Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Chair for
INFORMATICS I
Efficient Algorithms and
Knowledge-Based Systems

Institute for Informatics

# Bundled Crossings Revisited

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Nat. Acad. Sciences of Ukraine, Lviv, Ukraine

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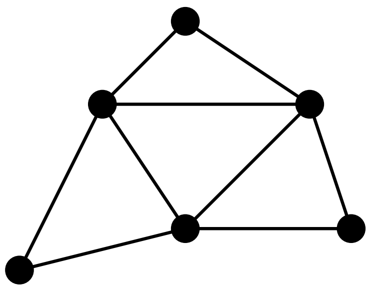

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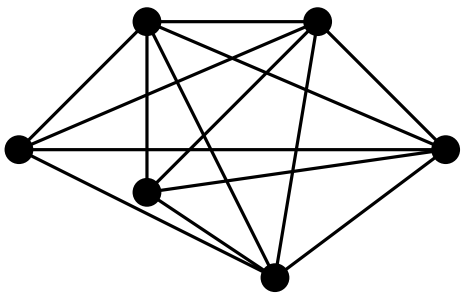

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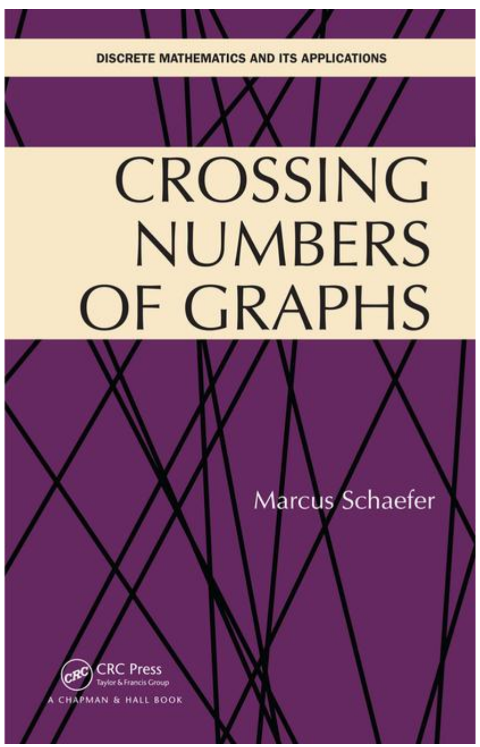


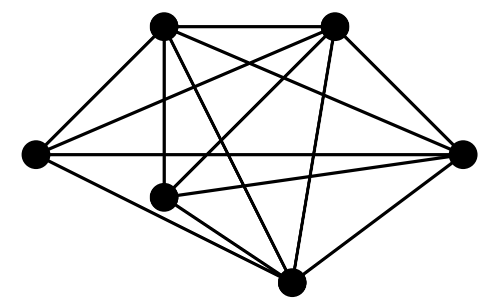

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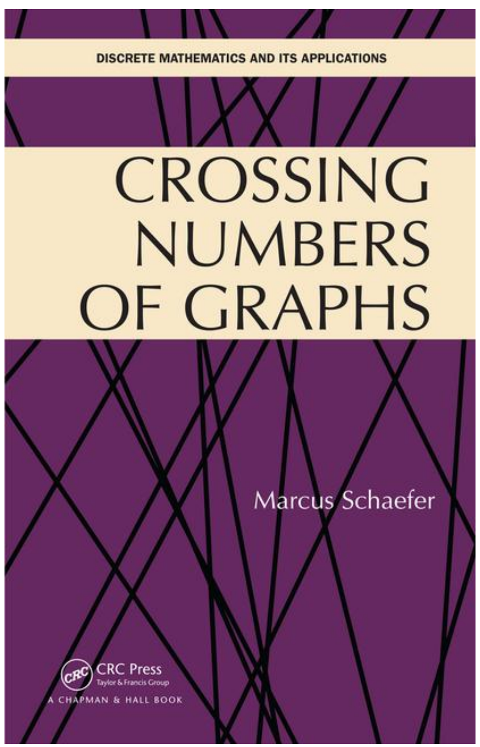


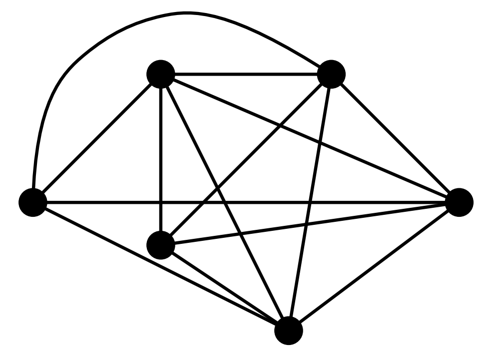

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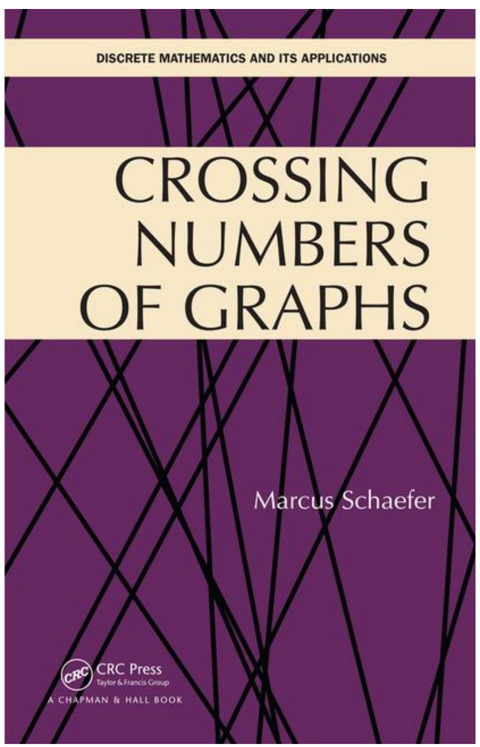


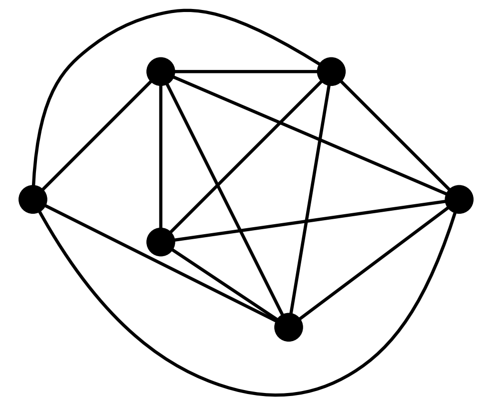

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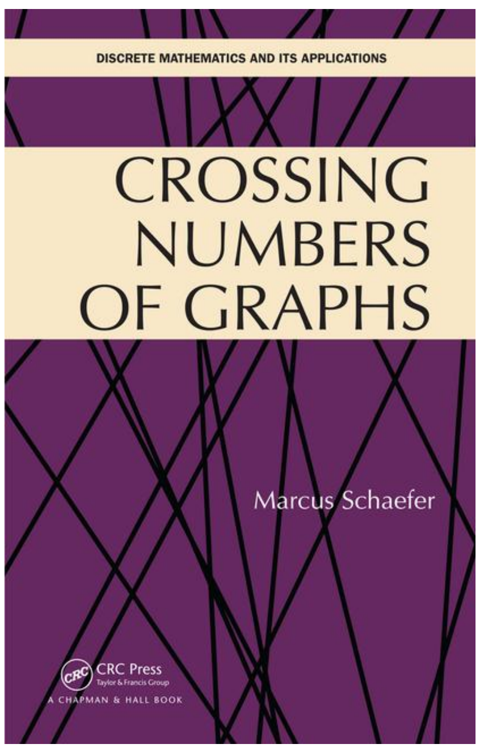


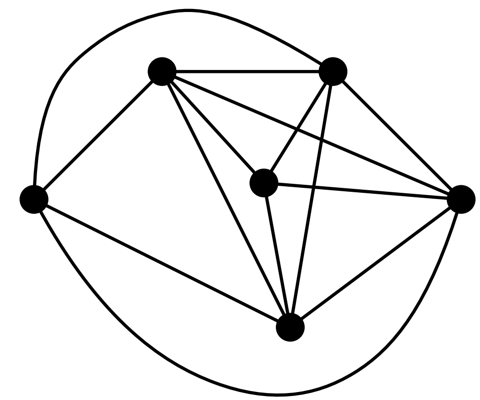

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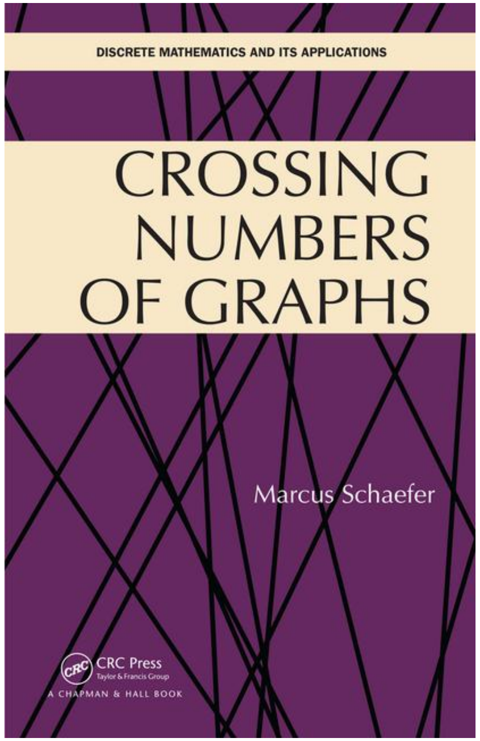


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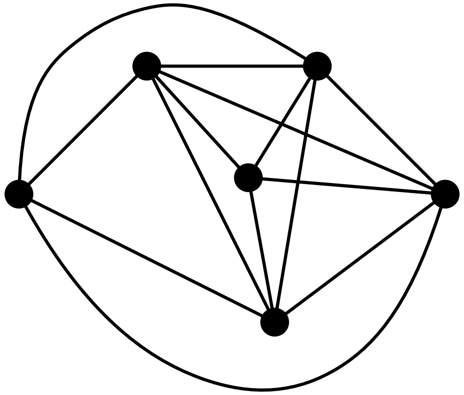

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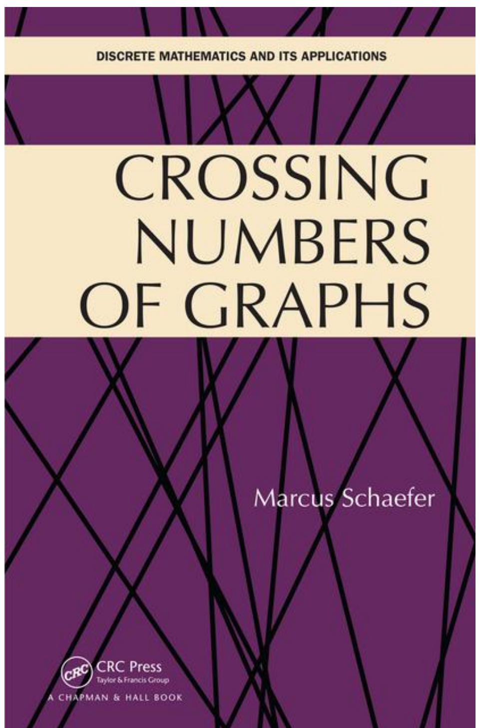
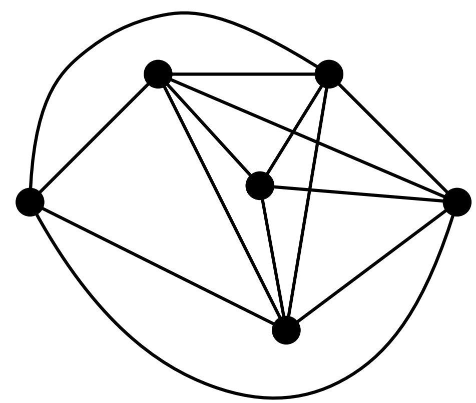
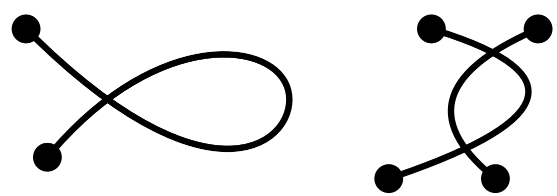


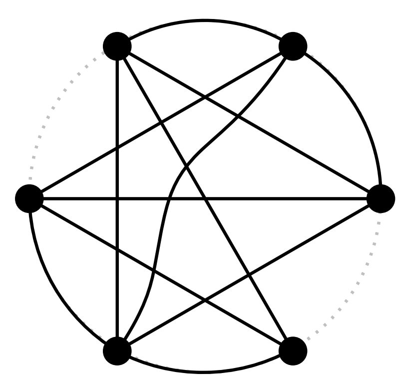



Lots of different variants.

Our main result concerns simple circular layouts.

simple avoids:





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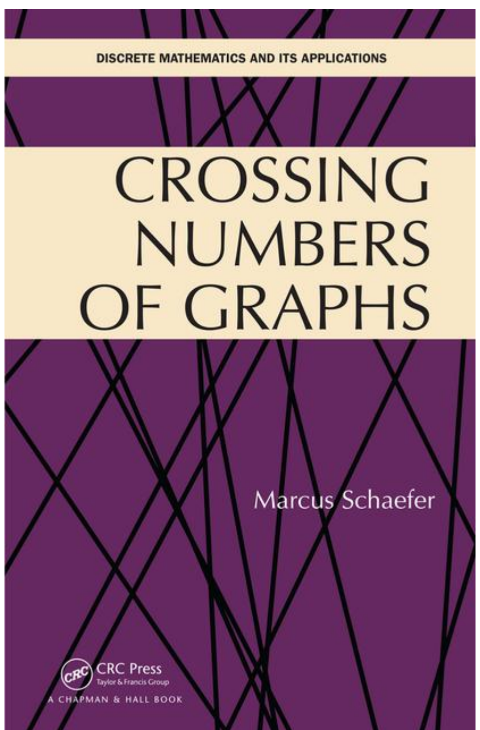


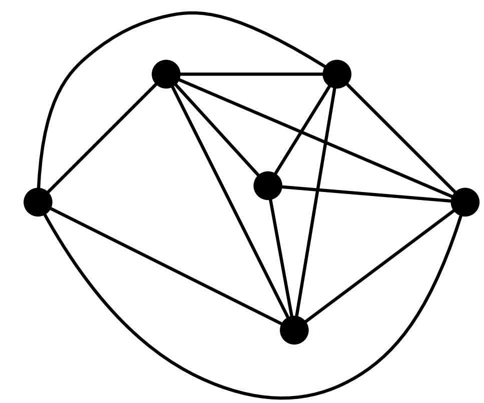

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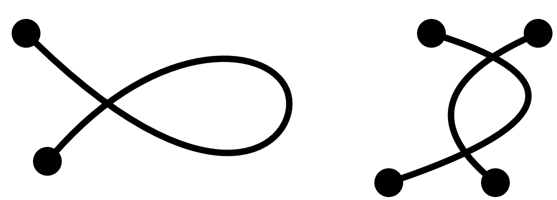
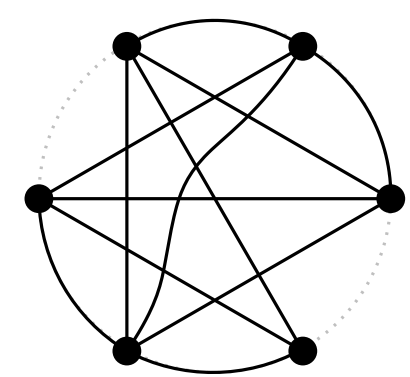
simple avoids: 



This talk concerns *bundled crossings*, def'd next.

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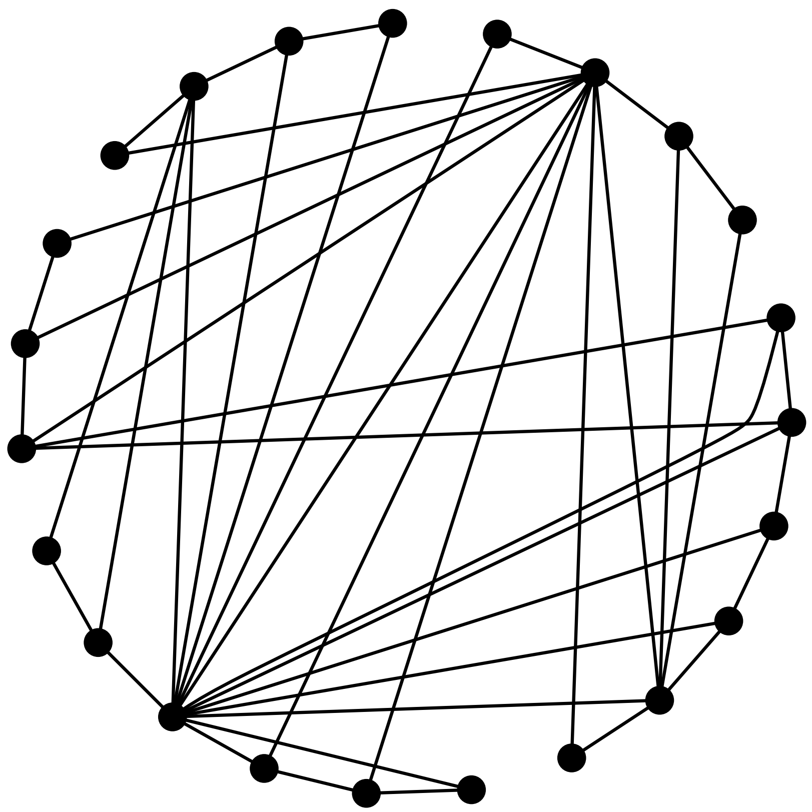

There is an FPT algorithm for deciding whether a graph admits a circular layout with $k$ crossings. [Bannister, Eppstein '14]

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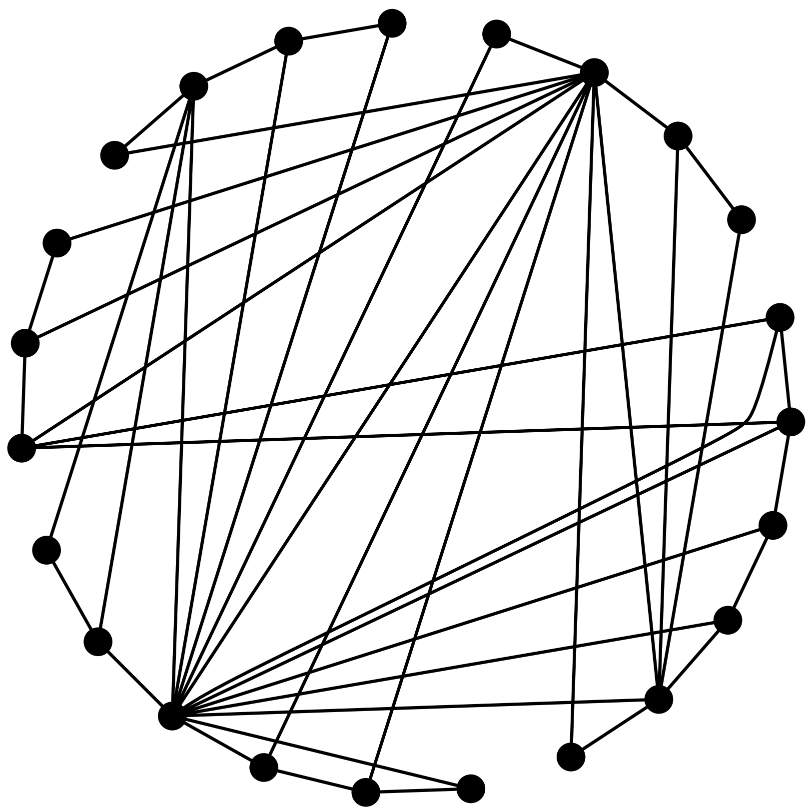
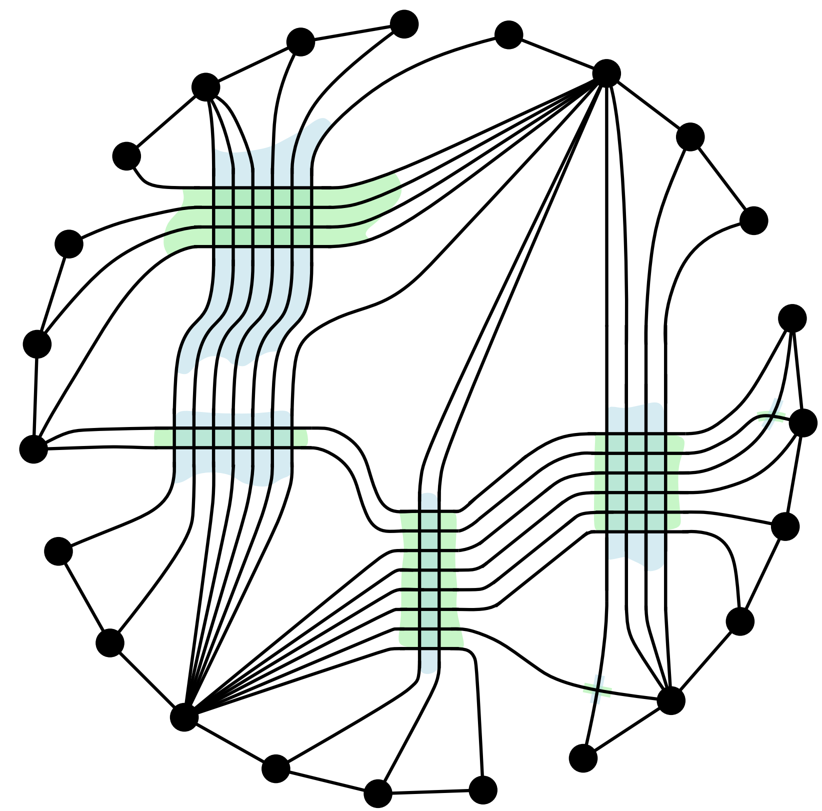


[Holten '06]

Bundle the drawing



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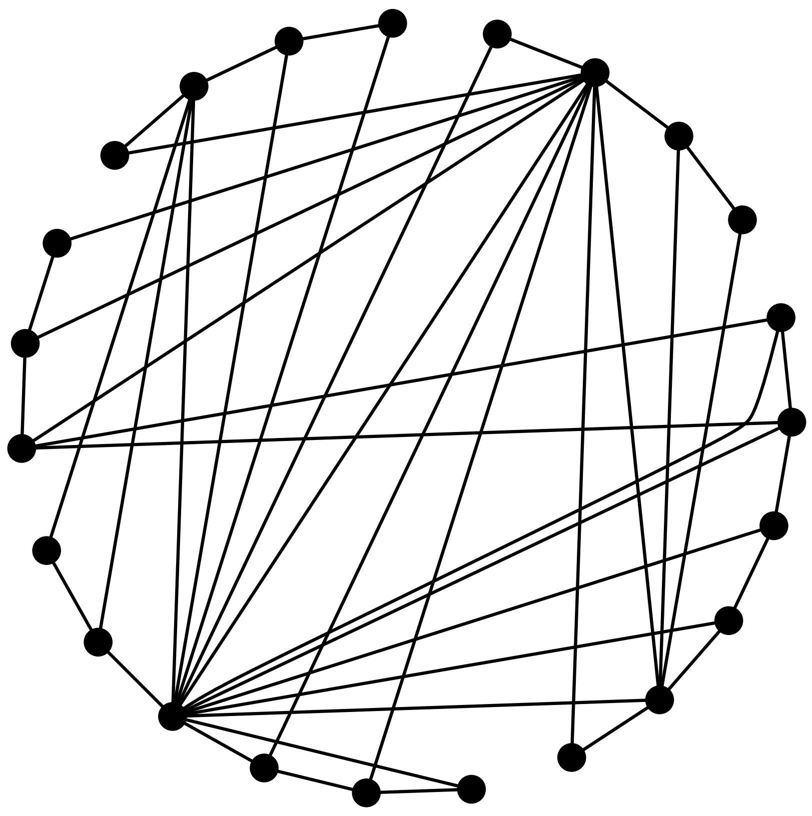

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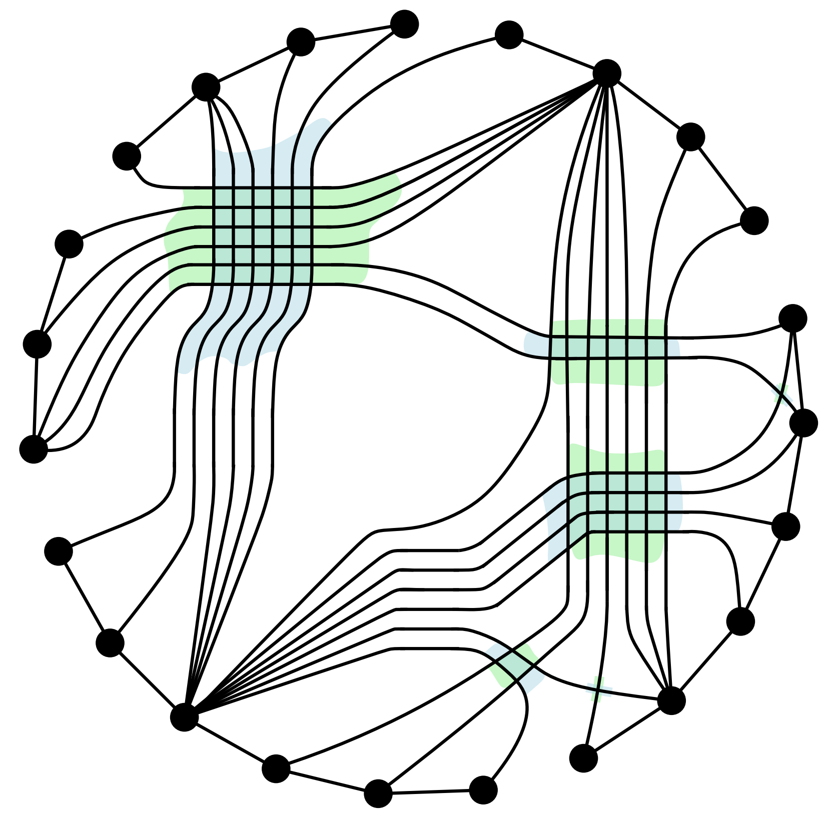

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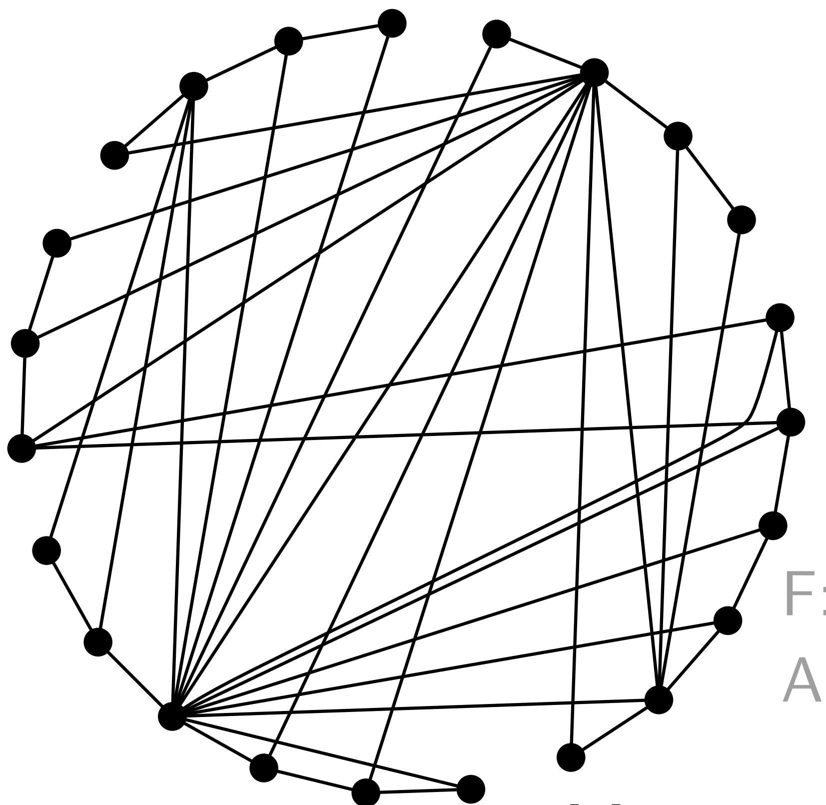

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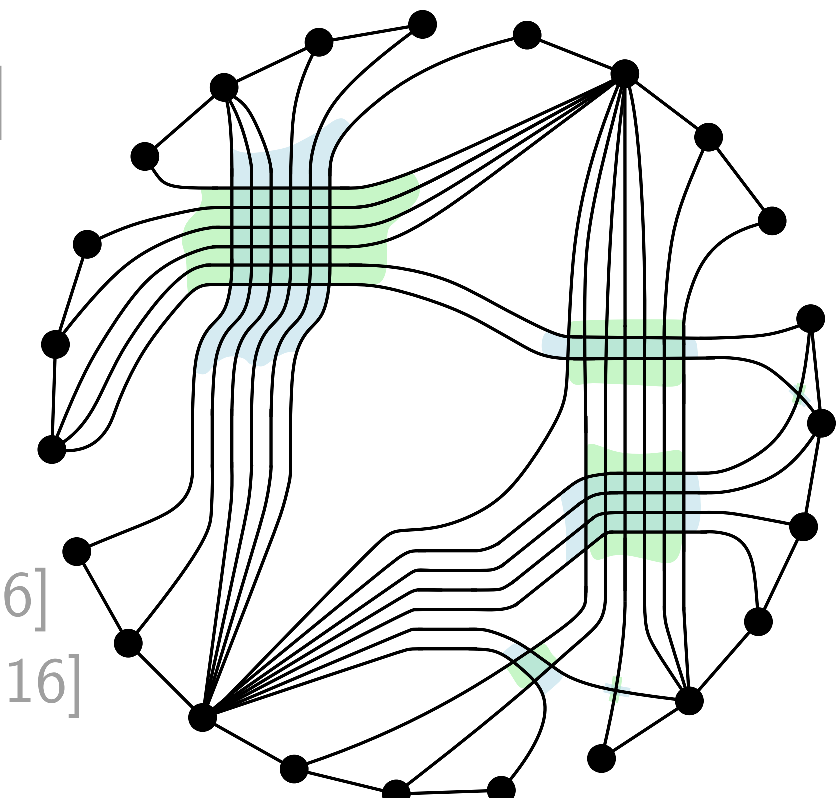
Bundle the drawing

F: [Fink et al. '16]

A: [Alam et al. '16]



Minimize crossings of bundles instead of edges!

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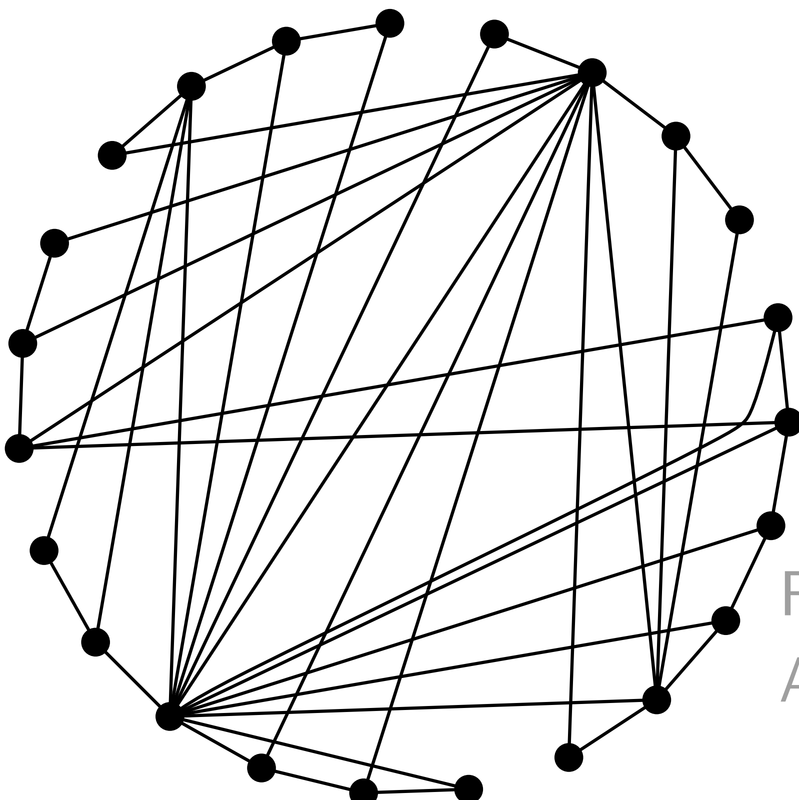

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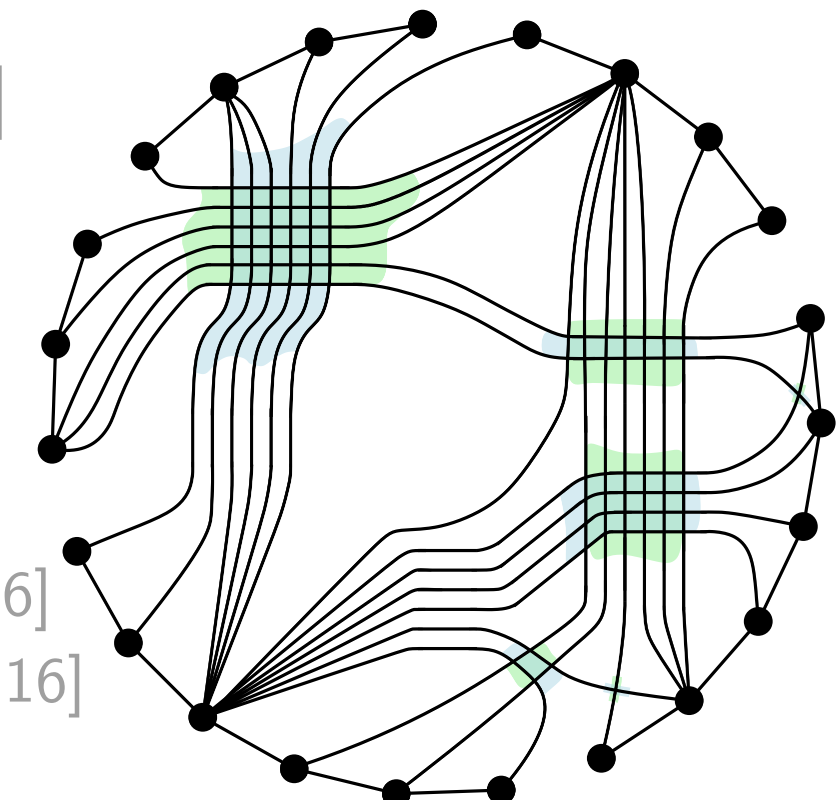

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Minimize crossings of bundles instead of edges!

- gen. layouts: NP-c for fixed [F] and variable [A] embeddings.
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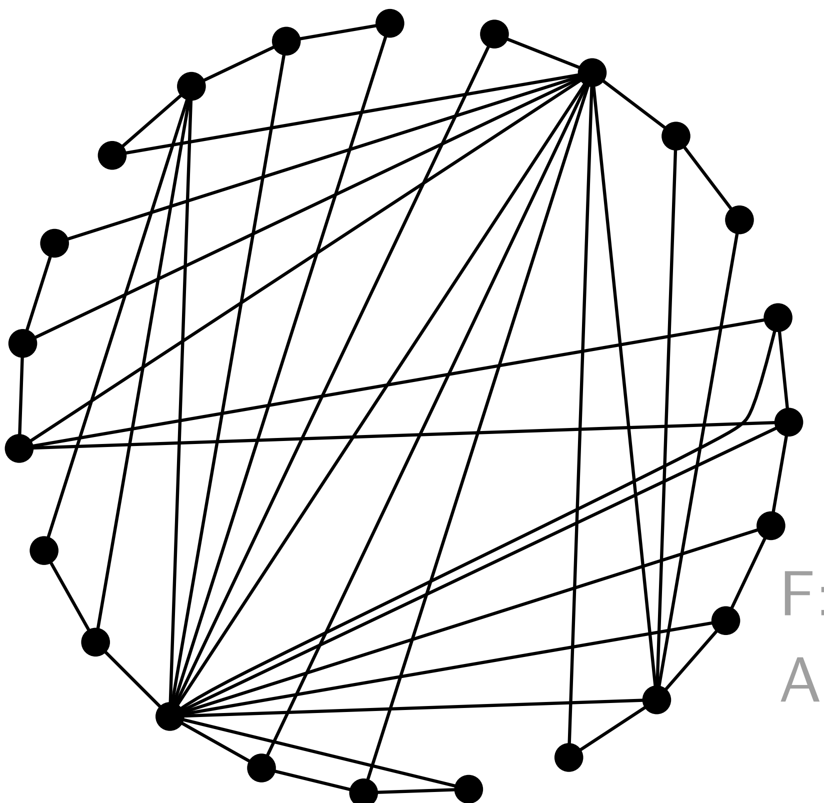

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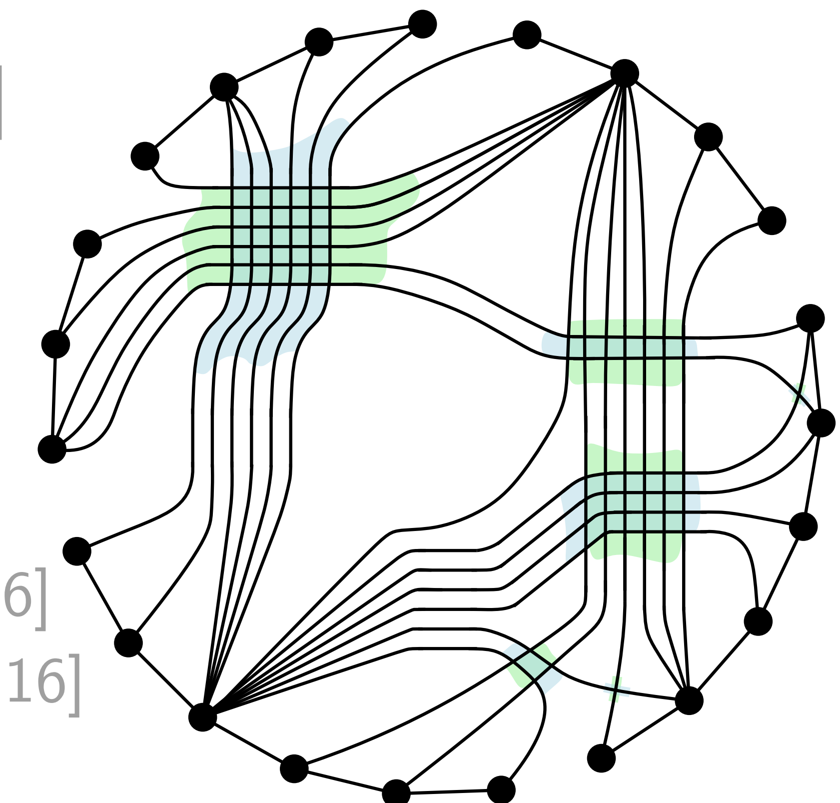

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Minimize crossings of bundles instead of edges!

gen. layouts: NP-c for fixed [F] and variable [A] embeddings.

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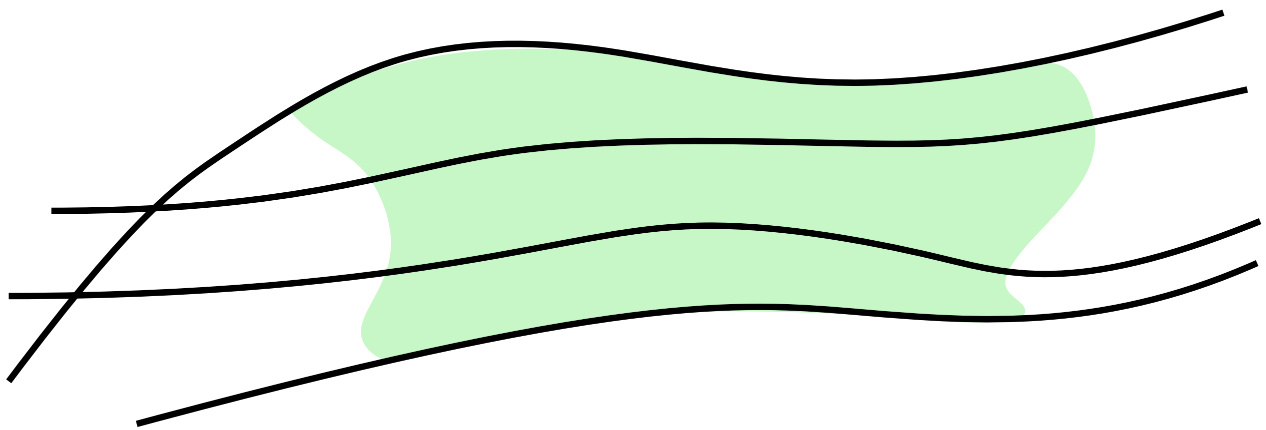
**Ques.** Is there an FPT algorithm for deciding whether a graph admits a circular layout with $k$ **bundled crossings**? [A]

# Bundled Crossing

A **bundle** is a set of pieces of edges that travel in parallel in the drawing.



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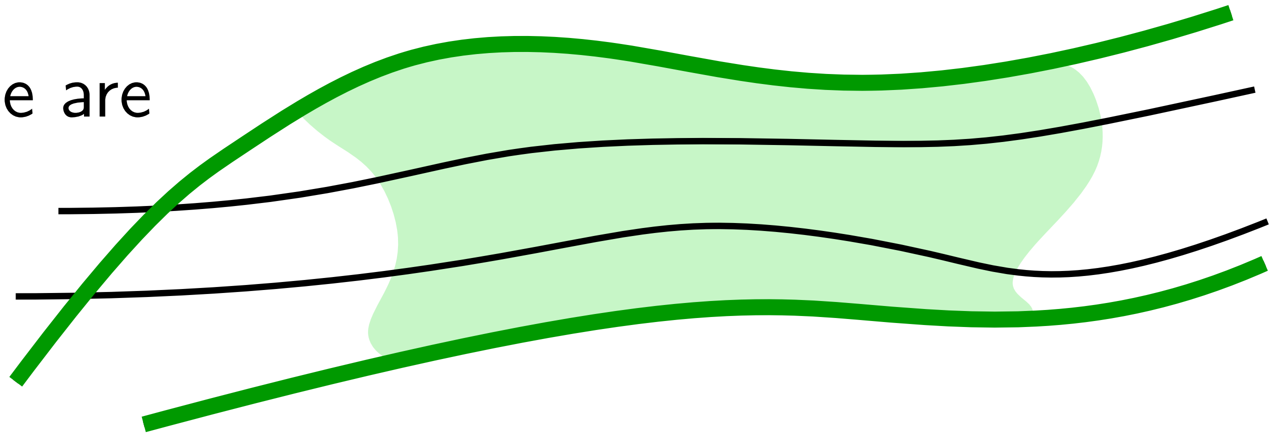

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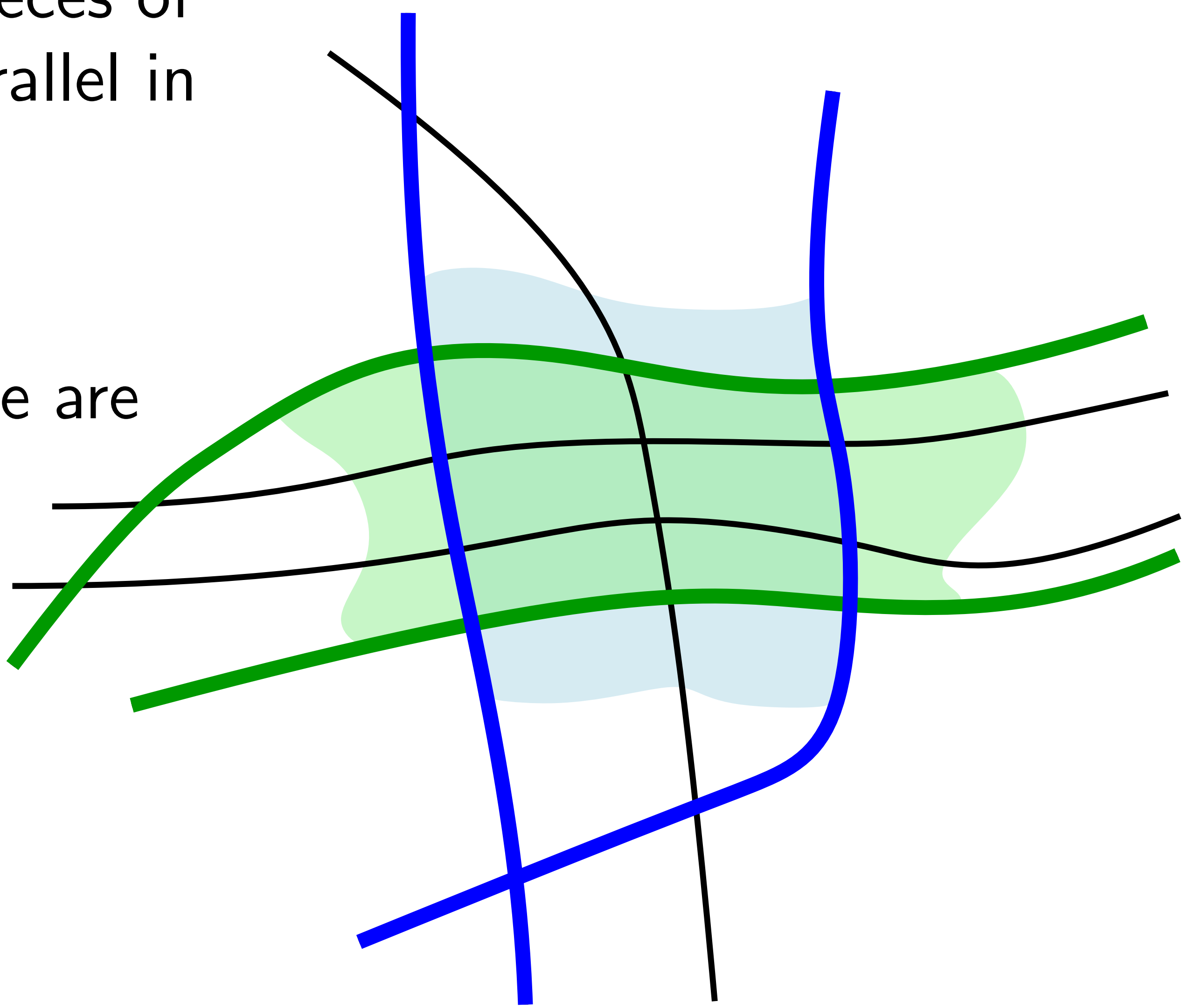

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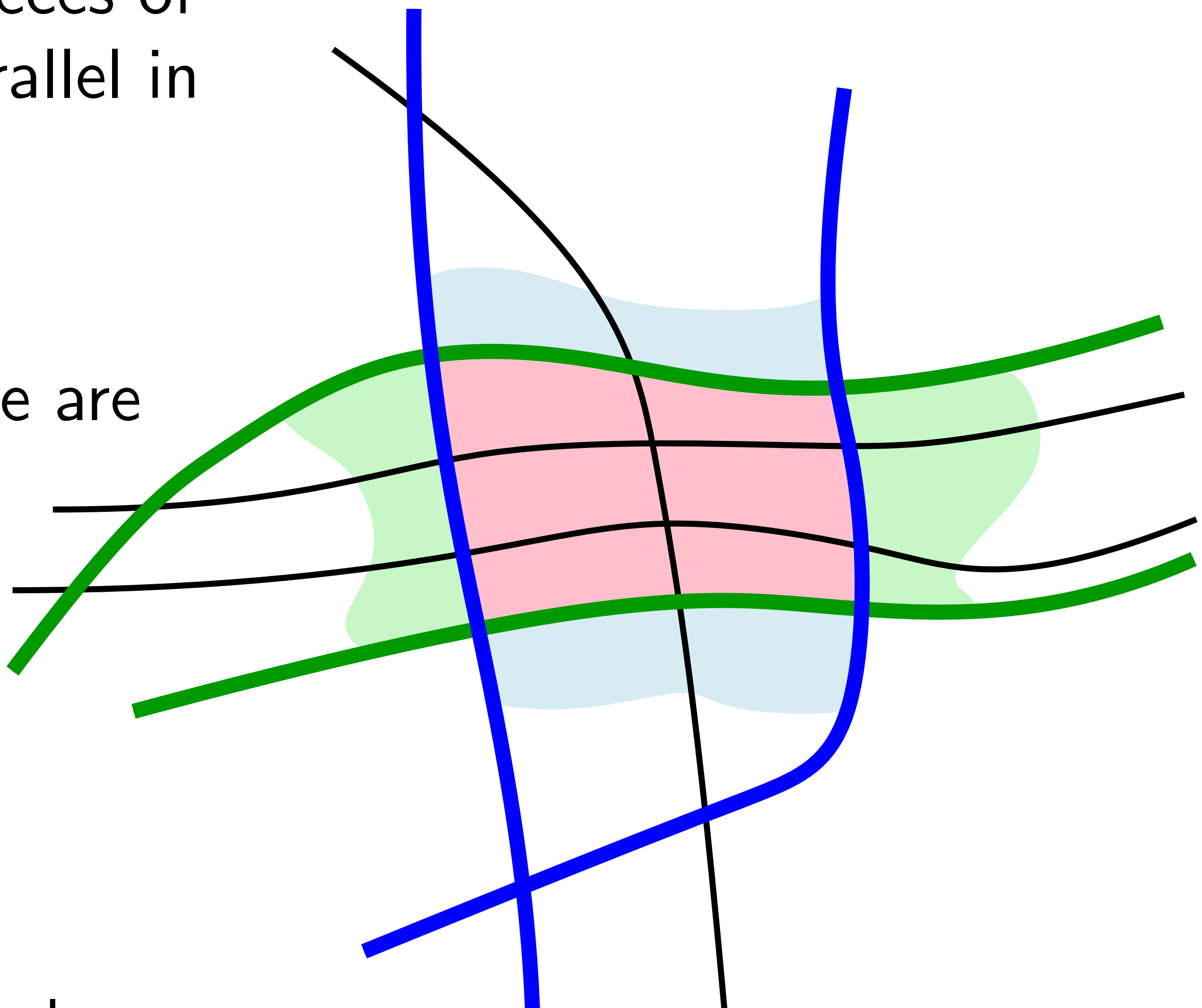
A **bundled crossing** is a set of crossings inside the region bounded by the frame edges.



# Bundled Crossing Minimization

**Def.** For a given graph $G$
the circular bundled crossing number $\mathrm{bc}^\circ(G)$ of $G$ is
the minimum number of bundled crossings
over all possible bundlings
of all possible simple circular layouts of $G$.

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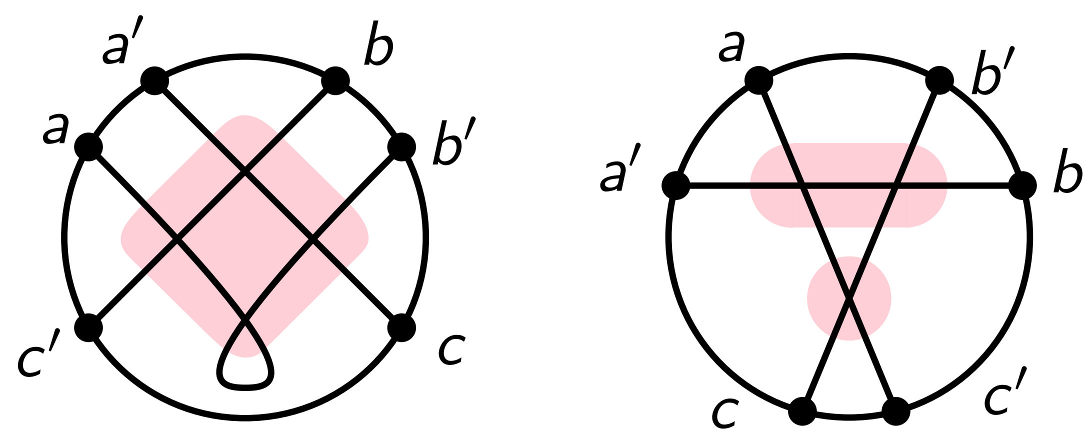

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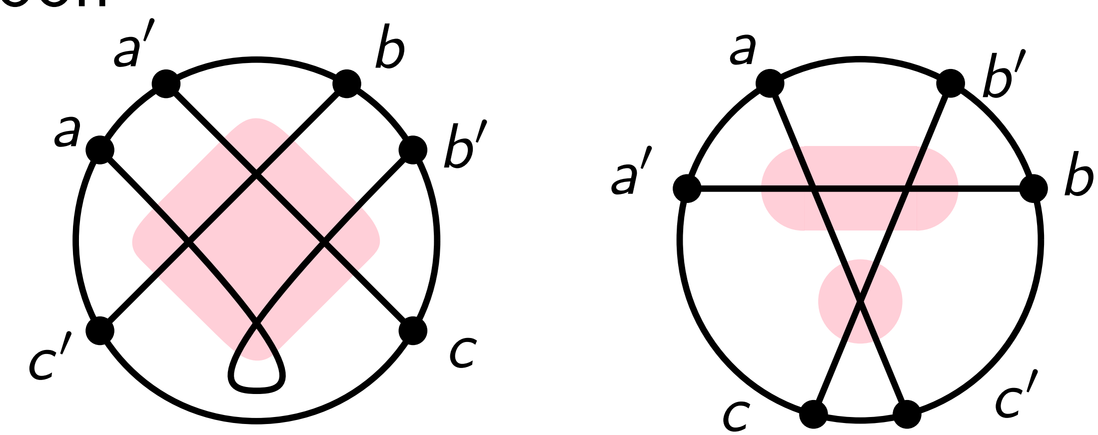
Remark on simple vs. non-simple: consider $K_{3,3}$

Non-simple $\rightsquigarrow$ orientable graph genus [Alam et al. 2016]

... more on this soon



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Other results (not covered in this talk, see the paper!):

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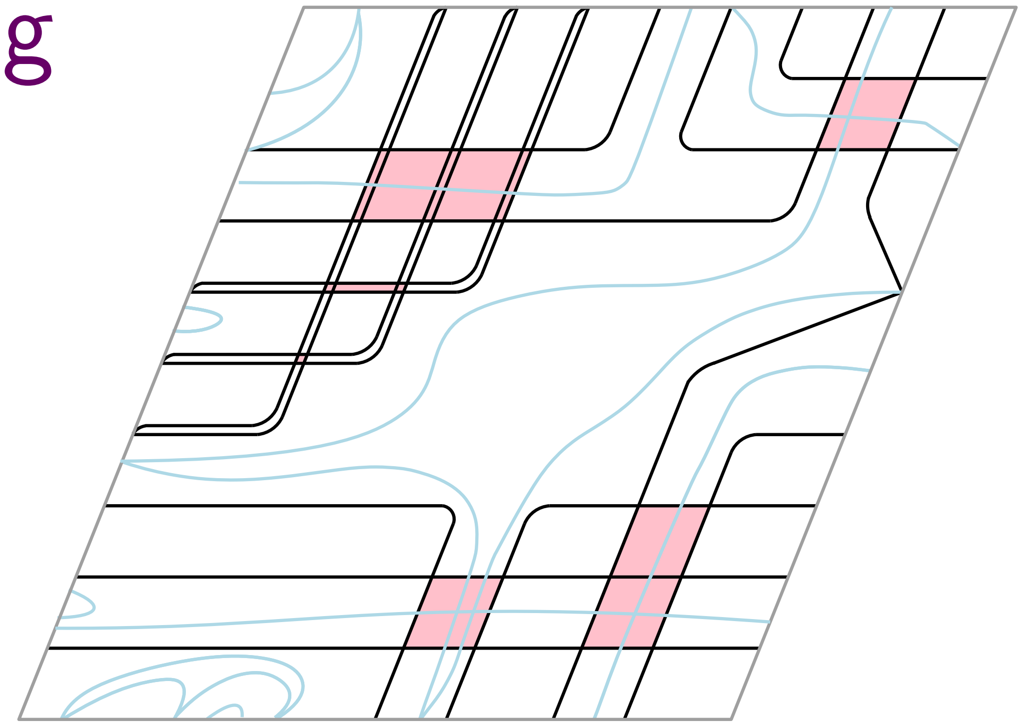
Other results (not covered in this talk, see the paper!):

**Thm.** For general layouts, on inputs $(G, k)$, deciding whether $G$ has a simple drawing with $k$ bundled crossings is NPc. For non-simple, this is FPT in $k$ (via genus).

**Obs.** For circular layouts, on inputs $(G, k)$, deciding whether $G$ has a (non-simple) circular drawing with $k$ bundled crossings is FPT in $k$ (via genus).

# Structure of a drawing

Consider a drawing with $k$ bundled crossings and observe that:



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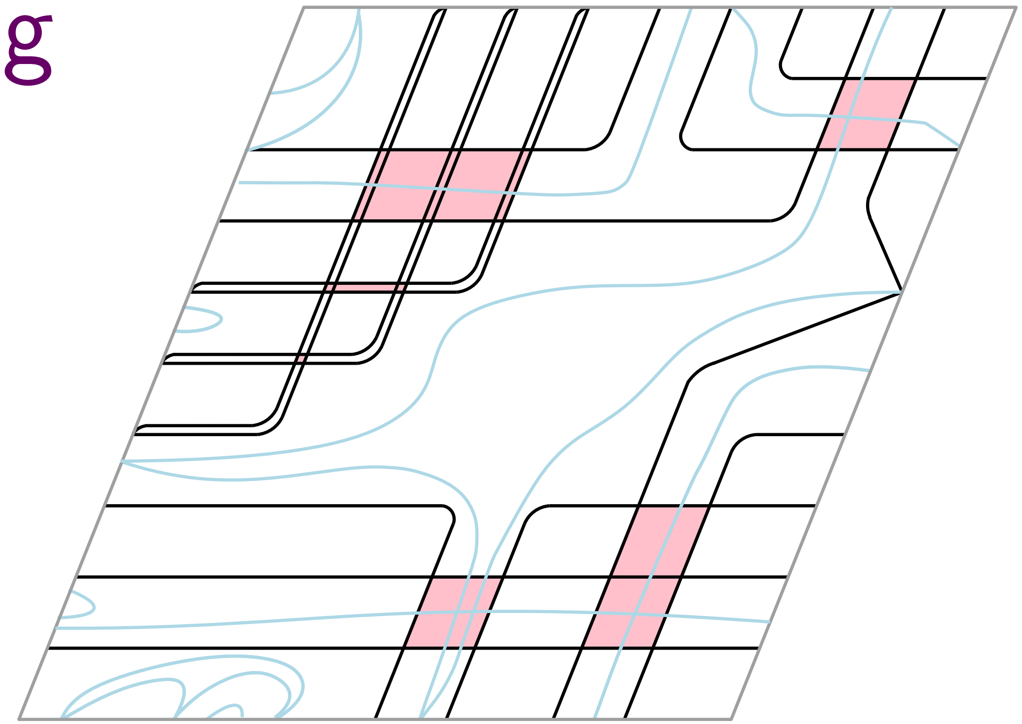

- At most $k$ bundled crossings $\implies$ at most $4k$ frame edges.

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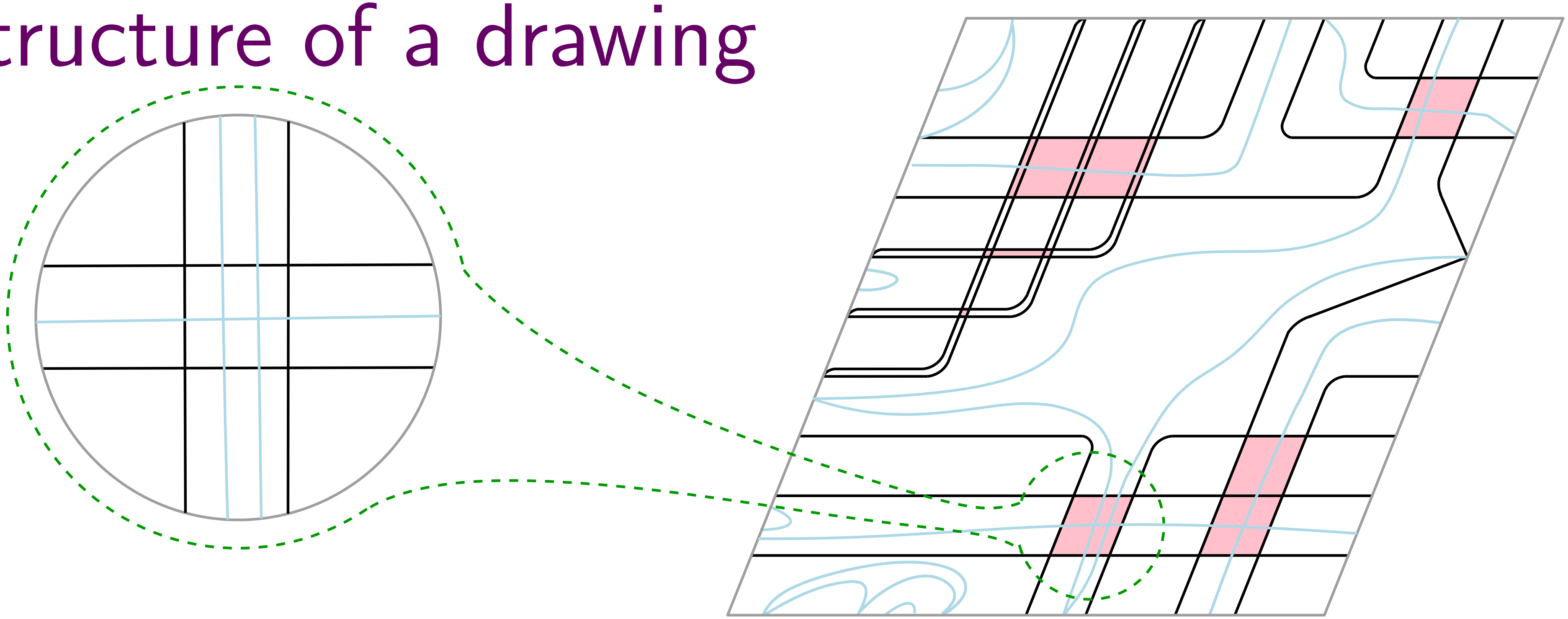

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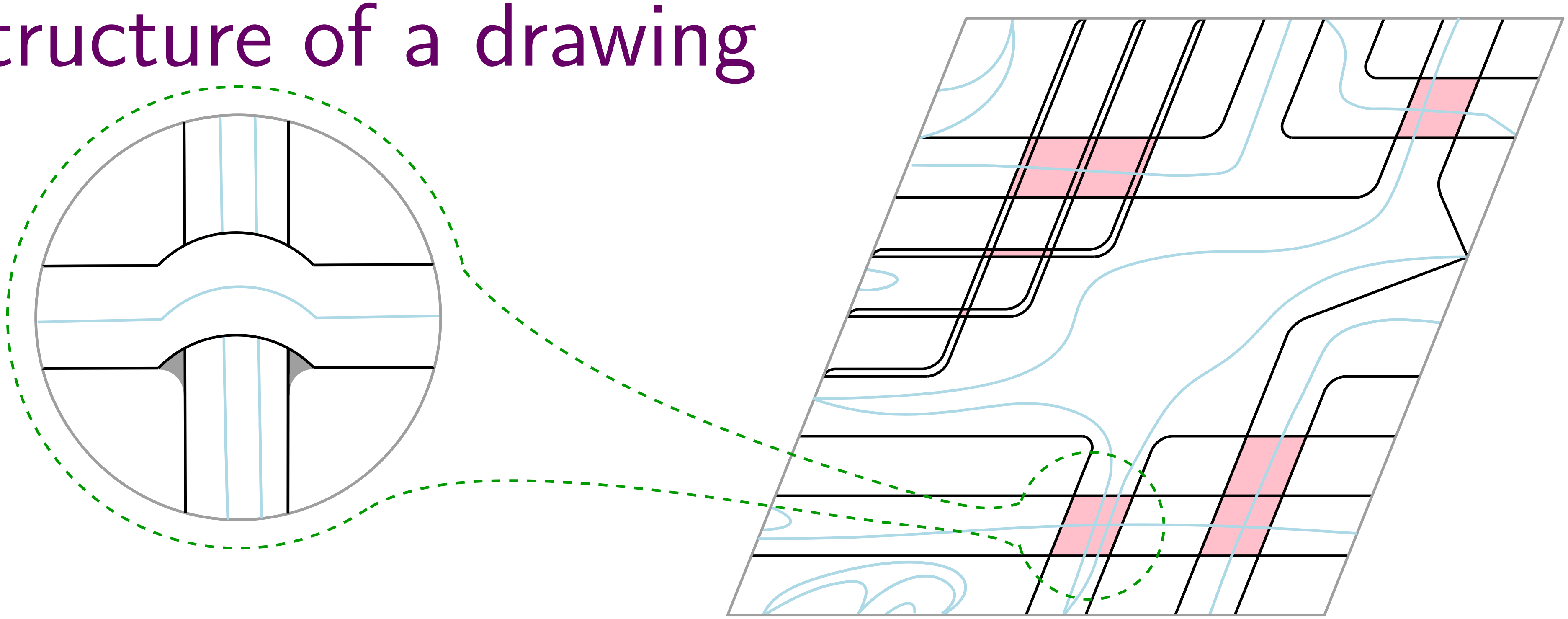

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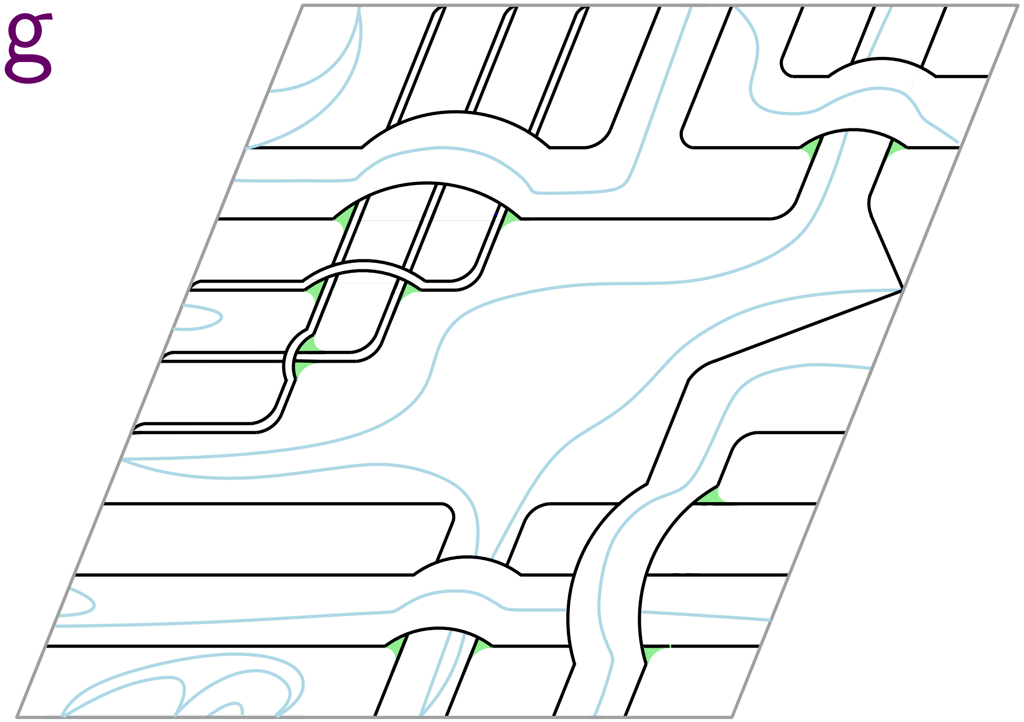

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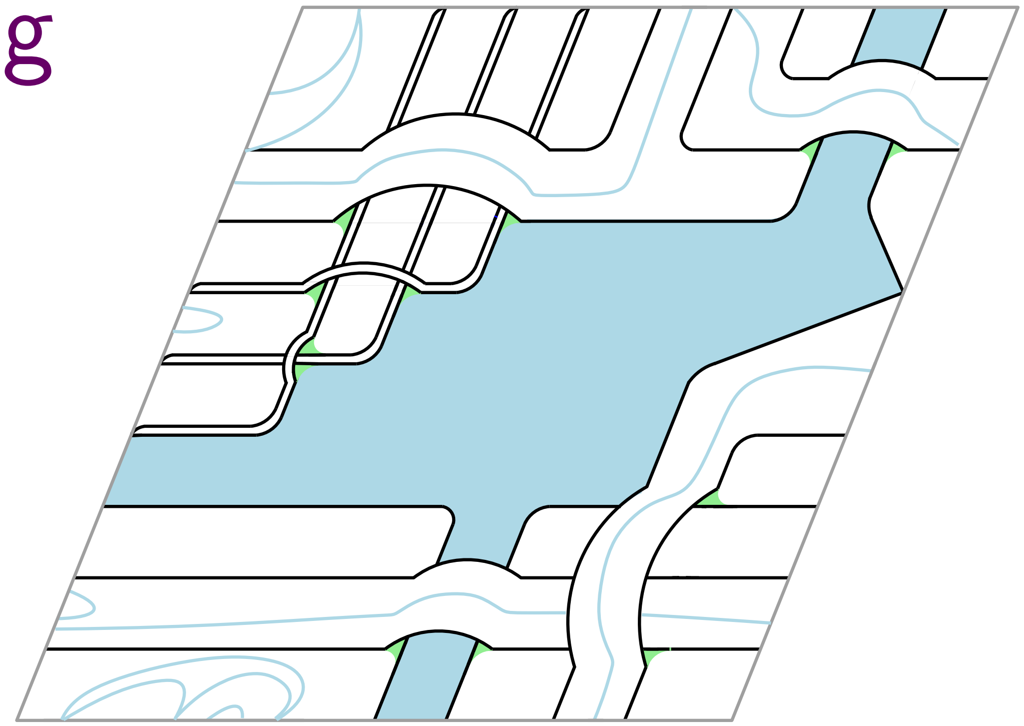

Consider a drawing with $k$ bundled crossings and observe that:

- At most $k$ bundled crossings $\implies$ at most $4k$ frame edges.
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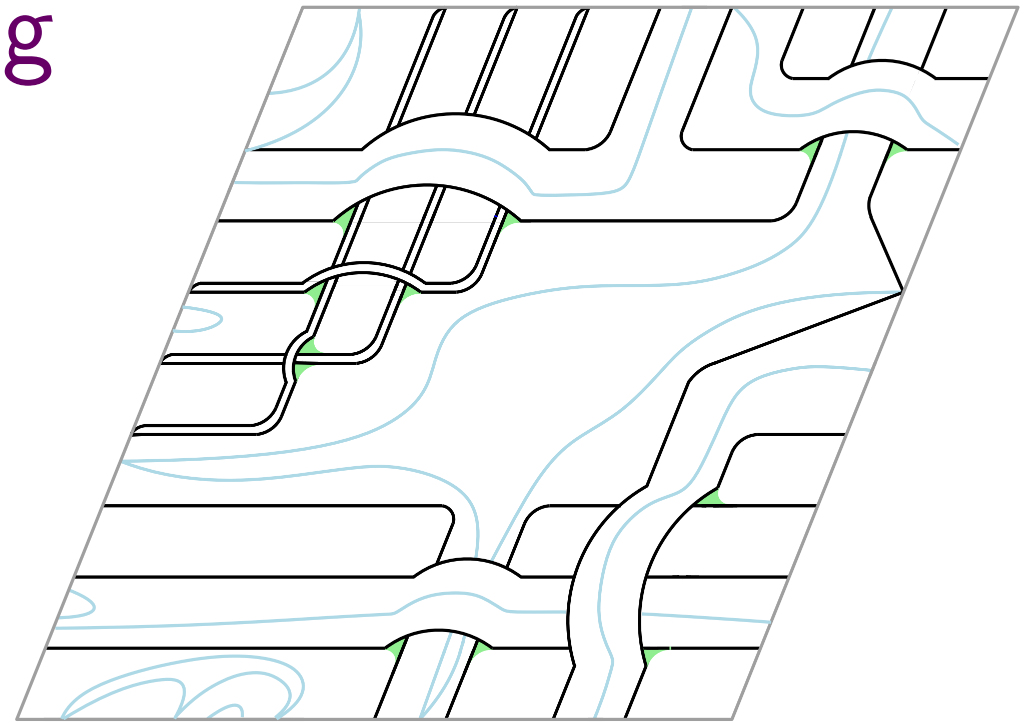

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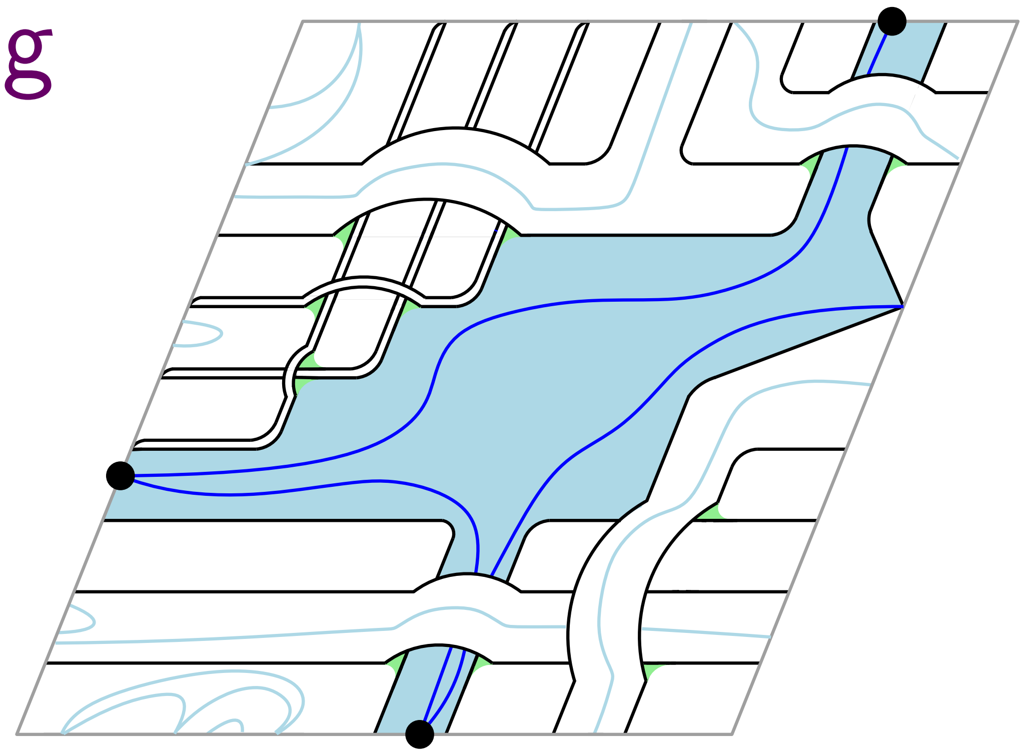

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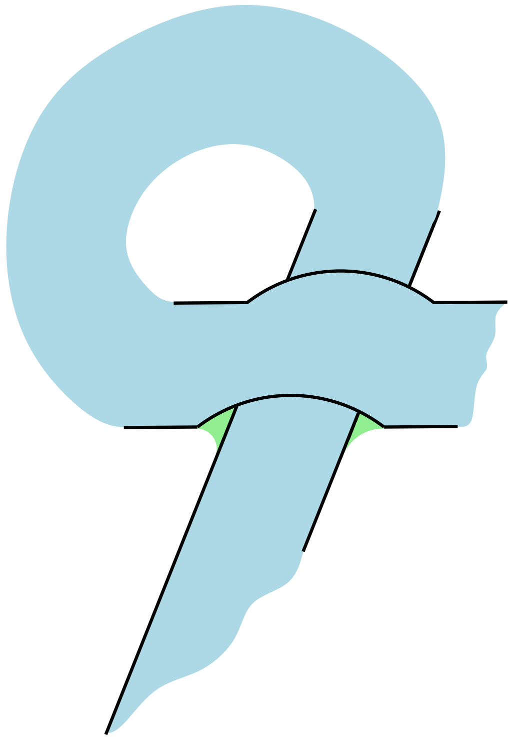
- The graph induced by edges inside a single region has a special outerplanar drawing.

# Structure of a drawing

What if a region has
a bridge and a tunnel
corresponding to
the same bundled crossing?



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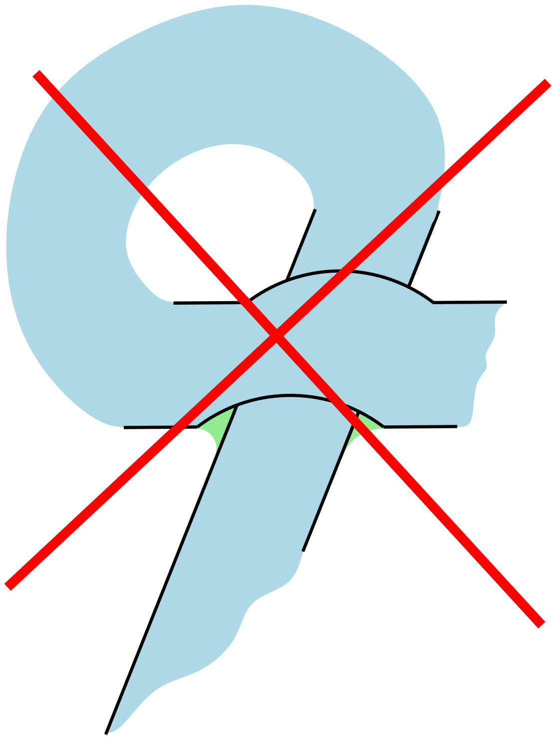

**Lem.** Each region is a topological disk.

- At most $k$ bundled crossings $\Longrightarrow$ at most $4k$ frame edges.
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# The Algorithm

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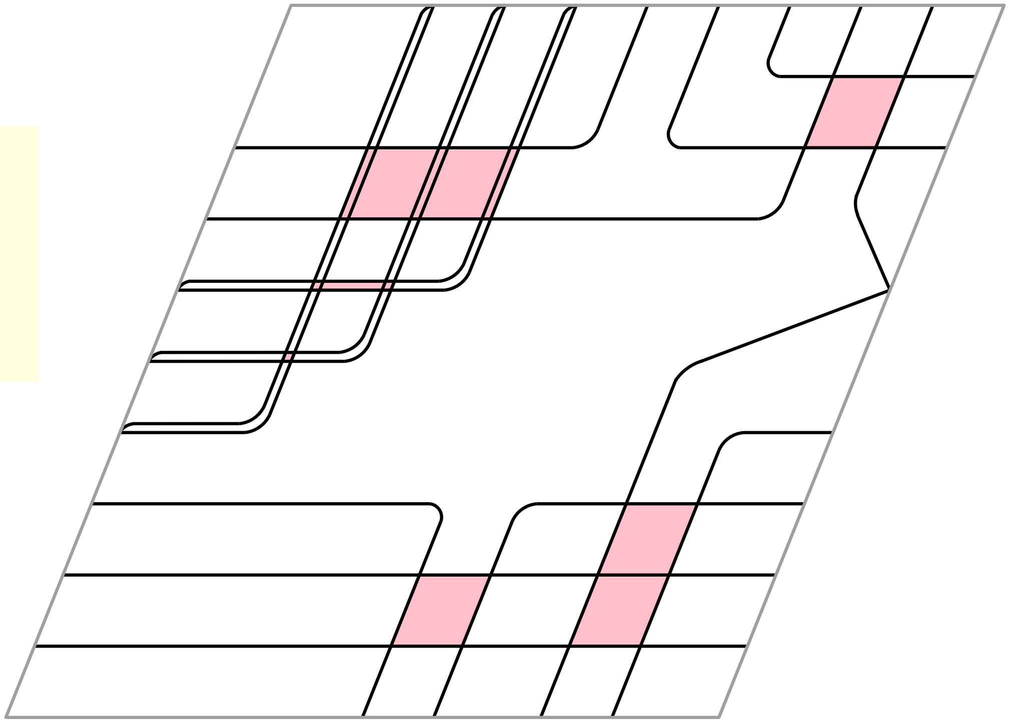
Deciding whether $\mathrm{bc}^\circ(G) = k$ is FPT in $k$.



- Guess the drawing of at most $4k$ frame edges and their bundling.

# The Algorithm

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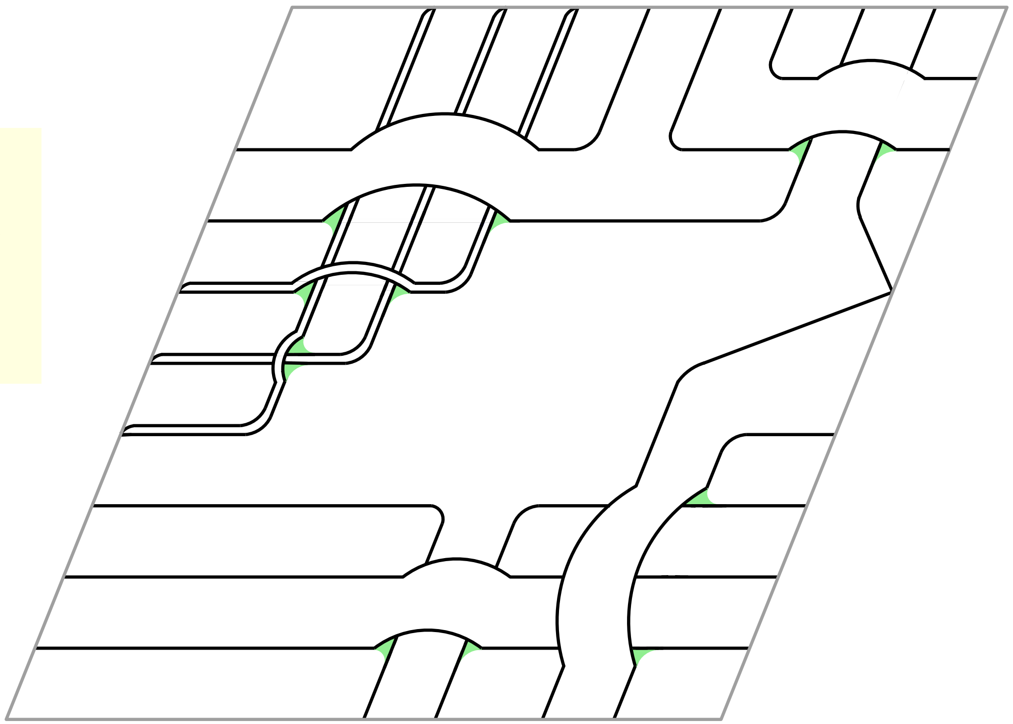

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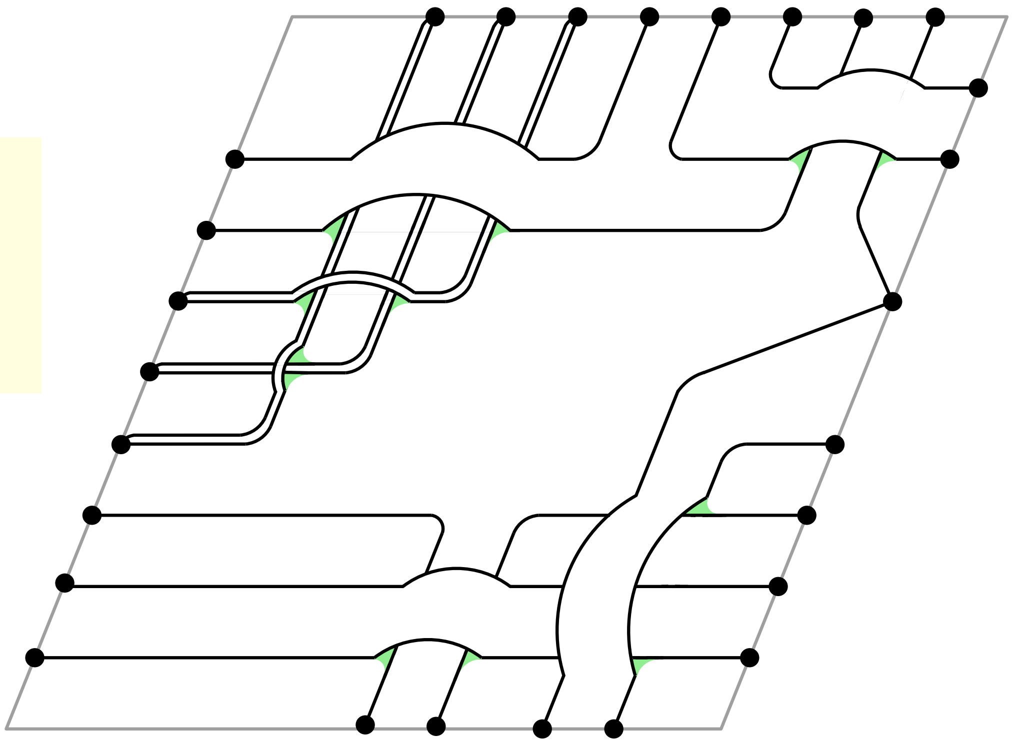

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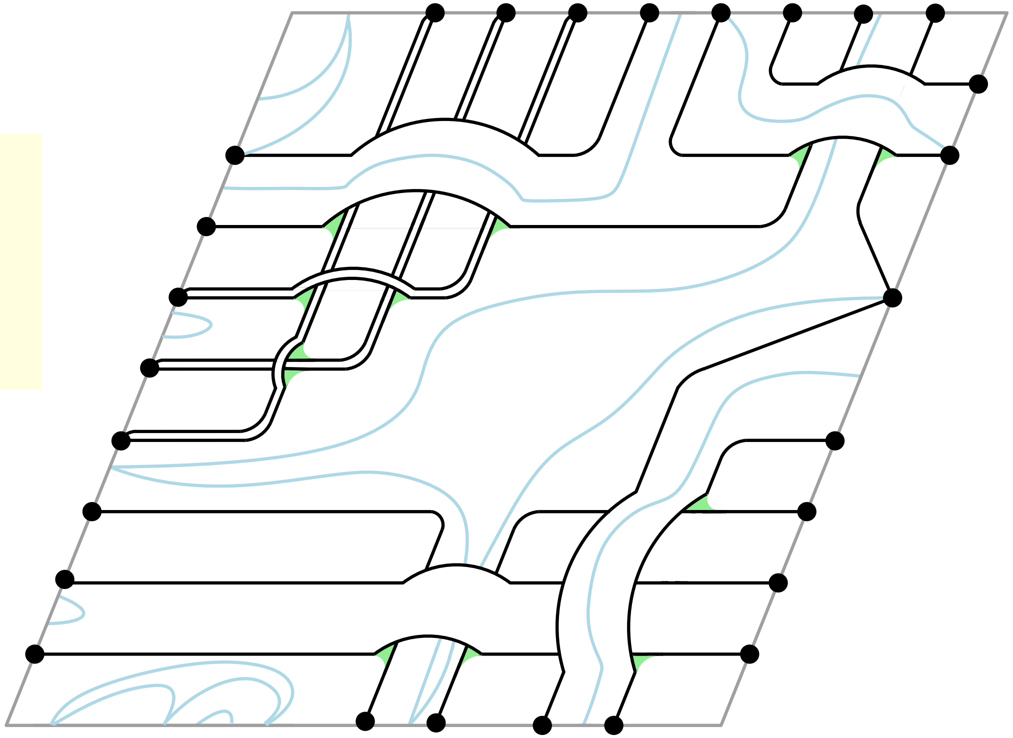

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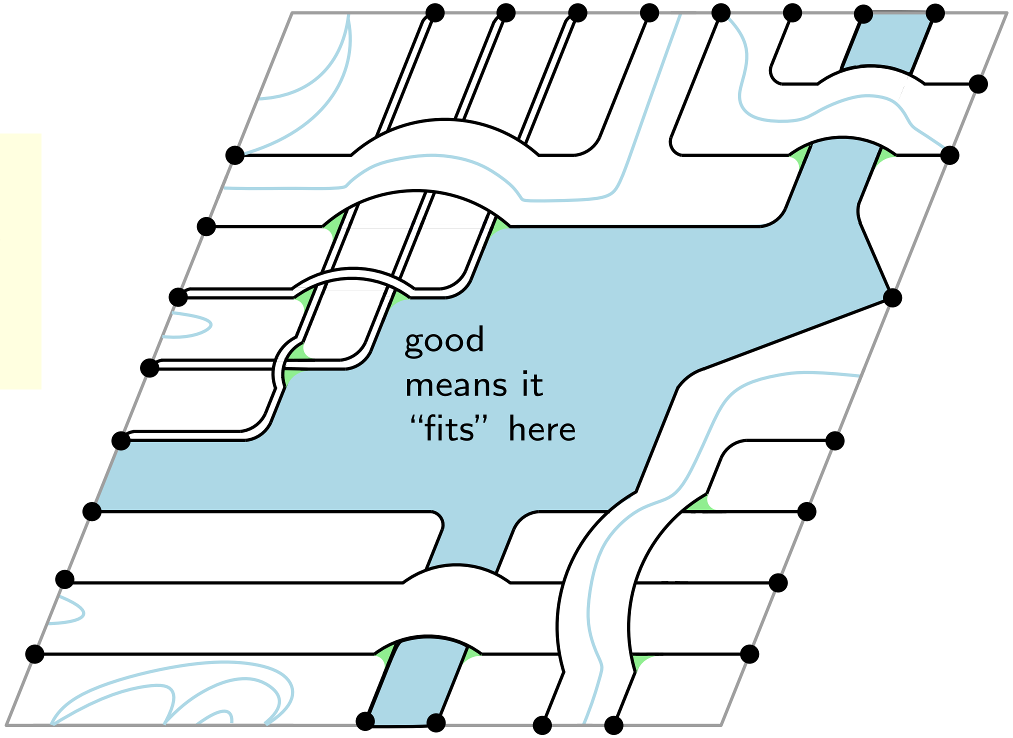


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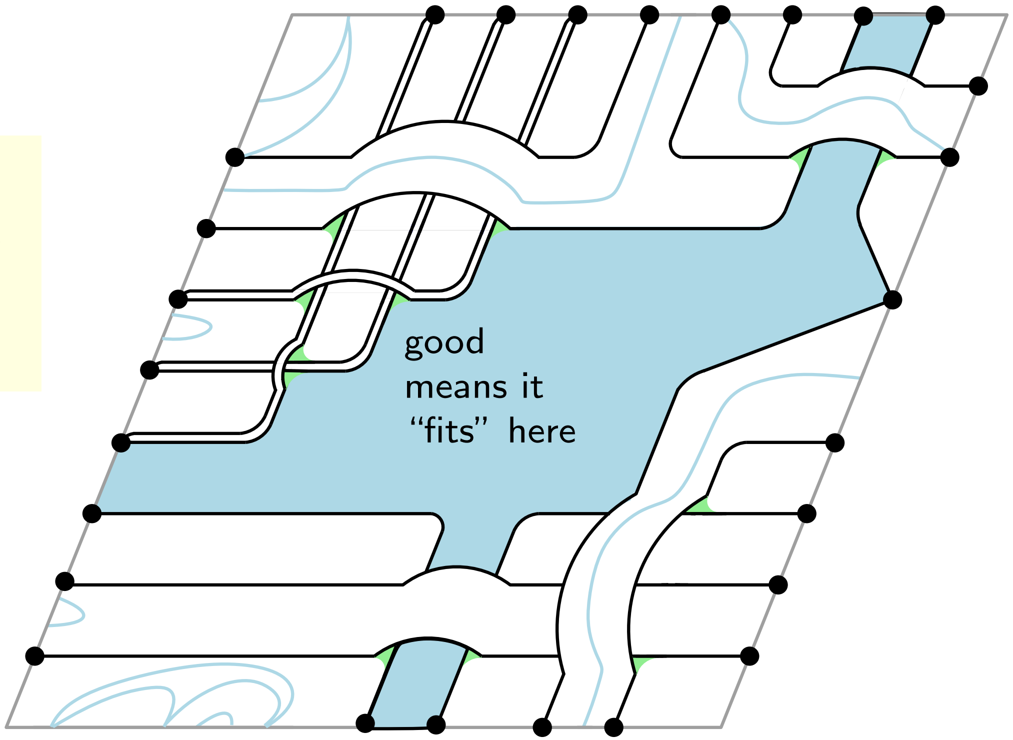


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$\mathrm{MSO}_2$

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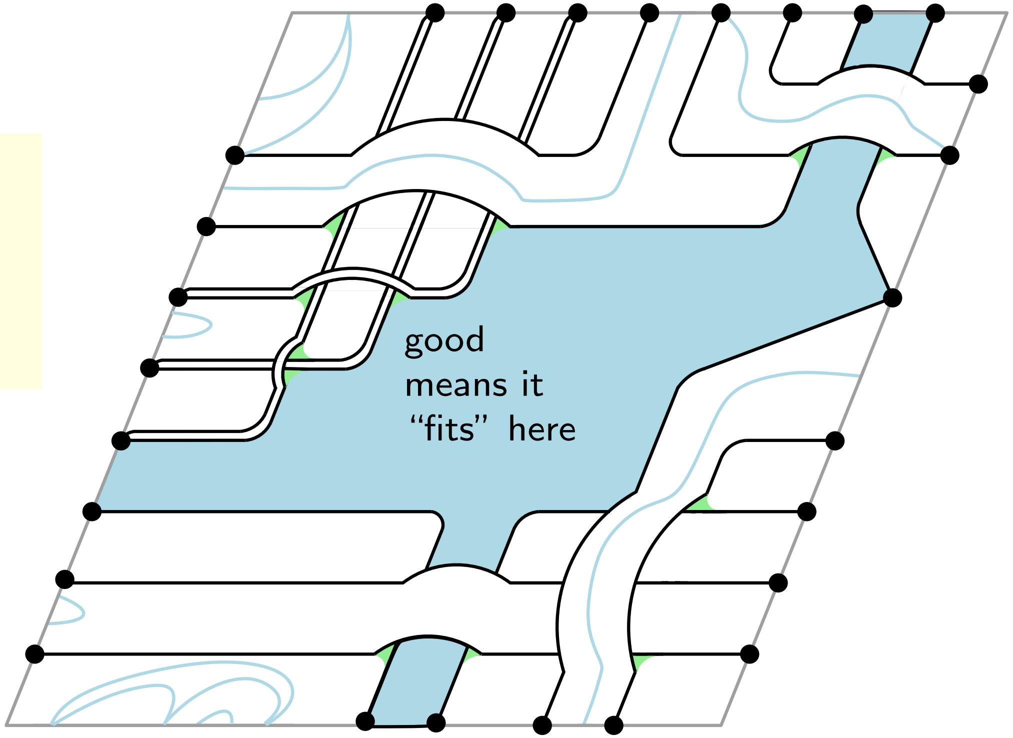


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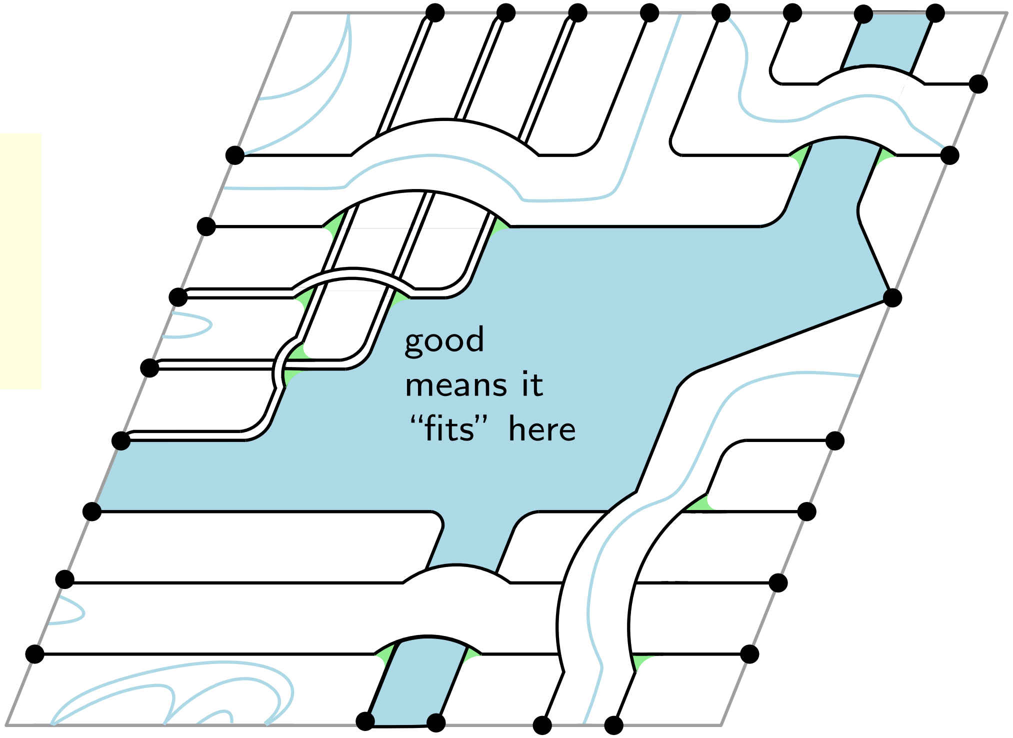


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# Monadic Second Order Logic ($\mathrm{MSO}_2$)

$$\textsc{Partition}(E; E_0, \ldots, E_\gamma) =$$
$$(\forall e \in E)\Big[\Big(\bigvee_{i=0}^{\gamma} e \in E_i\Big) \wedge \Big(\bigwedge_{i \neq j} \neg(e \in E_i \wedge e \in E_j)\Big)\Big].$$

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**Thm (Courcelle):** If a property $P$ is expressed as $\varphi \in \mathrm{MSO}_2$, then for every graph $G$ with treewidth at most $t$, $P$ can be tested in time $O(f(t, |\varphi|)(n+m))$ for a computable function $f$.

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Since our regions induce outerplanar graphs, we have treewidth at most $8k + 2$ where $k$ is the number of bundled crossings.

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But, what about?

- Test graphs in each region for a *good* outerplanar drawing.

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This can be stated in $MSO_2$ via a mechanism of *MSO-definition schemes*, and the *Backwards Translation Theorem* [Courcelle, Engelfriet; 2012]

# Monadic Second Order Logic ($MSO_2$)

**Theorem 7.10 (Backwards Translation Th**[illegible] Let $\mathcal{D}$ be a $k$-copying $C_r MS$-definition scheme of type $\mathcal{R} \to \mathcal{R}'$ [illegible] of parameters $\mathcal{W}$. Let $\mathcal{X}$ be a finite set of set variables and $\mathcal{Y} = \{y$ [illegible] be a set of first-order variables. For every $\beta \in C_r MS(\mathcal{R}', \mathcal{X} \cup \mathcal{Y})$ and [illegible] one can construct a formula $\beta_{\mathbf{i}}^{\mathcal{D}} \in C_r MS(\mathcal{R}, \mathcal{W} \cup \mathcal{X}^{(k)} \cup \mathcal{Y})$ such tha[illegible] $S \in STR^{c}(\mathcal{R})$, every $\mathcal{W}$-assignment $\gamma$, every $\mathcal{X}^{(k)}$-assignment $\eta$, and e[illegible]signment $\mu$, all of them in $S$, we have:

$(S, \gamma \cup \eta \cup \mu) \models \beta_{\mathbf{i}}^{\mathcal{D}}$ if a[illegible]

$\widehat{\mathcal{D}}(S, \gamma)$ is de[illegible]

$\eta^{[k]} \cup \mu_{\mathbf{i}}$ [illegible] $\mathcal{Y})$-assignment in $\widehat{\mathcal{D}}(S, \gamma)$, and

$(\widehat{\mathcal{D}}(S$ [illegible] $\mu_{\mathbf{i}}) \models \beta$.

The quantifier-height of $\beta_{\mathbf{i}}^{\mathcal{D}}$ is at most $k \cdot qh(\beta) + qh(\mathcal{D}) + 1$. □

It's a bit technical

This can be stated in $MSO_2$ via a mechanism of *MSO-definition schemes*, and the *Backwards Translation Theorem* [Courcelle, Engelfriet; 2012]

# Testing whether $\mathrm{bc}^\circ = k$

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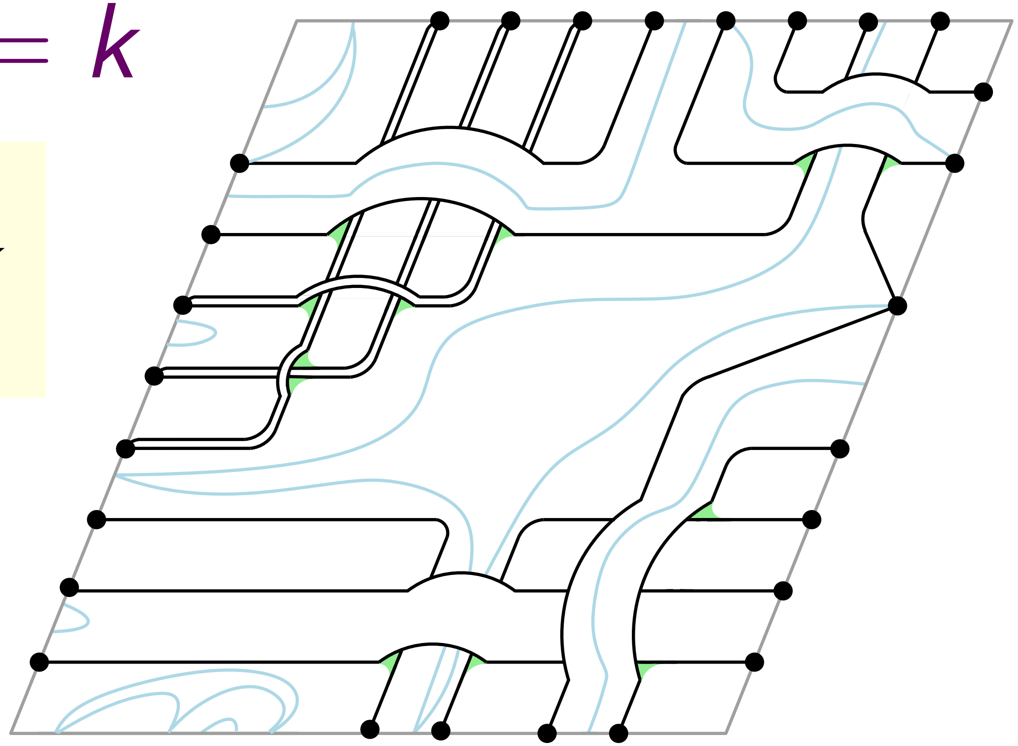

Runtime:

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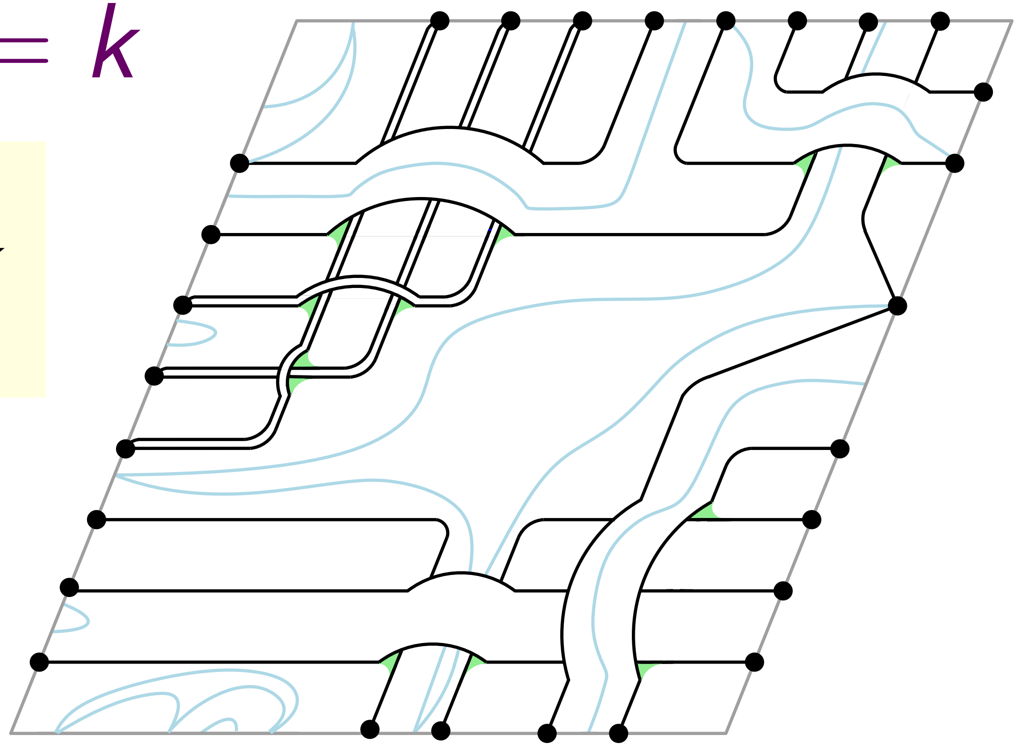

Runtime:

$2^{O(k^2)}$

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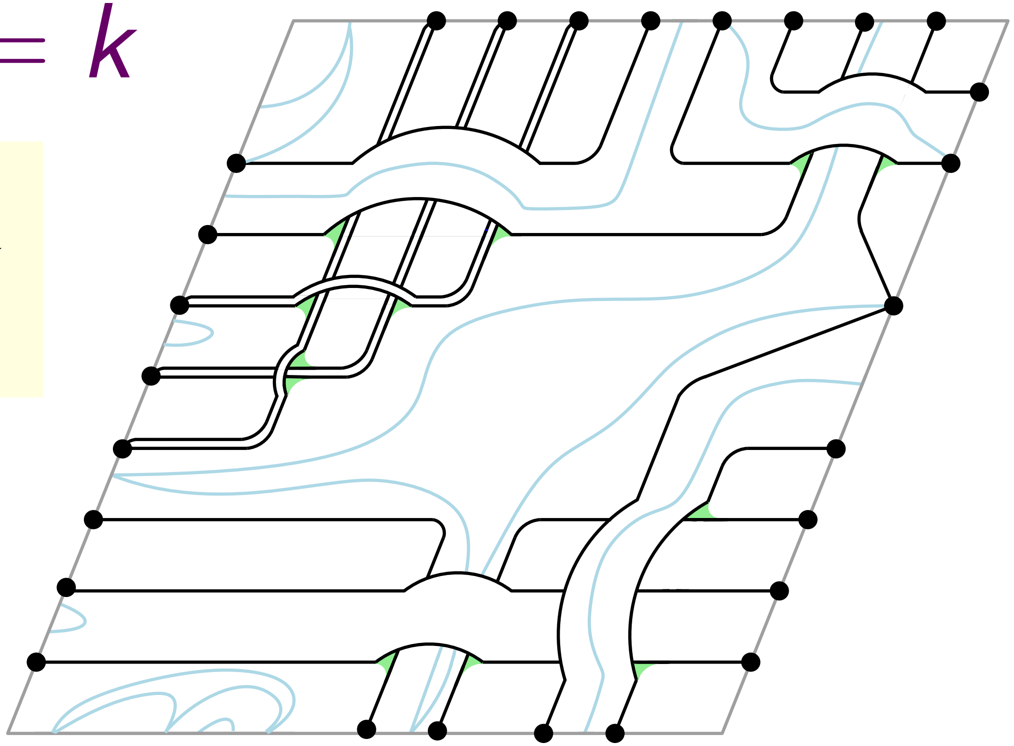

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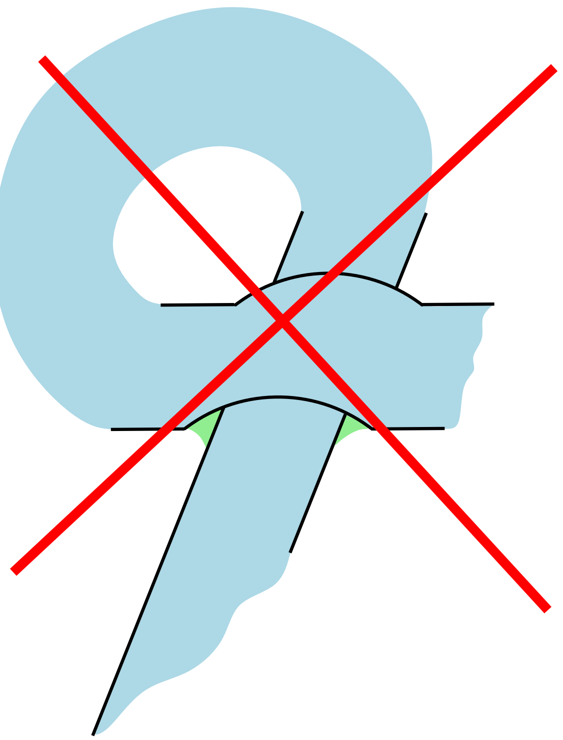

Recall that for correctness of the algorithm we need to show that

**Thm.** Each region is a topological disk.

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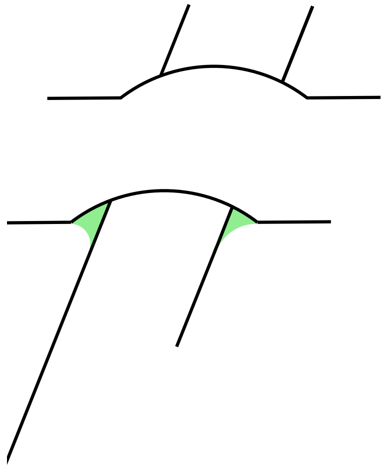
# A region is a topological disk

**Lem.** Each region is a topological disk.

Proof.



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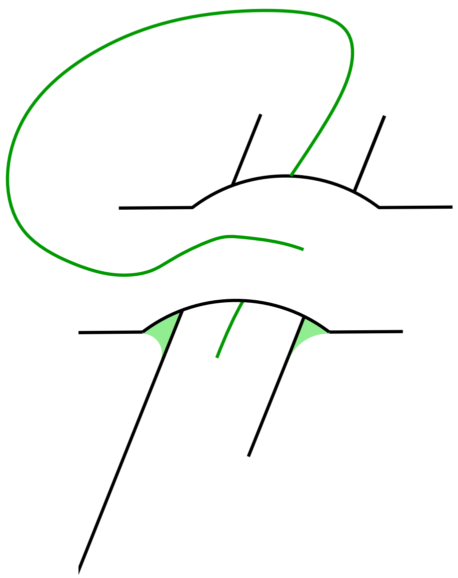

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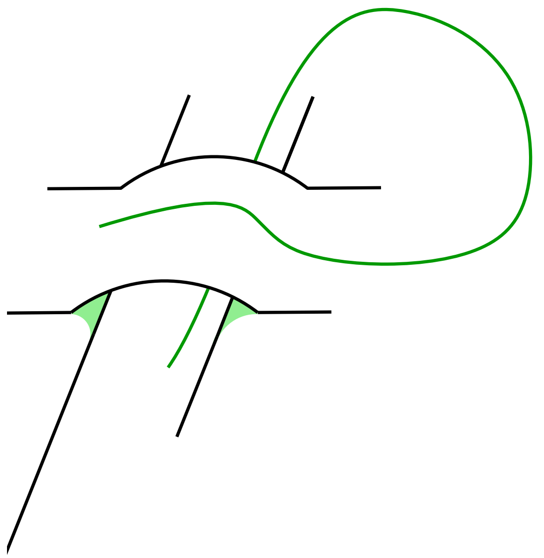

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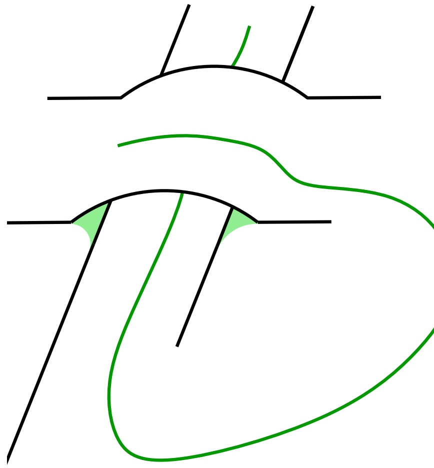

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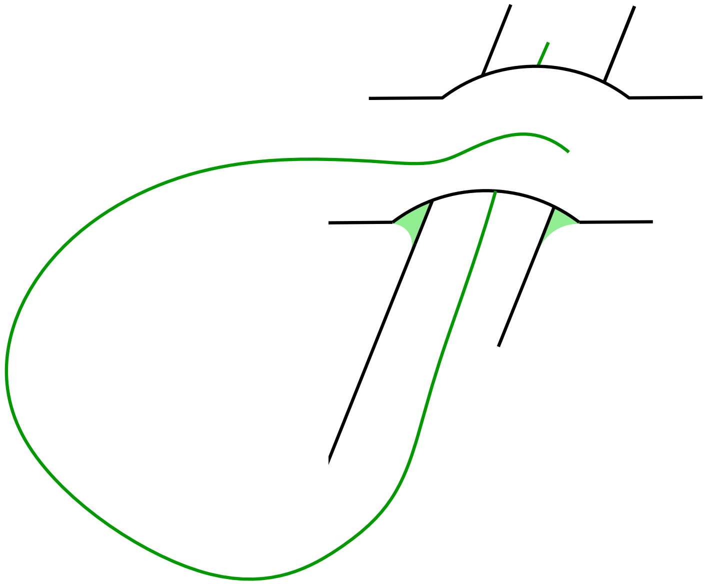

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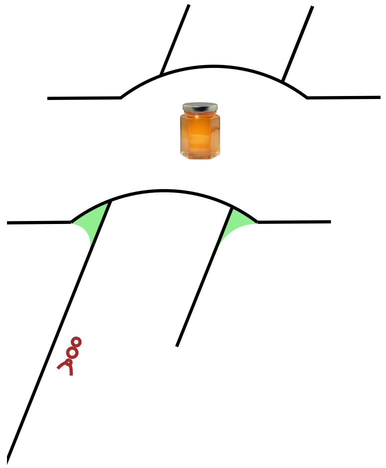

Stick to the right!

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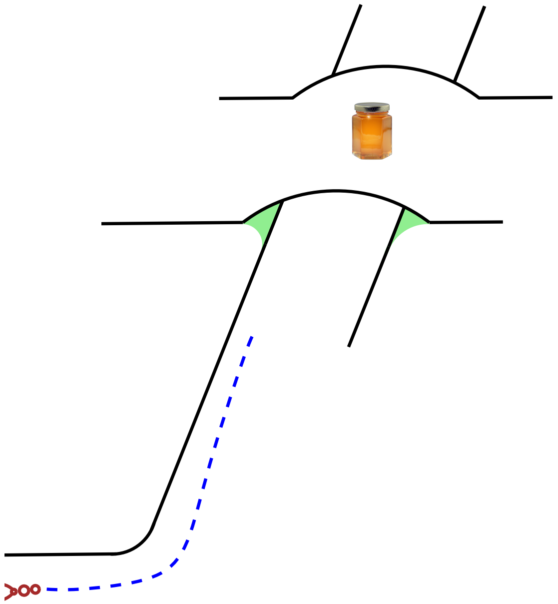

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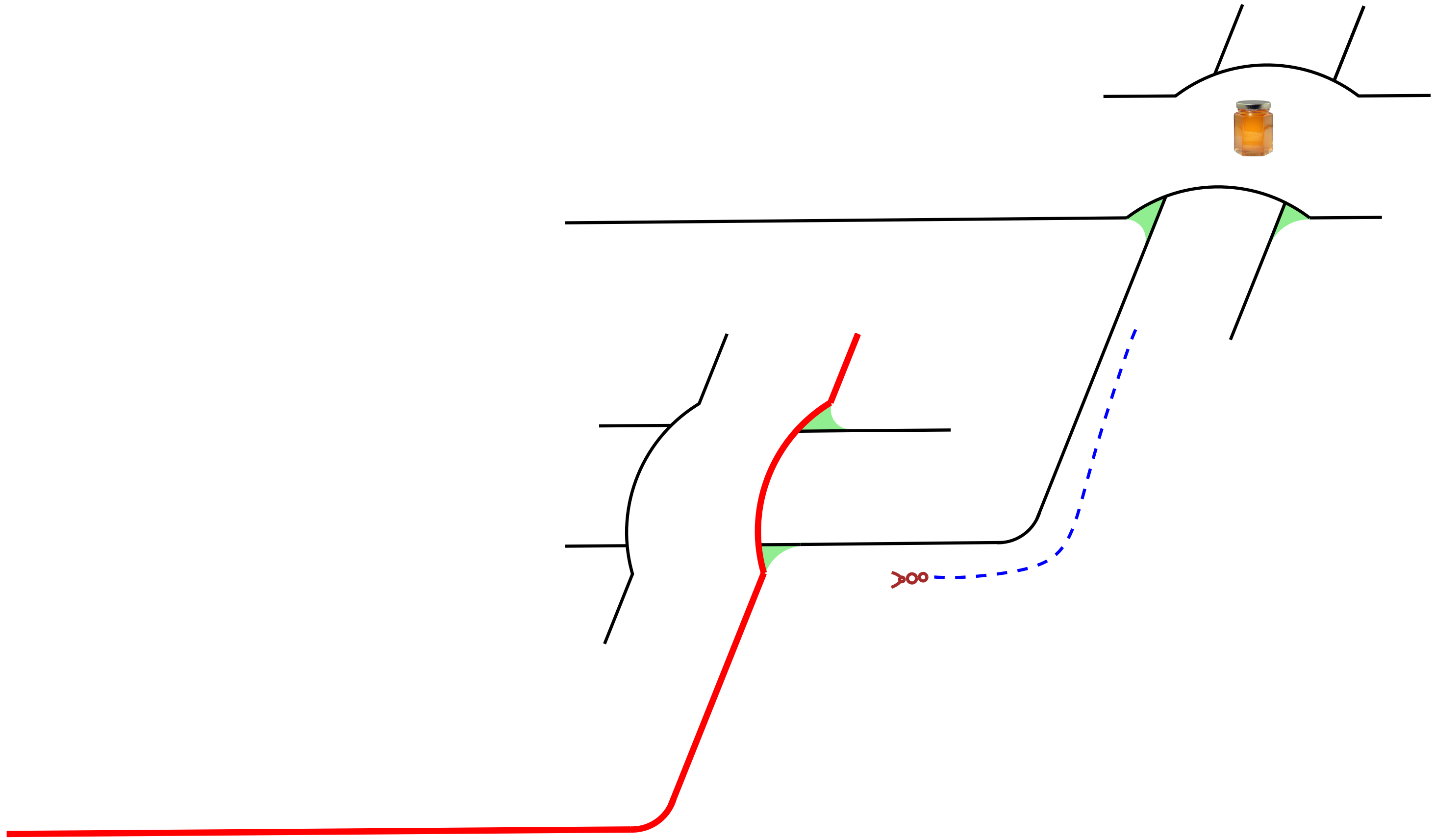

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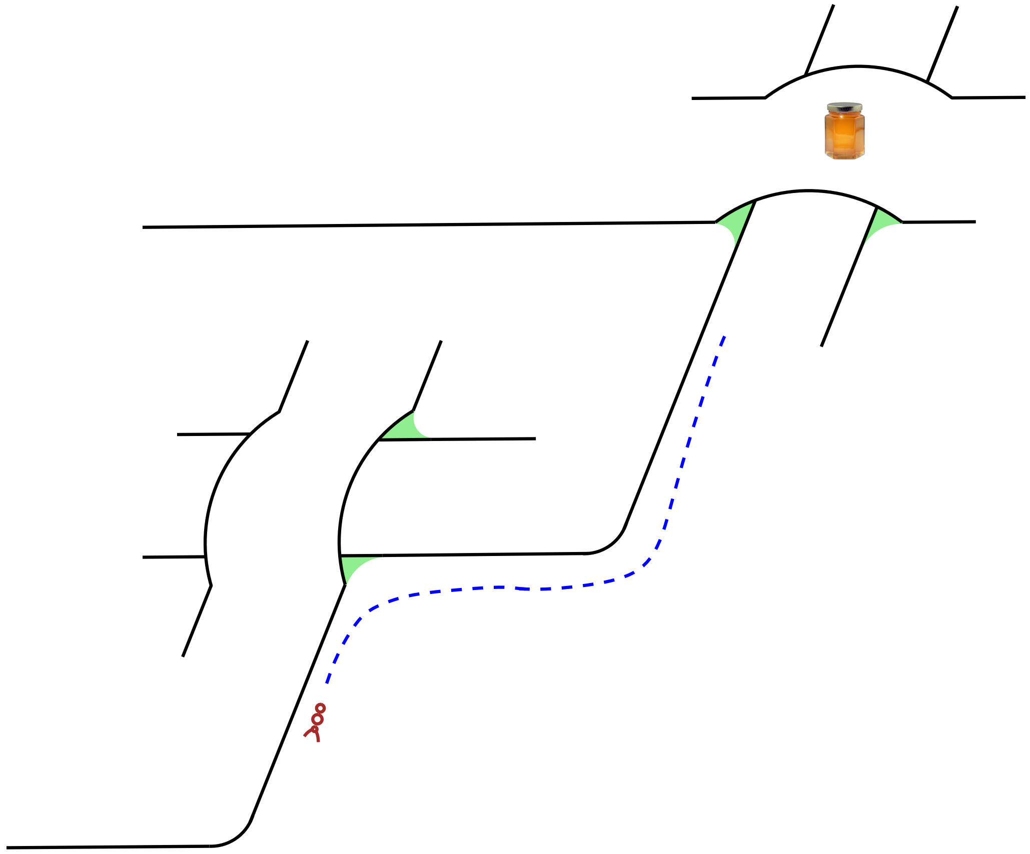

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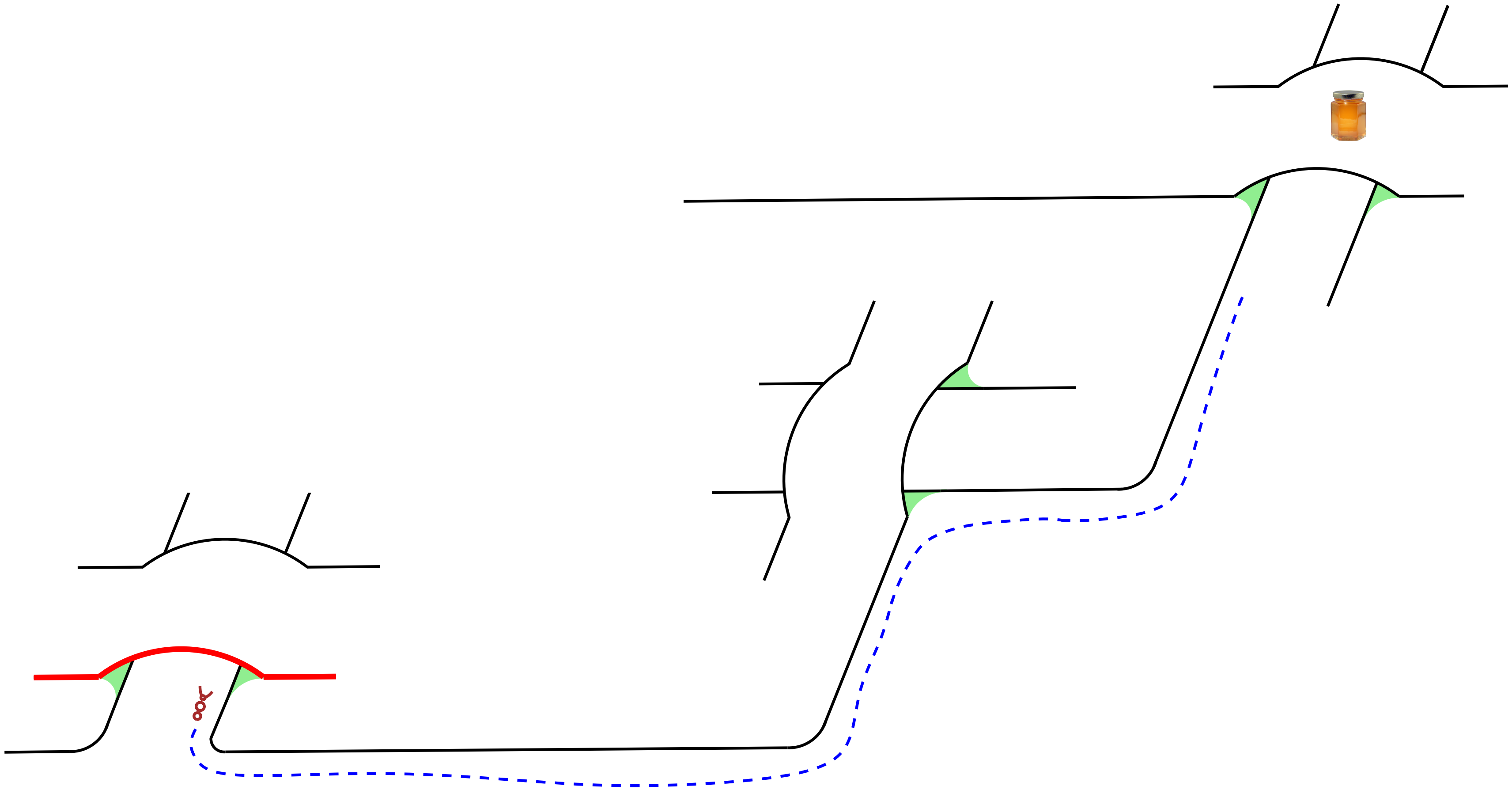

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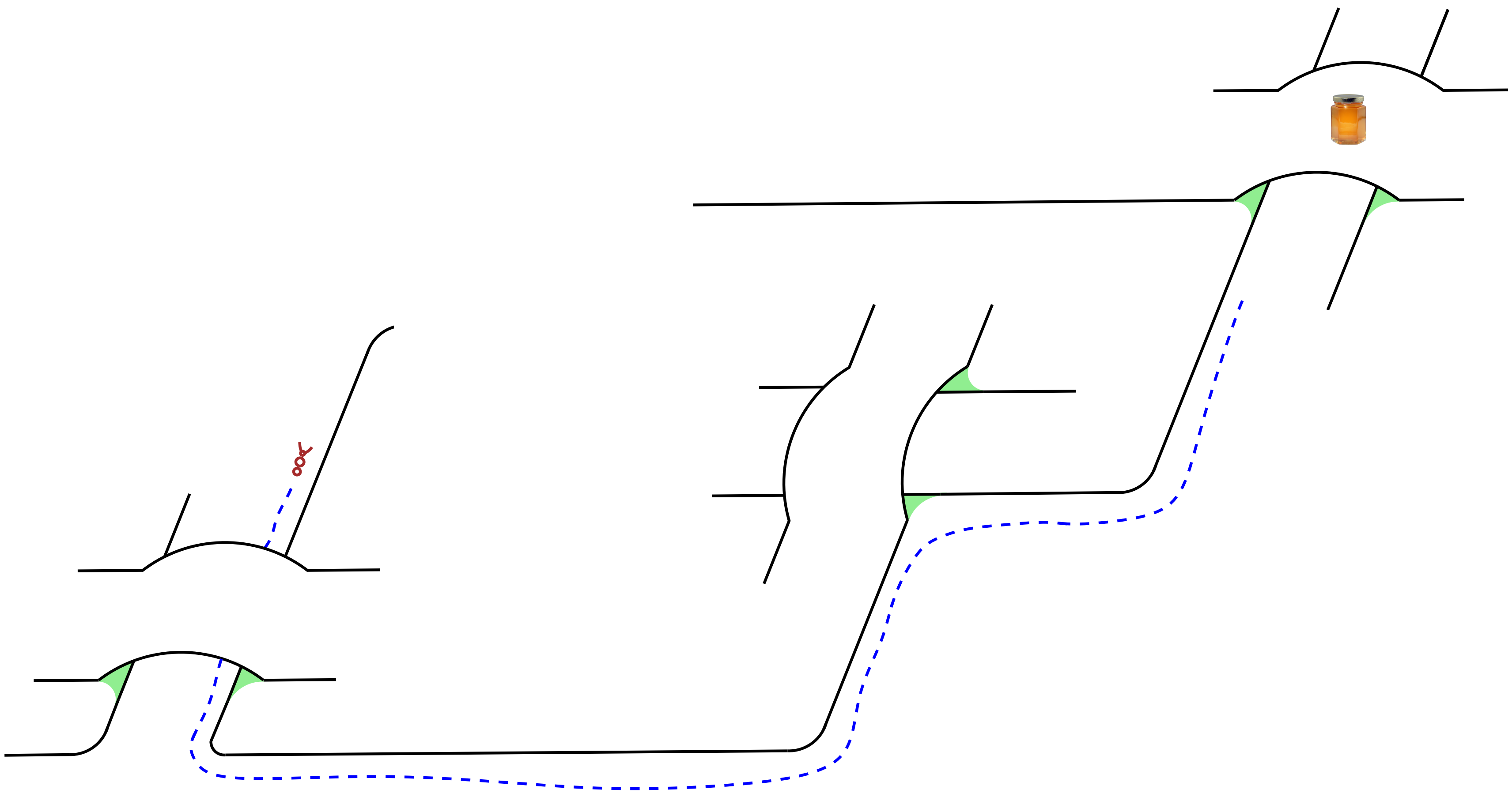

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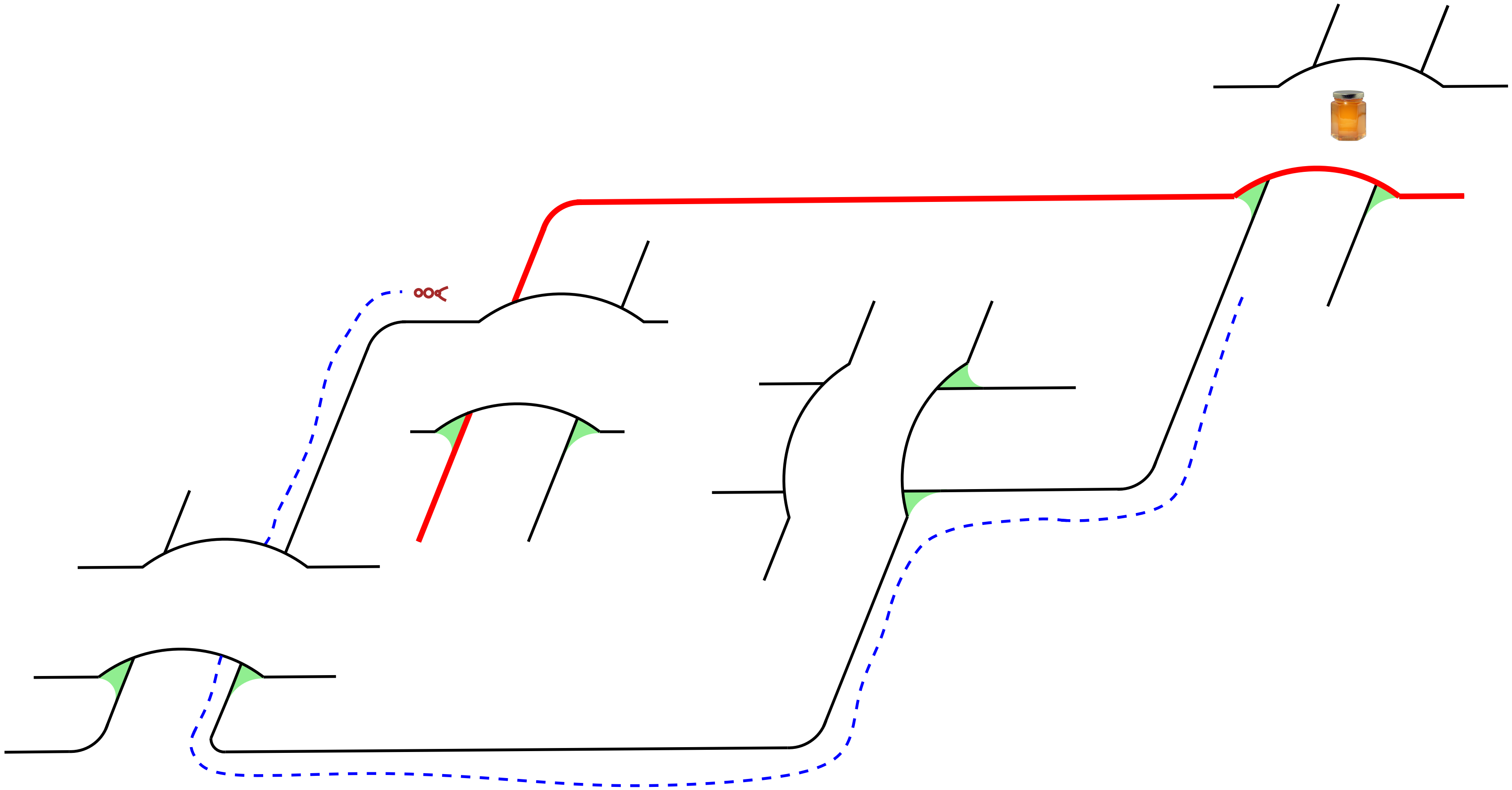

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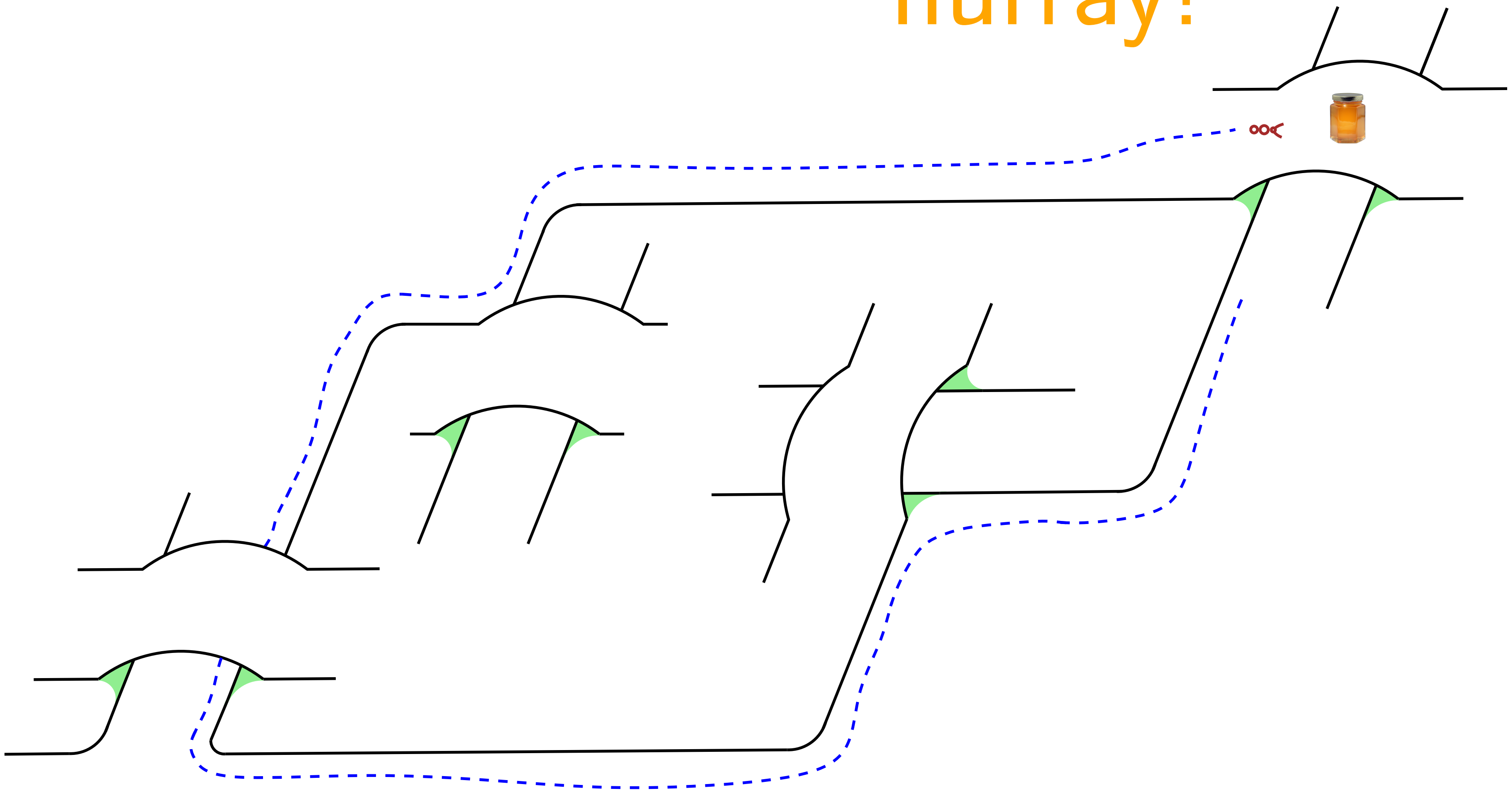
hurray!



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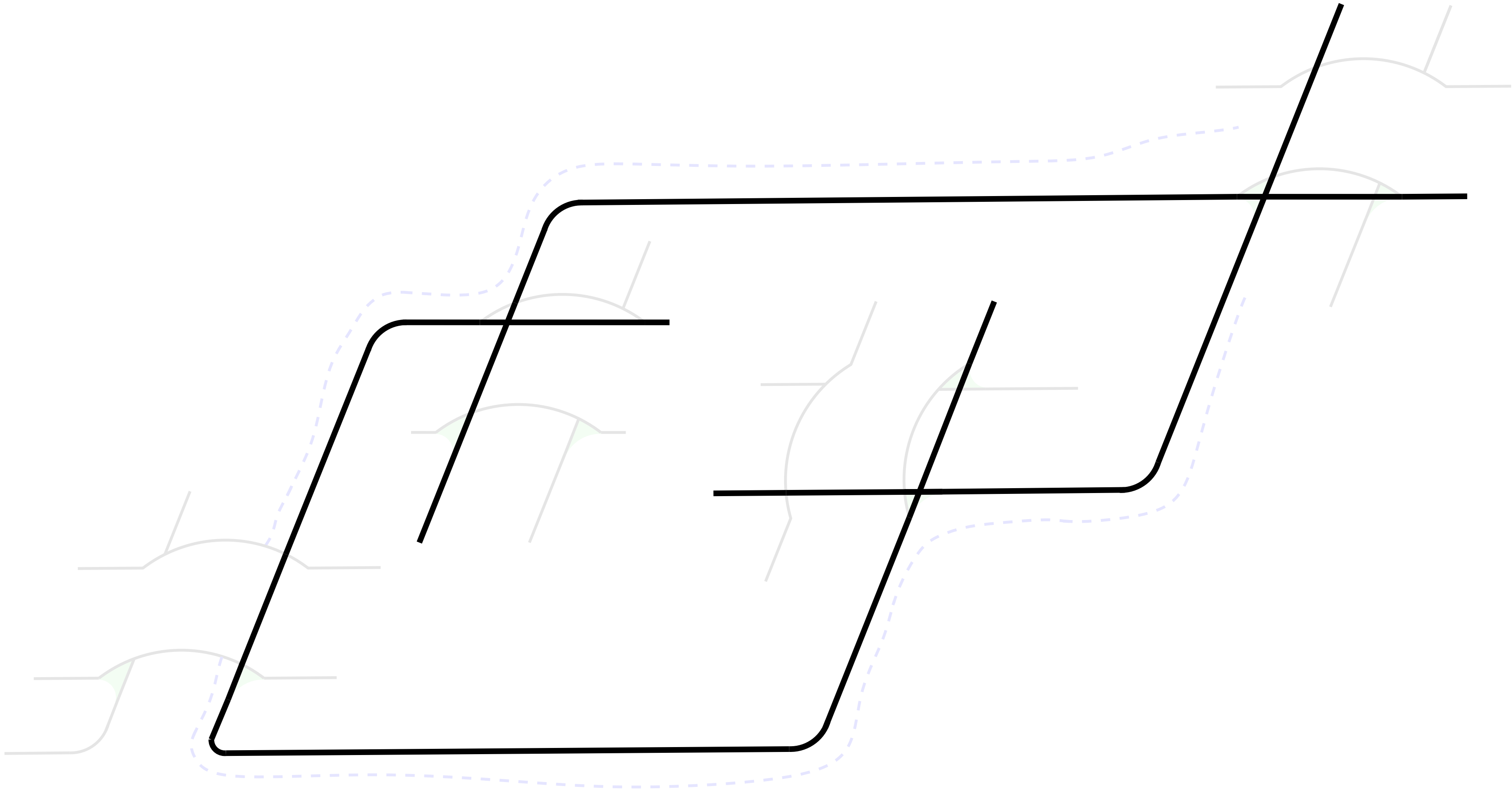

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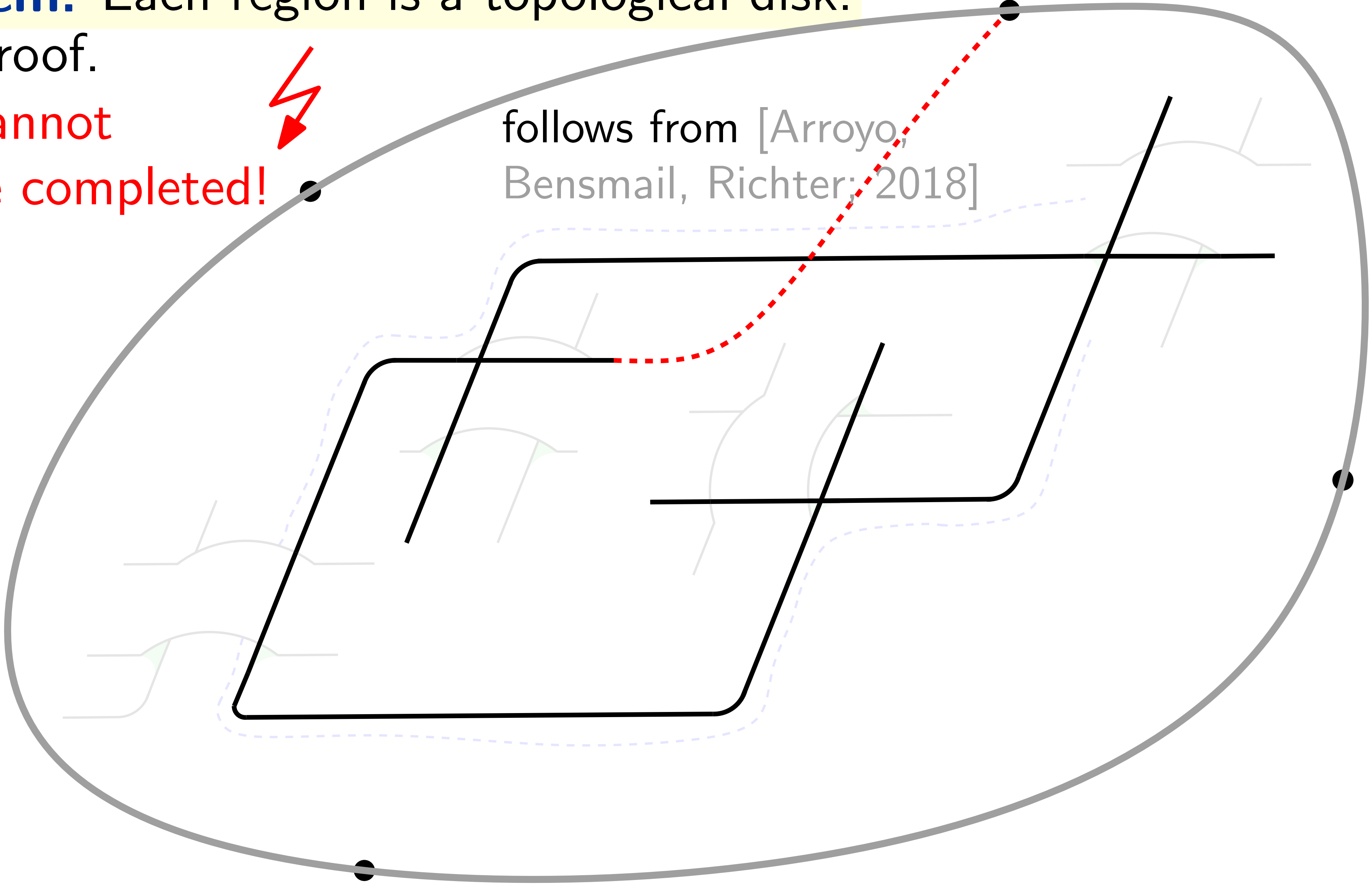

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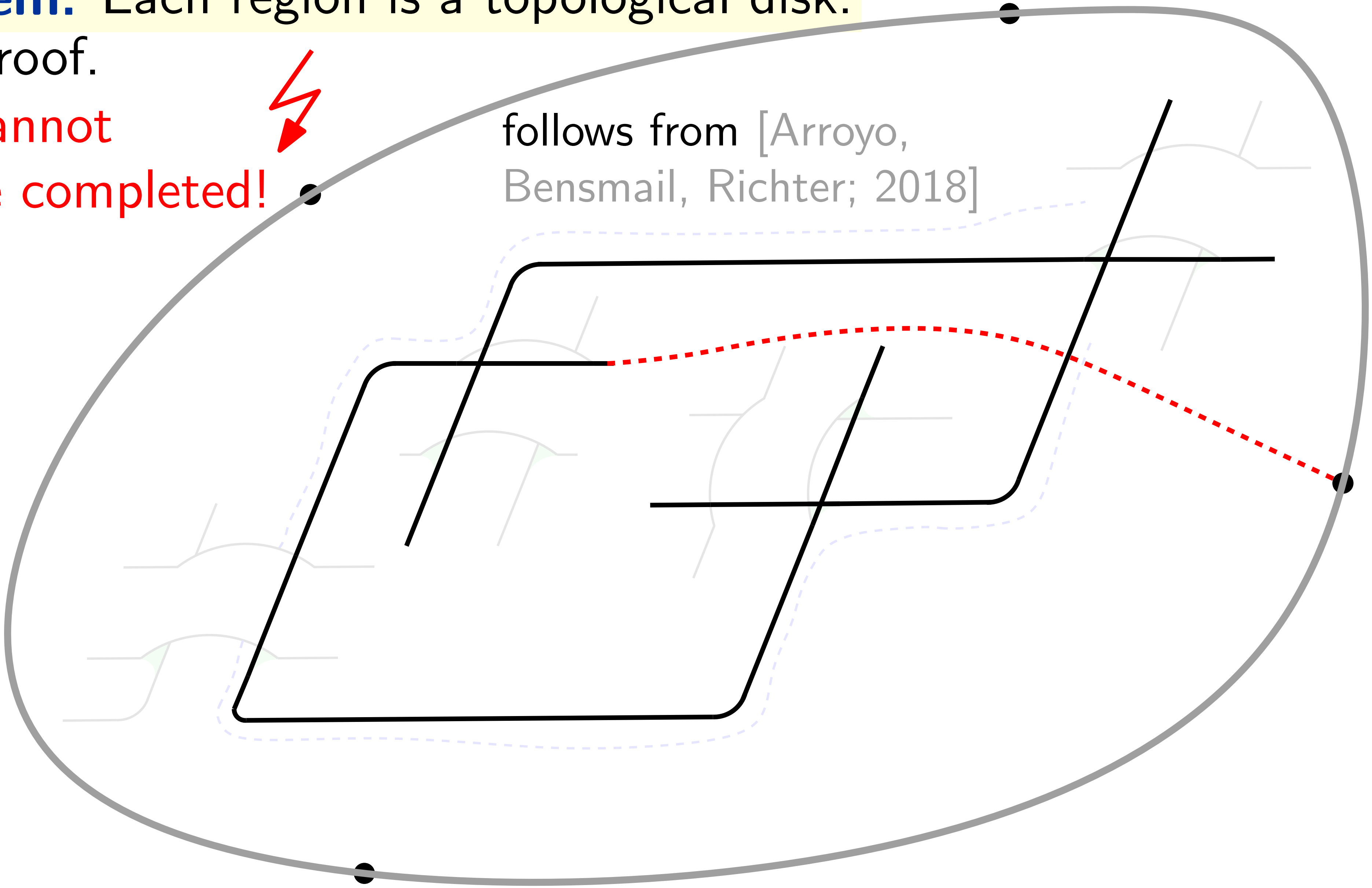

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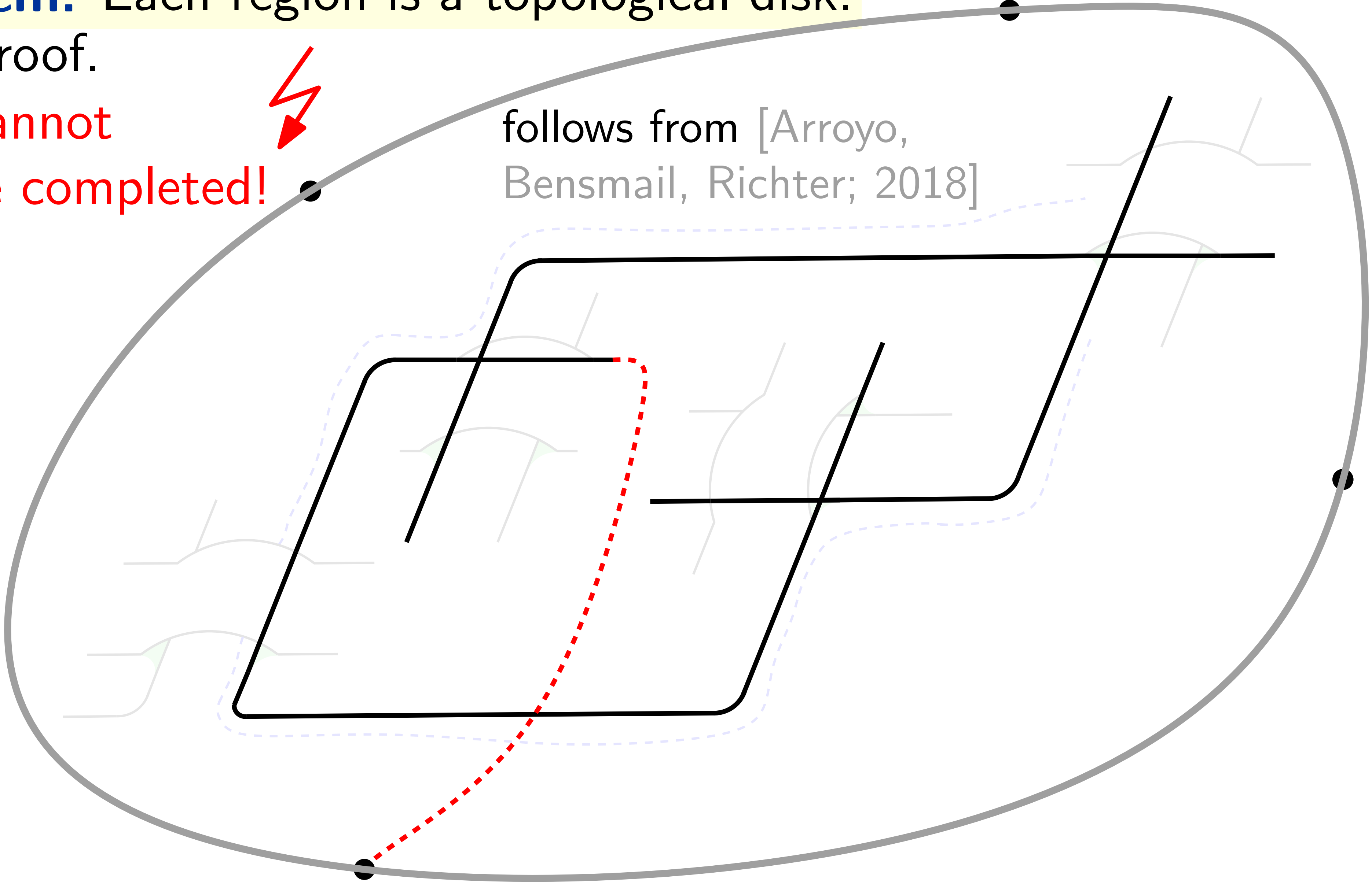

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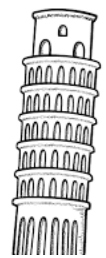 $(k) \times (|V| + |E|)$

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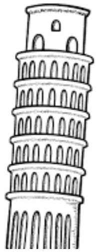 $(k)\times(|V|+|E|)$

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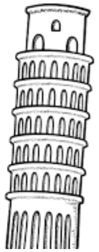

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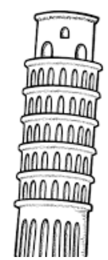 $(k)\times(|V| + |E|)$

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**Question 3**
Is bundle crossing min. also FPT for general simple layouts?