Pseudo-Convex Decomposition of Simple Polygons*

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Abstract

We extend a dynamic-programming algorithm of Keil and Snoeyink for finding a minimum convex decomposition of a simple polygon to the case when both convex polygons and pseudo-triangles are allowed. Our algorithm determines a minimum pseudo-convex decomposition of a simple polygon in $O(n^3)$ time where n is the number of the vertices of the polygon. In this way we obtain a well-structured decomposition with fewer polygons, especially if the original polygon has long chains of concave vertices.

1 Introduction

Pseudo-triangles are simple polygons with exactly three convex angles, i.e. interior angles of less than 180°. Recently they have emerged to have geometrical properties of interest for rigidity theory and ray-shooting problems [2]. This is why pseudo-triangles have been considered in relation with the *decomposition problem* of a set of points. It is defined as follows.

Given a set S of n points in the plane, decompose the convex hull of S into polygons of a given type such that the vertices of the polygons are in S and each point in S is a vertex of at least one of the polygons. The decomposition is called *convex* if only convex polygons are allowed, *pseudo-triangulation* if only pseudo-triangles are allowed, and *pseudo-convex* if both pseudo-triangles and convex polygons can be used. Convex decompositions have been considered by Fevens et al. [3]. Streinu [7] shows that the minimum number of edges needed to obtain a pseudo-triangulation is $2n-3$ and thus, by Euler, the number of pseudo-triangles is $n-2$, which does not depend on the structure of the point set but only on its size. This motivates research on the problem of enumerating all minimum pseudo-triangulations [2]. Aichholzer et al. [1] study pseudo-convex decompositions. They show that each minimum pseudo-convex decomposition of a set of n points consists of less than $7n/10$ polygons.

A related problem is the decomposition of *simple polygons* into convex polygons or pseudo-triangles, e.g. for point location or ray shooting. For decompositions of simple polygons the same terms as for decompositions of point sets apply. A decomposition is called *minimum* if it consists of the minimum number of regions.

In this paper we give an algorithm for computing minimum pseudo-convex decompositions of simple polygons. Given a simple polygon we use the same approach as Gudmundsson and Levcopoulos [4] to determine all geodesics in the polygon which can be sides of a pseudo-triangle and present a simple way to check whether three such geodesics form a pseudo-triangle. We use dynamic programming to solve proper subproblems which then can be combined to obtain a global solution. The resulting algorithm runs in $O(n^3)$ time and uses $O(n^2)$ space.

Our algorithm is based on a general technique for decomposing a simple polygon into polygons of a certain type proposed by Keil [5]. The technique is based on optimally decomposing subpolygons each of which is obtained from the original by drawing a single diagonal. This idea yields an $O(n^3 \log n)$-time algorithm for the *convex* decomposition problem [5]. Keil and Snoeyink [6] improve Keil's result by giving an $O(\min(nr^2, r^4))$-time algorithm, where r is the number of reflex vertices of the polygon.

2 Characterization of Pseudo-Triangles

We use $P^+(A_i, A_j)$ and $P^-(A_i, A_j)$ to denote the paths on the boundary ∂P from a vertex A_i to a vertex A_j of P in clockwise and anticlockwise direction, respectively. With $\text{vis}(A_i)$ we denote the list of all vertices of P which are visible from A_i in clockwise order starting with A_{i+1}. Unless stated otherwise, the vertices of a polygon will be given in clockwise order.

Definition 1 *Let $P = A_0A_1 \ldots A_{n-1}$ be a simple polygon. A path $p = B_1B_2 \ldots B_m$ from A_i to A_j is a* concave geodesic *with respect to the polygon P if it satisfies the following three conditions, see Fig. 1:*

(G1) $B_1 = A_i$ *and* $B_m = A_j$.

(G2) For each $k < m$ it holds that B_{k+1} is the last vertex on $P^+(B_k, A_j)$ which is visible from B_k.

(G3) $B_1B_2 \ldots B_m$ is a convex, anticlockwise oriented polygon.

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*Work supported by grant WO 758/4-2 of the German Research Foundation (DFG).

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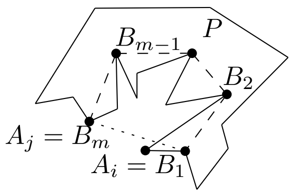

Figure 1: The geodesic $B_1B_2 \dots B_m$ from A_i to A_j is concave with respect to the simple polygon P.

Remark 1 *If $B_1B_2 \dots B_m$ is a concave geodesic from B_1 to B_m with respect to a simple polygon P then $B_2 \dots B_m$ is a concave geodesic from B_2 to B_m with respect to P.*

For our further considerations we will need the following fact [6]:

Fact 1 *Let A_i be a vertex of $P = A_0A_1 \dots A_{n-1}$. Then the cyclic order of the line segments A_iA_j with $A_iA_j \subseteq P$ around A_i is the same as the order of their other endpoints along ∂P.*

The following lemma states the relationship between the concave geodesics in a simple polygon and the pseudo-triangles that can participate in a decomposition of the polygon.

Lemma 1 *If a pseudo-triangle T is contained in a simple polygon $P = A_0A_1 \dots A_{n-1}$ with convex vertices at A_j, A_k and A_l, $j < k < l$, then the paths $T^+(A_j, A_k)$, $T^+(A_k, A_l)$ and $T^+(A_l, A_j)$ are concave geodesics with respect to P.*

Proof. (Sketch) Due to symmetry it suffices to prove that $T^+(A_j, A_k)$ is a concave geodesic. Properties (G1) and (G3) of a concave geodesic obviously hold. Thus we have to verify only property (G2).

First note that $T^+(A_j, A_k)$ contains only vertices of P that lie on $P^+(A_j, A_k)$, for otherwise T wouldn't be simple. Now assume that $T^+(A_j, A_k)$ does not satisfy property (G2), see Fig. 2. Let $k > i \geq j$ be such that $A_i \in T^+(A_j, A_k)$ violates the construction proposed in property (G2). Let A_s be the vertex on $T^+(A_j, A_k)$ after A_i and let $k \geq t > s$ be such that A_t is visible from A_i. It is clear that $s > i$. Due to Fact 1 we obtain that the edges A_iA_{i+1}, A_iA_s and A_iA_t appear in clockwise order around A_i. In particular, because of the convexity of $T^+(A_i, A_k)A_i$, A_iA_t intersects $T^+(A_i, A_k)$ only in A_i and A_iA_s is contained in the polygon $P^+(A_i, A_t)A_i$. However, A_k lies outside this polygon and thus $T^+(A_i, A_k)$ leaves $P^+(A_i, A_t)A_i$ in some point x which does not belong to A_iA_t, see Fig 2. Therefore $T^+(A_i, A_k)$ leaves P. Contradiction. □

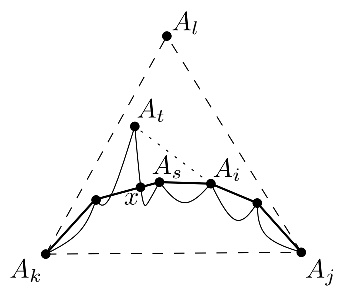

Figure 2: Each pseudo-triangle consists of three concave geodesics that connect its convex vertices. The arcs denote the boundary of $P^+(A_j, A_t)$ and the solid lines denote the edges of $T^+(A_j, A_k)$.

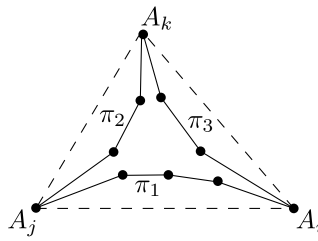

Figure 3: Testing whether three concave geodesics π_1, π_2, and π_3 define a pseudo-triangle

Next we establish the converse relation. Namely three concave geodesics determine a pseudo-triangle.

Lemma 2 *Let $P = A_0A_1 \dots A_{n-1}$ be a simple polygon. Further let $i < j < k$ and $\pi_1 = A_i \dots A_j$, $\pi_2 = A_j \dots A_k$ and $\pi_3 = A_k \dots A_i$ be concave geodesics with respect to P. If the triangle $A_iA_jA_k$ is clockwise oriented, then the polygon $\pi_1\pi_2\pi_3$ is a pseudo-triangle.*

Proof. (Idea) See Fig. 3. Using that, say π_1 and π_2 have only one common vertex, one can show that they have no other common points. Then the orientation of the triangle $A_iA_jA_k$ together with property (G2) from Definition 1 provide that in fact π_1 is contained in the triangle $A_iA_jA_k$. Similar considerations for the paths π_2 and π_3 show that $\pi_1\pi_2\pi_3$ is a pseudo-triangle. □

3 Algorithm

We use the same approach for finding a minimum pseudo-convex decomposition of a simple polygon as Keil and Snoeyink [6] for finding the minimum convex decomposition of a polygon. Namely we consider smaller simple polygons which are obtained from the original polygon by drawing a single diagonal. For each such polygon we make assumptions in what sort of polygon the diagonal can be included. In case the diagonal is a part of a convex polygon we use the algorithm of Keil and Snoeyink [6]. In case the diagonal is part of a pseudo-triangle we proceed as follows. Assume we have a precomputed list L of all concave geodesics w.r.t. P. Then we can filter L to find all pseudo-triangles that contain the diagonal as an edge. For each such pseudo-triangle T we compute the size of an optimal decomposition that contains T. The optimal solution is the minimum of the solutions obtained in the two cases. Finally we apply dynamic programming, just as Keil and Snoeyink [6].

Now we describe our ideas in detail. Let $P = A_0A_1 \dots A_{n-1}$ be a simple polygon. We use definitions similar to those in [6]. If $i < j$ and A_j is visible from A_i in P then we denote the line segment A_iA_j by d_{ij} and call it a *diagonal* of P. In particular each edge of P is a diagonal. For each such diagonal a simple polygon $P_{ij} = A_iA_{i+1} \dots A_j$ is defined.

Definition 2 *Let $\mathcal{D}$ denote the set of all pseudo-convex decompositions of a polygon P_{ij}. Then we introduce the following parameters:*

$$w_{ij} = \min\{|D| : D \in \mathcal{D}\}$$
$$cw_{ij} = \min\{|D| : D \in \mathcal{D}, \text{ the edge } d_{ij} \text{ is contained in a convex polygon}\}$$
$$pw_{ij} = \min\{|D| : D \in \mathcal{D}, \text{ the edge } d_{ij} \text{ is contained in a pseudo-triangle }\}$$

Clearly $w_{ij} = \min(cw_{ij}, pw_{ij})$.

Given the values w_{kl} for each k, l with $l - k < j - i$ and a list of all concave geodesics for the polygon P we first describe how to find pw_{ij}. We consider all concave geodesics which contain the edge A_iA_j and no vertex $A_k \in P$ with $k < i$ or $k > j$. For each such path $\pi_1 = B_1B_2 \dots B_m$ we go along $P^-(B_1, B_m)$ and for each vertex $A_l \in P^-(B_1, B_m)$ we check whether there exist concave geodesics $\pi_2 = B_m \dots A_l$ and $\pi_3 = A_l \dots B_1$. If π_2 and π_3 exist, we apply Lemma 2 to check whether the paths π_1,π_2 and π_3 determine a pseudo-triangle. If this is the case, an optimal decomposition of P_{ij} contains this pseudo-triangle if and only if for each pair $(k, l) \neq (i, j)$ such that A_kA_l is an edge of $\pi_1\pi_2\pi_3$ the polygon P_{kl} is optimally decomposed.

Thus if $w(\pi)$ denotes the sum of all w_{kl} where A_kA_l lies on a geodesic π, then it is clear that the optimal decomposition of P_{ij} using the pseudo-triangle $\pi_1\pi_2\pi_3$ consists of

$$s(\pi_1, A_l) = \sum_{A_kA_l \in \pi_1, A_kA_l \neq A_iA_j} w_{kl} + w(\pi_2) + w(\pi_3) + 1$$

polygons. Now we can compute pw_{ij} as the minimum of $s(\pi_1, A_l)$ over all pairs (π_1, A_l) that fulfill the above requirements.

To find the value cw_{ij} we consider all vertices A_k on the path $P^-(A_i, A_j)$ which are visible both from A_i and A_j. If A_iA_j is an edge of a convex polygon, then this polygon is either the triangle $T = A_iA_jA_k$ or a convex polygon $C = A_j \dots A_k \cup T$, where $A_j \dots A_k$ is a smaller convex polygon. In the former case, an optimal decomposition of P_{ij} consists of $w_{ik} + w_{kj} + 1$ polygons. In the latter case the decomposition of P_{ij} is the union of two pseudo-convex decompositions: (i) that of P_{ik} and (ii) that of P_{kj} under the condition that A_kA_j is an edge of a convex polygon C' with $C' \cup T$ convex. In (i) an optimal decomposition of P_{ik} consists of w_{ik} polygons. To determine an optimal decomposition of P_{kj} in (ii) we use the approach of Keil and Snoeyink [6], which relies on the following observation.

We call a *diagonal-convex* decomposition of P_{ij} a decomposition where d_{ij} is the diagonal of a convex polygon. For each polygon P_{ij} we store not only the value cw_{ij} but also a list CL_{ij} of *representatives* of diagonal-convex decompositions of P_{ij} which attain cw_{ij}. Given an optimal diagonal-convex decomposition Δ of P_{ij} the representative (s, t) of Δ is uniquely defined by a pair of vertices $\{A_s, A_t\} \cap \{A_i, A_j\} = \emptyset$. More precisely, A_s and A_t are those vertices of P_{ij} that are adjacent to A_i and A_j, respectively, in the only polygon $\Pi \in \Delta$ with d_{ij} being an edge of Π. We store only representatives (s, t) satisfying the property that for each other representative $(s', t') \neq (s, t)$ of P_{ij} either $s > s'$ or $t < t'$. Using the same arguments as Keil and Snoeyink [6, Section 3], one can show that in $O(n)$ time the value cw_{ij} can be correctly determined and the list CL_{ij} can be constructed—provided the lists CL_{kj} are available for all $i < k < j$.

4 Complexity

We now investigate the complexity of our algorithm. We first modify slightly Theorem 2 in [4].

Proposition 3 *Given a simple polygon $P = A_0A_1 \dots A_{n-1}$ we can construct in $O(n^2)$ time a data structure such that for any pair (i, j) it can decide in $O(1)$ time whether there is a concave geodesic π from A_i to A_j. If π exists, the data structure provides an $O(l)$-time walk along π, where l is the length of π.*

Proof. We first compute all lists vis(A_i) in $O(n^2)$ total time. Then we use dynamic programming to check whether there is a concave geodesic π from A_i to A_j. If π exists, we also compute the second and the second last vertex on π. We can walk on π by repeatedly jumping to the second vertex of the remaining path, which by Remark 1 is also a geodesic.

We consider the pairs (i, j) in increasing order of the number of vertices on the path $P^+(A_i, A_j)$. The edges A_iA_{i+1} obviously correspond to concave geodesics and it is easy to determine the second and second last vertex of these paths.

When the length of $P^+(A_i, A_j)$ is greater than 1 we use the list vis(A_i) to find the last vertex visible from

A_i on $P^+(A_i, A_j)$—this is either A_j or the last vertex visible from A_i on $P^+(A_i, A_{j-1})$. Fact 1 allows us to handle vis(A_i) in $O(1)$ time to obtain the desired information. Once we have found the last vertex A_l visible from A_i on $P^+(A_i, A_j)$ we check whether there is a concave geodesic π from A_l to A_j. If this is the case we use the second and the second last vertex on π to check whether A_i can be added to π without violating property (G2). According to Remark 1 this is the only way for obtaining a concave geodesic from A_i to A_j. Finally the second and the second last vertex on this path can also be computed in $O(1)$ time. Thus we need only $O(1)$ time per pair (i, j) in order to check whether there exists a concave geodesic from A_i to A_j and—in case it does—to find the second and the second last vertex on this path. Because the number of all pairs (i, j) is $O(n^2)$, this results in an $O(n^2)$-time algorithm with the desired properties. □

Notice that in the proof of Proposition 3 we can compute also the vertices with the greatest and smallest indices that lie on a given concave geodesic π without increasing the complexity of the algorithm. Moreover, we can check in constant time whether these two vertices are adjacent on π. We use this observation to compute a list PL_{ij} for each diagonal d_{ij}. In this list we store all pairs (k, l) such that there is a concave geodesic from A_k to A_l which contains d_{ij} but no vertex with index smaller than i or greater than j.

Theorem 4 *The number of polygons in a minimum pseudo-convex decomposition of a simple polygon* $P = A_0A_1 \dots A_{n-1}$ *can be computed in* $O(n^3)$ *time.*

Proof. We first set up the data structure of Proposition 3 and compute the lists PL_{ij}. This takes $O(n^2)$ total time. Then we implement the algorithm of Section 3. Using the technique of Keil and Snoeyink [6] the computation of all cw_{ij} can be carried out in total $O(n^3)$ time. To bound the time needed for the computation of pw_{ij} first note that each concave geodesic $\pi = B_1 \dots A_jA_i \dots B_m$ is contained in at most one list PL_{ij} and thus it is considered once only. We walk along π to determine the sum of the values w_{kl} over all $(k, l) \neq (i, j)$ with $A_kA_l \subseteq \pi$. This takes $O(n)$ time according to Proposition 3. Then for each point A_l on $P^+(B_m, B_1)$ we check whether there is a concave geodesic π_1 from A_l to B_1 and a concave geodesic π_2 from B_m to A_l. If this is the case, we use Lemma 2 to check whether $\pi\pi_1\pi_2$ is a pseudo-triangle. This takes $O(1)$ time. Finally we need the values $w(\pi_1)$ and $w(\pi_2)$ which can be computed in $O(n)$ time the first time we need them. Thus we walk along each geodesic only once and perform only $O(n)$ operations for each geodesic. The total number of concave geodesics is $O(n^2)$ which results in $O(n^3)$ time for determining all values pw_{ij}. Thus the number of polygons in a minimum pseudo-convex decomposition of a simple polygon P with n vertices can be computed in $O(n^3)$ time. □

Acknowledgments

We thank the anonymous referee whose comments helped us to improve the readability of this paper.

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