# Matching Points with Rectangles and Squares

Sergey Bereg, Nikolaus Mutsanas & Alexander Wolff

SOFSEM'06

# Outline

- Introduction
  - Matching in graphs and in the plane
  - Already known...
  - Open Problems
- Rectangles
  - General position
  - 1/2-Approximation
- Squares
  - Is there a strong realization?
  - Application to map-labeling
  - NP-Completeness

# Matching in graphs

Maximum Matching
[Micali & Vazirani] $O(\sqrt{n}m)$

Euclidean Minimum-Weight Perfect Matching
[Vaidya] $O(n^{2.5}\log^4 n)$
[Varadarajan & Agarwal] $O((n/\varepsilon^3)\log^6 n)$

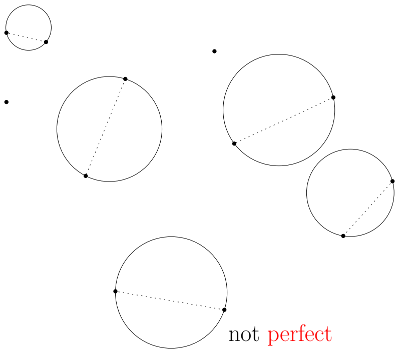
Matching with segments, rectangles, squares, disks...

# Matching in the plane

## Definition

- Matching is perfect: covers all points.
- Matching is strong: no overlap.

# Matching in the plane

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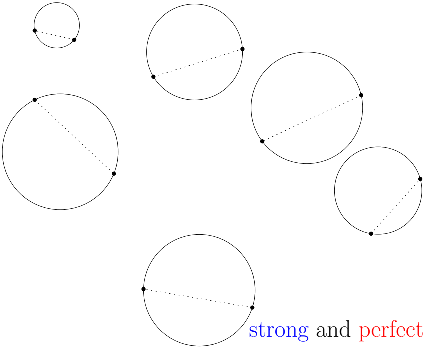


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# Already known...

Let $P$ be a set of $2n$ points in the plane.

**Theorem (Rendl & Woeginger)**

*It is NP-hard to decide whether $P$ admits a strong rectilinear segment matching.*

**Theorem (Ábrego et al.)**

*If $P$ is in general position (no two points on a horiz./vert. line), then $P$ admits*

- *a perfect* disk *matching and a perfect* square *matching.*
- *a strong* disk *matching covering at least 25% of $P$.*
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– How many points can be matched strongly?
– Does a given matching have a strong realization?

| | matching size | ex. strong realization? |
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| segments | 100% | $O(n \log n)$ |
| rectangles | ? | $O(n \log n)$ |
| squares | ? | ? |
| disks | ? | ? |

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| rectangles | 50% | $O(n \log n)$ |
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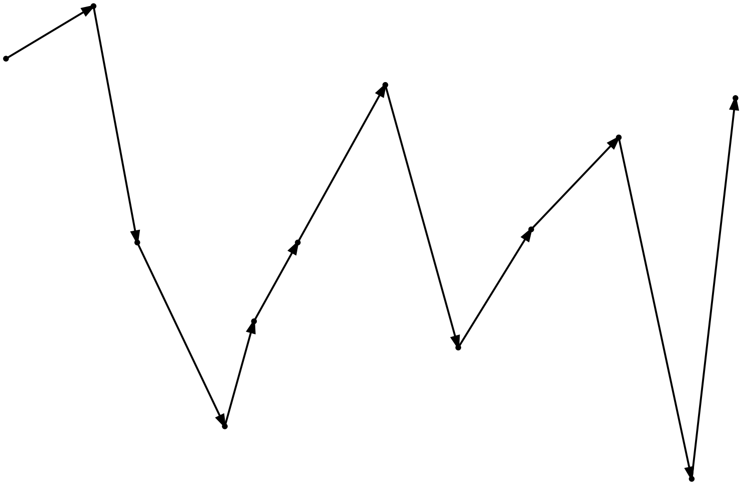

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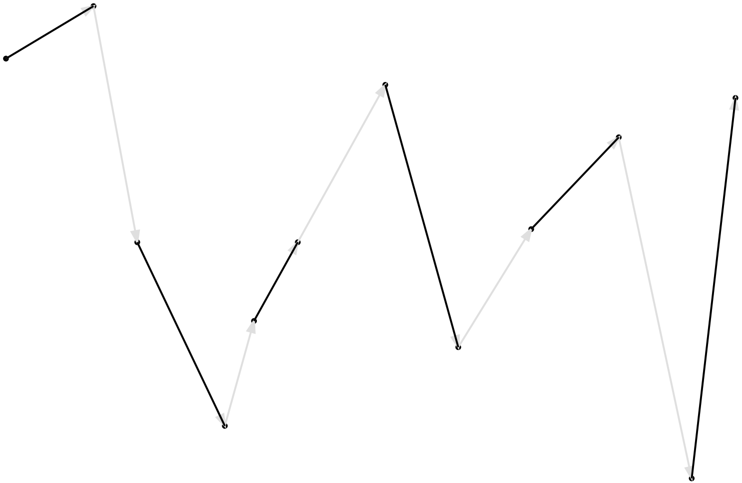

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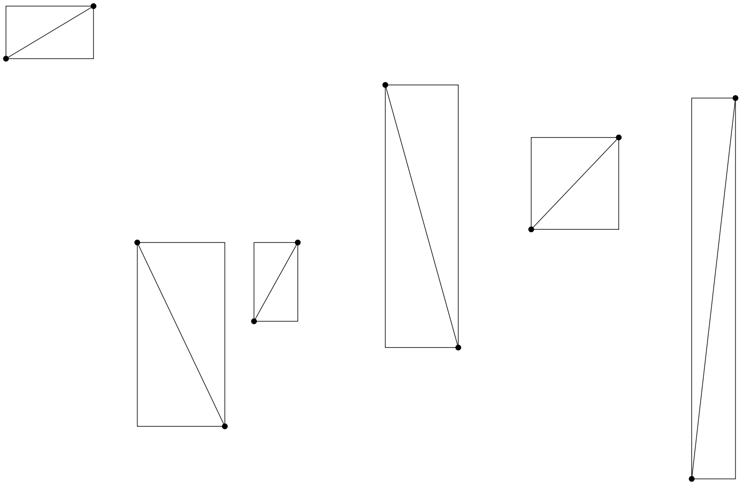

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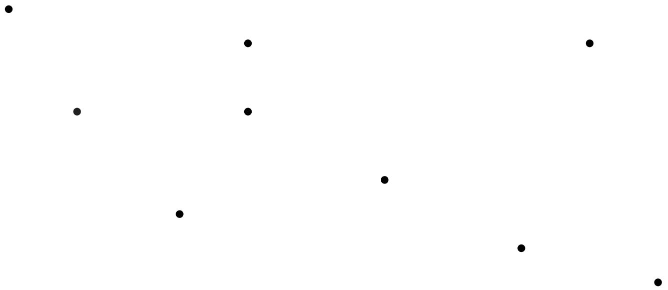

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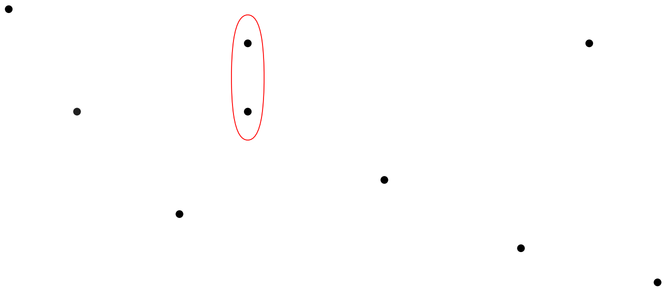

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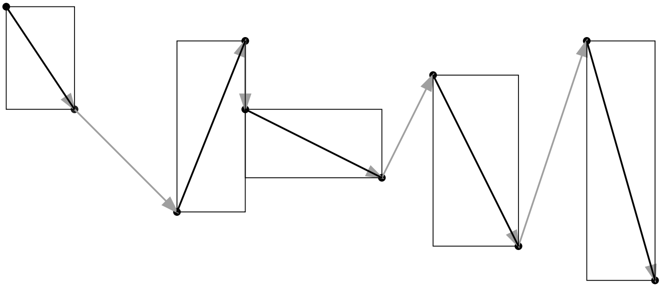

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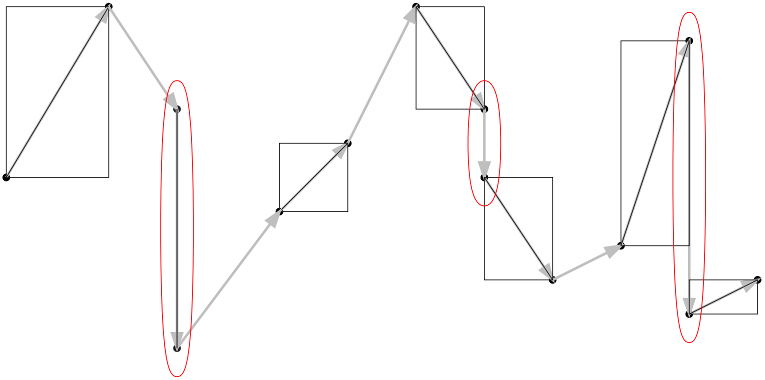

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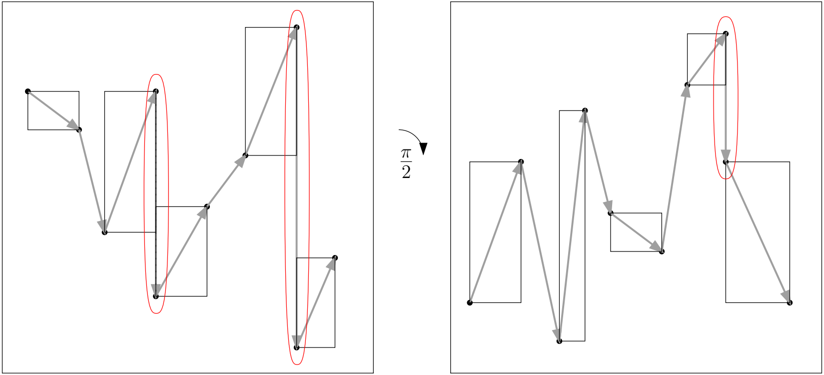

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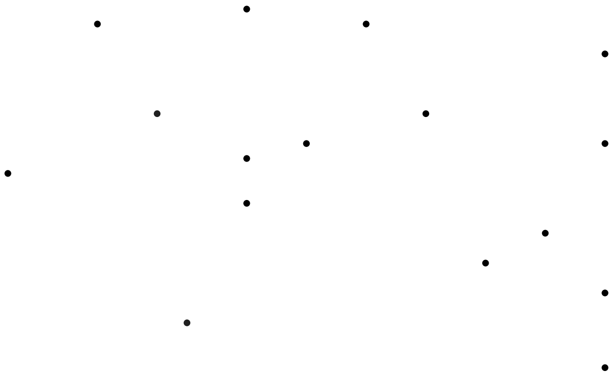
Divide into subsets → match subsets → put together



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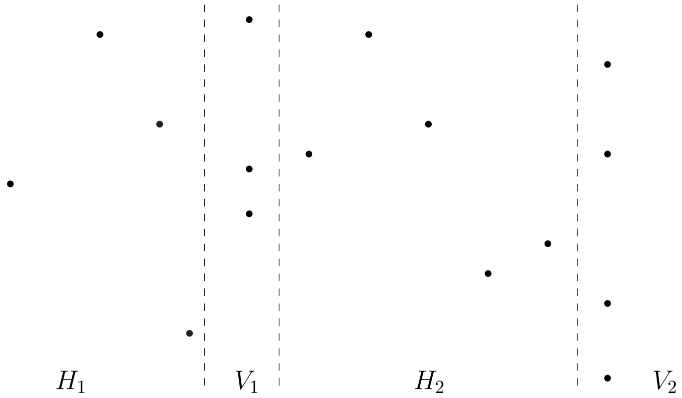

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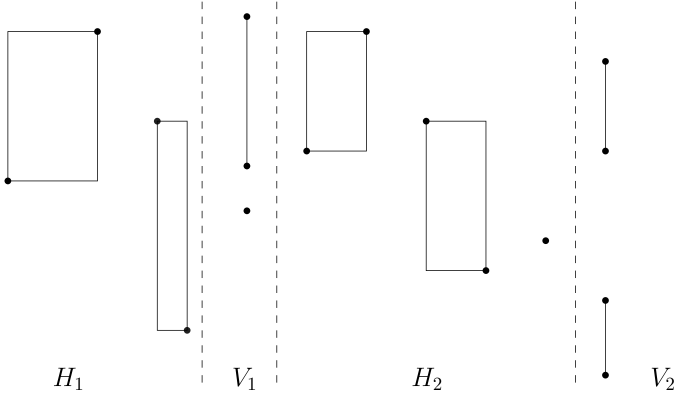

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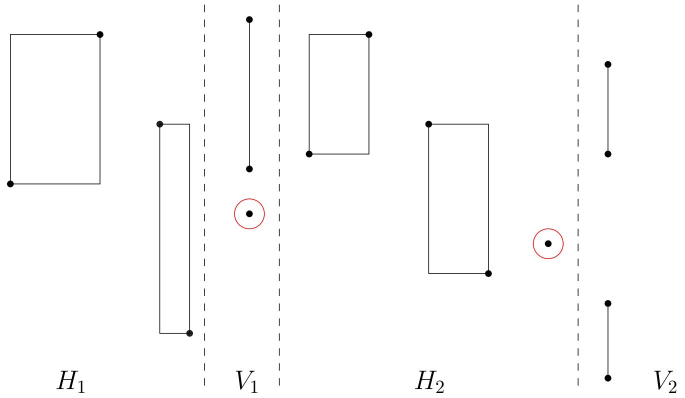

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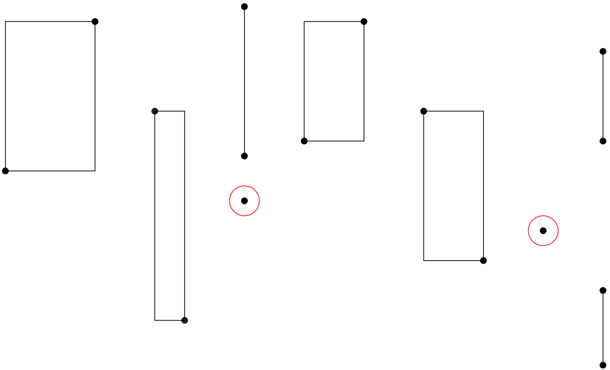

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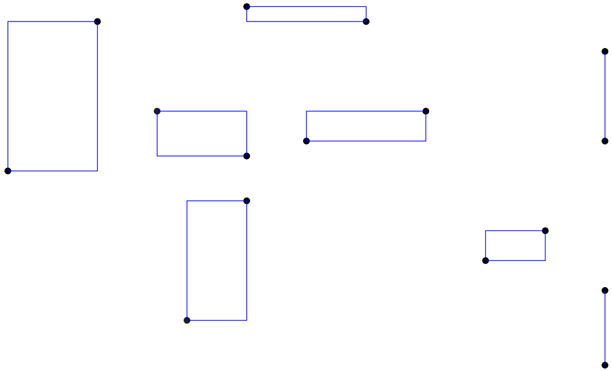

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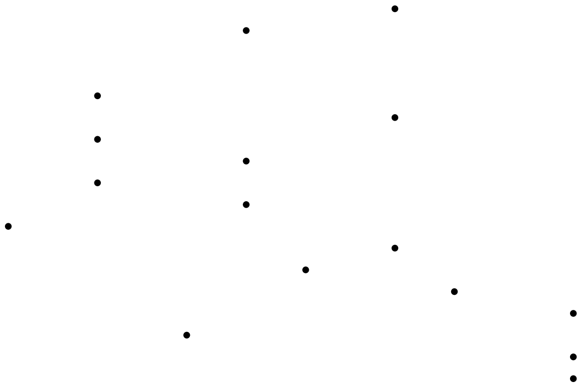
(almost) Worst Case



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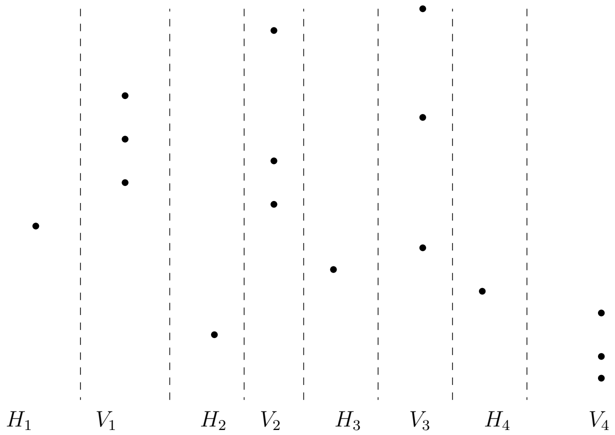

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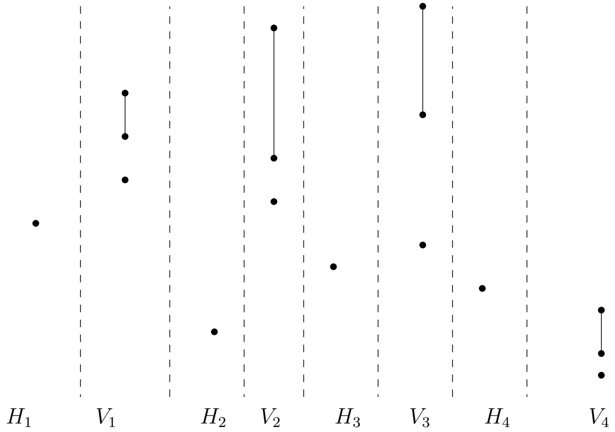

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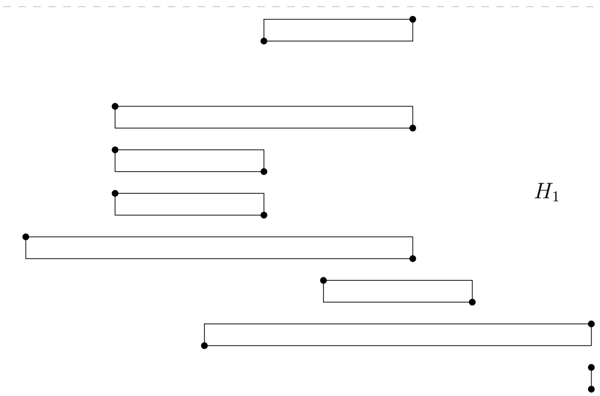

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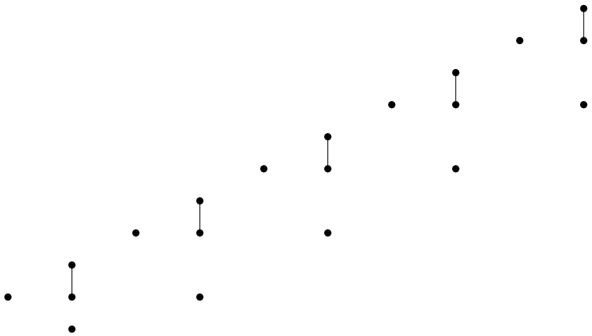

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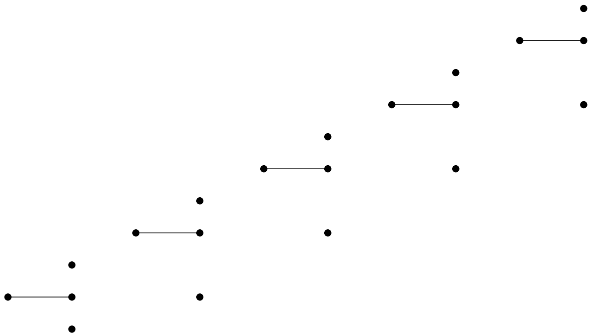

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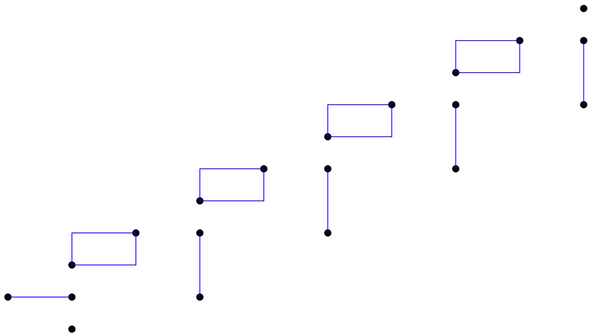
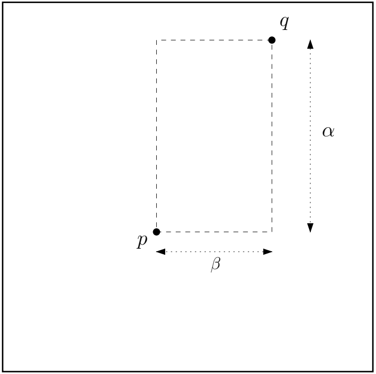


Optimal matching with $n - 2$ points.

# Minimal squares

Minimal squares: points lie on the boundary.



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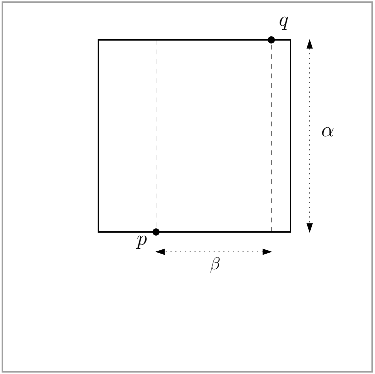
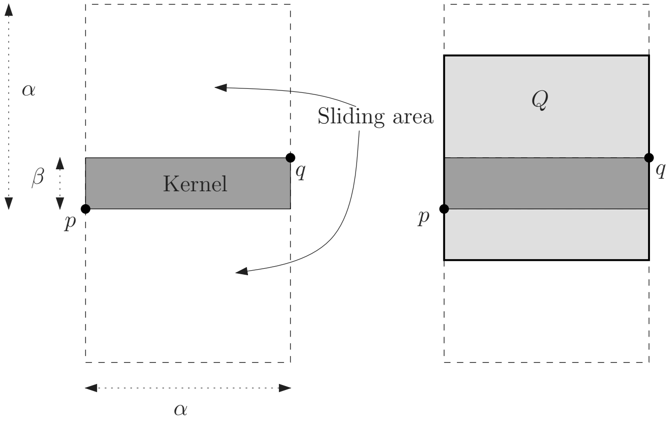


# Sliding squares



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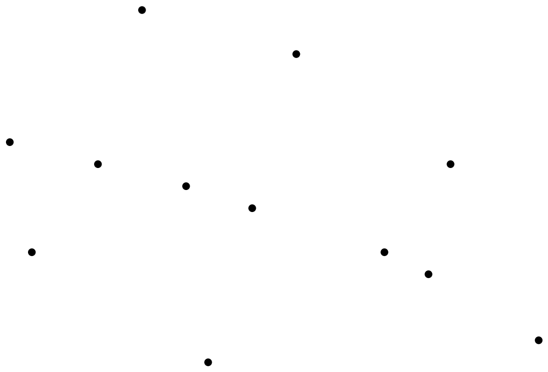

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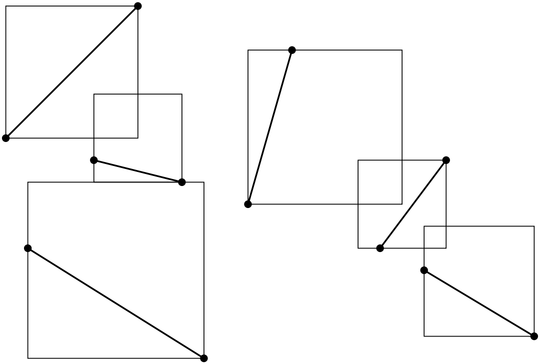

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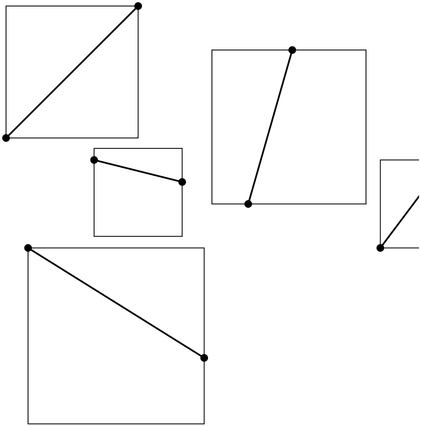

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## Labeling rectilinear segments

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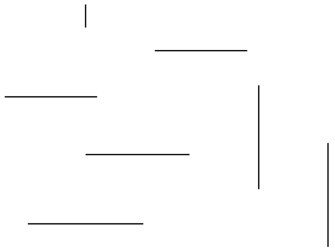

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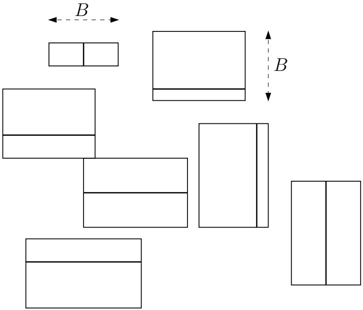

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## Theorem (Kim, Shin & Yang)

*Rectilinear segment labeling is solvable in* $O(n^2 \log n)$

# Squares - canonical form

Let squares slide

- for *vertical* kernels *leftwards* as far as possible.
- for *horizontal* kernels *downwards* as far as possible.

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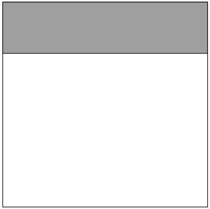

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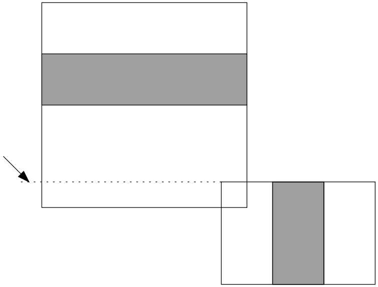

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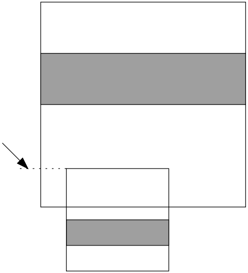

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## Observations

– The resulting positions can be computed in advance.
– Every square has $O(n)$ relevant positions.

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## Problem

Given: $P \subseteq \mathbb{R}^2$, matching $M \subseteq \binom{P}{2}$
Question: Is there a strong square realization of $M$?

- Do kernels overlap?
- Calculate relevant positions.
- Solve decision problem with 2-SAT.

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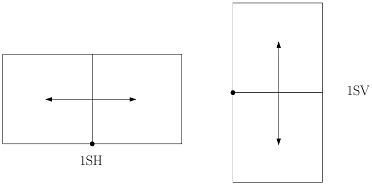
## Conclusion

The decision problem can be solved in $O(n^2 \log n)$.

# Labeling points with sliding labels

Labeling points with sliding labels:




- Labels also below / to the left of a point.
- Variable label sizes.
- *Sliding area can be shortened*.

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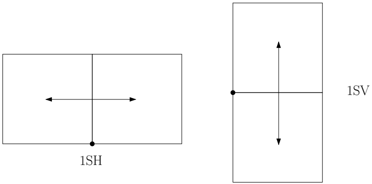


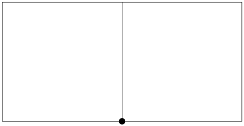

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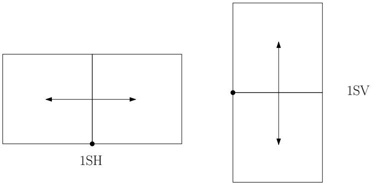


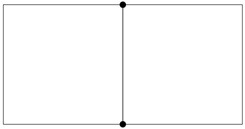

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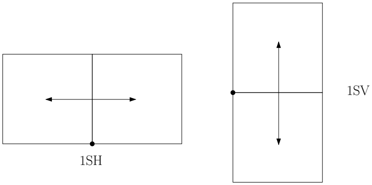


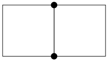

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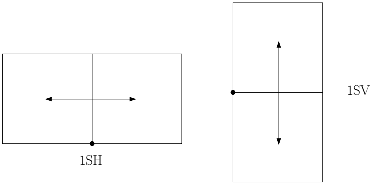


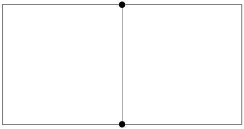

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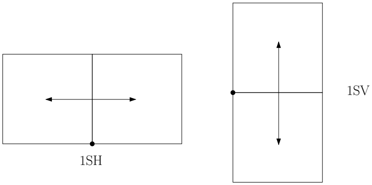


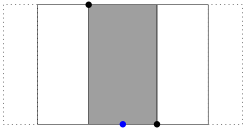

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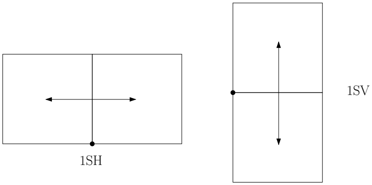


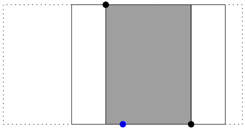

# NP-Completeness

**ESPSM**

Given: Point set $P \subseteq \mathbb{R}^2$
Question: Does a strong perfect square-matching exist?

**Theorem (Bereg, Mutsanas & Wolff '05)**

ESPSM *is NP-hard.*

**Proof.**

By reduction from PLANAR 3-SAT to ESPSM. □

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**ESPSM**

Given: Point set $P \subseteq \mathbb{R}^2$
Question: Does a strong perfect square-matching exist?

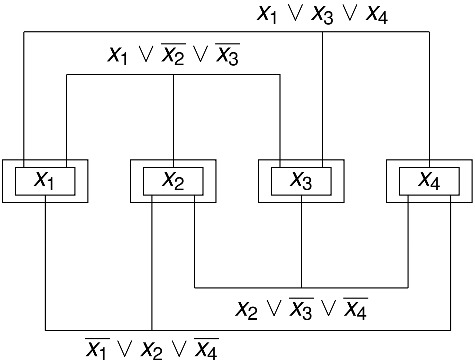
**Theorem (Bereg, Mutsanas & Wolff '05)**

ESPSM *is NP-hard.*

**Proof.**

By reduction from PLANAR 3-SAT to ESPSM. □

# Outline of the Reduction




Input: planar 3-SAT formula $\varphi =$
$(x_1 \vee x_3 \vee x_4) \wedge (x_1 \vee \overline{x_2} \vee \overline{x_3}) \wedge \dots$

Goal: Point set $P \subseteq \mathbb{R}^2$ with:
$P$ admits s. p. square-matching $\Leftrightarrow$ $\varphi$ satisfiable.

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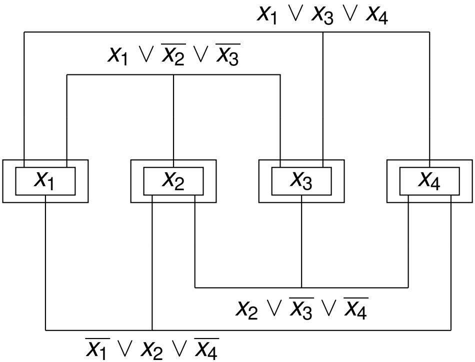


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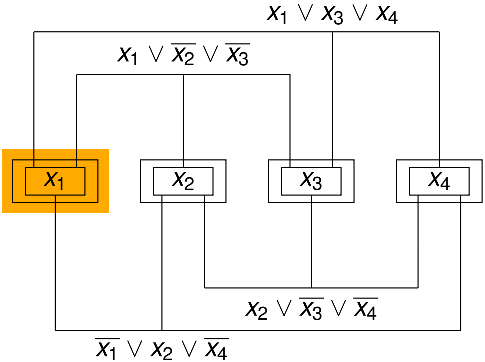


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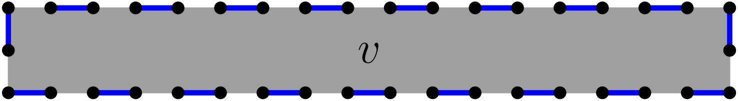
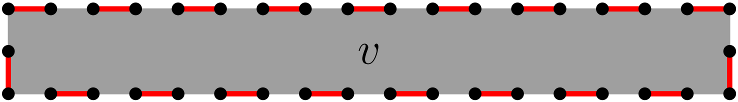
# Variable Gadget




$v =$ true

# Variable Gadget

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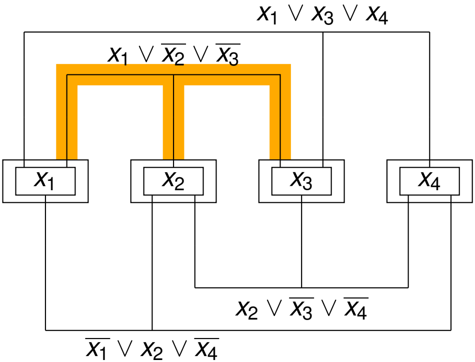


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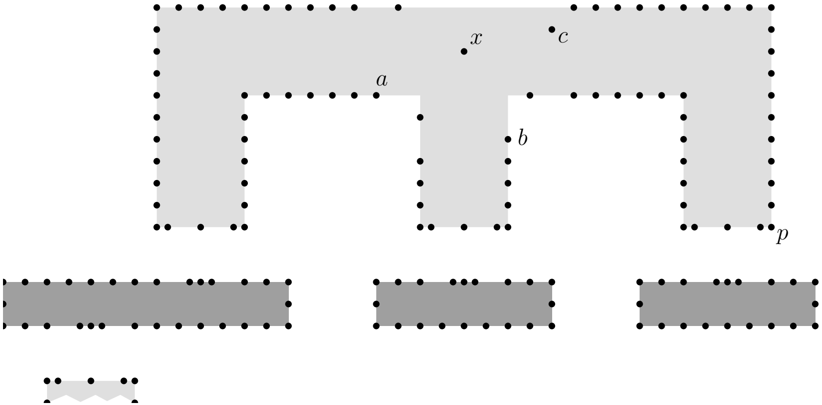
# Clause Gadget



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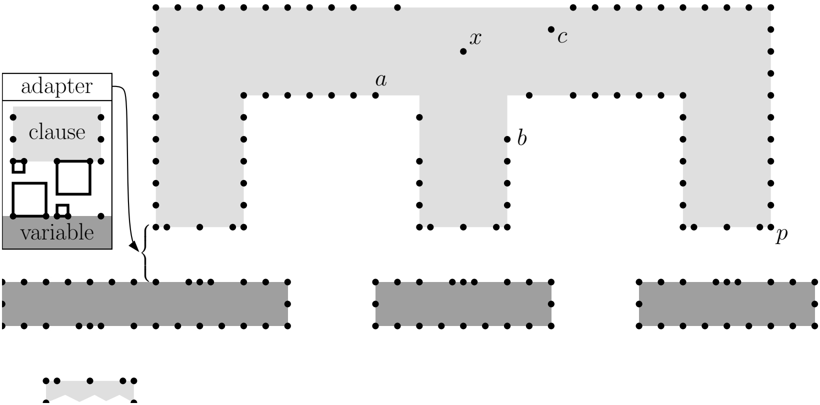

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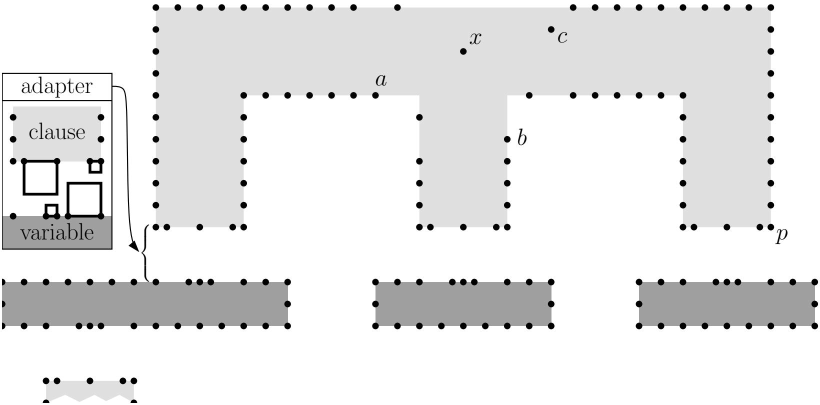

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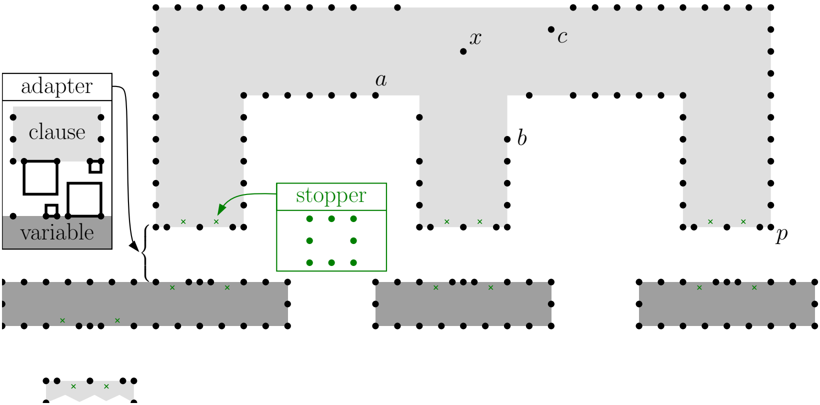

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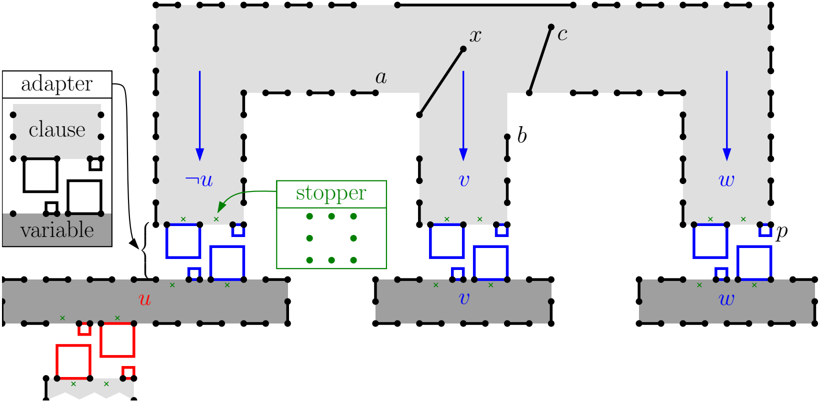

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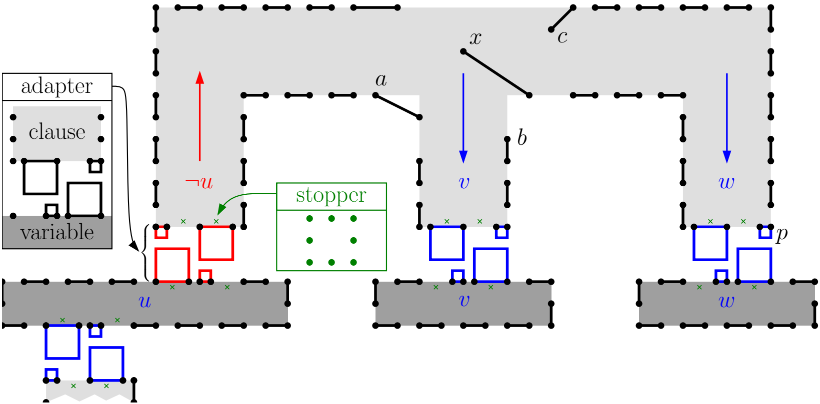

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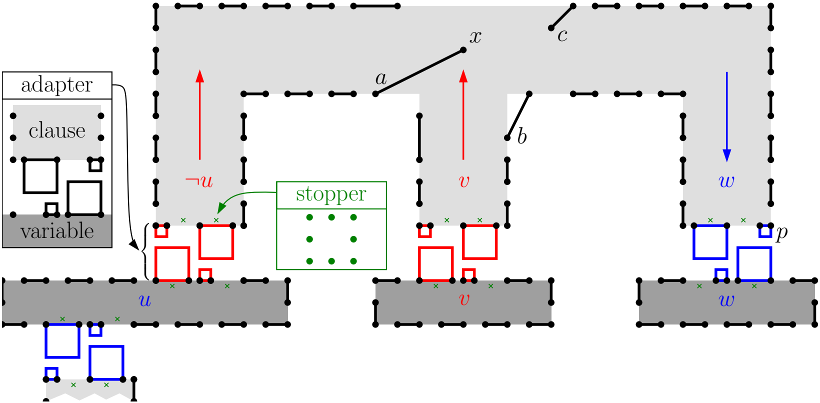

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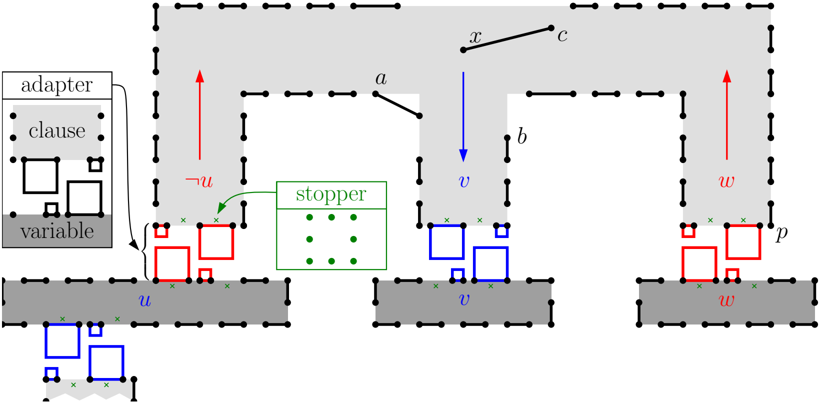

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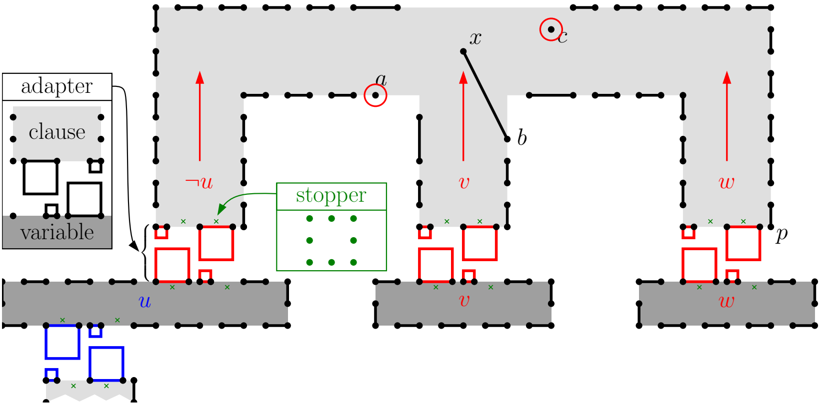

Thank you for your attention!