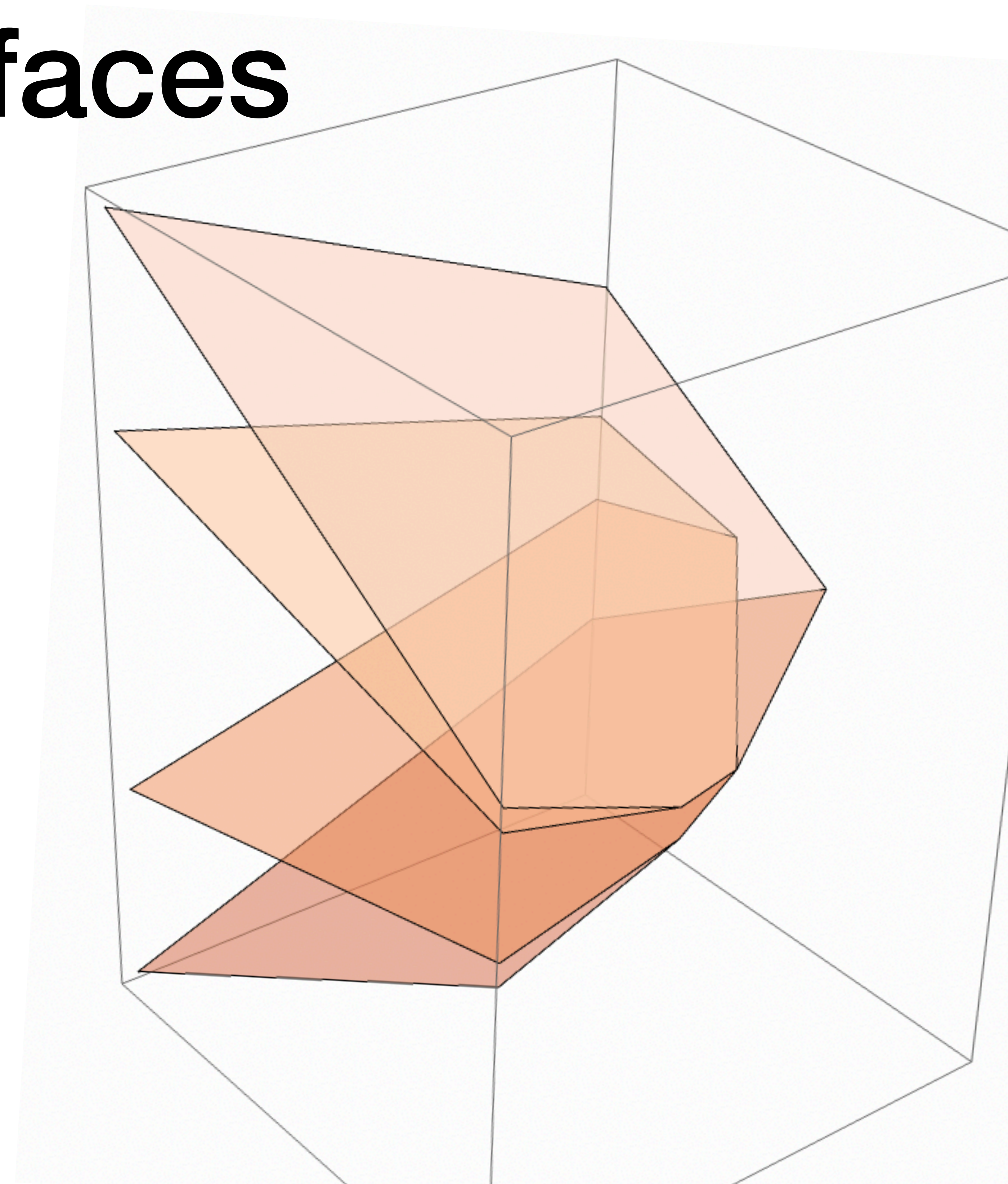
Adjacency Graphs of Polyhedral Surfaces



The 37th International Symposium on Computational Geometry

Elena Arseneva
Linda Kleist
Boris Klemz
Maarten Löffler
André Schulz
Birgit Vogtenhuber
Alexander Wolff

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- **given:** polyhedral surface in 3d
- **adjacency graph of the surface:**
  - vertices are the polygons
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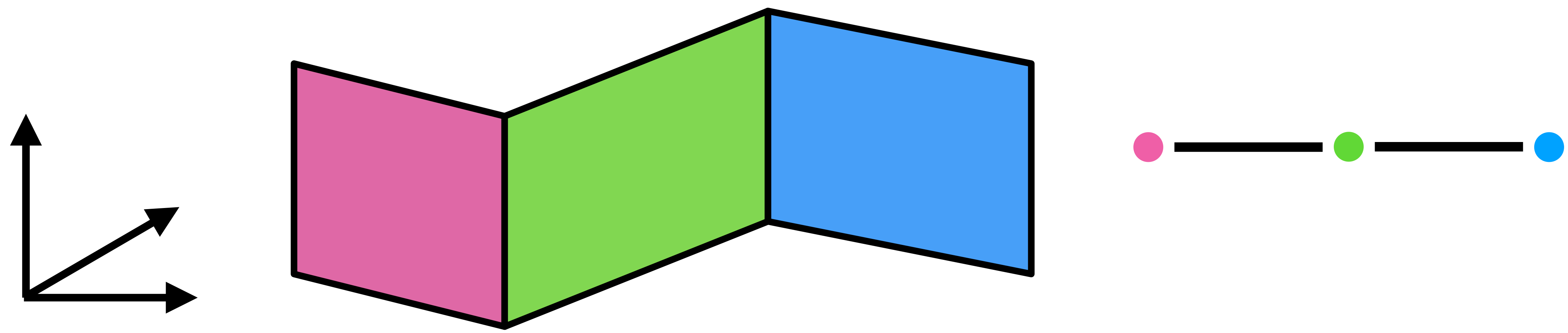

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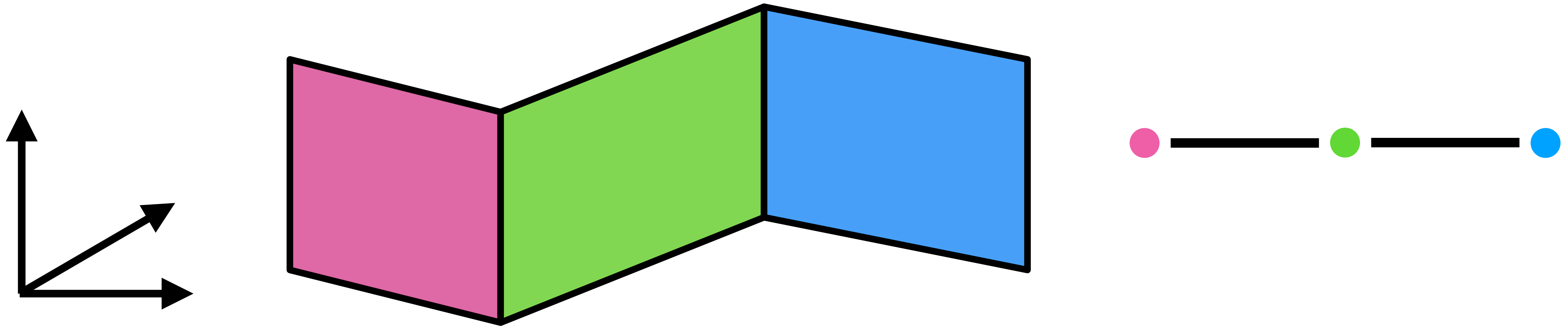

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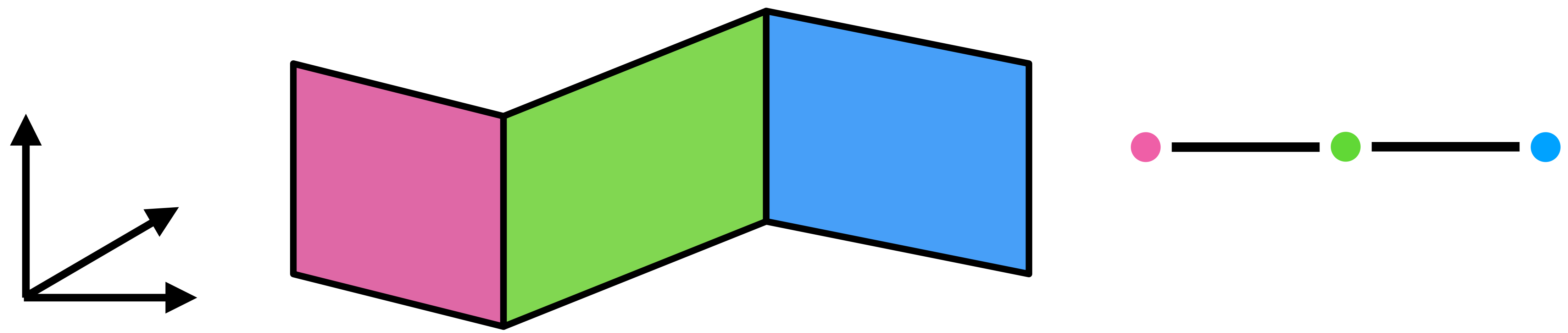

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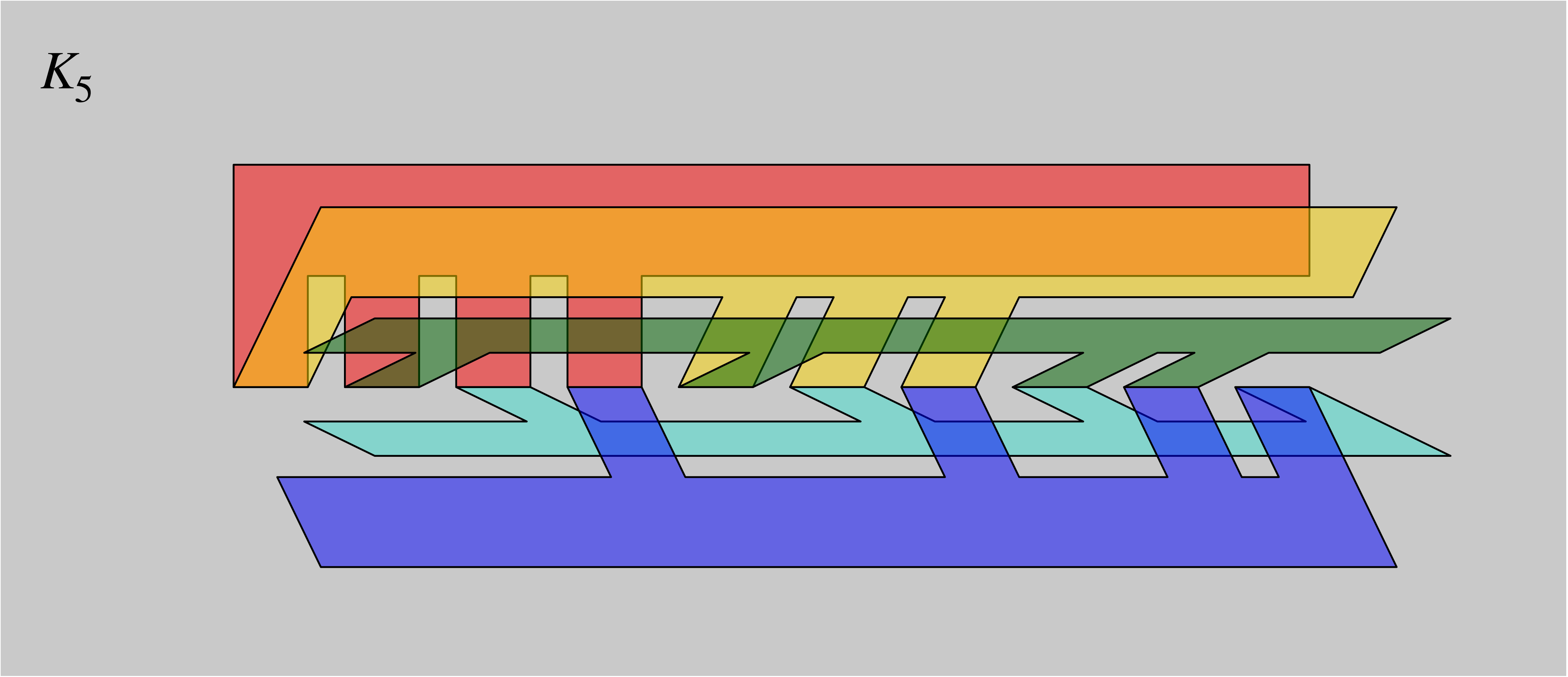
Q: What kind of graphs are possible?
(or: Which graphs have a realization?)

The Nonconvex Case



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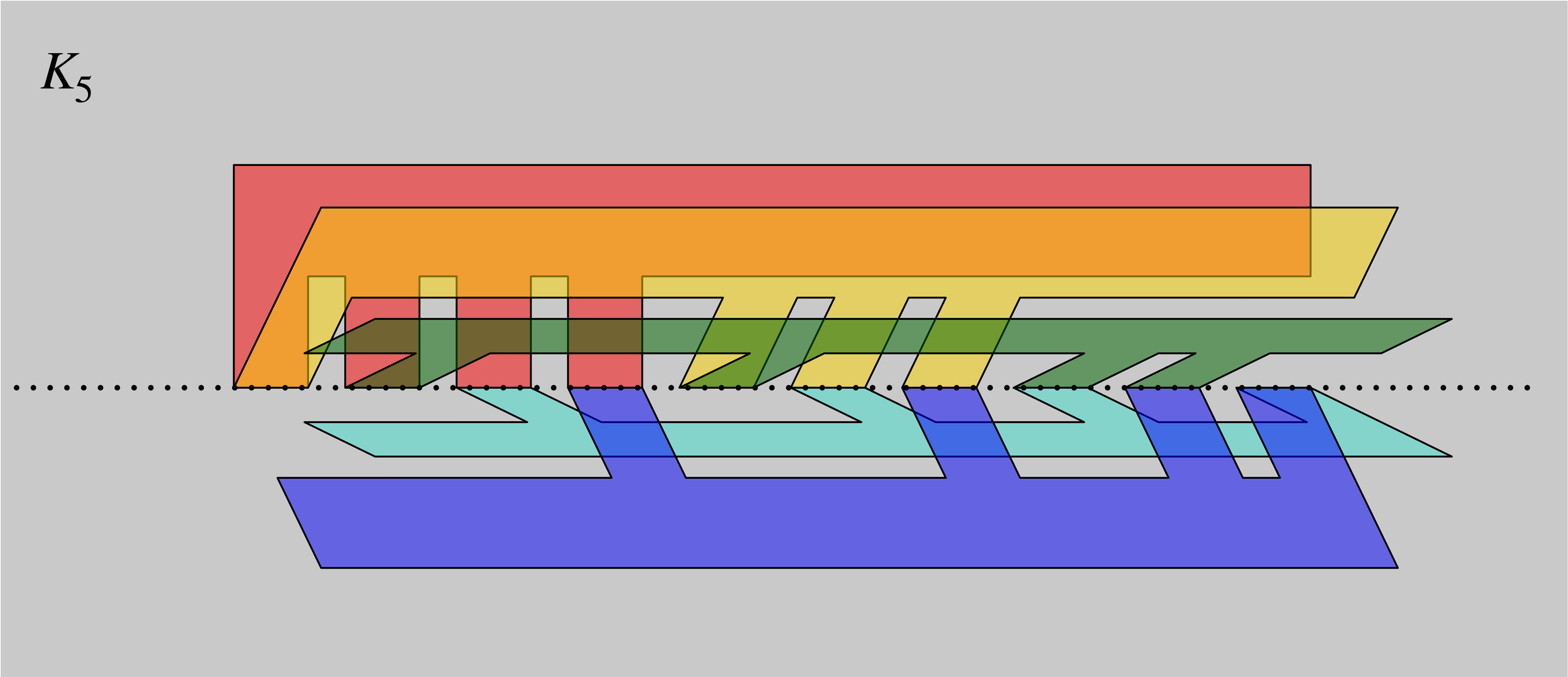

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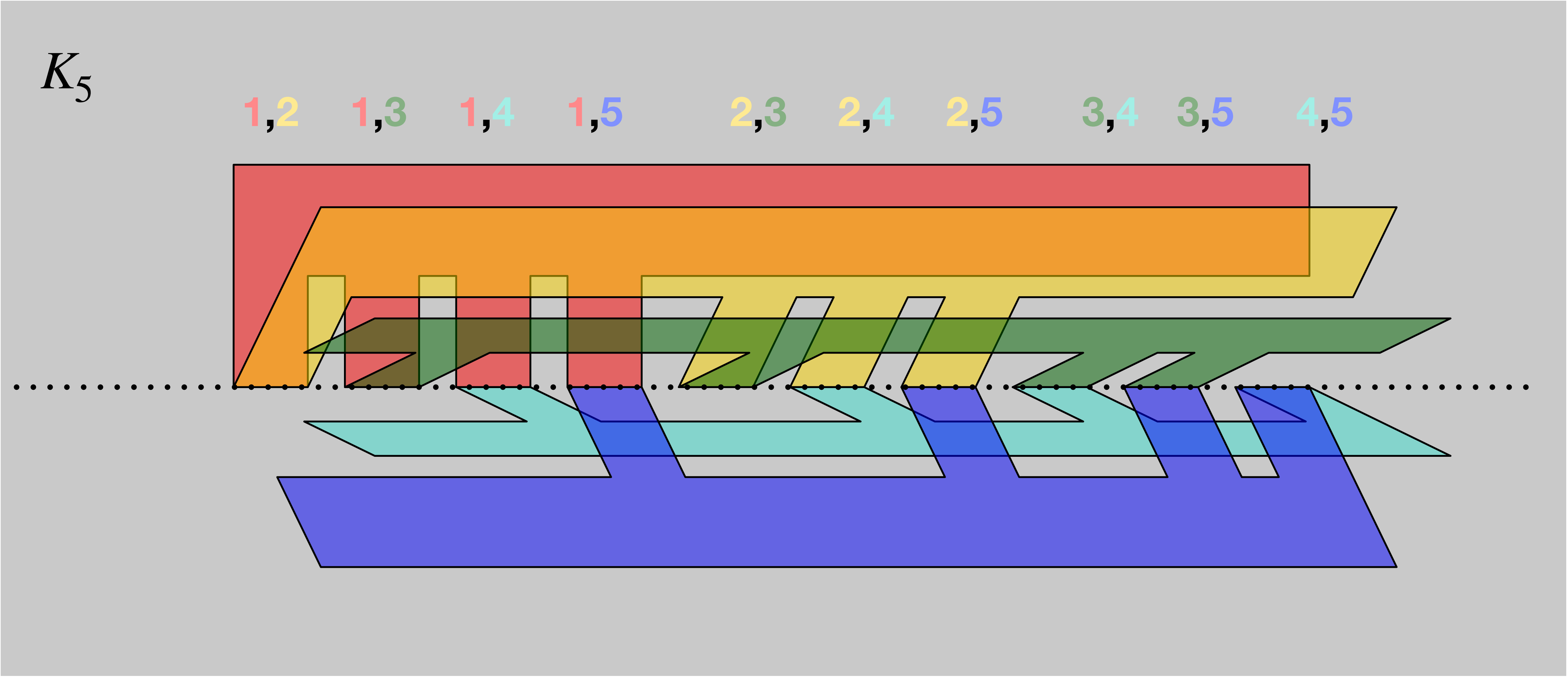

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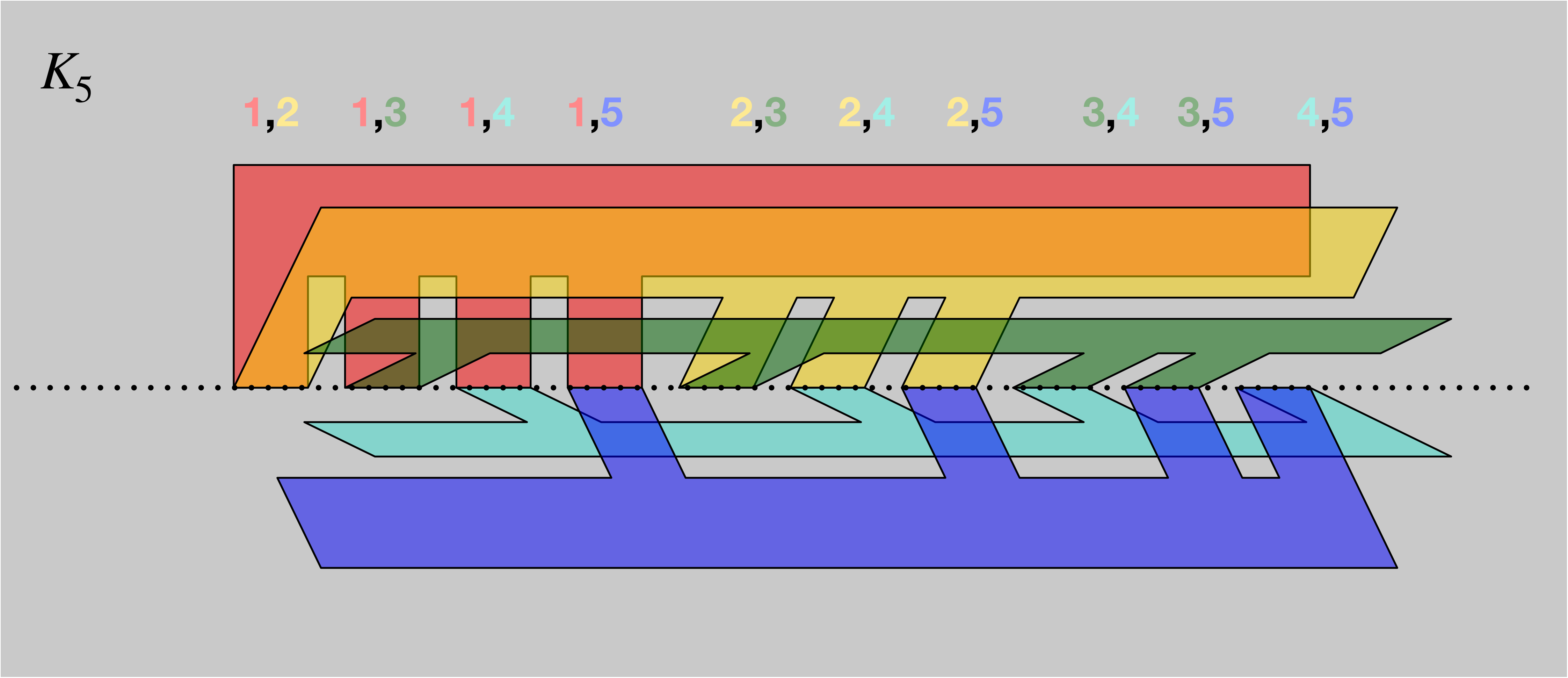

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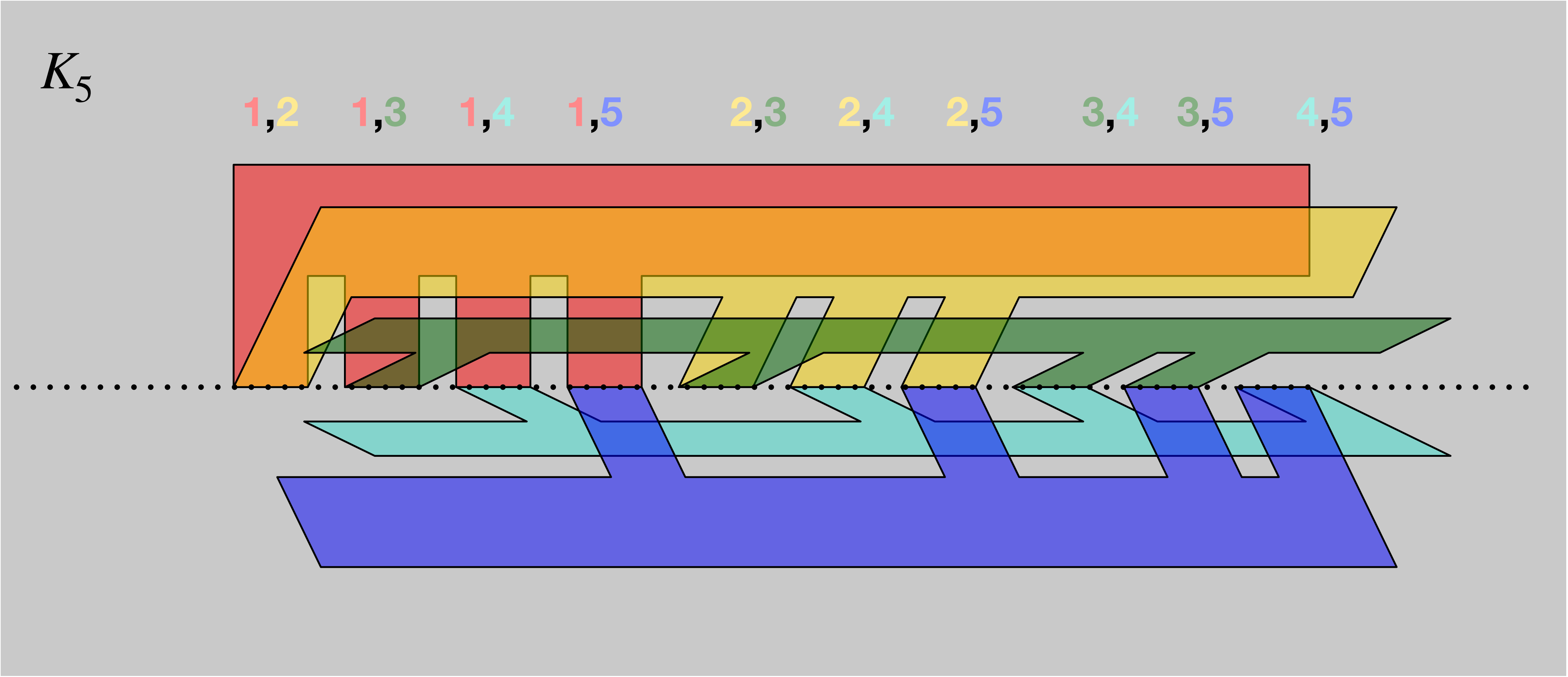

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(From now on convex polygons only!)

The Convex Case – Planar Graphs

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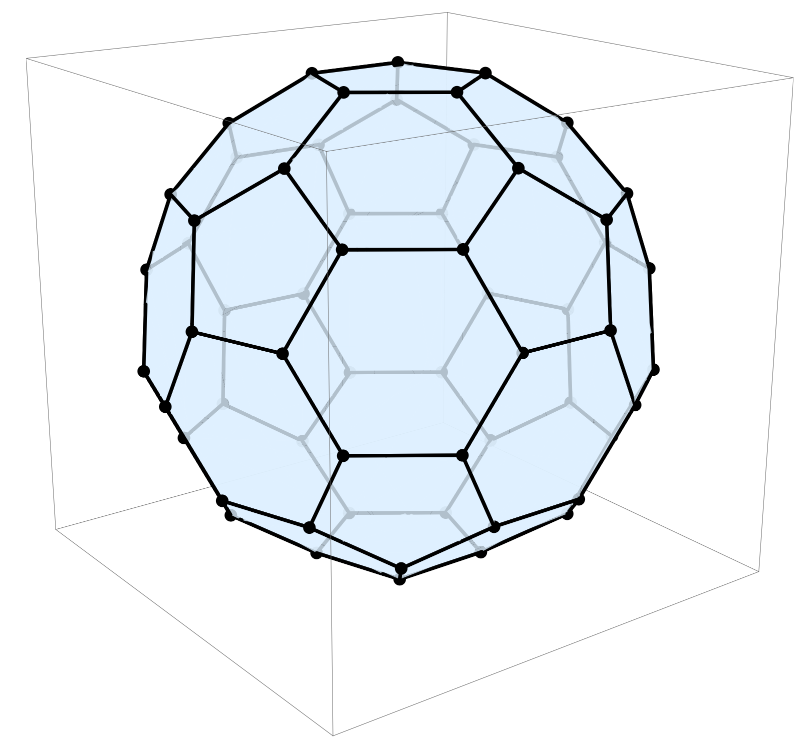
Steinitz's Theorem:
The 3-connected planar graphs are exactly the skeletons of convex polyhedra.



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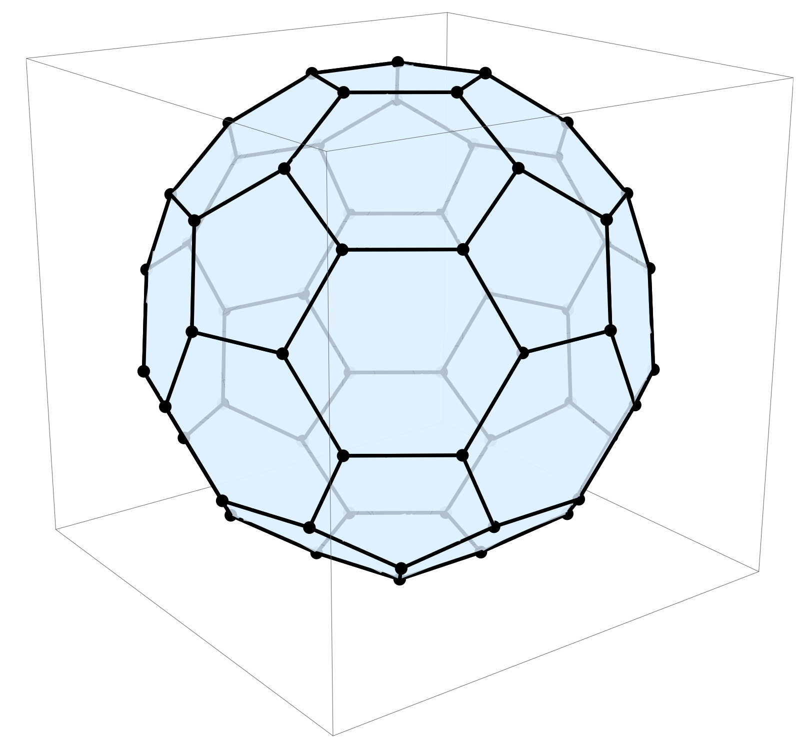

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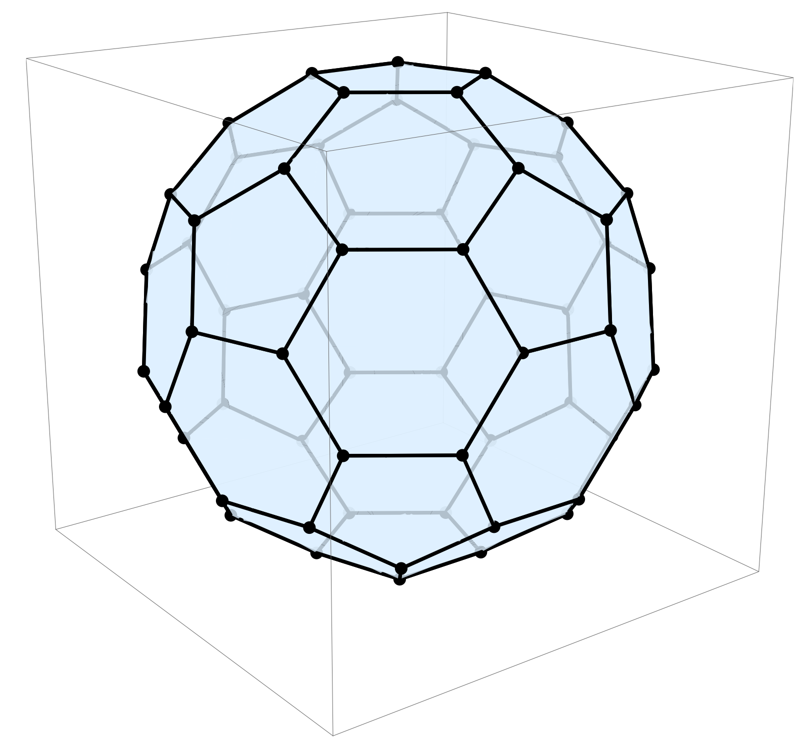

The dual of a 3-connected planar graph is again a 3-connected planar graph.

All planar graphs have a representation!

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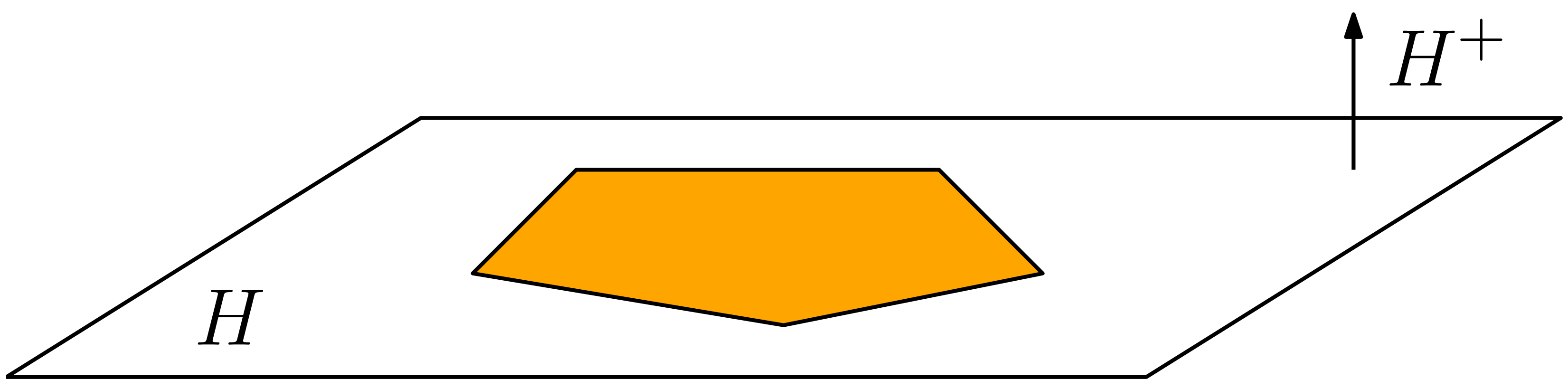

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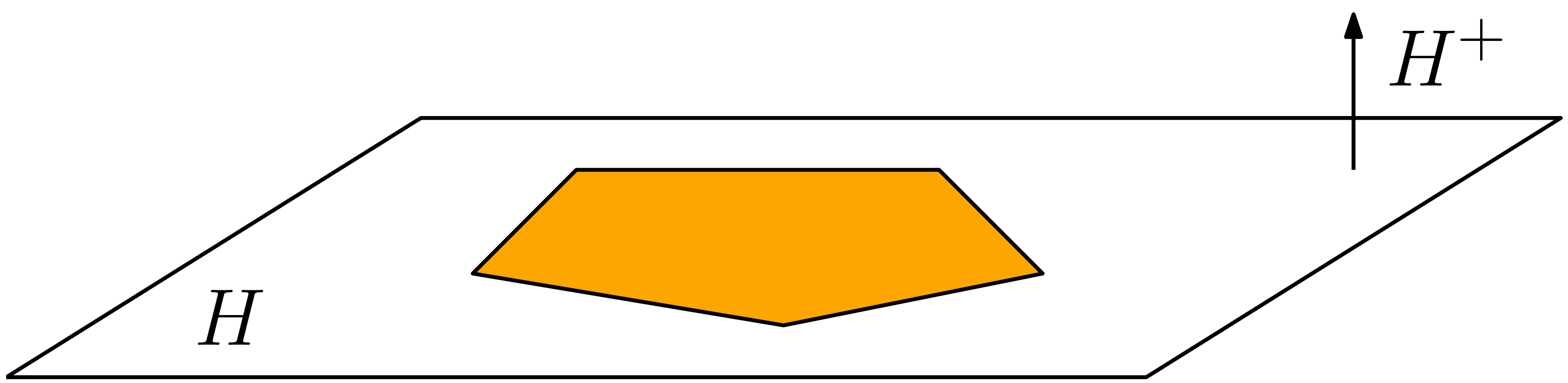

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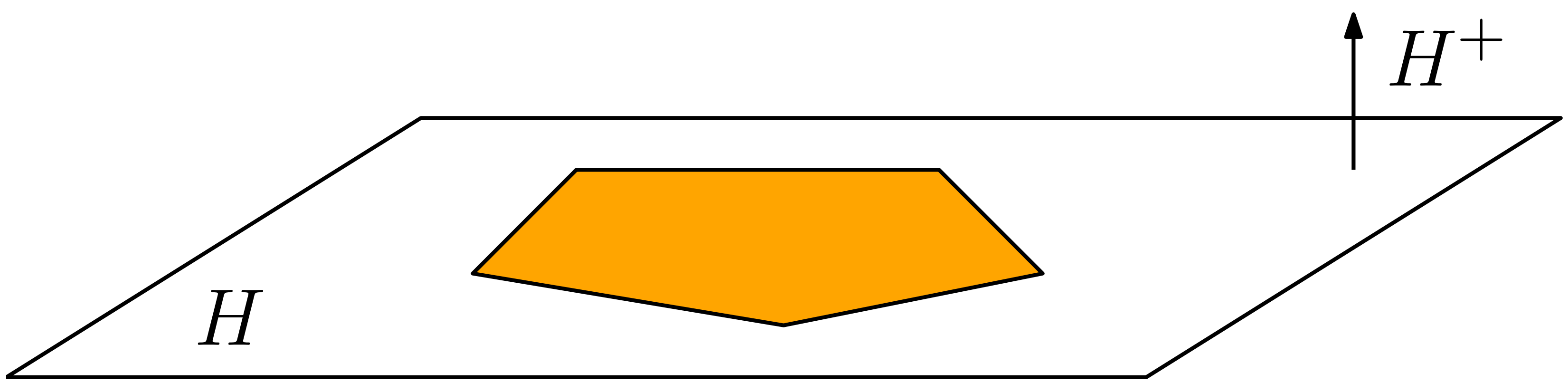

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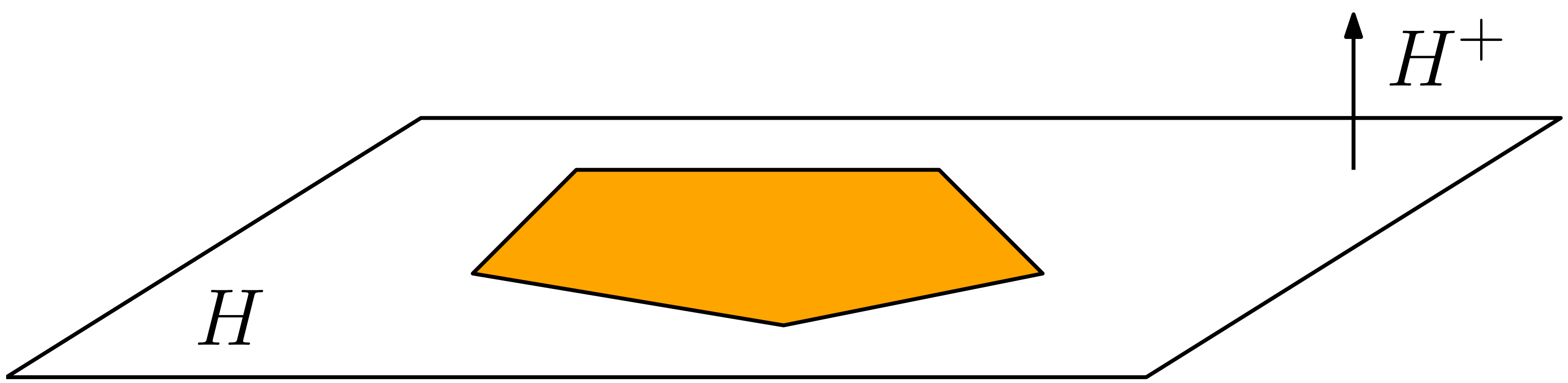

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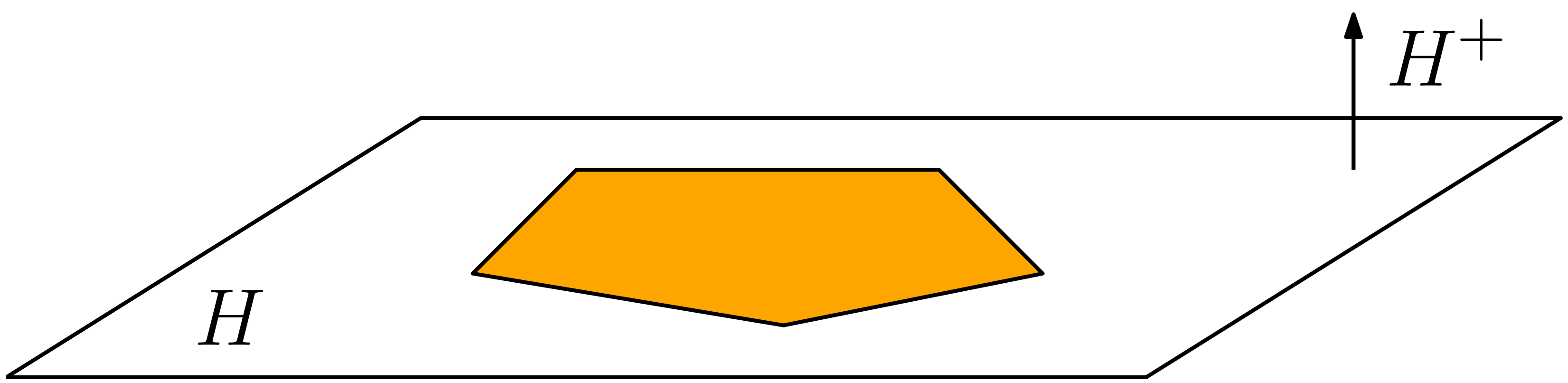
Impossible



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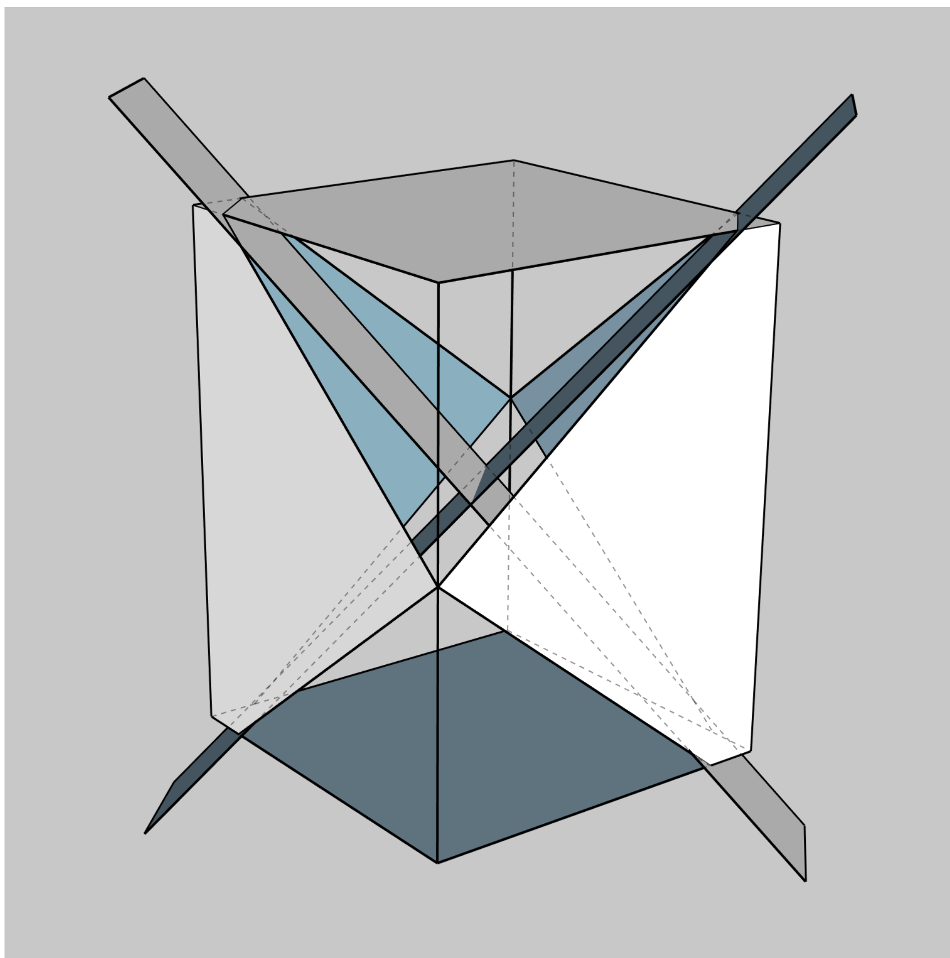

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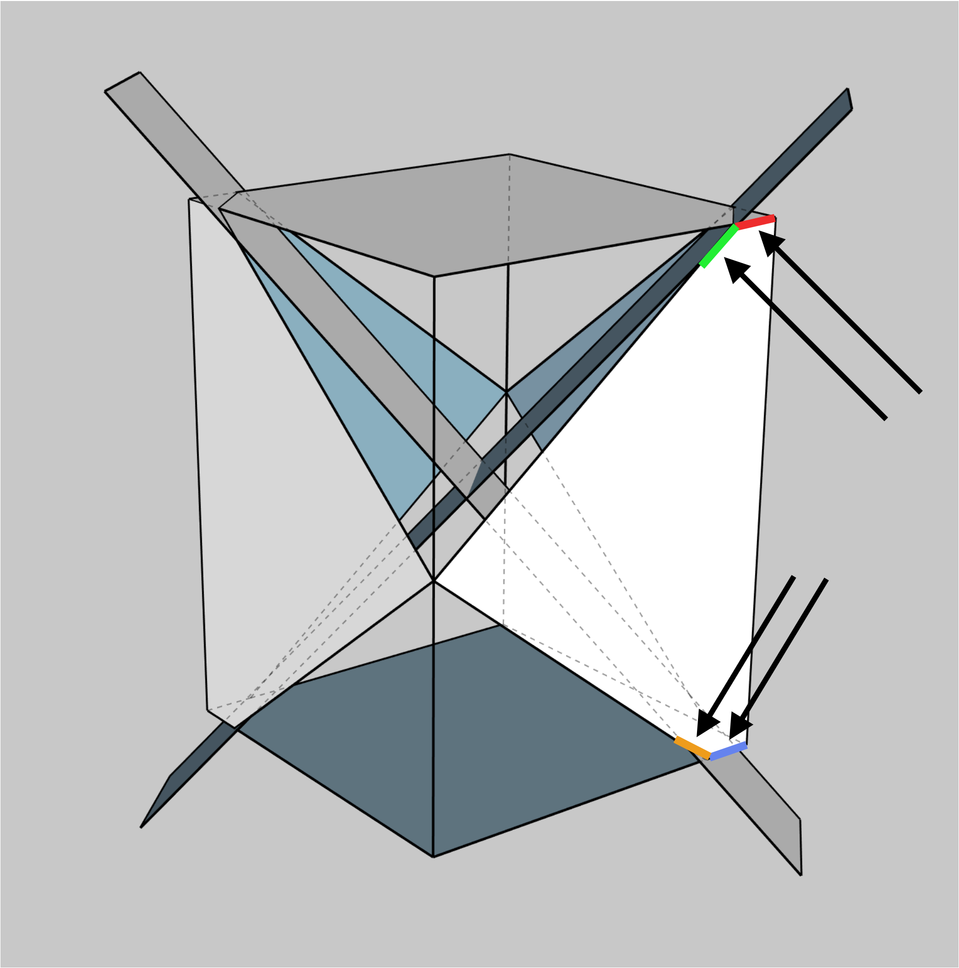

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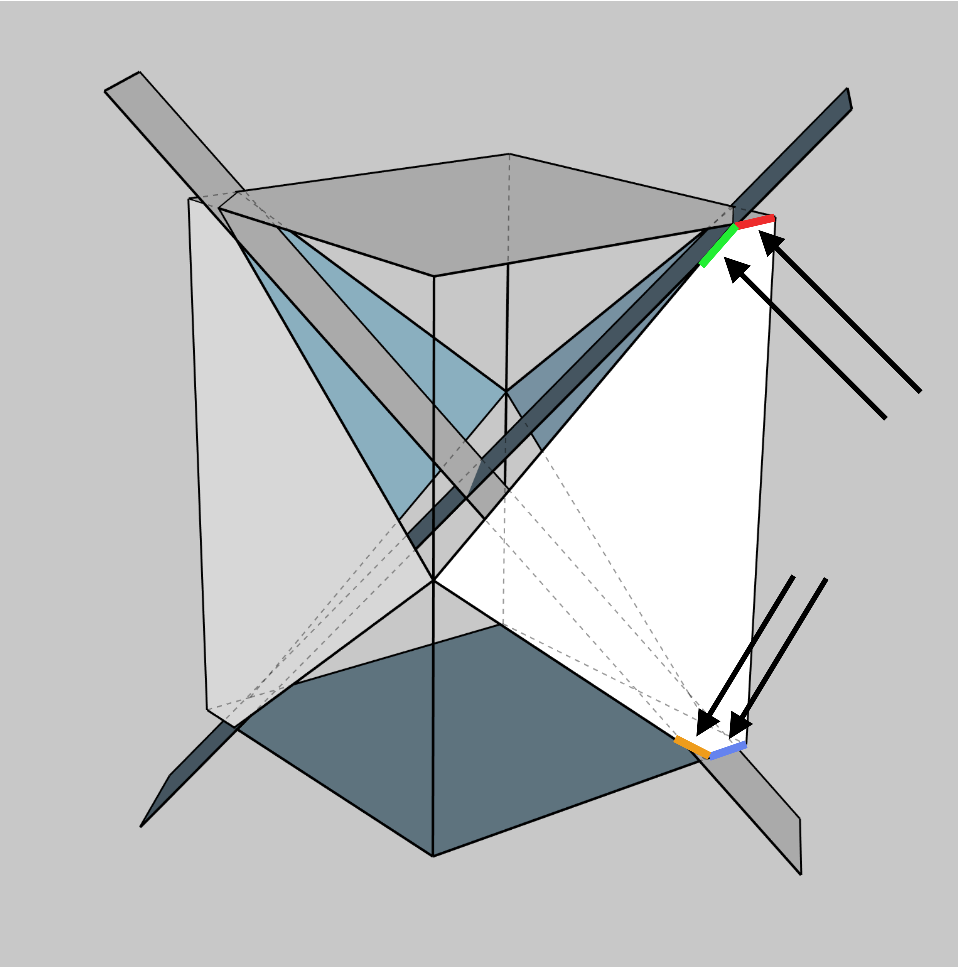

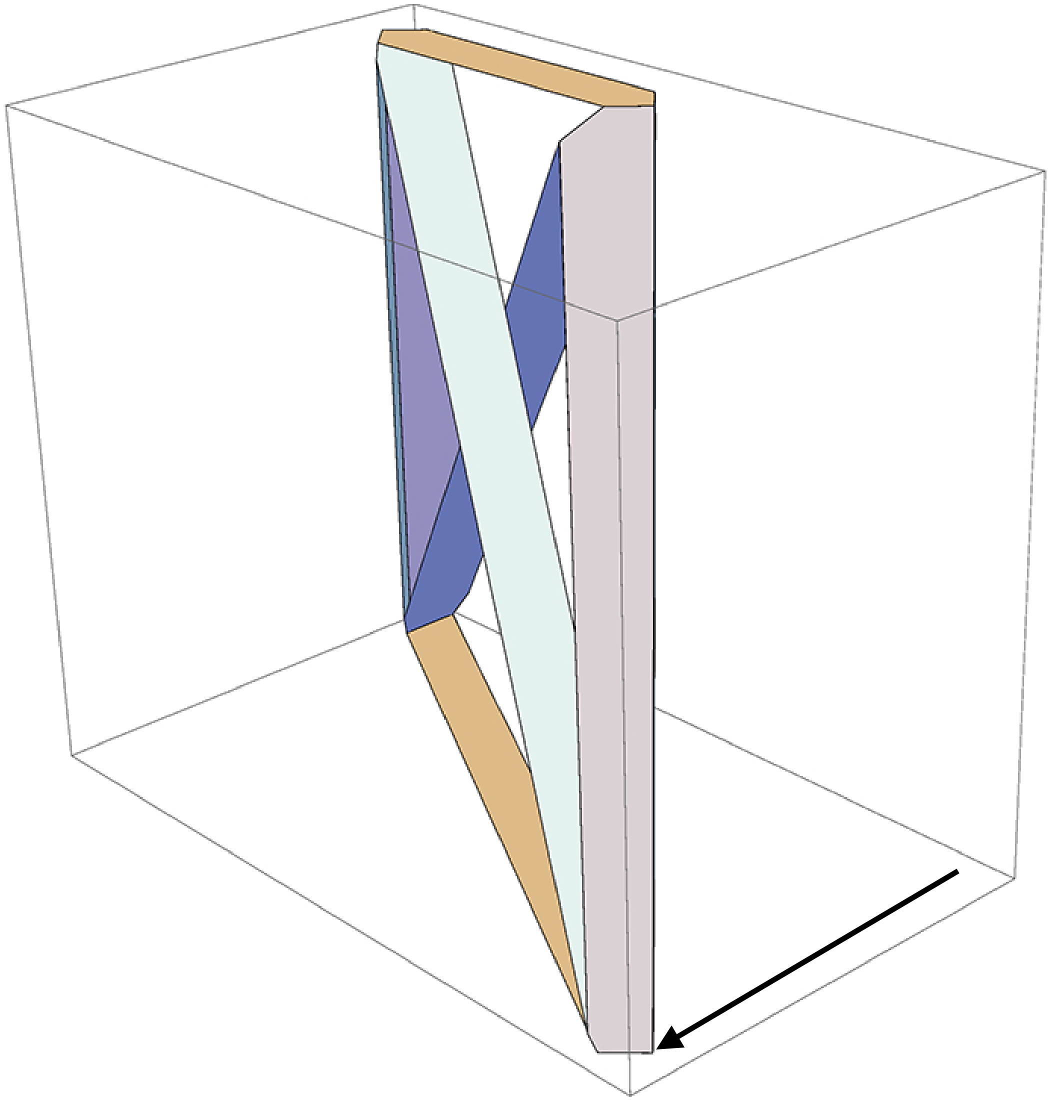

Hypercubes

Every graph of a *d*-hypercube has a realization with convex polygons.

by a construction of McMullen, Schulz, Wills 1983

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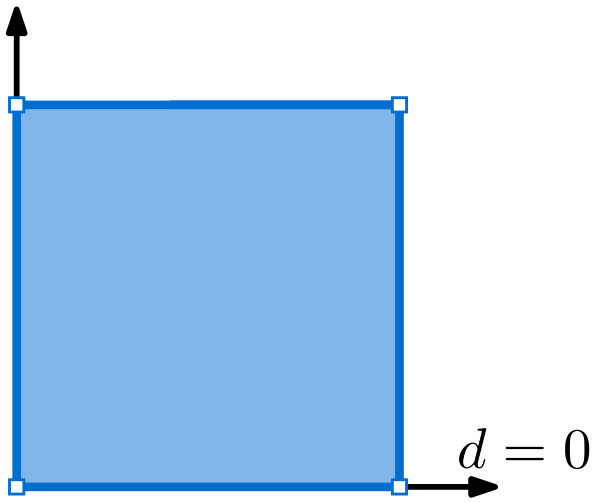
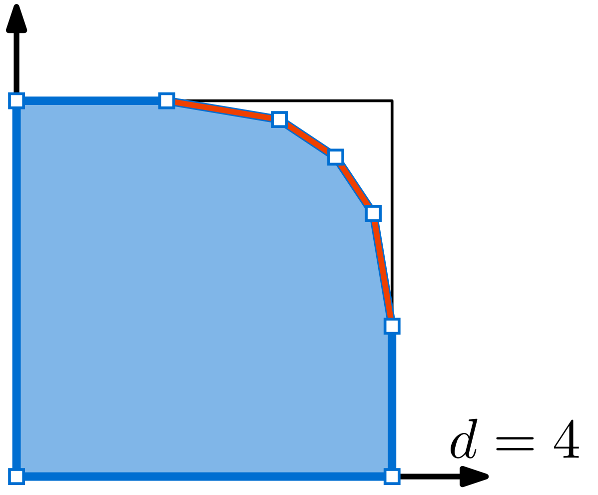
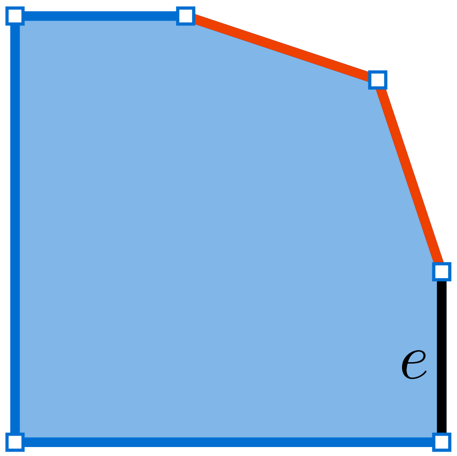


Incremental Construction with these invariants

- every face is a convex (*d*+4)-gon
- the projection in the xy-plane looks almost the same
  (only the convex chains differ)
- red edges have already 2 incident faces



Incremental Step



shear and translate such that for all polygons only the edge e lies below the xy-plane

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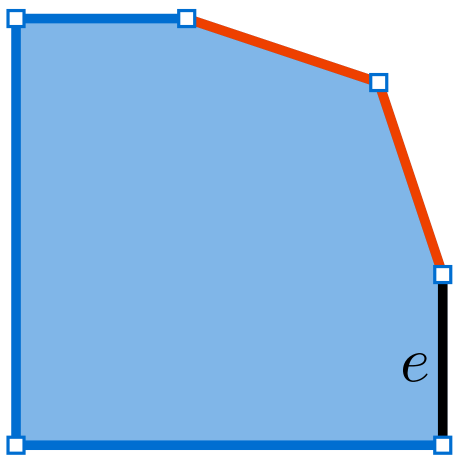

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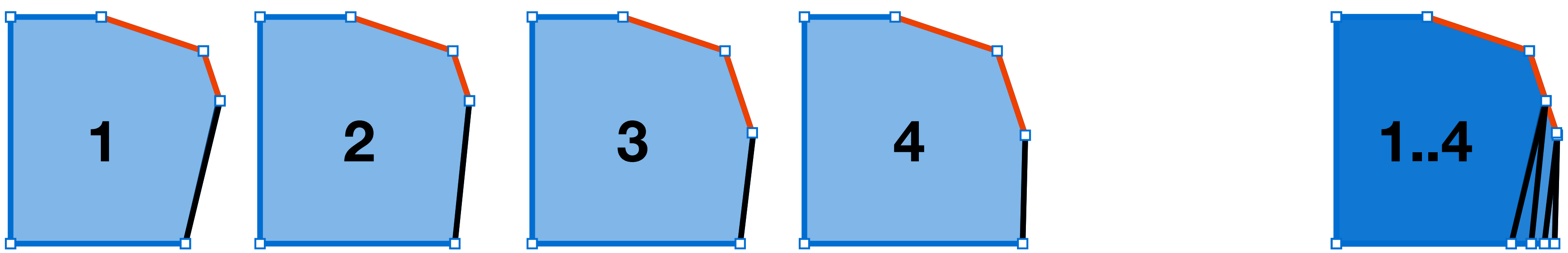

cut with xy-plane

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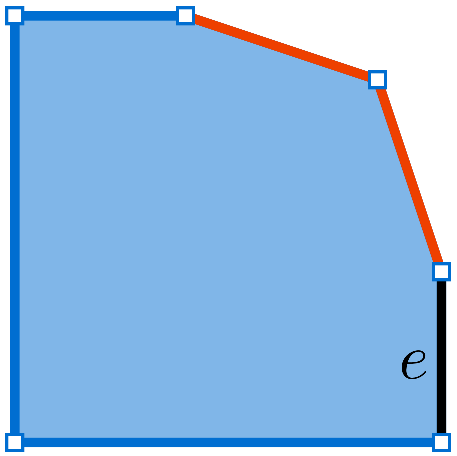

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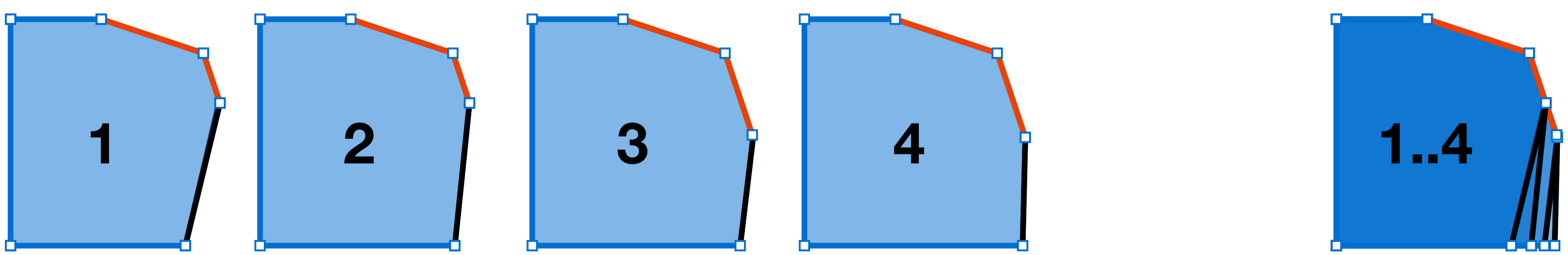

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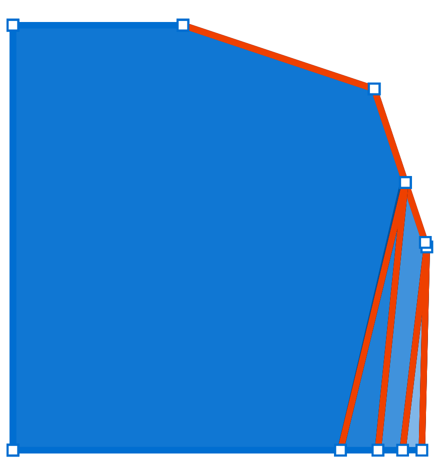
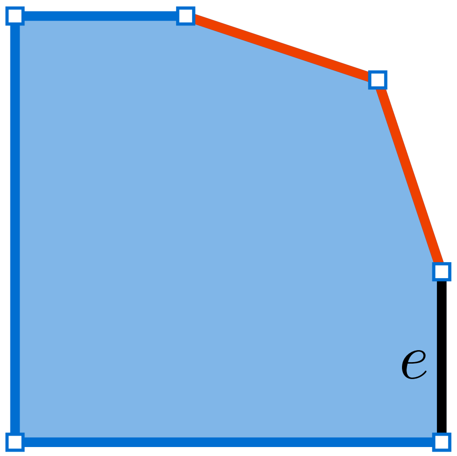


copy and reflect on xy-plane, then glue together

Incremental Step



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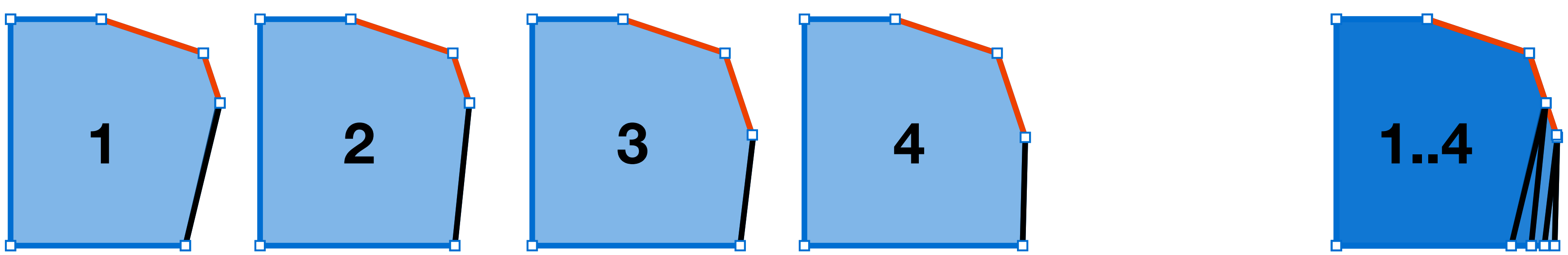

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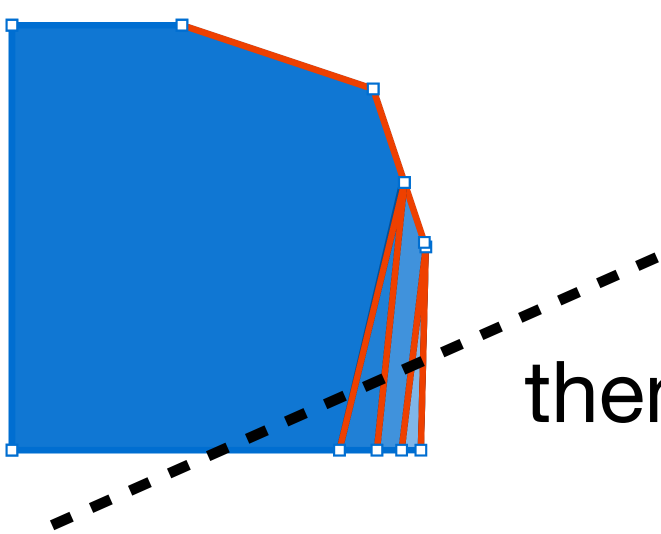

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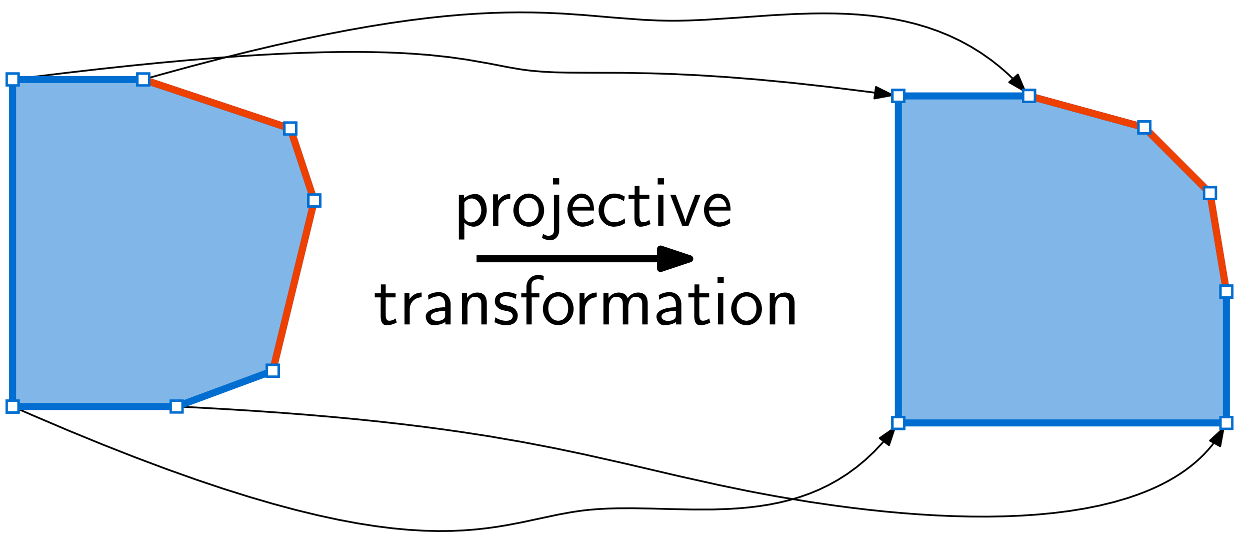
then cut with this plane

Incremental Step ctd.

current shape



required shape

Incremental Step ctd.

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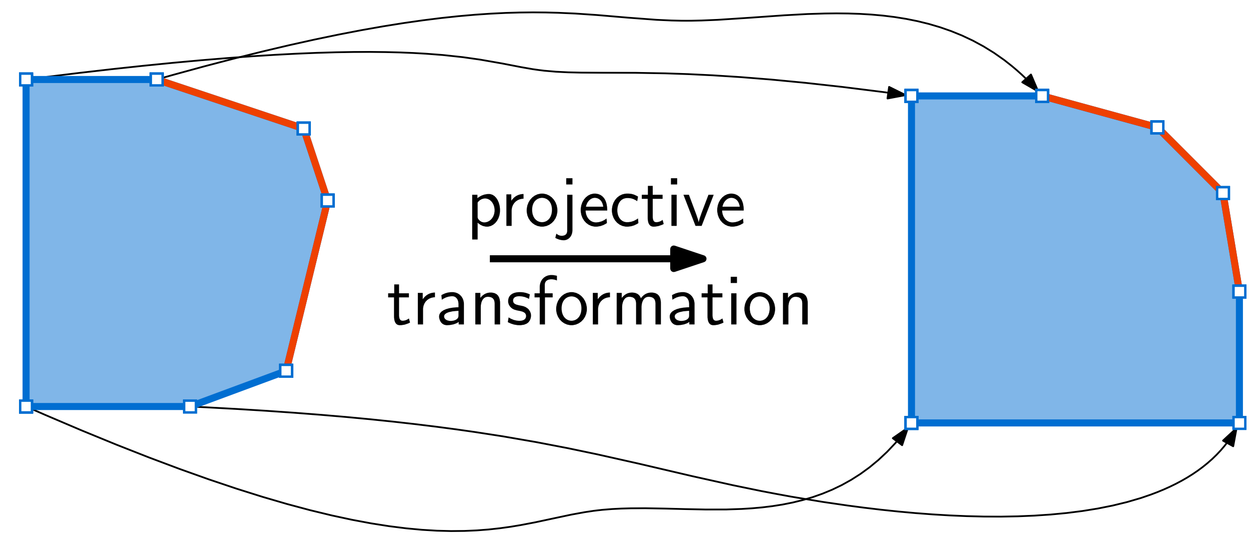

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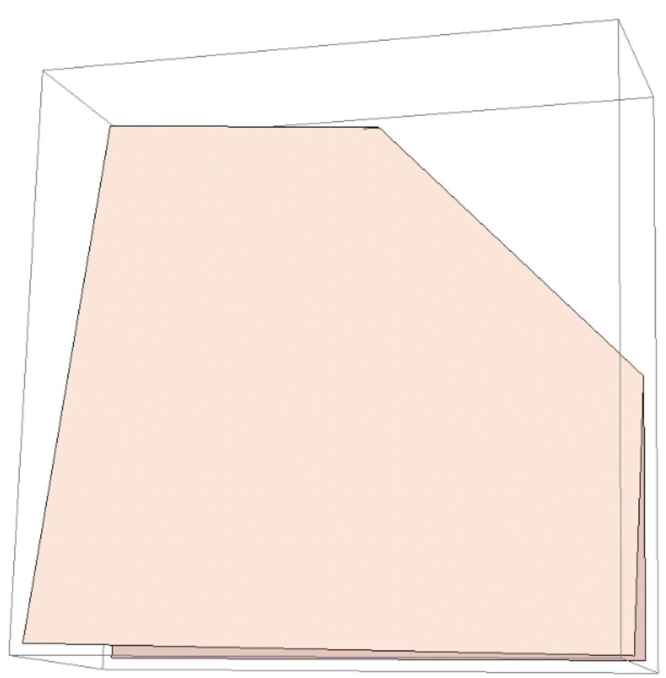

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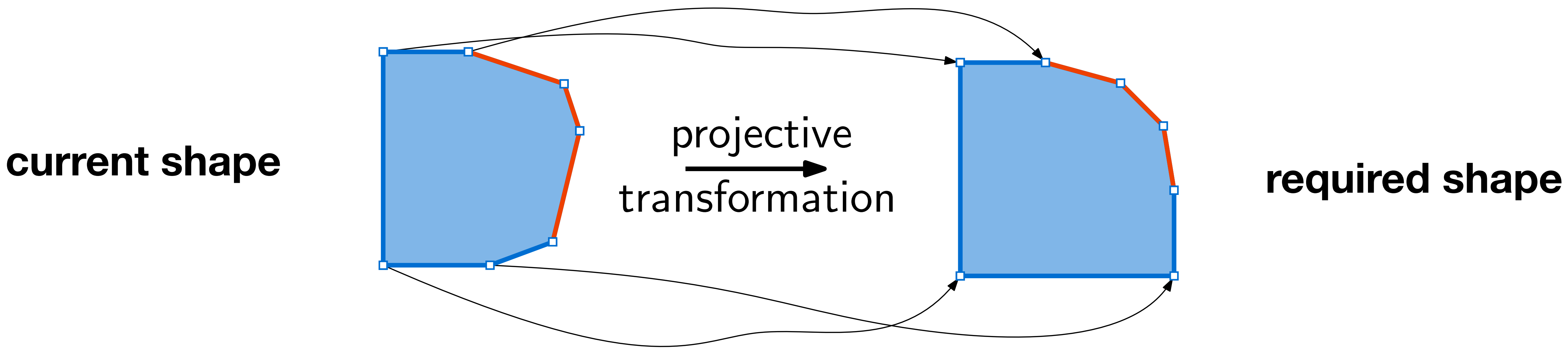

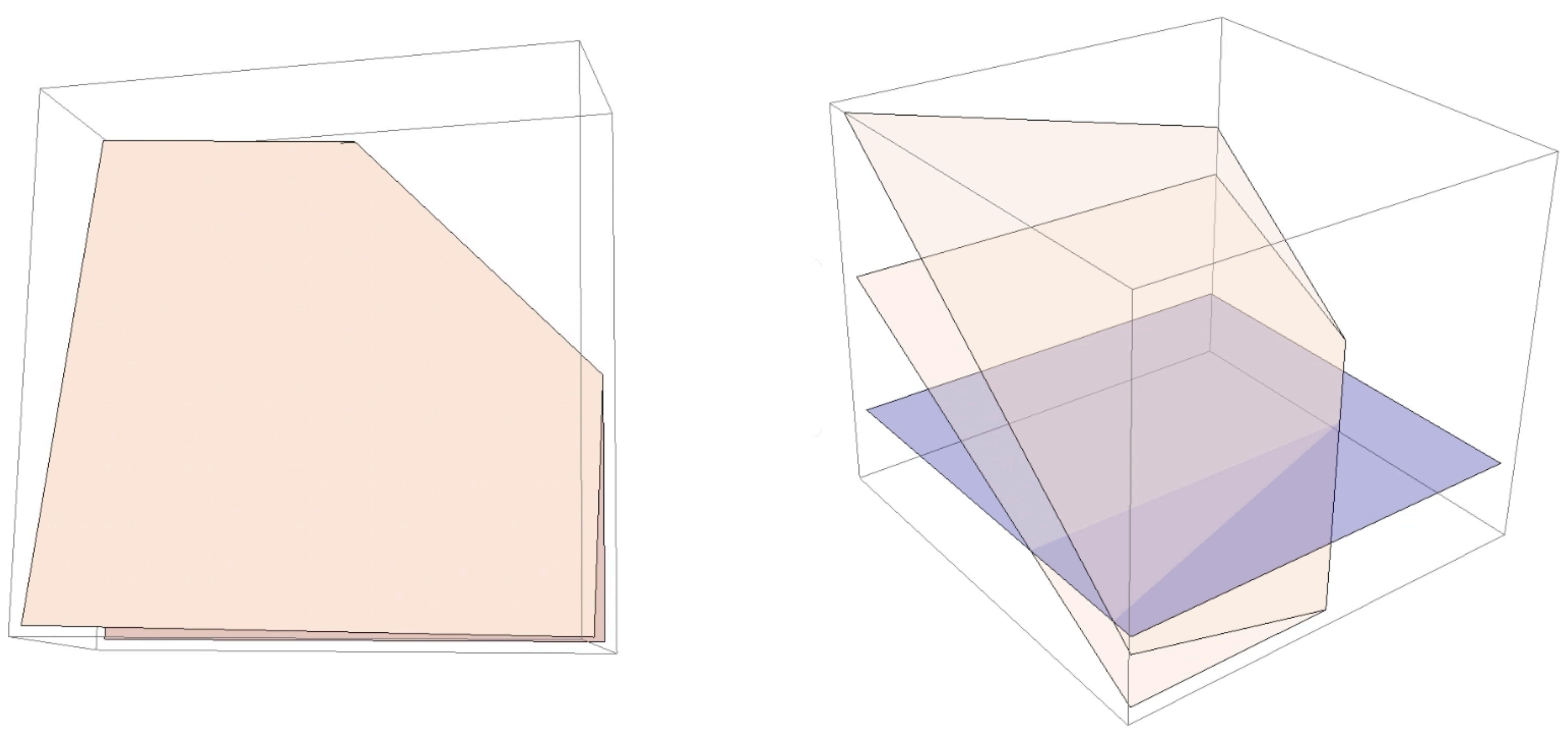

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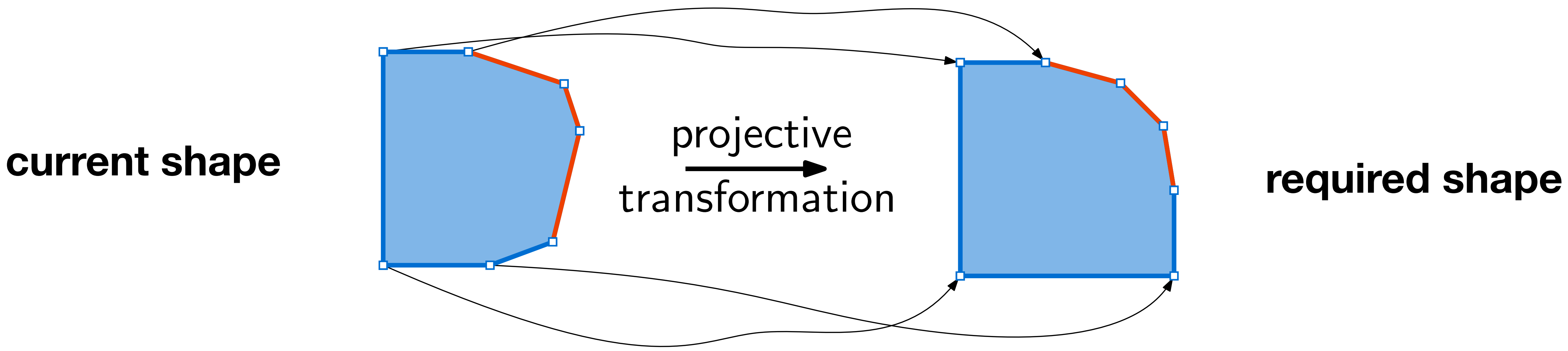
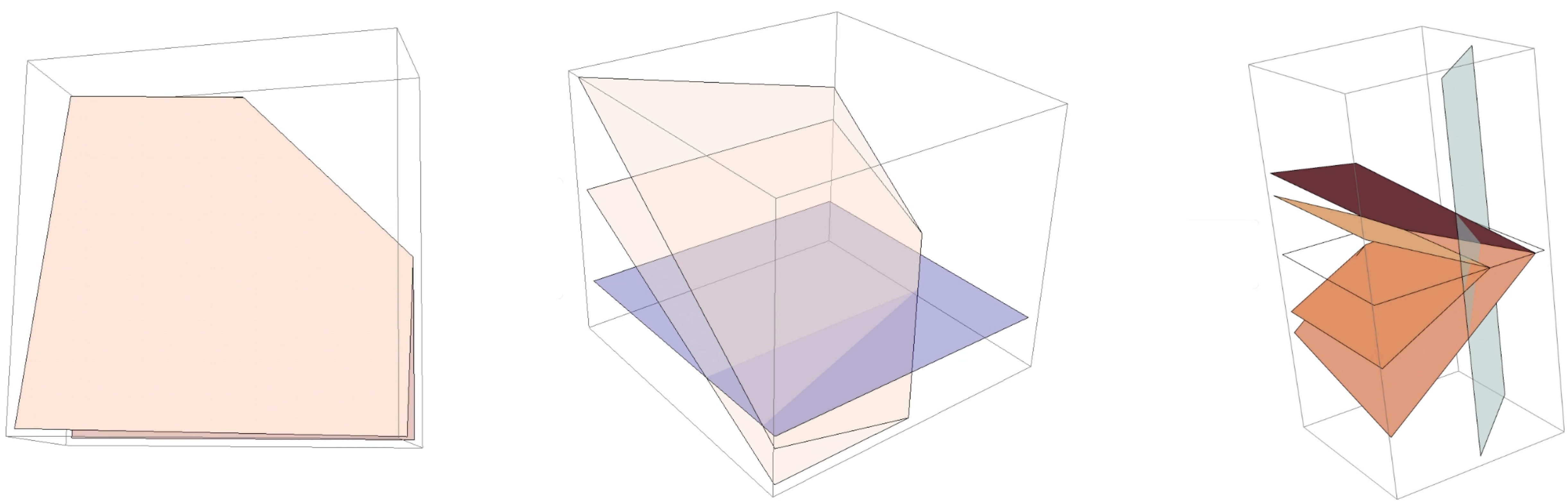
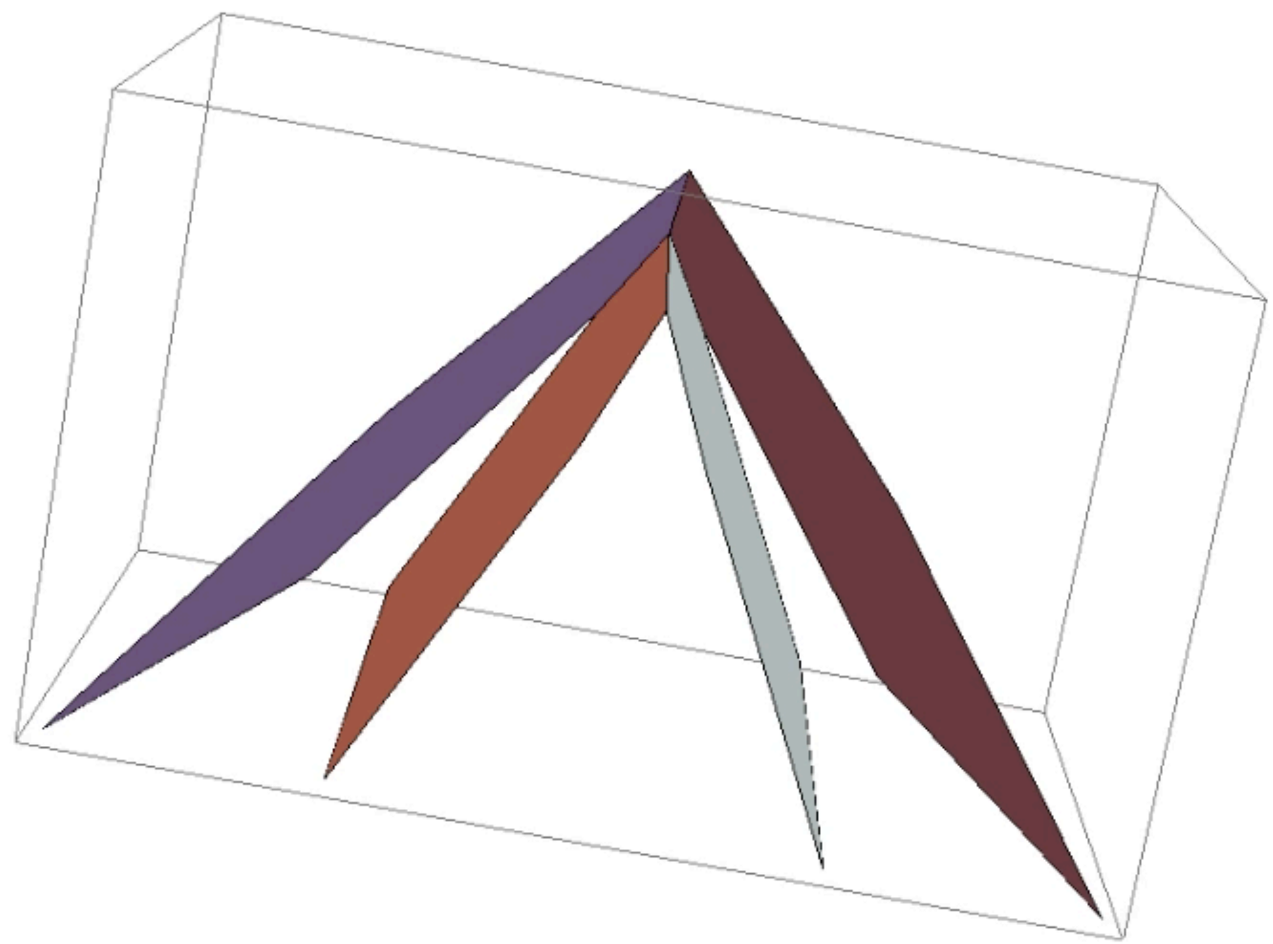




Example for d=2



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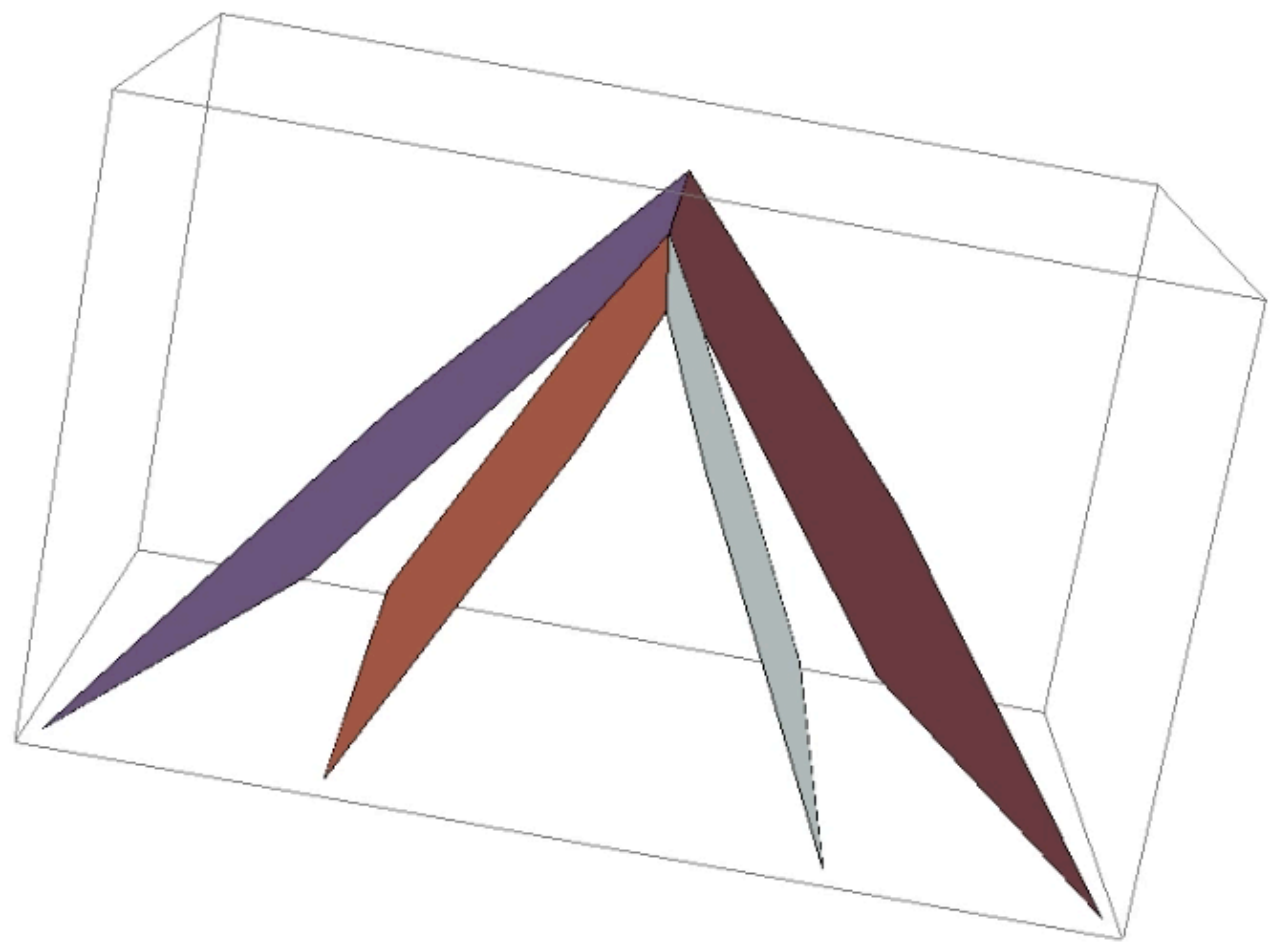

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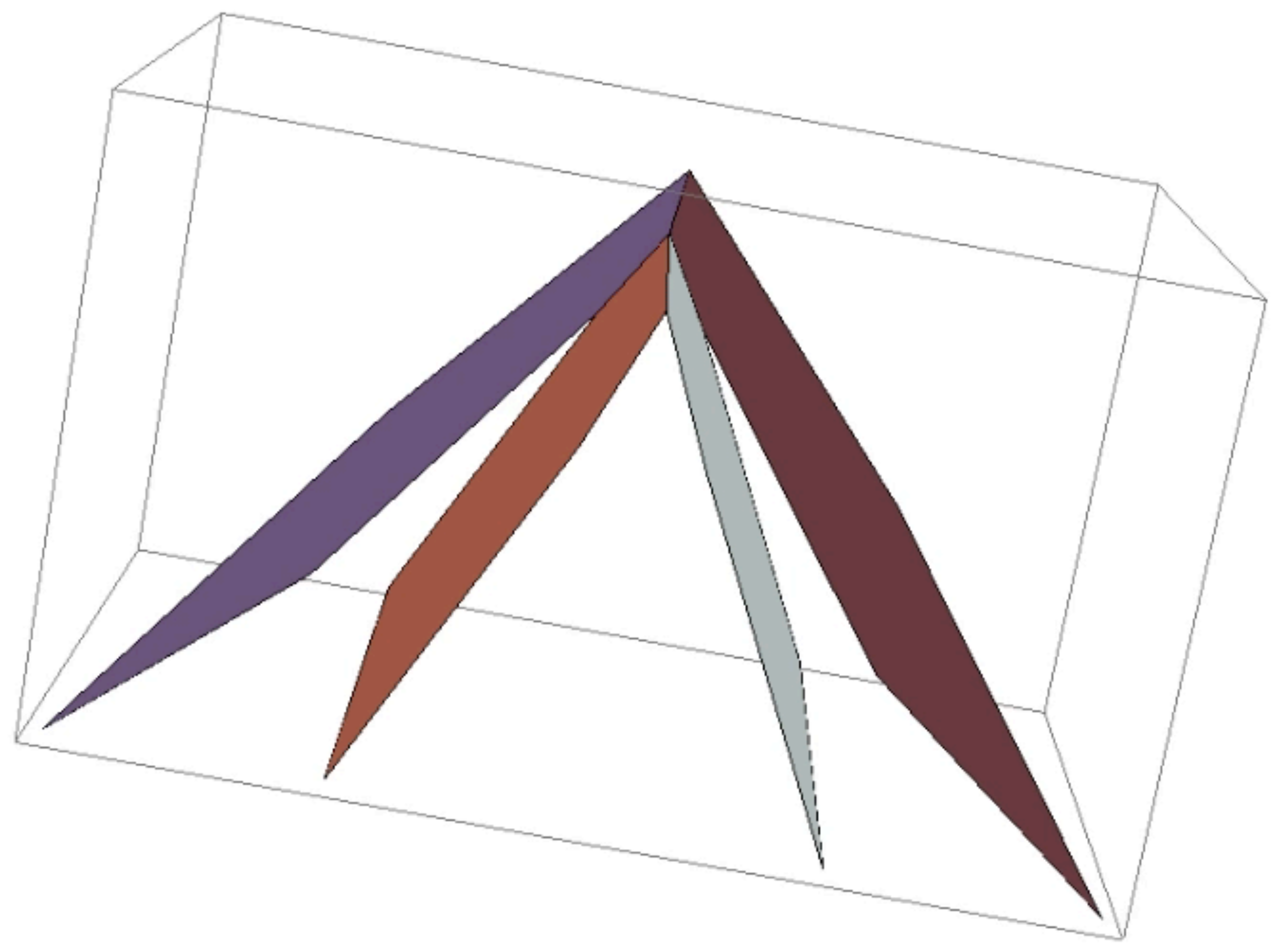

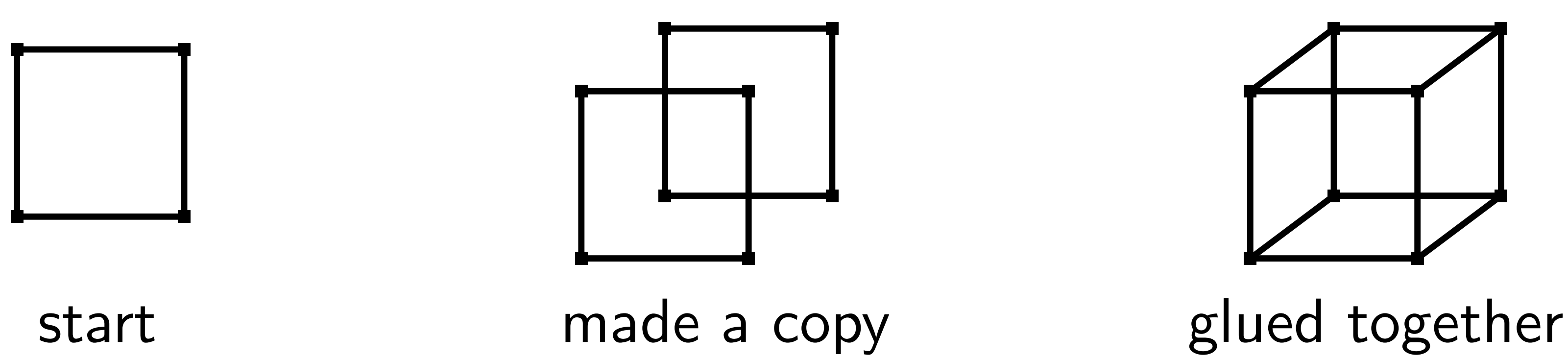

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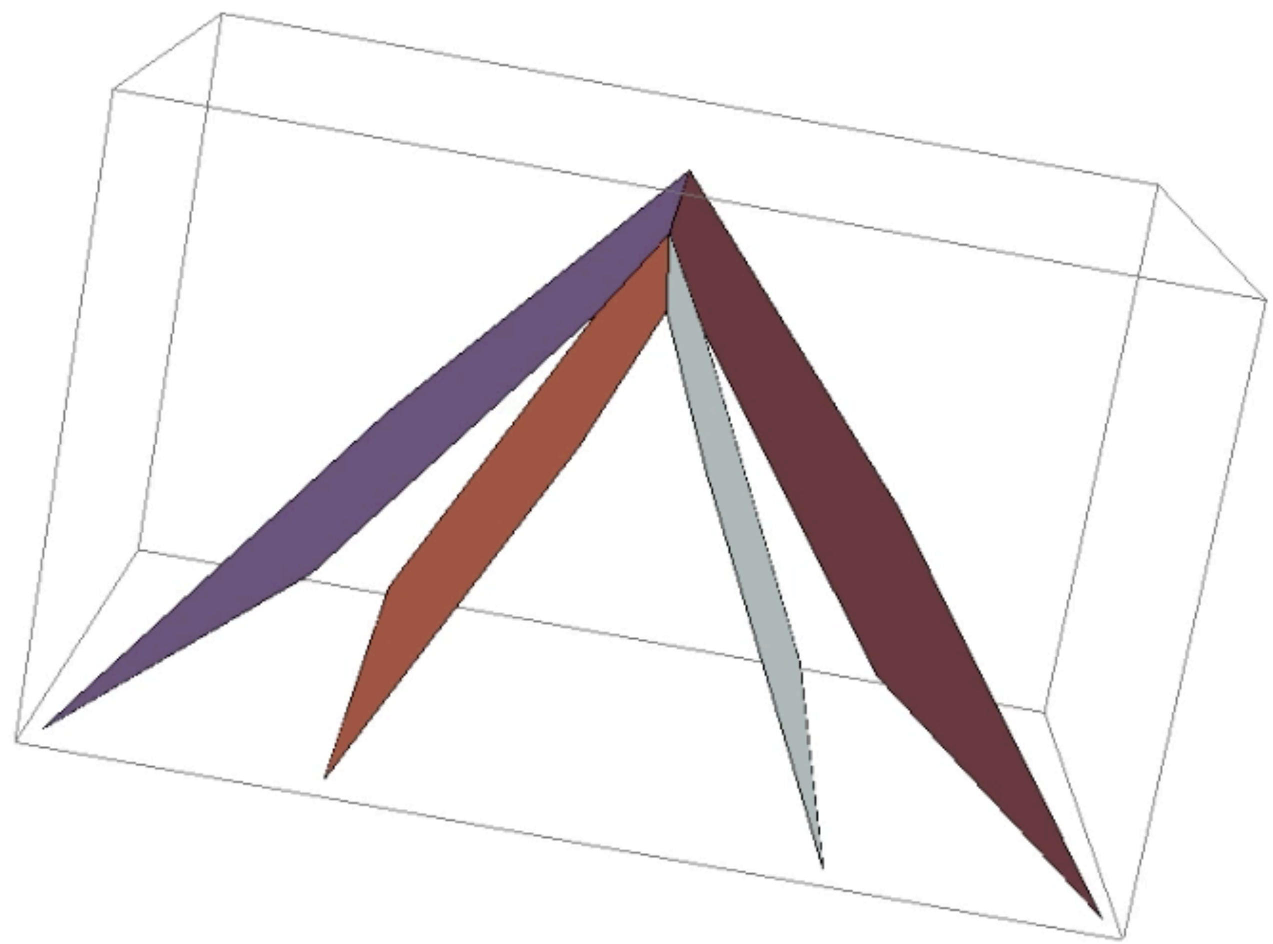

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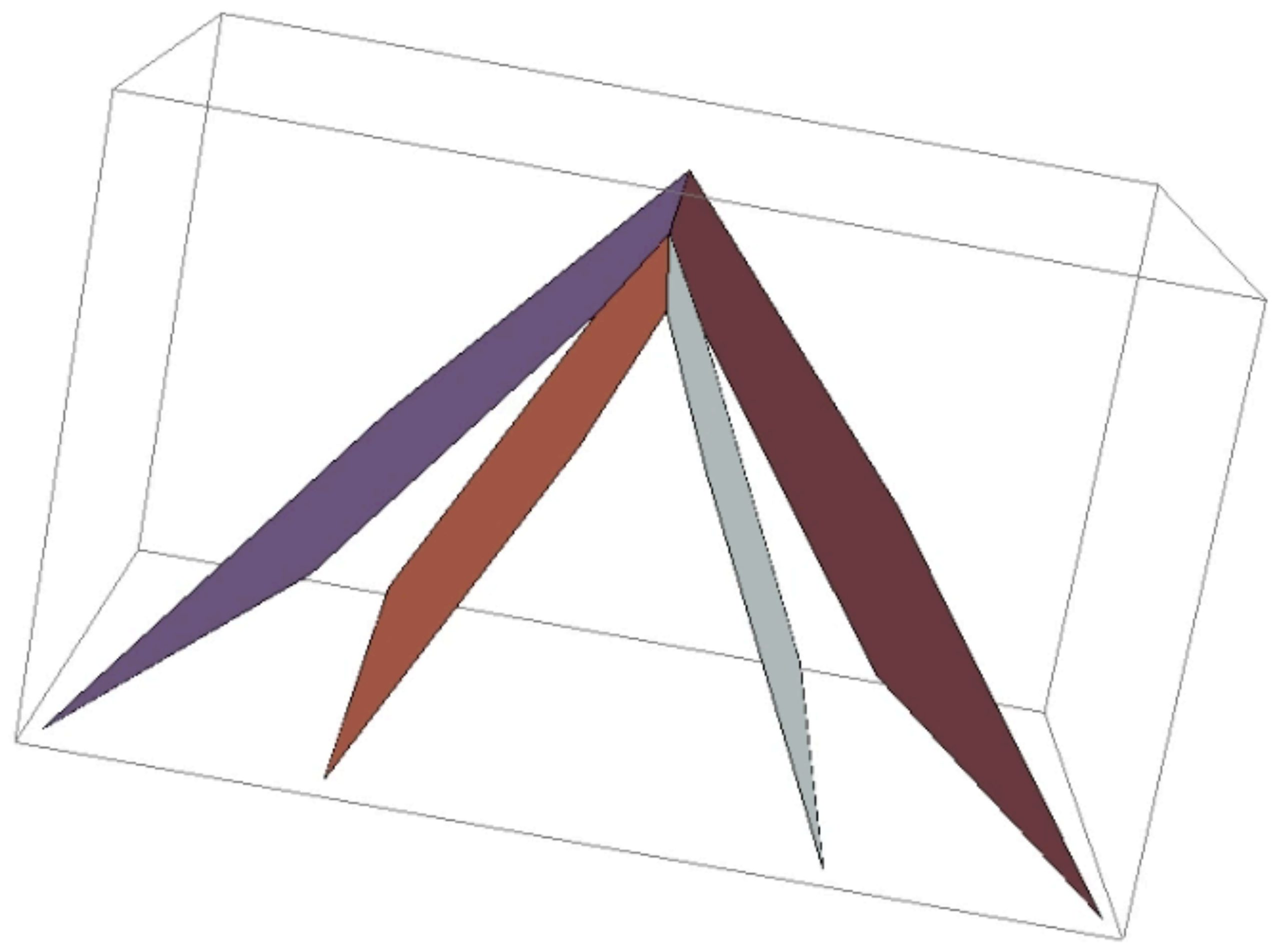

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➡**by the Kövari–Sós–Turán Theorem, the maximum edge density for a realizable graph is $O(n^{1.8})$**

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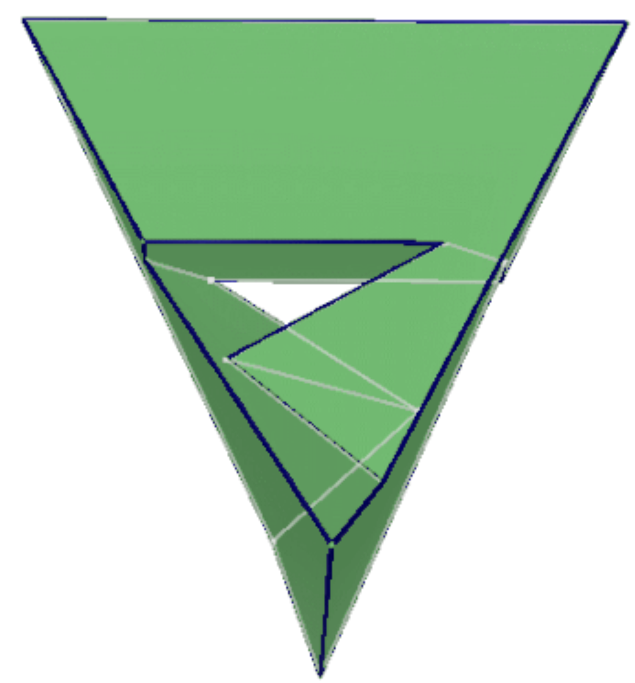

Model by user A2569875 @ wikimedia commons

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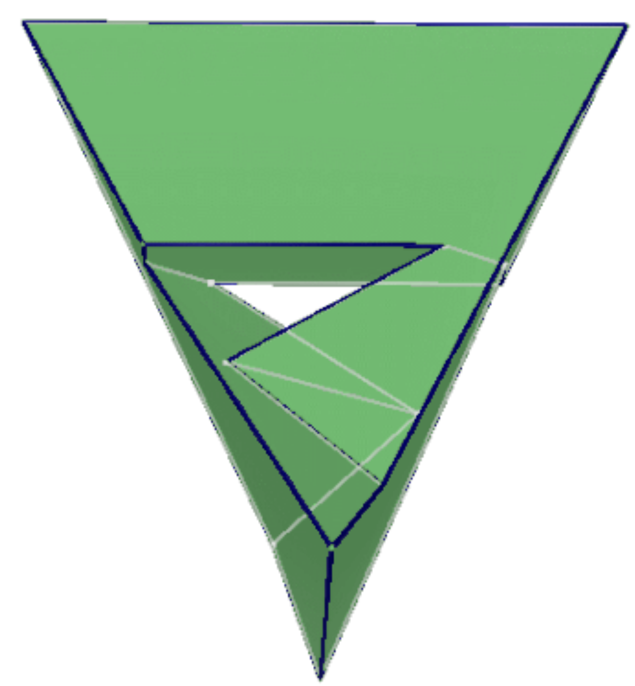

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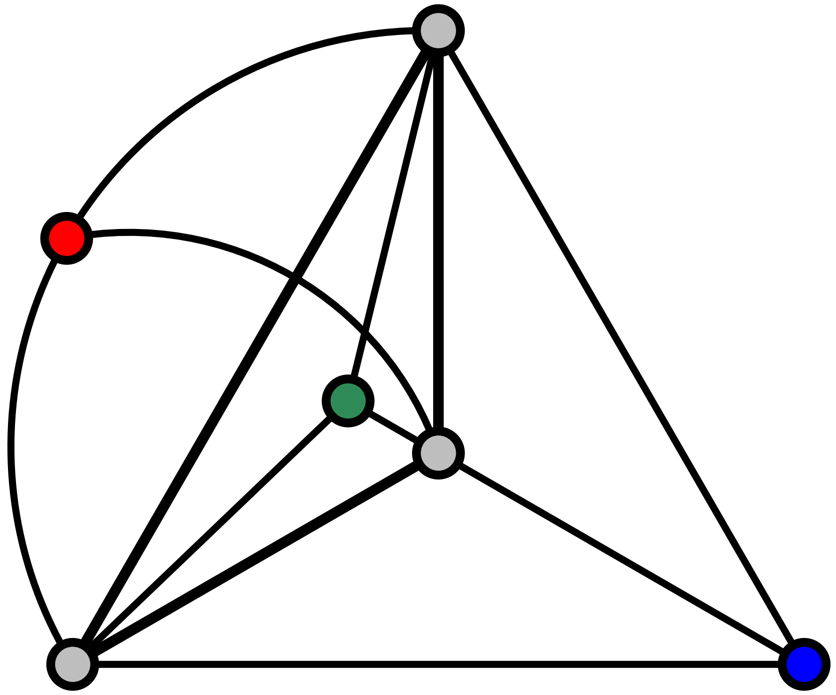
3-Trees



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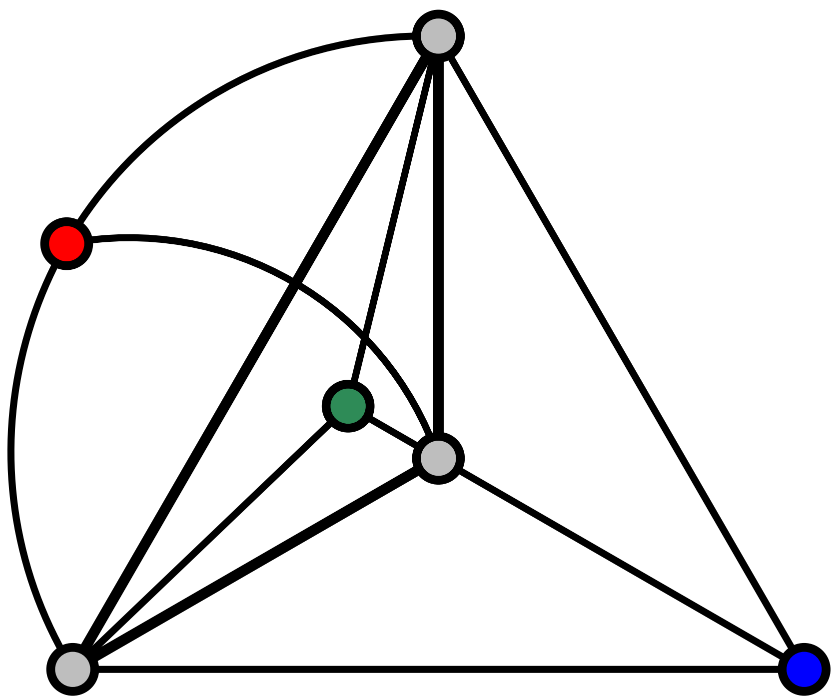

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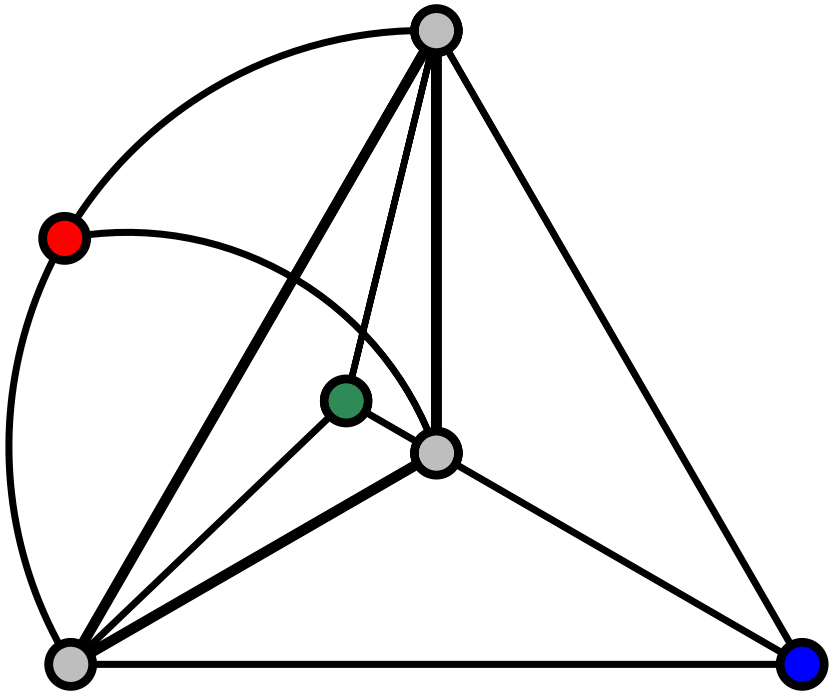

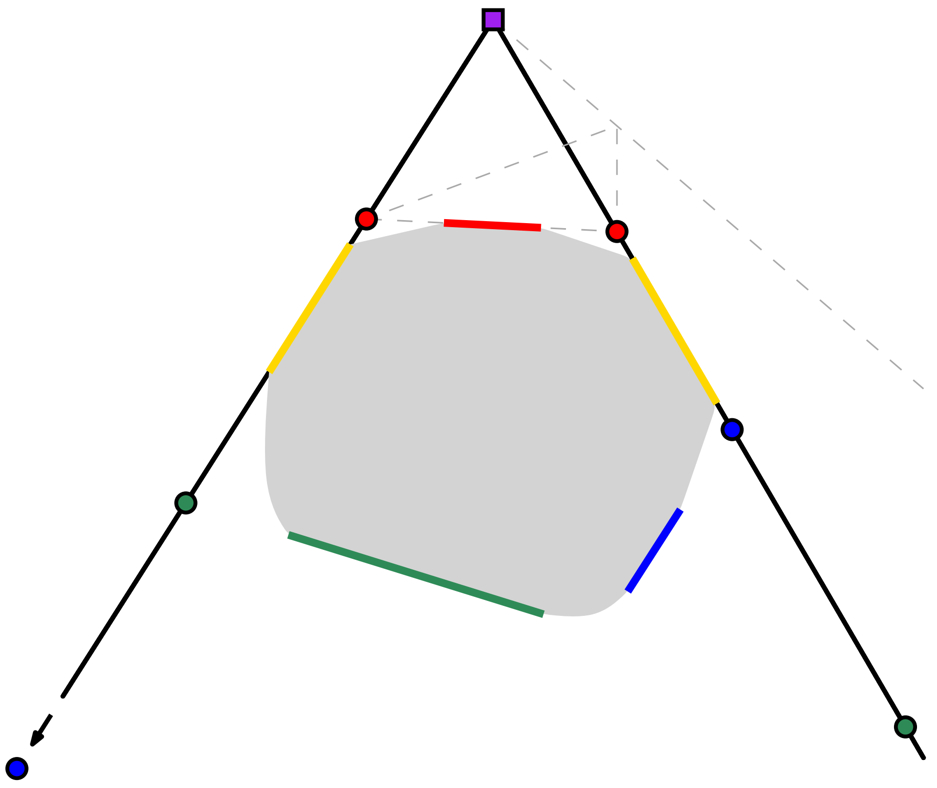

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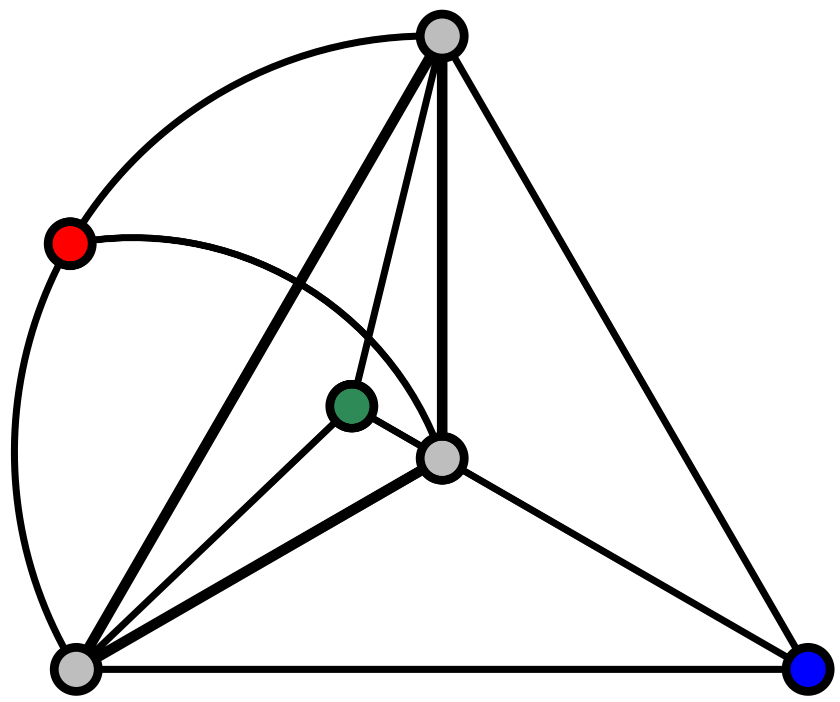

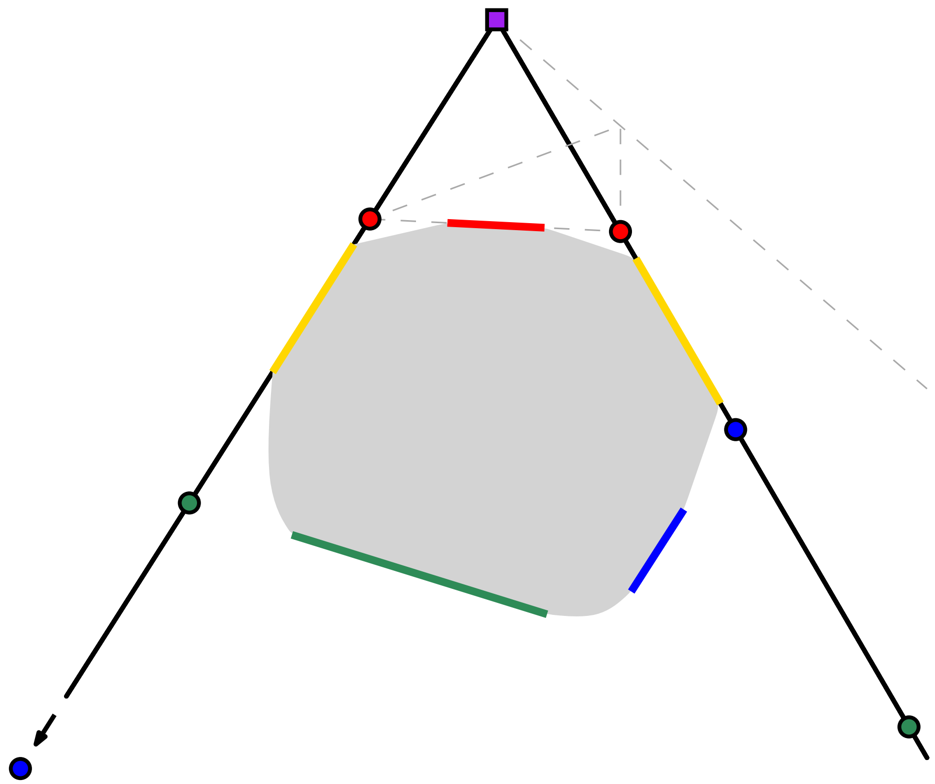

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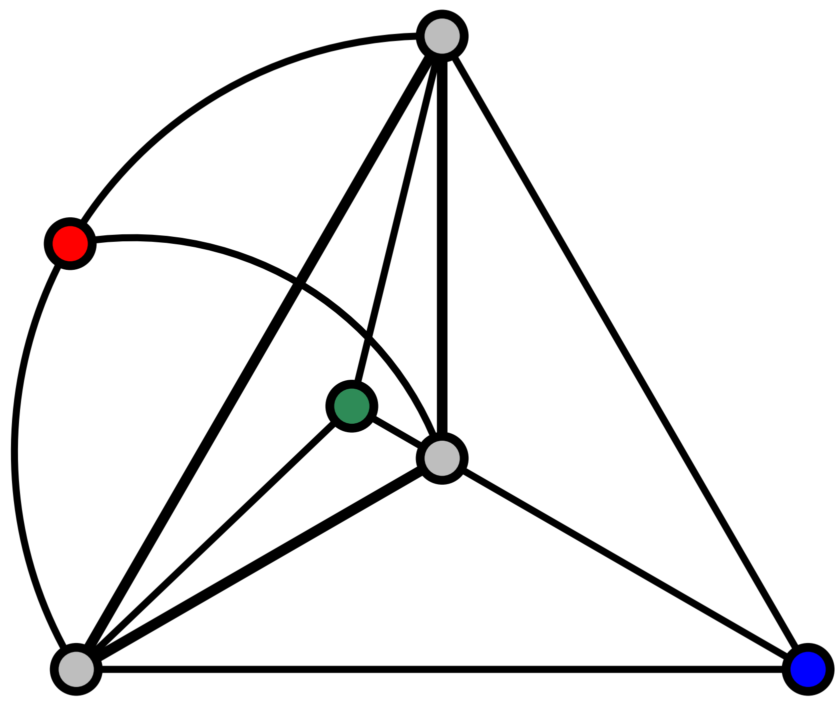

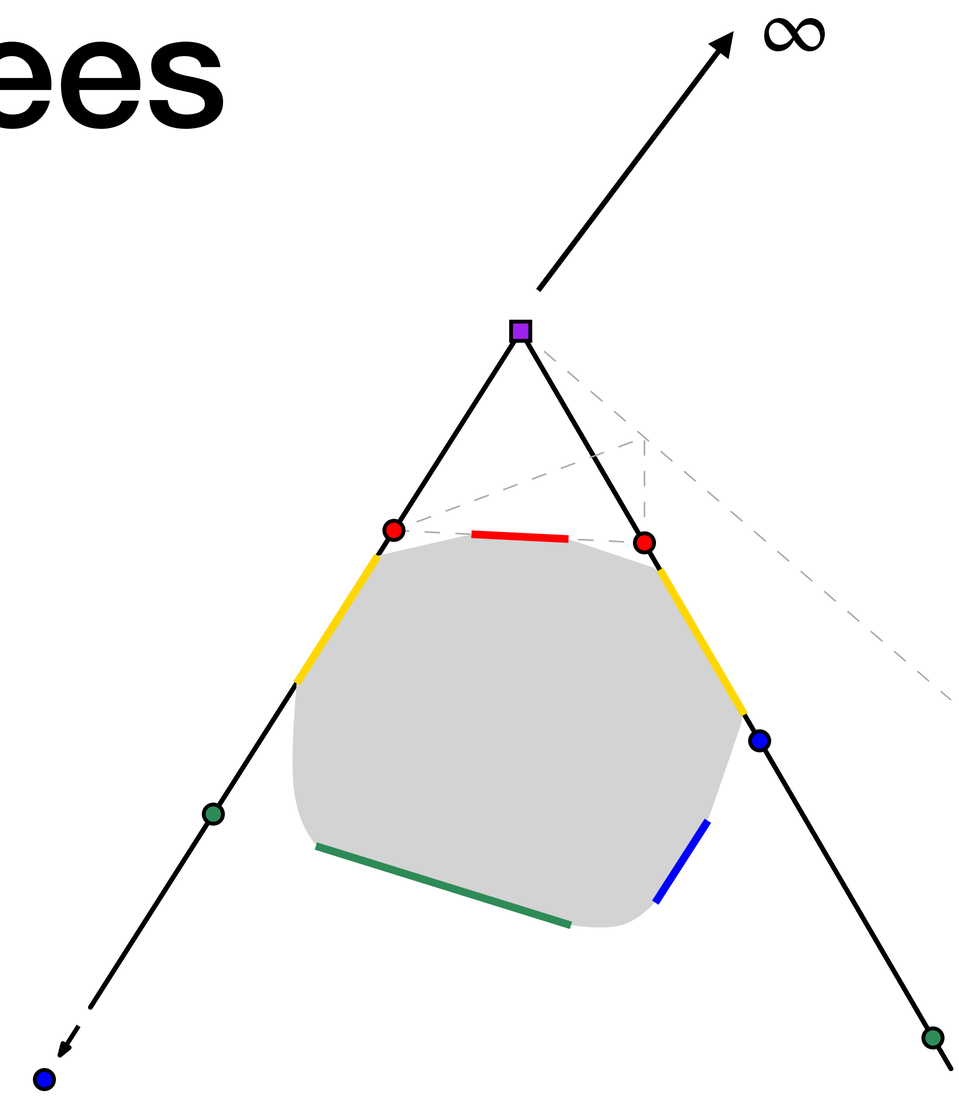

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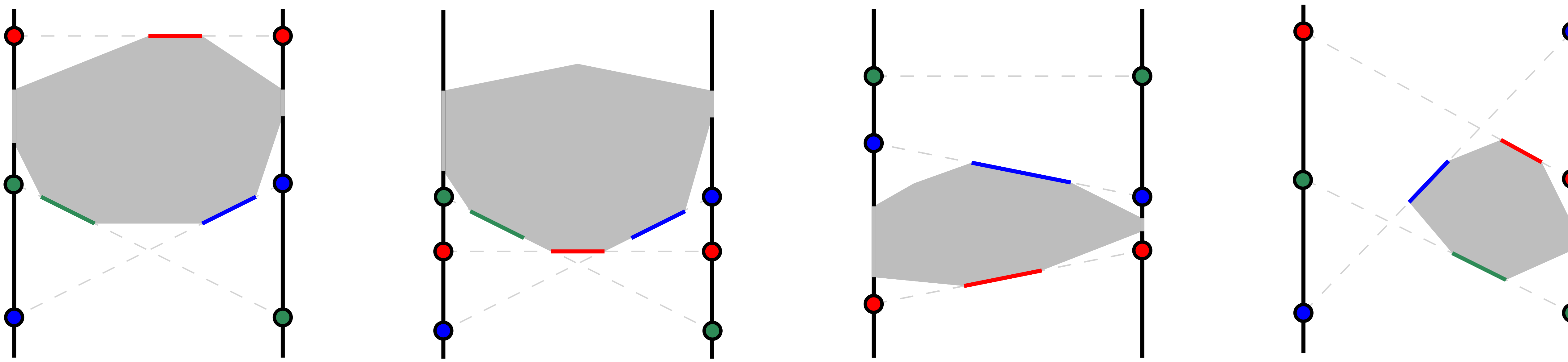
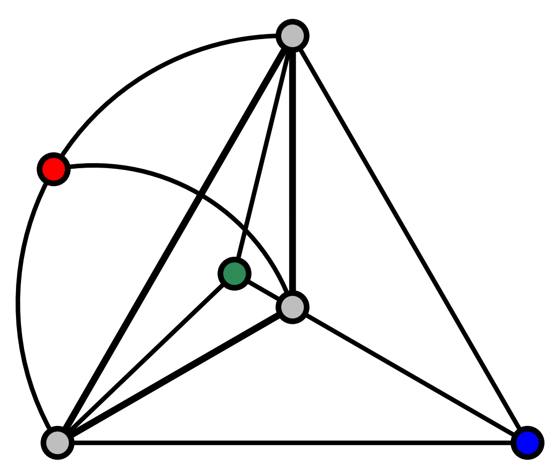
- apply projective transformation and move the apex to the point at infinity

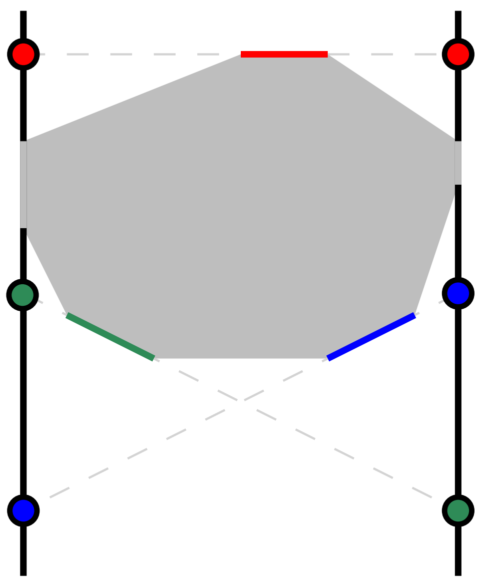
Permutations on Rays



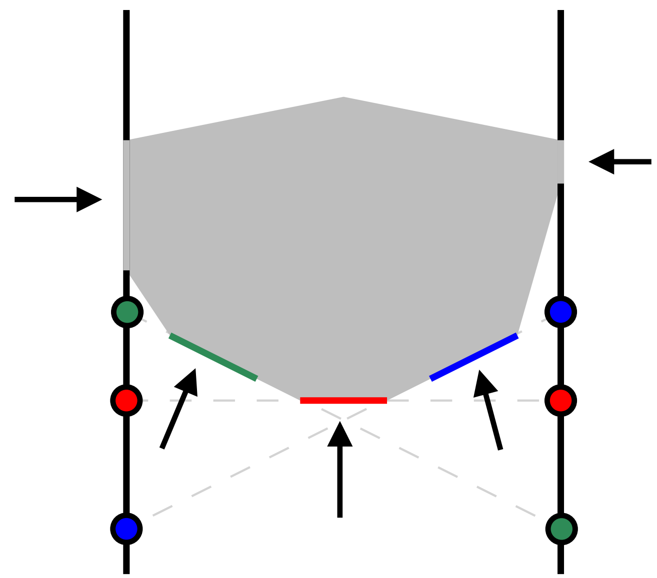
scenario 1 | scenario 2 | impossible | impossible



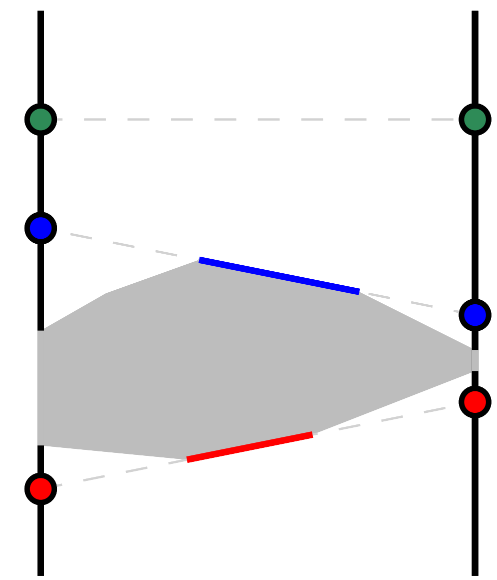
Permutations on Rays



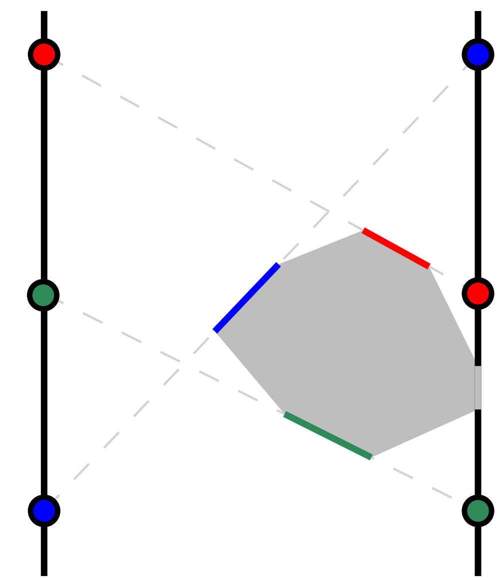
scenario 1



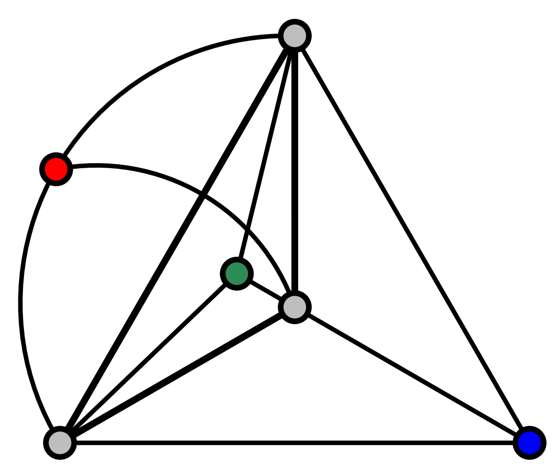
scenario 2



impossible



impossible



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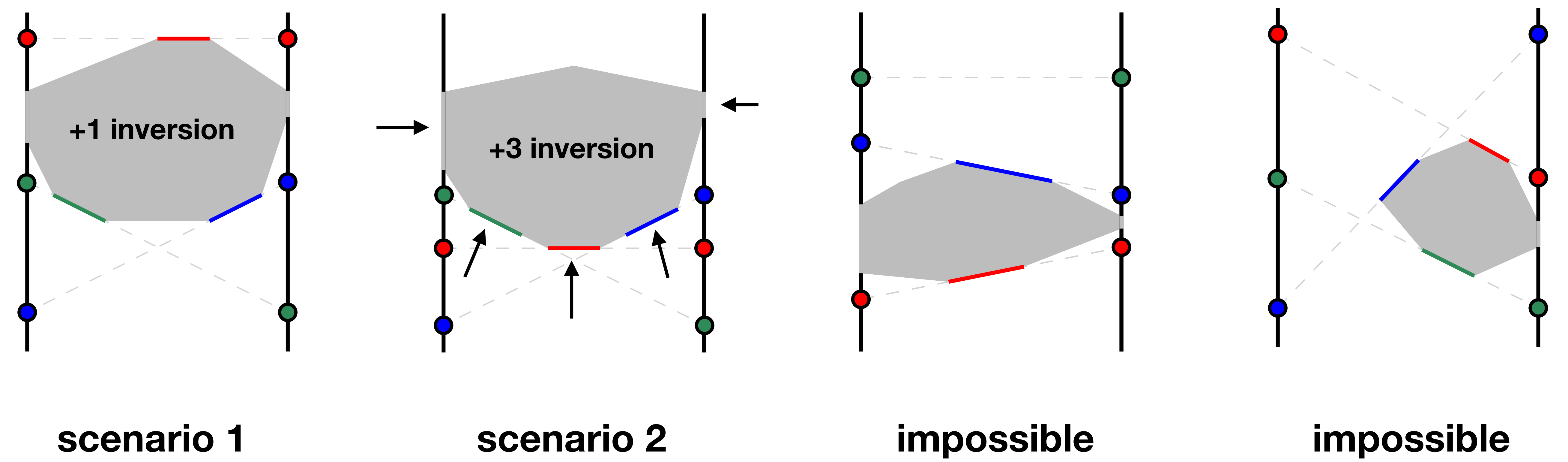

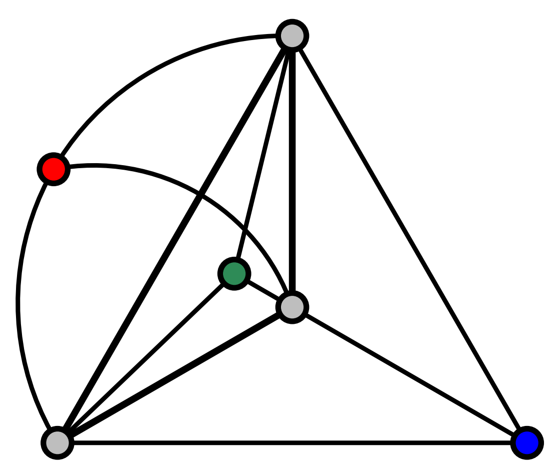

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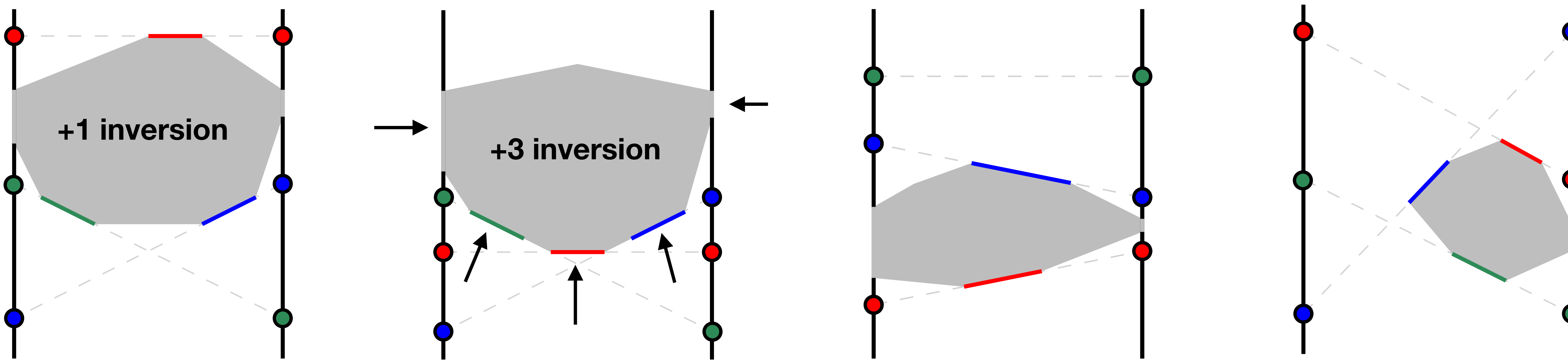
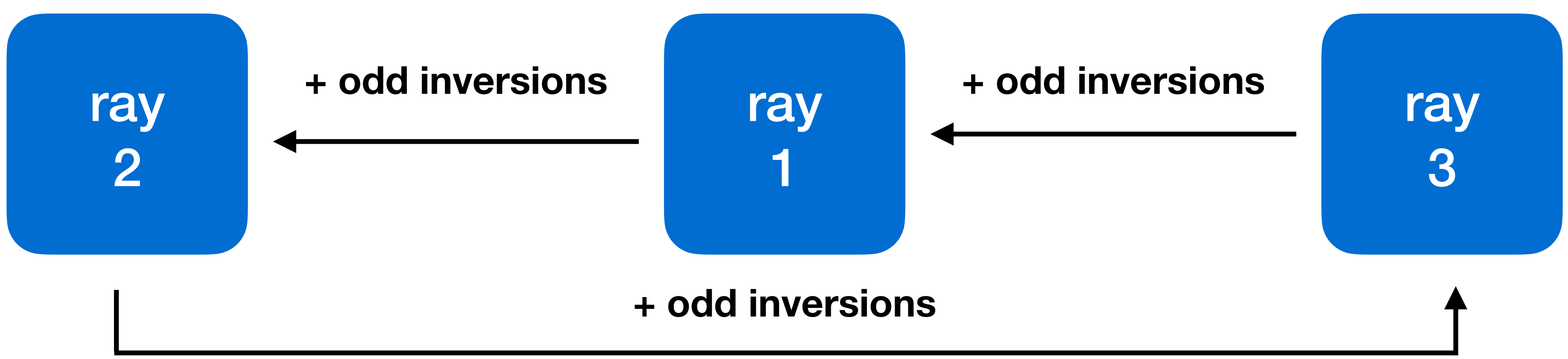
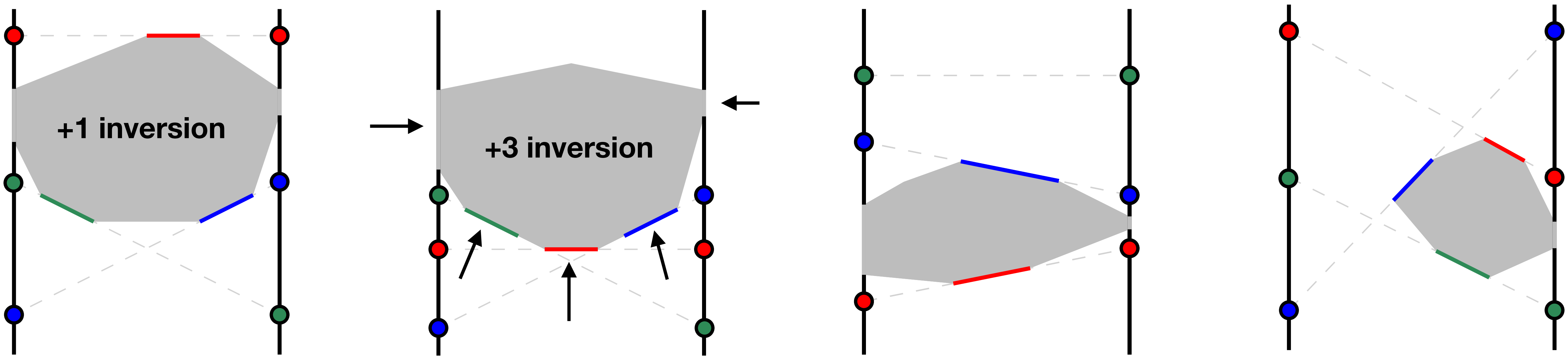


scenario 1 **scenario 2** **impossible** **impossible**



Permutations on Rays



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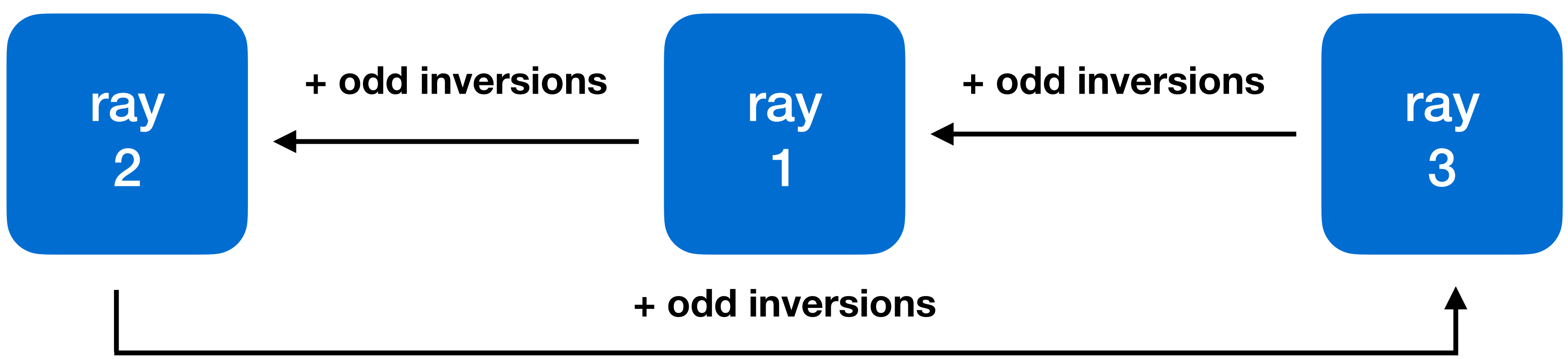
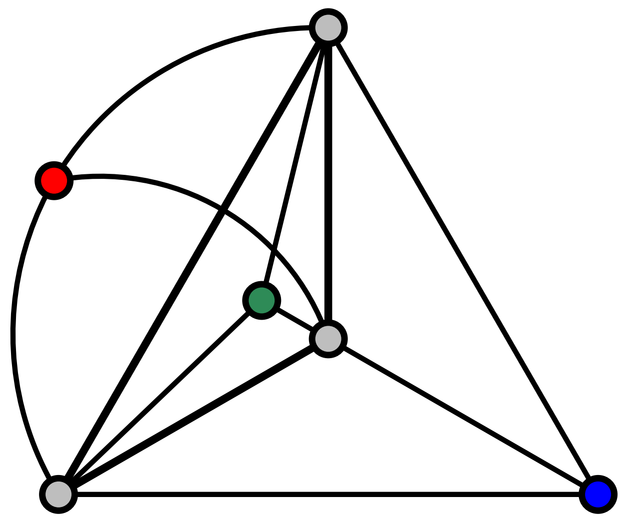


3 odd inversions cannot give identity !

3-Tree Summary



A 3-tree has representation with convex polygons, if and only if it is planar.

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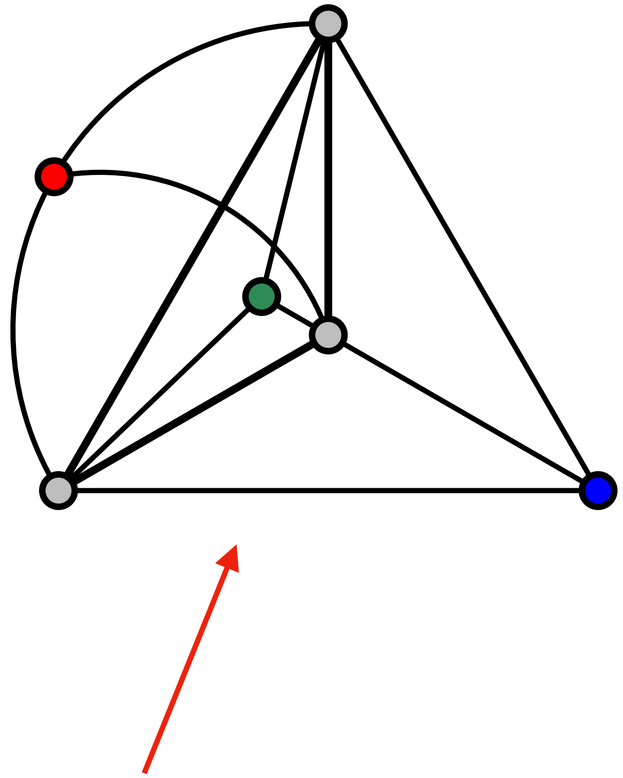

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this is a subgraph in all nonplanar 3-trees

A $K_{n,m}$ without Realization

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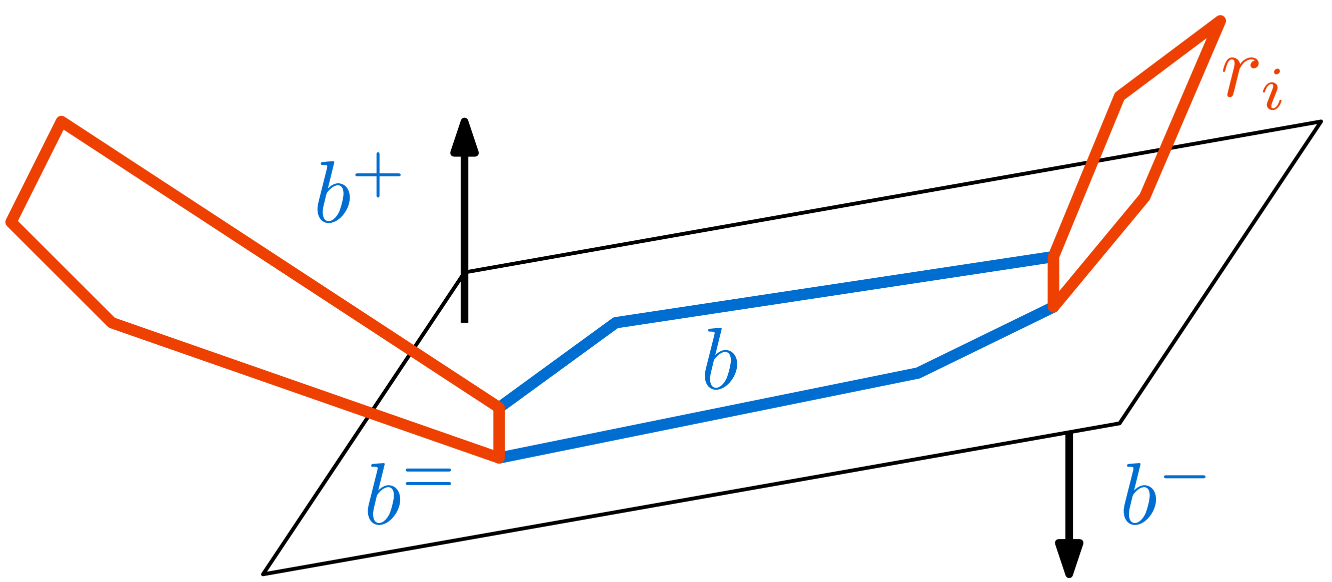

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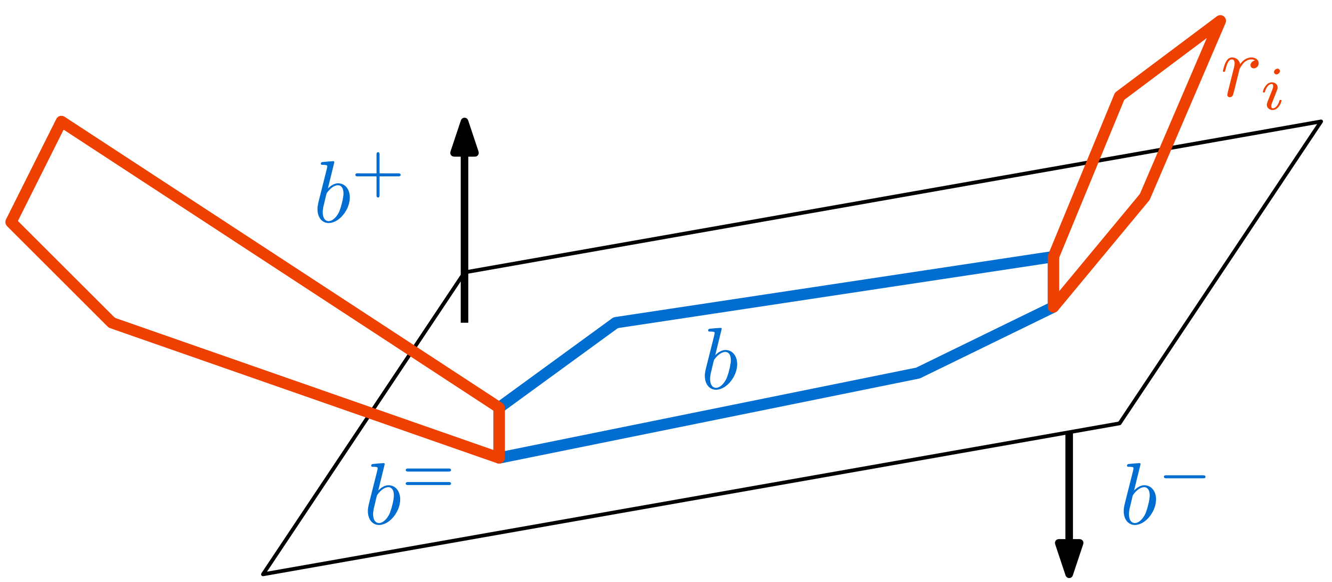
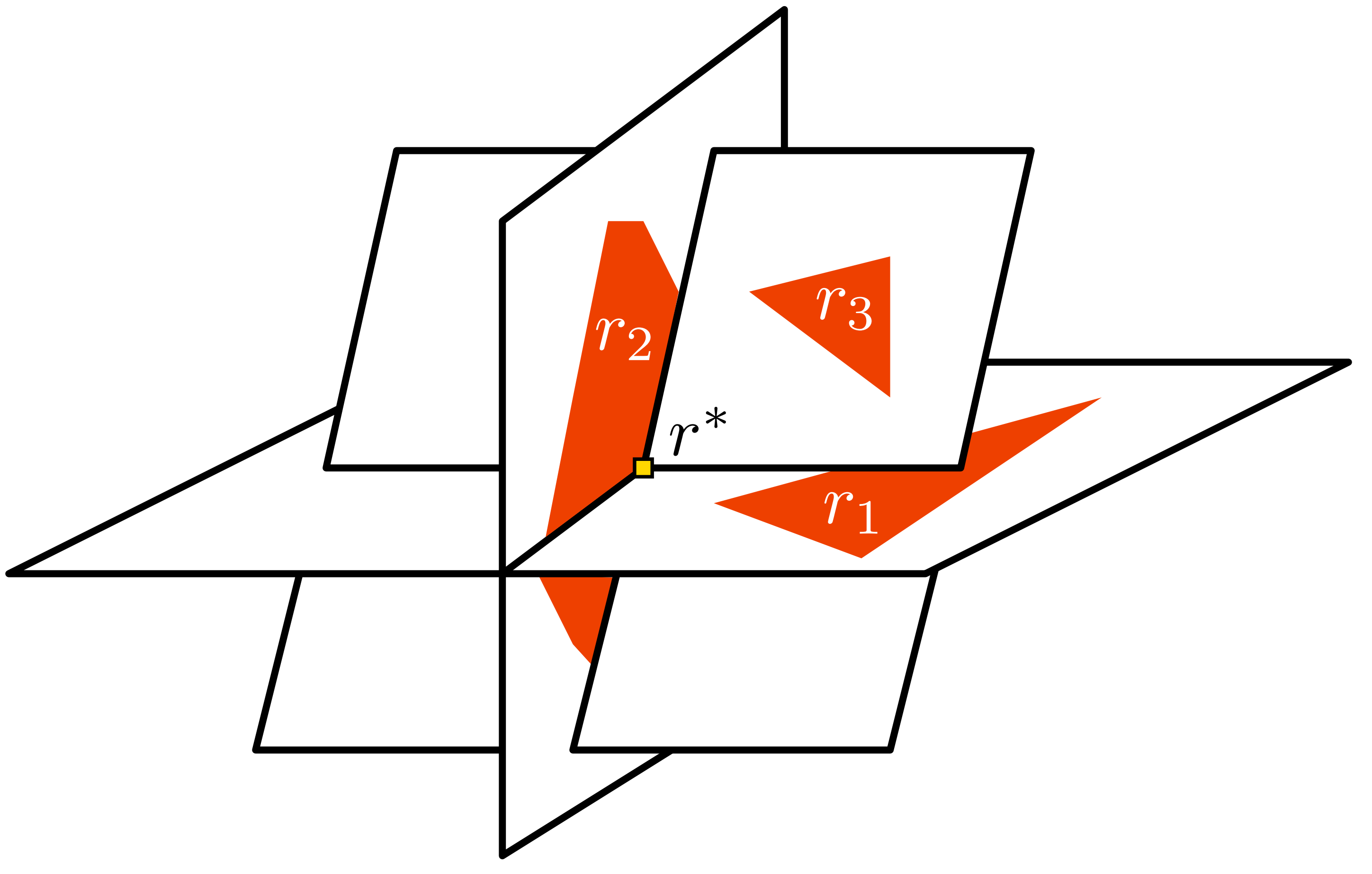
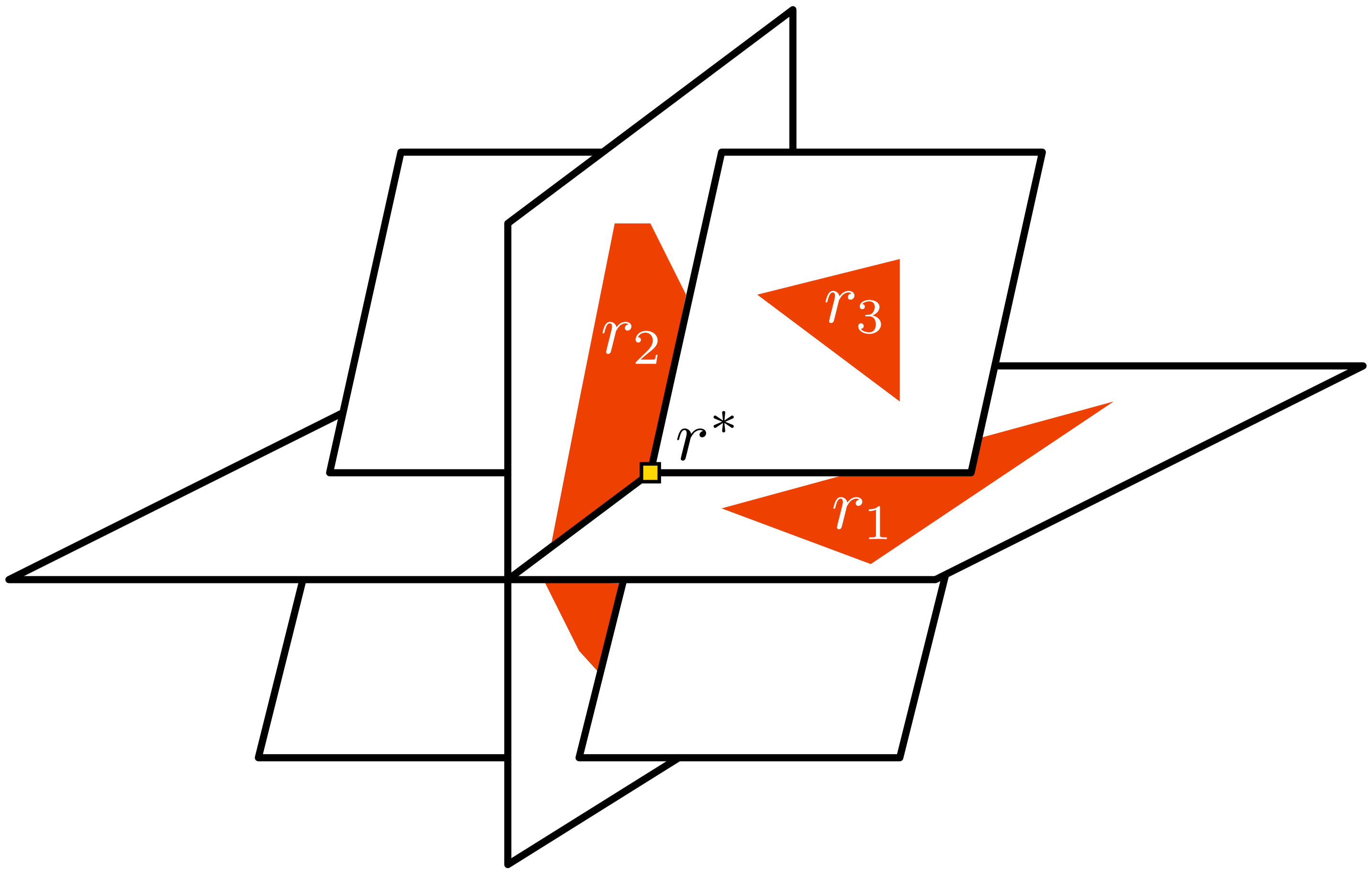


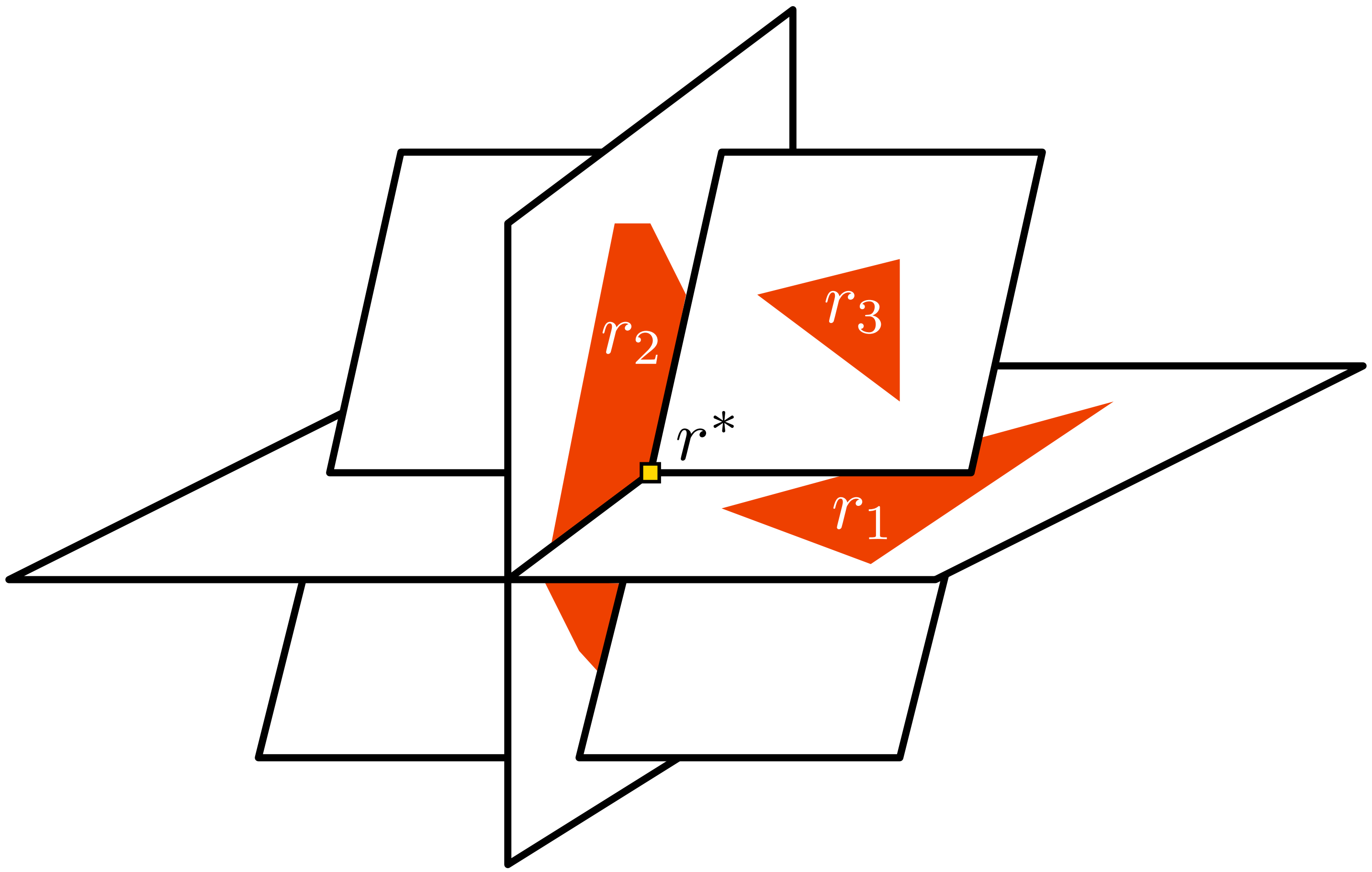
- Assume that we have 3 red polygons.



r_3
r_2
r^*
r_1



- arrangement of $\{r_1^=, r_2^=, r_3^=\}$ defines 8 octants (general case)



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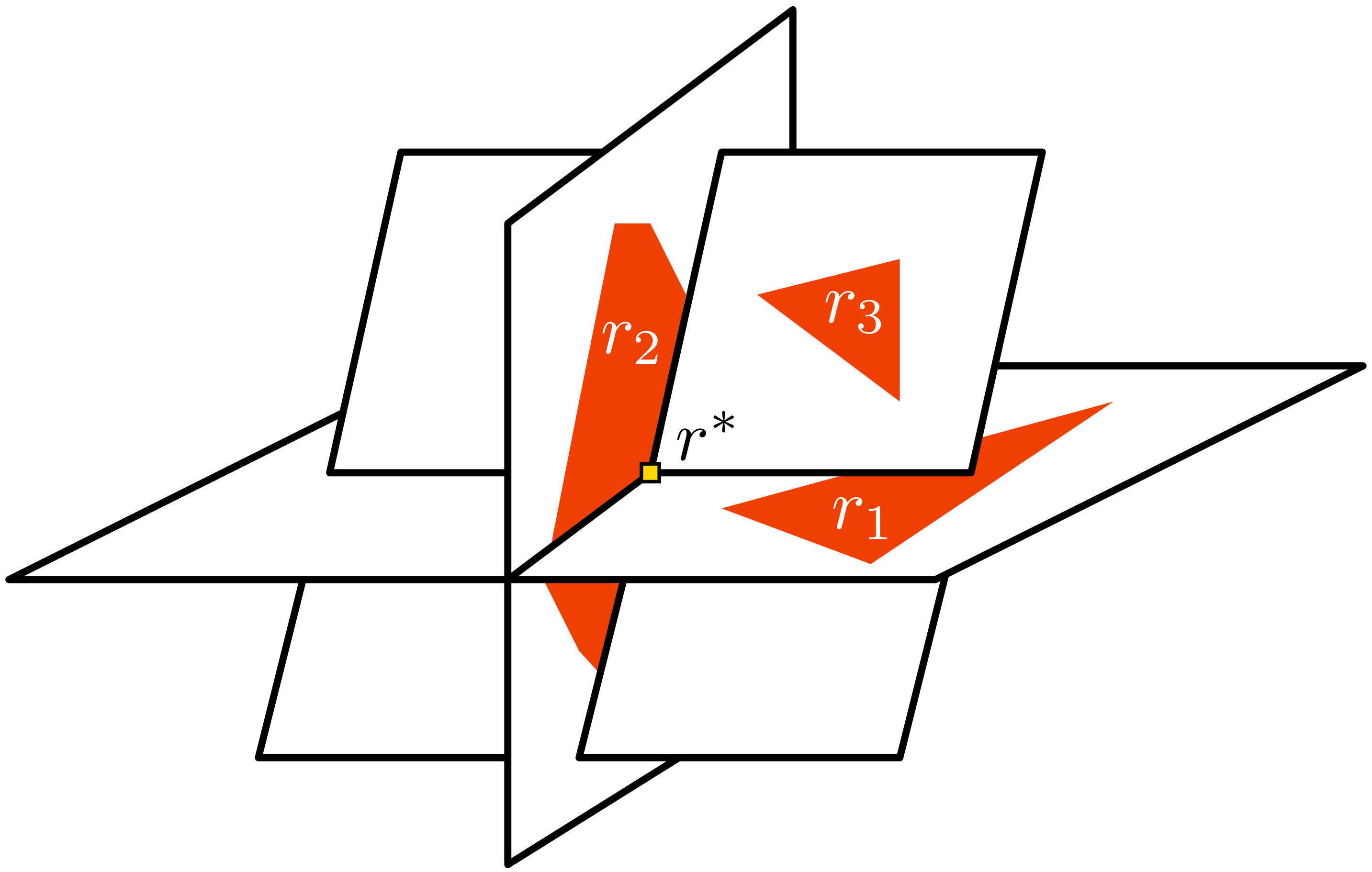

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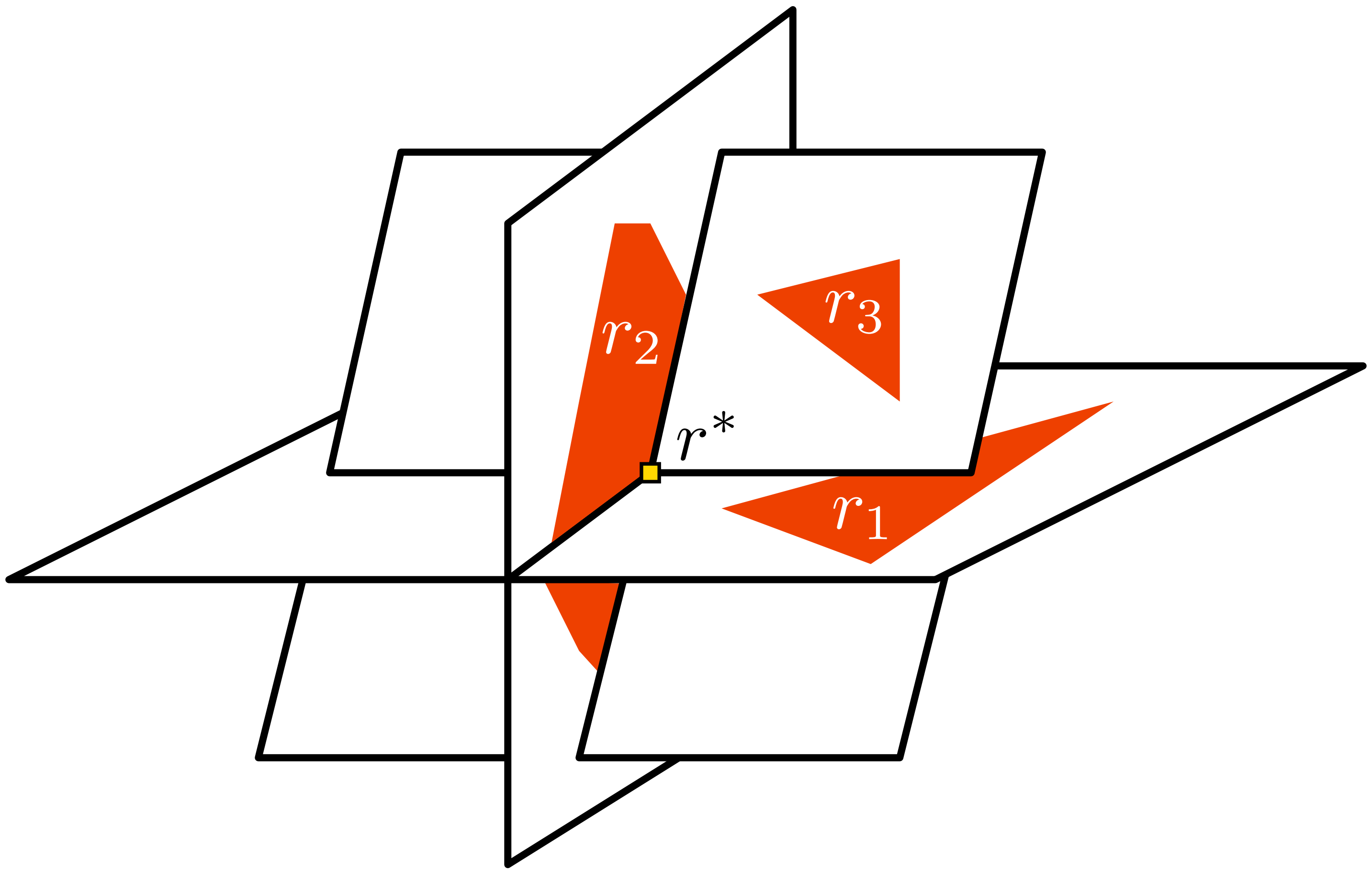

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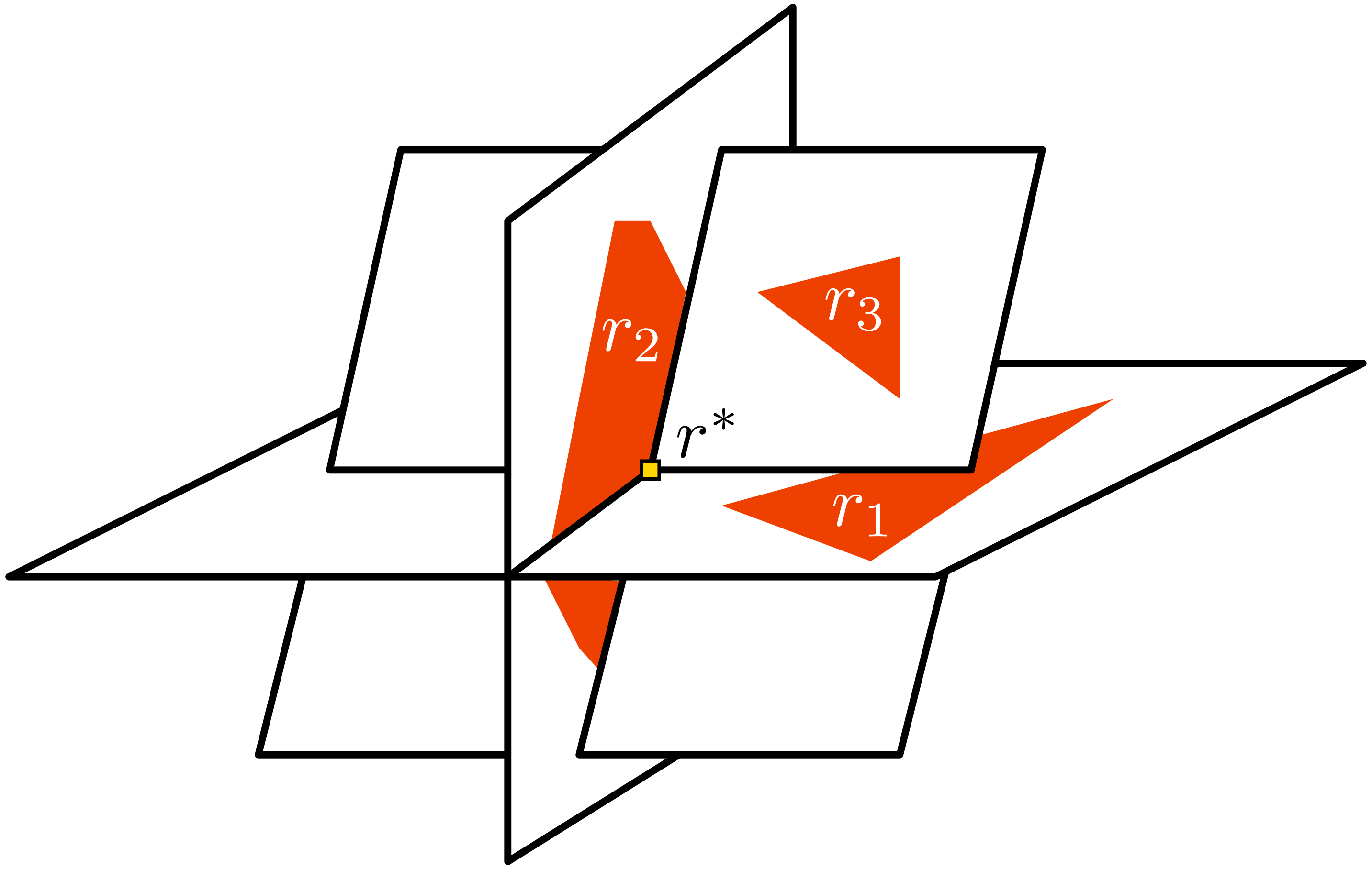

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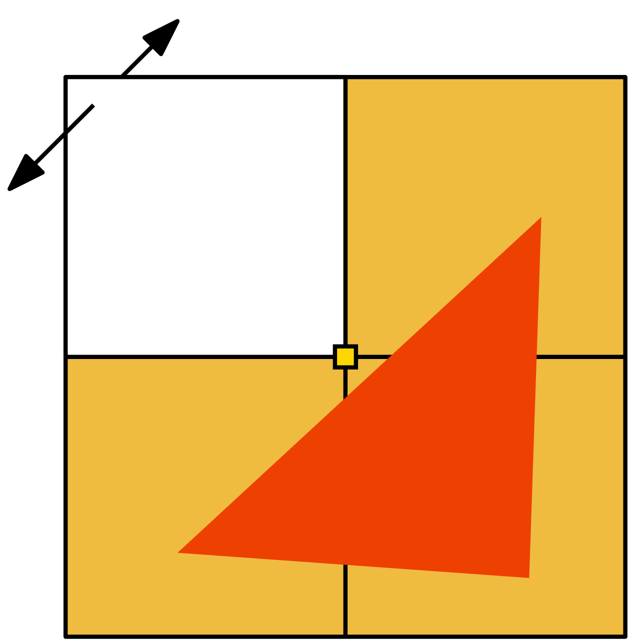
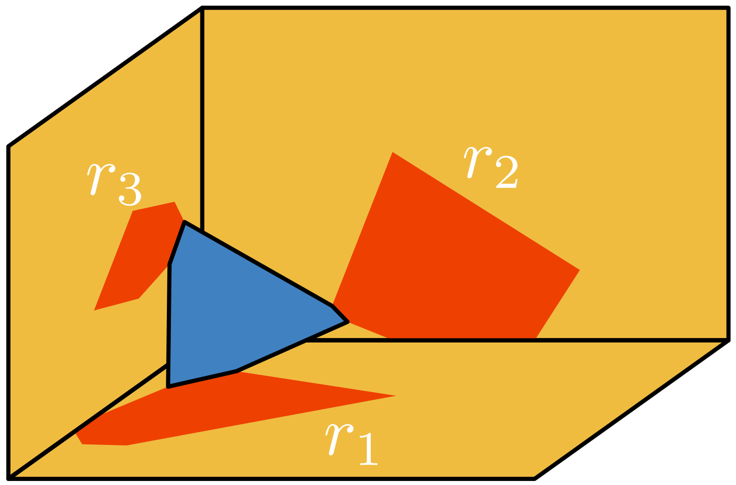
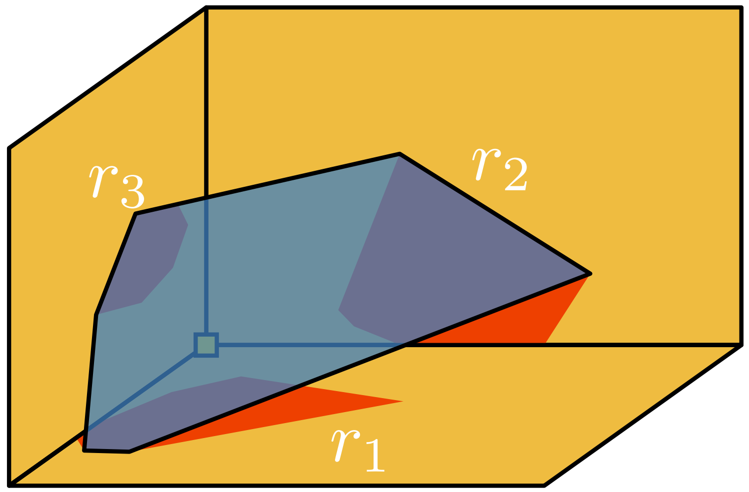


- every plane with $r^* \notin r_i$ rules out at least 2 adjacent octants as complete

Placing a blue polygon inside a complete octant



possible if $r_* \notin r_i$



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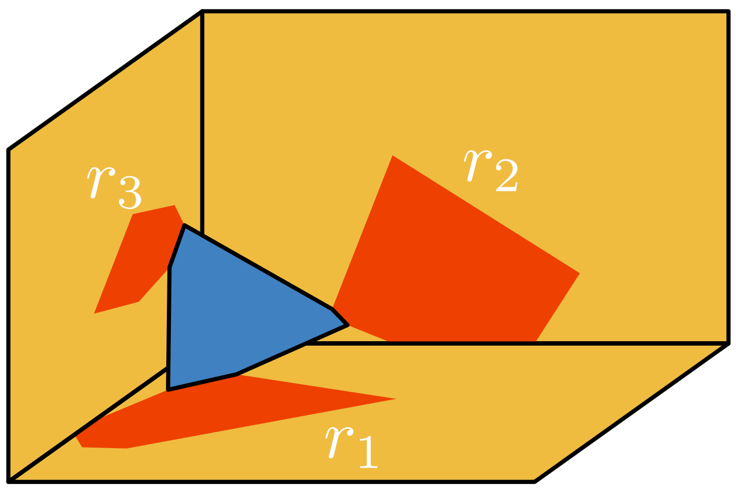

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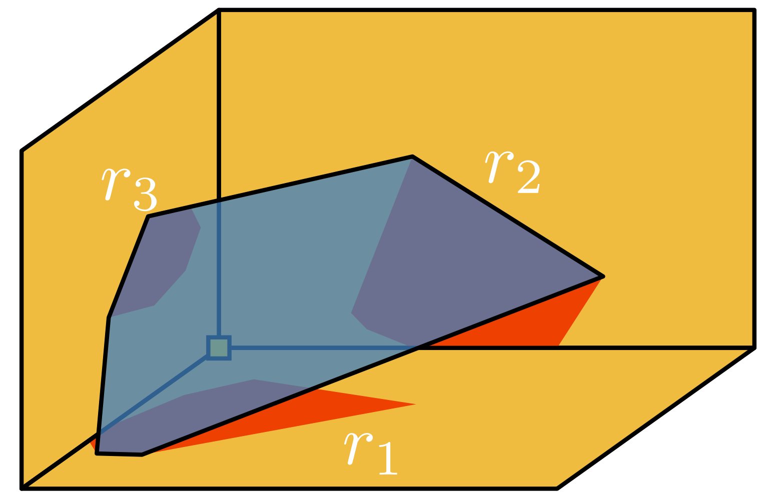

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- if $r^* \in r_i$: at most 5 complete octants with 1 blue polygon
- if $r^* \notin r_i$: at most 4 complete octants with 2 blue polygon

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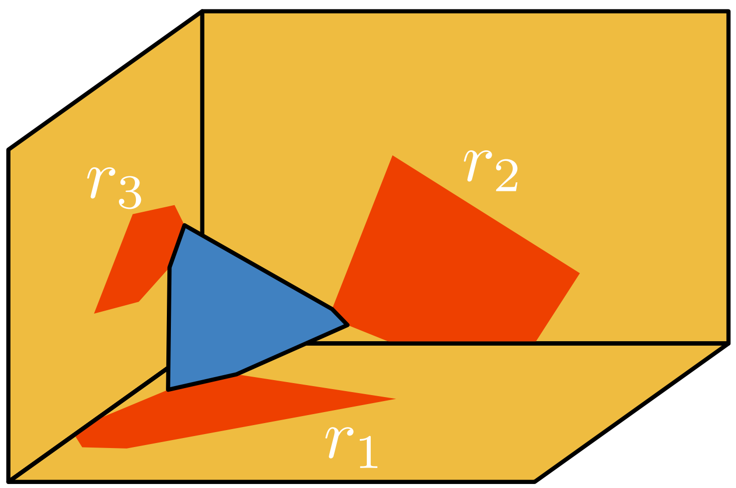

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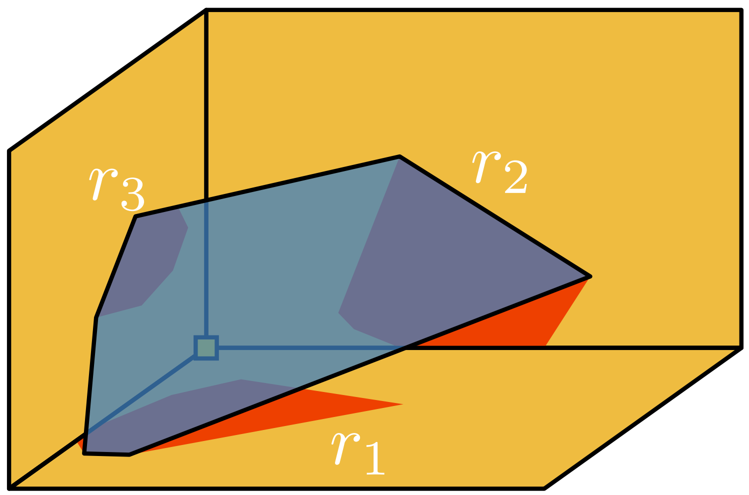

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➡ at most 8 blue polygons

➡ $K_{3,9}$ has no **1-sided realization.**

Form 1-Sided to General

$K_{5,81}$ has no realization with convex polygons.

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➡**By the Kövari–Sós–Turán theorem, the maximum edge density for a realizable graph is $O(n^{1.8n})$.**