Universal Point Sets for Planar Graph Drawings with Circular Arcs

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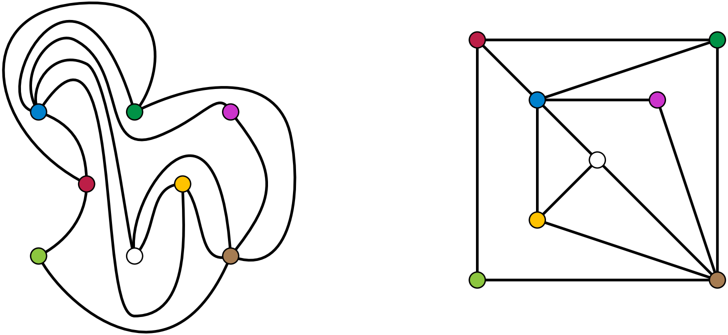
Fáry's theorem

Graphs that can be drawn with non-crossing curved edges can also be drawn with non-crossing straight edges
[Wagner 1936; Fáry 1948; Stein 1951]

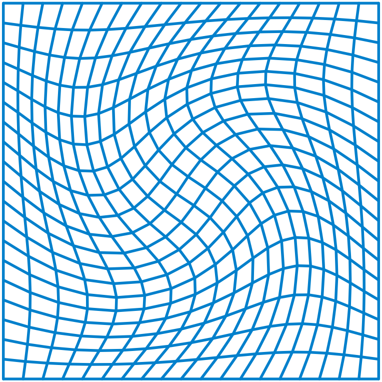


...but not necessarily with the same vertex positions!

The set of points in $\mathbb{R}^2$ is **universal** for straight drawings: it can be used to form the vertex set of any planar graph

Smaller universal sets than the whole plane?

Every set of n points is universal for topological drawings (edges drawn as arbitrary curves) of n-vertex graphs



Simply deform the plane to move the vertices where you want them, moving the edges along with them

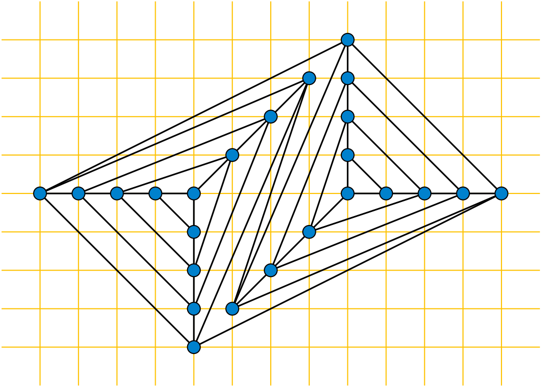
PD image File:Diffeomorphism of a square.svg by Oleg Alexandrov from Wikimedia commons

Universal grids for straight line drawings

$O(n) \times O(n)$ square grids are universal

[de Fraysseix et al. 1988; Schnyder 1990]

Some graphs require $\Omega(n^2)$ area when drawn in grids



Big gap for universal sets for straight line drawings

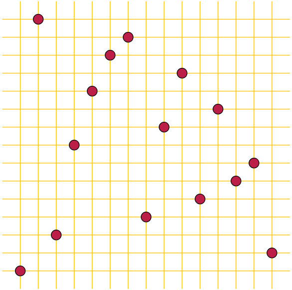
Best upper bound on universal point sets for straight-line drawing:

$$n^2/4 - O(n)$$

Based on *permutation patterns* [Bannister et al. 2013]

This 15-element permutation contains all 6-element 213-avoiding permutations

Exponential stretching produces an 18-point universal set for 9-vertex straight line drawings



Best lower bound: $1.098n - o(n)$ [Chrobak and Karloff 1989]

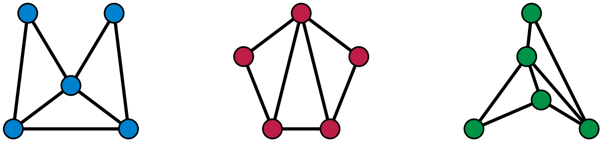
Two paths to perfection

Perfect universal set: exactly n points

Don't exist for straight drawings, $n \geq 15$ [Cardinal et al. 2012]
so have to relax either "straight" or "planar".

Every n-point set in general position is universal for

- paths (connect in coordinate order)
- trees
- outerplanar graphs [Gritzmann et al. 1991]



What about drawing all planar graphs but relaxing straightness?

Arc diagrams

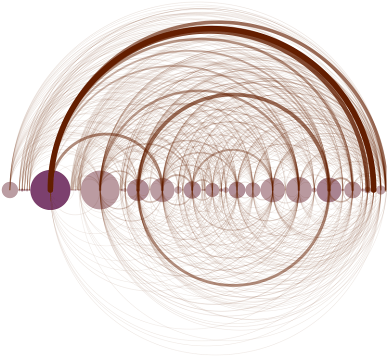
Vertices placed on a line; edges drawn on one or more semicircles

Initially used for drawing nonplanar graphs with few crossings

[Saaty 1964; Nicholson 1968]

Later named and popularized in information visualization

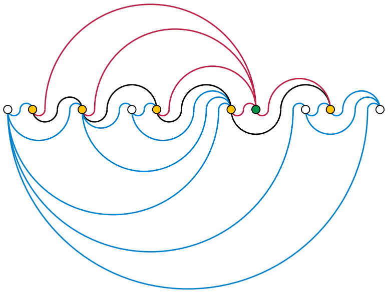
[Wattenberg 2002]



Visualization of internet chat connections, Martin Dittus, 2006, http://datavis.dekstop.de/irc_arcs/

Monotone topological 2-page book embeddings

Every planar graph has a planar arc diagram with each edge drawn as a two-semicircle “S” curve [Giordano et al. 2007; Bekos et al. 2013]



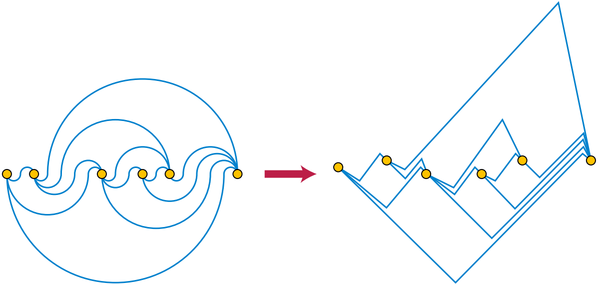
- Add edges to make the graph maximal
- Find canonical order (each vertex above earlier ones, neighbors form contiguous path on upper boundary)
- Add each vertex to the right of its penultimate neighbor

(Useful property: $\leq n-1$ inflections between consecutive vertices)

Perfect universal sets from monotone embeddings

Every *n* points on a line are universal for drawings in which edges are smooth curves formed from two circular arcs

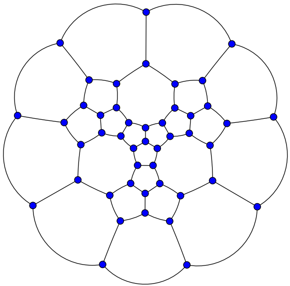
Every set of *n* points is universal for polyline drawings with two bends per edge (mimic semicircles with steep zigzags)



Every smooth convex curve contains *n* points that are universal for polyline drawings with **one** bend per edge [Everett et al. 2010]

Drawings with no bends and no inflections

What if we require each edge to be a single circular arc?



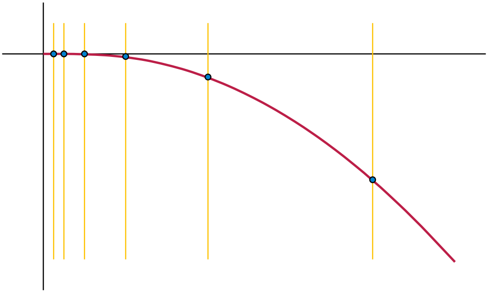
Lombardi drawing of a 46-vertex non-Hamiltonian graph with cyclic edge connectivity five [Grinberg 1968; Eppstein 2013]

Arc diagrams don't always exist and are NP-complete to find

Much recent interest in *Lombardi drawings* (evenly spaced edges at each vertex) [Duncan et al. 2012; Eppstein 2013] and *smooth orthogonal layouts* (axis-aligned arcs) [Bekos et al. 2013]

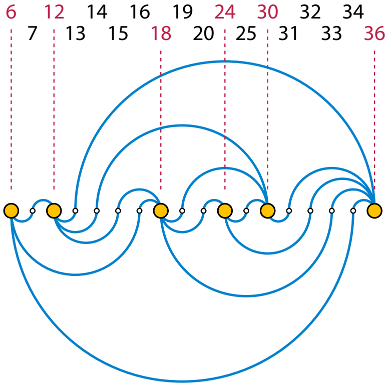
Our result

For every n, there exists a perfect universal point set
for drawings with circular-arc edges



Construction:
Choose n points on the parabola $y = -x^2$
at x-coordinates 2^n, 2^{2n}, 2^{3n}, $\ldots$ 2^{n^2}

How to draw a graph on this universal set

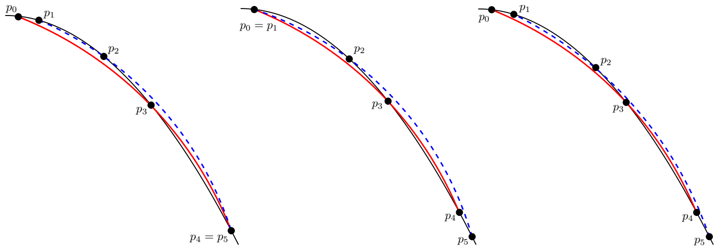


- Draw monotone topological book embedding
- Number vertices and inflection points from left to right, rounding vertex numbers up to multiples of n
- Map point i to point on parabola with $x = 2^i$
- Draw each edge as an arc through its three points

Why is the resulting drawing planar?

Key properties, proved with some algebra:

Arc through any three points on parabola crosses it once from below to above ⇒ edges pass above/below vertices correctly



For six points $x_0 \leq x_1 < x_2 < x_3 < x_4 \leq x_5$, spaced exponentially, arcs $x_0x_3x_4$ and $x_1x_2x_5$ are disjoint ⇒ edges do not cross

Conclusions

Perfect universal sets for circular-arc drawings

Purely a theoretical result—drawings are not usable

- Vertex placement requires exponential area
- Edges have very small angular resolution

In contrast, arc diagrams (with one arc per edge) are very usable and practical but can only handle a subset of planar graphs

Maybe some way of combining the advantages of both?

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