Drawing Graphs with Vertices at Specified Positions and Crossings at Large Angles⋆

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Abstract. Point-set embeddings and large-angle crossings are two areas of graph drawing that independently have received a lot of attention in the past few years. In this paper, we consider problems in the intersection of these two areas. Given the point-set-embedding scenario, we are interested in how much we gain in terms of computational complexity, curve complexity, and generality if we allow large-angle crossings as compared to the planar case.
We investigate two drawing styles where only bends or both bends and edges must be drawn on an underlying grid. We present various results for drawings with one, two, and three bends per edge.

1 Introduction

In point-set-embeddability (PSE) problems one is given not just a graph that is to be drawn, but also a set of points in the plane that specify where the vertices of the graph can be placed. The problem class was introduced by Gritzmann et al. [9] twenty years ago. They showed that any n-vertex outerplanar graph can be embedded on any set of n points in the plane (in general position) such that edges are represented by straight-line segments connecting the respective points and no two edge representations cross. Later on, the PSE question was also raised for other drawing styles, for example, by Pach and Wenger [14] and by Kaufmann and Wiese [12] for drawings with polygonal edges, so-called *polyline drawings*. In these and most other works, however, planarity of the output drawing was an essential requirement.

Recent experiments on the readability of drawings [10] showed that polyline drawings with angles at edge crossings close to 90° and a small number of bends per edge are just as readable as planar drawings. Motivated by these findings, Didimo et al. [5] recently defined *RAC* drawings where pairs of crossing edges must form a right angle and, more generally, αAC drawings (for $\alpha \in (0, 90°])$

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where the crossing angle must be at least α. As usual, edges may not overlap and may not go through vertices.

In this paper, we investigate the intersection of the two areas, PSE and RAC/αAC. Specifically, we consider the following problems.

Problems RAC PSE and αAC PSE. Given an n-vertex graph $G = (V, E)$ and a set S of n points in the plane, determine whether there exists a bijection μ between V and S, and a polyline drawing of G so that each vertex v is mapped to $\mu(v)$ and the drawing is RAC (or αAC). If such a drawing exists and the largest number of bends per edge in the drawing is b, we say that G *admits a* RAC_b (or an $\alpha\mathrm{AC}_b$) *embedding on S*.

If we insist on straight-line edges, the drawing is completely determined once we have fixed a bijection between vertex and point set. If we allow bends, however, PSE is also interesting *with mapping*, that is, if we are given a bijection μ between vertex and point set. We call an embedding using μ as the mapping *μ-respecting*. The maximum number of bends per edge is the *curve complexity*.

We now list three results that motivate the study of RAC and αAC point-set embeddings—even for planar graphs.

- Rendl and Woeginger [17] have already considered a special case of the question we investigate in this paper, that is, the interplay between planarity and RAC in PSE. They showed that, given a set S of n points in the plane, one can test in $O(n \log n)$ time whether a perfect matching admits a RAC_0 embedding on S. They required that edges are drawn as axis-aligned line segments. They also showed that if one additionally insists on planarity, the problem becomes NP-hard.
- Pach and Wenger [14] showed for the polyline drawing scenario with mapping that, if one insists on planarity, $\Omega(n)$ bends per edge are sometimes necessary even for the class of paths and for points in convex position.
- Cabello [2] proved that deciding whether a graph admits a planar straight-line embedding on a given point set is NP-hard even for 2-outerplanar graphs.

We concentrate on RAC PSE. In order to measure the size of our drawings, we assume that the given point set S lies on a grid Γ of size $n \times n$ where $n = |S|$. We further assume that the points in S are in *general position*, that is, no two points lie on the same horizontal or vertical line. We call S an *$n \times n$ grid point set*. We require that, in our output drawings, bends lie on grid points of a (potentially larger of finer) grid containing Γ. We concentrate on two variants of the problem. We either restrict the edges, which are drawn as polygonal lines, to grid lines or we don't. We refer to the restricted version of the problem as *restricted* RAC PSE. We treat the restricted version in Section 2 and the unrestricted version in Section 3. Note that restricted RAC_0 PSE does not make sense on $n \times n$ grid point sets. The graphs we study are always undirected.

Our results concerning restricted RAC PSE are as follows.

- Every n-vertex binary tree admits a restricted RAC_1 embedding on any $n \times n$ grid point set (Theorem 1). This is not known for the planar case—see our list of open problems in Section 4. We slightly extend this result to graphs of

maximum degree 3 that arise when replacing the vertices of a binary tree by cycles. In the case of a single cycle, the statement even holds if the mapping is prescribed. This is not true in the planar case: take the 4-vertex cycle and the four points $(2,2),(4,4),(1,1),(3,3)$, in this order.
- Given a graph, a point set on the grid, and a mapping μ, we can test in linear time whether the graph admits a μ-respecting restricted RAC_1 point-set embedding (Theorem 2). The same simple 2-SAT based test works in the planar case but, of course, fails more often.
- Every n-vertex graph of maximum degree 3 admits a restricted RAC_2 embedding on any $n \times n$ grid point set even if the mapping is prescribed (Theorem 3). Given a matching with n vertices, a set of n points on the y-axis, and a mapping μ, we can compute, in $O(n^2)$ time, a μ-respecting restricted RAC_2 embedding of minimum area (to the right of the y-axis, see Theorem 4).

Concerning unrestricted RAC and αAC PSE, we show the following results which all hold even if the mapping is prescribed.

- We modify Cabello's result [2] to show that RAC_0 (and $\alpha\mathrm{AC}_0$) PSE is NP-hard. Hence, we focus on the case with bends. Due to space limitation, we omit the proof here and refer to an extended version [7].
- Every graph with n vertices and m edges admits a RAC_3 embedding on any $n \times n$ grid point set within area $O\big((n+m)^2\big)$ (Theorem 5). To RAC draw arbitrary graphs, curve complexity 3 is needed—even without PSE [1]. In the planar case (with mapping), the curve complexity for PSE is $\Omega(n)$ [14].
- For any $\varepsilon > 0$, we get a $(\pi/2-\varepsilon)\mathrm{AC}_2$ drawing within area $O(nm)$ (Theorem 6). On a grid refined by a factor of $O(1/\varepsilon^2)$, we get a $(\pi/2-\varepsilon)\mathrm{AC}_1$ drawing within area $O(n^2)$ (Theorem 7), which is optimal [3]. In the planar case, it is NP-hard to decide the existence of a 1-bend point-set embedding—both with [8] and without [12] prescribed mapping.

Related work. Besides the above-mentioned work of Rendl and Woeginger [17], the study of PSE has primarily focussed on the planar case, in connection with the drawing conventions straight-line and polyline. A special case of the polyline drawings are *Manhattan-geodesic* drawings which require that the edges are drawn as monotone chains of axis-parallel line segments. This convention was recently introduced by Katz et al. [11]. They proved that Manhattan-geodesic PSE is NP-hard (even for subdivisions of cubic graphs). On the other hand, they provided an $O(n \log n)$ decision algorithm for the n-vertex cycle. They also showed that Manhattan-geodesic PSE with mapping is NP-hard even for perfect matchings—if edges are restricted to the grid.

Although RAC and αAC drawings have been introduced very recently, there is already a large body of literature on the problem. Regarding the area of RAC drawings, Didimo et al. [5] proved that an unrestricted RAC_3 drawing of an n-vertex graph uses area $\Omega(n^2) \cap O(m^2)$. Di Giacomo et al. [3] showed that, for RAC_4 drawings, area $O(n^3)$ suffices and that, for any $\varepsilon > 0$, every n-vertex graph admits a $(\pi/2-\varepsilon)\mathrm{AC}_1$ drawing within area $\Theta(n^2)$. Our results for RAC_3 and AC_1 drawings (in Theorems 5 and 7) match the ones cited here, in spite of the fact that vertex positions are prescribed in our case.

2 Restricted RAC Point-Set Embeddings

In this section, we study restricted RAC point-set embeddings. It is clear that only graphs with maximum degree 4 may admit a restricted RAC embedding on a point set. We start with the study of RAC_1 drawings.

2.1 Restricted RAC_1 point-set embeddings

We can RAC_1 embed every n-vertex path or cycle on any $n \times n$ grid point set, even with mapping: we simply leave each point horizontally and enter the next one vertically in the order prescribed by the mapping.

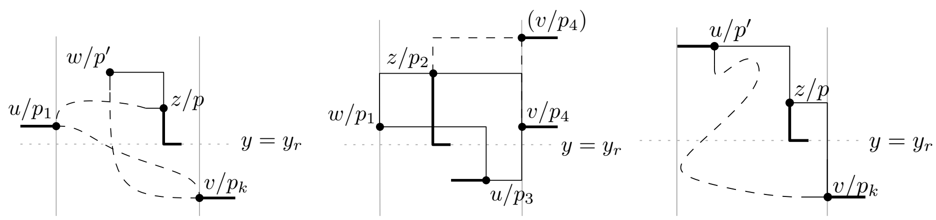
Without a given mapping, it is not hard to see that every binary tree has a restricted RAC_1 embedding on every $n \times n$ grid point set. The idea is to map the root to the point that has as many points to its left as the number of nodes in the left subtree of the root. When applying this idea recursively and drawing the first leg of the outgoing edges horizontally (one to the left, one to the right) and the second leg vertically, no two edges overlap. This was independently observed by Di Giacomo et al. [4]. We extend the result to a slightly larger class of graphs.

Theorem 1. *Let G be an n-vertex graph of maximum degree 3 that arises when replacing the vertices of a binary tree by cycles, and let S be an $n \times n$ grid point set. Then G admits a restricted RAC_1 embedding on S.*

Proof. The basic idea for extending the construction for binary trees to our new class of graphs is to treat each cycle as a single node of a binary tree. We do this by reserving the adequate number of consecutive columns for the nodes of the cycle in the middle of the drawing area for the current subtree when splitting into the drawing areas for the subtrees. The subtrees are connected to the cycle by leaving one point to the right, and one point to the left, respectively. The most difficult part is to connect the reserved points to a cycle in such a way that the point representing the vertex that is the connector to the parent vertex (or cycle), which was embedded before, can be connected by entering the node with a vertical segment such that the connections to the left and the right are possible.

Let C with $k := |C| \geq 3$ be the cycle representing the root of the current subtree. Let u and v be the vertices of C that connect C to its left and right child respectively, and let z be the vertex of C that connects C to its parent r. Let $S' = \{p_1, \ldots, p_k\}$ be the set of points reserved for C in consecutive columns ordered from left to right. The edges connecting C to its left and right subtrees leave the points representing u and v to the left and right, respectively, while the edge connecting z to r enters z from above or below, depending on the y-coordinate of the point chosen to represent z. Let y_r be the y-coordinate of r. We analyze three cases.

1. Vertex z has a neighbor $w \neq u, v$ in C and $k \geq 5$:
   Set $\mu(u) = p_1$ and $\mu(v) = p_k$. Either above or below the line $y = y_r$ we find two points $p, p' \in S' \setminus \{p_1, p_k\}$. Let p be the one closer to the line $y = y_r$.



(a) case 1 (b) case 2 (c) case 3a

Fig. 1: Drawings of cycle C in the three different cases.

We set $\mu(z) = p, \mu(w) = p'$ and draw the edge wz such that p is entered vertically. Then we can complete the cycle such that each point is incident to a horizontal and a vertical segment, see Figure 1a. It is easy to see that the connections to r and its children can now be drawn without overlap.

2. Vertex z has a neighbor $w \neq u, v$ in C and $k = 4$:
   Let $C = (u, w, z, v)$; other case is symmetric. If p_2 and p_3 both lie either below or above $y = y_r$, we can proceed as in case 1. Suppose p_2 lies above r and p_3 below; the other case is, again, symmetric. We have two subcases depending on the position of p_4:
   – p_4 lies above p_3: We can draw C as shown in Figure 1b.
   – p_4 lies below p_3: Similarly to case 1, we can draw C such that each point is incident to both a vertical and a horizontal segment.
3. The two neighbors of z are u and v.
   If there is a point $p \in S' \setminus \{p_1, p_k\}$ that is vertically between p_1 and p_k, then we can set $\mu(u) = p_1$, $\mu(v) = p_k$ and $\mu(z) = p$ and draw C having a vertical and a horizontal segment incident to each point.
   Otherwise, there is no such point vertically between p_1 and p_k.
   a) If $k \geq 5$ we find, similarly to case 1, two points $p, p' \in S' \setminus \{p_1, p_k\}$ both below or above r such that p is the one closer to the line $y = y_r$. Again we set $\mu(z) = p$; if p' is left of p, we set $\mu(u) = p'$ and $\mu(v) = p_k$, see Figure 1c; otherwise, we symmetrically set $\mu(v) = p'$ and $\mu(u) = p_1$. Now we can draw the cycle without overlap such that each point is incident to a vertical and a horizontal segment.
   b) It is not hard to see that we can draw C without overlap if $k \in \{3, 4\}$.

This completes the case analysis and, hence, the proof. □

In the proofs of the previous theorems we exploited the fact that we could choose the vertex–point mapping as needed. Figure 2 shows a 6-vertex binary tree that does *not* have a restricted RAC_1 drawing on the given point set if the vertex–point mapping is fixed as indicated by the edges. Hence, we turn to the corresponding decision problem. We characterize situations when a restricted RAC_1 point-set embedding with mapping exists.

Theorem 2. *Let G be an n-vertex graph of maximum degree 4, let S be an $n \times n$ grid point set, and let μ be a vertex–point mapping. We can test in $O(n)$ time whether G admits a μ-respecting restricted RAC_1 embedding on S and, if yes, construct such an embedding within the same time bound.*

Proof. We use a 2-SAT encoding to solve the problem. A similar approach was used by Raghavan et al. [16] to deal with the planar case. We associate each edge uv of G with a boolean variable x_{uv}. The two possible drawings of edge uv correspond to the two literals x_{uv} and $\neg x_{uv}$. As S is in general position, only drawings of edges incident to the same vertex can possibly overlap. Now we construct a 2-SAT formula ϕ as follows. Consider a pair of drawings of edges uv and uw that overlap. Assume that x_{uv} and $\neg x_{vw}$ are the literals corresponding to the two edge drawings. Then we add the clause $\neg(x_{uv} \wedge \neg x_{uw}) = \neg x_{uv} \vee x_{uw}$ to ϕ.

It is clear that ϕ is satisfiable if and only if G has a μ-respecting RAC_1 embedding on S without overlapping edges. Recall that the maximum degree of G is 4. Hence, ϕ contains at most $n \cdot \binom{4}{2} \cdot 4$ clauses. Since the satisfiability of a 2-SAT formula can be decided in time linear in the number of clauses [6], the testing can be done in $O(n)$ time. □

2.2 Restricted RAC_2 point-set embeddings

Consider, for a moment, a specialized restricted RAC_2 drawing convention that requires the first and the last (of the three) segments of an edge to go in the same direction—a *bracket* drawing. If we do not restrict the drawing area, then the problem of bracket embedding a graph G on an $n \times n$ grid point set is equivalent to 4-edge coloring G. The idea is that the four colors encode the direction of the first and last edge segment (going up, down, left, or right) and that the second segment is drawn sufficiently far away. The edge coloring ensures that no two edges incident to a vertex overlap. For any graph of maximum degree 3 a 4-edge-coloring exists and can be found in linear time [18]. Let us summarize.

Theorem 3. *Every graph G of maximum degree 3 admits a restricted RAC_2 embedding on any $n \times n$ grid point set with any vertex–point mapping.*

Note that graphs of maximum degree 4 that do not admit a 4-edge coloring, but admit a restricted RAC_2 embedding for some grid point sets, can easily be found, e.g., K_4.

Now we turn to the problem of minimizing the drawing area. Observe that there are examples of a graph G, a grid point set S, and a mapping μ such that G does not admit a restricted RAC_2 point-set embedding on S with mapping μ if we insist that the drawing lies within the bounding box of S, see Fig. 3.

We conjecture that restricted RAC_2 PSE is NP-hard. Therefore, we consider the special case where S is one-dimensional, i.e., all points lie on a common (e.g., vertical) straight line. More precisely, we are looking for a *one-page RAC_2 book embedding* with given mapping, i.e., we want to use only space on one side of the line.

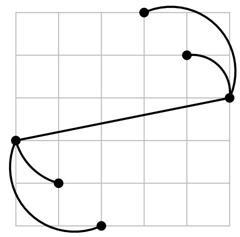


Fig. 2: Tree without restricted RAC_1 drawing.

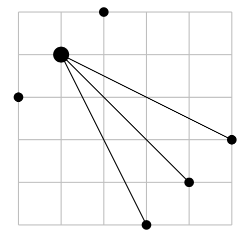


Fig. 3: Counterexample.

Clearly, in this setting, if we forget about straight-line edges, each vertex can only have degree 1, hence the given graph must be a (perfect) matching. Given these restrictions, we can minimize the area of the drawing.

Theorem 4. *Let S be a set of n points on the y-axis, let G be a matching consisting of $n/2$ edges, and let μ be a vertex–point mapping. A minimum-area μ-respecting restricted RAC_2 drawing of G to the right of the y-axis can be computed in $O(n^2)$ time.*

Proof. Pairs of neighboring points corresponding to edges of the matching can simply be connected by (vertical) straight-line segments. To draw any of the remaining edges of the matching in a restricted RAC_2 fashion, we must connect its endpoints by two horizontal segments leaving the y-axis to the right and a vertical segment that joins the horizontal segments. As G is a matching, only vertical segments can overlap. In order to minimize the drawing area, we, thus, have to minimize the number of vertical lines, the *layers*, needed to draw the vertical segments of all edges without overlap.

It is not hard to see that finding an assignment of these vertical segments to a minimum number of layers is equivalent to coloring the conflict graph $G' = (V', E')$ of these middle segments—which is an interval graph—with a minimum number of colors. This can be done in $O(|V'| + |E'|) = O(n^2)$ time [13]. □

If we are not given a prescribed mapping, then the problem can be solved straightforwardly in linear time for all graphs of maximum degree 2. If we abandon the restriction to draw edges on the grid and relax the constraint on the crossing angle, we can find, for *any* graph, an αAC_2 embedding on any point set on the y-axis with an arbitrary mapping, see the comment after the proof of Theorem 6.

3 Unrestricted RAC and αAC Point-Set Embeddings

Didimo et al. [5] have shown that any graph with n vertices and m edges admits a RAC_3-drawing within area $O(m^2)$. Their proof uses an algorithm of Papakostas and Tollis [15] for drawing graphs such that each vertex is represented by an axis-aligned rectangle and each edge by an *L-shape*, that is, an axis-aligned 1-bend polyline. Didimo et al. turn such a drawing into a RAC_3-drawing by replacing each rectangle with a point. In order to make the edges terminate at these points, they add at most two bends per edge. We now show how to compute a RAC_3-drawing of the same size (assuming $n \in O(m)$)—although we are restricted to the given point set.

Note that curve complexity 3 is actually necessary for RAC drawing arbitrary graphs—even without a prescribed point set: Arikushi et al. [1] showed that RAC_2 drawings only exist for graphs with a linear number of edges.

Theorem 5. *Let G be a graph with n vertices and m edges and let S be an $n \times n$ grid point set. Then G admits a RAC_3-drawing on S (with or without given vertex–point mapping) within area $O\left((n+m)^2\right)$.*

Proof. If the vertex–point mapping μ is not given, let μ be an arbitrary mapping. Let $v_1, \dots, v_n$ be an ordering of V so that $p_i := \mu(v_i)$ has x-coordinate i. We construct a RAC_3-drawing as follows. Each edge has—after insertion of "virtual" bends—exactly three bends and four straight-line segments. We ensure that intersections involve only the "middle" segments of edges, and that these middle segments have only slope $+1$ or -1.

For an edge uv, we call the bend directly connected to u a *u-bend*, the bend directly connected to v a *v-bend*, and the remaining bend the *middle bend*. We start constructing the drawing by placing the v-bends for each vertex v, starting with v_n. We set the y-coordinate y_n of the first v_n-bend to 0. Then, for $i = n, n-1, \dots, 1$, observe that there are exactly $\deg v_i$ many v_i-bends, which we place in column $i+1$ starting at y-coordinate y_i below the $n \times n$ grid using positions $\{(i+1, y_i), (i+1, y_i-2), (i+1, y_i-4), \dots, (i+1, y_i-2\cdot(\deg v_i - 1)\}$, see Figure 4. We connect each vertex with its associated bends without introducing any intersection since we stay inside the area between columns i and $i+1$. We set $y_{i-1} = y_i - 2 \cdot (\deg v_i - 1) - 3$. If v_i has degree 0, we do not place bends but set $y_{j-1} = y_j - 3$ to avoid overlaps and crossings. Then we continue with v_{i-1}.

Since we place the bends from right to left and from top to bottom by moving our "pointer" by L_1- (or Manhattan) distances 2 or 4, each pair of these bends has even Manhattan distance. To draw an edge uv, we first select a "free" u-bend position and a free v-bend position. The two middle segments go to the right at slopes $+1$ and -1. Since u- and v-bend have even Manhattan distance, the middle bend has integer coordinates.

Let u and v be two vertices with u-bend b_u and v-bend b_v, respectively. The segments $\overline{ub_u}$ and $\overline{vb_v}$ cannot intersect; we want to see that the middle segment starting at b_u also cannot intersect $\overline{vb_v}$. Such an intersection can only occur if u lies to the left of v.By our construction, b_v lies, in this case, above b_u with a y-distance that is greater than their x-distance. As all middle segments have a slope of at most $+1$, b_v lies above the relevant middle segment, which can, hence, not intersect $\overline{vb_v}$.

It remains to show the space limitation. Clearly, the drawing of any edge requires not more horizontal than vertical space. On the other hand, for any vertex v, we need at most $2 \cdot \deg v + 3$ rows below the grid, resulting in a total vertical space requirement of $O(n+m)$. This completes the proof. ◻

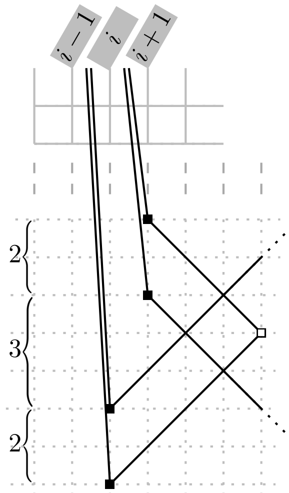


Fig. 4: Construction of a RAC_3 drawing.

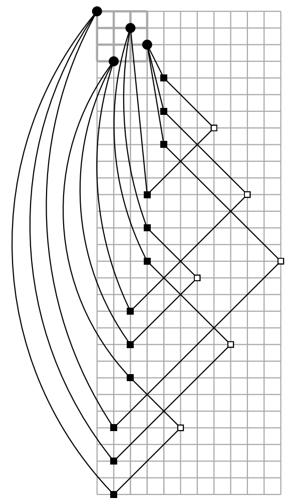


Fig. 5: RAC_3-drawing of K_4; some straight-line segments have been replaced by circular arcs for the sake of clarity.

In the remainder of this section, we focus on αAC point-set embeddings. We show that both area and curve complexity can be significantly improved if we soften the restriction on the crossing angles. Our results hold for both scenarios, with and without vertex–point mapping.

Theorem 6. *Let G be a graph with n vertices and m edges, let S be a $n \times n$ grid point set, and let $0 < \varepsilon < \frac{\pi}{2}$. Then G admits a $(\frac{\pi}{2} - \varepsilon)AC_2$ embedding on S (with or without given vertex–point mapping) within area $O(n(m + \cot \varepsilon)) = O(n(m + 1/\varepsilon^2))$.*

Proof. If the vertex–point mapping μ is not given, let μ be arbitrary. Let $v_1, \ldots, v_n$ be an ordering of V so that $p_i := \mu(v_i)$ has x-coordinate i. Each edge $e = uv$ has exactly two bends, a u-bend and a v-bend (with the obvious meanings). For $i = 1, \ldots, n$, we place all v_i-bends in column $i + 1$. We make all middle segments of edges horizontal. Thus, the bends for an edge $e = v_i v_j$ are at positions $(i + 1, y)$ and $(j + 1, y)$ in some row $y < 0$ below the original grid, see Figure 6. By using a dedicated row for each edge, we achieve that no two middle segments intersect. By construction, no two first or last edge segments intersect. Hence, crossings occur only between the horizontal middle segments and first or last segments. By making the y-coordinates of the middle segments small enough, we will achieve that all crossing angles are at least $\pi/2 - \varepsilon$.

Let $\{e_1, \ldots, e_m\}$ be the set of edges of G, and let $uv := e_k$ be one of these edges. We set the y-coordinates of the middle segment of e_k to $-k - \lceil \cot \varepsilon \rceil$. Let $e_{k'}$ be an edge whose horizontal segment intersects the first segment of e_k. The crossing angle is $\pi/2 - \delta$, where δ is the angle between the vertical line through the u-bend and the first segment of uv, see Figure 7. We have $\delta \leq$

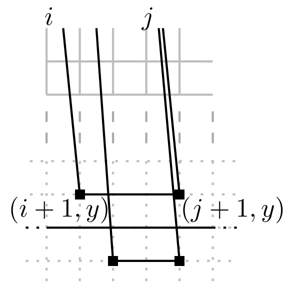


Fig. 6: $(\frac{\pi}{2} - \varepsilon)$AC construction.

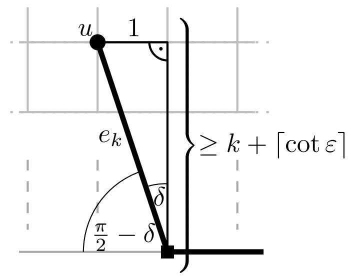


Fig. 7: Angles in the 2-bend drawing.

$\operatorname{arccot}(k + \lceil \cot \varepsilon \rceil) \leq \varepsilon$. Thus, the crossing angle is at least $\pi/2 - \varepsilon$. Note that $\cot \varepsilon \in O(1/\varepsilon^2)$. □

Note that the proof only requires the fact that no two points lie in the same *column.*

In Theorem 6, we required the bends to lie on points of the given grid. The following result shows that we need only one bend per edge if we allow the bends to lie on points of a *refined* grid. For fixed $\epsilon > 0$, our new drawings need less area than those of Theorem 6; even in terms of the refined grid.

Theorem 7. *Let G be a graph with n vertices, let S be an $n \times n$ grid point set, and let $0 < \varepsilon < \frac{\pi}{2}$. Then G admits a $(\frac{\pi}{2} - \varepsilon)AC_1$ embedding on S (with or without given vertex–point mapping) on a grid that is finer than the original grid by a factor of $\lambda \in O(\cot \varepsilon) = O(1/\varepsilon^2)$.*

Proof. If the mapping μ is not given, let μ be an arbitrary mapping. The idea of our construction is as follows. For each edge, we first choose one of the two possible drawings on the grid lines with one bend. This gives us a drawing of the graph with many overlaps of edges. Then, we slightly twist each edge such that its horizontal segment becomes *almost horizontal*, meaning it gets a negative slope close to 0. At the same time, we make the vertical segment *almost vertical*, meaning it gets a very large positive slope, see Figure 8.

As we want all bends to be on grid points, we first refine the grid by an integral factor of $\lambda = \lceil 1 + \cot \varepsilon \rceil$. We do this by inserting, at equal distances, $\lambda - 1$ new rows or columns between two consecutive grid rows or columns, respectively. Now, a point $s = (a, b) \in S$ lies at $(\lambda a, \lambda b)$ w.r.t. the new $\lambda n \times \lambda n$ grid.

Let e be an edge and let (e_x, e_y) be the original position of the bend of e w.r.t. the new grid. We choose the new position of the bend to be the unique grid point diagonally next to (e_x, e_y) such that the horizontal and vertical segments of e become almost horizontal and almost vertical, respectively. If we apply this construction to all edges, we get a drawing in which none of the almost horizontal and almost vertical segments belonging to some vertex v can overlap. Moreover, two almost horizontal or two almost vertical segments belonging to different vertices neither overlap nor intersect due to S being in general position. Thus, each crossing involves an almost horizontal and an almost vertical segment.

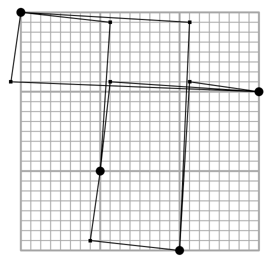


Fig. 8: Drawing of K_4 on a grid refined by factor $\lambda = 8$.

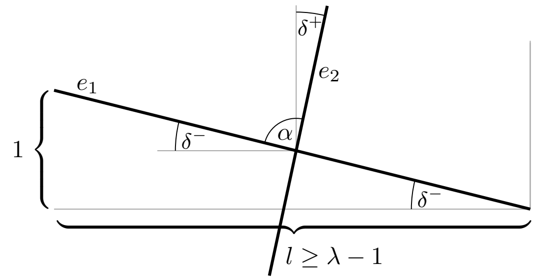


Fig. 9: Analyzing the crossing angle.

Let e_1 and e_2 be two crossing edges such that the almost horizontal segment involved in the crossing belongs to e_1. We can assume—up to symmetry—that the smaller angle of the crossing occurs to the top left of the crossing. Let δ^- be the angle formed by the almost horizontal segment of e_1 and a horizontal line, and let δ^+ be the angle formed by the almost vertical segment of e_1 and a vertical line, see Figure 9. Then the crossing angle of e_1 and e_2 is $\alpha = \pi/2 - \delta^- + \delta^+ \geq \pi/2 - \delta^-$. For δ^- to be maximal, the horizontal length l of the almost horizontal segment has to be minimal. As this length cannot be less than $\lambda - 1$, we get $\delta^+ \leq \operatorname{arccot}(\lambda - 1) \leq \varepsilon$. Hence, the crossing angle α is at least $\pi/2 - \varepsilon$. □

Note that we leave the original grid by at most one row or column of the refined grid in each direction. Hence, the area requirement is $O((n \cdot \cot \varepsilon)^2)$ in terms of the finer grid. We argue that our area bounds are quite reasonable: for a minimum crossing angle of 70°, the drawings provided by Theorems 6 and 7 use grids of sizes at most $n(m + 3)$ and $(3n)^2$, respectively.

4 Open Problems

In this paper, we have opened an interesting new area: the intersection of point-set embeddability and drawings with crossings at large angles. We have done a few first steps, but we leave open a large number of questions.

1. Does every n-node binary tree have a restricted *planar* 1-bend embedding on any $n \times n$ grid point set?
2. Does every n-node ternary tree have a restricted RAC_1 embedding on any $n \times n$ grid point set?
3. What about outerplanar graphs?
4. Can we efficiently test whether a given graph has a restricted RAC_1 or RAC_2 embedding on a given $n \times n$ grid point set (without mapping)?
5. Can we efficiently test whether a given graph has a (unrestricted) RAC_2 embedding on a given $n \times n$ grid point set? If yes, can we minimize its area?
6. Di Giacomo et al. [3] have shown that any n-vertex graph admits a RAC_4-drawing that uses area $O(n^3)$. Can we achieve the same in the PSE setting?

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