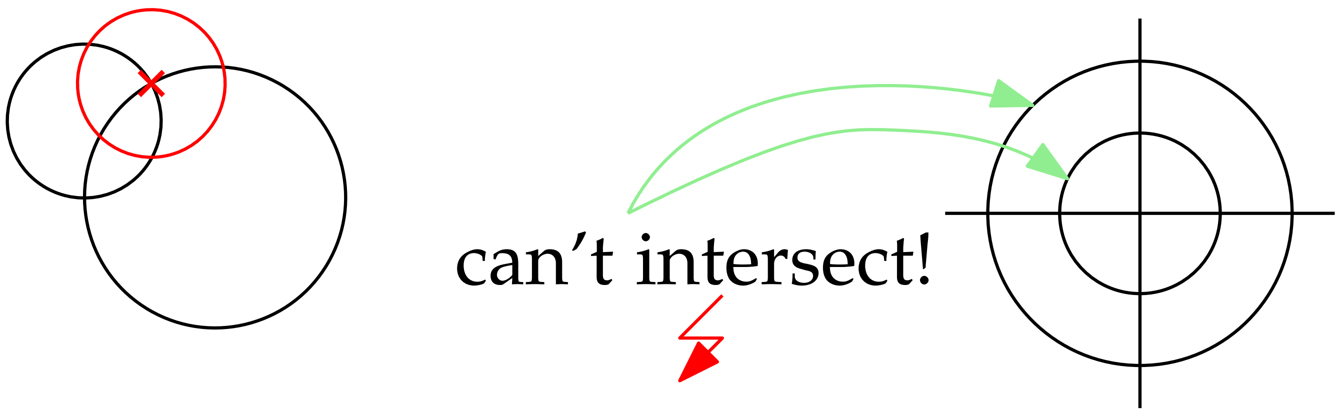

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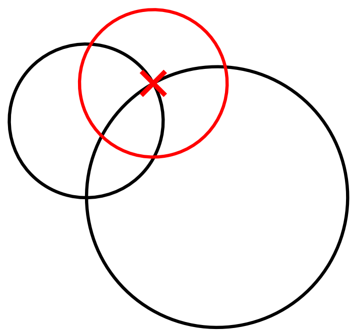

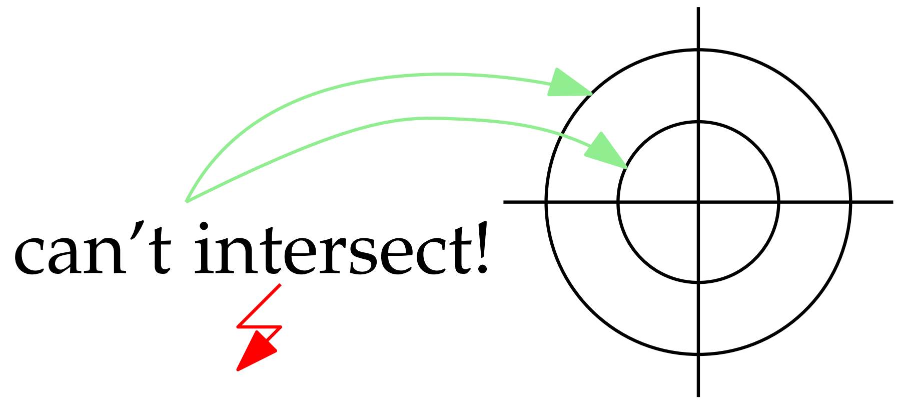

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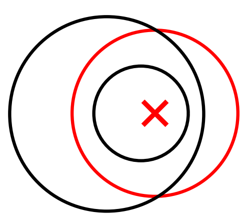

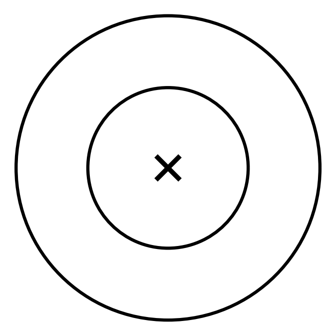

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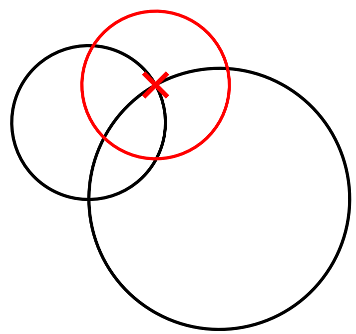

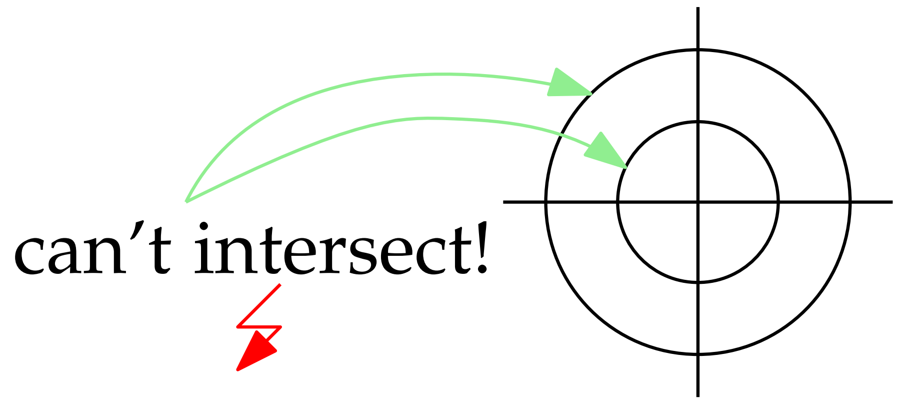

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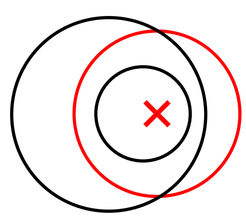

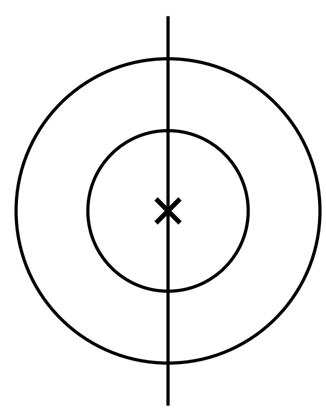

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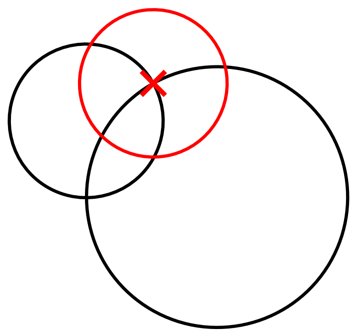

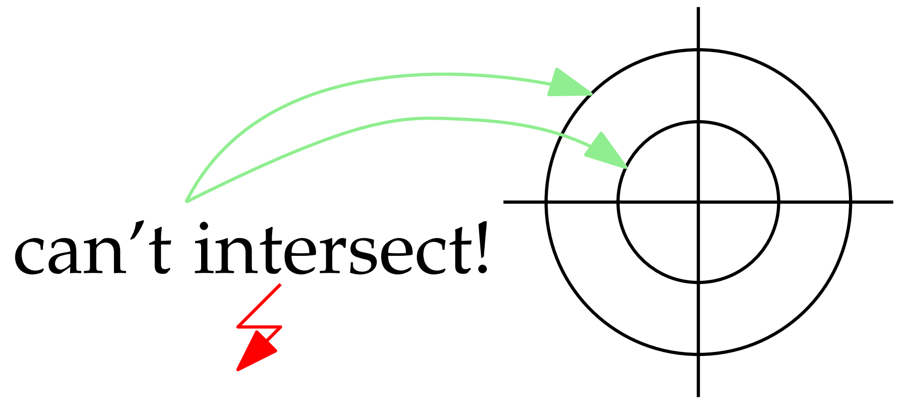

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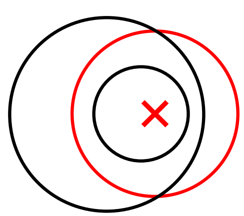
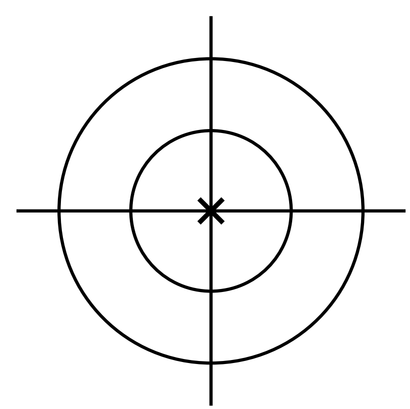
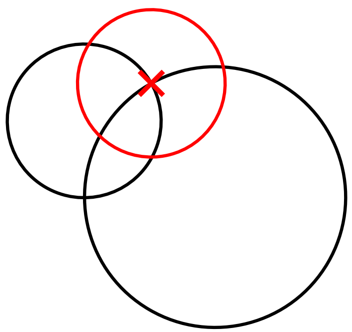


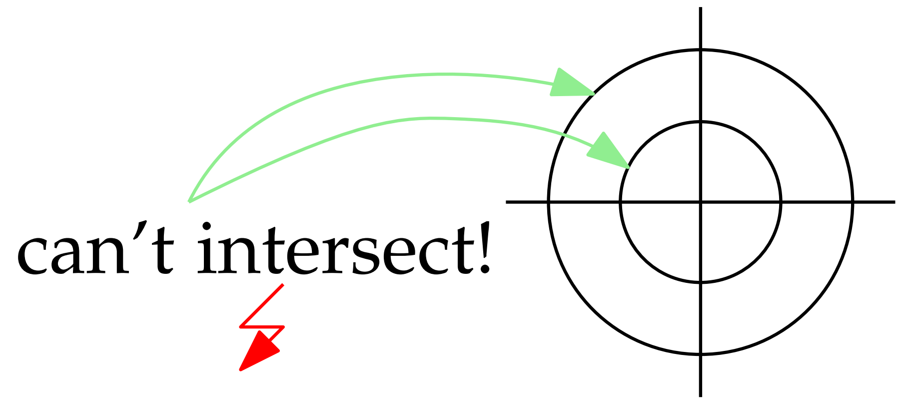


Local Properties

Lem. There are no four pairwise orthogonal circles.

Proof: Assume for contradiction there exist such four circles.





Lem. Two disjoint circles can be orthogonal to at most two other circles.

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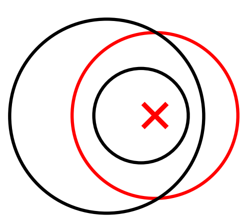
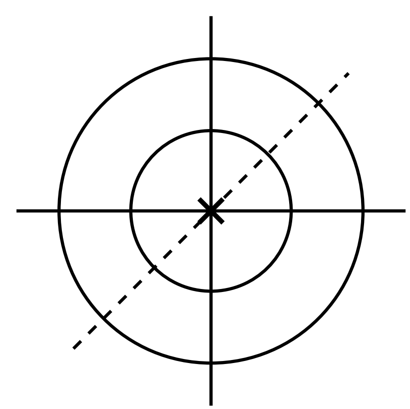


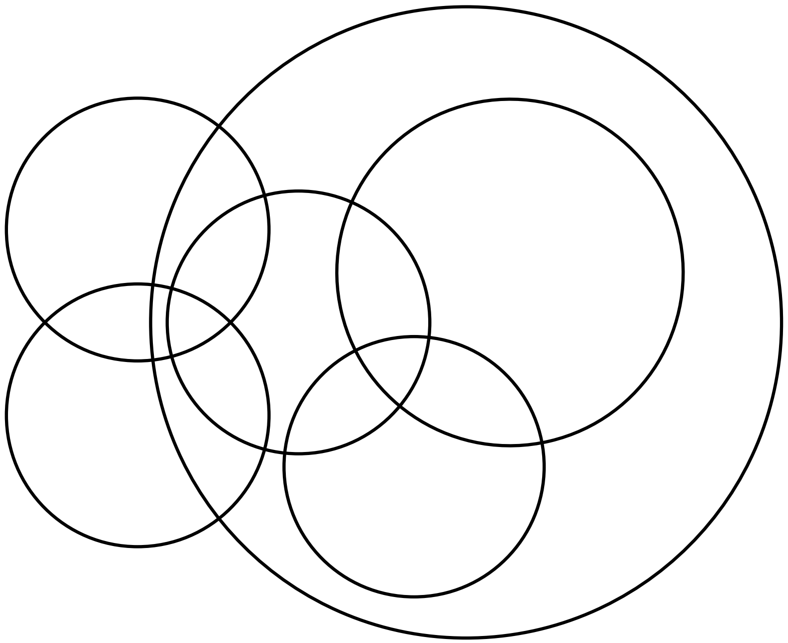


□

Main Lemma

Consider an arrangement $\mathcal{A}$ of orthogonal circles.

Def. A *Smallest* circle in $\mathcal{A}$ is a circle with the smallest radius.



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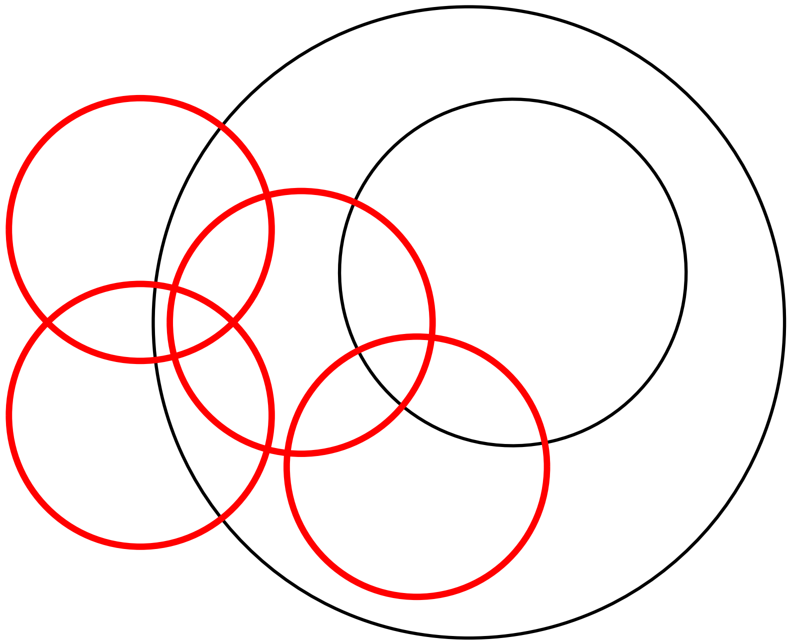

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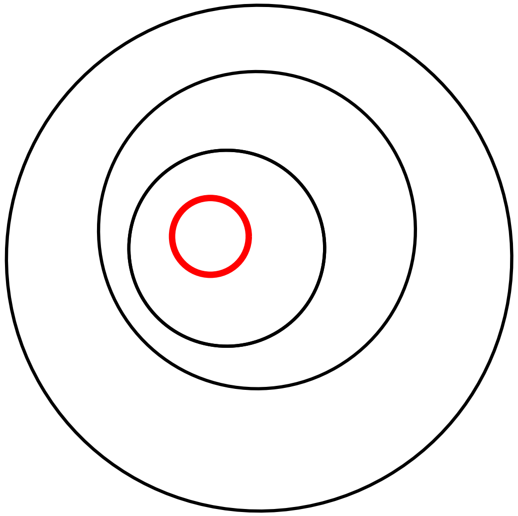

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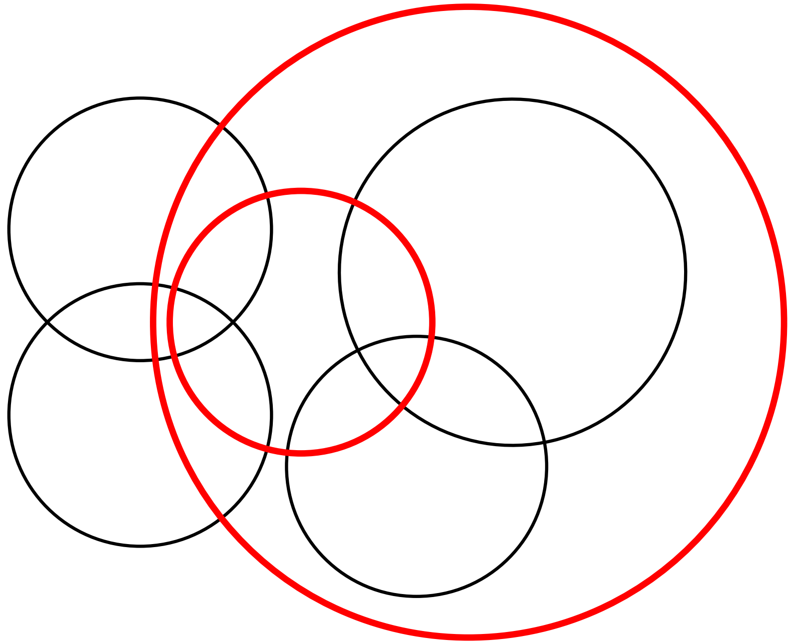

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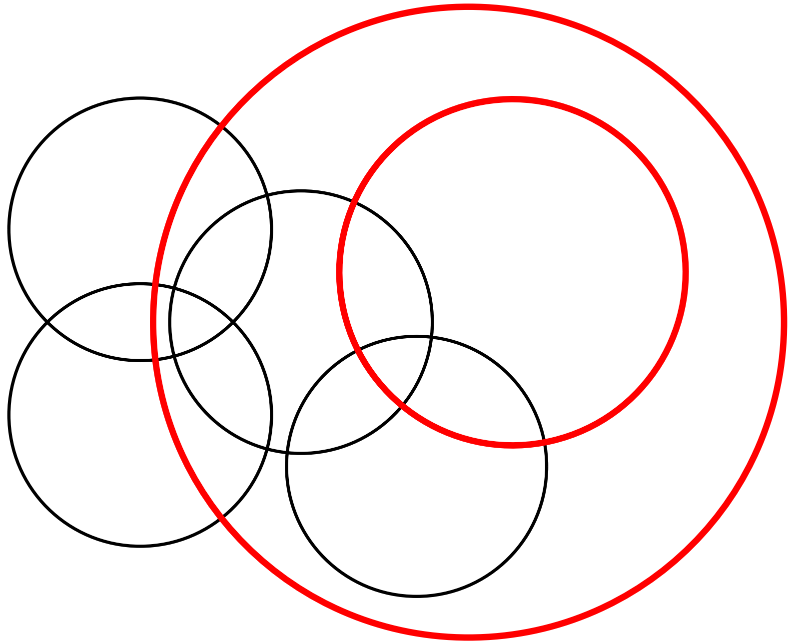

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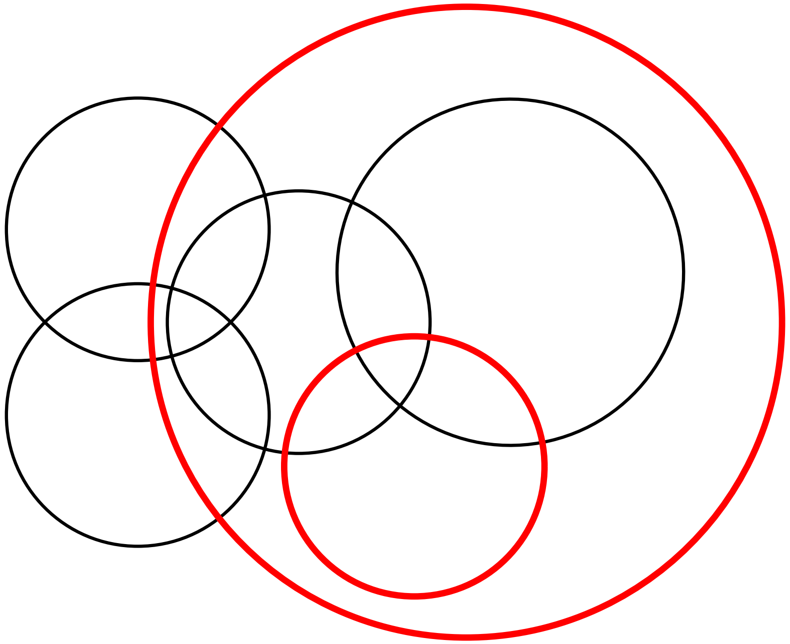

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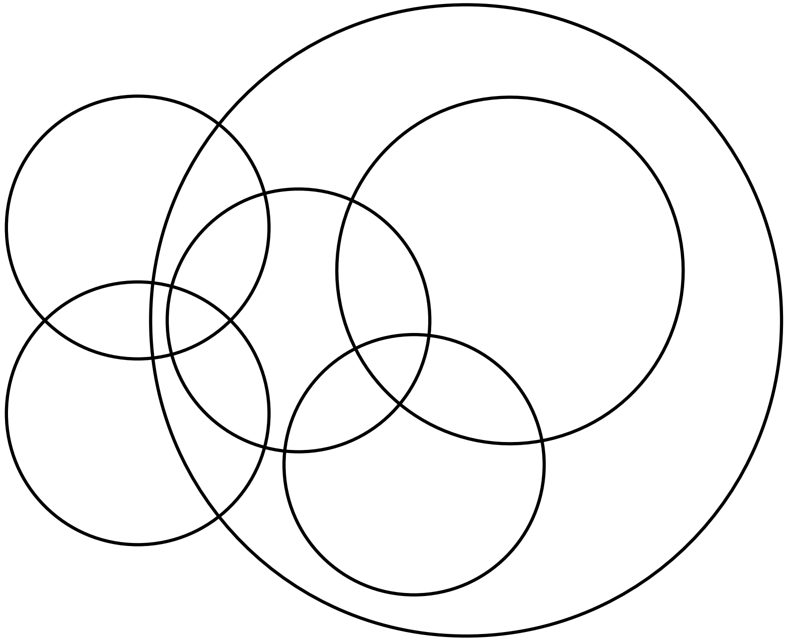

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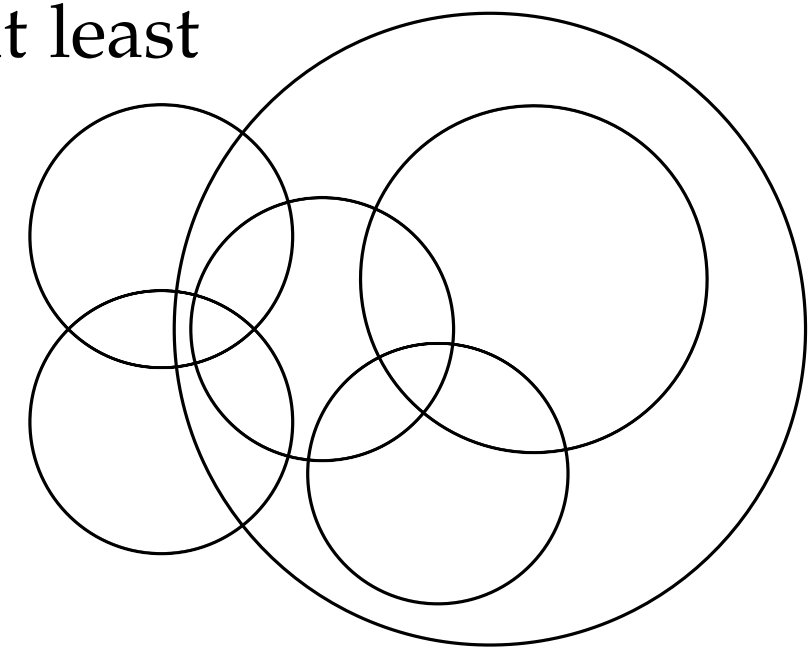
Lem. Among the deepest circles a smallest one has at most 8 neighbours.

Main Lemma

Lem. $\star$ Let S be the set of neighbours of α s.t. S does not contain nested circles and each circle in S has radius at least as large as α, then $|S| \leq 6$.



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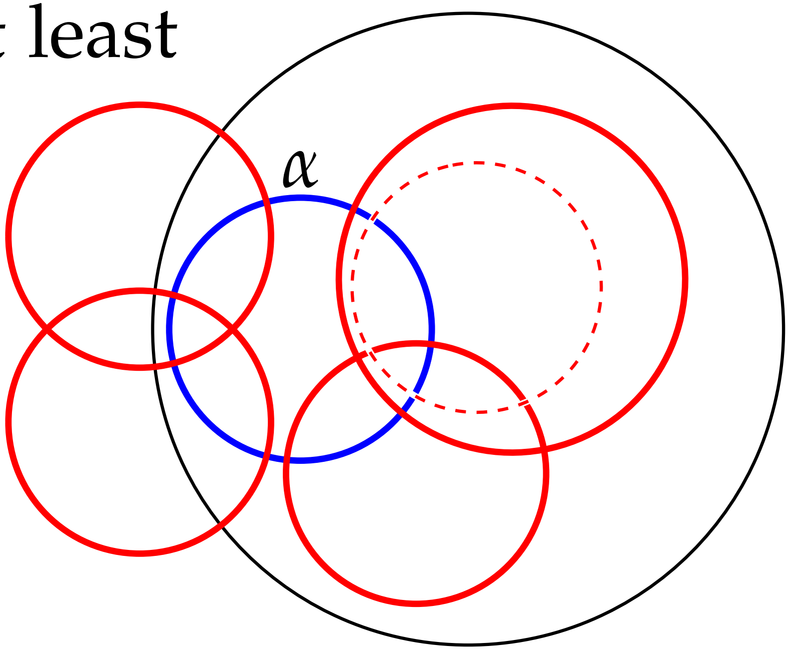

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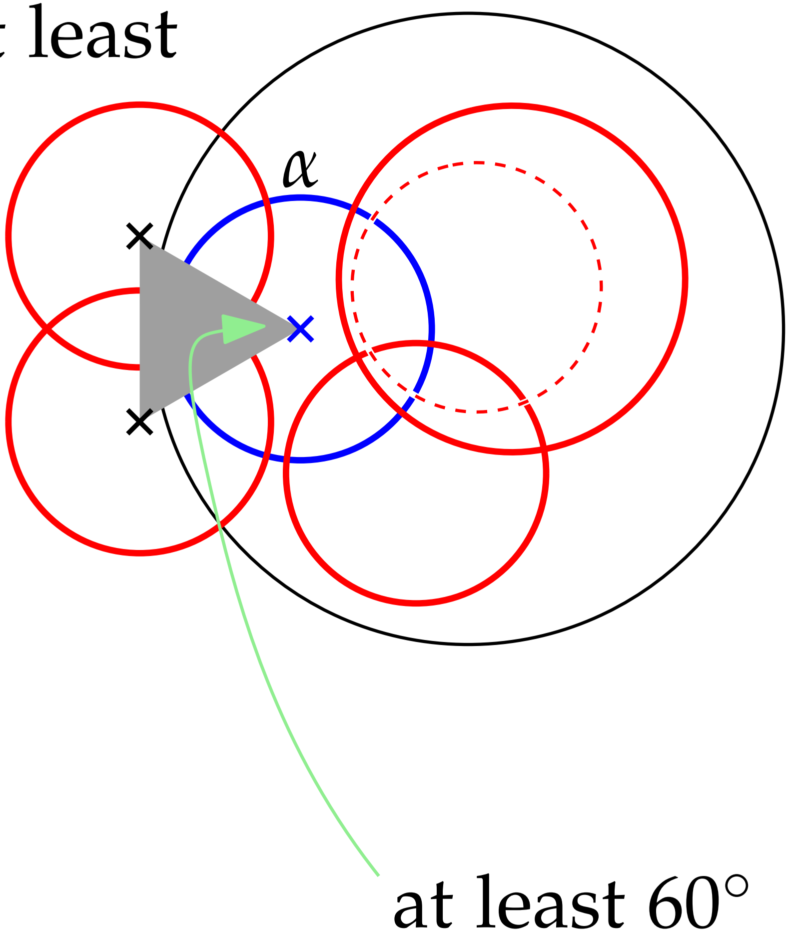

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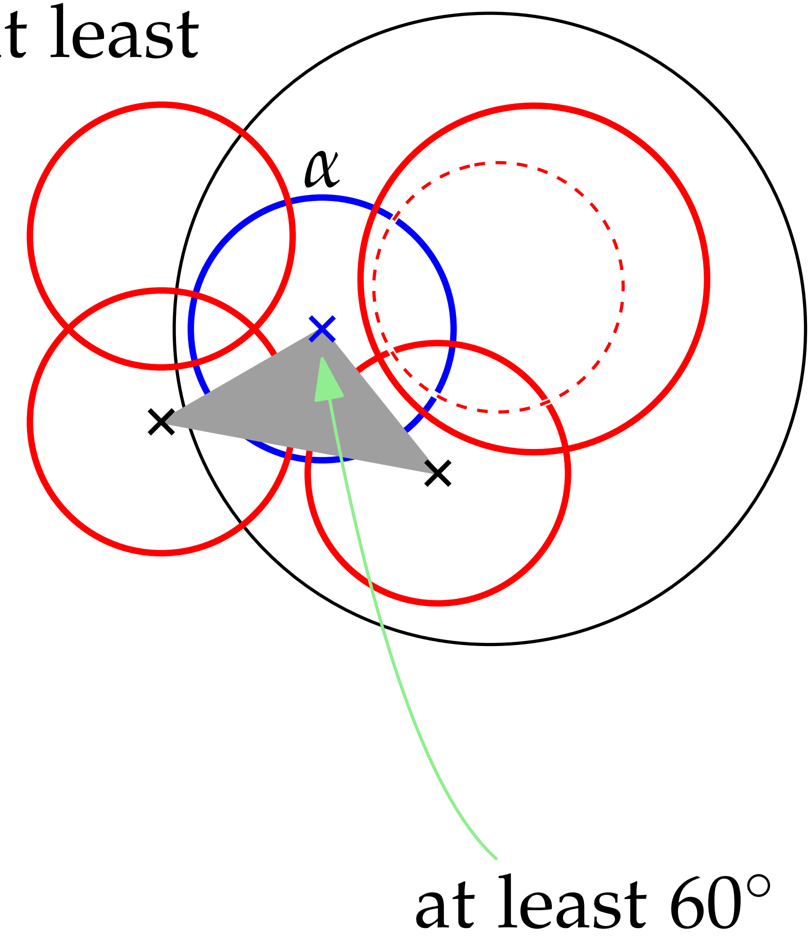

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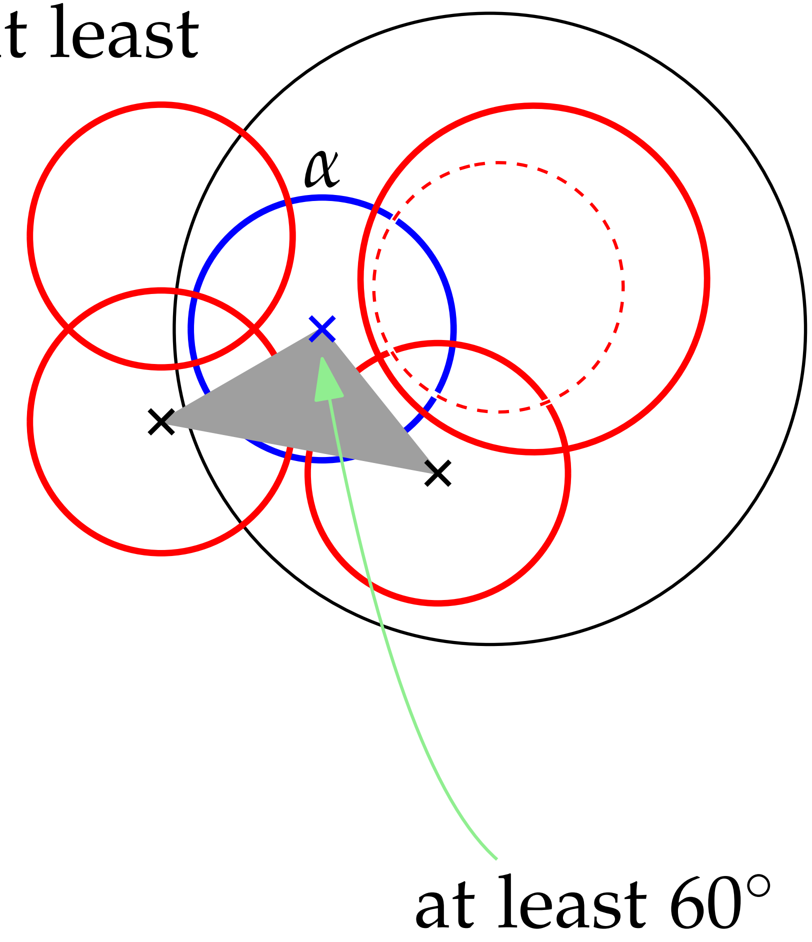
Proof: α can have at most $\frac{360\circ}{60^\circ} = 6$ neighbours. □



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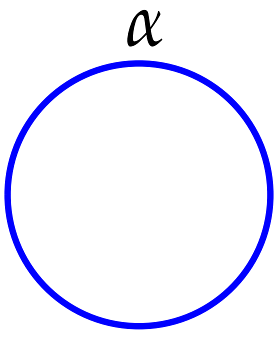

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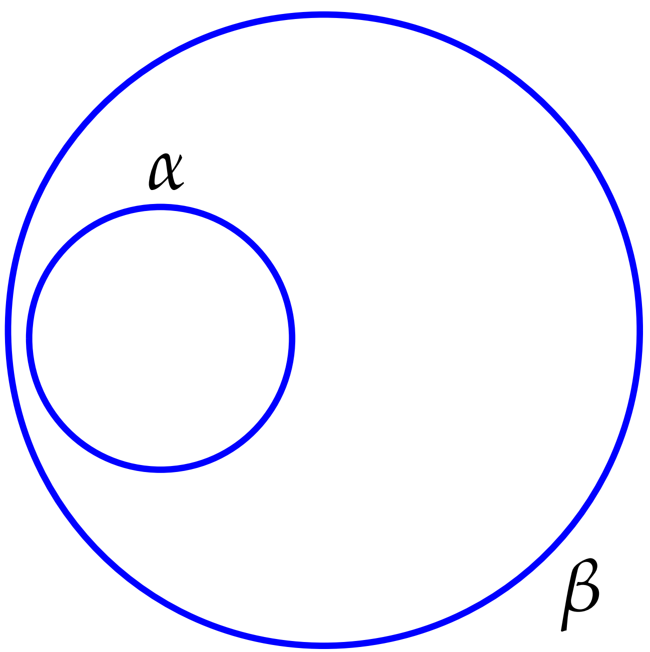

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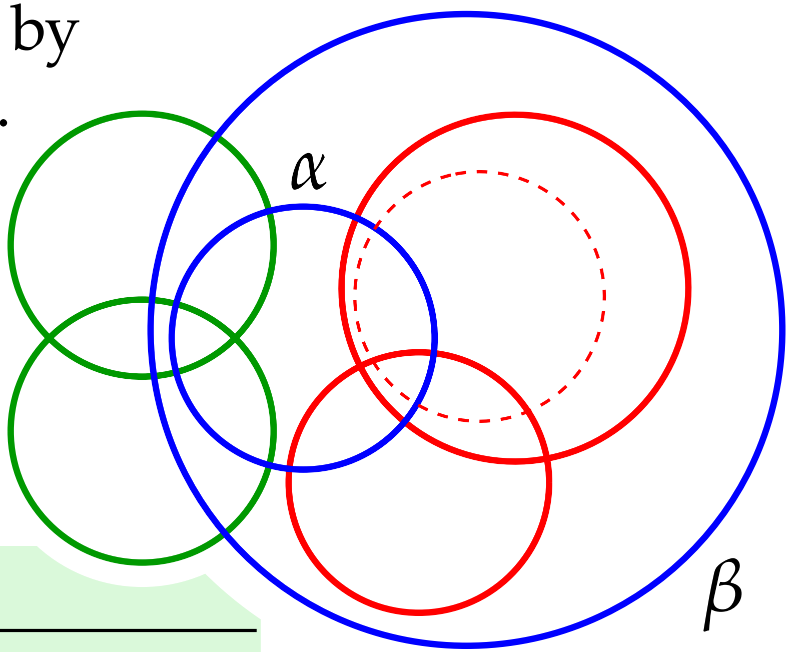

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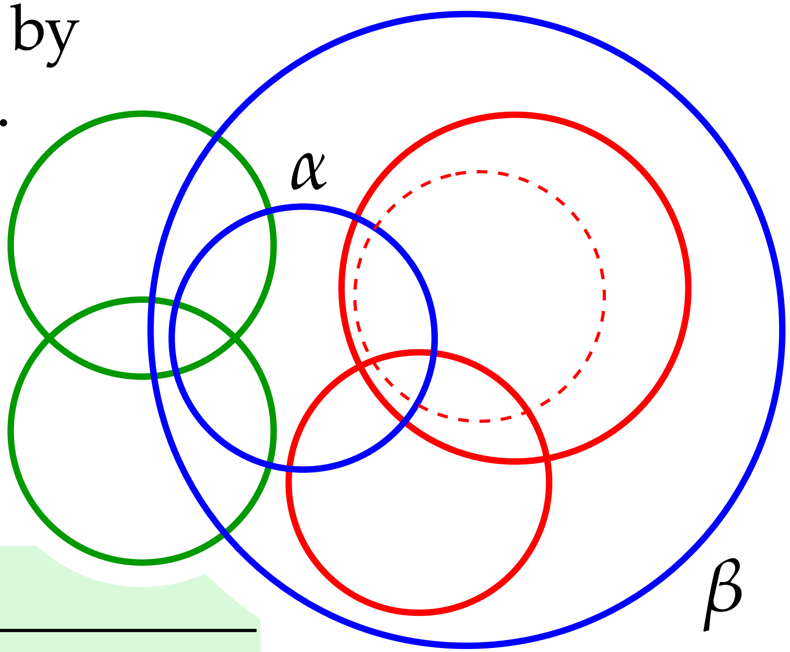

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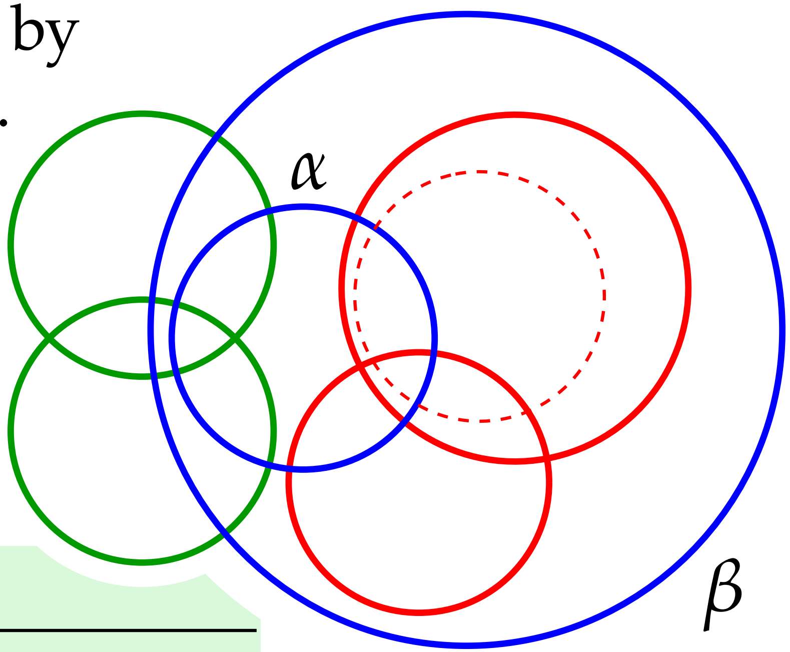

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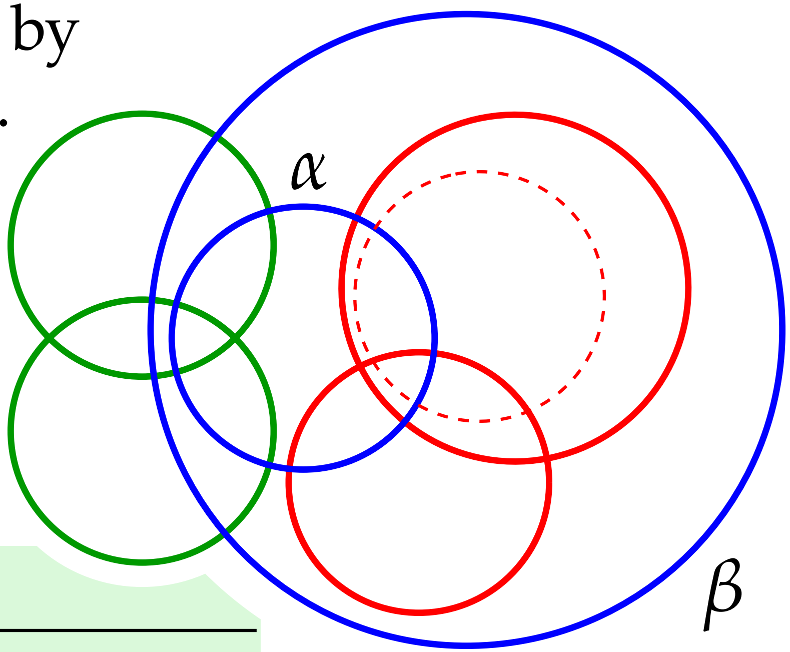

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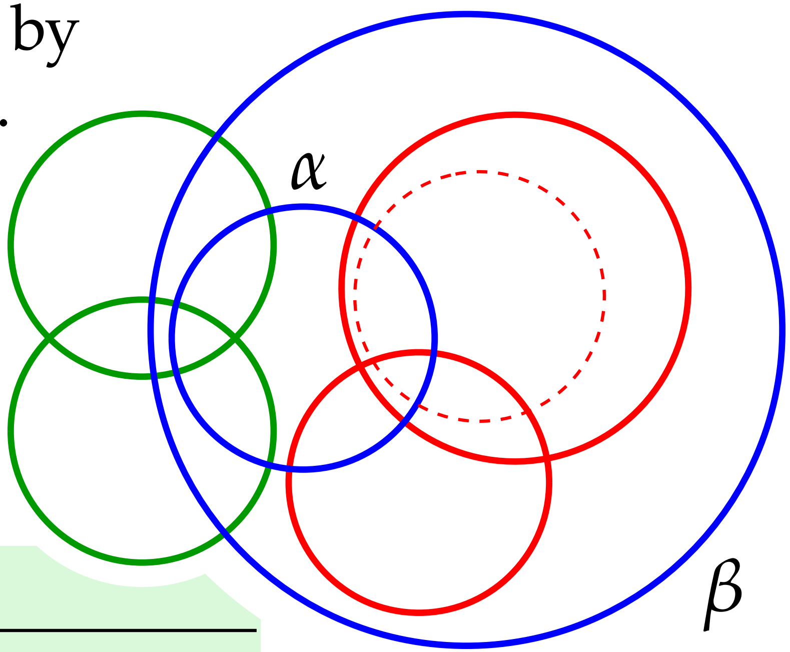

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Recall:

Lem. Two disjoint circles can be orthogonal to at most two other circles.

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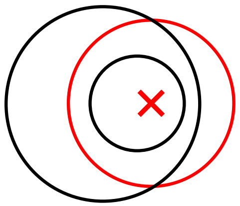

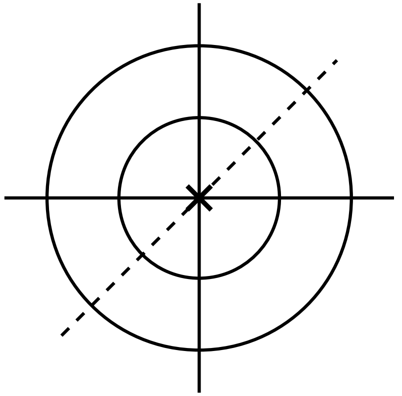

□

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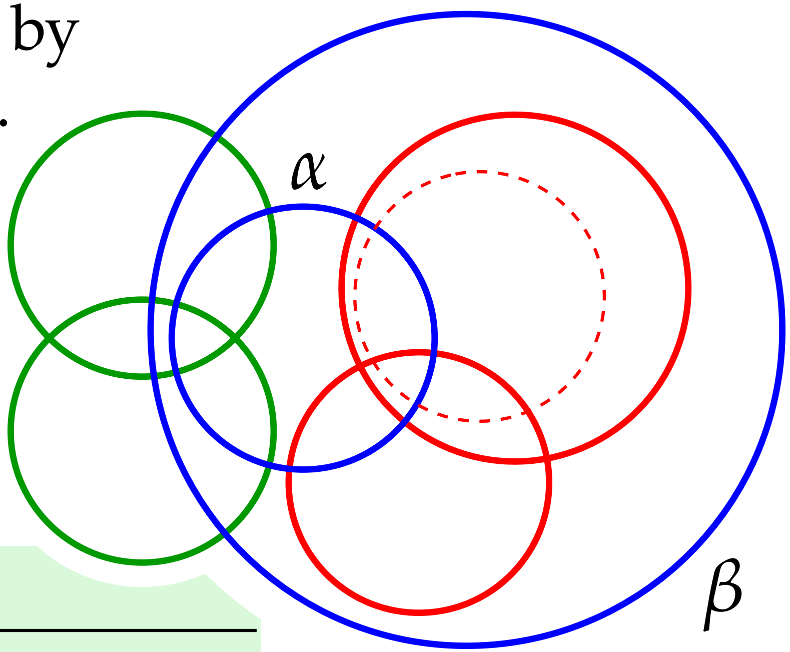

do not intersect β	do intersect β
• are not nested • are larger than α $\Downarrow$ by Lem.$\star$ • at most 6	• are orthogonal to two disjoint circles, that is, α and β • at most 2

Main Result

Thm. Every arrangement of n orthogonal circles has at most $16n$ intersection points and $17n + 2$ faces.

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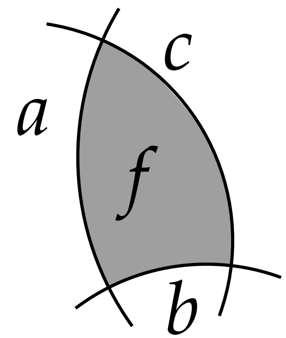
Proof: The number of intersection points follows by inductively applying the main lemma.

The bound on the number of faces follows then by Euler's formula.

Estimating the Number of Small Faces

Consider a face f with sides formed by circular arcs a, b, c.



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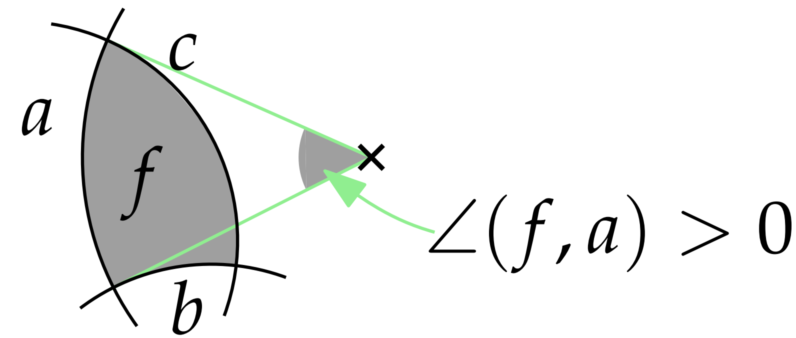

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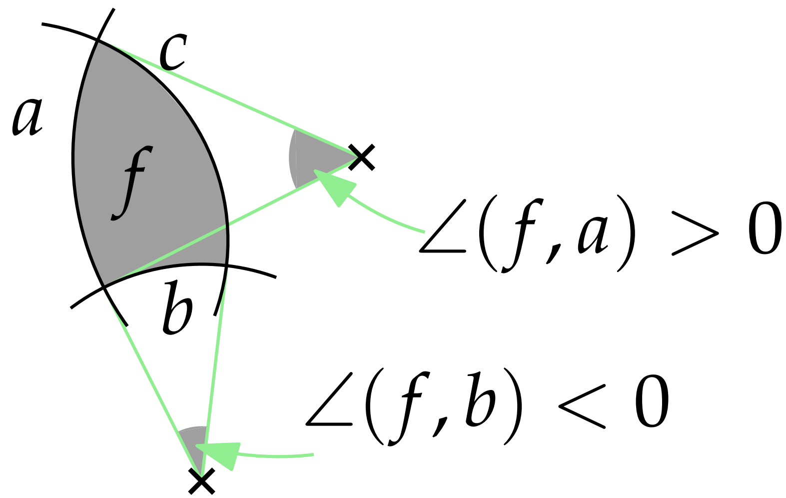

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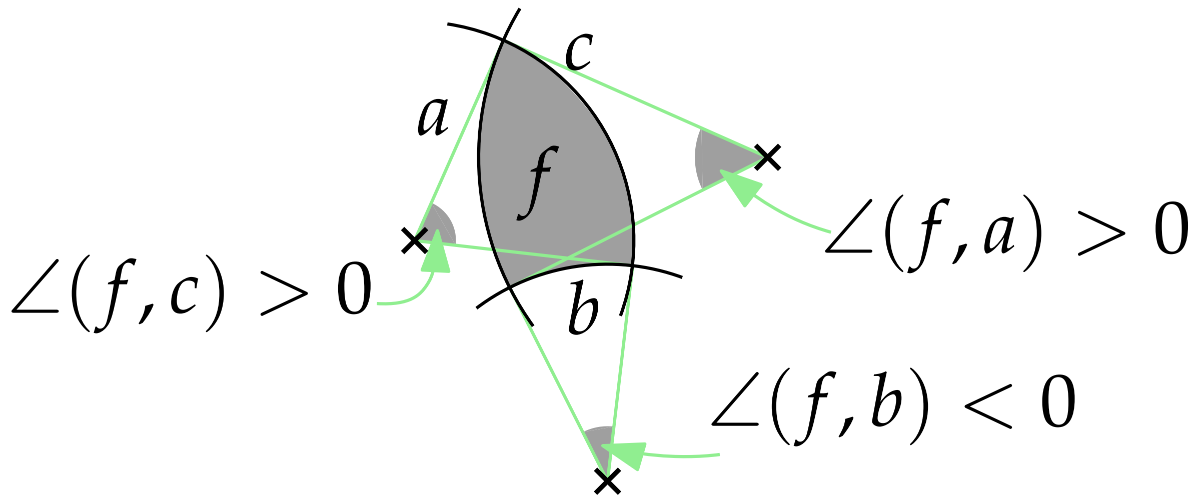

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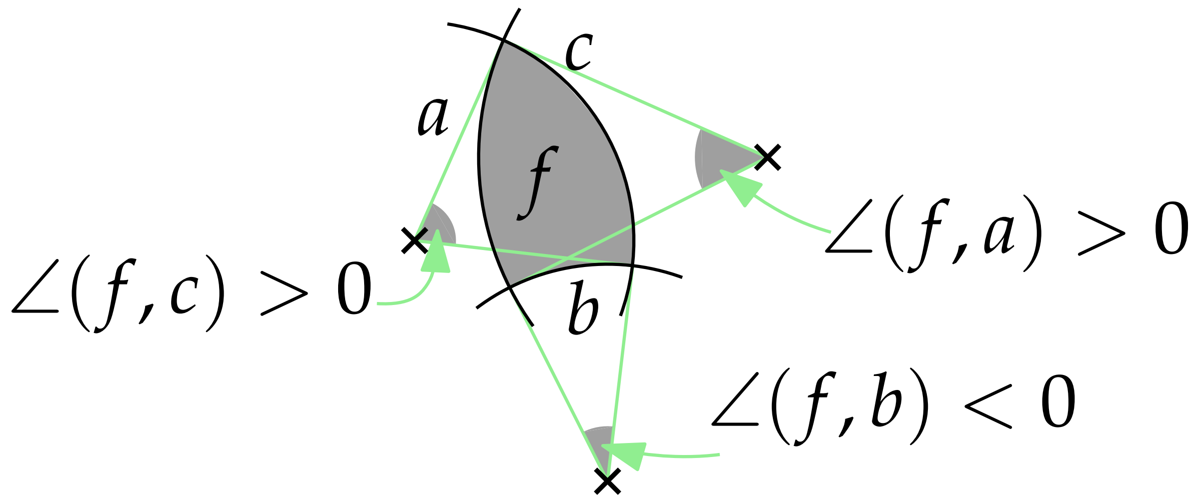

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For every face f in an arrangement of orthogonal circles its total angle is $2\pi - \frac{|f|\pi}{2}$.

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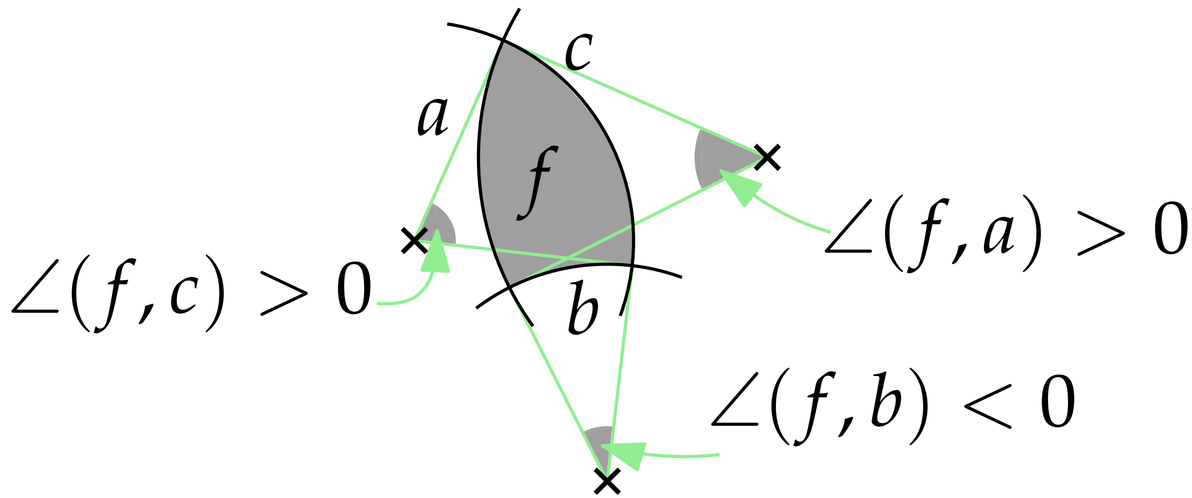

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Counting Small Faces

Thm. For every arrangement $\mathcal{A}$ of n orthogonal circles $2p_2(\mathcal{A}) + p_3(\mathcal{A}) \leq 4n$.

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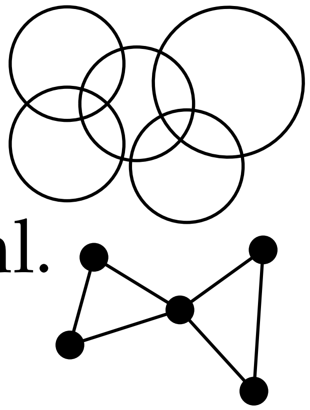
$$p_2(\mathcal{A})\pi + p_3(\mathcal{A})\frac{\pi}{2} \leq 2n\pi.$$

□

Intersection Graphs

[Hliněný and Kratochvíl, 2001]



Orthogonal circle intersection graph:
– a vertex for each circle
– an edge between two circles if they are orthogonal.

Intersection Graphs

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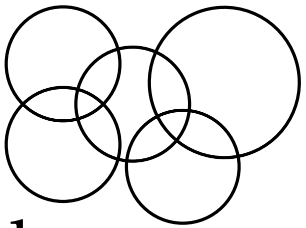

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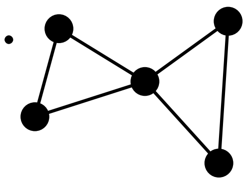

Properties:

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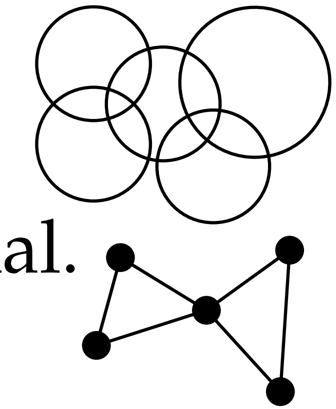

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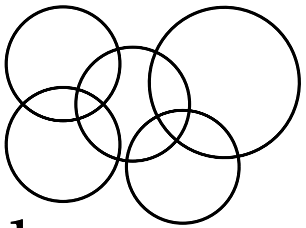

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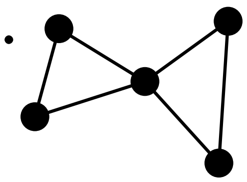

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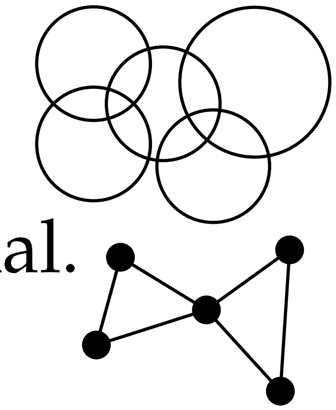

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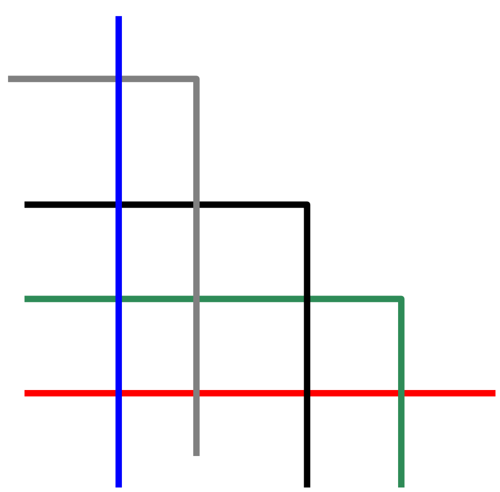

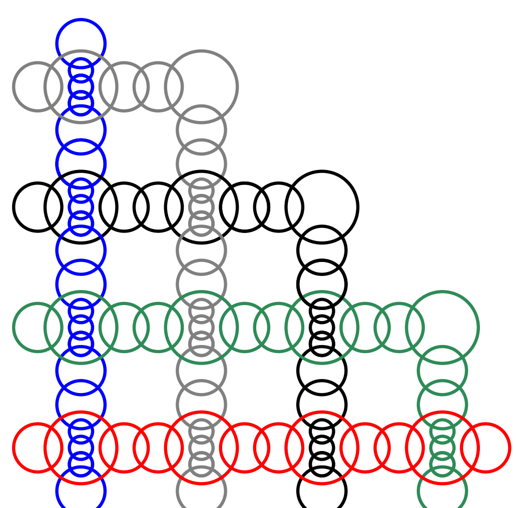

Intersection Graphs of Unit Circles

Orthogonal unit circle intersection graph:
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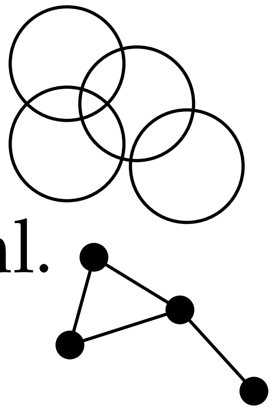

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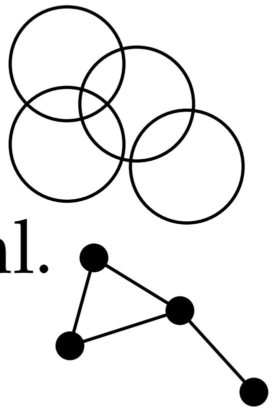
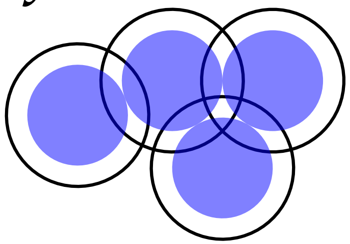


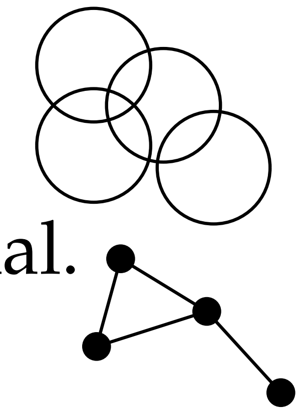
Properties:

- are a subclass of the *contact graphs of unit circles* also known as *penny* graphs.



Intersection Graphs of Unit Circles

Orthogonal unit circle intersection graph:
– a vertex for each unit circle
– an edge between two circles if they are orthogonal.



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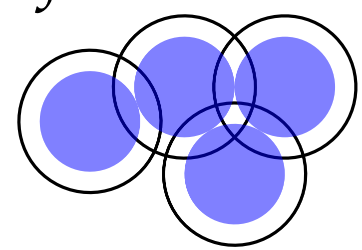
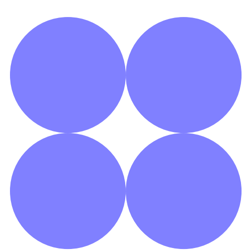
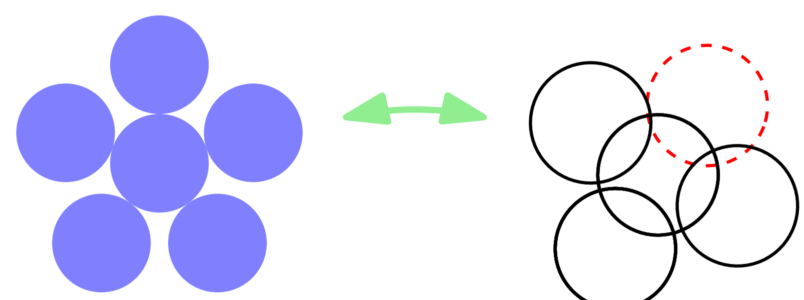


- ∃ penny graphs which are not orthogonal unit circle intersection graphs.



an induced C_4



a k-star, $k \geq 4$

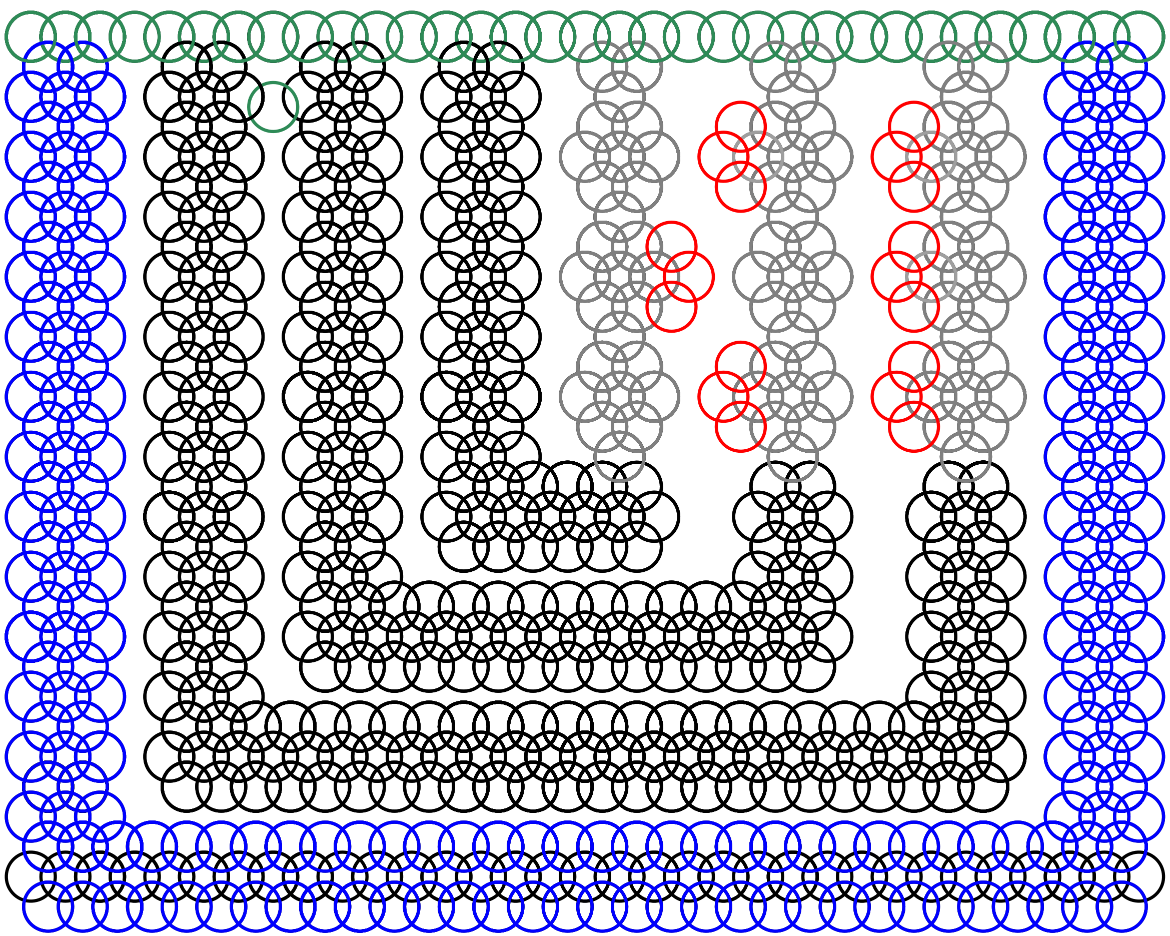
Intersection Graphs of Unit Circles

Thm. Recognizing orthogonal unit circle intersection graphs is NP-hard.

Intersection Graphs of Unit Circles

Thm. Recognizing orthogonal unit circle intersection graphs is NP-hard. [Di Battista et al., GD'99]

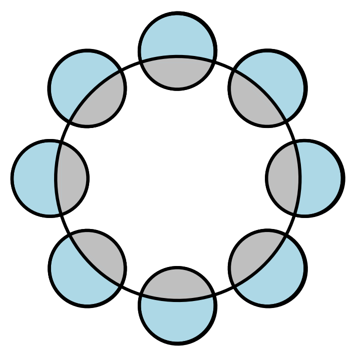
Proof (idea): Logic engine which emulates Not-All-Equal-3-Sat (NAE3SAT) problem.



$$x \wedge \neg y \wedge \neg z$$
$$\neg x \wedge y \wedge \neg z$$
$$x \wedge \neg y \wedge \neg z$$

Open Problems

Bounds on the # of faces that we have so far:

	digonal faces	triangular faces	all faces
upper bound	$2n$ [Gauss–Bonnet]	$4n$ [Gauss–Bonnet]	$17n+2$ Main Thm.
lower bound	$2n-2$ 		

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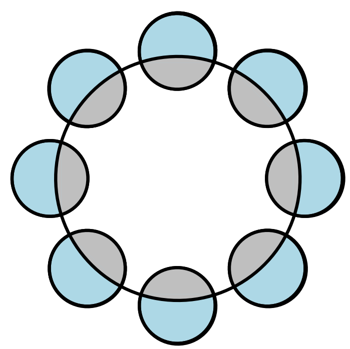
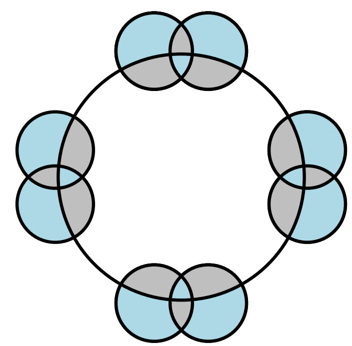
lower bound	$2n-2$ 	$3n-3$ 	

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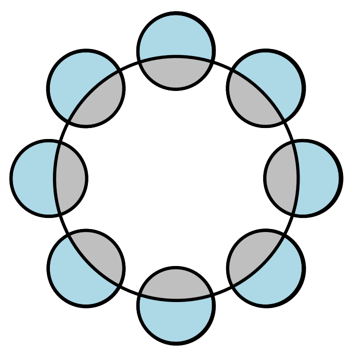
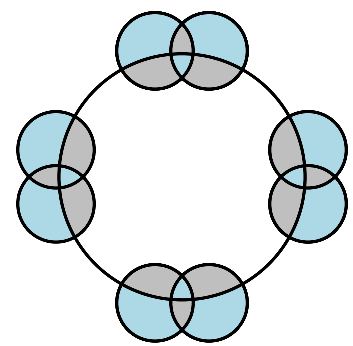
lower bound	$2n-2$ 	$3n-3$ 	?

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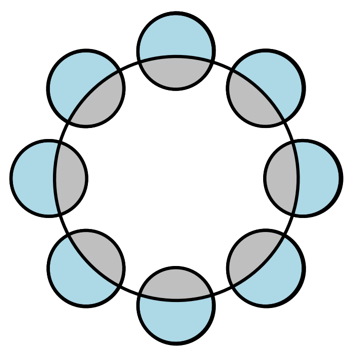
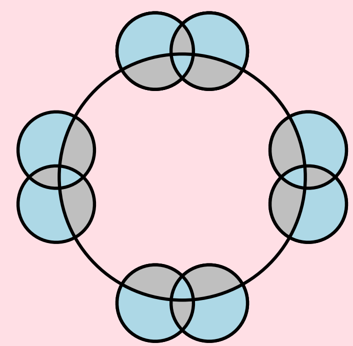
lower bound	$2n-2$ 	$3n-3$ 	?

Open Problems

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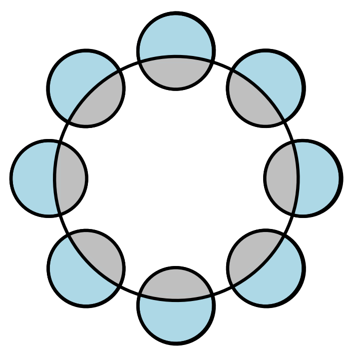
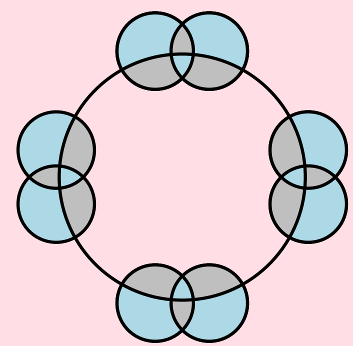
lower bound	$2n - 2$ 	$3n - 3$ 	?

What is the complexity of recognizing general orthogonal circle intersection graphs?