Pixel & Voxel Representations of Graphs

Md. Jawaherul Alam

Thomas Bläsius
Ignaz Rutter
Torsten Ueckerdt

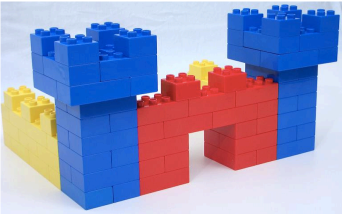
Alexander Wolff

The University of Arizona
Karlsruher Institut für Technologie
Julius-Maximilians-Universität Würzburg

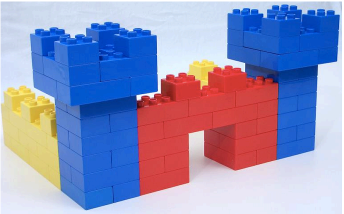
Graph Drawing
Northridge, Los Angeles – September 26, 2015

Motivation

Motivation

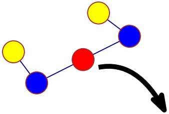


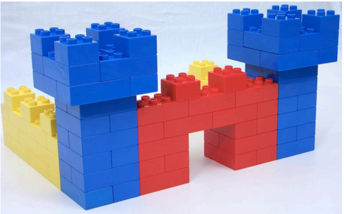
Motivation



Build contact representation of graphs

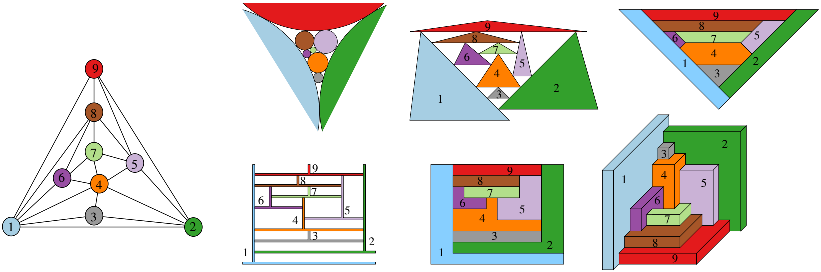
Motivation





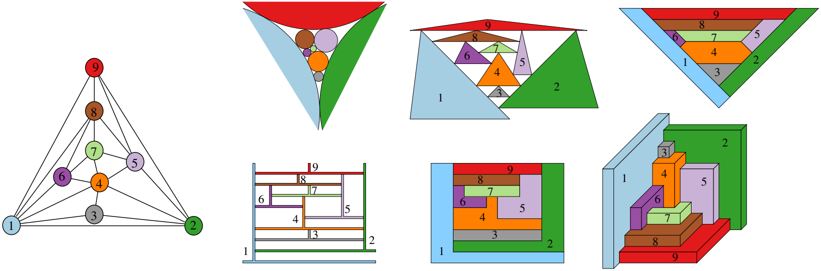
Build contact representation of graphs

Contact Representations



- Vertices ⇒ Geometric objects (polygons, arcs, polyhedra)
- Edges ⇒ Contacts

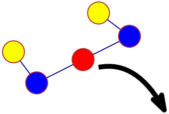
Contact Representations

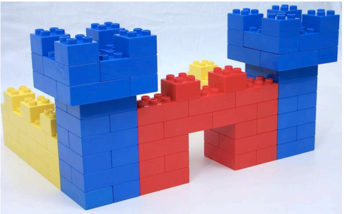


- Vertices ⇒ Geometric objects (polygons, arcs, polyhedra)
- Edges ⇒ Contacts

Goal: minimize polygonal complexity

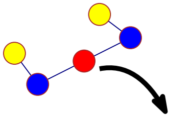
Motivation

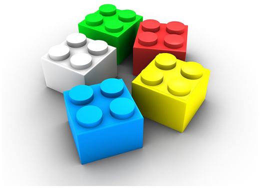


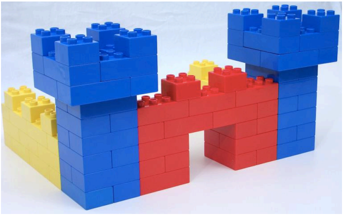


Build contact representation of graphs

Motivation

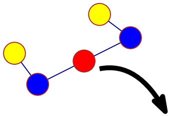


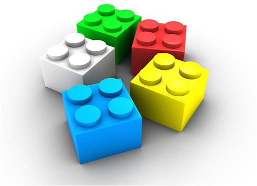


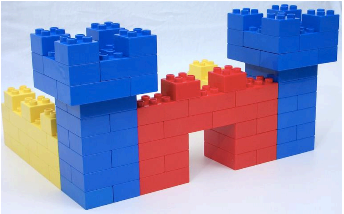


Build contact representation of graphs from unit blocks

Motivation







Build contact representation of graphs from unit blocks

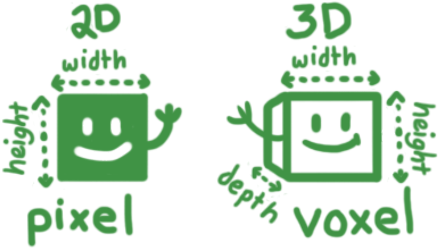
How many unit blocks are required?

Pixel and Voxel Representations

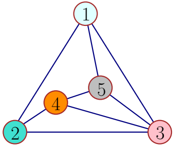
- Building contact representation from unit blocks

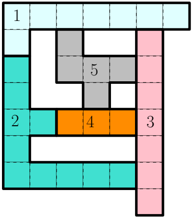
Pixel and Voxel Representations

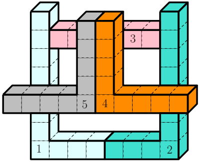
- Building contact representation from unit blocks
- Pixel in 2D, Voxel in 3D



Pixel and Voxel Representations

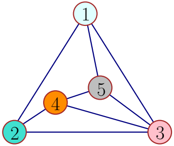


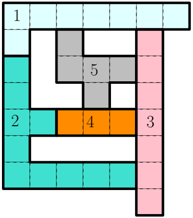


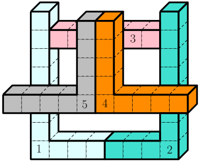


- Vertices ⇒ Blobs (connected sets of pixels/voxels)
- Edges ⇒ Adjacent (face-to-face) pixels/voxels in two blobs

Pixel and Voxel Representations



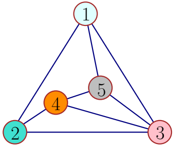


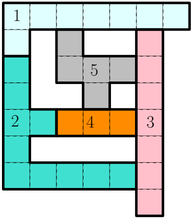


- Vertices ⇒ Blobs (connected sets of pixels/voxels)
- Edges ⇒ Adjacent (face-to-face) pixels/voxels in two blobs

Goal: minimize total number of pixels/voxels

Pixel Representations

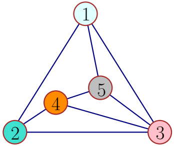


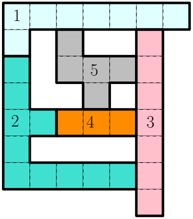


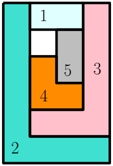
- Vertices ⇒ Blobs (connected sets of pixels)
- Edges ⇒ Adjacent (face-to-face) pixels in two blobs

Goal: minimize total number of pixels

Pixel Representations



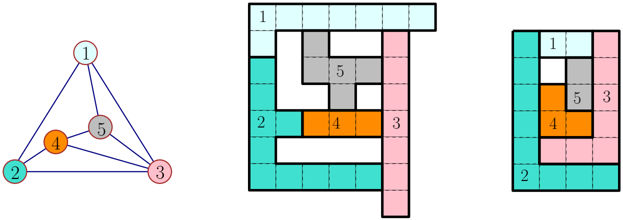




- Vertices ⇒ Blobs (connected sets of pixels)
- Edges ⇒ Adjacent (face-to-face) pixels in two blobs

Goal: minimize total number of pixels

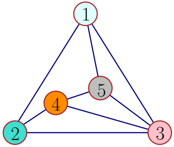
Pixel Representations

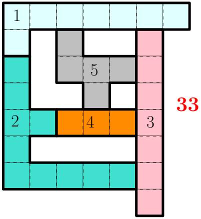


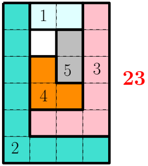
- Vertices ⇒ Blobs (connected sets of pixels)
- Edges ⇒ Adjacent (face-to-face) pixels in two blobs

Goal: minimize total number of pixels

Pixel Representations



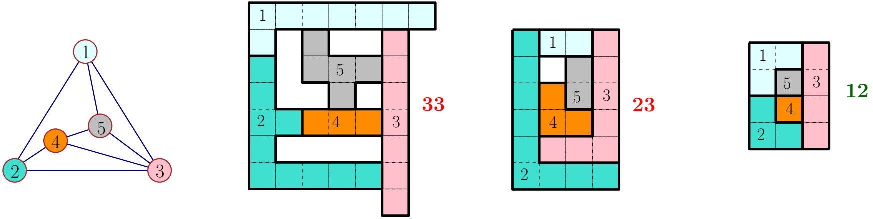




- Vertices ⇒ Blobs (connected sets of pixels)
- Edges ⇒ Adjacent (face-to-face) pixels in two blobs

Goal: minimize total number of pixels

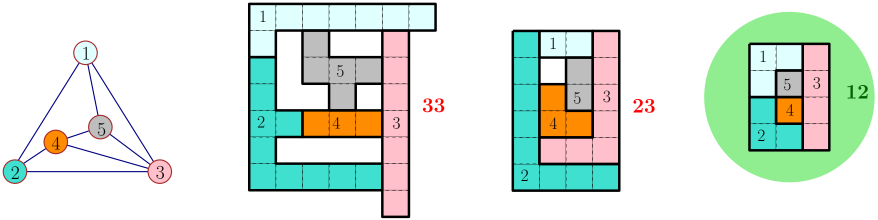
Pixel Representations



- Vertices ⇒ Blobs (connected sets of pixels)
- Edges ⇒ Adjacent (face-to-face) pixels in two blobs

Goal: minimize total number of pixels

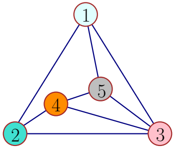
Pixel Representations



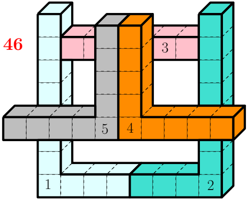
- Vertices ⇒ Blobs (connected sets of pixels)
- Edges ⇒ Adjacent (face-to-face) pixels in two blobs

Goal: minimize total number of pixels

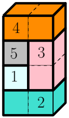
Voxel Representations



46



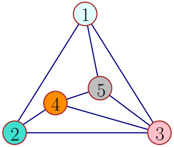
8

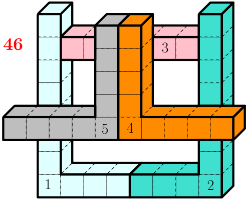


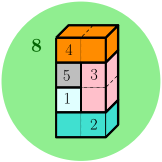
- Vertices ⇒ Blobs (connected sets of voxel)
- Edges ⇒ Adjacent (face-to-face) voxels in two blobs

Goal: minimize total number of voxels

Voxel Representations







- Vertices ⇒ Blobs (connected sets of voxel)
- Edges ⇒ Adjacent (face-to-face) voxels in two blobs

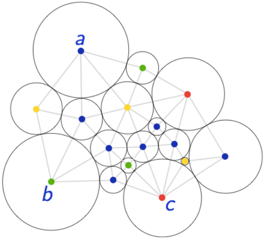
Goal: minimize total number of voxels

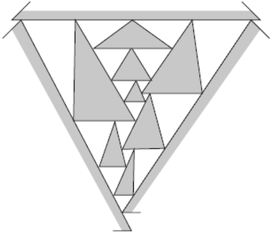
Related Work

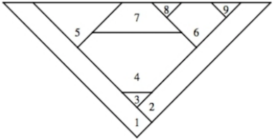
Related Work

Contact Representations

- Point-contact with circles [Koebe, 1936]
- Point-contact with triangles [De Fraysseix et al., 1994]
- Side-contact with hexagons
  [Gansner et al., 2010], [Bonichon et al., 2010]



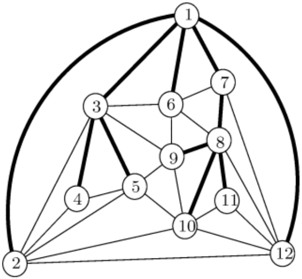


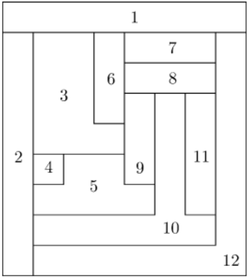


Related Work

Contact Representations with Rectilinear Polygons

- Contact with 8-sided rectilinear polygons:
  [Yeap and Sarrafzadeh, 1993], [He, 1999], [Liao et al., 2003]

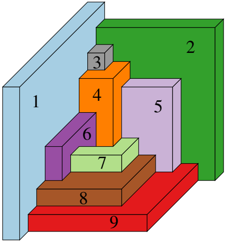


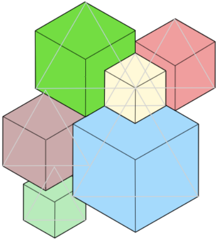


Related Work

Contact Representations in 3D

- Contact representation of planar graphs with cuboids
  [Thomassen, 1986], [Bremner et al., 2012]
- Improper contact representation of planar graphs with cubes
  [Felsner and Francis, 2011]

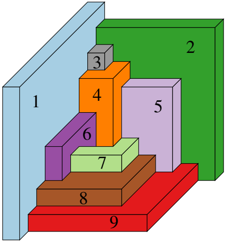


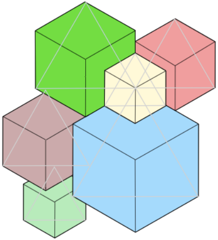


Related Work

Contact Representations in 3D

- Contact representation of planar graphs with cuboids
  [Thomassen, 1986], [Bremner et al., 2012]
- Improper contact representation of planar graphs with cubes
  [Felsner and Francis, 2011]
- Contact Representation of nonplanar graphs

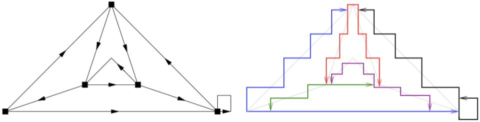




Related Work

Vertex Contact Graphs of Paths on a Grid (VCPG)

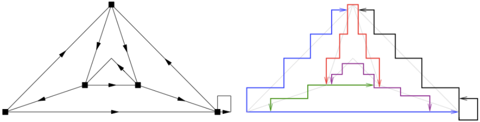
- Contact graphs of grid paths [Aerts and Felsner, 2014]



Related Work

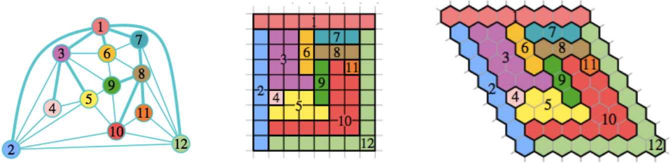
Vertex Contact Graphs of Paths on a Grid (VCPG)

- Contact graphs of grid paths [Aerts and Felsner, 2014]



Mosaic Drawing

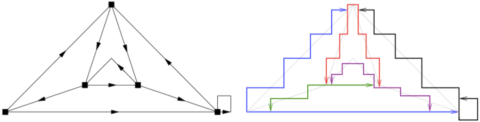
- Contact of square or hexagonal tilies [Cano et al., 2015]



Related Work

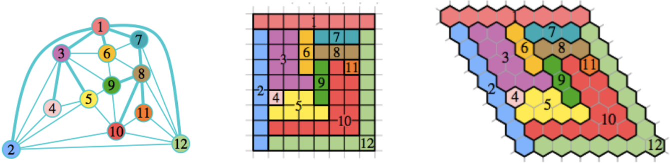
Vertex Contact Graphs of Paths on a Grid (VCPG)

- Contact graphs of grid paths [Aerts and Felsner, 2014]



Mosaic Drawing

- Contact of square or hexagonal tilies [Cano et al., 2015]

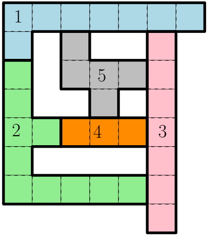


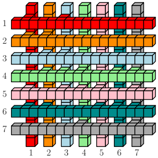
Same representation, different objective!

Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

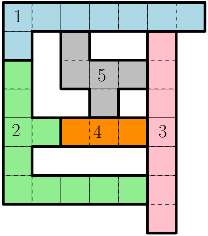




Our Result

Computational Complexity

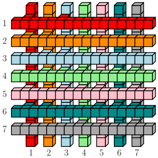
- Finding minimum-size representation is NP-complete in both 2D and 3D



Reduction from: $\mathcal{P}$

Input: a planar max-degree-4 graph G Find a grid drawing with unit edge lengths

$$\mathcal{P}$$
$$\Downarrow$$
Min-Pixel-Representation
$$\Downarrow$$
Min-Voxel-Representation



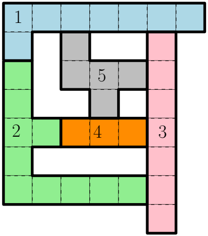
Our Result

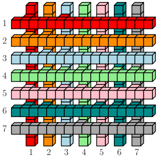
Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient





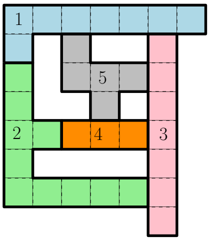
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

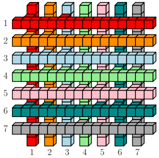
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Voxel Representation

- $O(n^2)$ voxels are sufficient



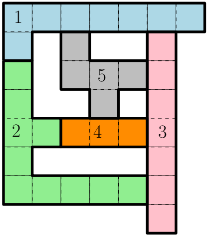
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

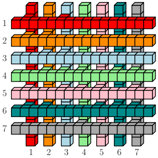
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient



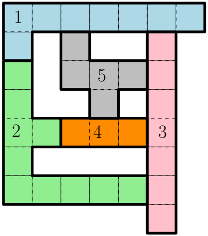
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

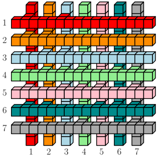
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient

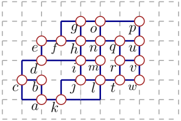


Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
$\Rightarrow$ Pixel/voxel representation of G with size $2l + n - m$.

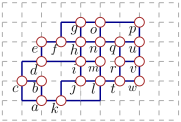
Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
$\Rightarrow$ Pixel/voxel representation of G with size $2l + n - m$.

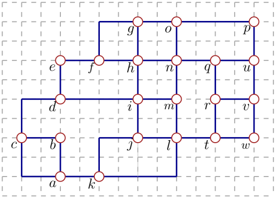


Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
$\Rightarrow$ Pixel/voxel representation of G with size $2l + n - m$.

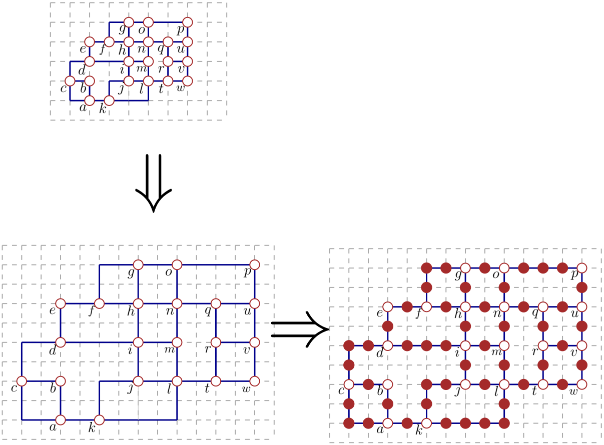


$\Downarrow$



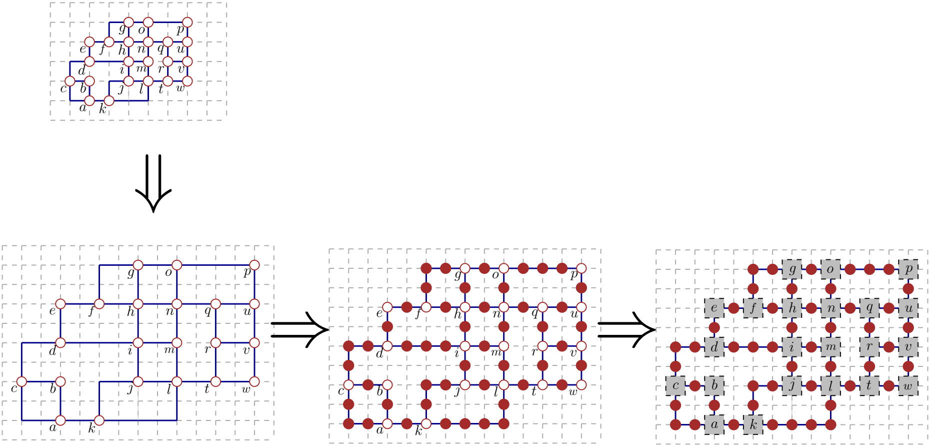
Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
$\Rightarrow$ Pixel/voxel representation of G with size $2l + n - m$.



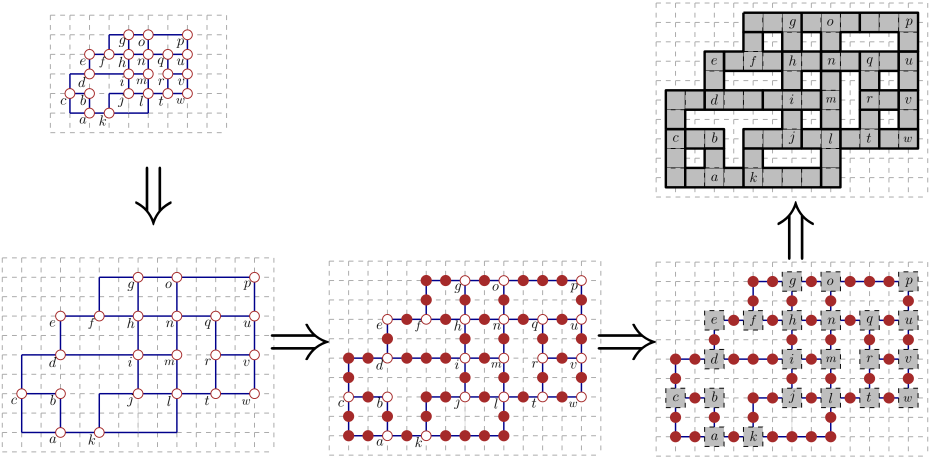
Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
$\Rightarrow$ Pixel/voxel representation of G with size $2l + n - m$.



Orthogonal Drawing to Pixel/Voxel Representation

A graph G with n vertices, m edges, and an orthogonal drawing of total edge length l
$\Rightarrow$ Pixel/voxel representation of G with size $2l + n - m$.



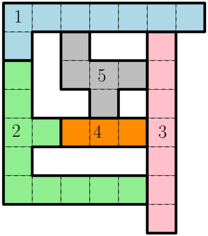
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

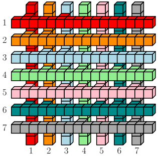
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient



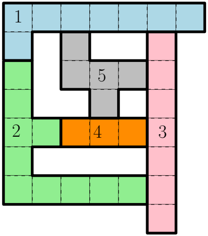
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

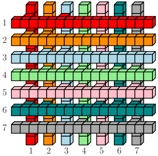
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

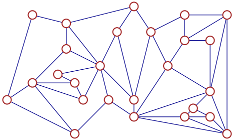


Voxel Representation

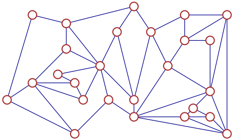
- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient



Pixel Representations for k-Outerplanar Graphs

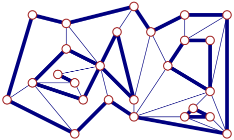


Pixel Representations for k-Outerplanar Graphs



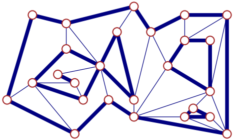
- An outerplanar graph is a 1-Outerplanar graph.
- Removing outervertices from a k-outerplanar graph yields $(k-1)$-outerplanar graphs

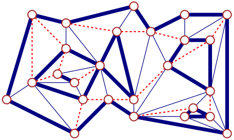
Pixel Representations for k-Outerplanar Graphs



- An outerplanar graph is a 1-Outerplanar graph.
- Removing outervertices from a k-outerplanar graph yields $(k-1)$-outerplanar graphs

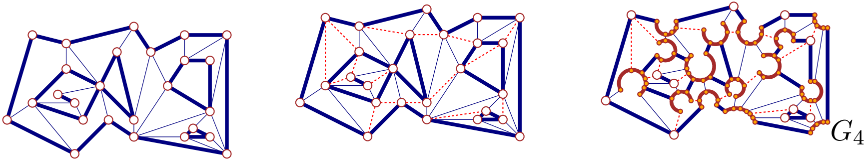
Pixel Representations for k-Outerplanar Graphs





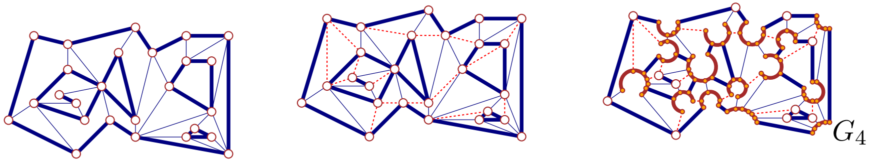
- triangulate

Pixel Representations for k-Outerplanar Graphs



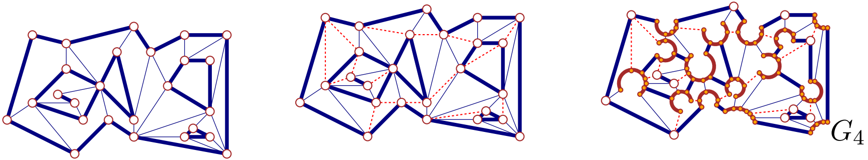
- G_4 : k-outerplanar with max-degree 4

Pixel Representations for k-Outerplanar Graphs



- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

Pixel Representations for k-Outerplanar Graphs

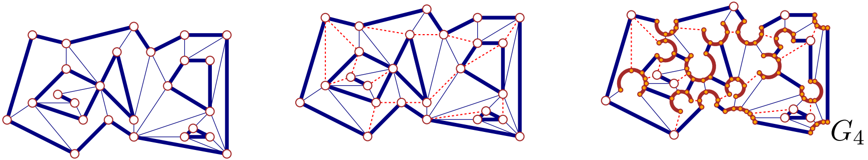


- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

$\Rightarrow$ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$

[D. Dolev, T. Leighton, H. Trickey, 1984]

Pixel Representations for k-Outerplanar Graphs



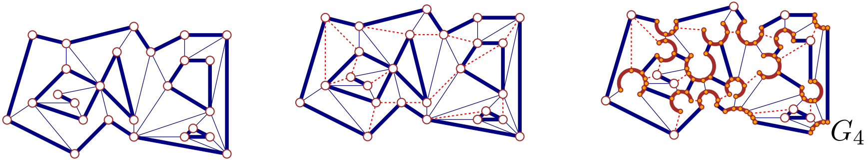
- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

$\Rightarrow$ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$

[D. Dolev, T. Leighton, H. Trickey, 1984]

$\Rightarrow$ G_4 has a pixel representation with size $\Theta(kn)$

Pixel Representations for k-Outerplanar Graphs



- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

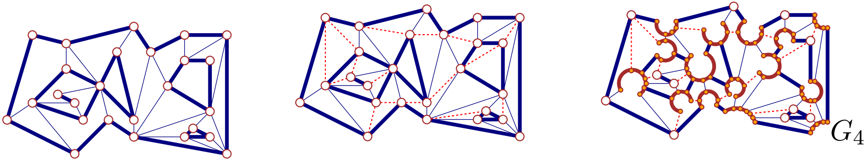
$\Rightarrow$ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$

[D. Dolev, T. Leighton, H. Trickey, 1984]

$\Rightarrow$ G_4 has a pixel representation with size $\Theta(kn)$

- Representation for G?

Pixel Representations for k-Outerplanar Graphs



- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

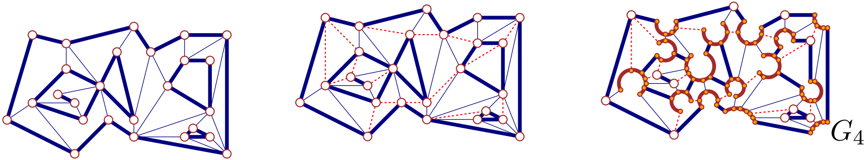
$\Rightarrow$ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$

[D. Dolev, T. Leighton, H. Trickey, 1984]

$\Rightarrow$ G_4 has a pixel representation with size $\Theta(kn)$

- Representation for G?
  - Contract edges
  - Delete extra edges

Pixel Representations for k-Outerplanar Graphs



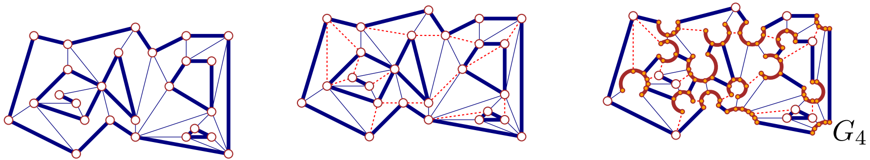
- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

$\Rightarrow$ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$
[D. Dolev, T. Leighton, H. Trickey, 1984]

$\Rightarrow$ G_4 has a pixel representation with size $\Theta(kn)$

- Representation for G?
  - Contract edges: identify blobs
  - Delete extra edges

Pixel Representations for k-Outerplanar Graphs



- G_4 : k-outerplanar with max-degree 4
- Maximum shortest path length to reach outerface = $\Theta(k)$

$\Rightarrow$ G_4 has an orthogonal drawing with total edge length $\Theta(kn)$

[D. Dolev, T. Leighton, H. Trickey, 1984]

$\Rightarrow$ G_4 has a pixel representation with size $\Theta(kn)$

- Representation for G?
  - Contract edges: identify blobs
  - Delete extra edges: remove contact pixels

Pixel Representations for k-Outerplanar Graphs

Lower Bound

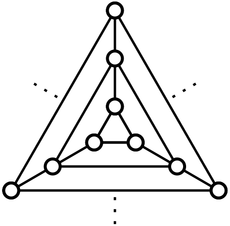
- Any k-outerplane pixel representation has size at least $4k^2 - 4k$.

Pixel Representations for k-Outerplanar Graphs

Lower Bound

- Any k-outerplane pixel representation has size at least $4k^2 - 4k$.

$\Rightarrow$ Some k-outerplanar graphs require $\Omega(kn)$ pixels

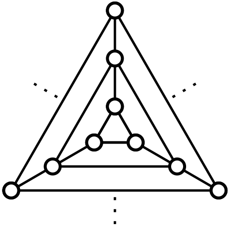


Pixel Representations for k-Outerplanar Graphs

Lower Bound

- Any k-outerplane pixel representation has size at least $4k^2 - 4k$.

$\Rightarrow$ Some k-outerplanar graphs require $\Omega(kn)$ pixels



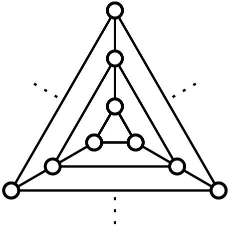
$\Rightarrow$ $\Theta(kn)$ pixels are sometimes necessary and always sufficient

Pixel Representations for k-Outerplanar Graphs

Lower Bound

- Any k-outerplane pixel representation has size at least $4k^2 - 4k$.

$\Rightarrow$ Some k-outerplanar graphs require $\Omega(kn)$ pixels



$\Rightarrow$ $\Theta(kn)$ pixels are sometimes necessary and always sufficient

- Linear pixels for outerplanar, quadratic for planar graphs.

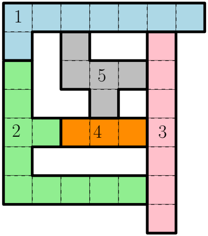
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

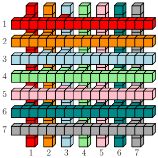
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient



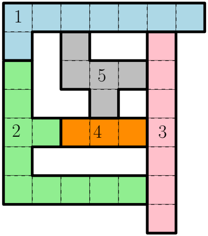
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

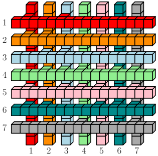
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

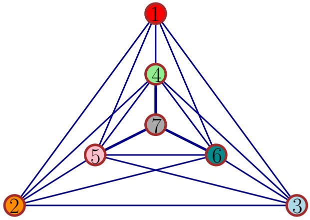


Voxel Representation

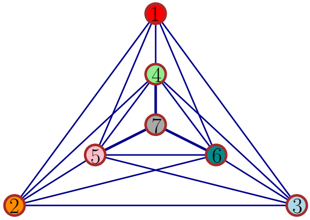
- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient

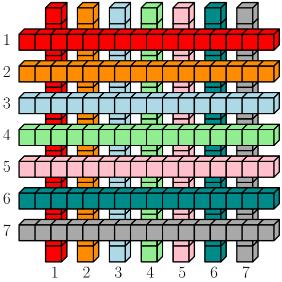


Voxel Representations for Graphs

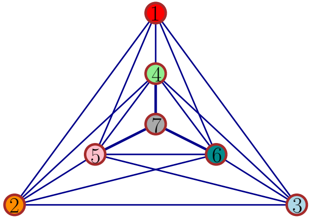


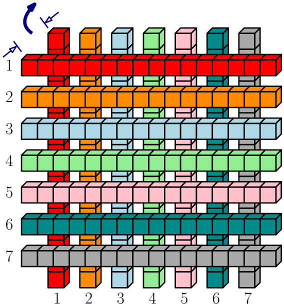
Voxel Representations for Graphs



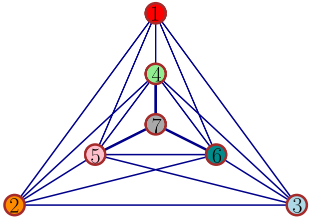


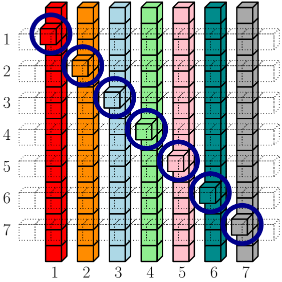
Voxel Representations for Graphs





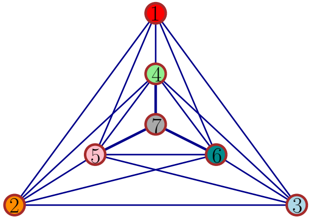
Voxel Representations for Graphs

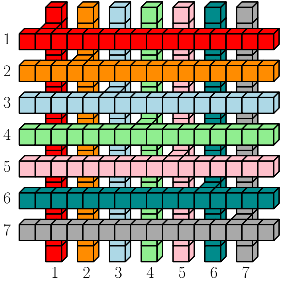




- Add diagonal voxels

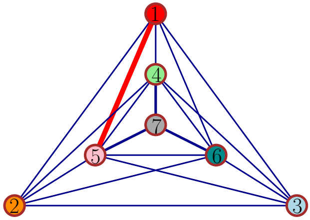
Voxel Representations for Graphs

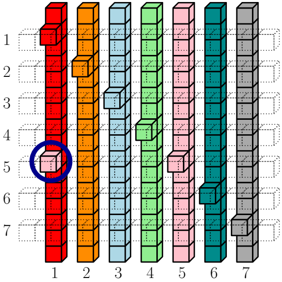




- Add diagonal voxels

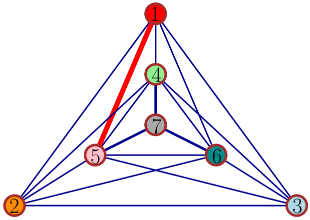
Voxel Representations for Graphs

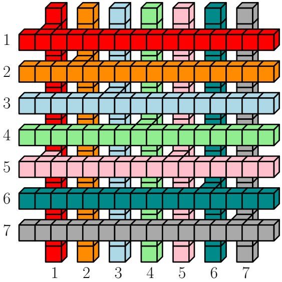




- Add diagonal voxels
- Add voxels for edges

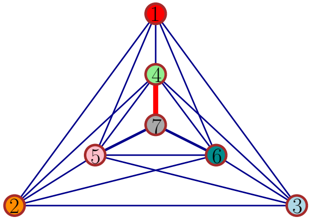
Voxel Representations for Graphs

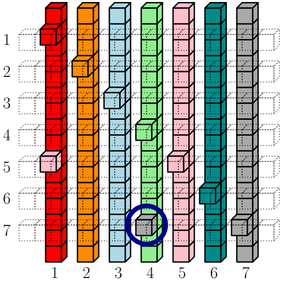




- Add diagonal voxels
- Add voxels for edges

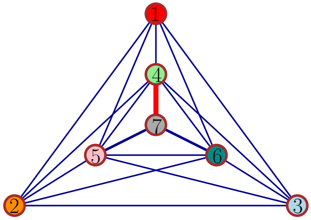
Voxel Representations for Graphs

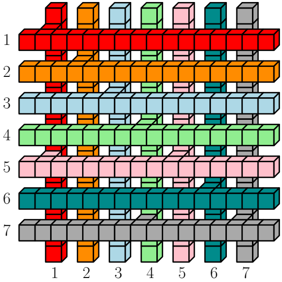




- Add diagonal voxels
- Add voxels for edges

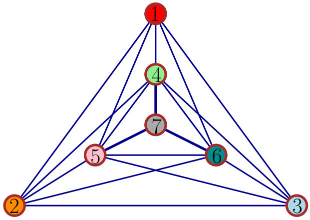
Voxel Representations for Graphs

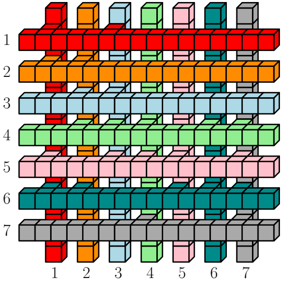




- Add diagonal voxels
- Add voxels for edges

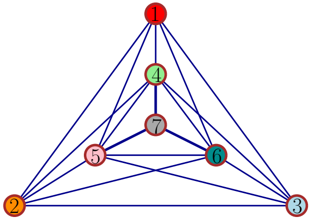
Voxel Representations for Graphs

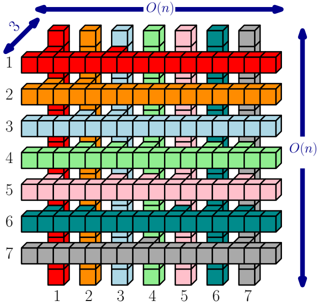




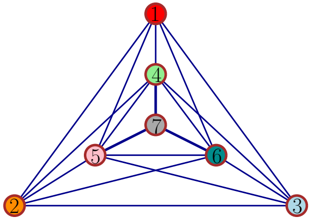
- Add diagonal voxels
- Add voxels for edges

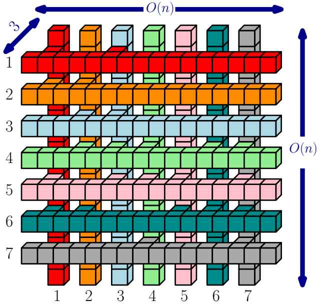
Voxel Representations for Graphs





Voxel Representations for Graphs





Better bound for constant treewidth or constant genus

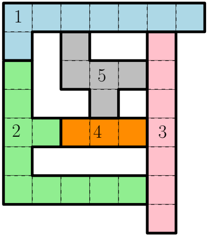
Our Result

Computational Complexity

- Finding minimum-size representation is NP-complete in both 2D and 3D

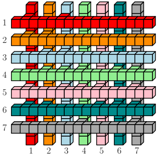
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient



Voxel Representation

- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient



Our Result

Computational Complexity

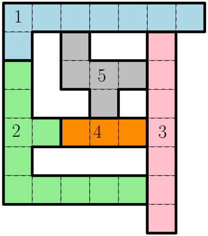
- Finding minimum-size representation is NP-complete in both 2D and 3D

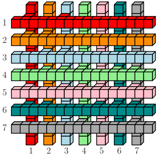
Pixel Representation

- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

Voxel Representation

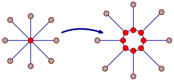
- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient





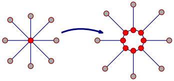
Voxel Representations for Genus-g Graphs

- Make the maximum degree 4



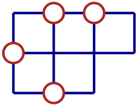
Voxel Representations for Genus-g Graphs

- Make the maximum degree 4



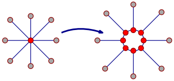
$\Downarrow$

Orthogonal drawing on the plane (with crossing)
with total edge length $O((g+1)^2 n \log^2 n)$ [Leiserson, 1980]



Voxel Representations for Genus-g Graphs

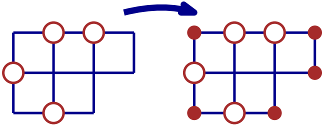
- Make the maximum degree 4



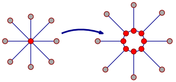
$$\Downarrow$$

Orthogonal drawing on the plane (with crossing)
with total edge length $O((g+1)^2 n \log^2 n)$ [Leiserson, 1980]

- Subdivide at bend points



Voxel Representations for Genus-g Graphs

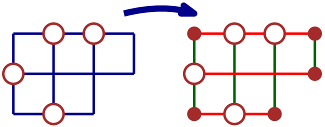


- Make the maximum degree 4

$\Downarrow$

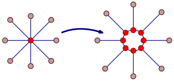
Orthogonal drawing on the plane (with crossing)
with total edge length $O((g+1)^2 n \log^2 n)$ [Leiserson, 1980]

- Subdivide at bend points
- Split horizontal and vertical graphs



Voxel Representations for Genus-g Graphs

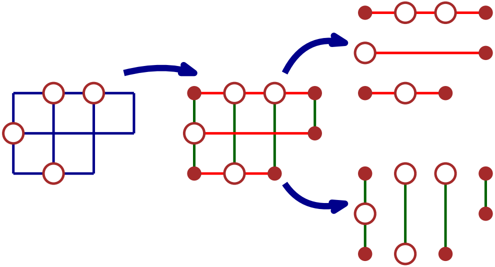
- Make the maximum degree 4



$\Downarrow$

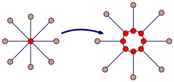
Orthogonal drawing on the plane (with crossing)
with total edge length $O((g+1)^2 n \log^2 n)$ [Leiserson, 1980]

- Subdivide at bend points
- Split horizontal and vertical graphs



Voxel Representations for Genus-g Graphs

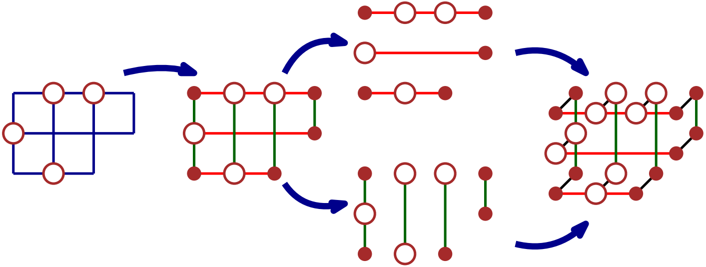
- Make the maximum degree 4



$\Downarrow$

Orthogonal drawing on the plane (with crossing)
with total edge length $O((g+1)^2 n \log^2 n)$ [Leiserson, 1980]

- Subdivide at bend points
- Split horizontal and vertical graphs
- Combine horizontal and vertical graphs

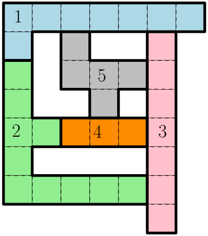


Summary

Computational Complexity

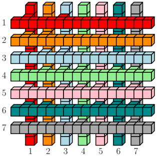
- Finding minimum-size representation is NP-complete in both 2D and 3D

Pixel Representation



- For a k-outerplanar graph, $\Theta(kn)$ pixels are necessary and sufficient

Voxel Representation



- $O(n^2)$ voxels are sufficient
- For a graph with treewidth τ, $\Theta(n \cdot \tau)$ voxels are necessary and sufficient
- For a graph with genus g, $O((g+1)^2 n \log^2 n)$ voxels are sufficient

Future Work and Open Problems

- Approximation for minimum-size representation
  - Approximation algorithm or hardness

Future Work and Open Problems

- Approximation for minimum-size representation
  - Approximation algorithm or hardness
- Tighten bound for constant genus graphs
  - improve upper bound of $O((g+1)^2 n \log^2 n)$?

Future Work and Open Problems

- Approximation for minimum-size representation
  - Approximation algorithm or hardness
- Tighten bound for constant genus graphs
  - improve upper bound of $O((g+1)^2 n \log^2 n)$?
- Hexagonal or other shapes for pixels/voxels?

Future Work and Open Problems

- Approximation for minimum-size representation
  - Approximation algorithm or hardness
- Tighten bound for constant genus graphs
  - improve upper bound of $O((g+1)^2 n \log^2 n)$?
- Hexagonal or other shapes for pixels/voxels?

Acknowledgements

- Giuseppe Liotta and Walter Didimo, University of Perugia

  Organizers, 2014 Bertinoro Workshop on Graph Drawing
- Sue Whitesides, University of Victoria