Simple Algorithms for Partial and Simultaneous Rectangular Duals with Given Contact Orientations*

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Abstract. A rectangular dual of a graph G is a contact representation of G by axis-aligned rectangles such that (i) no four rectangles share a point and (ii) the union of all rectangles is a rectangle. The *partial representation extension problem* for rectangular duals asks whether a given partial rectangular dual can be extended to a rectangular dual, that is, whether there exists a rectangular dual where some vertices are represented by prescribed rectangles. The *simultaneous representation problem* for rectangular duals asks whether two (or more) given graphs that share a subgraph admit rectangular duals that coincide on the shared subgraph. Combinatorially, a rectangular dual can be described by a regular edge labeling (REL), which determines the orientations of the rectangle contacts.

We describe linear-time algorithms for the partial representation extension problem and the simultaneous representation problem for rectangular duals when each input graph is given together with a REL. Both algorithms are based on formulations as linear programs, yet they have geometric interpretations and can be seen as extensions of the classic algorithm by Kant and He that computes a rectangular dual for a given graph.

Keywords: rectangular dual · partial representation extension · simultaneous representation

1 Introduction

A *geometric intersection representation* of a graph G is a mapping $\mathcal{R}$ that assigns to each vertex w of G a geometric object $\mathcal{R}(w)$ such that two vertices u and v are adjacent in G if and only if $\mathcal{R}(u)$ and $\mathcal{R}(v)$ intersect. In a *contact representation* we further require that, for any two vertices u and v, the objects

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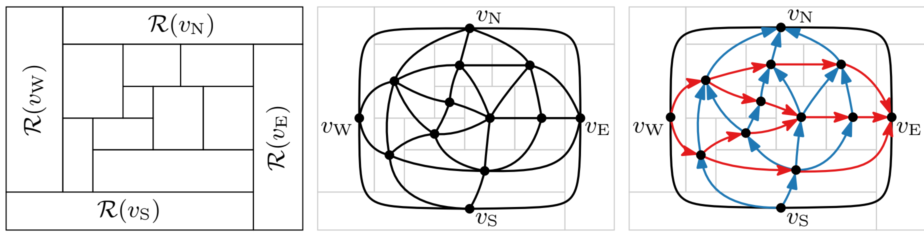
$\mathcal{R}(u)$ and $\mathcal{R}(v)$ have disjoint interiors. The *recognition problem* asks whether a given graph admits an intersection or contact representation whose sets have a specific geometric shape. Classic examples are interval graphs [4], where the objects are intervals of $\mathbb{R}$, or coin graphs [28], where the objects are interior-disjoint disks in the plane. The *partial representation extension problem* is a natural generalization of this question where, for each vertex u of a given subset of the vertex set, the geometric object is already prescribed, and the question is whether this partial representation can be extended to a full representation of the input graph. In the last decade the partial representation extension problem has been intensely studied for various classes of intersection graphs, such as (unit or proper) interval graphs [25,26], circle graphs [9], trapezoid graphs [30], as well as for contact representations [8] and bar-visibility representations [10].

A different generalization is the *simultaneous representation problem*, where, given several input graphs $G_1, \dots, G_k$, one asks whether there exist representations $\mathcal{R}_1, \dots, \mathcal{R}_k$ of $G_1, \dots, G_k$ such that each vertex v contained in G_i and in G_j satisfies $\mathcal{R}_i(v) = \mathcal{R}_j(v)$, i.e., any two representations coincide on the shared vertices. Most frequently, this problem is studied in the *sunflower case*, where one additionally assumes that the pairwise intersection of any two graph G_i, G_j with $i \neq j$ is the same subgraph S, which is usually called the *shared graph*. The question is equivalent to asking whether there exists a representation of S that simultaneously extends to each of the input graphs $G_1, \dots, G_k$. Simultaneous representation problems have long been studied for planar graphs; see Bläsius et al. [2] and Rutter [36] for surveys. For intersection representations, the problem was originally introduced by Jampani and Lubiw, who gave polynomial-time algorithms for interval graphs [22] as well as for comparability and permutation graphs [23]. They also proved NP-completeness for chordal graphs. Bläsius and Rutter later improved the running time for interval graphs to linear [3]. Recently, Rutter et al. [37] gave efficient algorithms for proper and unit interval graphs. Previous work on simultaneous contact representations has focused on representing planar graphs and their duals, for example, with triangles in the plane [19] or with boxes in 3D [1].

Rectangular duals. In this paper we consider the partial representation extension problem and the simultaneous representation for the following type of representation. A *rectangular dual* of a graph G is a contact representation $\mathcal{R}$ of G by axis-aligned rectangles such that (i) no four rectangles share a point and (ii) the union of all rectangles is a rectangle; see Fig. 1. We observe that G may admit a rectangular dual only if it is planar and internally triangulated. Furthermore, a rectangular dual can always be augmented with four additional vertices (one on each side) so that only four rectangles touch the outer face of the representation. It is customary that the four vertices on the outer face are denoted by v_{S}, v_{W}, v_{N}, and v_{E} corresponding to the geographic directions, and to require that $\mathcal{R}(v_{\mathrm{W}})$ is the leftmost rectangle, $\mathcal{R}(v_{\mathrm{E}})$ is rightmost, $\mathcal{R}(v_{\mathrm{S}})$ is bottommost, and $\mathcal{R}(v_{\mathrm{N}})$ is topmost; see Fig. 1. We call these vertices the *outer vertices* and the remaining ones the *inner vertices*. It is known that a plane internally-triangulated graph has a representation with only four rectangles touching the outer face if and



(a) $\mathcal{R}$ (b) G (c) (L_1, L_2)

Fig. 1. A rectangular dual $\mathcal{R}$ for the graph G; the REL (L_1, L_2) induced by $\mathcal{R}$. In the REL, the red edges cross vertical rectangle edges from left to right, and the blue edges cross horizontal rectangle edges from bottom to top.

only if its outer face is a 4-cycle and it has no separating triangles, that is, a triangle whose removal disconnects the graph [29]. Such a graph is called a *properly-triangulated planar (PTP)* graph. Kant and He [24] have shown that a rectangular dual of a given PTP graph G can be computed in linear time.

Historically, rectangular duals have been studied due to their applications in architecture [38], VLSI floor-planning [33, 39], and cartography [18]. Besides the question of an efficient construction algorithm [24], other problems concerning rectangular duals are area minimization [7], sliceability [32], and area-universality, that is, rectangular duals where the rectangles can have any given areas [14]. The latter question highlights the close relation between rectangular duals and rectangular cartograms. Rectangular cartograms were introduced in 1934 by Raisz [35] and combine statistical and geographical information in thematic maps, where geographic regions are represented as rectangles and scaled in proportion to some statistic. There has been lots of work on efficiently computing rectangular cartograms [6, 21, 31]; Nusrat and Kobourov [34] recently surveyed this topic. As a dissection of a rectangle into smaller rectangles, a rectangular dual is also related to other types of dissections, for example with squares [5] or hexagons [13]; see also Felsner's survey [15].

Contribution and outline. A *regular edge labeling (REL)* of a graph describes, for every edge (u, v), the type of contact between $\mathcal{R}(u)$ and $\mathcal{R}(v)$ in a rectangular dual $\mathcal{R}$, that is, the side of the rectangle $\mathcal{R}(u)$ that touches $\mathcal{R}(v)$. We consider the case where a graph is given together with a fixed REL. First, we formally define the problems that we consider, mention the tools that we use to solve them, and give a brief overview about previous results to obtain rectangular duals for graphs with given REL; see Section 2. Then, we describe a linear program (LP) in the form of a system of difference constraints to compute rectangular duals for PTP graphs with given RELs; see Section 3. Next, we show how to use the LP (i) to construct a rectangular dual, (ii) to solve the partial representation extension problem, and (iii) to solve the respective simultaneous representation problem in linear time; see Section 4. Finally, we present some open problems; see Section 5.

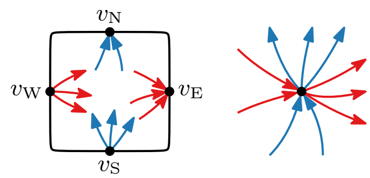


Fig. 2. Edge order at the four outer vertices and an inner vertex.

Our main results are the following two theorems.

Theorem 1. *The partial representation extension problem for rectangular duals with a fixed regular edge labeling can be solved in linear time. For yes-instances, an explicit rectangular dual can be constructed within the same time bound.*

Theorem 2. *The simultaneous representation problem for rectangular duals with fixed regular edge labelings can be solved in linear time. For yes-instances, simultaneous rectangular duals can be constructed within the same time bound.*

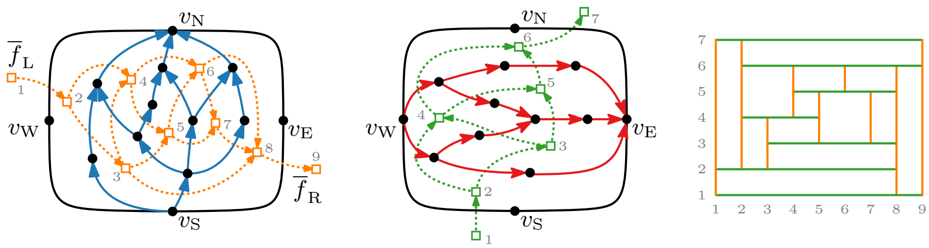
2 Preliminaries

In this section, we formally define regular edge labelings and partial rectangular duals. We give a brief overview about existing results for the relationship between regular edge labelings and rectangular duals.

Topological Numberings. Let G be a directed graph with n vertices. A *topological numbering* of G is a function $d\colon V \to \mathbb{N}_0^+$ such that, for any edge (u, v) in G, $d(v) > d(u)$. A *topological sorting* is a topological numbering that bijectively maps the vertices to the numbers $1, \dots, n$. For a weighted directed graph, that is, a directed graph with a weight function $w\colon E \to \mathbb{R}_0^+$, a *weighted topological numbering* of G is a function $d\colon V \to \mathbb{R}_0^+$ such that, for any edge (u, v) in G, $d(v) \geq d(u) + w(u, v)$.

Regular Edge Labelings. The combinatorial aspects of a contact representation of a graph G can often be described with a coloring and orientation of the edges of G. For example, Schnyder woods describe contact representations of planar graphs by triangles [16]. Such a description also exists for contact representations by rectangles, for example for triangle-free rectangle arrangements [27] or rectangular duals [24]. More precisely, a rectangular dual $\mathcal{R}$ gives rise to a 2-coloring and an orientation of the inner edges of G as follows. We color an edge $\{u, v\}$ blue if the contact between $\mathcal{R}(u)$ and $\mathcal{R}(v)$ is a horizontal line segment, and we color it red otherwise. We orient a blue edge $\{u, v\}$ as (u, v) if $\mathcal{R}(u)$ lies below $\mathcal{R}(v)$, and we orient a red edge $\{u, v\}$ as (u, v) if $\mathcal{R}(u)$ lies to the left of $\mathcal{R}(v)$; see Fig. 1. The resulting coloring and orientation has the following properties (see Fig. 2):

(1) All inner edges incident to v_W, v_S, v_E, and v_N are red outgoing, blue outgoing, red incoming, and blue incoming, respectively.



(a) $L_1(G)$ and $L_1^*(G)$ (b) $L_2(G)$ and $L_2^*(G)$ (c) $\mathcal{R}$

Fig. 3. Illustration of the algorithm by He [20]. (a+b) Topological numberings (gray) for the weak duals $L_1^*(G)$ and $L_2^*(G)$ (dotted) of $L_1(G)$ and $L_2(G)$; (c) resulting rectangular dual $\mathcal{R}$.

(2) The edges incident to each inner vertex form four counterclockwise ordered non-empty blocks of red incoming, blue incoming, red outgoing, and blue outgoing, respectively.

A coloring and orientation with these properties is called a *regular edge labeling (REL)* or *transversal structure* [17]. Let (L_1, L_2) denote a REL, where L_1 is the set of blue edges and L_2 is the set of red edges. Let $L_1(G)$ and $L_2(G)$ denote the two subgraphs of G induced by the outer four-cycle and L_1 and L_2, respectively. Note that both $L_1(G)$ and $L_2(G)$ are st-graphs, that is, directed acyclic graphs with exactly one source and exactly one sink. It is well known that every PTP graph admits a REL and thus a rectangular dual [24]. A rectangular dual $\mathcal{R}$ *realizes* a REL (L_1, L_2) if the REL induced by $\mathcal{R}$ is (L_1, L_2). Note that while a rectangular dual uniquely defines a REL, there exist different rectangular duals that realize any given REL.

Algorithms by He and by Kant and He. Kant and He [24] introduced RELs and described two linear-time algorithms that compute a REL for a given PTP graph; one algorithm is based on edge contractions, the other is based on canonical orderings. They then use the algorithm by He [20] to construct in linear time a rectangular dual that realizes this REL (with integer coordinates).

The algorithm by He works as follows; see Figure 3. Given a PTP graph G with REL (L_1, L_2). Consider the weak dual $L_1^*(G)$ of $L_1(G)$. In $L_1^*(G)$, there is a vertex f for every interior face f in the plane embedding of $L_1(G)$, and there are two vertices $\overline{f}_L$ and $\overline{f}_R$ for the outer face $\overline{f}$ in the plane embedding of $L_1(G)$. There is an edge (f, f') if f is the interior face to the left and f' is the interior face to the right of some edge in $L_1(G)$; an edge $(\overline{f}_L, f')$ if $\overline{f}$ is the face to the left and f' is the interior face to the right of some edge in $L_1(G)$; and an edge $(f, \overline{f}_R)$ if f is the interior face to the left and $\overline{f}$ is the face to the right of some edge in $L_1(G)$. Then $L_1^*(G)$ is a planar st-graph with source $\overline{f}_L$ and sink $\overline{f}_R$. Compute a topological numbering d of $L_1^*(G)$. For any vertex v of G, let left(v) be the face to the left of v in $L_1(G)$, and let right(v) be the face to the right of v in $L_1(G)$; these are the two faces between an incoming and an outgoing edge of v.

Then the algorithm sets the x-coordinate of the left side of $\mathcal{R}(v)$ to $d(\text{left}(v))$ and the x-coordinate of the right side of $\mathcal{R}(v)$ to $d(\text{right}(v))$. The y-coordinates for the bottom and top sides of $\mathcal{R}(v)$ are calculated analogously from the weak dual $L_2^*(G)$ of $L_2(G)$.

Note that the maximal vertical segments in the resulting rectangular dual $\mathcal{R}$ are in bijection with the vertices of $L_1^*(G)$ and the maximal horizontal segments are in bijection with the vertices of $L_2^*(G)$; see Fig. 3(c). Further note that a rectangular dual can also be computed via two linear programs (LPs); He's algorithm solves these LPs efficiently by exploiting the simple (acyclic) structure of the graphs $L_1^*(G)$ and $L_2^*(G)$. The vertices of these graphs correspond to the LP variables for the x- and y-coordinates of the rectangles, respectively. The edges of the two graphs correspond to the LP constraints.

He [20] did not show this explicitly, but for a given REL, his linear-time algorithm yields a rectangular dual of minimum width and height, and thus of minimum area and perimeter. He posed the open question whether it is possible to find a rectangular dual of minimum area or perimeter for a given PTP graph if the REL is not fixed.

Partial Rectangular Duals. For a graph G, let $E(G)$ denote the set of edges and $V(G)$ the set of vertices of G. Let $U \subseteq V(G)$. Then $G[U]$ denotes the subgraph of G induced by U. The pair $(U, \mathcal{P})$ is a *partial rectangular dual* of G if $\mathcal{P}$ is a contact representation of $G[U]$ that maps each $u \in U$ to an axis-aligned rectangle $\mathcal{P}(u)$. We call the vertices in U *fixed* and, for $u \in U$, we call $\mathcal{P}(u)$ a *fixed rectangle*. We further define $|\mathcal{P}| = |U|$. For the sake of readability, we refer from now on to a partial rectangular dual $(U, \mathcal{P})$ simply with $\mathcal{P}$ and consider the domain U of $\mathcal{P}$ as implicitly given.

For a given graph G and a partial rectangular dual $\mathcal{P}$, the *partial rectangular dual extension problem* asks whether $\mathcal{P}$ can be extended to a rectangular dual $\mathcal{R}$ of G. In particular, for such an extension $\mathcal{R}$ and each fixed vertex u, we require that $\mathcal{P}(u) = \mathcal{R}(u)$. In this paper, we study the variant of this problem where we are not only given G and $\mathcal{P}$, but also a REL (L_1, L_2) of G and ask whether there is an extension $\mathcal{R}$ of $\mathcal{P}$ that realizes (L_1, L_2).

Closely related work includes partial representation extension of segment contact graphs [8] and bar-visibility representations [10]. Both problems are NP-complete. However, the hardness reductions crucially rely on low connectivity for choices in the planar embedding. Since PTP graphs are triconnected, they have a unique planar embedding and hence these results cannot be easily transferred.

Boundary Paths. In the conference version of this paper [11], we gave a geometric proof of Theorem 1. The main idea of that algorithm is that the left/right sides of the prescribed rectangles cut the bounding rectangle of the partial rectangular dual into vertical strips, and likewise, the top/bottom sides determine horizontal strips. We then gave a combinatorial description how to cut the given REL along so-called boundary paths into horizontal and vertical strips that can be used to fill the unprescribed parts of the rectangular dual. A key issue here is that there may be $\Omega(n)$ horizontal strips and $\Omega(n)$ vertical strips, which naively leads to $\Omega(n^2)$ rectangular cells, formed by the intersection of a horizontal and a vertical strip, that need to be filled. On the other hand, since there are only $O(n)$ vertices, only $O(n)$ of the cells can be intersected by more than one rectangle. Linear running time was achieved by constructing a compressed version of the boundary paths that contains only those cells explicitly that are intersected by at least two rectangles. In the present paper, this algorithm has been replaced by a simpler approach that is based on linear programming and topological numberings. Besides being conceptually simpler, it additionally generalizes to the simultaneous representation problem.

3 Rectangular Duals by Linear Programming

In this section, we describe how a rectangular dual of a given PTP graph and a given REL can be computed with the help of an LP. Felsner [15] used an LP to compute square duals, that is, a rectangular dual where each rectangle is actually a square. While Felsner's LP can be adapted to compute rectangular duals, we formulate our LP differently such that we can also use it for the partial representation extension problem and the simultaneous representation problem. As we will see, our LP has strong parallels with the linear-time algorithm of He [20].

Let G be a PTP graph, let (L_1, L_2) be a REL of G, and let $\varepsilon > 0$. We call our LP $\texttt{RecDual}(G, (L_1, L_2), \varepsilon)$. We first describe the variables and then the constraints. We associate four variables with each vertex u of G. The variables $x_{1,u}$ and $x_{2,u}$ denote the x-coordinates of the left side and the right side of $\mathcal{R}(u)$, respectively, and the variables $y_{1,u}$ and $y_{2,u}$ denote the y-coordinates of the bottom side and the top side of $\mathcal{R}(u)$, respectively. In what follows, we treat only the constraints regarding the x-variables. The constraints regarding the y-variables are analogous. There are no constraints regarding both types of variables. We require that each rectangle has width (and height) at least ε, i.e.,

$$x_{2,u} - x_{1,u} \geq \varepsilon \quad \text{for each vertex } u \text{ of } G.$$

We have two types of constraints for the edges.

First, for every edge $(u, v) \in E(L_2(G))$, we ensure that the left side of $\mathcal{R}(v)$ touches the right side of $\mathcal{R}(u)$; see Fig. 4(a). In other words,

$$x_{2,u} - x_{1,v} = 0 \quad \text{for each edge } (u, v) \in E(L_2(G)).$$

(We treat the two edges $(v_\mathrm{S}, v_\mathrm{W})$ and $(v_\mathrm{E}, v_\mathrm{N})$ as edges of $L_1(G)$ and the two edges $(v_\mathrm{W}, v_\mathrm{N})$ and $(v_\mathrm{S}, v_\mathrm{E})$ as edges of $L_2(G)$; see Fig. 1. Thus, the lower left corner of the rectangular dual belongs to $\mathcal{R}(v_\mathrm{S})$.)

Second, for every edge $(u, v) \in E(L_1(G))$, we enforce that the rectangles $\mathcal{R}(u)$ and $\mathcal{R}(v)$ overlap horizontally; see Fig. 4(b). To this end, for a vertex u in G, let v and v' be the (clockwise) first and last outgoing neighbors of u in $L_1(G)$. (Note that v and v' do not necessarily have to be distinct.) Then,

$$\begin{aligned} x_{2,v} - x_{1,u} &\geq \varepsilon \quad \text{and} \\ x_{2,u} - x_{1,v'} &\geq \varepsilon \quad \text{for each vertex } u \text{ of } G \text{ with outgoing neighbors } v \text{ and } v'. \end{aligned}$$

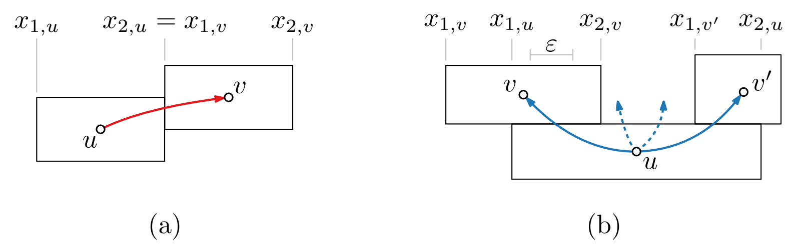


Fig. 4. The relation between the LP variables such that the edges of the REL are represented correctly.

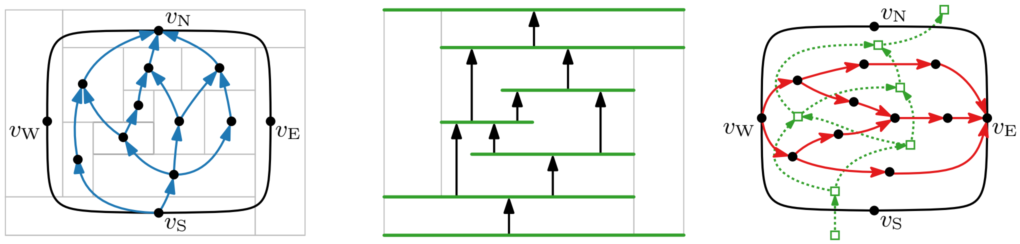
As a result, $\mathcal{R}(v)$ and $\mathcal{R}(v')$ overlap with $\mathcal{R}(u)$ horizontally by at least ε. Rectangles corresponding to other outgoing neighbors of u overlap with $\mathcal{R}(u)$ because they lie between $\mathcal{R}(v)$ and $\mathcal{R}(v')$ by the first type of constraint. We have analogous inequalities for the first and last *incoming* neighbor of u in $L_1(G)$, and for the two analogous cases in $L_2(G)$.

If the given graph G has n vertices, our LP has $\mathcal{O}(n)$ variables and constraints.

Solving the LP. Our LP is a so-called *system of difference constraints (SDC)*. This means that, if we write the LP in the standard form $Ax \leq b$, every entry of the matrix A is in $\{-1, 0, 1\}$ and in each row of A at most one entry is a 1 and at most one entry is a -1. The advantage of an SDC is that the Bellman–Ford algorithm can be used to find a solution (if one exists) in $\mathcal{O}(N^2 + NM)$ time, where N is the number of variables and M is the number of constraints [12].

More precisely, the LP can be transformed to an auxiliary weighted directed graph X that contains N vertices and M edges, such that the solution to the LP corresponds to a distance labeling with respect to a specific start vertex s. Then the LP has a feasible solution if and only if X does not contain a negative cycle. As it turns out, our LP has the additional property that the constants on the right-hand sides all have the same sign, and therefore all edges of X have negative weight. Then the LP is feasible if and only if X is acyclic. If X is indeed acyclic, shortest paths can be computed in linear time by topological numbering. In our case, the graph X is basically the REL, except that every REL vertex v is split into two vertices v^- and v^+; v^- receives the incoming edges and v^+ receives the outgoing edges, and there is an edge from v^- to v^+. Hence, X is acyclic, and our LP is feasible. Note that X has two weakly connected components, one for the x-coordinates and one for the y-coordinates. For each component, we compute a topological numbering. Then we find sets of variables that get equal values due to equality constraints. In each such set, we identify all variables into a single variable. The remaining variables correspond to maximal vertical and horizontal line segments in the rectangular dual to be computed; see Fig. 5. In other words, this is then exactly the algorithm of He [20] for computing rectangular duals.

It is worth noting that, if we set $x_{1,v_\mathrm{S}} = 0$ and $\varepsilon = 1$, all constraints use integer values. Then the Bellman–Ford algorithm (or topological numbering) will yield a solution (that is, a rectangular dual) with integer coordinates. This



(a) $L_1(G)$ (b) maximal horizontal segments (c) $L_2^*(G)$

Fig. 5. Solving the LP that corresponds to the blue edges in a REL.

solution (as the output of the algorithm of He [20]) has minimum width and height among all rectangular duals on the integer grid.

4 Applications

In this section we show how the LP from the previous section and thus He's algorithm [20] can be extended to solve the partial representation extension and the simultaneous representation problem for rectangular duals when each graph comes with a REL.

4.1 Partial Representation Extension

Let G be a PTP graph with REL (L_1, L_2) and let $\mathcal{P}$ be a partial rectangular dual of G. Recall that the partial rectangular dual extension problem asks for whether $\mathcal{P}$ admits a rectangular dual extension for G and (L_1, L_2); see Fig. 6. We modify $\texttt{RecDual}(G, (L_1, L_2), \varepsilon)$ to solve this problem as follows. For each fixed vertex u of $\mathcal{P}$, we set $x_{1,u}$, $x_{2,u}$, $y_{1,u}$, and $y_{2,u}$ according to the fixed rectangle $\mathcal{R}(u)$ of $\mathcal{P}$. We call this LP $\texttt{RepEx}(G, (L_1, L_2), \mathcal{P}, \varepsilon)$.

We need to set ε such that rectangles can be placed between fixed rectangles with sizes and overlaps of at least ε. Let D_x be the set of x-coordinates $x_{1,u}$ and $x_{2,u}$ of all fixed vertices u. Define D_y analogously. Let d be the minimum distance between any pair in D_x or in D_y. Observe that in a rectangular dual of G at most n distinct x-coordinates are used for the x-coordinates of the left and right sides of the rectangles. The analogous statement holds for the y-coordinates. Hence, we get the following lemma.

Lemma 3. *If there exists any $\varepsilon > 0$ such that* $\texttt{RepEx}(G, (L_1, L_2), \mathcal{P}, \varepsilon)$ *has a solution, then* $\texttt{RepEx}(G, (L_1, L_2), \mathcal{P}, d/n)$ *has a solution as well.*

We show that the LP $\texttt{RepEx}(G, (L_1, L_2), \mathcal{P}, d/n)$ can be solved in linear time via topological numbering just like $\texttt{RecDual}(G, (L_1, L_2), \varepsilon)$. Note that, like above, this LP can be transformed into an auxiliary weighted directed graph X. The only difference to above is that some variables are now already set. If we identify

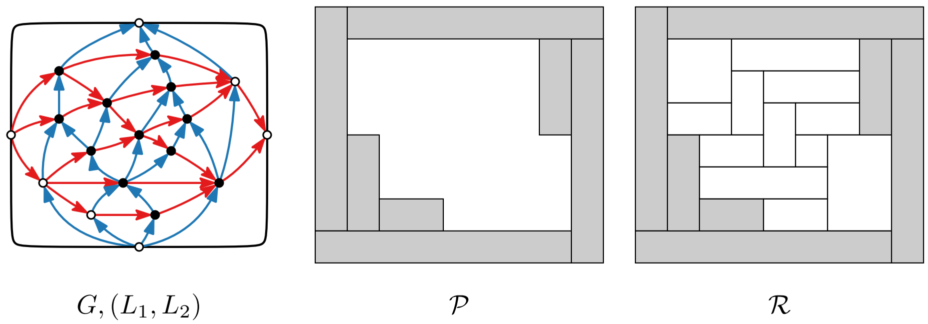


Fig. 6. The PTP graph G with REL (L_1, L_2) and the partial representation $\mathcal{P}$ are an instance of the particular rectangular dual extension problem. The rectangular dual $\mathcal{R}$ is a valid extension of $\mathcal{P}$.

sets of variables that must receive the same values, we again obtain variables for the maximal segments of G under (L_1, L_2); see Fig. 7a–b. Note that if this is not possible because variables within one set have been assigned different values, then the given representation extension instance has no solution. Furthermore, this is also the case if X does not admit a topological numbering. This can happen only if maximal horizontal segments (or maximal vertical segments) are in an order due to the fixed values that is not compatible with (L_1, L_2). Hence, we conclude that there exist topological numberings for X if and only if there exists a solution to the given partial extension problem; see Fig. 7c. Since the two topological numberings can be computed in linear time, we get Theorem 1.

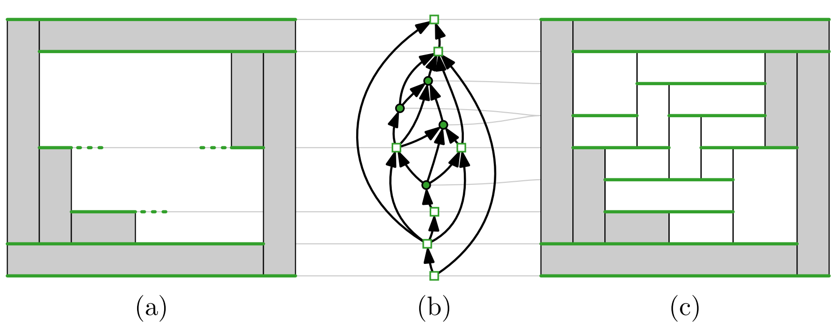


Fig. 7. (a) The partial rectangular dual $\mathcal{P}$ from Fig. 6 fixes the y-coordinates of some maximal horizontal segments; (b) the vertices representing the unfixed maximal horizontal segments in X need to get y-coordinates based on a topological numbering, (c) which results in the extension $\mathcal{R}$ of $\mathcal{P}$.

Theorem 1. *The partial representation extension problem for rectangular duals with a fixed regular edge labeling can be solved in linear time. For yes-instances, an explicit rectangular dual can be constructed within the same time bound.*

It remains open how to solve `RecDual`$(G, (L_1, L_2), \varepsilon)$ such that the resulting rectangular dual $\mathcal{R}$ has the best possible resolution in terms of the grid size it uses.

4.2 Simultaneous Representations

Let $G_1, \ldots, G_k$ be PTP graphs with RELs $(L_1, L_2)_1, \ldots, (L_1, L_2)_k$, respectively. For distinct i and j in $\{1, \ldots, k\}$, let $H_{i,j}$ be the common subgraph of G_i and G_j. Recall that the simultaneous rectangular dual representation problem for $G_1, \ldots, G_k$ asks for rectangular duals $\mathcal{R}_1(G_1), \ldots, \mathcal{R}_k(G_k)$ where each $H_{i,j}$ is represented identically in $\mathcal{R}_i(G_i)$ and $\mathcal{R}_j(G_j)$. An example with $k = 2$ is shown in Fig. 8.

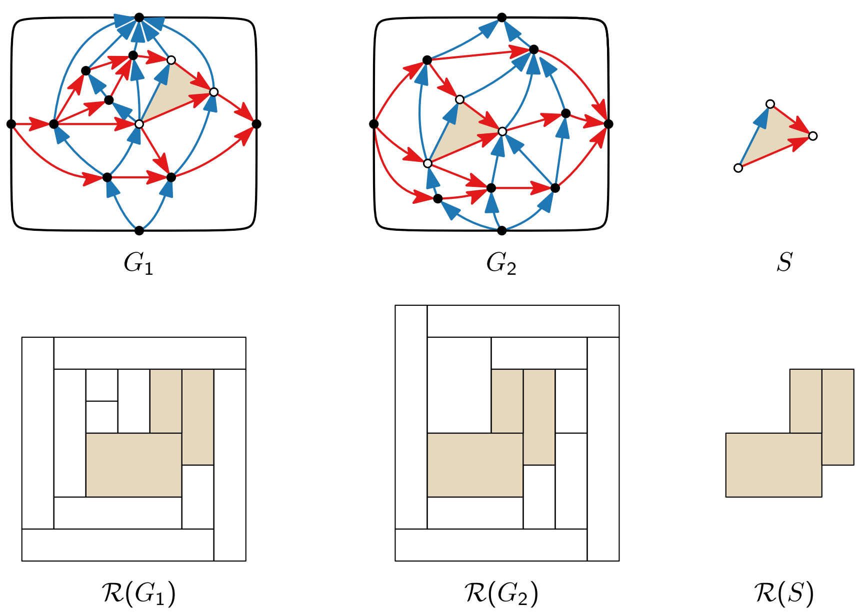


Fig. 8. The two PTP graphs G_1 and G_2 that share the subgraph S (indicated by the brown triangle) have the simultaneous representations $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$.

We show how to modify the LP `RecDual` to solve the simultaneous rectangular dual representation problem for $G_1, \ldots, G_k$. The idea is to generate one LP per graph and then, for each vertex shared between two graphs, to identify the respective variables. More precisely, for each $i \in \{1, \ldots, k\}$, we generate the LP `RepEx`$(G_i, (L_1, L_2)_i, 1)$ and let its variables have superscript i. We then merge the k LPs into a single LP. To ensure a simultaneous representation, for each $u \in V(H_{i,j})$, we set $x^i_{1,u} = x^j_{1,u}$, $x^i_{2,u} = x^j_{2,u}$, $y^i_{1,u} = y^j_{1,u}$, and $y^i_{2,u} = y^j_{2,u}$. The rectangles $\mathcal{R}_i(u)$ and $\mathcal{R}_j(u)$ in the constructed rectangular duals of G_i and G_j, respectively, will thus be identical. Overall, the result is an LP of size linear in the total number of vertices of $G_1, \dots, G_k$.

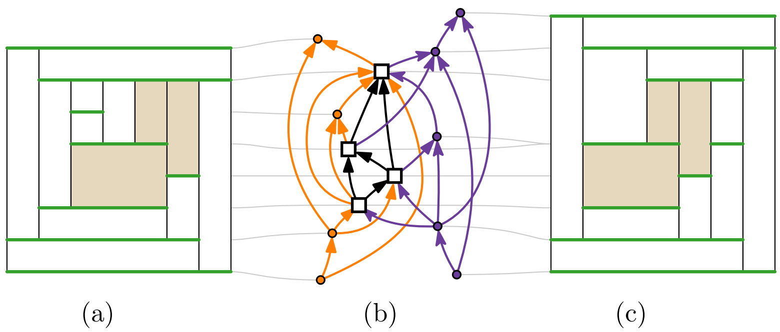


Fig. 9. A topological numbering of the joined duals $L^*_2(G_1)$ (orange) and $L^*_2(G_2)$ (purple) from Fig. 8 yields y-coordinates for the maximal horizontal segments of $\mathcal{R}(G_1)$ and $\mathcal{R}(G_2)$. The brown rectangles represent the joint part that must be identical in both representations.

The technique from above solves this LP in linear time, too, as follows. For $i \in \{1, \dots, k\}$, we identify sets of variables corresponding to maximal line segments in $\mathcal{R}(G_i)$. Then we identify further variables for the shared representations between pairs of graphs. In this way, we transform the LP into an auxiliary weighted directed graph X. Assuming that for $i \in \{1, \dots, k\}$, the graph G_i has a nonempty joint subgraph with at least one other subgraph G_j, the auxiliary graph X has exactly two weakly connected components; one for the vertical and one for the horizontal maximal line segments (see Fig. 9). The simultaneous representation extension problem has a solution if and only if there exist topological numberings for both components of X. This yields Theorem 2.

Theorem 2. *The simultaneous representation problem for rectangular duals with fixed regular edge labelings can be solved in linear time. For yes-instances, simultaneous rectangular duals can be constructed within the same time bound.*

5 Concluding Remarks

In this paper, we have formulated a system of difference constraints (a special kind of LP) that can handle slightly more general versions of the partial rectangular dual extension problem. Furthermore, the LP can also be used to solve the simultaneous rectangular dual representation problem for PTP graphs with given RELs. One can simply formulate an LP for each graph separately and then concatenate them into a single LP where the variables for shared vertices are merged. As far as we know, this is the first result concerning the simultaneous representation of contact representations. It would be interesting to see this approach applied to other contact representations.

The partial rectangular dual extension problem remains open for the case that no REL is specified. Eppstein et al. [14] gave algorithms that compute constrained area-universal rectangular duals and solved the extension problem for RELs. A partial rectangular dual induces a partial REL. Hence an extension of a partial rectangular dual $\mathcal{P}$ can be found by computing every extension of this partial REL and by testing for each whether it admits an extension of $\mathcal{P}$, using our linear-time algorithm. There can, however, be exponentially many extensions of a partial REL. Naturally, we are interested in a faster approach.

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