The Complexity of Drawing Graphs on Few Lines and Few Planes

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Abstract. It is well known that any graph admits a crossing-free straight-line drawing in $\mathbb{R}^3$ and that any planar graph admits the same even in $\mathbb{R}^2$. For a graph G and $d \in \{2,3\}$, let $\rho^1_d(G)$ denote the minimum number of lines in $\mathbb{R}^d$ that together can cover all edges of a drawing of G. For $d = 2$, G must be planar. We investigate the complexity of computing these parameters and obtain the following hardness and algorithmic results.

- For $d \in \{2,3\}$, we prove that deciding whether $\rho^1_d(G) \le k$ for a given graph G and integer k is $\exists\mathbb{R}$-complete.
- Since NP $\subseteq \exists\mathbb{R}$, deciding $\rho^1_d(G) \le k$ is NP-hard for $d \in \{2,3\}$. On the positive side, we show that the problem is fixed-parameter tractable with respect to k.
- Since $\exists\mathbb{R} \subseteq$ PSPACE, both $\rho^1_2(G)$ and $\rho^1_3(G)$ are computable in polynomial space. On the negative side, we show that drawings that are optimal with respect to ρ^1_2 or ρ^1_3 sometimes require irrational coordinates.
- Let $\rho^2_3(G)$ be the minimum number of planes in $\mathbb{R}^3$ needed to cover a straight-line drawing of a graph G. We prove that deciding whether $\rho^2_3(G) \le k$ is NP-hard for any fixed $k \ge 2$. Hence, the problem is not fixed-parameter tractable with respect to k unless P = NP.

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1 Introduction

As is well known, any graph can be drawn in $\mathbb{R}^3$ without crossings so that all edges are segments of straight lines. Suppose that we have a supply $\mathcal{L}$ of lines in $\mathbb{R}^3$, and the edges are allowed to be drawn only on lines in $\mathcal{L}$. How large does $\mathcal{L}$ need to be for a given graph G? For planar graphs, a similar question makes sense also in $\mathbb{R}^2$, since planar graphs admit straight-line drawings in $\mathbb{R}^2$ by the Wagner–Fáry–Stein theorem. Let $\rho^1_3(G)$ denote the minimum size of $\mathcal{L}$ which is sufficient to cover a drawing of G in $\mathbb{R}^3$. For a planar graph G, we denote the corresponding parameter in $\mathbb{R}^2$ by $\rho^1_2(G)$. The study of these parameters was posed as an open problem by Durocher et al. [10]. The two parameters are related to several challenging graph-drawing problems such as small-area or small-volume drawings [9], layered or track drawings [8], and drawing graphs with low visual complexity. Recently, we studied the extremal values of $\rho^1_3(G)$ and $\rho^1_2(G)$ for various classes of graphs and examined their relations to other characteristics of graphs [6]. In particular, we showed that there are planar graphs where the parameter $\rho^1_3(G)$ is much smaller than $\rho^1_2(G)$. Determining the exact values of $\rho^1_3(G)$ and $\rho^1_2(G)$ for particular graphs seems to be tricky even for trees.

In fact, the setting that we suggested is more general [6]. Let $1 \le l < d$. We define the *affine cover number* $\rho^l_d(G)$ as the minimum number of l-dimensional planes in $\mathbb{R}^d$ such that G has a straight-line drawing that is contained in the union of these planes. We suppose that $l \le 2$ as otherwise $\rho^l_d(G) = 1$.

Moreover, we can focus on $d \le 3$ as every graph can be drawn in 3-space as efficiently as in higher dimensions, that is, $\rho^l_d(G) = \rho^l_3(G)$ if $d \ge 3$ [6]. This implies that, besides the *line cover numbers* in 2D and 3D, $\rho^1_2(G)$ and $\rho^1_3(G)$, the only interesting affine cover number is the *plane cover number* $\rho^2_3(G)$. Note that $\rho^2_3(G) = 1$ if and only if G is planar. Let K_n denote the complete graph on n vertices. For the smallest non-planar graph K_5, we have $\rho^2_3(K_5) = 3$. The parameters $\rho^2_3(K_n)$ are not so easy to determine even for small values of n. We have shown that $\rho^2_3(K_6) = 4$, $\rho^2_3(K_7) = 6$, and $6 \le \rho^2_3(K_8) \le 7$ [6]. It is not hard to show that $\rho^2_3(K_n) = \Theta(n^2)$, and we determined the asymptotics of $\rho^2_3(K_n)$ up to a factor of 2 using the relations of these numbers to Steiner systems.

The present paper is focused on the computational complexity of the affine cover numbers. A good starting point is to observe that, for given G and k, the statement $\rho^l_d(G) \le k$ can be expressed by a first-order formula about the reals of the form $\exists x_1 \ldots \exists x_m \Phi(x_1, \ldots, x_m)$, where the quantifier-free subformula Φ is written using the constants 0 and 1, the basic arithmetic operations, and the order and equality relations. If, for example, $l = 1$, then we just have to write that there are k pairs of points, determining a set $\mathcal{L}$ of k lines, and there are n points representing the vertices of G such that the segments corresponding to the edges of G lie on the lines in $\mathcal{L}$ and do not cross each other. This observation shows that deciding whether or not $\rho^l_d(G) \le k$ reduces in polynomial time to the decision problem (Hilbert's *Entscheidungsproblem*) for *the existential theory of the reals*. The problems admitting such a reduction form the complexity class $\exists\mathbb{R}$ introduced by Schaefer [23], whose importance in computational geometry has been recognized recently [4,16,24]. In the complexity-theoretic hierarchy, this class occupies a position between NP and PSPACE. It possesses natural complete problems like the decision version of the rectilinear crossing number [1], the recognition of segment intersection graphs [15] or unit disk graphs [13].

Below, we summarize our results on the computational complexity of the affine cover numbers.

The complexity of the line cover numbers in 2D and 3D. We begin by showing that it is $\exists\mathbb{R}$-hard to compute, for a given graph G, its line cover numbers $\rho^1_2(G)$ and $\rho^1_3(G)$; see Section 2.

Our proof uses some ingredients from a paper of Durocher et al. [10] who showed that it is NP-hard to compute the *segment number* $\mathrm{segm}(G)$ of a graph G. This parameter was introduced by Dujmović et al. [7] as a measure of the visual complexity of a planar graph. A *segment* in a straight-line drawing of a graph G is an inclusion-maximal connected path of edges of G lying on a line, and the *segment number* $\mathrm{segm}(G)$ of a planar graph G is the minimum number of segments in a straight-line drawing of G in the plane. Note that while $\rho^1_2(G) \le \mathrm{segm}(G)$, the parameters can be far apart, e.g., as shown by a graph with m isolated edges. For connected graphs, we have shown earlier [6] that $\mathrm{segm}(G) \in O(\rho^1_2(G)^2)$ and that this bound is optimal as there exist planar triangulations with $\rho^1_2(G) \in O(\sqrt{n})$ and $\mathrm{segm}(G) \in \Omega(n)$. Still, we follow Durocher et al. [10] to some extent in that we also reduce from ARRANGEMENT GRAPH RECOGNITION (see Theorem 1).

Parameterized complexity of computing the line cover numbers in 2D and 3D. It follows from the inclusion NP $\subseteq \exists\mathbb{R}$ that the decision problems $\rho^1_2(G) \le k$ and $\rho^1_3(G) \le k$ are NP-hard if k is given as a part of the input. On the positive side, in Section 3, we show that both problems are fixed-parameter tractable. To this end, we first describe a linear-time kernelization procedure that reduces the given graph to one of size $O(k^4)$. Then, in $k^{O(k^2)}$ time, we carefully solve the problem on this reduced instance by using the exponential-time decision procedure for the existential theory of the reals by Renegar [20,21,22] as a subroutine. To the best of our knowledge, this is the first application of Renegar's algorithm for obtaining an FPT result, in particular, in the area of graph drawing where FPT algorithms are widely known.

The space complexity of ρ^1_d-optimal drawings. Since $\exists\mathbb{R}$ belongs to PSPACE (as shown by Canny [3]), the parameters $\rho^1_d(G)$ for both $d = 2$ and 3 are computable in polynomial space. On the negative side, we construct a graph G with a ρ^1_2-optimal drawing requiring irrational coordinates; we provide a more complex argument to show that *any* ρ^1_2-optimal drawing of G requires irrational coordinates; for details see the full version [5].

The complexity of the plane cover number. Though the decision problem $\rho^2_3(G) \le k$ also belongs to $\exists\mathbb{R}$, its complexity status is different from that of the line cover numbers. In Section 4, we establish the NP-hardness of deciding whether $\rho^2_3(G) \le k$ for any fixed $k \ge 2$, which excludes an FPT algorithm for this problem unless P = NP. To show this, we first prove NP-hardness of Positive Planar Cycle 1-in-3-Sat (a new problem of planar 3-SAT type), which we think is of independent interest.

Weak affine cover numbers. We previously defined the *weak affine cover number* $\pi^l_d(G)$ of a graph G similarly to $\rho^l_d(G)$ but under the weaker requirement that the l-dimensional planes in $\mathbb{R}^d$ whose number has to be minimized contain the *vertices* (and not necessarily the edges) of G [6]. Based on our combinatorial characterization of π^1_3 and π^2_3 [6], we show in Section 5 that the decision problem $\pi^l_3(G) \leq 2$ is NP-complete, and that it is NP-hard to approximate $\pi^l_3(G)$ within a factor of $O(n^{1-\epsilon})$, for any $\epsilon > 0$. Asymmetrically to the affine cover numbers ρ^1_2, ρ^1_3, and ρ^2_3, here it is the parameter π^1_2 (for planar graphs) whose complexity remains open. For more open problems, see Section 6.

2 Computational Hardness of the Line Cover Numbers

In this section, we show that deciding, for a given graph G and integer k, whether $\rho^1_2(G) \leq k$ or $\rho^1_3(G) \leq k$ is an $\exists\mathbb{R}$-complete problem. The $\exists\mathbb{R}$-hardness results are often established by a reduction from the Pseudoline Stretchability problem: Given an arrangement of pseudolines in the projective plane, decide whether it is *stretchable*, that is, equivalent to an arrangement of lines [17,18]. Our reduction is based on an argument of Durocher et al. [10] who designed a reduction of the Arrangement Graph Recognition problem, defined below, to the problem of computing the segment number of a graph.

A *simple line arrangement* is a set $\mathcal{L}$ of k lines in $\mathbb{R}^2$ such that each pair of lines has one intersection point and no three lines share a common point. In the following, we assume that every line arrangement is simple. We define the *arrangement graph* for a set of lines as follows [2]: The vertices correspond to the intersection points of lines and two vertices are adjacent in the graph if and only if they are adjacent along some line. The Arrangement Graph Recognition problem is to decide whether a given graph is the arrangement graph of some set of lines.

Bose et al. [2] showed that this problem is NP-hard by reduction from a version of Pseudoline Stretchability for the Euclidean plane, whose NP-hardness was proved by Shor [25]. It turns out that Arrangement Graph Recognition is actually an $\exists\mathbb{R}$-complete problem [11, page 212]. This stronger statement follows from the fact that the Euclidean Pseudoline Stretchability is $\exists\mathbb{R}$-hard as well as the original projective version [16,23].

Theorem 1. *Given a planar graph G and an integer k, it is $\exists\mathbb{R}$-hard to decide whether $\rho^1_2(G) \leq k$ and whether $\rho^1_3(G) \leq k$.*

Proof. We first treat the 2D case. We show hardness by a reduction from Arrangement Graph Recognition. Let G be an instance of this problem. If G is an arrangement graph, there must be an integer ℓ such that G consists of $\ell(\ell-1)/2$ vertices and $\ell(\ell-2)$ edges, and each of its vertices has degree d where

$d \in [2, 4]$. So, we first check these easy conditions to determine ℓ and reject G if one of them fails. Let G' be the graph obtained from G by adding one tail (i. e., a degree-1 vertex) to each degree-3 vertex and two tails to each degree-2 vertex. So every vertex of G' has degree 1 or 4. Note that, if G is an arrangement graph, then there are exactly 2ℓ tails in G' (2 for each line) – if this is not true we can already safely reject G. We now pick $k = \ell$, and show that G is an arrangement graph if and only if $\rho^1_2(G') \leq k$.

For the first direction, let G be an arrangement graph. By our choice of k, it is clear that G corresponds to a line arrangement of k lines. Clearly, all edges of G lie on these k lines and the tails of G' can be added without increasing the number of lines. Hence, $\rho^1_2(G') \leq k$.

For the other direction, assume $\rho^1_2(G') \leq k$ and let Γ' be a straight-line drawing of G' on $\rho^1_2(G')$ lines. The graph G' contains $\binom{k}{2}$ degree-4 vertices. As each of these vertices lies on the intersection of two lines in Γ', we need k lines to get enough intersections, that is, $\rho^1_2(G') = k$. Additionally, there are no intersections of more than two lines. The most extreme points on any line have degree 1, that is, they are tails, because degree 4 would imply a more extreme vertex. We can assume that there are exactly $2k$ tails, otherwise G would have been rejected before as it could not be an arrangement graph. Each line contains exactly two of them. Let n_2 (resp. n_3) be the number of degree-2 (resp. degree-3) vertices. As we added 2 (resp. 1) tails to each of these vertices, we have $2k = 2n_2 + n_3$. By contradiction, we show that the edges on each line form a single segment. Otherwise, there would be a line with two segments. Note that the vertices at the ends of each segment have degree less than 4 (that is, degree 1). This would imply more than two degree-1 vertices on one line, a contradiction. So Γ' is indeed a drawing of G' using k segments. By removing the tails, we obtain a straight-line drawing of G using $k = n_2 + n_3/2$ segments. The result by Durocher et al. [10, Lemma 2] implies that G is an arrangement graph.

Now we turn to 3D. Let G be a graph and let G' be the augmented graph as above. We show that $\rho^1_3(G') = \rho^1_2(G')$, which yields that deciding $\rho^1_3(G')$ is also NP-hard. Clearly, $\rho^1_3(G') \leq \rho^1_2(G')$. Conversely, assume that G' can be drawn on k lines in 3-space. Since G' has $\binom{k}{2}$ vertices of degree 4, each of them must be a crossing point of two lines. It follows that each of the k lines crosses all the others. Fix any two of the lines and consider the plane that they determine. Then all k lines must lie in this plane, which shows that $\rho^1_2(G') \leq \rho^1_3(G')$. □

It remains to notice that the decision problems under consideration lie in the complexity class $\exists\mathbb{R}$. To this end, we transform the inequalities $\rho^l_d(G) \leq k$ into first-order existential expressions about the reals. For details, see the full version [5].

Lemma 2. *Each of the following decision problems belongs to the complexity class* $\exists\mathbb{R}$

(a) deciding, for a planar graph G and an integer k, whether $\rho^1_2(G) \leq k$;
(b) deciding, for a graph G and an integer k, whether $\rho^1_3(G) \leq k$;

(c) deciding, for a graph G and an integer k, whether $\rho^2_3(G) \leq k$.

3 Fixed-Parameter Tractability of the Line Cover Numbers

In this section we show that, for an input graph G and integer k, both testing whether $\rho^1_2(G) \leq k$, and testing whether $\rho^1_3(G) \leq k$ are decidable in FPT time (in k). Moreover, for both the 2D and 3D cases, for positive instances (G, k), we can compute the *combinatorial description* of a solution also in time FPT in k. One subtle point here is that there are graphs where *each* ρ^1_2-optimal drawing requires irrational coordinates; see the full version for details [5]. Thus, in some sense, a combinatorial description of a solution can be seen as a best possible output from an algorithm for these problems. Note that, by a k-line cover in $\mathbb{R}^d$ of a graph G, we mean a drawing D of G together with a set $\mathcal{L}$ of k lines such that $(D, \mathcal{L})$ certifies $\rho^1_d(G) \leq k$.

Our FPT algorithm follows from a simple *kernelization/pre-processing* procedure in which we reduce a given instance (G, k) to a reduced instance (H, k) where H has $O(k^4)$ vertices and edges, and G has a k-line cover if and only if H does as well. After this reduction, we can then apply any decision procedure for the existential theory of the reals since we have shown in Lemma 2 that both k-line cover problems are indeed both members of this complexity class. Our kernelization approach is given as Theorem 3 and our FPT result follows as described in Corollary 4. We denote the number of vertices and the number of edges in the input graph by n and m respectively.

Theorem 3. *For each $d \in \{2, 3\}$, graph G, and integer k, the problem of deciding whether $\rho^1_d(G) \leq k$ admits a kernel of size $O(k^4)$, i.e., we can produce a graph H such that H has $O(k^4)$ vertices and edges and $\rho^1_d(G) \leq k$ if and only if $\rho^1_d(H) \leq k$. Moreover, H can be computed in $O(n + m)$ time.*

Proof. For a graph G, if G is going to have a k-line cover $(D, \mathcal{L})$, then there are several necessary conditions about G which we can exploit to shrink G. First, notice that any connected components of G which are paths can easily be placed on any line in $\mathcal{L}$ without interfering with the other components, i.e., these can be disregarded. This provides a new instance G'. Second, there are at most $\binom{k}{2}$ intersection points among the lines in $\mathcal{L}$. Thus, G has at most $\binom{k}{2}$ vertices with degree larger than two. Moreover, each line $\ell \in \mathcal{L}$ will contain at most $k - 1$ of these vertices. Thus, the total number of edges which are incident to vertices with degree larger than two, is at most $2 \cdot (k - 1)$ per line, or $2 \cdot (k^2 - k)$ in total. Thus, G' contains at most $2 \cdot (k^2 - k)$ vertices of degree one (since each one occurs at the end of a path originating from a vertex of degree larger than two where all the internal vertices have degree 2). Similarly, G' contains at most $2 \cdot (k^2 - k)$ paths where every internal vertex has degree two and the end vertices either have degree one or degree larger than two. Finally, for each such path, at most $\binom{k}{2}$ vertices are mapped to intersection points in $\mathcal{L}$. Thus, any path with more than $\binom{k}{2}$ vertices can be safely contracted to a path with at most $\binom{k}{2}$ vertices. This results in our final graph G'' which can easily be seen to have $O(k^4)$ vertices and $O(k^4)$ edges (when G has a k-line cover). Now, if G'' does not satisfy one of the necessary conditions described above, we use the graph $K_{1,2k+1}$ as H, i.e., this way H has no k-line cover.

We conclude by remarking that this transformation of G to G'' can be performed in $O(n+m)$ time. The transformation from G to G' is trivial. The transformation from G' to G'' can be performed by two traversals of the graph (e.g., breadth first searches) where we first measure the lengths of the paths of degree-2 vertices, then we shrink them as needed. □

In the notation of the above proof, note that the statement $\rho^1_d(G'') \le k$ can be expressed as a prenex formula Φ in the existential first-order theory of the reals. The proof of Lemma 2 shows that such a formula can be written using $O(k^4)$ first-order variables and involving $O(k^4)$ polynomial inequalities, each of total degree at most 4 and with coefficients ± 1. We could now directly apply the decision procedure of Renegar [20,21,22] to Φ and obtain an FPT algorithm for deciding whether $\rho^1_d(G) \le k$, but that would only provide a running time of $(k^{O(k^4)} + O(n+m))$. We can be a little more clever and reduce the exponent from $O(k^4)$ to $O(k^2)$. This is described in the proof of the following corollary.

Corollary 4. *For each $d \in \{2,3\}$, graph G, and integer k, we can decide whether $\rho^1_d(G) \le k$ in $k^{O(k^2)} + O(n+m)$ time, i.e., FPT time in k.*

Proof. First, we apply to the given graph G the kernelization procedure from the proof of Theorem 3 to obtain a reduced graph G''. Now, notice that G'' has at most $O(k^4)$ vertices of degree two, but only $\binom{k}{2}$ of these can be bend points and are actually important in a solution, i.e., at most $\binom{k}{2}$ of these vertices are mapped to intersection points of the lines. Thus, we can simply enumerate all possible $O\left(\binom{k^4}{\binom{k}{2}}\right)$ subsets which will occur as intersection points, and, for each of these, test whether this further reduced instance has a k-line cover using Renegar's decision algorithm. This leads to a total running time of $k^{O(k^2)} + O(n+m)$ as needed. □

We have now seen how to decide if a given graph G has a k-line cover in both 2D and 3D. Moreover, when G is a positive instance, our approach provides a reduced graph G''' where G''' also has a k-line cover, G''' has $O(k^2)$ vertices and edges, and any k-line cover of G''' naturally induces a k-line cover of G. In the following theorem, whose proof can be found in the full version [5], we show that we can further determine the combinatorial structure of some k-line cover of G''' in $k^{O(k^2)}$ time and use this to recover a corresponding combinatorial structure for G. Here, the combinatorial structure is a set of k *linear forests* since each line in a k-line cover naturally induces a linear forest in G. Recall that a linear forest is a forest whose connected components are paths.

Theorem 5. *For each $d \in \{2,3\}$, graph G, and integer k, in $2^{O(k^3)} + O(n+m)$ time we can not only decide whether $\rho^1_d(G) \le k$ but, if so, also partition the edge set of G into linear forests accordingly to a k-line cover of G in $\mathbb{R}^d$.*

4 Computational Complexity of the Plane Cover Number

While graphs with ρ_3^2-value 1 are exactly the planar graphs, recognizing graphs with ρ_3^2-value k, for any $k > 1$, immediately becomes NP-hard. This requires a detour via the NP-hardness of a new problem of planar 3-SAT type, which we think is of independent interest. Full proofs for this section are given in the full version [5].

Definition 6 ([19]). *Let Φ be a Boolean formula in 3-CNF. The* associated graph *of Φ, $G(\Phi)$, has a vertex v_x for each variable x in Φ and a vertex v_c for each clause c in Φ. There is an edge between a variable-vertex v_x and a clause-vertex v_c if and only if x or $\neg x$ appears in c. The Boolean formula Φ is called* planar *if $G(\Phi)$ is planar.*

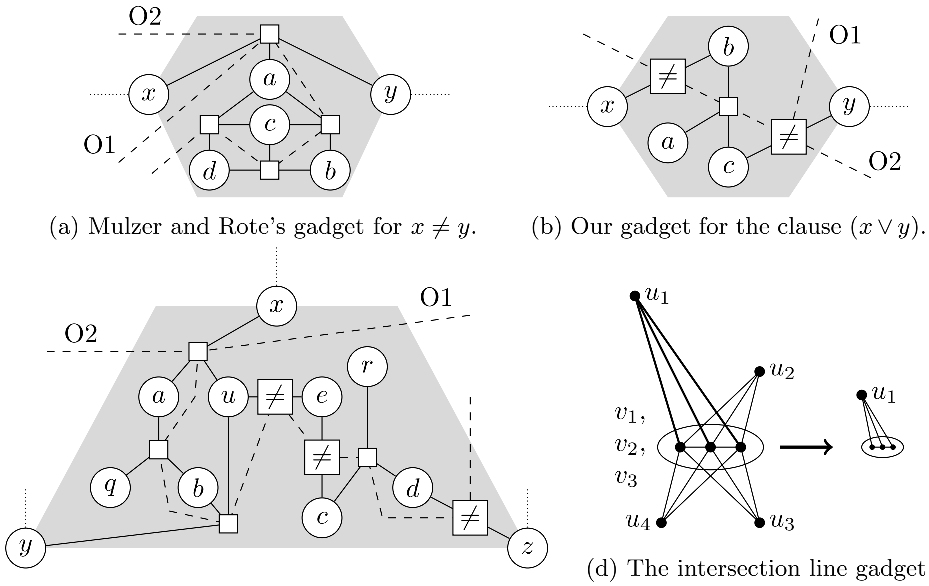
Kratochvíl et al. [14] proved NP-hardness of Planar Cycle 3-Sat, which is a variant of Planar 3-Sat where the clauses are connected by a simple cycle in the associated graph without introducing crossings. Their reduction even shows hardness of a special case, where all clauses consist of at least two variables. We consider only this special case. Mulzer and Rote [19] proved NP-hardness of Positive Planar 1-in-3-Sat, another variant of Planar 3-Sat where all literals are positive and the assignment must be such that, in each clause, exactly one of the three variables is true. We combine proof ideas from the two to show NP-hardness of the following new problem.

Definition 7. *In the* Positive Planar Cycle 1-in-3-Sat *problem, we are given a collection Φ of clauses each of which contains exactly three variables, together with a planar embedding of $G(\Phi) + C$ where C is a cycle through all clause-vertices. Again, all literals are positive. The problem is to decide whether there exists an assignment of truth values to the variables of Φ such that exactly one variable in each clause is true.*

Lemma 8. Positive Planar Cycle 1-in-3-Sat *is NP-complete.*

Proof (sketch). We reduce from Planar Cycle 3-Sat. We iteratively replace the clauses by positive 1-in-3-Sat clauses while maintaining the cycle through these clauses. Our reduction uses some of the gadgets from the proof of Mulzer and Rote [19]. We show how to maintain the cycle when inserting these gadgets.

We consider the interaction between the cycle and the clauses. Every clause consists of two or three literals and thus there are two or three faces around a clause in the drawing. There are two options for the cycle: (O1) it can "touch" the clause, that is, the incoming and the outgoing edge are drawn in the same face; (O2) it can "pass through" the clause, that is, incoming and outgoing edge are drawn in different faces. As an example, Fig. 1a shows how we weave the cycle through the inequality gadget by Mulzer and Rote. As a replacement for the clauses with 2 variables we cannot use the gadget described by Mulzer and Rote as it does not allow us to add a cycle through the clauses. Therefore, we use a new gadget that is depicted in Fig. 1b. □



(a) Mulzer and Rote's gadget for $x \neq y$.

(b) Our gadget for the clause $(x \vee y)$.

(c) Mulzer and Rote's gadget for the clause: $x \vee y \vee z$.

(d) The intersection line gadget and how it is depicted in Fig. 2.

Fig. 1: Gadgets for our NP-hardness proof. Variables are drawn in circles, clauses are represented by squares. The boxes with the inequality sign represent the inequality gadget. The dashed line shows how we weave the cycle through the clauses. There are two variants of the cycle, which differ only in one edge: (O1) The cycle touches the gadget; (O2) the cycle passes through the gadget.

We now introduce what we call the *intersection line gadget*; see Fig. 1d. It consists of a $K_{3,4}$ in which the vertices in the smaller set of the bipartition—denoted by v_1, v_2, and v_3—are connected by a path. We denote the vertices in the other set by u_1, u_2, u_3, and u_4.

Lemma 9. *If a graph containing the intersection line gadget can be embedded on two non-parallel planes, the vertices v_1, v_2, and v_3 must be drawn on the intersection line of the two planes while the vertices u_1, u_2, u_3, and u_4 cannot lie on the intersection line.*

Theorem 10. *Let G be a graph. Deciding whether $\rho^2_3(G) = 2$ is NP-hard.*

Proof (sketch). We show NP-hardness by reduction from Positive Planar Cycle 1-in-3-Sat. We build the graph $G^*(\Phi) = (V, E)$ for formula Φ that consists of n clauses as follows: Each clause c is represented by a clause gadget that consists of three vertices v_c^1, v_c^2, and v_c^3 that are connected by a path. Let x be a variable that occurs in the clauses $c_{i_1}, c_{i_2}, \dots, c_{i_l}$ with $i_1 < i_2 < \dots < i_l$. Each variable x is represented by a tree with the vertices $w_1^x, w_2^x, \dots, w_l^x$ that are connected to the relevant clauses, and the vertices $v_1^x, v_2^x, \dots, v_l^x$ that lie on a path and are connected to these vertices. To each of the vertices $v_1^x, v_2^x, \dots, v_l^x$ one

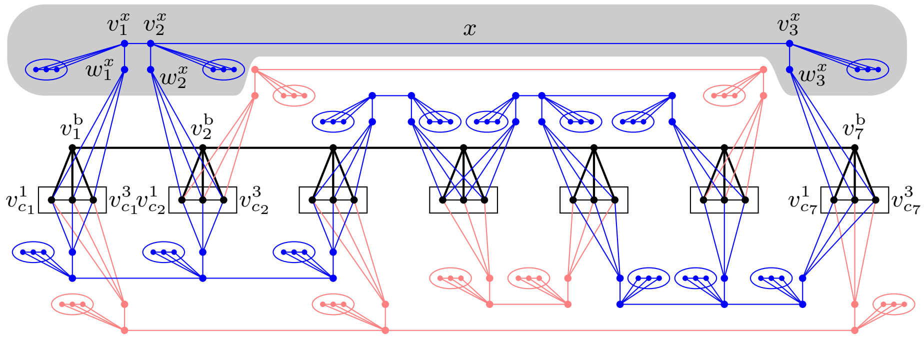


Fig. 2: Example for the graph $G^*(\Phi)$ constructed from a Positive Planar Cycle 1-in-3-Sat instance Φ. The clauses are depicted by the black boxes with three vertices inside and denoted by $c_1, \ldots, c_7$ from left to right. The variables are drawn in pale red (true) and blue (false). The variable x is highlighted by a shaded background. The ellipses attached to variable-vertices stand for the intersection line gadget (see Fig. 1d). The depicted vertices incident to the gadget correspond to u_1 in Fig. 1d; u_2 to u_4 are not shown. If Φ is true, one plane covers the blue variable gadgets and one plane covers the blocking caterpillar (bold black) and the pale red variable gadgets.

instance of the intersection line gadget is connected. Finally, we add a blocking caterpillar consisting of the vertices $v_1^{\mathrm{b}}, \ldots, v_n^{\mathrm{b}}$ that are connected to the clauses in their cyclic order, which exists for the Positive Planar Cycle 1-in-3-Sat instance by definition. See Fig. 2 for an example of this construction.

We show that the formula Φ has a truth assignment with exactly one true variable in each clause if and only if the graph $G^*(\Phi)$ can be drawn onto two planes. The idea of our construction is that only two variables can be connected to a clause gadget on each of the planes. One plane contains the blocking caterpillar and one variable per clause (corresponding to true variables). The other plane contains two variables per clause (corresponding to false variables). Our construction ensures that the vertices of a variable cannot be partitioned onto both planes in any drawing. □

Corollary 11. *Deciding whether $\rho_3^2(G) = k$ is NP-hard for any $k \geq 2$.*

Proof (sketch). We add a blocking gadget for each additional plane. □

5 Complexity of the Weak Affine Cover Numbers π_3^1 / π_3^2

Recall that a *linear forest* is a forest whose connected components are paths. The *linear vertex arboricity* $\mathrm{lva}(G)$ of a graph G equals the smallest size r of a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that every V_i induces a linear forest. The *vertex thickness* $\mathrm{vt}(G)$ of a graph G is the smallest size r of a partition $V(G) = V_1 \cup \cdots \cup V_r$ such that $G[V_1], \ldots, G[V_r]$ are all planar. Obviously, $\mathrm{vt}(G) \leq \mathrm{lva}(G)$. We recently used these notions to characterize the 3D weak affine cover numbers in purely combinatorial terms [6]: $\pi_3^1(G) = \mathrm{lva}(G)$ and $\pi_3^2(G) = \mathrm{vt}(G)$.

Theorem 12. *For $l \in \{1, 2\}$,*
(a) deciding whether or not $\pi_3^l(G) \le 2$ is NP-complete, and
(b) approximating $\pi_3^l(G)$ within a factor of $O(n^{1-\epsilon})$, for any $\epsilon > 0$, is NP-hard.

Proof. (a) The membership in NP follows directly from the above combinatorial characterization [6], which also allows us to deduce NP-hardness from a much more general hardness result by Farrugia [12]: For any two graph classes $\mathcal{P}$ and $\mathcal{Q}$ that are closed under vertex-disjoint unions and taking induced subgraphs, deciding whether the vertex set of a given graph G can be partitioned into two parts X and Y such that $G[X] \in \mathcal{P}$ and $G[Y] \in \mathcal{Q}$ is NP-hard unless both $\mathcal{P}$ and $\mathcal{Q}$ consist of all graphs or all empty graphs. To see the hardness of our two problems, we set $\mathcal{P} = \mathcal{Q}$ to the class of linear forests (for $l = 1$) and to the class of planar graphs (for $l = 2$).

(b) The combinatorial characterization [6] given above implies that $\chi(G) \le 4\,\mathrm{vt}(G) = 4\pi_3^2(G)$ (by the four-color theorem). Note that each color class can be placed on its own line, so $\pi_3^1(G) \le \chi(G)$. As $\pi_3^2(G) \le \pi_3^1(G)$, both parameters are linearly related to the chromatic number of G. Now, the approximation hardness of our problems follows from that of the chromatic number [26]. □

6 Conclusion and Open Problems

1. We have determined the computational complexity of the affine cover numbers ρ_2^1 and ρ_3^1. The corresponding decision problems $\rho_2^1(G) \le k$ and $\rho_3^1(G) \le k$ turn out to be $\exists\mathbb{R}$-complete. On the positive side, these problems admit an FPT algorithm (Corollary 4). This is impossible for the plane cover number ρ_3^2, unless P = NP, because the decision problem $\rho_3^2(G) \le k$ is NP-hard even for $k = 2$ (Theorem 10 in Section 4). If k is arbitrary and given as a part of the input, then this problem is in $\exists\mathbb{R}$ (Lemma 2)—but is it $\exists\mathbb{R}$-hard?
2. Is the segment number $\mathrm{segm}(G)$ introduced in [7] fixed-parameter tractable?
3. Our proof of Theorem 1 implies that computing $\rho_2^1(G)$ and $\rho_3^1(G)$ is hard even for planar graphs of maximum degree 4. Can $\rho_2^1(G)$ and $\rho_3^1(G)$ be computed efficiently for trees? This is true for the segment number $\mathrm{segm}(G)$ [7].
4. How hard is it to approximate ρ_2^1, ρ_3^1, and ρ_3^2?

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