Minimum Rectilinear Polygons for Given Angle Sequences

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In this paper, we consider the problem of computing, for a given rectilinear angle sequence, a "small" rectilinear polygon. A *rectilinear angle sequence* S is a sequence of left (90°) turns and right (270°) turns, that is, $S = (s_1, \ldots, s_n) \in \{\mathtt{L}, \mathtt{R}\}^n$, where n is the *length* of S. As we consider only rectilinear angle sequences, we usually drop the term "rectilinear." A polygon P *realizes* an angle sequence S if there is a counterclockwise walk along the interior boundary of P such that the turns at the vertices of P occur in the same order as in S.

In order to measure the size of a polygon, we only consider polygons that lie on the integer grid. Then, the *area* of a polygon P corresponds to the number of grid cells that lie in the interior of P. The *bounding box* of P is the smallest axis-parallel enclosing rectangle of P. The *perimeter* of P is the sum of the lengths of the edges of P. The task is, for a given angle sequence S, to find a polygon that realizes S and minimizes (i) (the area of) its bounding box, (ii) its area, or (iii) its perimeter. Fig. 1 shows that, in general, the three criteria cannot be minimized simultaneously.

Obviously, the angle sequence of a polygon is unique (up to rotation), but the number of polygons that realize a given angle sequence is unbounded. The formula for the angle sum of a polygon implies that, in any angle sequence, $\#\mathtt{L} = \#\mathtt{R}+4$, where $\#$ counts the numbers of L and R symbols.

Bae et al. [1] considered, for a given angle sequence S, the polygon $P(S)$ that realizes S and minimizes its area. They studied the following question: Given a length n, find the angle sequence S such that the area of $P(S)$ is minimized ($\delta(n)$) or maximized ($\Delta(n)$). They proved that $\delta(n) \in \{n/2-1; n/2\}$ and $\Delta(n) = (n-2)(n+4)/8$.

In graph drawing, the standard approach to drawing a graph of maximum degree 4 orthogonally (that is, with rectilinear edges) is the topology–shape–metrics approach of Tamassia [4]: (1) Compute a planar(ized) embedding; (2) compute an *orthogonal representation*, that is, an angle sequence for each edge and an angle for each vertex; (3) *compact* the graph, that is, draw it inside a bounding box of minimum area. Step (3) has been shown to be NP-complete by Patrignani [3]. Note that an orthogonal representation computed in step (2) is essentially an angle sequence for each face of the planarized embedding, so our problem corresponds to step (3) in the special case that the input graph is a simple cycle.

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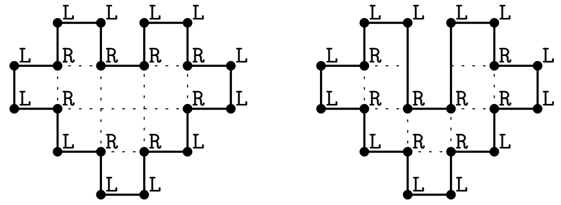

(a) Area 11, perimeter 20. (b) Area 10, perimeter 22.

Fig. 1: Two polygons realizing the same angle sequence. The bounding box of both polygons has area 20, but (a) has minimum perimeter and (b) has minimum area.

Our contribution is as follows. First, we show that finding a minimum polygon that realizes a given angle sequence is NP-hard for any of the three measures; see Section 1. This extends the result of Patrignani [3] and settles an open question that he posed. Then, we give efficient algorithms for special types of angle sequences, namely *xy*- and *x-monotone sequences*, which induce *xy*-monotone and *x*-monotone polygons, respectively. (For example, LLRRLLRLLRLRLLRLRLLR is an *x*-monotone sequence, see Fig. 1.) Our algorithms minimize area (Section 2) and perimeter (Section 3).

1 NP-Hardness of the General Case

Theorem 1. *Given an angle-sequence S, it is NP-hard to find a polygon that realizes S and minimizes (i) its bounding box, (ii) its area, or (iii) its perimeter.*

Proof sketch. Our sketch is for minimizing the area of the bounding box. We reduce from 3-Partition: Given a multiset A of $n = 3m$ positive integers with $\sum A = mB$, is there a partition of A into m subsets $A_1, \ldots, A_m$ such that $\sum A_i = B$ for each i? 3-Partition is NP-hard [2] even if $|A_1| = \cdots = |A_m| = 3$.

Given an instance $A = \{a_1, \ldots, a_{3m}\}$ of 3-Partition, we construct an angle sequence S that can be realized inside a bounding box of size at most $(2m^3 + Bm^2 + 12m) \times (2m^3 + (3m + C)m + 2)$ iff A is a yes-instance (C is a constant). Fig. 2 gives the main idea. There is an $m \times 3$ grid of boxes inside a large rectangle R. The height of box i is forced to be at least $m^2 a_i$. Within a column, the boxes are in order of increasing index so that they can be connected to the outside of R by crossing-free tunnels. All tunnels have the same number of L- and R-turns, but we have drawn only some. □

2 The Monotone Case: Minimum Area

In this section, we show how to compute, for a monotone angle sequence, a polygon of minimum area.

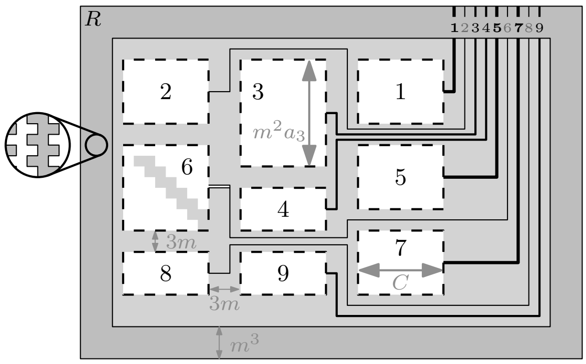

Fig. 2: NP-hardness by reduction from 3-Partition.

An xy-monotone polygon consists of (up to) four xy-monotone chains (*stairs*) TL, TR, BL, BR between its four *extreme edges*, that is, the leftmost, rightmost, topmost, and bottommost edge; see Fig. 3(a). The extreme edges correspond to the exactly four `L L`-sequences in an xy-monotone angle sequence and are unique up to rotation. Let $T = TL \oplus TR$, $R = TR \oplus BR$, $B = BL \oplus BR$, and $L = TL \oplus BL$. For a chain C, let its *length* $r(C)$ be the number of reflex vertices on C.

Theorem 2. *Given an xy-monotone angle sequence S of length n, we can find a polygon P that realizes S and minimizes its (i) bounding box or (ii) area in $O(n)$ time, and in constant time if the stair lengths are given.*

Proof sketch. (i) The bounding box of every polygon that realizes S has width at least $\max\{r(T), r(B)\} + 1$ and height at least $\max\{r(L), r(R)\} + 1$. By drawing three stairs with edges of unit length, we can meet these lower bounds.

(ii) The hardest case is that P consists of four stairs. Then, P is either point symmetric (if n is small) or in one of the two configurations depicted in Fig. 3(b) and 3(c). For both configurations, there is only a constant number of ways to connect the pair of opposite stairs to the triangles described by the other two stairs. We partition P into three polygons; a polygon for the pair of opposite stairs and a polygon for each of the two other stairs. Since these polygons contain at most two stairs, they can be optimized easily. We check every configuration of S and then output a polygon of minimum area. □

An x-monotone polygon consists of a unique leftmost and rightmost edge, and a chain of stairs T above and a chain of stairs B below its extremal edges.

Theorem 3. *Given an x-monotone angle sequence S of length n, we can find a polygon P that realizes S and minimizes its (i) bounding box in $O(n^3)$ time or (ii) area in $O(n^4)$ time.*

Proof sketch. (i) There is always an optimal solution in which every edge of length greater than 1 is either an `LL` edge or an edge incident to an `RR` edge. We guess the height ($O(n)$ possibilities) and then, with a dynamic program (DP), find out for each horizontal `LL` edge which edge is lying on the opposite side ($O(n^2)$ combinations).

(ii) Since P is x-monotone and area-minimal, every grid column contains at most two horizontal edges, and two horizontal edges have at most one common column; hence, such a pair of horizontal edges uniquely identifies a column. With a DP, we reconstruct P by traversing all possible pairs from left to right. For each pair, we try out every vertical distance, and combine it with the minimum solution to all the $O(1)$ consistent pairs of $O(n)$ possible vertical distances in the column to the left. Thus, our DP table consists of $O(n^3)$ entries and is computed in $O(n^4)$ time. □

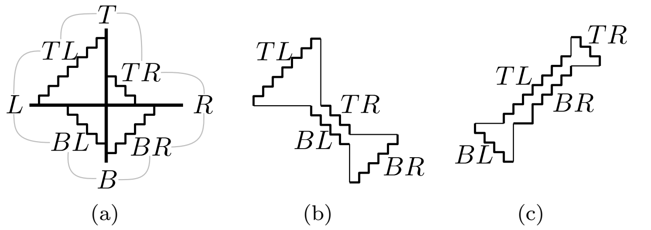

Fig. 3: (a) The four stairs of an xy-monotone polygon. (b–c) Possible optimal configurations of the polygon.

3 The Monotone Case: Min. Perimeter

In this section, we show how to compute, for a monotone angle sequence, a polygon of minimum perimeter. Recall the decomposition of xy-monotone polygons into stairs (Section 2). Let e_L^P be the leftmost edge and let e_R^P be the rightmost edge of a polygon P.

Lemma 1. *Given an xy-monotone angle sequence S, there is a minimum-perimeter polygon P realizing S with $r(T) \geq r(B)$ and $\mathrm{peri}(P) = 3r(T) + r(B) + 2 + |e_L^P| + |e_R^P|$.*

Using the above lemma, we get the following result.

Theorem 4. *Given an xy-monotone angle sequence S of length n, we can find a polygon that realizes S and minimizes its perimeter in $O(n)$ time.*

Note that this algorithm runs in constant time if the lengths of the stairs are given. Now, observe the following properties of x-monotone angle sequences.

Lemma 2. *Given an x-monotone angle sequence S, there is a minimum-perimeter polygon P realizing S such that (i) every vertical edge except e_R^P and e_L^P has unit length, and (ii) every horizontal edge in the longer chain between T and B has unit length.*

Using DP and the lemma above yields the following.

Theorem 5. *Given an x-monotone angle sequence S of length n, we can find a polygon P that realizes S and minimizes the perimeter in $O(n^2)$ time.*

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