Eliminating Crossings in Ordered Graphs

TCS Colloquium @ UJ — SWAT 2024

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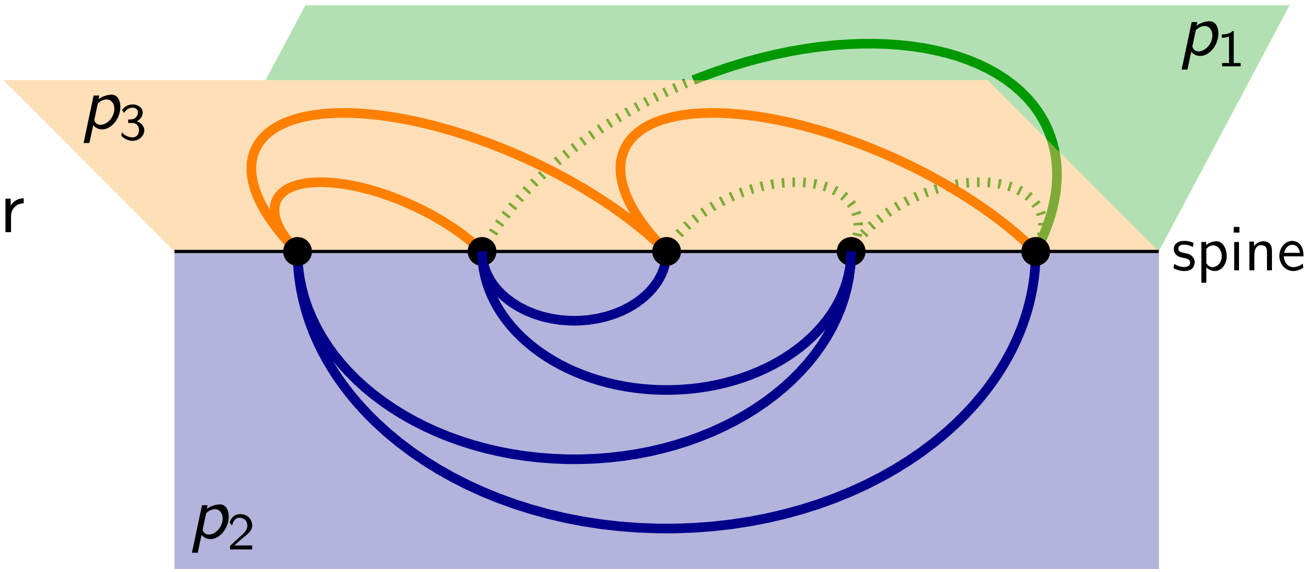
book embedding
with fixed vertex order



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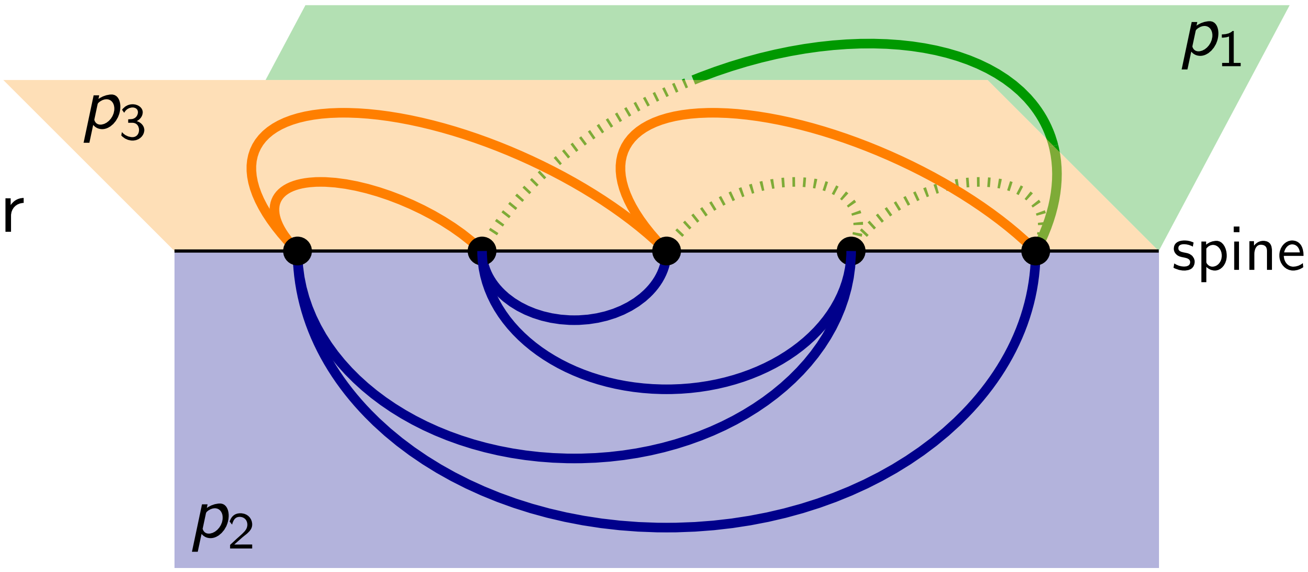

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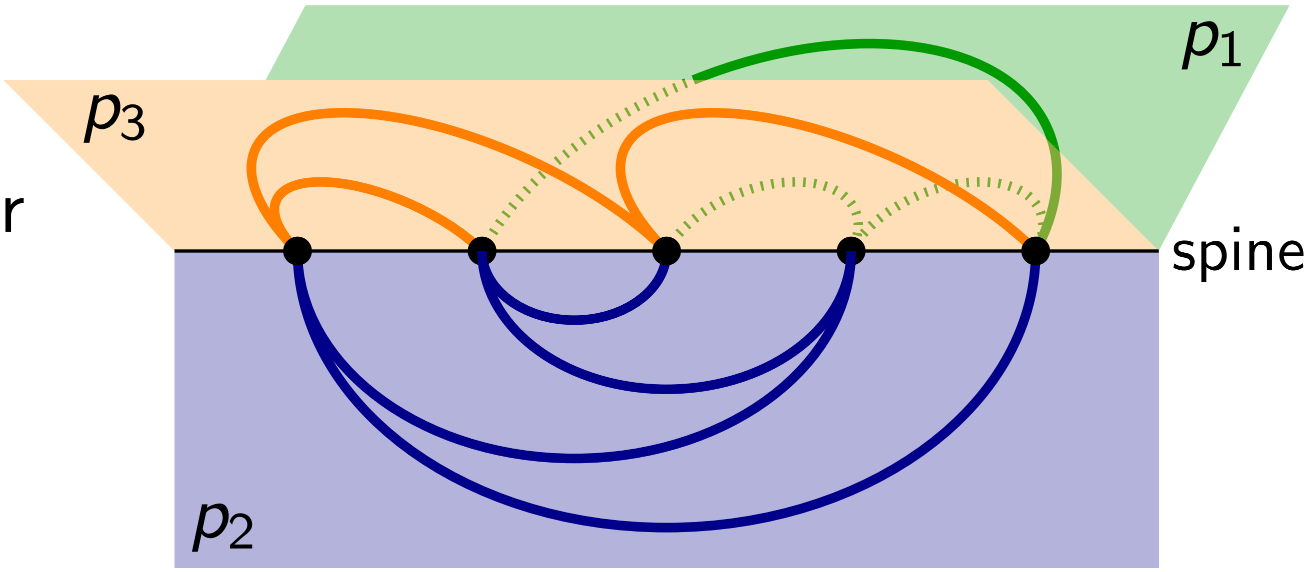

Yet another option: Remove part of every edge (e.g., middle half) → *partial edge drawings* (not today).

The Problem

EDGE DELETION TO p-PAGE d-PLANAR

Input:	ordered graph (G, σ), positive integers k, p, d.
Parameters:	k, p, d
Question:	Does there exist a set S of at most k edges of G such that $(G - S, \sigma)$ is p-page d-planar?

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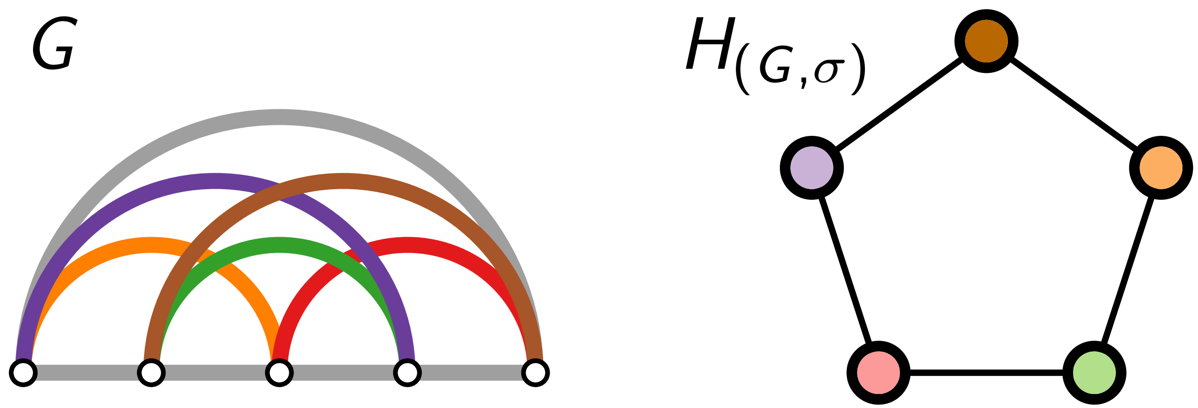
Given an ordered graph (G, σ), its *conflict graph* $H_{(G,\sigma)}$ is the graph that has a vertex for each edge of G and an edge for each pair of crossing edges of G.



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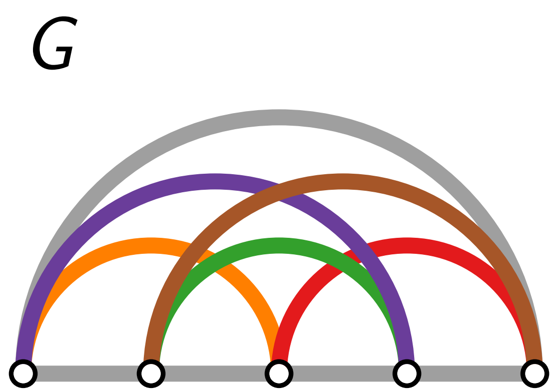

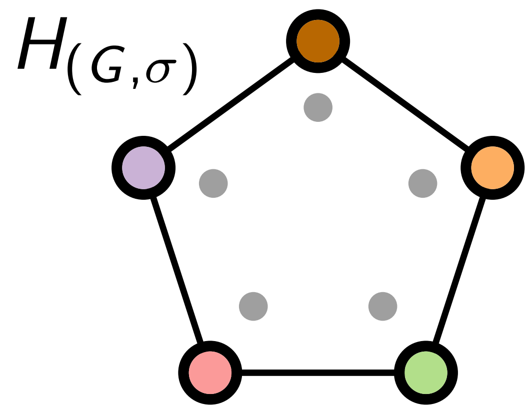

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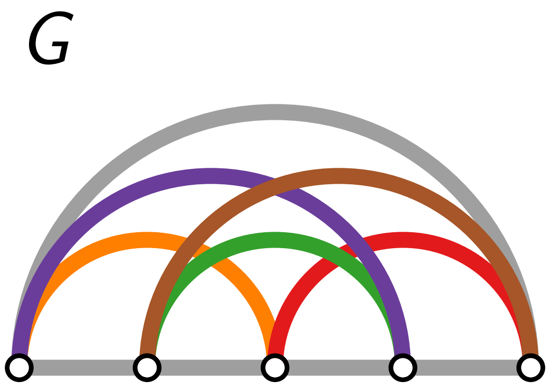

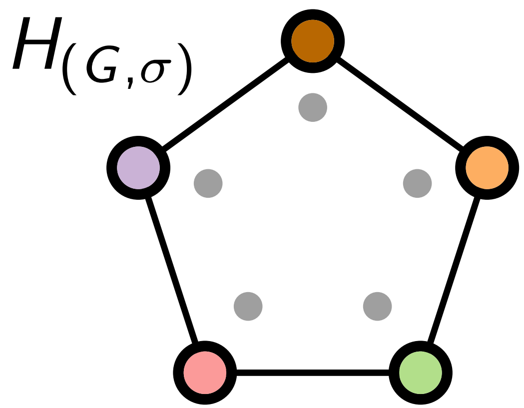

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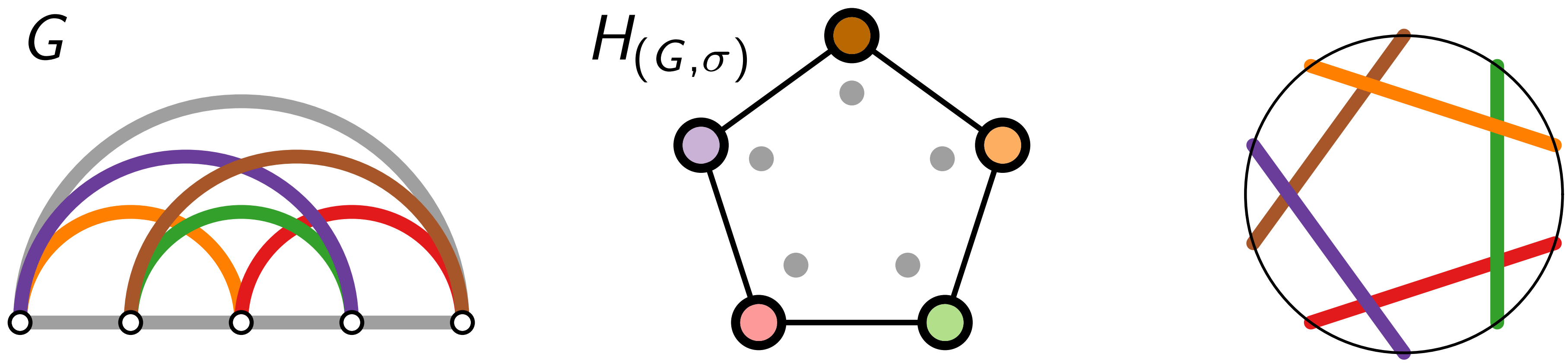

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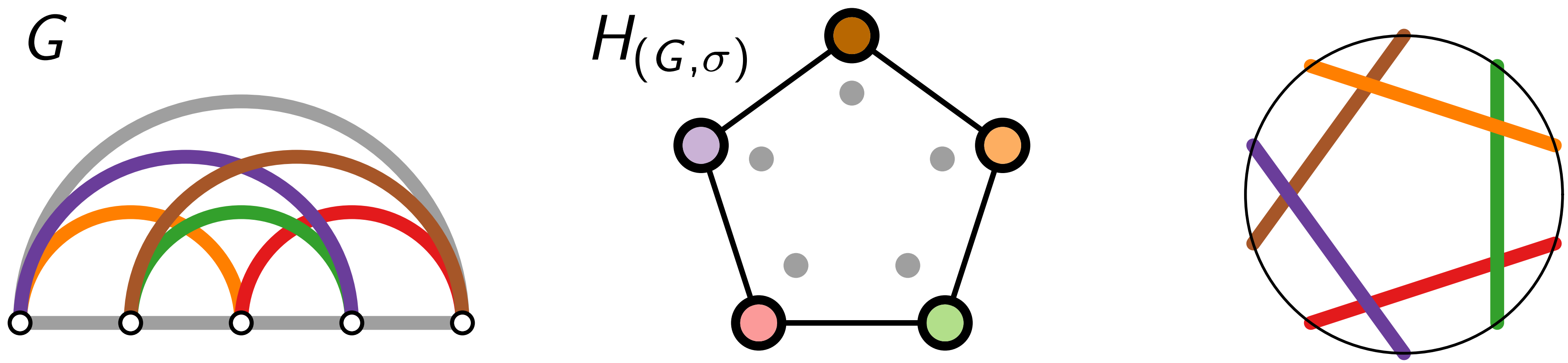

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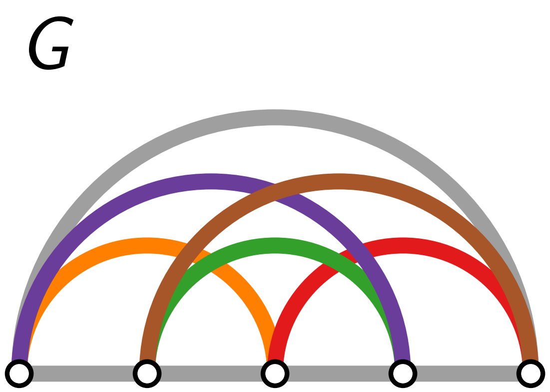

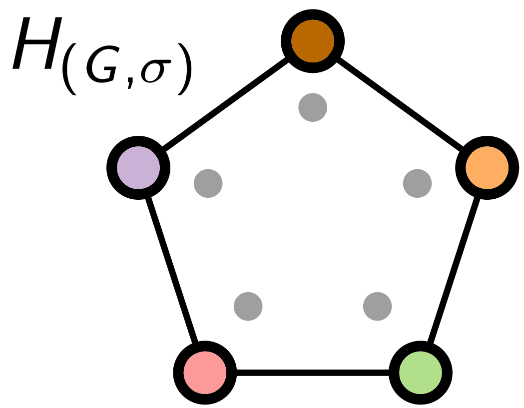

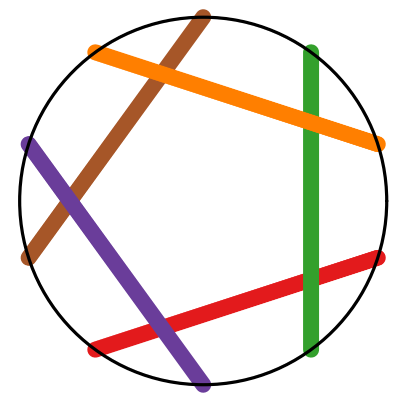

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For general graphs, this admits a quadratic kernel. [Xiao, 2017]

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$p = 2$: ODD CYCLE TRANSVERSAL in circle graphs – FPT [Reed, Smith, Vetta, 2004]

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Bhore et al. [2020] also study the flexible vertex-order case: They solve Page Number in $2^{\mathrm{vc}^{O(\mathrm{vc})}} + \mathrm{vc} \log \mathrm{vc} \cdot n$ time.

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- We can compute the fixed-vertex-order page number of an ordered graph with m edges & n vertices in $2^m \cdot n^{O(1)}$ time. Alternatively, given a budget p of pages, we can compute a p-page book embedding with the min. number of crossings.

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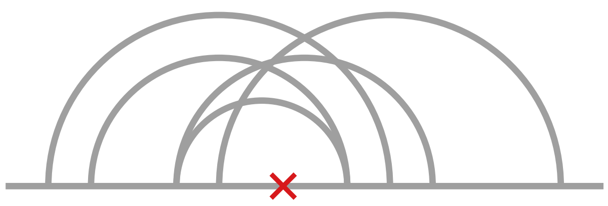

- Let h be the size of a *hitting set*.
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Theorem. Given $p \geq 1$ and an ordered graph (G, σ) with n vertices and m edges, we can compute the values $\mathrm{cr}_1(G, \sigma), \ldots, \mathrm{cr}_p(G, \sigma)$ in $2^m \cdot n^{O(1)}$ time.

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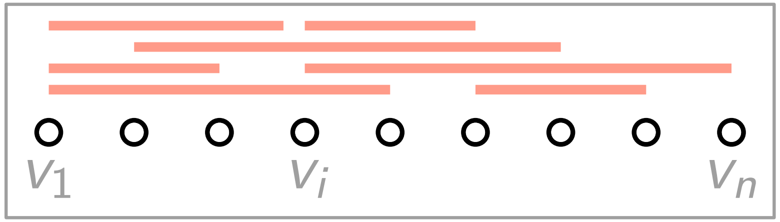

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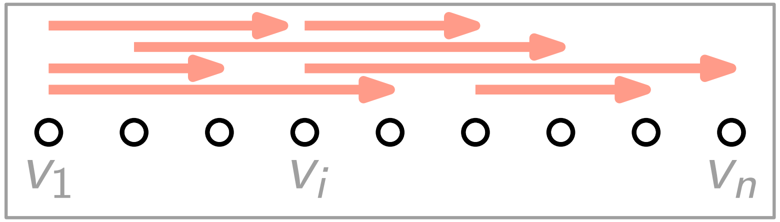

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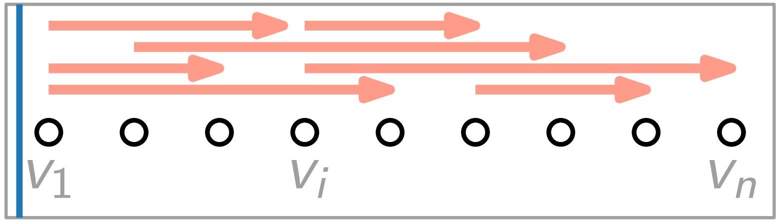

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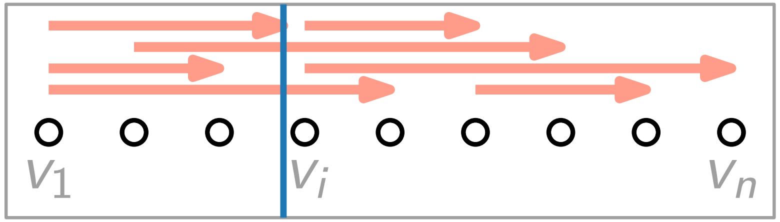

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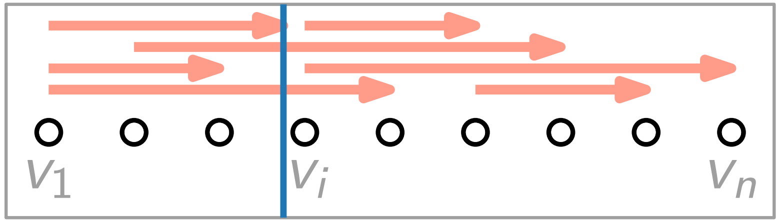

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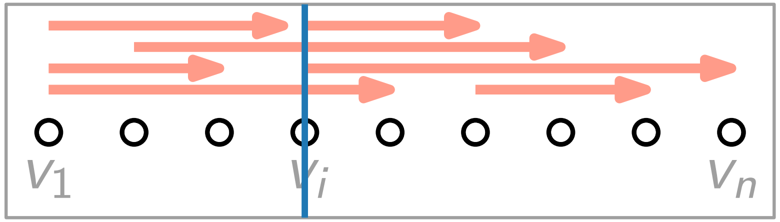

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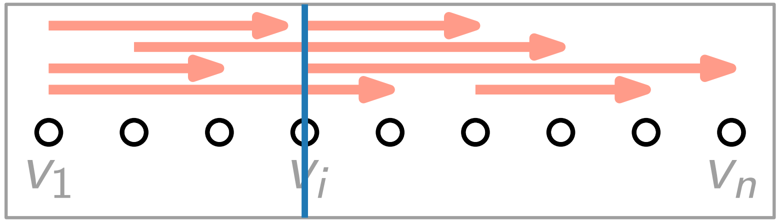

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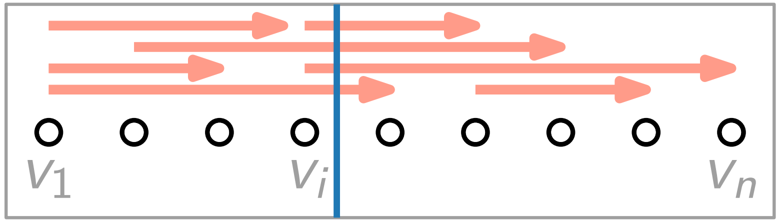

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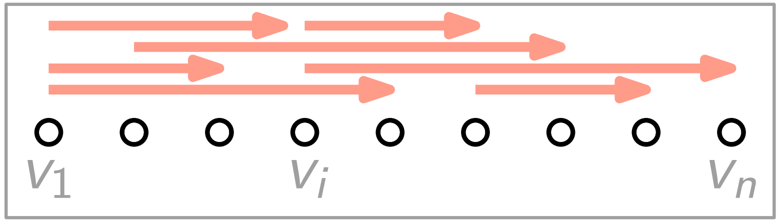

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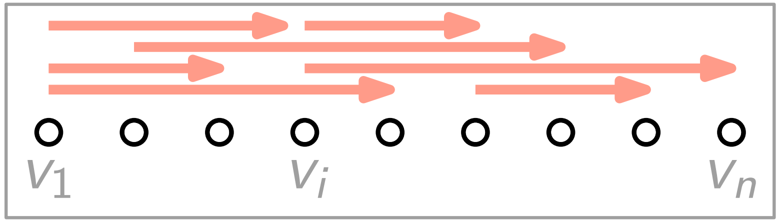

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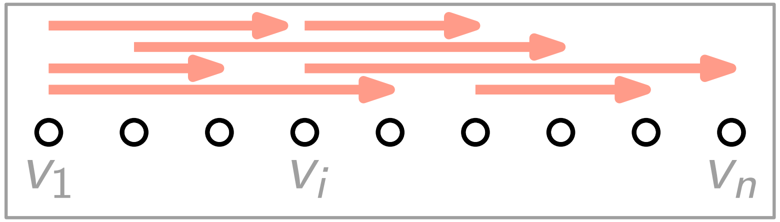

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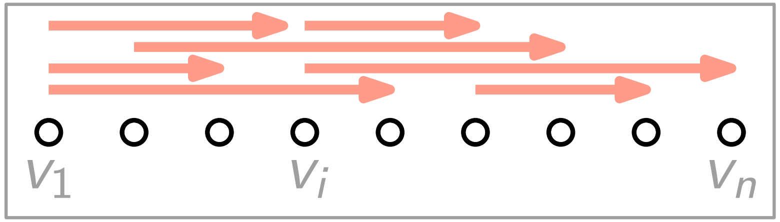

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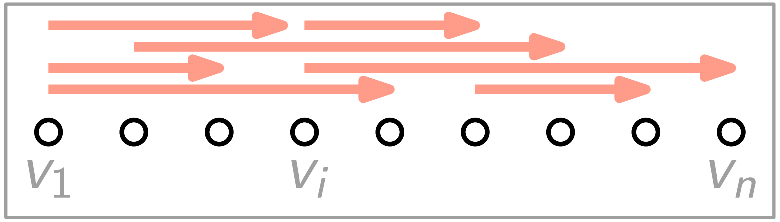

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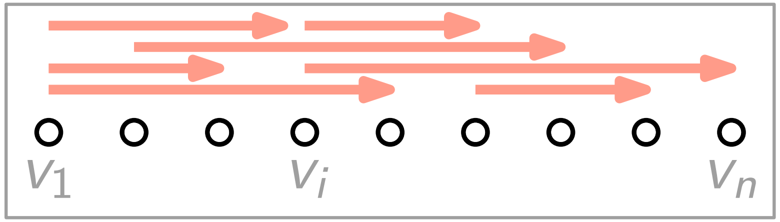

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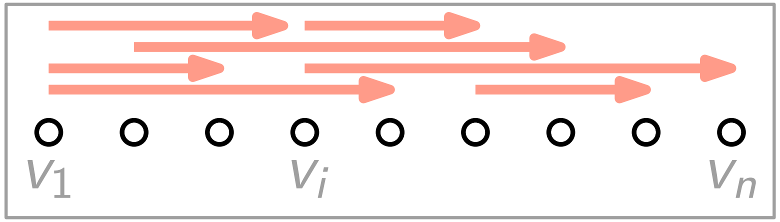

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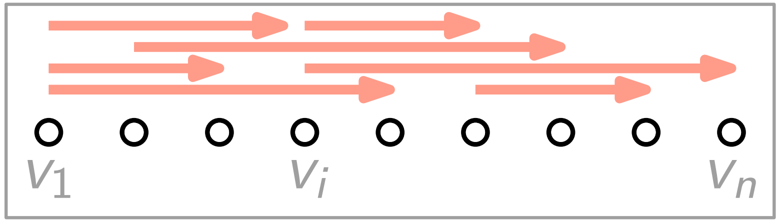

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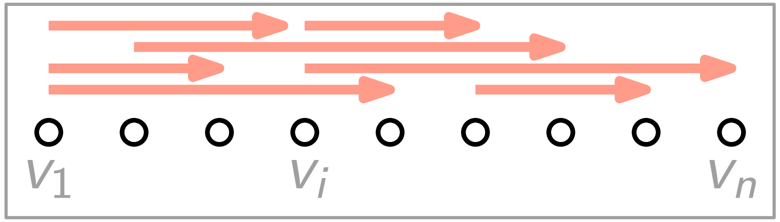

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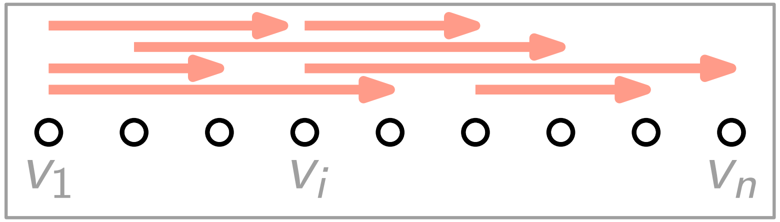

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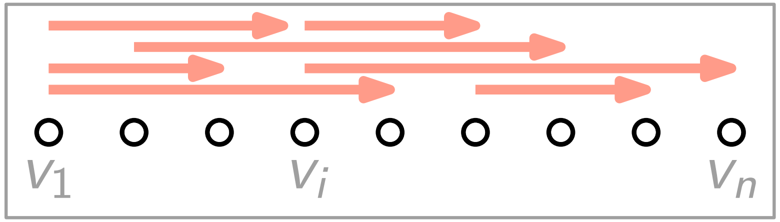

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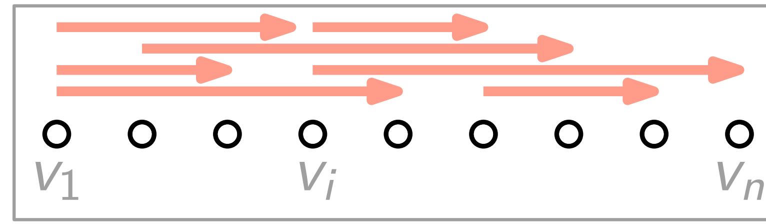

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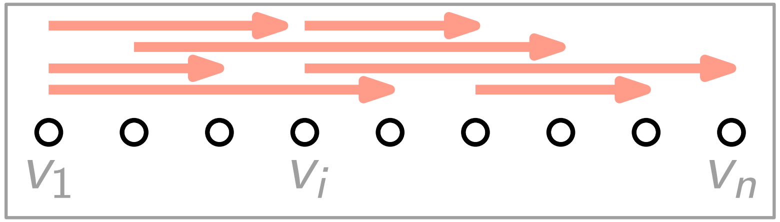

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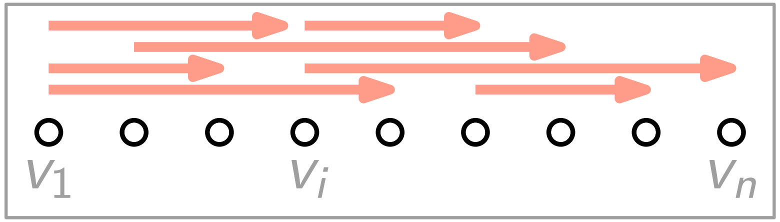

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Theorem. Given $p \geq 1$ and an ordered graph (G, σ) with n vertices and m edges, we can compute the values $\mathrm{cr}_1(G, \sigma), \ldots, \mathrm{cr}_p(G, \sigma)$ in $\tilde{O}(p \cdot m^2 2^m)$ time.

So we can compute $\mathrm{cr}_1(G[F], \sigma)$ for all $F \subseteq E(G)$ in $\tilde{O}(m2^m)$ total time.

For $q > 1$ and $F \subseteq E(G)$, we have the recurrence

$$\mathrm{cr}_q(G[F], \sigma) \; = \; \min_{F' \subseteq F} \{ \mathrm{cr}_1(G[F'], \sigma) + \mathrm{cr}_{q-1}(G[F \setminus F'], \sigma) \}.$$

Brute-force computation takes time $O(m) \cdot \sum_{i=1}^{m} \binom{m}{i} 2^i = \tilde{O}(m3^m)$.

Instead, do *subset convolution*! [Björklund, Husfeldt, Kaski, Koivisto, '07]

Define two functions $f, g \colon 2^{E(G)} \to R$ to the $(\min, +)$-ring R:

$f \colon F \mapsto \mathrm{cr}_1(G[F], \sigma)$ and $g \colon F \mapsto \mathrm{cr}_{q-1}(G[F], \sigma)$ with $f, g(\cdot) \leq m^2$

Then $(f * g)(F) = \sum_{F' \subseteq F} f(F') \cdot g(F \setminus F')$ can be computed in $\tilde{O}(m^2 2^m)$.

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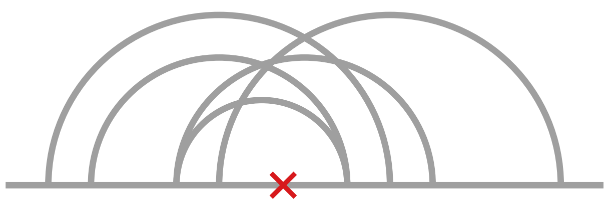
Compute solution $\mathcal{S}$ by greedily adding the set F in $\mathcal{F}$ that maximizes $|F \setminus \bigcup \mathcal{S}|$. Apply lemma to $G - \bigcup \mathcal{S}$.

Our Contribution

- We can compute the fixed-vertex-order page number of an ordered graph with m edges & n vertices in $2^m \cdot n^{O(1)}$ time.
  Alternatively, given a budget p of pages, we can compute a p-page book embedding with the min. number of crossings.
- We obtain an $O((d+1)\log n)$-approximation algorithm for the fixed-vertex-order d-planar page number.
- We show how to decide in $2^{O(c\sqrt{k}\log(c+k))} \cdot n^{O(1)}$ time whether deleting k edges of an ordered graph suffices to obtain a d-planar layout on *one* page.



- Let h be the size of a *hitting set*.
  $h = 1$: We can *efficiently* compute the smallest set of edges whose deletion yields fixed-vertex-order page number p.
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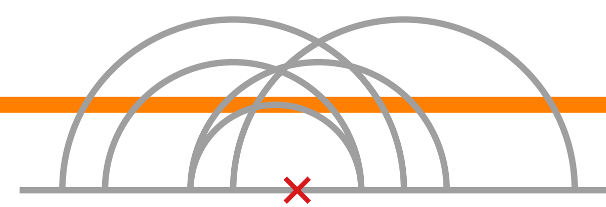

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For each mapping of the m edges to the p pages (allowing also for edge deletion), check for each page whether the edges assigned to it form an outerplanar graph.

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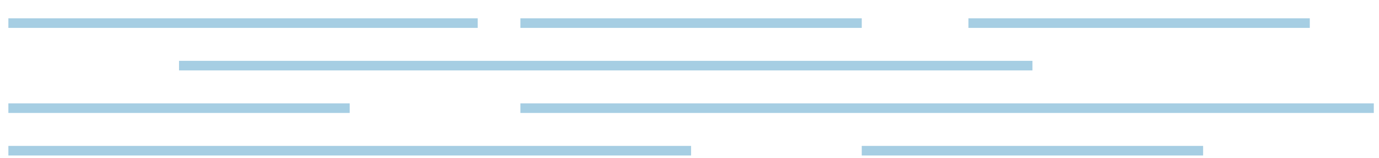

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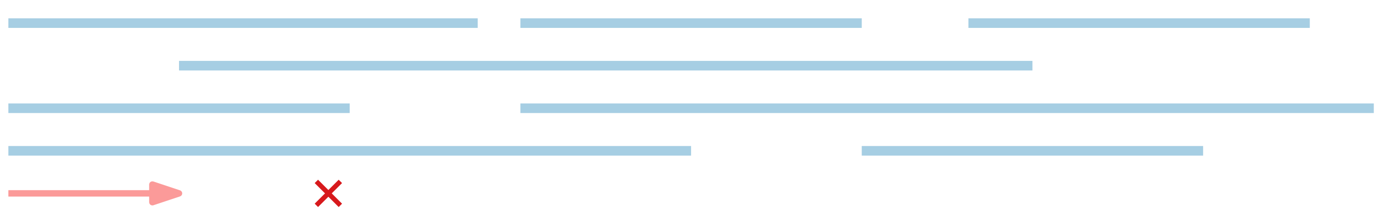

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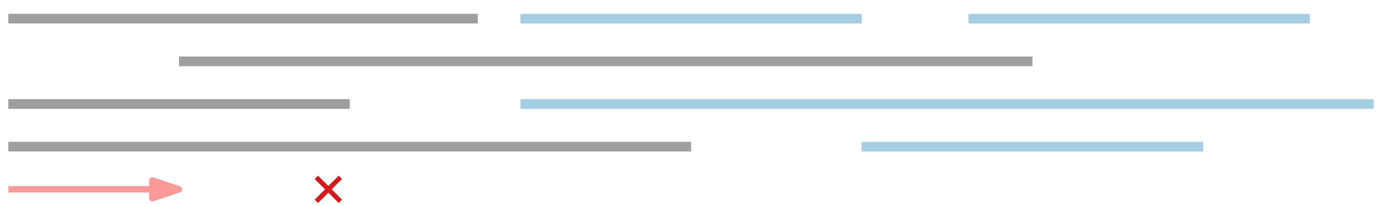

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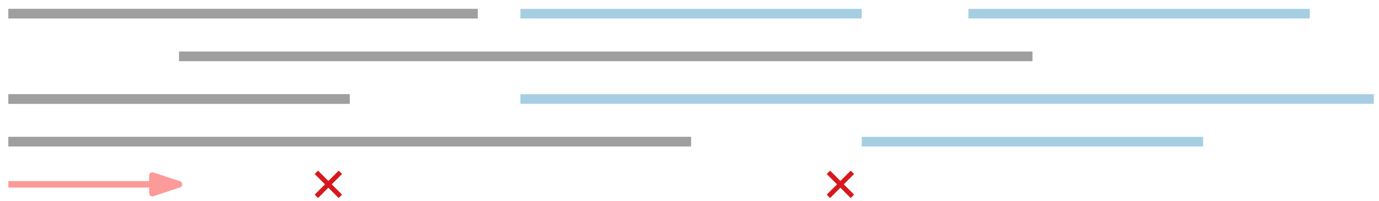

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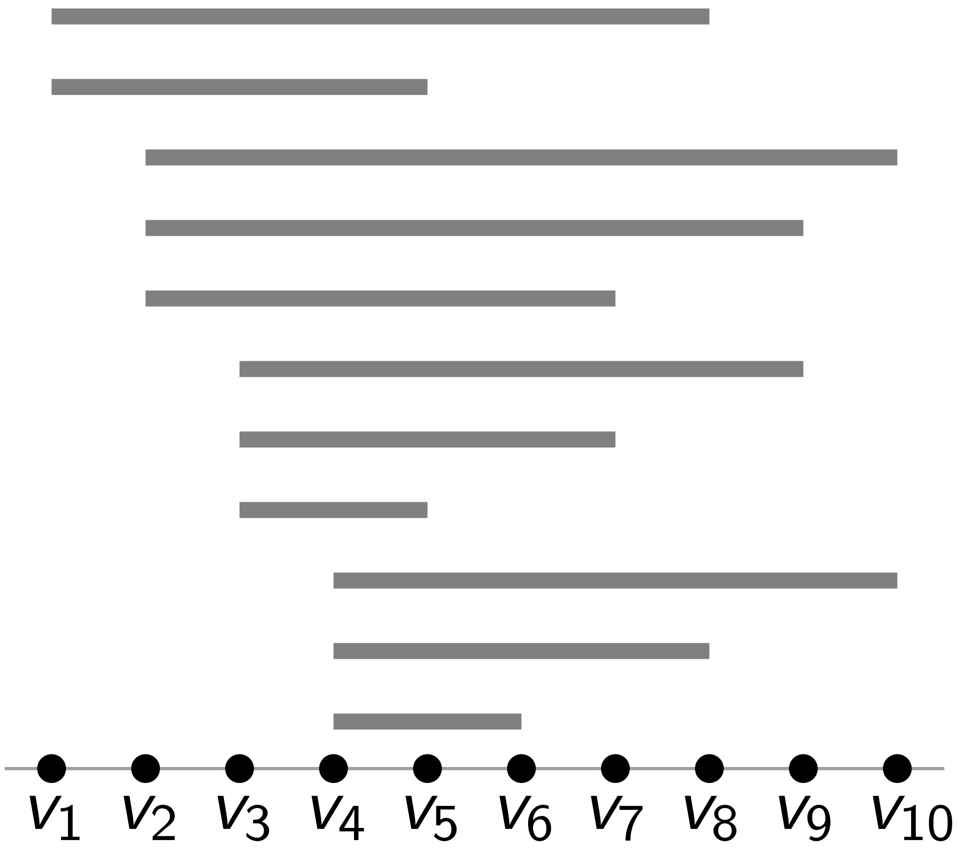

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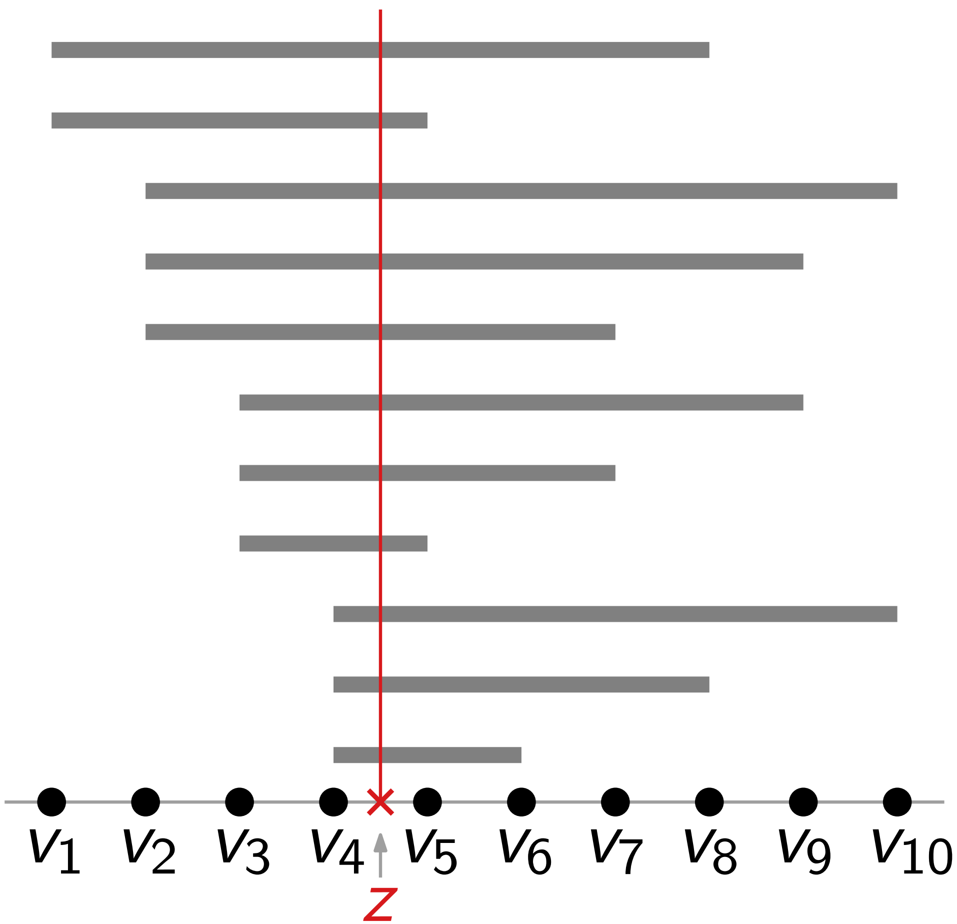

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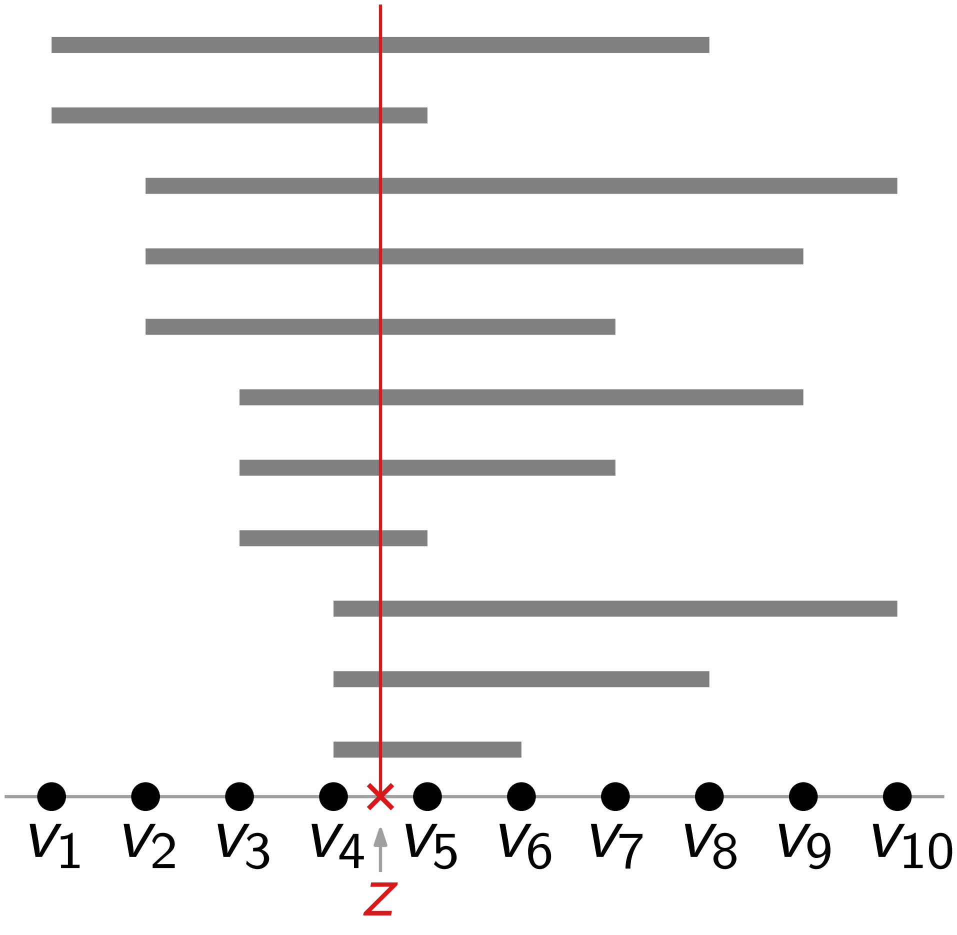

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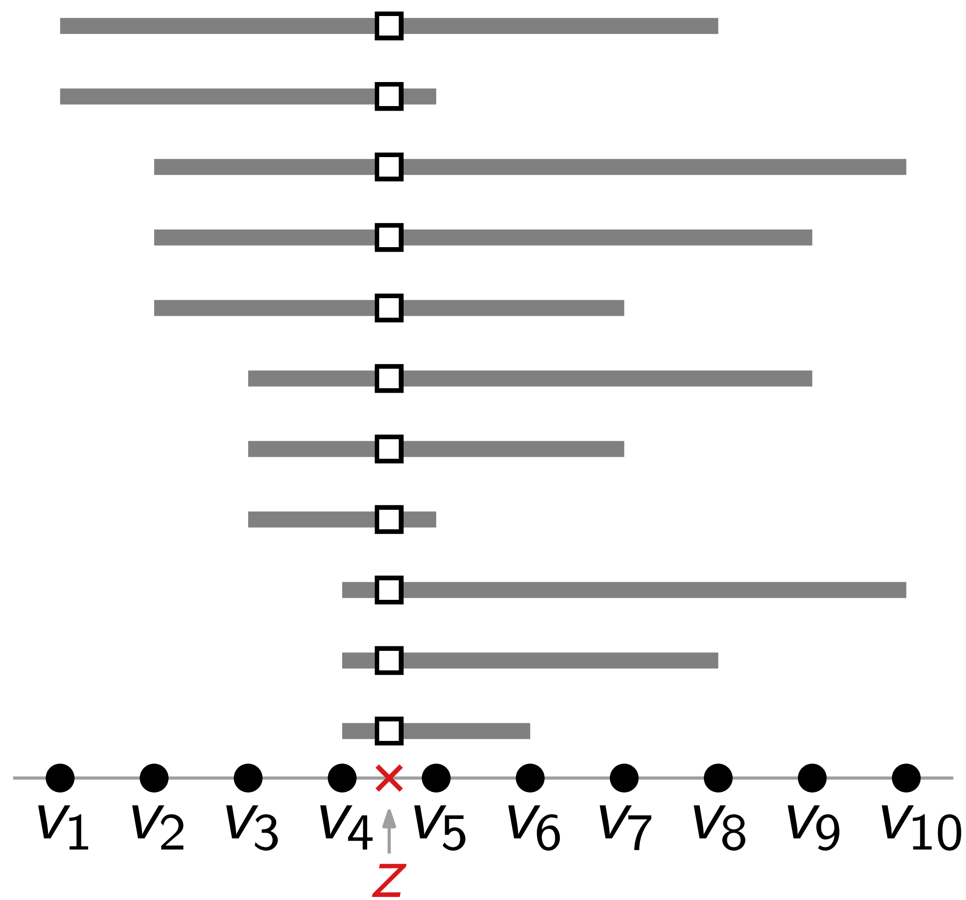

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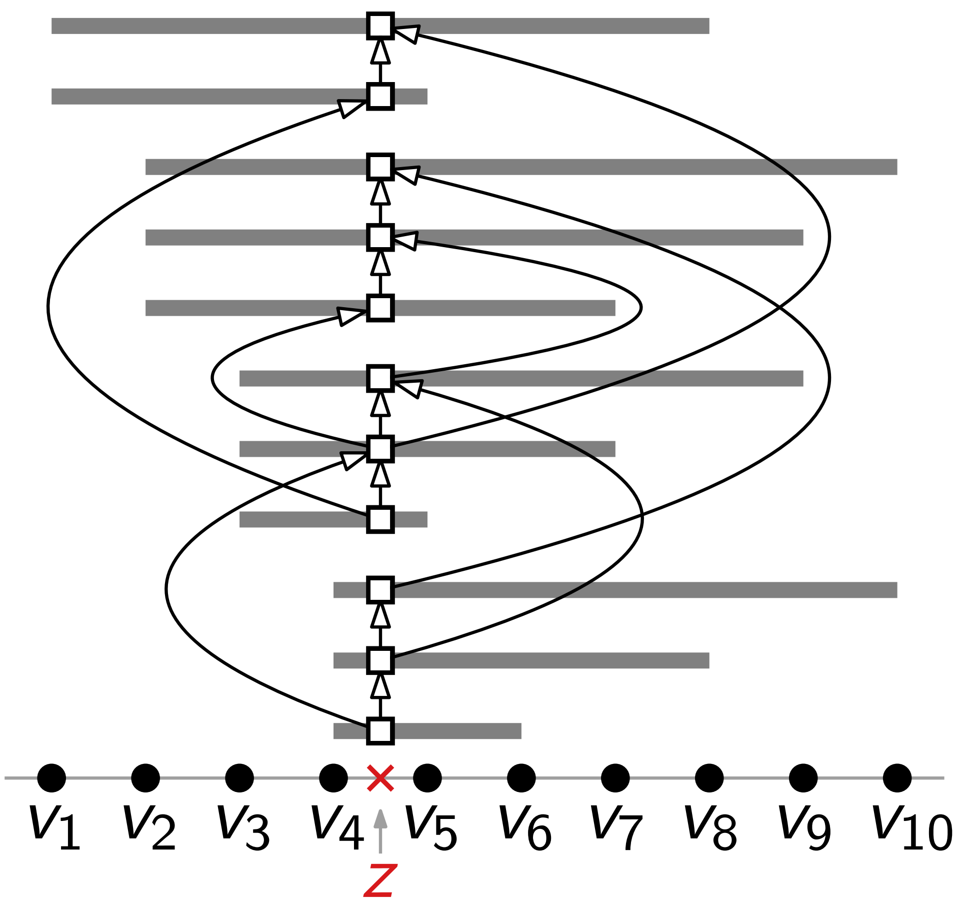

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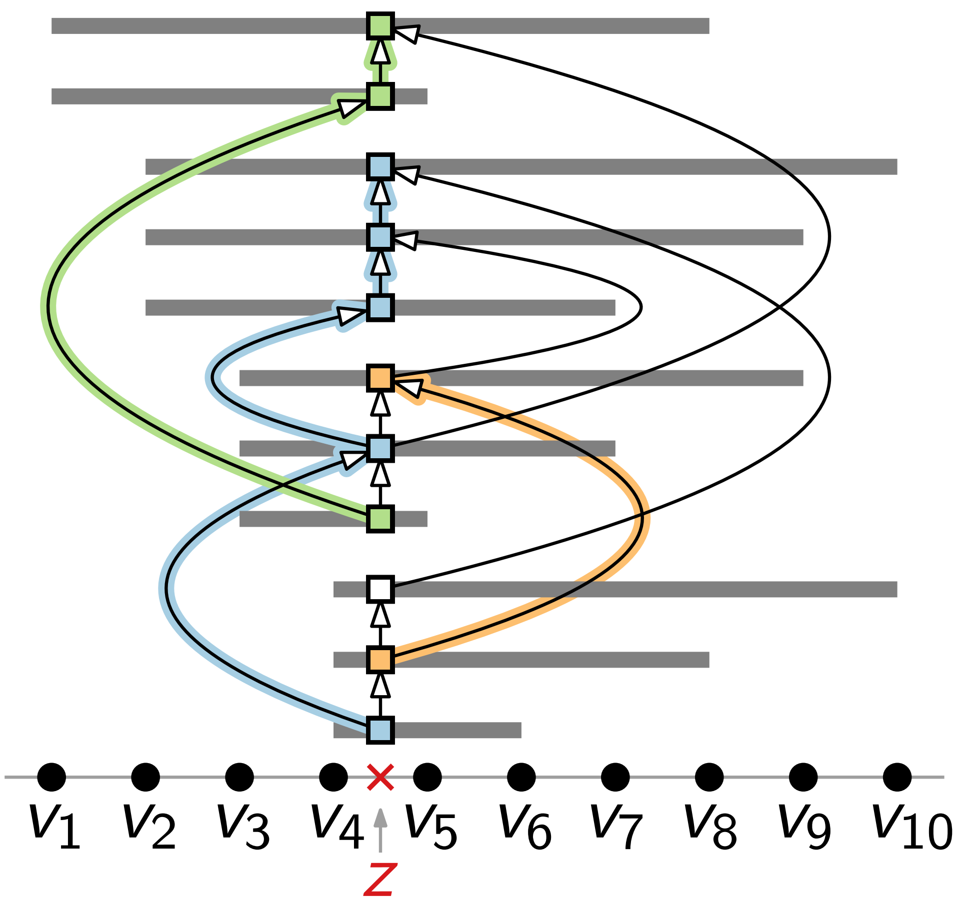

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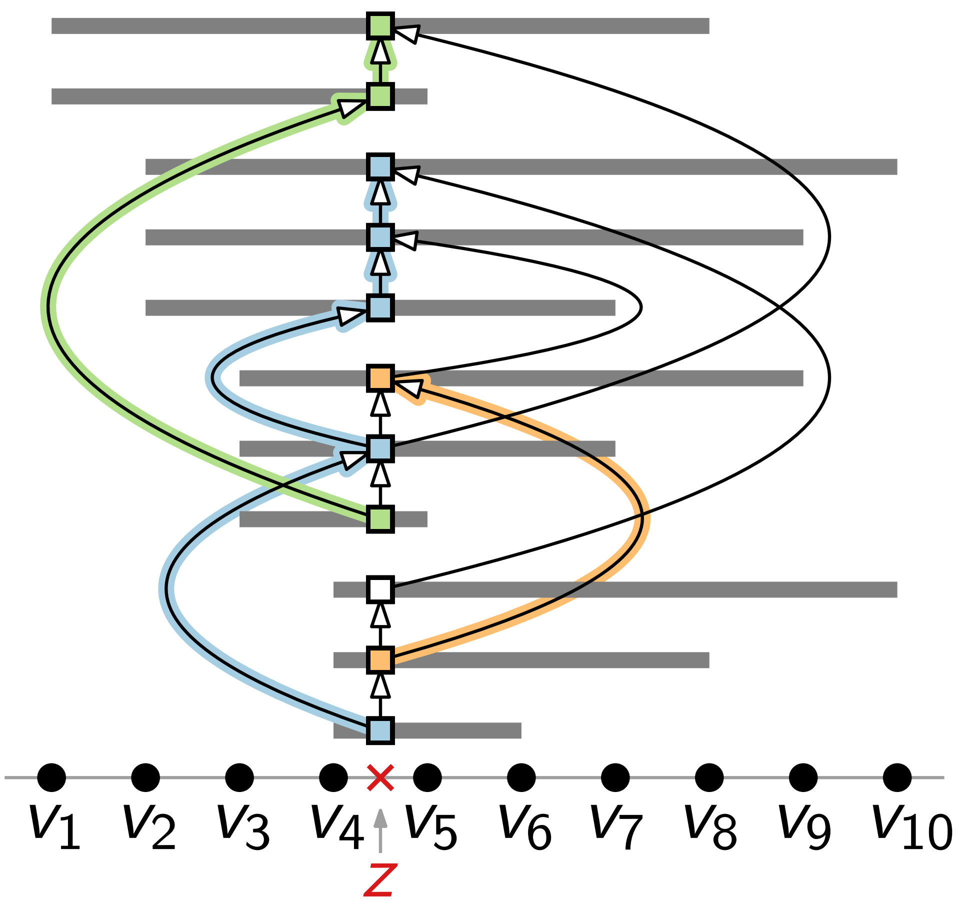

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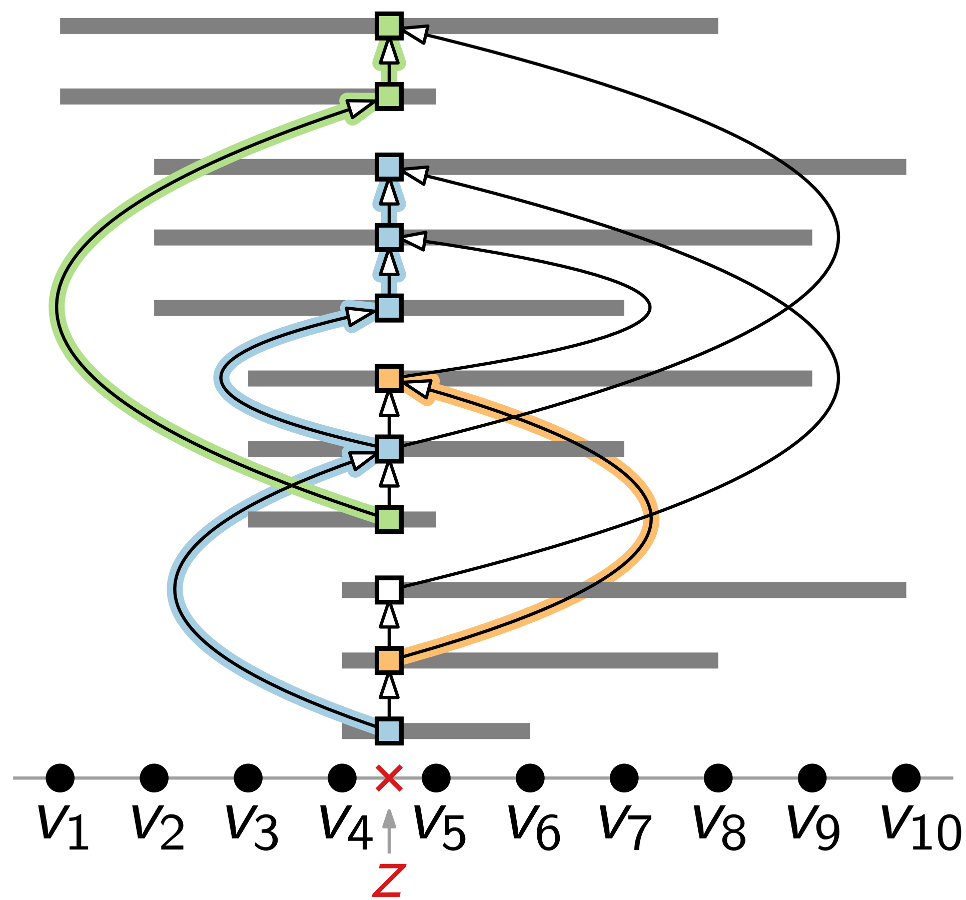

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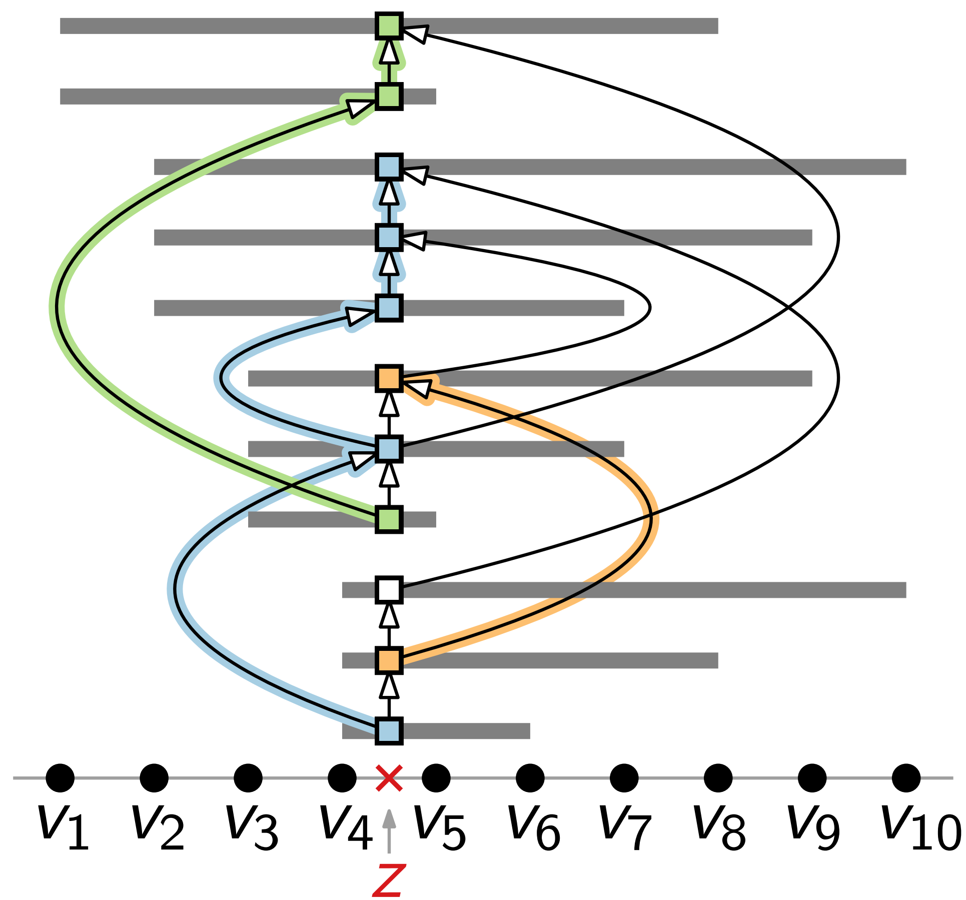

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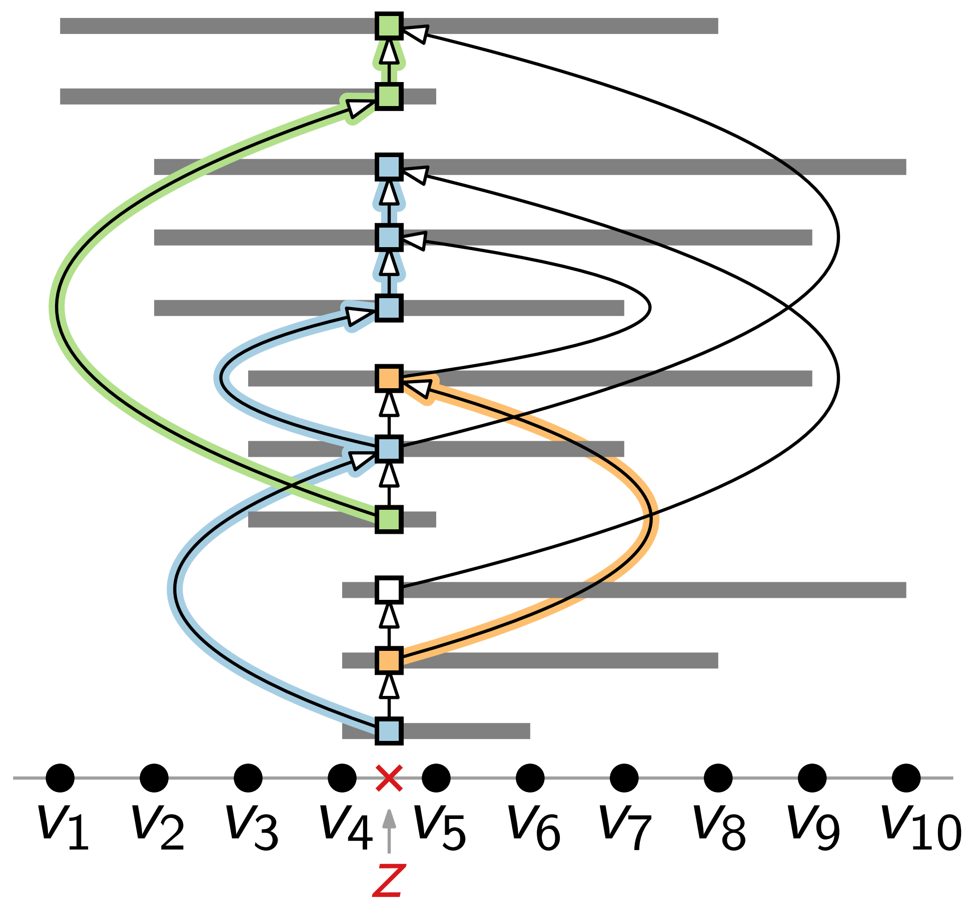
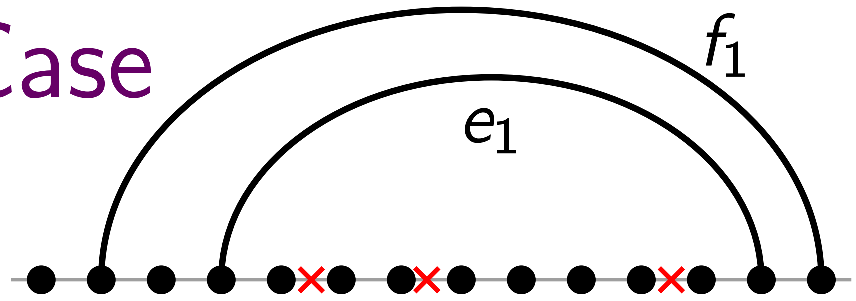


Preparing for the General Case



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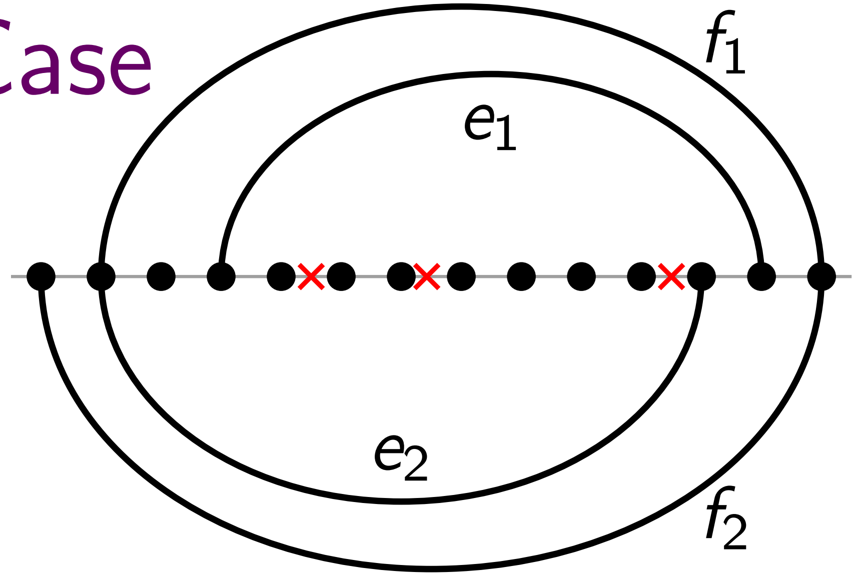

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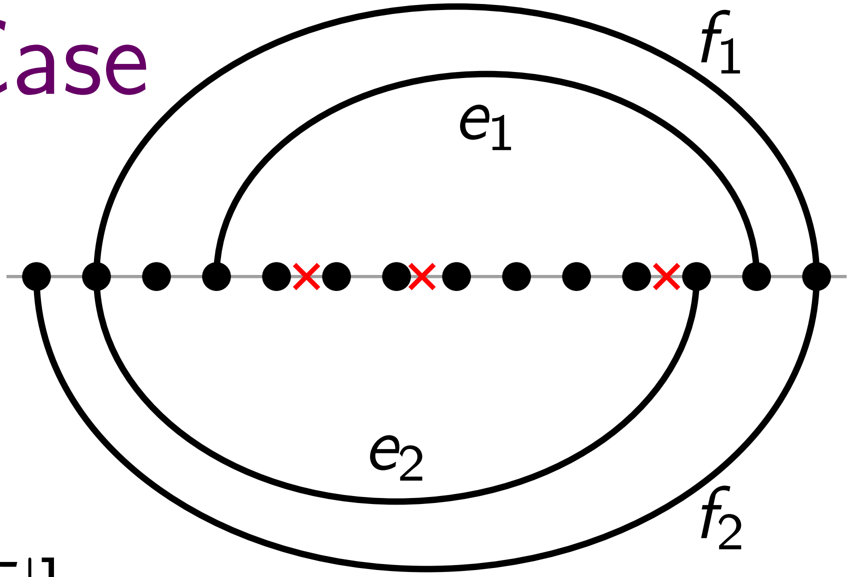

Two subsets $E, F \subseteq E(G)$ are *compatible* if $|E| = |F|$ and there is an enumeration $e_1, \ldots, e_{|F|}$ of E and an enumeration $f_1, \ldots, f_{|F|}$ of F s.t. e_i is contained in f_i for each $i \in [|F|]$.

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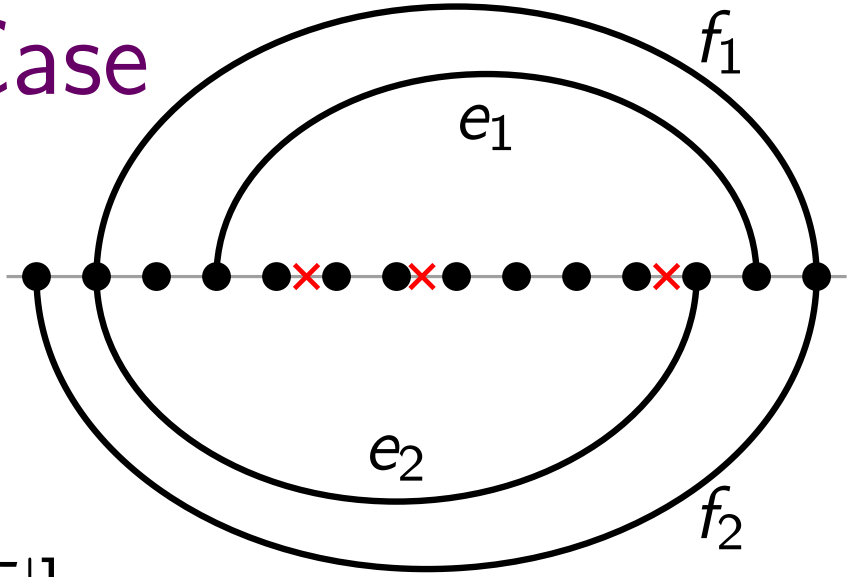

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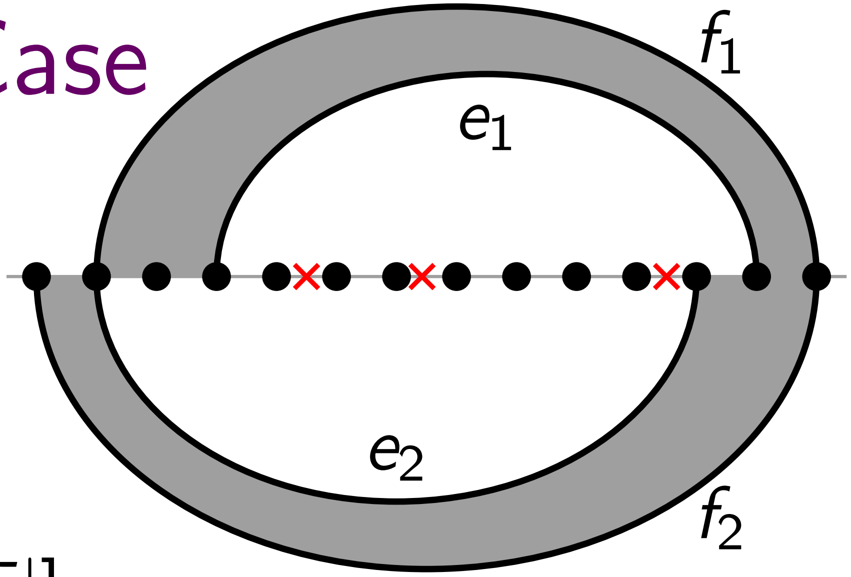

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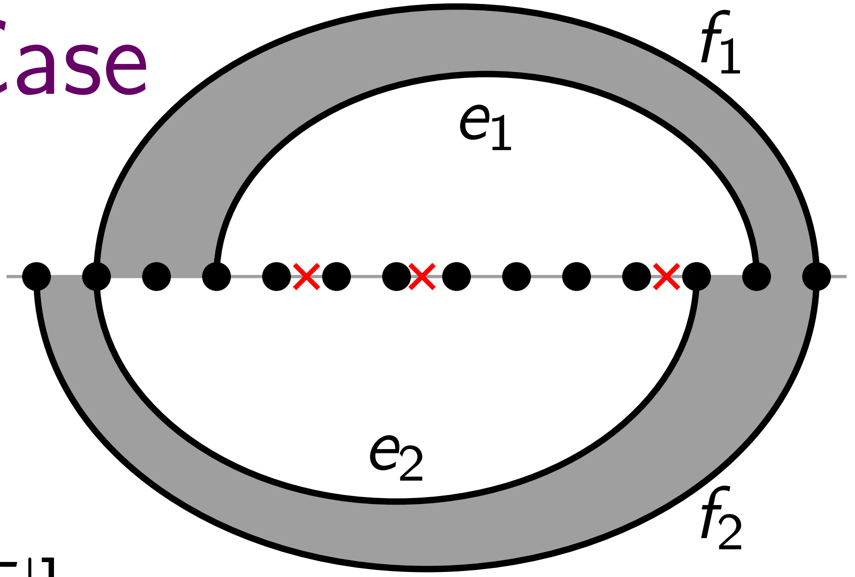

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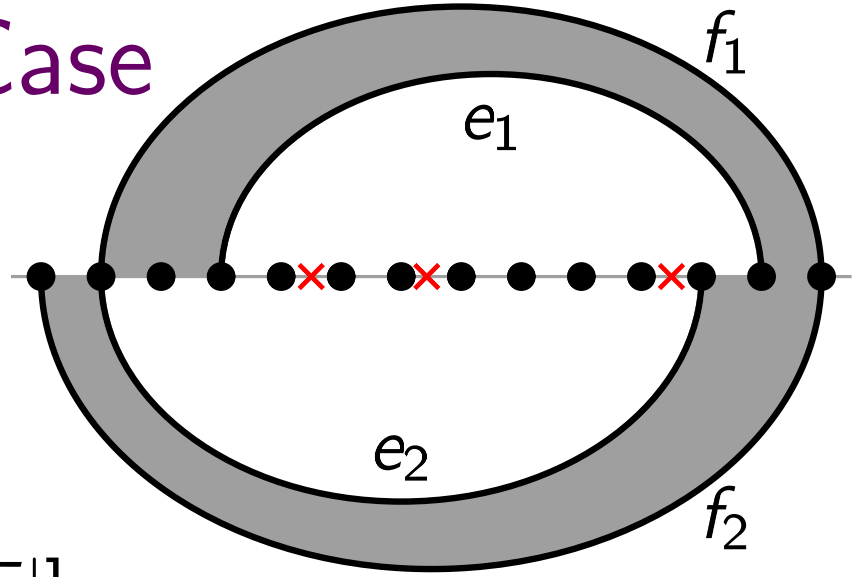

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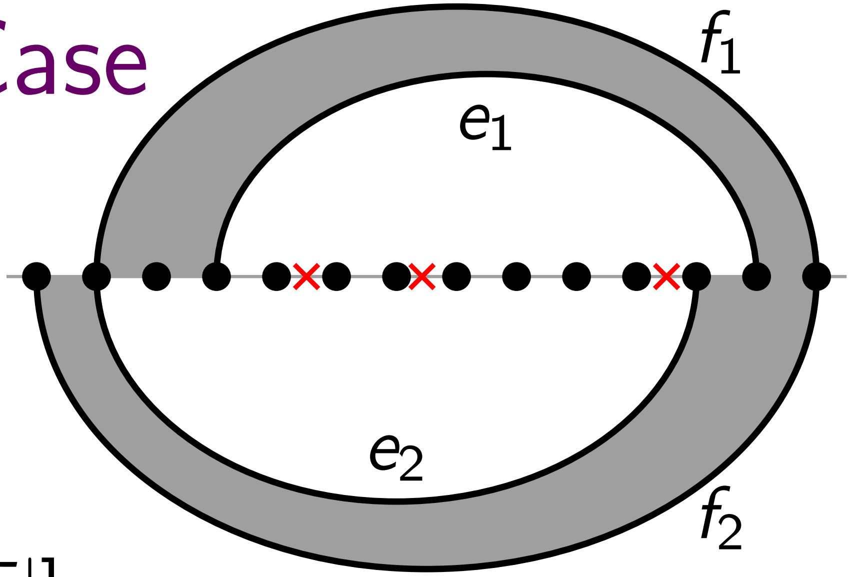
Preparing for the General Case



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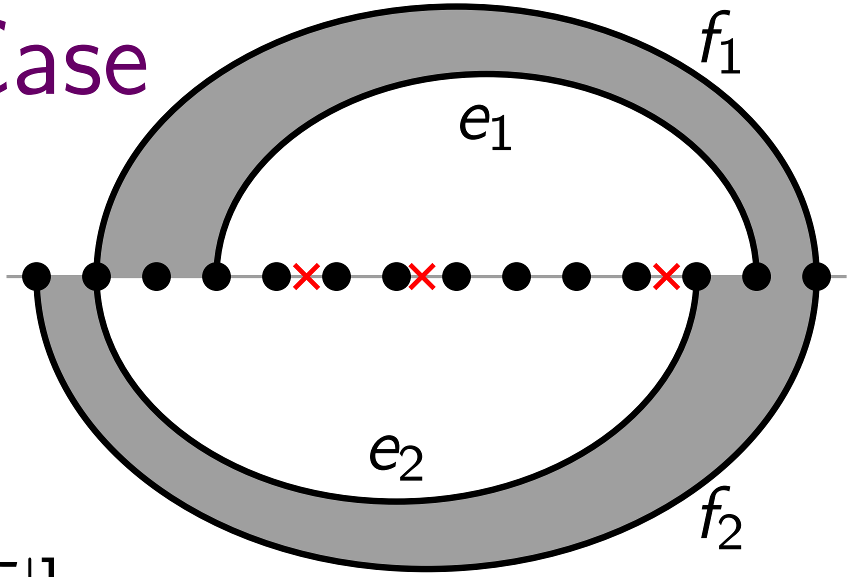

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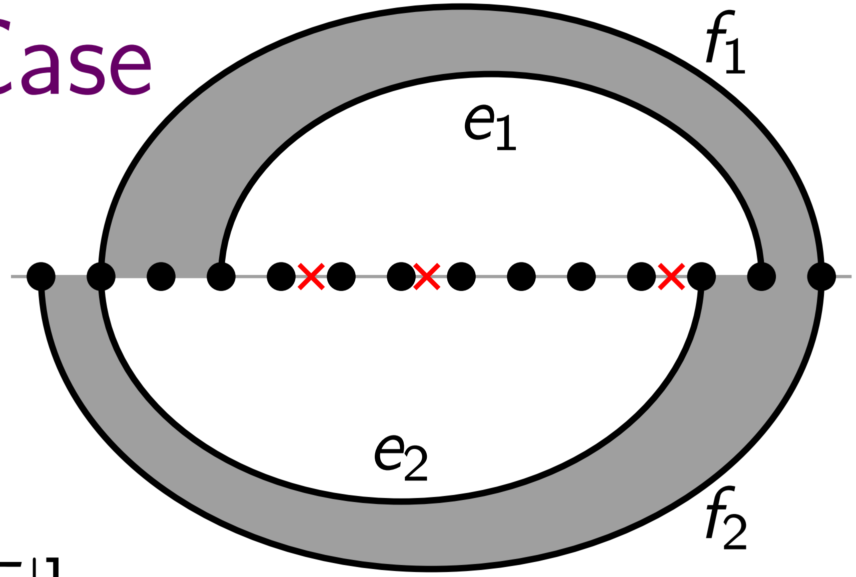

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E and E compatible $\Leftrightarrow$ maximum flow has value p.

The General Case

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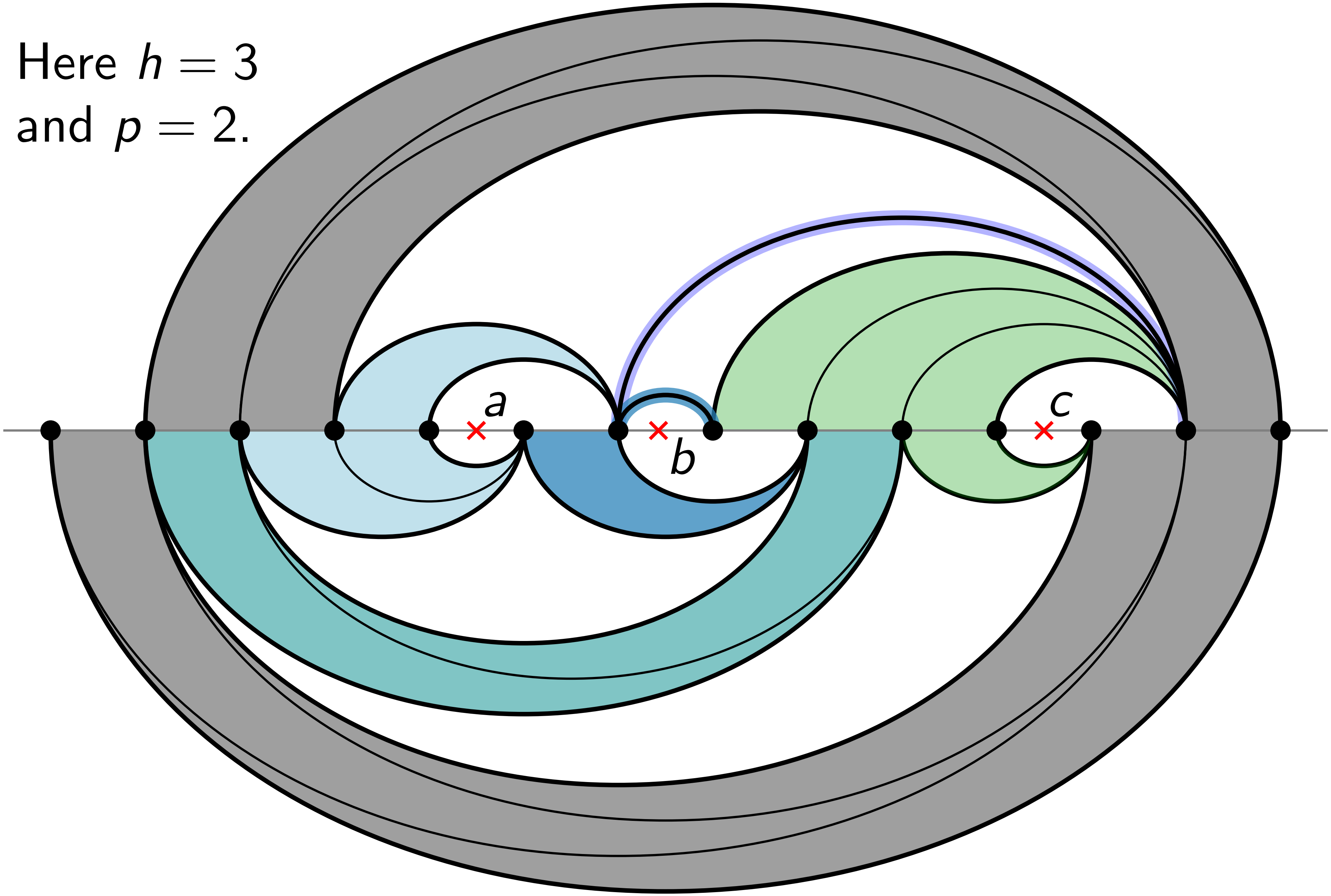

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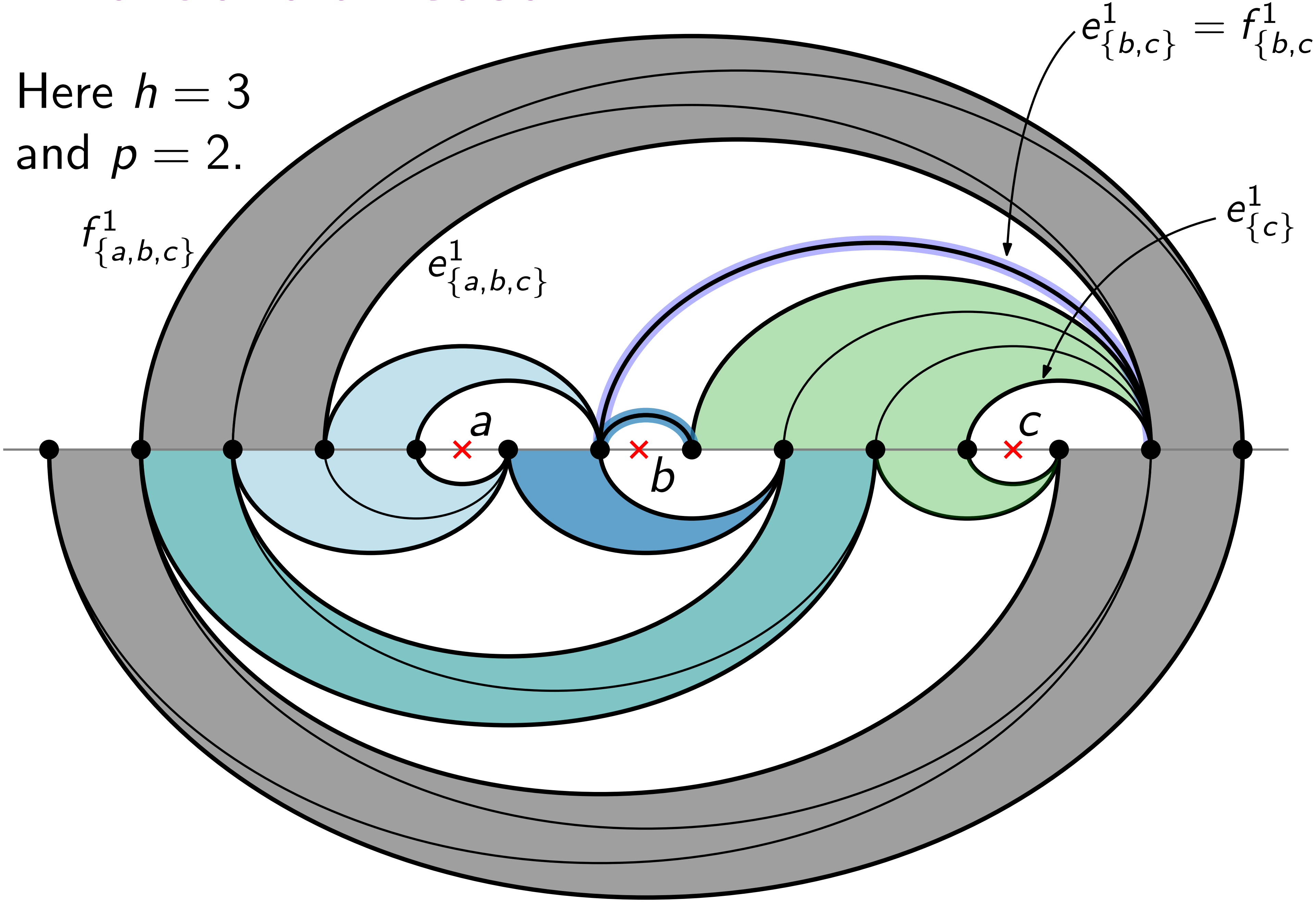

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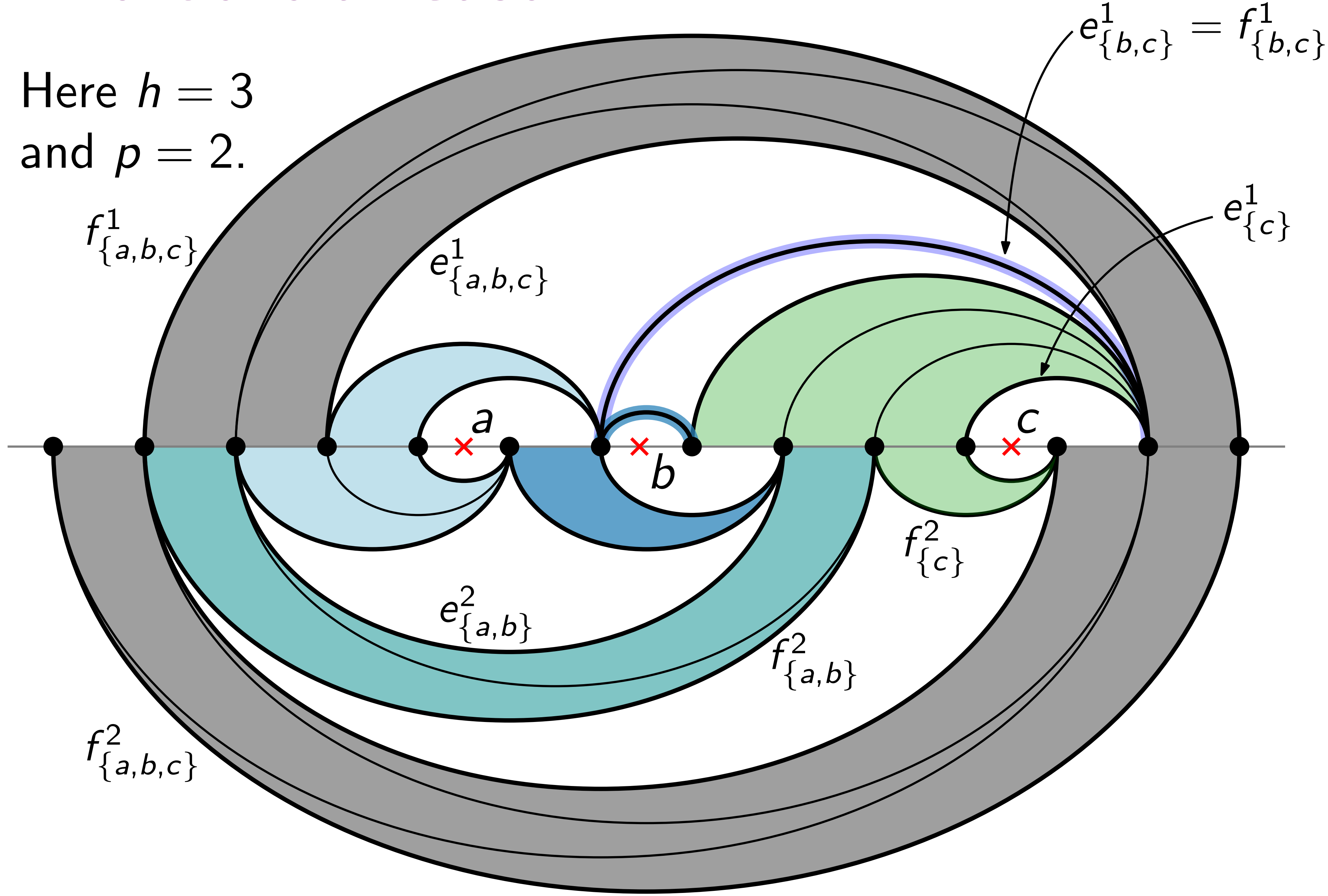

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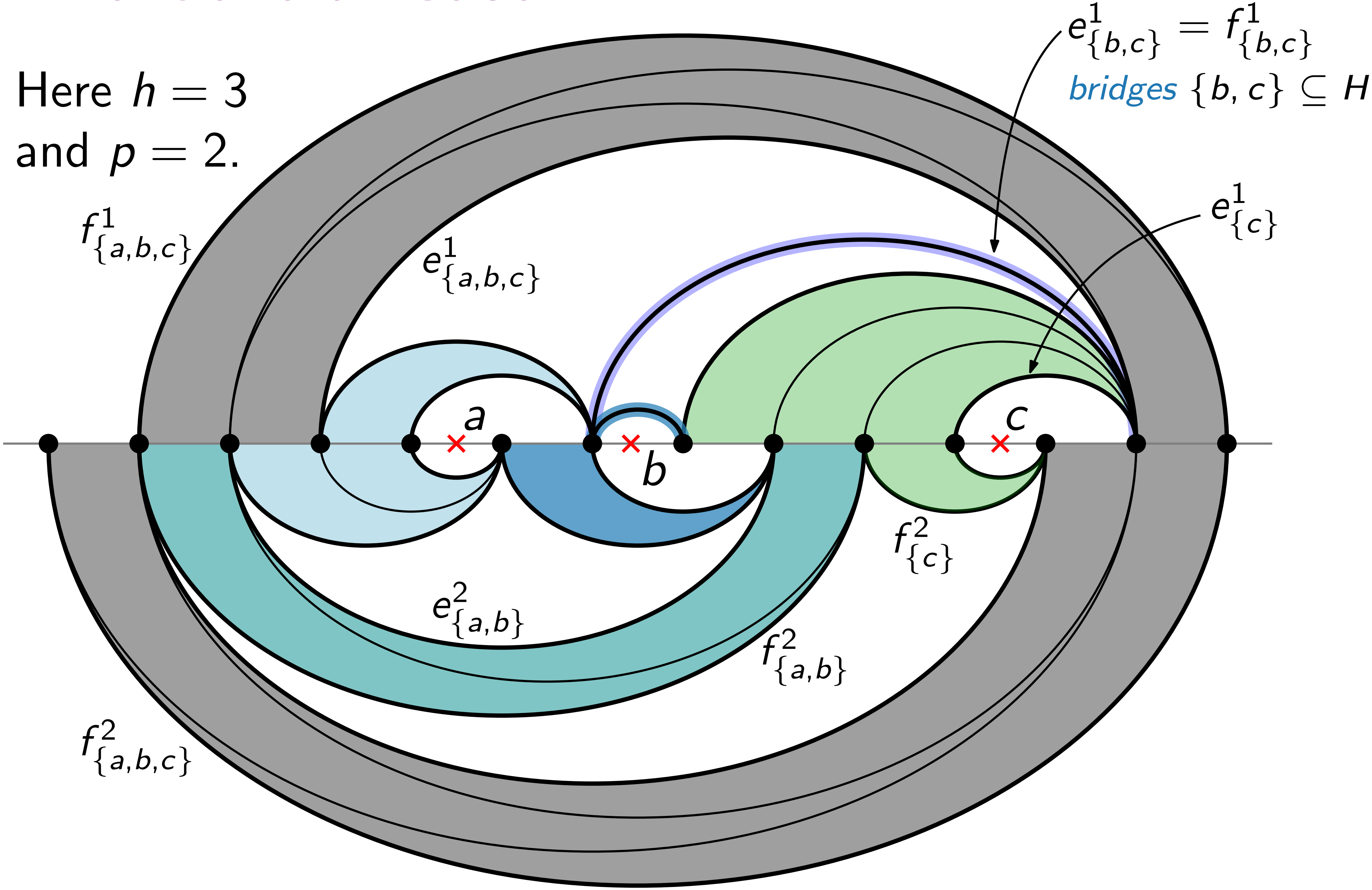

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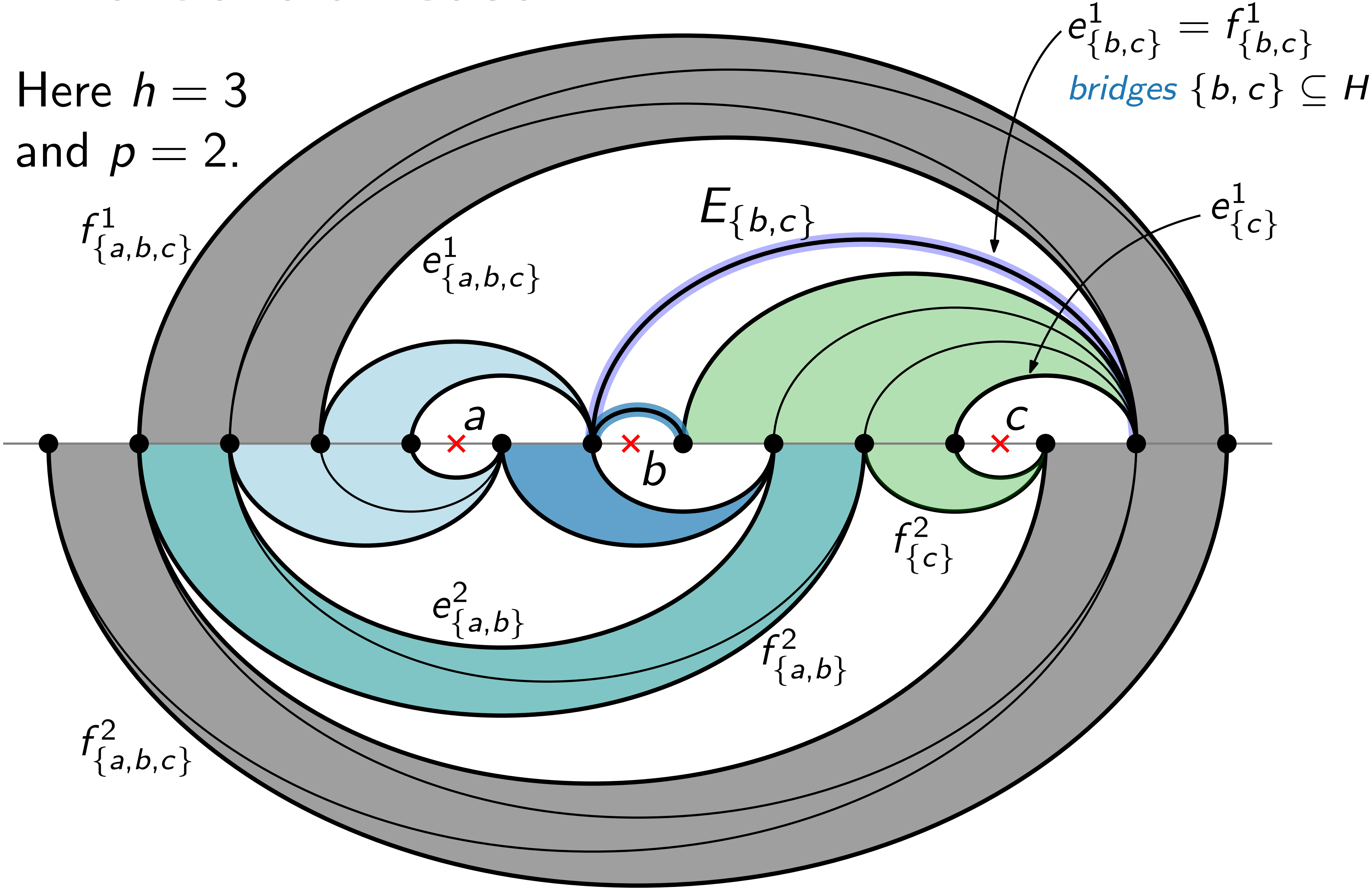

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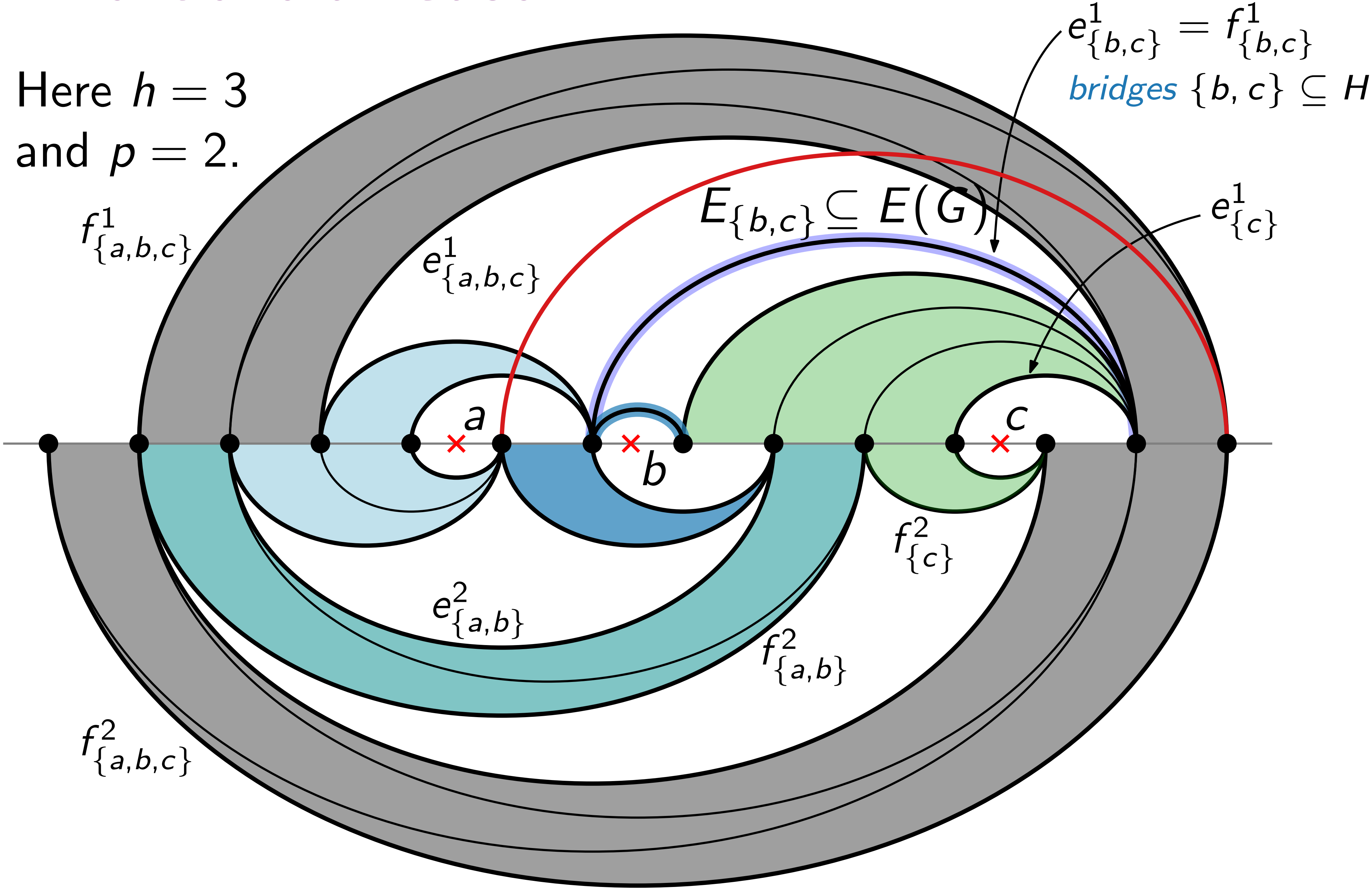

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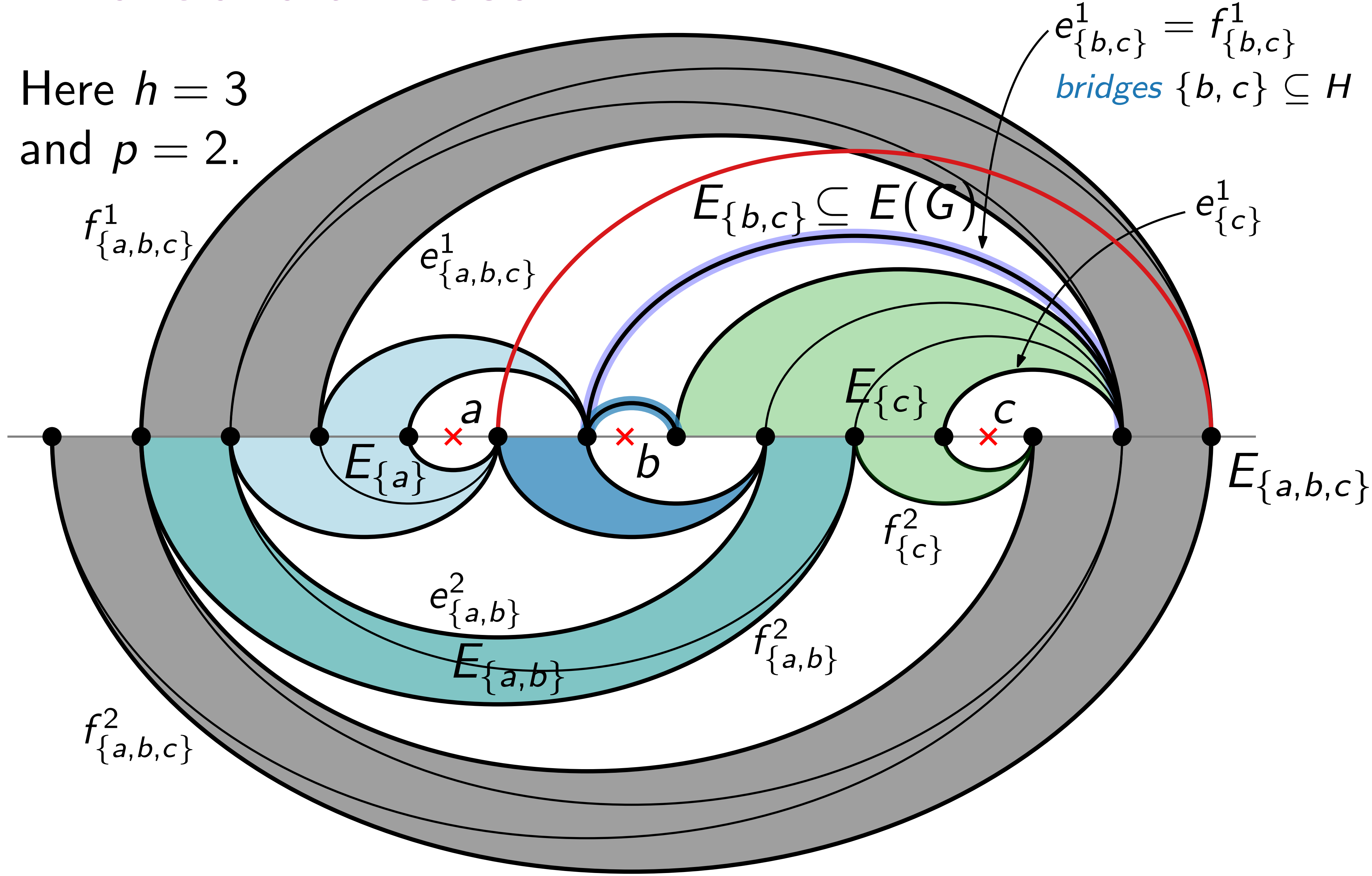

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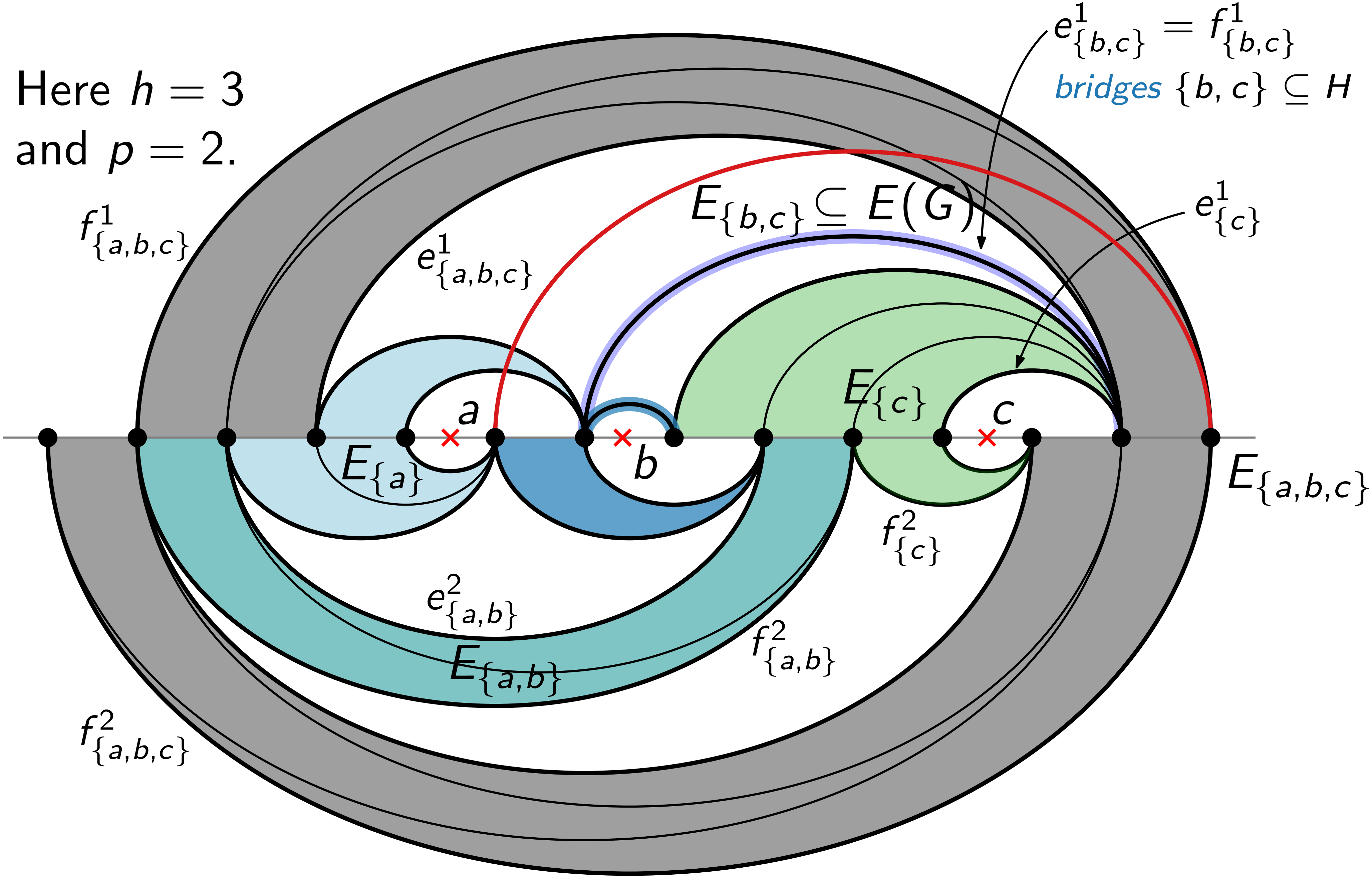

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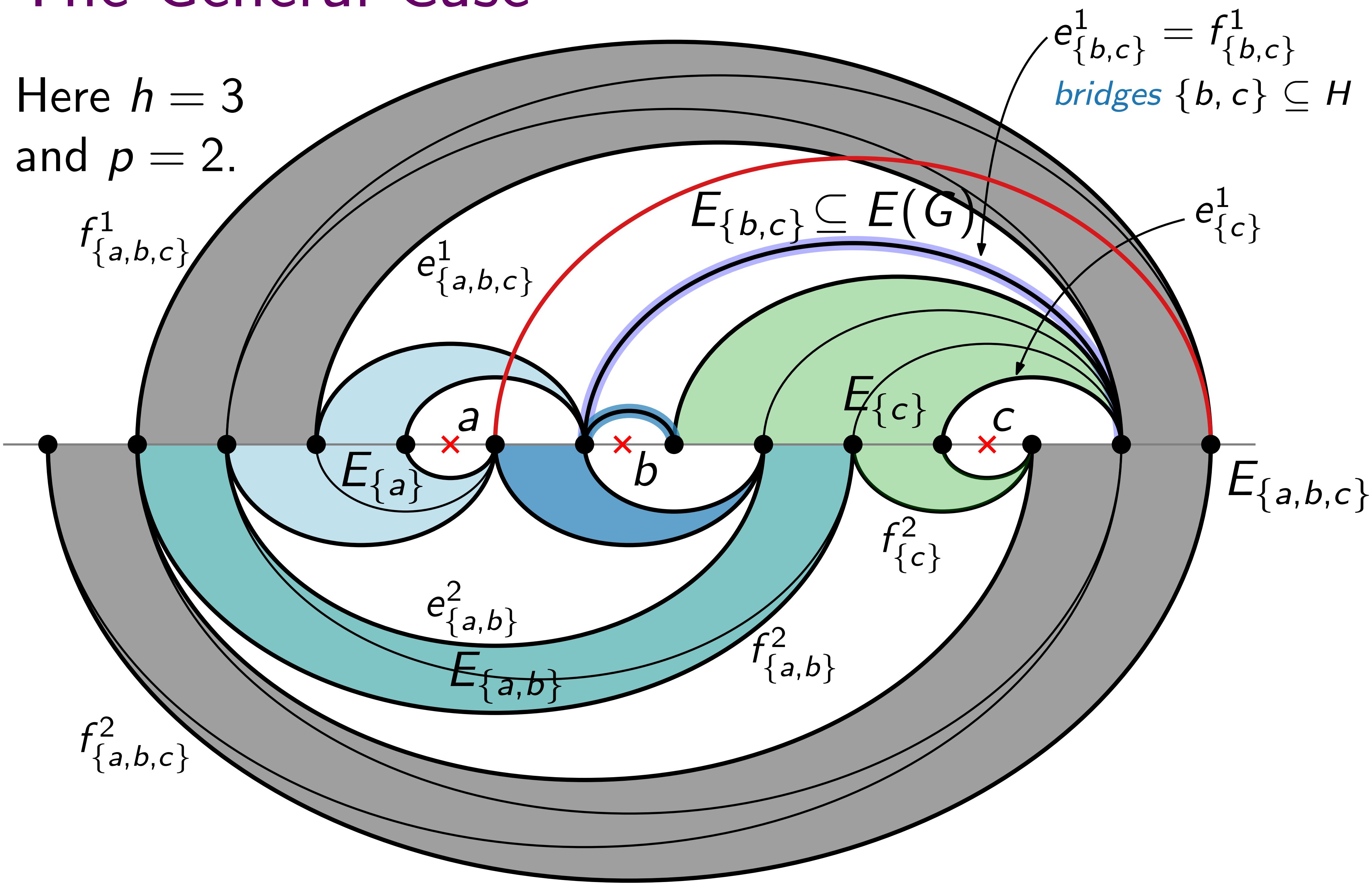

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Challenge: If $|Q_X| > 1$, the choices of which edges are drawn on which of these pages are not independent.

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We make a drawing of $\hat{E} := (E \setminus E_X) \cup S_X$ on p pages by assigning edges to pages, as follows.

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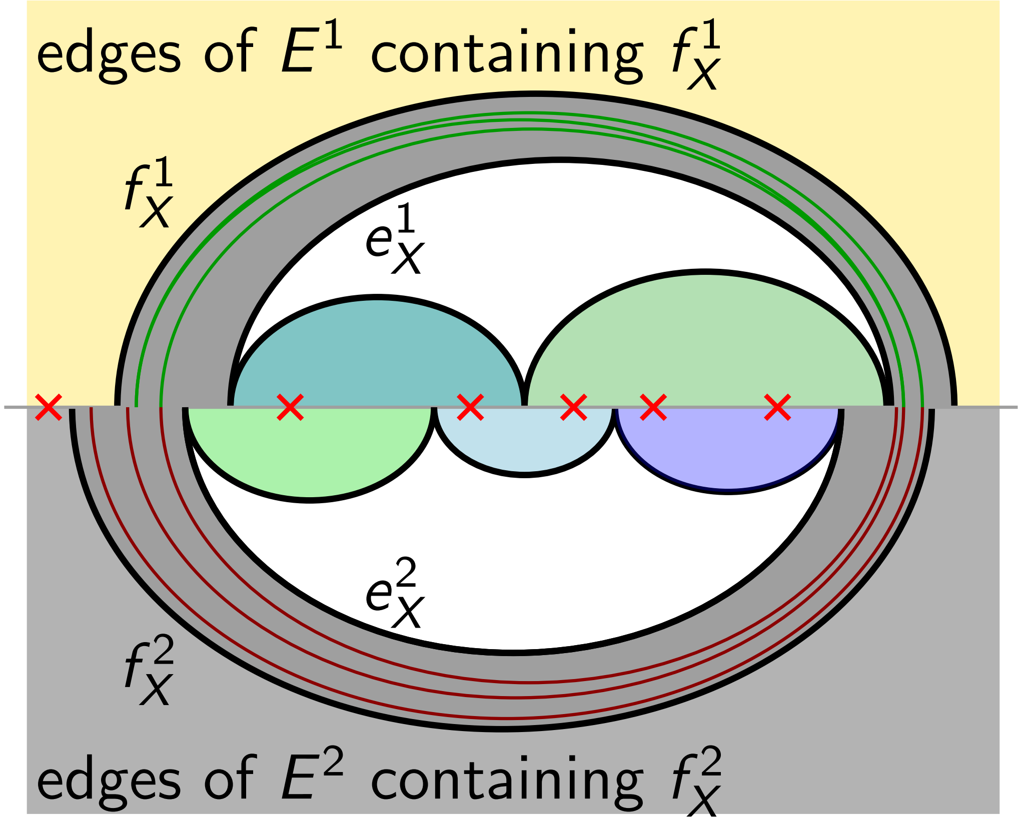

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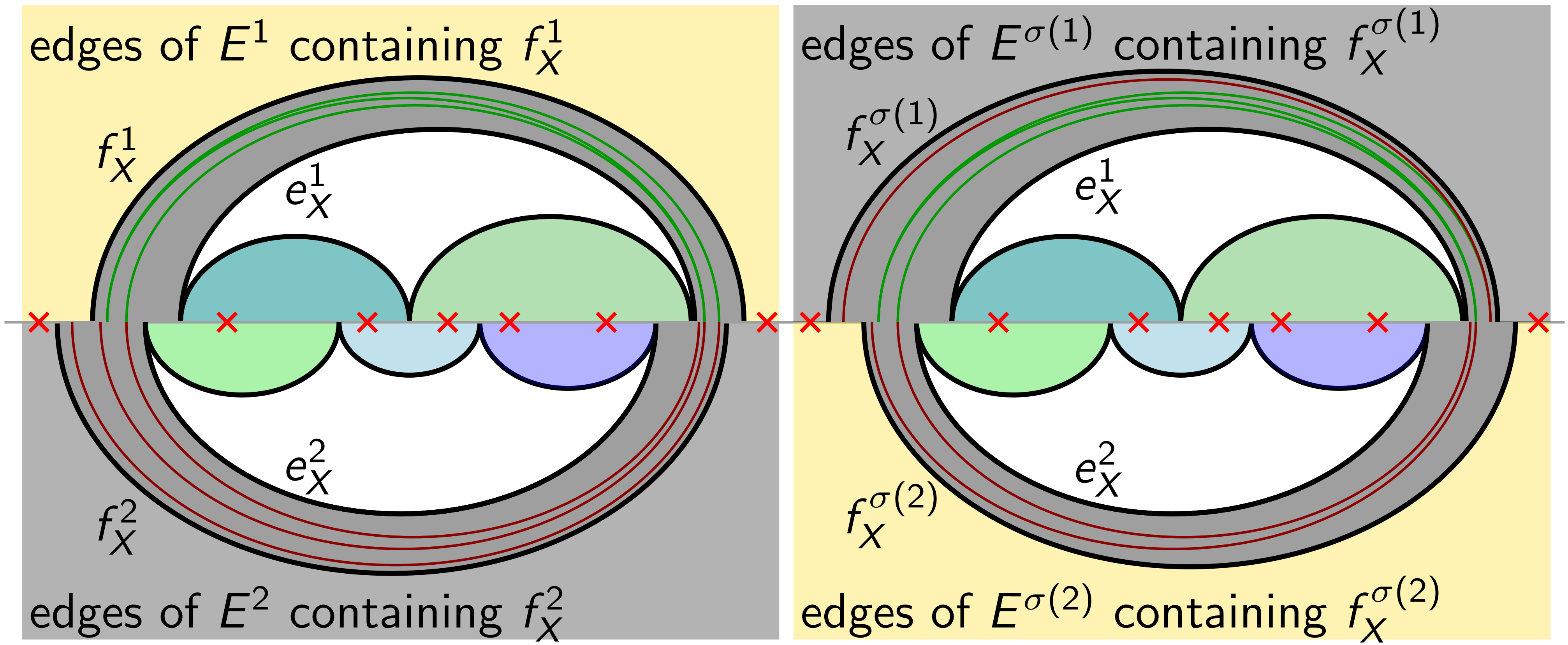

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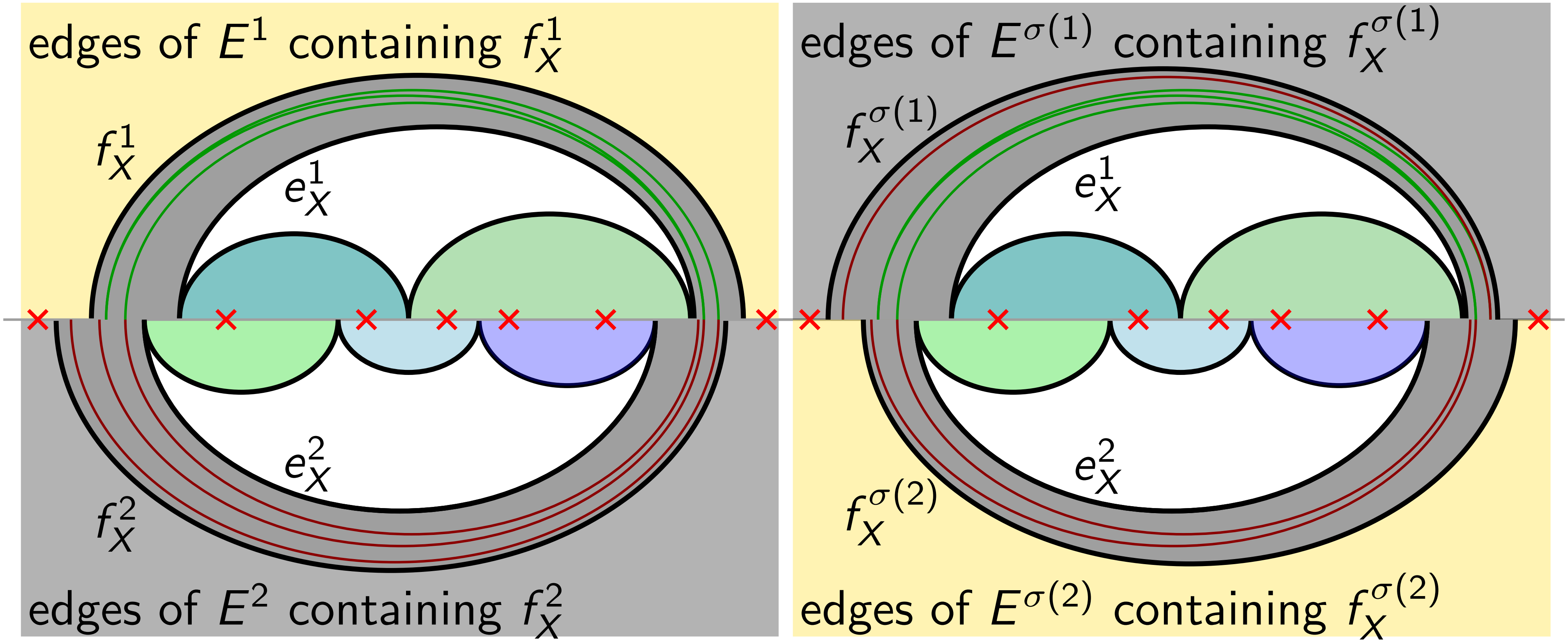

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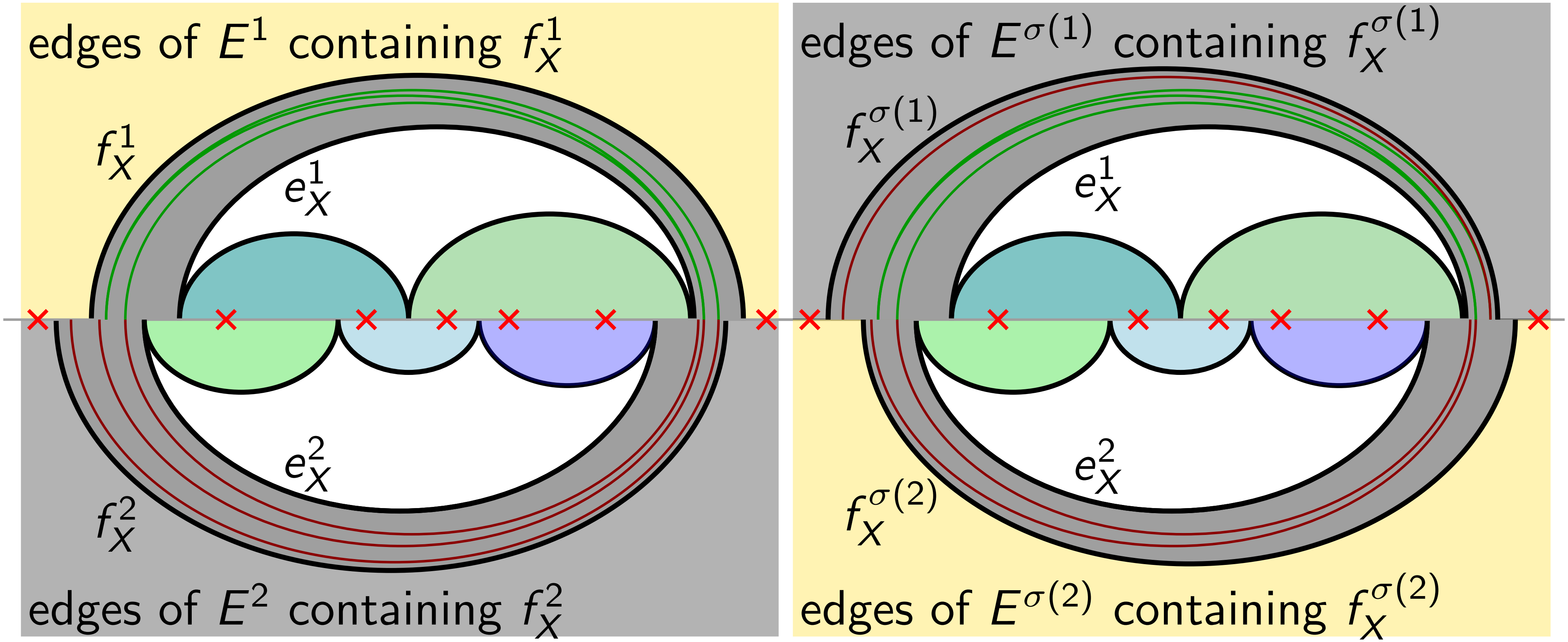

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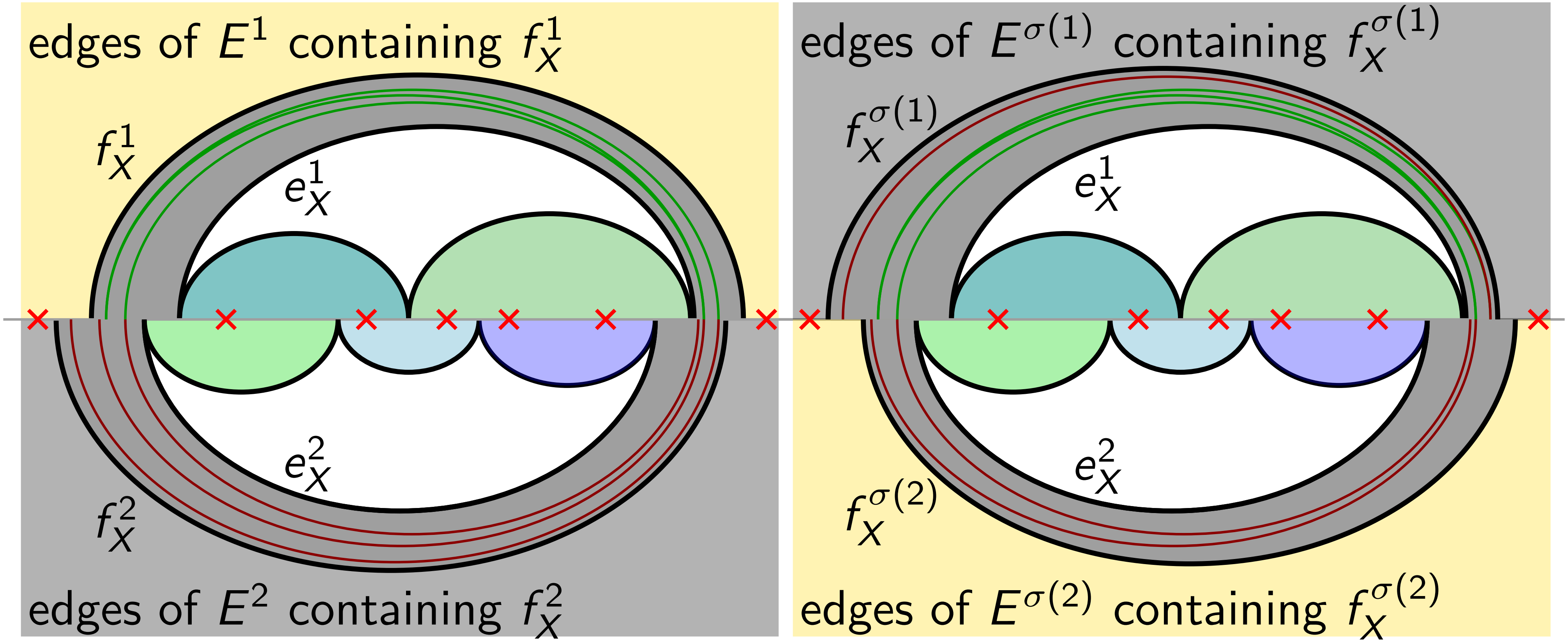

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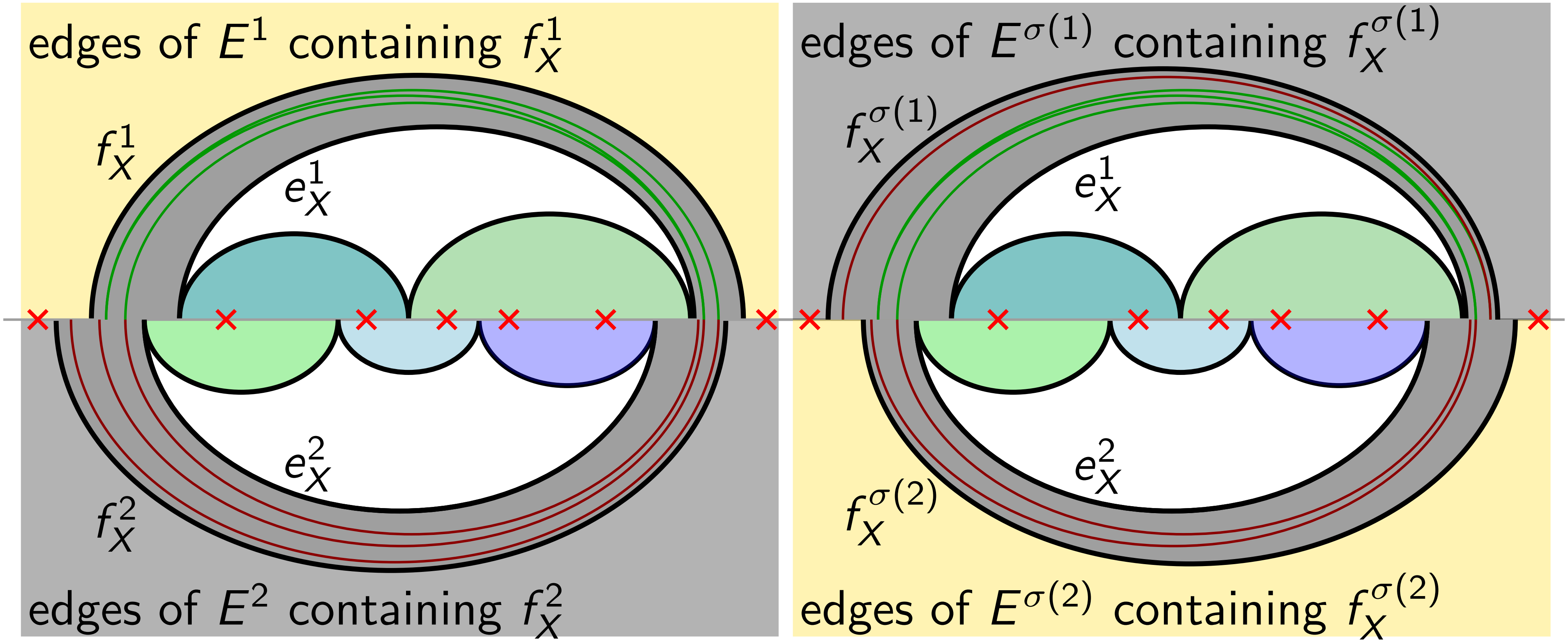

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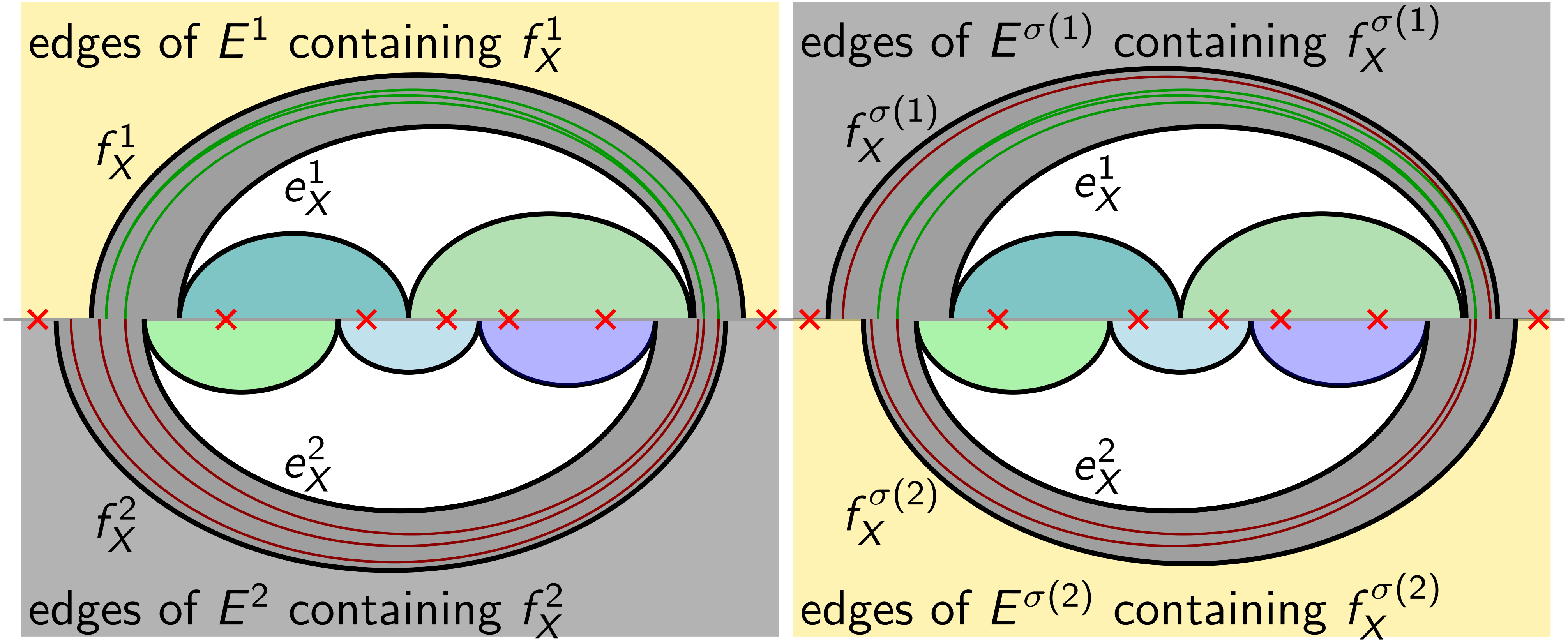

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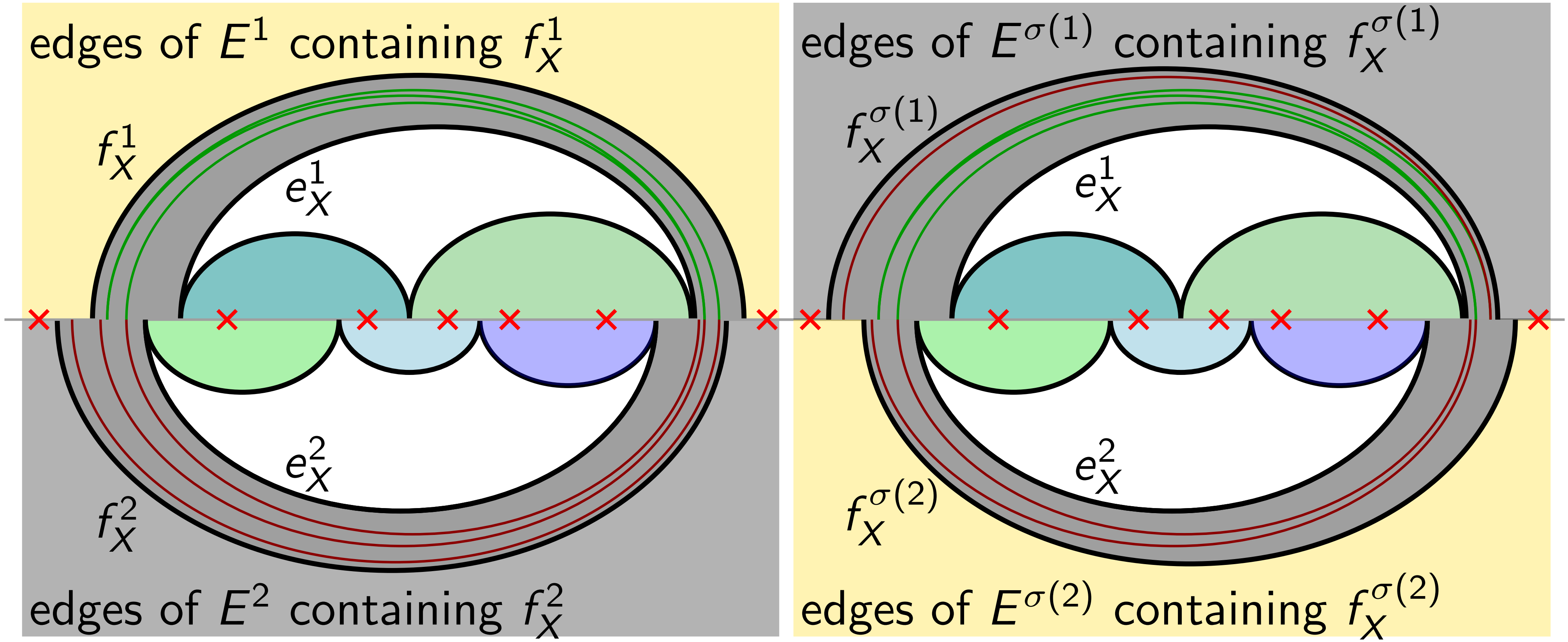

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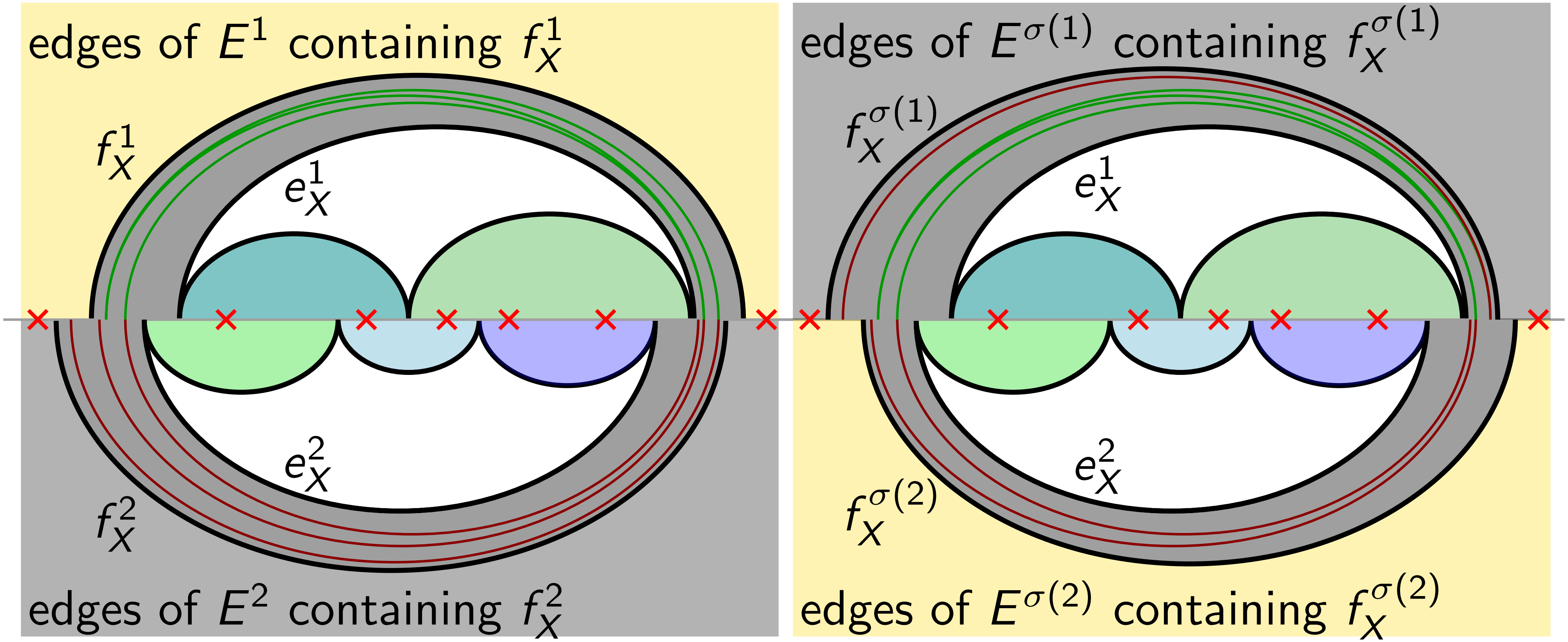

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Finally, $\hat{E} = \bigcup_X S_X = S$. $\square$

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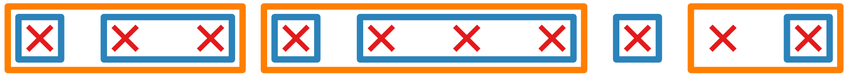

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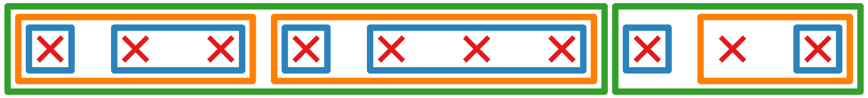

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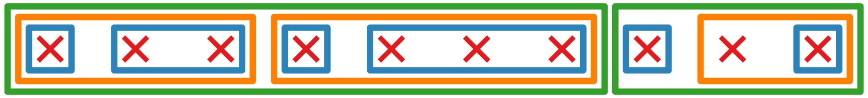

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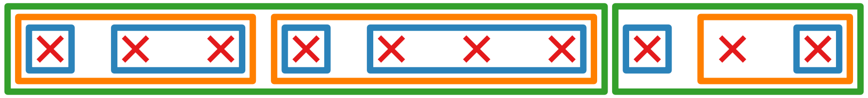

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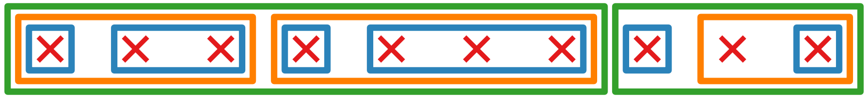
Then $|\mathcal{X}^p| \leq 2h - 1$. 

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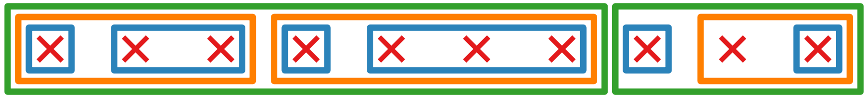

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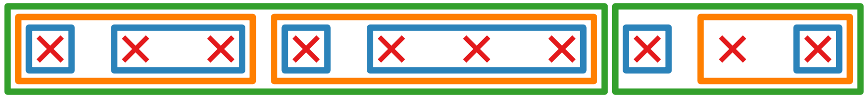

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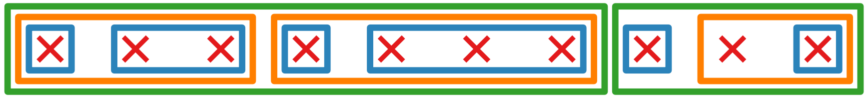

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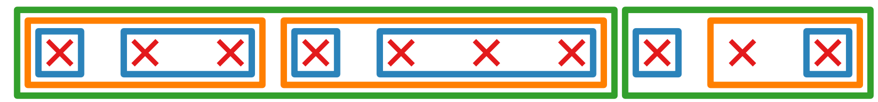

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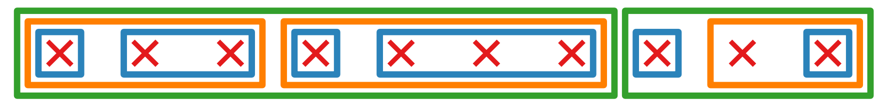

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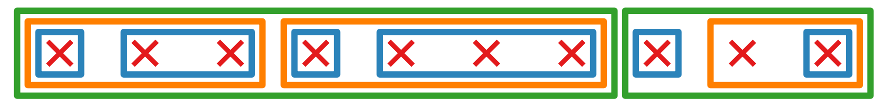

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- What is the parameterized complexity of Edge Deletion to Outer d-Planarity (that is, for unordered graphs)?