On the Weak Line Cover Numbers

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Abstract

For a given graph, we want to find crossing-free straight-line drawings of low visual complexity. A measure for the visual complexity of a drawing that has been considered before is the minimum number of lines needed to cover all vertices. In 3D, this number, the *3D weak line cover number*, is denoted by $\pi^1_3(G)$ for a given graph G. In 2D, for any planar graph G, the *2D weak line cover number* is denoted by $\pi^1_2(G)$.

We inductively construct an infinite family of polyhedral graphs with maximum degree 6, treewidth 3, and unbounded π^1_2-value. We also determine the π^1_2- and π^1_3-values of the Platonic graphs. We pose a number of open questions about the 2D and 3D weak line cover numbers.

1 Introduction

Recently, there has been considerable interest in representing graphs with as few objects as possible. The idea behind this objective is to keep the visual complexity of a drawing low for the observer. The types of objects that have been used are straight-line segments [4–7] and circular arcs [6, 10].

Chaplick et al. [1] considered *covering* straight-line drawings of graphs by unbounded objects (lines, planes) and defined the following new graph parameters. Let $1 \le l < d$, and let G be a graph. The *l-dimensional affine cover* number of G in $\mathbb{R}^d$, denoted by $\rho^l_d(G)$, is defined as the minimum number of l-dimensional planes in $\mathbb{R}^d$ such that G has a crossing-free straight-line drawing that is contained in the union of these planes. The *weak* l-dimensional affine cover number of G in $\mathbb{R}^d$, denoted by $\pi^l_d(G)$, is defined similarly to $\rho^l_d(G)$, but under the weaker restriction that the vertices (and not necessarily the edges) of G are contained in the union of the planes.

Clearly, for any suitable combination of l and d, it holds that $\pi^l_d(G) \le \rho^l_d(G)$. For any graph G, if $l' \le l$ and $d' \le d$ then $\pi^l_d(G) \le \pi^{l'}_{d'}(G)$ and $\rho^l_d(G) \le \rho^{l'}_{d'}(G)$. Chaplick et al. showed that it suffices to study the parameters ρ^1_2, ρ^1_3, ρ^2_3, and π^1_2, π^1_3, π^2_3:

▶ **Theorem 1** (Collapse of the Affine Hierarchy [1]). *For any integers $1 \le l < 3 \le d$ and for any graph G, it holds that $\pi^l_d(G) = \pi^l_3(G)$ and $\rho^l_d(G) = \rho^l_3(G)$.*

We call $\pi^1_2(G)$ and $\pi^1_2(G)$ also the 2D and 3D *weak line cover number* of G, respectively. For a given graph G, $\pi^1_2(G)$ is at most as large as $\rho^1_2(G)$ but it can be much smaller. For instance, Chaplick et al. showed that for the nested-triangles graph $T_k = C_3 \times P_k$ (shown in Fig. 2 for $k = 4$) with $n = 3k$ vertices it holds that $\rho^1_2(T_k) \ge n/2$, whereas clearly $\pi^1_2(T_k) \le 3$.

Chaplick et al. [2] also investigated the complexity of computing the (weak) affine cover numbers. Among others, they showed that in 3D, for $l \in \{1, 2\}$, it is NP-complete to decide whether $\pi^l_3(G) \le 2$ for a given graph G. In 2D, the question is still open.

▶ **Open Problem 1.** Is it NP-hard to compute, for a given planar graph G, its weak line cover number $\pi^1_2(G)$?

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Independently of this complexity issue, Chaplick et al. also asked the following question:

▶ **Open Problem 2.** Does the class of planar graphs admit a sublinear upper bound for π_2^1?

Even for restricted graph classes, the problem remains open. So far only two graph families with unbounded π_2^1-value are known [1, 9]. The first graph family, by Ravsky and Verbitsky [9], has treewidth 8 (which we define below). As Da Lozzo et al. [3] noted, the construction can be modified so that the treewidth of the graphs in the family becomes 5. In the second graph family, by Chaplick et al. [1], the maximum degree is bounded by 12 and $\pi_2^1(G) \geq n^{0.01}$. This yields a new type of open question:

▶ **Open Problem 3.** How small can we make the maximum degree in a family of planar graphs such that their π_2^1-value is still unbounded?

In this paper, we'll improve upon the result of Chaplick et al. in this respect.

We can also ask the opposite question – if we restrict the maximum degree of a graph family, how large can we make its π_2^1-value?

▶ **Open Problem 4.** Does the class of planar graphs with constant maximum degree admit a sublinear upper bound on π_2^1? In particular, is there a constant upper bound on π_2^1 for the class of planar graphs of maximum degree 3?

Another thread of research concerns the (un)boundedness of π_2^1 in the class of graphs of bounded treewidth. A graph has treewidth at most k if it is a subgraph of a k-tree. The class of *k-trees* (consisting of not necessarily planar graphs) is defined recursively as follows. The complete graph K_{k+1} is a k-tree; if G is a k-tree and H is obtained from G by adding a new vertex and connecting it to a k-clique of G then H is also a k-tree. Observe that the 1-trees are exactly the usual trees.

It is well known that the treewidth of any outerplanar graph is at most 2, and all graphs of treewidth 2 are planar. Chaplick et al. [1] proved that $\pi_2^1(G) \leq 2$ for any outerplanar graph G and asked the following question, which is still open.

▶ **Open Problem 5.** Does the class of treewidth-2 graphs have constant π_2^1-value?

Our contribution. As a warm-up, we compute the weak line cover numbers π_2^1 and π_3^1 of the Platonic graphs (1-skeletons of the Platonic solids) in 2D and 3D, respectively; see Section 2. Our main result is the construction of an infinite family of polyhedral graphs with maximum degree 6, treewidth 3, and unbounded π_2^1-value; see Section 3.

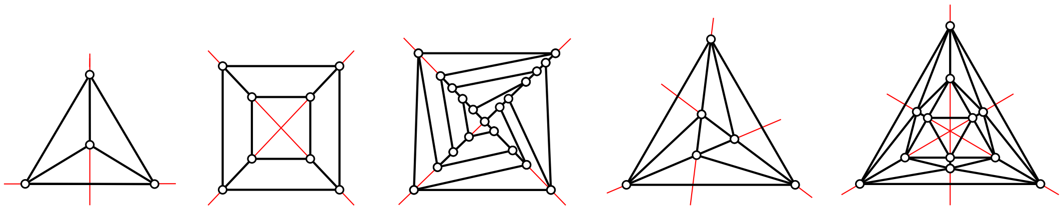
2 Optimal Weak Line Covers of the Platonic Graphs

The Platonic solids are convex polyhedra; hence, the Platonic graphs are planar. Kryven et al. [8] computed various parameters of visual complexity for these graphs. We compute the weak line cover numbers of the Platonic graphs in 2D and 3D. Interestingly, two graphs in this family behave differently in 2D and 3D.

Recall that a *linear forest* is a forest whose connected components are paths.

▶ **Proposition 1.** *Let G be a Platonic graph. Then*

(a) $\pi_2^1(G) = 2$ if G is the graph of the tetrahedron, the cube, or the dodecahedron;
(b) $\pi_2^1(G) = 3$ if G is the graph of the octahedron or the icosahedron.



(a) tetrahedron **(b)** cube **(c)** dodecahedron **(d)** octahedron **(e)** icosahedron

Figure 1 π_2^1-optimal drawings of the Platonic graphs.

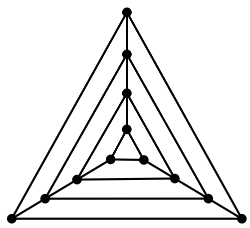


Figure 2 The nested-triangles graph $T_4 = C_3 \times P_4$.

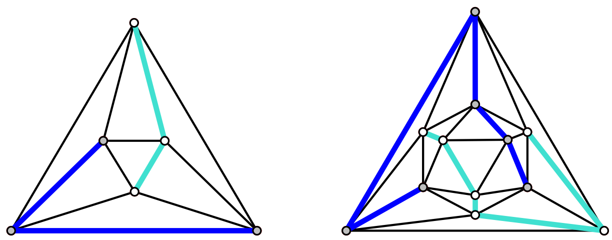


Figure 3 For any Platonic graph, its vertex set can be partitioned into two subsets that each induces a linear forest.

Proof. (a) See Figs. 1a, 1b, and 1c and note that only linear forests have π_2^1-value 1.

(b) Let G be the graph of the octahedron or the icosahedron. Then $\pi_2^1(G) \leq 3$; see Figs. 1d and 1e. On the other hand, $\pi_2^1(G) \geq 3$. Indeed, assume that there exists a plane drawing of G such that all vertices of G are covered by two straight lines ℓ_1 and ℓ_2. Since the outer face of G is a triangle, one of these straight lines, say ℓ_1, contains two vertices of the outer face. Thus ℓ_1 contains no other vertices of G; all of them are placed on ℓ_2. But this is impossible since the subgraph induced by these vertices is not a linear forest (in fact, it even contains a triangle), a contradiction. Hence, $\pi_2^1(G) = 3$. ◀

Chaplick et al. [1] related the affine cover numbers to standard combinatorial characteristics of graphs. The *linear vertex arboricity* $\mathrm{lva}(G)$ of a graph G is the smallest size r of a partition $V(G) = V_1 \cup \dots \cup V_r$ such that every V_i induces a linear forest. Chaplick et al. showed that the combinatorial parameter $\mathrm{lva}(G)$ actually coincides with the geometric parameter $\pi_3^1(G)$.

▶ **Theorem 2** ([1]). *For any graph G, it holds that $\pi_3^1(G) = \mathrm{lva}(G)$.*

We exploit this to compute the π_3^1-values of the Platonic graphs.

▶ **Proposition 2.** *For any Platonic graph G, it holds that $\pi_3^1(G) = 2$.*

Proof. If G is the graph of the tetrahedron, the cube or the dodecahedron, the claim follows from the fact that $\pi_3^1(G) \leq \pi_2^1(G)$. If G is the graph of the octahedron or the icosahedron, we use Theorem 2 and note that $\mathrm{lva}(G) = 2$; see Fig. 3. ◀

3 A Family of Graphs with Unbounded Weak Line Cover Number

We say that two plane graphs are *strongly equivalent* (have the same combinatorial embedding) if they are obtainable from one another by a plane homeomorphism, and are *equivalent* if they are obtainable from one another by a plane homeomorphism, up to the choice of the outer face.

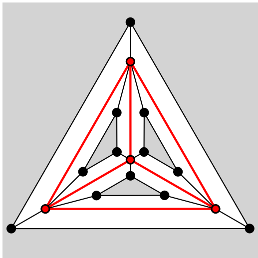


Figure 4 The graph G_1 from the proof of Proposition 3. The graph H_1 is obtained from G_1 by removing the vertices of the outer face.

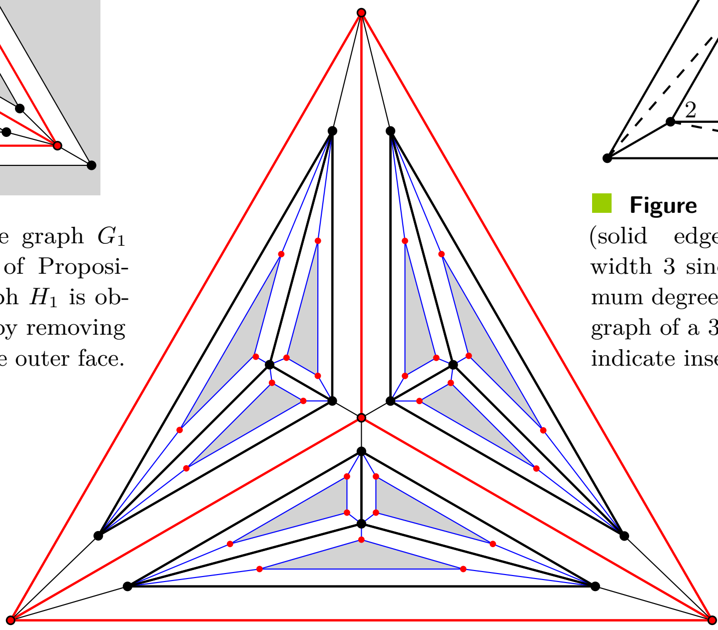


Figure 5 The plane graph H_2 is constructed by replacing each of the special faces of H_1 by another copy of H_1.

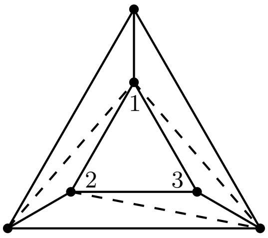


Figure 6 The prism (solid edges) has treewidth 3 since it has minimum degree 3 and is a subgraph of a 3-tree (numbers indicate insertion order).

▶ **Proposition 3.** *There is an infinite family of polyhedral graphs with maximum degree* 6*, treewidth* 3*, and unbounded (logarithmic)* π_2^1*-value.*

Proof. The base of our inductive construction is the graph G_1 depicted in Fig. 4. It has four *special* gray *faces* (the three triangles and the outer face); they are disjoint from the unique K_4-subgraph (red in Fig. 4), which we imagine to be a tetrahedron with four equal faces. Let H_1 be the *plane* graph obtained by removing from G_1 the vertices of its outer face. Assume that, at the induction step $i \geq 1$, we are given a graph G_i and a plane graph H_i with $4 \cdot 3^{i-1}$ and 3^i special faces, respectively. We construct a graph G_{i+1} from G_i and a plane graph H_{i+1} from H_i by replacing each of their special faces by a copy of the graph H_1; see Fig. 5. Then G_{i+1} and H_{i+1} have $4 \cdot 3^i$ and 3^{i+1} special faces, respectively. Note that G_i can be naturally interpreted as a 1-skeleton of a convex polyhedron, and the construction of G_{i+1} preserves this property. Thus, the graph G_{i+1} is polyhedral.

Using the fact that the special faces of G_{i+1} are disjoint from G_i, it is easy to see that $\Delta(G_{i+1}) = 6$. Since G_i is polyhedral, by a well-known result of Whitney [11], all its plane embeddings are equivalent. But, independently of the choice of the outer face in this equivalence, each plane embedding of G_i contains a subgraph strongly equivalent to H_i.

It is easy to check that we can build a 1-skeleton of a triangular prism from a triangle keeping treewidth 3; see Fig. 6. So during both actions, (a) attaching to a tetrahedron gray triangles in order to construct G_1 and (b) replacing each of the special faces of the graph G_i by a copy of H_1, the treewidth of the resulting graph remains at most 3. On the other hand, the treewidth of G_i is lowerbounded by the minimum degree of G_i, which is 3.

We need at least two straight lines to cover all vertices of a graph strongly equivalent to H_1. For $i \geq 1$, let the *central vertex* of H_i be the unique vertex that is incident to all vertices on the outer face. If $i \geq 1$, it is not difficult to check that, for each straight line ℓ

containing the central vertex of a graph strongly equivalent to H_i, there exists a subgraph strongly equivalent to H_{i-1} drawn inside a special face of H_i disjoint from ℓ. Taking into account this subgraph, we need at least one more line to cover the central vertex of the graph strongly equivalent to H_i. By induction we see that we need at least $i+1$ straight lines to cover all vertices of a graph strongly equivalent to H_i. Since any drawing of G_i contains such a graph, we have $\pi_2^1(G_i) \geq i+1$. The graph G_i has $n_i = 20 \cdot 3^{i-1} - 4$ vertices, thus $\pi_2^1(G_i) \in \Omega(\log n_i)$. ◀

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