Untangling a Planar Graph

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Abstract. In John Tantalo's on-line game *Planarity* the player is given a non-plane straight-line drawing of a planar graph. The aim is to make the drawing plane as quickly as possible by moving vertices. Pach and Tardos have posed a related problem: can any straight-line drawing of any planar graph with n vertices be made plane by vertex moves while keeping $\Omega(n^{\varepsilon})$ vertices fixed for some absolute constant $\varepsilon > 0$? It is known that three vertices can always be kept (if $n \geq 5$).
We still do not solve the problem of Pach and Tardos, but we report some progress. We prove that the number of vertices that can be kept actually grows with the size of the graph. More specifically, we give a lower bound of $\Omega(\sqrt{\log n / \log \log n})$ on this number. By the same technique we show that in the case of outerplanar graphs we can keep a lot more, namely $\Omega(\sqrt{n})$ vertices. We also construct a family of outerplanar graphs for which this bound is asymptotically tight.

1 Introduction

At the 5th Czech-Slovak Symposium on Combinatorics in Prague in 1998, Mamoru Watanabe asked the following question. Is it true that every polygon P with n vertices can be untangled, i.e., turned into a non-crossing polygon, by moving at most εn of its vertices for some absolute constant $\varepsilon < 1$? Pach and Tardos [8] have answered this question in the negative by showing that there must be polygons where at most $O((n \log n)^{2/3})$ of the vertices can be kept fixed. In their paper, Pach and Tardos in turn asked the following question: can any straight-line drawing of any planar graph with n vertices be made plane by vertex moves while keeping $\Omega(n^{\varepsilon})$ vertices fixed for some absolute constant $\varepsilon > 0$? It is known [14, 4] that at least three vertices can always be kept (assuming $n \geq 5$). We still do not know the answer to the question of Pach and Tardos, but we report further progress. We show that $\Omega(\sqrt{\log n / \log \log n})$ vertices can always be kept. For outerplanar graphs our method keeps a lot more, namely $\Omega(\sqrt{n})$ vertices, and we show that there are drawings of outerplanar graphs where only $O(\sqrt{n})$ vertices can be kept fixed, i.e., our bound is asymptotically tight.

There is a popular on-line game that is related to the problem of Pach and Tardos. In John Tantalo's game *Planarity* [12] the player is given a non-plane straight-line drawing of a planar graph. The player can move vertices, which always keep straight-line connections to their neighbors. The aim is to make the drawing plane as quickly as possible.

Let's formalize our problem. Given a planar graph $G = (V, E)$ a straight-line drawing of G in the plane is uniquely defined by an injective map $\delta : V \to \mathbb{R}^2$ of the vertices of G into the plane. It will be convenient to identify the map δ with the straight-line drawing of G that is defined by δ. A drawing of G is plane if no two edges in the drawing cross each other, that is, they only share points which are endpoints of both edges. Given a drawing δ of G let

$$\text{fix}(G, \delta) = \max_{\delta' \text{ plane drawing of } G} |\{v \in V \mid \delta(v) = \delta'(v)\}|,$$

denote the maximum number of vertices of G that can be kept fixed when making δ plane. Let $\text{fix}(G) = \min_{\delta \text{ drawing of } G} \text{fix}(G, \delta)$ denote the maximum number of vertices of G that can be kept fixed when starting with the worst-possible drawing of G. In this paper we show $\text{fix}(G) \in \Omega(\sqrt{\log n / \log \log n})$, where n is the number of vertices of G.

Our approach is as follows. Our main theorem (see Section 4) guarantees that $\text{fix}(G) \in \Omega(\sqrt{l})$ for all triangulated planar graphs G that contain a simple length-l path of a special structure. In terms of the diameter d and the maximum degree Δ of G our main theorem yields bounds of $\Omega(\sqrt{d})$ and $\Omega(\sqrt{\Delta})$, respectively, for $\text{fix}(G)$. The former is achieved with the help of so-called *Schnyder woods*. Moore's bound—a trade-off between d and Δ—then yields the bound $\Omega(\sqrt{\log n / \log \log n})$ for $\text{fix}(G)$ in terms of n, see Section 5. The bound $\Omega(\sqrt{\Delta})$ immediately yields a lower bound of $\Omega(\sqrt{n})$ for outerplanar graphs. We complement this result by an asymptotically tight upper bound in Section 6. We start by reviewing previous work in Section 2 and outlining our method in Section 3.

2 Previous and Related Work

Pach and Tardos [8] have shown that $\sqrt{n} < \text{fix}(C_n) \le c(n \log n)^{2/3}$ where C_n is the cycle with n vertices and c is some positive constant. They used a probabilistic method based on the crossing lemma.

Verbitsky [14] has considered two graph parameters; the *obfuscation complexity* $\text{obf}(G)$ of a graph G, which is the maximum number of edge crossings in any drawing of $G = (V, E)$, and the *shift complexity* $\text{shift}(G) = |V| - \text{fix}(G)$ of G. Concerning the shift complexity he observed that $\text{fix}(G) \ge 3$ for planar graphs with $n \ge 5$ vertices. Further he gave two linear lower bounds on $\text{shift}(G)$ depending on the connectivity of G. By reduction from independent set in line-segment intersection graphs he showed that computing the shift complexity $\text{shift}(G, \delta)$ of a fixed drawing is NP-hard even if the given graph is restricted to a matching. This explains why Tantalo's game Planarity is difficult and shows that computing $\text{fix}(G, \delta)$ is hard, too.

Independently, Goaoc et al. [4] have also shown that $\text{fix}(G) \ge 3$ for any planar graph G with $n \ge 5$ vertices and that computing the shift complexity is NP-hard. Their (more complicated) reduction is from planar 3-SAT. A variant of their reduction also shows that $\text{shift}(G, \delta)$ is hard to approximate. More precisely, if $\mathcal{P} \neq \mathcal{NP}$ then for no $\varepsilon \in (0, 1]$ there is a polynomial-time $(n^{1-\varepsilon})$-approximation algorithm for $\text{shift}(G, \delta) + 1$. Note that this does *not* imply hardness of approximation for computing $\text{fix}(G, \delta)$. On the combinatorial side, Goaoc et al. showed that $\text{fix}(T) \geq \sqrt{n/3}$ for any tree T with n vertices and that there exist planar graphs G with an arbitrary large number n of vertices such that $\text{fix}(G) \leq \lceil \sqrt{n-2}\, \rceil + 1$. Note that the graphs in their construction are not outerplanar and, therefore, this does not imply our result presented in Section 6.

Kang et al. [7] have investigated an interesting related problem. They start with a plane drawing of a graph and want to make it straight-line, again by vertex moves. For any positive integers s and k they construct a graph $G_{s,k}$ with $n = k(s + k)$ vertices and a plane drawing $\delta_{s,k}$ of $G_{s,k}$ such that $M \geq s(k - 1)$ moves are needed to make $\delta_{s,k}$ straight-line. The bound on M is maximized for $k \in O(n^{1/3})$ and thus shows that $\text{fix}(G, \delta_{s,k}) \in O(n^{2/3})$, which is weaker than the upper bound of $2\sqrt{n}$ proved by Goaoc et al. Note, however, that the drawings of Goaoc et al. are not plane.

Very recently—after submission of this paper and its long version [11]—Bose et al. [2] answered the question of Pach and Tardos [8] in the affirmative by showing that for any planar graph with n vertices at least $\sqrt[4]{n/9}$ vertices can be kept, thus improving our bound. They also showed that the $\Omega(\sqrt{n})$ lower bound of Goaoc et al. for trees is asymptotically tight.

3 Preliminaries and Overview

Definitions and notation. A(n abstract) *plane embedding* of a planar graph is given by the circular order of the edges around each vertex and by the choice of the outer face. A plane embedding of a planar graph can be computed in linear time [6]. If G is triangulated, a plane embedding of G is determined by the choice of the outer face. Recall that an edge of a graph is called *chord* with respect to a path Π if the edge does not lie on Π but both its endpoints are vertices of Π.

For a point $p \in \mathbb{R}^2$ let $x(p)$ and $y(p)$ be the x- and y-coordinates of p, respectively. We say that p lies *vertically below* $q \in \mathbb{R}^2$ if $x(p) = x(q)$ and $y(p) \leq y(q)$. For a polygonal path $\Pi = v_1, \ldots, v_k$, we denote by $V_\Pi = \{v_1, \ldots, v_k\}$ the set of vertices of Π and by $E_\Pi = \{v_1v_2, \ldots, v_{k-1}v_k\}$ the set of edges of Π. We call a polygonal path $\Pi = v_1, \ldots, v_k$ *x-monotone* if $x(v_1) < \cdots < x(v_k)$. In addition, we say that a point $p \in \mathbb{R}^2$ lies *below* an x-monotone path Π if p lies vertically below a point p' (not necessarily a vertex!) on Π. Analogously, a line segment $\overline{pq}$ lies below Π if every point $r \in \overline{pq}$ lies below Π. We do not always strictly distinguish between a vertex v of G and the point $\delta(v)$ to which this vertex is mapped in a particular drawing δ of G. Similarly, we write vw both for the edge $\{v, w\}$ of G and the straight-line segment connecting $\delta(v)$ with $\delta(w)$.

The basic idea. Note that in order to establish a lower bound on $\text{fix}(G)$ we can assume that the given graph G is triangulated. Otherwise we can triangulate G

arbitrarily (by fixing an embedding of G and adding edges until all faces are 3-cycles) and work with the resulting triangulated planar graph. A plane drawing of the latter trivially yields a plane drawing of G. So let G be a triangulated planar graph, and let δ_0 be a drawing of G, e.g., one with $\text{fix}(G, \delta_0) = \text{fix}(G)$.

The basic idea of our algorithm is to find a plane embedding β of G such that there exists a long simple path Π connecting two vertices s and t of the outer triangle stu with the property that all chords of Π lie on one side of Π (with respect to β) and u lies on the other. For an example of such an embedding β, see Fig. 1(b). We describe how to find β and Π depending on the maximum degree and the diameter of G in Section 5. For the time being, let's assume they are given. Now our goal is to produce a drawing of G according to the embedding β and at the same time keep many of the vertices of Π at their positions in δ_0. Having all chords on one side is the crucial property of Π we use to achieve this. We allow ourselves to move all other vertices of G to any location we like. This gives us a lower bound on $\text{fix}(G, \delta)$ in terms of the number l of vertices of Π. Our method is illustrated in Fig. 1.

Algorithm outline. Now we sketch our three-step method. Let C denote the set of chords of Π. We assume that these chords lie to the right of Π in the embedding β. (Note that "below" is not defined in an embedding.) Let V_{bot} denote the set of vertices of G that lie to the right of Π in β and let $V_{\text{top}} = V \setminus (V_\Pi \cup V_{\text{bot}})$. Note that u lies in V_{top}.

In step 1 of our algorithm we bring the vertices in V_Π from the position they have in δ_0 into the same ordering according to increasing x-coordinates as they appear along Π in β. This yields a new (usually non-plane) drawing δ_1 of G that maps Π on an x-monotone polygonal path Π_1. Now we can apply the Erdős–Szekeres theorem [3] that basically says that a sequence of l distinct integers always contains a monotone (increasing or decreasing) subsequence of length at least $\sqrt{l-1}+1 \geq \lceil \sqrt{l}\, \rceil$. Thus we can choose δ_1 such that at least $\sqrt{l}$ vertices of Π remain fixed. Let $F \subseteq V_\Pi$ be the set of the fixed vertices. Note that $\delta_1|_{V \setminus V_\Pi} = \delta_0$, see Fig. 1(c).

Once we have constructed Π_1 we have to find suitable positions for the vertices in $V_{\text{top}} \cup V_{\text{bot}}$. This is simple for the vertices in V_{top}: if we move vertex u, which lies on the outer face, far enough above Π_1, then the polygon P_1 bounded by Π_1 and by the edges us and ut will be *star-shaped*. Recall that a polygon P is called *star-shaped* if the interior of its *kernel* is non-empty, and the *kernel* of a clockwise-oriented polygon P is the intersection of the right half-planes induced by the edges of P. Now if P_1 is star-shaped, we have fulfilled one of the assumptions of the following result of Hong and Nagamochi [5] for drawing *triconnected* graphs, i.e., graphs that cannot be decomposed by removing two vertices. We would like to use their result in order to draw into P_1 the subgraph G^+_{top} of G induced by $V_{\text{top}} \cup V_\Pi$ *excluding* the chords in C.

Theorem 1 ([5]). *Given a triconnected plane graph H, every drawing δ^* of the outer facial cycle of H on a star-shaped polygon P can be extended in linear time to a plane drawing of H (even one where all inner faces are convex).*

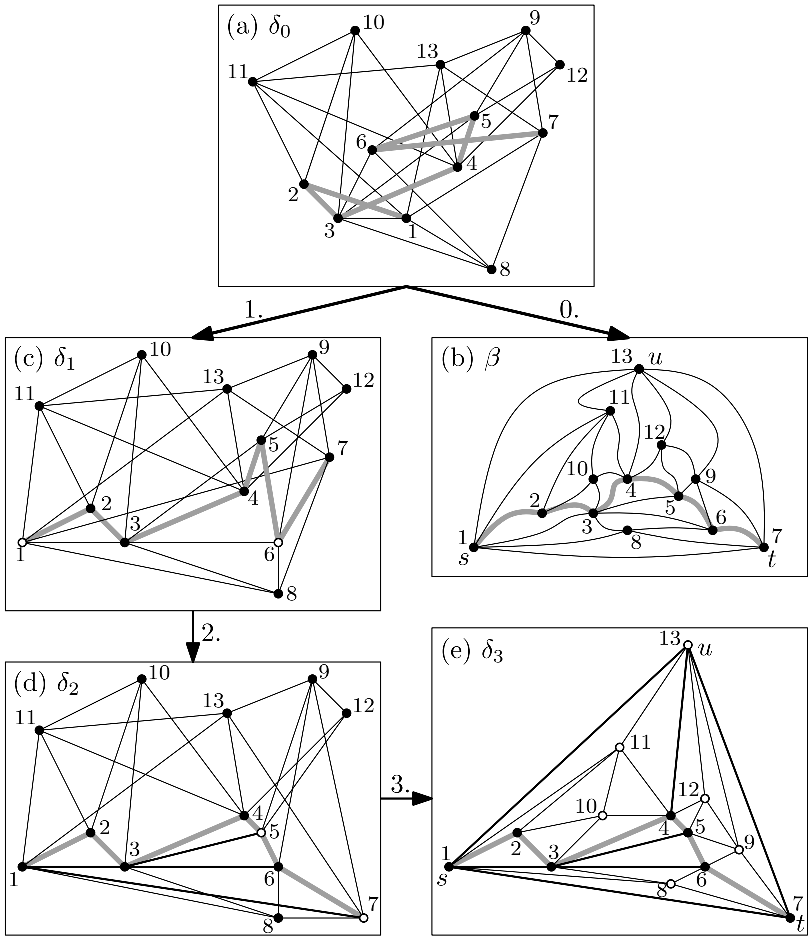


Fig. 1: An example run of our algorithm: (a) input: the given non-planar drawing δ_0 of a triangulated planar graph G. (b) Plane embedding β of G with path Π (drawn gray) that connects two vertices on the outer face. To make δ_0 plane we first make Π x-monotone (c), then we bring all chords (bold segments) to one side of Π (d), move u to a position where u sees all vertices in V_Π, and finally move the vertices in $V \setminus (V_\Pi \cup \{u\})$ to suitable positions within the faces bounded by the bold gray and black edges (e). Vertices that (do not) move from δ_{i-1} to δ_i are marked by circles (disks).

Observe that G^+_{top} is *not* necessarily triconnected: vertex u may be adjacent to vertices on Π other than s and t. But what about the subgraphs of G^+_{top} bounded (in β) by Π and edges of type uw_i, where $(s =)w_1, w_2, \ldots, w_l(= t)$ is the sequence of vertices of Π? Recall that a planar graph H is called a *rooted triangulation* [1] if in every plane drawing of H there exists at most one facial cycle with more than three vertices. According to Avis [1] the result stated in the following lemma is well known. It can be shown using Tutte's characterization of triconnected graphs [13].

Lemma 1 ([1]). *A rooted triangulation is triconnected if and only if no facial cycle has a chord.*

Now it is clear that we can apply Theorem 1 to draw each subgraph of $G^+_{\rm top}$ bounded by Π and by the edges of type uw_i. By the placement of u, each drawing region is star-shaped, and by construction, each subgraph is chordless and thus triconnected. However, to draw the graph $G^+_{\rm bot}$ induced by $V_{\rm bot} \cup V_\Pi$ (*including* the chords in C) we must work a little harder.

In step 2 of our algorithm we once more change the embedding of Π. We first carefully pick a subset V^* of vertices of Π. On the one hand V^* contains at least one endpoint of each chord in C. On the other hand V^* contains only a fixed fraction of the vertices in F, the subset of V_Π that δ_1 leaves fixed. Then we go through the vertices in V^* in a certain order, moving each vertex vertically down as far as necessary (see vertices 5 and 7 in Fig. 1(d)) to achieve two goals: (a) all chords in C move below the resulting polygonal path Π_2, and (b) the faces bounded by Π_2, the edge st, and the chords become star-shaped polygons. This defines a new drawing δ_2, which leaves a third of the vertices in F and all vertices in $V \setminus V_\Pi$ fixed.

In step 3 we use that Π_2 is still x-monotone. This allows us to move vertex u to a location above Π_2 where it can see every vertex of Π_2. Now Π_2, the edges of type uw_i (with $1 < i < l$) and the chords in C partition the triangle ust into star-shaped polygons with the property that the subgraphs of G that have to be drawn into these polygons are all rooted triangulations, and thus triconnected. This means that we can apply Theorem 1 to each of them. The result is our final—and plane—drawing δ_3 of G, see Fig. 1(e).

4 The Main Theorem

Recall that F is the set of vertices in V_Π we kept fixed in the step 1, i.e., in the construction of the x-monotone polygonal path Π_1. Our goal is to keep a constant fraction of the vertices in F fixed when we construct Π_2, which also is an x-monotone polygonal path, but has two additional properties: (a) all chords in C lie below Π_2 and (b) the faces induced by Π, w_1w_l, and the chords in C are star-shaped polygons. The following lemmas form the basis for the proof of our main theorem (Theorem 2), which shows that this can be achieved. For the proofs refer to the long version [11].

Lemma 2. *Let $\Pi = v_1, \dots, v_k$ be an x-monotone polygonal path such that (i) the segment v_1v_k lies below Π and (ii) the polygon P bounded by Π and v_1v_k is star-shaped. Let v'_k be any point vertically below v_k. Then the polygon $P' = v_1, \dots, v_{k-1}, v'_k$ is also star-shaped.*

Lemma 3. *Let $\Pi = v_1, \dots, v_k$ be an x-monotone polygonal path and let D be a set of pairwise non-crossing straight-line segments with endpoints in V_Π that all* *lie below* Π*. Let* v'_k *be a point vertically below* v_k*, let* $\Pi' = v_1, \ldots, v_{k-1}, v'_k$*, and finally let* D' *be a copy of* D *with each segment* $wv_k \in D$ *replaced by* wv'_k*.*

Then the segments in D' *are pairwise non-crossing and all lie below* Π'*.*

Lemma 4. *Let* $\Pi = v_1, \ldots, v_k$ *be an* x*-monotone polygonal path. Let* C_Π *be a set of chords of* Π *that can be drawn as non-crossing curved lines below* Π*. Let* G_Π *be the graph with vertex set* V_Π *and edge set* $E_\Pi \cup C_\Pi$*. Let* V^* *be a vertex cover of the edges in* C_Π*. Then there is a way to modify* Π *by decreasing the* y*-coordinates of the vertices in* V^* *such that the resulting straight-line drawing* δ^* *of* G_Π *is plane, the bounded faces of* δ^* *are star-shaped, and all edges in* C_Π *lie below the modified polygonal path* Π*.*

Note that the vertices in the complement of V^* remain fixed and that the modified polygonal path Π is x-monotone, too.

Proof. We use induction on the number m of chords. If $m = 0$, we need not modify Π. So, suppose $m > 0$. We first choose a chord $vw \in C_\Pi$ with $x(v) < x(w)$ such that there is no other edge $v'w' \in C_\Pi$ with the property that $x(v') \leq x(v)$ and $x(w') \geq x(w)$. Clearly, such an edge always exists. Then we apply the induction hypothesis to $C_\Pi \setminus \{vw\}$. This yields a modified path Π' of Π such that all edges in the resulting straight-line drawing of $G_\Pi - vw$ lie below Π' and all bounded faces in this drawing are star-shaped. Now consider the chord vw and let f be the new bounded face that results from adding vw. Without loss of generality we assume that $v \in V^*$. According to Lemmas 2 and 3 we can move v downwards from its position in Π' as far as we like. Hence, we can make the face f star-shaped without destroying this property for the other faces. □

Now suppose we have modified the x-monotone path Π_1 according to Lemma 4. Then the resulting x-monotone path Π_2 admits a straight-line drawing of the chords in C below Π_2 such that the bounded faces are star-shaped polygons, see for the example in Fig. 1(d). Recall that $u \in V_{\text{top}}$ is the vertex of the outer triangle in β that does not lie on Π. We now move vertex u to a position above Π_2 such that all edges $uw \in E$ with $w \in V_\Pi$ can be drawn without crossing Π_2 and such that the resulting faces are star-shaped polygons. Since Π_2 is x-monotone, this can be done. As an intermediate result we obtain a plane straight-line drawing of a subgraph of G where all bounded faces are star-shaped. It remains to find suitable positions for the vertices in $(V_{\text{top}} \setminus \{u\}) \cup V_{\text{bot}}$. For every star-shaped face f there is a unique subgraph G_f of G that must be drawn inside this face. Note that by our construction every edge of G_f that has both endpoints on the boundary of f must actually be an edge of the boundary. Therefore, G_f is a rooted triangulation where no facial cycle has a chord. Now Lemma 1 yields that G_f is triconnected. Finally, we can use the result of Hong and Nagamochi [5] (see Theorem 1) to draw each subgraph of type G_f and thus finish our construction of a plane straight-line drawing of G, see the example in Fig. 1(e). Let's summarize.

Theorem 2. *Let* G *be a triangulated planar graph that contains a simple path* $\Pi = w_1, \ldots, w_l$ *and a face* uw_1w_l*. If* G *has an embedding* β *such that* uw_1w_l *is* *the outer face, u lies on one side of Π, and all chords of Π lie on the other side, then* $\mathrm{fix}(G) \geq \sqrt{l}/3$.

Proof. We continue to use the notation introduced earlier in this section. Recall that F is the set of vertices that we kept fixed in the first step, that is in the construction of the x-monotone path Π_1. It follows from [8, Proposition 1] that we can make sure that $|F| \geq \sqrt{l}$, where l is the number of vertices of the path Π we started with. Further recall that C is the set of chords of Π. Now let C' be the subset of those chords in C that have both endpoints in F. Consider the graph H induced by the edges in C' on F. For example, in Fig. 1(c), $C = \{17, 13, 35, 36\}$, $F = \{2, 3, 4, 5, 7\}$, $C' = \{35\}$ and $H = (\{3, 5\}, C')$. Since H is outerplanar, it is easy to color H with three colors: the dual of H without the vertex for the outer face consists of trees each of which can be processed by, say, breadth-first search. The union U of the smallest two color classes is a vertex cover of H of size at most $2|F|/3$. Now let $V^* = (V_\Pi \setminus F) \cup U$. Then every chord in C has at least one of its endpoints in V^* and $|V^* \cap F| = |U| \leq 2/3|F|$. Hence, by Lemma 4 at least a third of the vertices in F remain fixed when we construct the x-monotone path Π_2. In the remaining steps of our construction, i.e., when placing the vertices in V_{top} and V_{bot}, none of the vertices in $F \setminus U$ is moved. Hence, $\mathrm{fix}(G) \geq |F \setminus U| \geq |F|/3 \geq \sqrt{l}/3$. □

5 Finding a Suitable Path

In this section we present two strategies for finding a suitable path Π. They both do not depend on the geometry of the given drawing δ_0 of G. Instead, they exploit the graph structure of G. The first strategy works well if G has a vertex of large degree and, even though it is very simple, yields asymptotically tight bounds for outerplanar graphs.

Lemma 5. *Let G be a triangulated planar graph with maximum degree Δ. Then* $\mathrm{fix}(G) \geq \sqrt{\Delta}/3$.

Proof. Let u be a vertex of degree Δ and consider a plane embedding β of G where vertex u lies on the outer face. Since G is planar, such an embedding exists. Let $W = \{w_1, \dots, w_\Delta\}$ be the neighbors of u in β sorted clockwise around u. This gives us the desired polygonal path $\Pi = w_1, \dots, w_\Delta$ that has no chords on the side that contains u. Thus Theorem 2 yields $\mathrm{fix}(G) \geq \sqrt{\Delta}/3$. □

Lemma 5 yields a lower bound for outerplanar graphs that is asymptotically tight as we will see in the next section.

Corollary 1. *Let G be an outerplanar graph with n vertices. Then* $\mathrm{fix}(G) \geq \sqrt{n-1}/3$.

Proof. We select an arbitrary vertex u of G. Since G is outerplanar, we can triangulate G in such a way that in the resulting triangulated planar graph G' vertex u is adjacent to every other vertex in G'. Thus the maximum degree of a vertex in G' is $n - 1$. □

Our second strategy works well if the diameter d of G is large.

Lemma 6. *Let G be a triangulated planar graph of diameter d. Then* $\operatorname{fix}(G) \geq \sqrt{2d-1}/3$.

Proof. We choose two vertices s and v such that a shortest s–v path has length d. We compute any plane embedding of G that has s on its outer face. Let t and u be the neighbors of s on the outer face. Recall that a *Schnyder wood* (or *realizer*) [10] of a triangulated plane graph is a (special) partition of the edge set into three spanning trees each rooted at a different vertex of the outer face. Edges can be viewed as being directed to the corresponding roots. The partition is special in that the cyclic pattern in which the spanning trees enter and leave a vertex is the same for all inner vertices. Schnyder [10] showed that this cyclic pattern ensures that the three unique paths from a vertex to the three roots are vertex-disjoint and chordless. Let π_s, π_t, and π_u be the "Schnyder paths" from v to s, t, and u, respectively. Note that the length of π_s is at least d, and the lengths of π_t and π_u are both at least $d-1$. Let Π be the path that goes from s along π_s to v and from v along π_t to t. The length of Π is at least $2d-1$. Note that due to the existence of π_u the path Π has no chords on the side that contains u. Thus, Theorem 2 yields $\operatorname{fix}(G,\delta) \geq \sqrt{2d-1}/3$. □

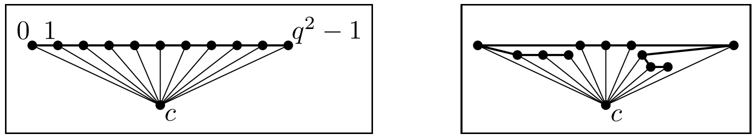
Next we determine the trade-off between the two strategies above.

Theorem 3. *Let G be a planar graph with $n \geq 4$ vertices. Then* $\operatorname{fix}(G) \geq \frac{1}{3}\sqrt{\frac{2(\log n)-2}{\log\log n} - 1}$.

Proof. Let G' be an arbitrary triangulation of G. Note that the maximum degree Δ of G' is at least 3 since $n \geq 4$ and G' is triangulated. To relate Δ to the diameter d of G' we use a very crude counting argument—Moore's bound: starting from an arbitrary vertex of G we bound the number of vertices we can reach by a path of a certain length. Let j be the smallest integer such that $1 + (\Delta-1) + (\Delta-1)^2 + \cdots + (\Delta-1)^j \geq n$. Then $d \geq j$. By the definition of j we have $n \leq (\Delta-1)^{j+1}/(\Delta-2)$, which we can simplify to $n \leq 2(\Delta-1)^j$ since $\Delta \geq 3$. Hence we have $d \geq j \geq \frac{(\log n)-1}{\log(\Delta-1)}$.

Now, if $\Delta \geq \log n$, then Lemma 5 yields $\operatorname{fix}(G') \geq \sqrt{\log n}/3$. Otherwise $2d-1 \geq \frac{2(\log n)-2}{\log\log n} - 1$, and we can apply Lemma 6. Observing that $\operatorname{fix}(G) \geq \operatorname{fix}(G')$ yields the desired bound. □

Remark 1. The proof of Theorem 3 (together with the auxiliary results stated earlier) yields an efficient algorithm for making a given straight-line drawing of a planar graph G with n vertices plane by moving some of its vertices to new positions. The running time is dominated by the time spent in the first step, i.e., computing the x-monotone path Π_1, which takes $O(n \log n)$ time [9]. The remaining steps of our method can be implemented to run in $O(n)$ time, including the computation of a Schnyder wood [10] needed in the proof of Lemma 6.



(a) (b)

Fig. 2: The outerplanar graph H_q that we use in our upper-bound construction.

6 An Upper Bound for Outerplanar Graphs

In this section we want to show that the lower bound for outerplanar graphs in Corollary 1 is asymptotically tight. For a given positive integer q let H_q be the outerplanar graph that consists of a path $0, 1, \ldots, q^2 - 1$ and an extra vertex $c = q^2$ that is connected to all other vertices, see Fig. 2(a). Note that H_q has many plane embeddings—e.g., Fig. 2(b)—but only two outerplane embeddings: Fig. 2(a) and its mirror image.

Let δ_q be the drawing of H_q where all vertices are placed on a horizontal line ℓ as follows. While vertex c can go to any (free) spot, vertices $0, \ldots, q^2 - 1$ are arranged in the order σ_q, namely

$$(q-1)q, (q-2)q, \ldots, 2q, q, \underline{0}, 1+(q-1)q, \ldots, 1+q, \underline{1}, \ldots, q^2-1, \ldots, (q-1)+q, \underline{q-1}.$$

The same sequence has been used by Goaoc et al. [4] to construct a planar (but not outerplanar) n-vertex graph G with $\mathrm{fix}(G) \leq \lceil \sqrt{n-2}\, \rceil + 1$.

We now make two observations about the structure of σ_q.

Observation 1 ([4]) *The longest increasing or decreasing subsequence of σ_q has length q.*

For the second observation let's define that two sequences Σ and Σ' of numbers *overlap* if $[\min(\Sigma), \max(\Sigma)] \cap [\min(\Sigma'), \max(\Sigma')] \neq \emptyset$.

Observation 2 *Let Σ and Σ' be two non-overlapping decreasing or two non-overlapping increasing subsequences of σ_q. Then $|\Sigma \cup \Sigma'| \leq q + 1$.*

Proof. First consider the case that Σ and Σ' are both decreasing. Since they do not overlap we can assume without loss of generality that $\max(\Sigma) < \min(\Sigma')$. We define $V_i = \{iq + j : 0 \leq j \leq q - 1\}$ for $i = 0, \ldots, q - 1$. Then, since Σ and Σ' are both decreasing, they can each have at most one element in common with every V_i. Now suppose they have both one element in common with some V_{i_0}. Then, since $\max(\Sigma) < \min(\Sigma')$, Σ cannot have an element in common with any V_i, $i > i_0$, and Σ' cannot have an element in common with any V_i, $i < i_0$. Therefore, $|\Sigma \cup \Sigma'| \leq q + 1$.

Due to the symmetry of σ_q the case that Σ and Σ' are both increasing can be analyzed analogously. □

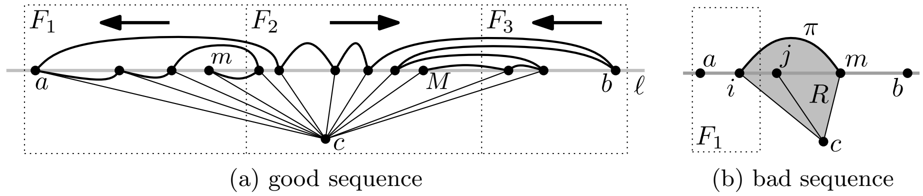


Fig. 3: Analyzing the sequence of fixed vertices along the line ℓ.

Given these observations we can now prove our upper bound on $\mathrm{fix}(H_q, \delta_q)$.

Theorem 4. *For any $q \geq 2$ it holds that $\mathrm{fix}(H_q, \delta_q) \leq 2q + 1 = 2\sqrt{n-1} + 1$, where $n = q^2 + 1$ is the number of vertices of H_q.*

Proof. Let δ' be a plane drawing of H_q that maximizes the number of fixed vertices with respect to δ_q. Let F be the set of fixed vertices. Our proof exploits the fact that the simple structure of H_q forces the left-to-right sequence of the fixed vertices to also have a very simple structure.

Consider the drawing δ'. Clearly vertex c does not lie on ℓ. Thus we can assume that c lies below ℓ. Let a and b be the left- and rightmost vertices in F, respectively, and let m and M be the vertices with minimum and maximum index in F, respectively. Without loss of generality we can assume that m lies to the left of M, see Fig. 3(a).

We go through the vertices of F from left to right along ℓ. Let F_1 be the longest uninterrupted decreasing sequence of vertices in F starting from a. We claim that m is the last vertex in F_1. Assume to the contrary that $i \neq m$ is the last vertex of F_1, and let $j \in F$ be its successor on ℓ, see Fig. 3(b). If m is not the last vertex of F_1, then F_1 does not contain m. Thus m lies to the right of j. Consider the path $\pi = i, i-1, \ldots, m$. Since $j > i > m$, j is not a vertex of π. Clearly j lies below π, otherwise the edge jc would intersect π. Let R be the polygon bounded by π and by the edges ci and cm. Since δ' is plane, R is simple. Observe that j lies in the interior of R, which is shaded in Fig. 3(b). On the other hand, neither a nor b lies in the interior of R, otherwise the edge ac or the edge bc would intersect π.

We consider two cases. First suppose $j < a$. Then H_q contains the path $j, j+1, \ldots, a$ and we know that $a \neq i$ (since by definition of i and j we have $i < j$). Thus the path $j, j+1, \ldots, a$ does not contain any vertex incident to R. So it crosses some edge on the boundary of R. This contradicts δ' being plane. Now suppose $j > a$. Then H_q contains the path $j, j+1, \ldots, b$. In this case we can argue analogously since $m < b$ (otherwise m would lie to the right of M), reaching the same contradiction. Thus our assumption $i \neq m$ is wrong, and m is indeed the last vertex of F_1.

Now let F_2 be the longest uninterrupted *increasing* sequence of vertices in F starting from the successor of m. With similar arguments as above we can show that M is the last vertex in F_2. Finally let F_3 be the sequence of the remaining

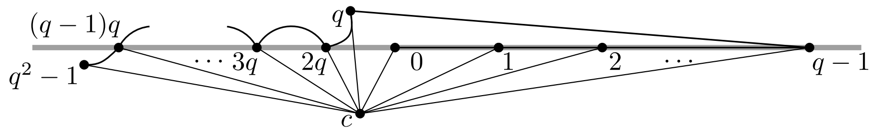


Fig. 4: This plane drawing of H_q shows that $\text{fix}(H_q, \delta_q) \geq 2q-2$ since it keeps the vertices $0, 1, 2, \ldots, q-1, 2q, 3q, \ldots, (q-1)q$ of δ_q fixed.

vertices from the successor of M to b. Again with similar arguments as above we can show that F_3 is decreasing.

The set F is partitioned by F_1, F_2, and F_3; F_2 is increasing, while F_1 and F_3 are decreasing. Thus Observations 1 and 2 yield $|F| \leq 2q+1$ as desired. □

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