A polynomial-time approximation algorithm for a geometric dispersion problem

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Abstract

We consider the problem of placing a set of disks in a region containing obstacles such that no two disks intersect. We are given a bounding polygon P and a set $\mathcal{R}$ of possibly intersecting unit disks whose centers are in P. The task is to find a set $\mathcal{B}$ of m disks of maximum radius such that no disk in $\mathcal{B}$ intersects a disk in $\mathcal{B} \cup \mathcal{R}$, where m is the maximum number of unit disks that can be packed.

Baur and Fekete showed that the problem cannot be solved efficiently for radii that exceed 13/14, unless $P = NP$. In this paper we present a 2/3-approximation algorithm.

1 Introduction

The problem of packing objects into a bounded region is one of the classic problems in mathematics and theoretical computer science, see for example the monographs [7, 9] which are solely devoted to this problem, and the survey by Tóth [8].

In this paper we consider a problem related to packing disks into a polygonal region. As pointed out by Baur and Fekete in [1], even when the structure of the region and the objects is simple, only very little is known, see for example [4, 6]. We consider the following geometric dispersion problem:

Problem 1 (ApproxSize) *Given a bounding polygon P and a set $\mathcal{R}$ of, possibly intersecting, unit disks whose centers are in P, the aim is to pack m non-intersecting disks of maximum radius in P, where m is the maximal number of unit disks that can be packed in P.*

Note that we do not know the value of m a priori. In 1985 Hochbaum and Maas gave a PTAS for the problem of packing a maximal number of unit disks in a region in their pioneering work [5]. The problem is known to be NP-complete [3]. Even though the corresponding geometric dispersion problem looks very similar much stronger inapproximability results have been shown. Baur and Fekete [1] proved hardness results for a variety of geometric dispersion problems, and their results can be modified to our setting with a bit of effort. Specifically, they show that ApproxSize cannot be solved in polynomial time for disks of radius exceeding 13/14. Furthermore, for the case when the objects are squares, Baur and Fekete gave a 2/3-approximation algorithm. However, since a square is a simpler shape and easier to pack than a disk their approach cannot be generalized to disks. The main contribution of this paper is a polynomial time 2/3-approximation algorithm. Actually, we conjecture that 2/3 is indeed the largest value for which the problem can be solved, but so far we have been unable to prove it.

ApproxSize has applications in non-photorealistic rendering system, where 3D models are to be rendered in an oil painting style, as well as in random examinations of, e.g., soil ground.

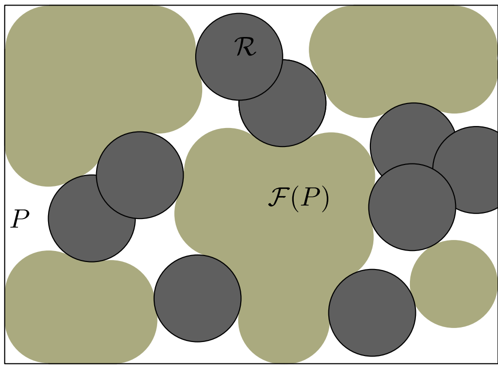

Figure 1: The freespace $\mathcal{F}(P)$ (light shaded) is the region that could be covered by a unit disk not intersecting any disk of $\mathcal{R}$.

2 The approximation algorithm

We will use the term r-disk to refer to a disk of radius r. The main idea of our approximation algorithm is described next, see also Algorithm 1. First compute the space $\mathcal{F}(P)$, denoted *freespace*, that could potentially be covered by a unit disk not intersecting any disk of $\mathcal{R}$. Then, apply the PTAS of Hochbaum and Maas [5] for the problem of packing unit disks in

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Algorithm 1: DISKPACKING

1. Compute the freespace $\mathcal{F}(P)$.
2. Use HM's algorithm [5] to compute a set $\mathcal{B}$ of at least $\frac{12}{13}m$ unit disks in $\mathcal{F}(P)$.
3. Introduce a metric d on the set $\mathcal{B}$ of unit disks.
4. Compute the nearest neighbor graph $G = (\mathcal{B}, E)$ with respect to d.
5. Find a sufficiently large matching in G.
6. **For** each matching pair $\{C, D\}$ of 1-disks **do**
7. Place three $\frac{2}{3}$-disks in $T_{2/3}(C, D)$.
8. **For** each unmatched unit disk D **do**
9. Place one $\frac{2}{3}$-disk in D.

$\mathcal{F}(P)$. If we set $\varepsilon = 1/13$ in the PTAS this yields a set $\mathcal{B}$ of at least $12/13 \cdot m$ unit disks and it requires $O(n^{625})$ time to compute. Here, n is the minimum number of unit squares that cover P.

Note that the approximation scheme by Hochbaum and Maas can be modified such that the algorithm is strongly polynomial with respect to the size of the input. If the number of disks that can be packed is not polynomial in the size of P and $\mathcal{R}$ then there must exist a huge empty square region within P. This can be "cut out" and packed almost optimally by using a naîve approach. The added error obtained is bounded by $O(1/\tilde{n}^2)$ where $\tilde{n}$ is the optimal number of disks that can be packed in the square. This step can be repeated until there are no more huge empty squares.

Let m' be the number of unit disks in the set $\mathcal{B}$. Starting from $\mathcal{B}$ we compute a set $\mathcal{B}_{2/3}$ of disks of radius $2/3$ that has cardinality at least $13/12 \cdot m'$ which in turn yields that $\mathcal{B}_{2/3}$ contains at least m disks. We obtain $\mathcal{B}_{2/3}$ by computing a sufficiently large matching in the nearest neighbor graph of $\mathcal{B}$. Then, we define a region for each matching pair such that one can insert three 2/3-disks in each region and all regions are pairwise disjoint. For each unmatched unit disk we insert one 2/3-disk, see Figure 2.

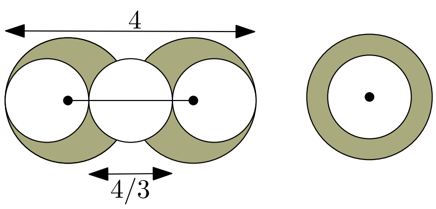

Figure 2: Packing three $\frac{2}{3}$-disks in a matching pair of unit disks (left) and one in a single disk (right).

In the next sections we describe each step of Algorithm 1 more detailed.

3 The freespace and a metric on it

We briefly recall the setting. We are given a set $\mathcal{R}$ of unit disks whose centers lie in a polygon P. The disks in $\mathcal{R}$ are allowed to intersect.

Definition 1 *The* freespace $\mathcal{F}(P)$ *is the union of all unit disks D in P such that $D \cap \bigcup_{D' \in \mathcal{R}} D' = \emptyset$.*

For an example of a freespace see Figure 1. Obviously, $\mathcal{F}(P)$ can be computed in $O(n^{625})$ time. From now on we will w.l.o.g. assume that $\mathcal{F}(P)$ consists of only one connected component, since each component can be handled separately. Next, we introduce a metric for a set of non-intersecting disks in $\mathcal{F}(P)$.

Definition 2 *Let C and D be two non-intersecting unit disks in $\mathcal{F}(P)$. There is a shortest movement of a unit disk from the position of C to the position of D that keeps entirely within $\mathcal{F}(P)$. The distance $d(C, D)$ is the length of the center-point curve $\tilde{c}(C, D)$ of this movement. We call the orbit that is induced by the transformation of the unit disk the* 1-transformation tunnel $T(C, D)$.

The curve $\tilde{c}(C, D)$ can consist of straight-line segments (the disk can be moved arbitrarily in the freespace without hitting any obstacles) and of arcs of radius 2 (the disk hits a disk $R \in \mathcal{R}$ on the boundary of the freespace), see Figure 3.

Next, we define a transformation for 2/3-disks. The transformation tunnels of this movement yield us the regions in which we will place the m 2/3-disks.

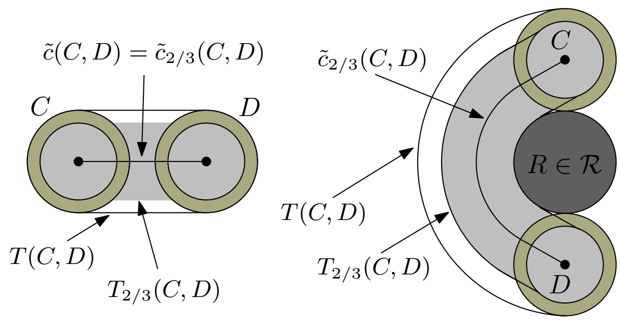

Figure 3: The minimum transformations of disks $\{C, D\}$ and $\{C', D'\}$ in $\mathcal{F}(P)$. Left: unrestricted case, right: a disk $R \in \mathcal{R}$ as obstacle.

Definition 3 *Let C and D be two non-intersecting unit disks in $\mathcal{F}(P)$. Let $C_{2/3}$ and $D_{2/3}$ be the $\frac{2}{3}$-disks centered at the centers of C and D, respectively. There is a shortest movement of a 2/3-disk from the position of $C_{2/3}$ to the position of $D_{2/3}$ that keeps entirely within $T(C, D)$. We call the orbit that is induced by the transformation of the 2/3-disk the* $\frac{2}{3}$-transformation tunnel $T_{2/3}(C, D)$ *and its center-point curve $\tilde{c}_{2/3}(C, D)$.*

4 The set $\mathcal{B}$ and its nearest neighbor graph

Now, the freespace $\mathcal{F}(P)$ is the region for which we compute a 12/13-approximation algorithm for the problem of packing the maximum number of unit disks. We do this by applying the PTAS of Hochbaum and Maas for $\varepsilon = 1/13$, with running time of $O(n^{625})$. Let the resulting set of unit disks be $\mathcal{B}$. By a post-processing step we can assume that $\mathcal{F}(P) \setminus \bigcup_{B \in \mathcal{B}} B$ does not offer enough space for another unit disk (inserting unit disks in a greedily manner until no more disks can be added). We need this to ensure that the nearest neighbor graph of $\mathcal{B}$ (w.r.t. d) is planar and of bounded degree. Using a similar argument as in the proof [2] showing that the nearest neighbor graph of a point set in the plane (w.r.t. the Euclidean metric) has degree at most 6, we can show that the degree of the nearest neighbor graph G of $\mathcal{B}$ is also bounded by 6. Furthermore, G is planar since no two edges in a nearest neighbor graph can intersect. Obviously, G can be computed in $O(n^{625})$ time.

From now on we will call a pair $\{C, D\} \subseteq \mathcal{B}$ a *nearest pair* if $\{C, D\}$ is an edge in G, i.e., either D is the nearest disk to C (in $\mathcal{B}$) or C is the nearest disk to D (in $\mathcal{B}$). For every nearest pair $\{C, D\}$ we define the region $\mathcal{A}(C, D)$ to be $C \cup D \cup T_{2/3}(C, D)$. As the nearest pair $\{C, D\}$ is a potential candidate to become a matching pair, we want to ensure that we can use $\mathcal{A}(C, D)$ to pack three 2/3-disks in it such that all these packed 2/3-disks are pairwise disjoint. Thus, we have to prove: (i) three 2/3-disks fit into $\mathcal{A}(C, D)$ and (ii) for any nearest pair $\{E, F\}$ where C, D, E and F are pairwise disjoint, $\mathcal{A}(C, D)$ does not intersect $\mathcal{A}(E, F)$. Note that we do not have to care whether, e.g., $\mathcal{A}(C, D)$ intersects $\mathcal{A}(C, E)$ because the matching will choose at most one pair out of $\{C, D\}$ and $\{C, E\}$. Clearly, three 2/3-disks fit into $\mathcal{A}(C, D)$ since C and D do not intersect. Thus, (i) is fulfilled but the second part (ii) requires much more work.

We split the proof into two parts. The first part shows that $T_{2/3}(C, D)$ is not intersected by any other disk than C and D. The second part shows that no two tunnels $T_{2/3}(C, D)$ and $T_{2/3}(E, F)$ can intersect.

We start with a technical lemma that will help to prove the first part. The notation $|p, q|$ will indicate the Euclidean distance between two points p and q.

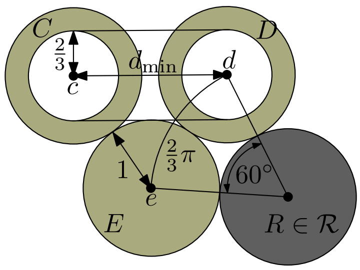

Figure 4: Illustration for Lemmas 1 and 2.

Lemma 1 *Let C and D be two unit disks in $\mathcal{F}(P)$ with centers c and d. If $|c, d|$ is less than $d_{\min} := \frac{2}{3} \cdot \sqrt{11}$, the center-point curve $\tilde{c}_{2/3}(C, D)$ is straight and each unit disk that does not intersect $C \cup D$ cannot intersect $T_{2/3}(C, D)$.*

Proof. We compute $d_{\min}$ as the minimum value of $|c, d|$ when a disk E, disjoint from C and D, intersects $T_{2/3}(C, D)$. Clearly, $d_{\min}$ is attained if E and $T_{2/3}(C, D)$ only intersect in a single point and furthermore, also E and C as well as E and D only intersect in a single point, see Figure 4. This means that $|c, e| = |d, e| = 2$, where e is the center of E. Moreover, the Euclidean distance between e and the straight-line segment cd is $1 + \frac{2}{3} = \frac{5}{3}$. By Pythagoras' theorem we calculate $d_{\min} = |c, d|$ to be $\frac{2}{3}\sqrt{11}$. □

We make the following observation:

Observation 1 *Let C and D be two unit disks in $\mathcal{F}(P)$ that are infinitesimally close to each other. Then $d(C, D) \leq \frac{2}{3}\pi$.*

Proof. For simplification we assume that C and D touch. The curve $\tilde{c}(C, D)$ attains its longest length if both C and D touch an obstacle disk that has to be overcome. In this case $\tilde{c}(C, D)$ describes an arc of radius 2 and 60°. Its length is $\frac{1}{6} \cdot 2 \cdot 2\pi = \frac{2}{3}\pi$. □

Lemma 2 *Let $\{C, D\} \subseteq \mathcal{B}$ be a nearest pair with the center-point curve $\tilde{c}(C, D)$. Then, no disk of $\mathcal{B} \cup \mathcal{R} \setminus \{C, D\}$ intersects $T_{2/3}(C, D)$.*

Proof. It immediately follows from Definitions 1 and 3 that neither $T(C, D)$ nor $T_{2/3}(C, D)$ are intersected by a disk in $\mathcal{R}$. Thus, it remains to prove that apart from C and D no disk in $\mathcal{B}$ intersects $T_{2/3}(C, D)$.

W.l.o.g. let C be the nearest disk to D (in $\mathcal{R}$). The proof is done by contradiction: assume that there is a disk $E \in \mathcal{B}$ that intersects $T_{2/3}(C, D)$.

First, we move a unit disk from the position of D on the center-point curve $\tilde{c}(C, D)$ to the first position in which it hits E, denote the disk in this position by $\overline{D}$, see Figure 4 where $D = \overline{D}$ holds. According to Observation 1, we know that $\frac{2}{3}\pi \approx 2.09$ is an upper bound on $d(\overline{D}, E)$. As C is closer to $\overline{D}$ than E is, the center of C has to lie within a disk of radius $\frac{2}{3}\pi$ with center $\overline{d}$. However then, according to Lemma 1, $\tilde{c}_{2/3}(C, D)$ must be a straight-line because $\frac{2}{3}\pi < \frac{2}{3}\sqrt{11} \approx 2.21$ holds. Thus, also according to Lemma 1, E cannot intersect $T_{2/3}(C, D)$ which yields the contradiction. □

Lemma 2 settles that no other disks apart from C and D intersect $T_{2/3}(C, D)$. We still have to show that any two $\frac{2}{3}$-transformation tunnels $T_{2/3}(C, D)$ and $T_{2/3}(E, F)$ do not intersect.

Lemma 3 *Let $\{C, D\}, \{E, F\} \subseteq \mathcal{B}$ be two nearest pairs such that C, D, E and F are pairwise disjoint. Then $T_{2/3}(C, D) \cap T_{2/3}(E, F) = \emptyset$.*

Proof. The proof is by contradiction again. Obviously, we can w.l.o.g assume that $T_{2/3}(C, D)$ and $T_{2/3}(E, F)$ only intersect in a single point p, see Figure 5. The basic idea of the proof is to show that then, $\{C, D\}$ and $\{E, F\}$ cannot be nearest pairs at the same time. Note that at least one of the disks $\{C, D, E, F\}$ intersects the unit disk P with center p: otherwise there would be another disk in $\mathcal{B}$ located in the space between C, D, E and F which would immediately contradict $\{C, D\}$ as well as $\{E, F\}$ being nearest pairs.

W.l.o.g. let C be a disk that intersects P. We can show that $d(C, E) < d(E, F)$ holds, i.e. that F is not the nearest neighbor of E, thus E has to be the nearest neighbor of F in order for $\{E, F\}$ to be a nearest pair. Under this assumption we can then show that $d(C, E) < d(C, D)$ and $d(D, F) < d(C, D)$ holds. However, this contradicts $\{C, D\}$ being a nearest pair because neither D is the nearest neighbor of C nor C is the nearest neighbor of D. □

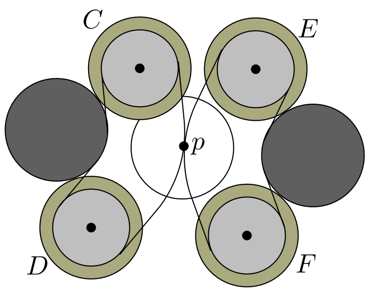

Figure 5: Illustration for the proof of Lemma 3. If $T_{2/3}(C, D)$ and $T_{2/3}(E, F)$ intersect, not both pairs $\{C, D\}$ and $\{E, F\}$ can be nearest pairs.

5 Placing the 2/3-disks

After computing $\mathcal{B}$ and the nearest neighbor graph $G = (\mathcal{B}, E)$, we compute a matching in G. Let $m' = |\mathcal{B}|$ be the number of unit disks in $\mathcal{B}$. Recall that G is planar and of bounded degree 6. We show that we can find a matching in which the number of matched disks is at least $1/6 \cdot m'$. Observe that G can consist of more than one connected component. We look at each connected component separately. Let C be a connected component and c be the number of disks that it contains. Clearly, C contains a spanning tree of bounded degree 6. It is easy to see that there is a matching in C that matches at least $1/6 \cdot c$ disks. Doing this for each component yields a matching in G that contains at least $1/6 \cdot m'$ matched disks.

According to Lemmas 2 and 3 we can pack three 2/3-disks in $\mathcal{A}(C, D)$ for every matched pair $\{C, D\}$ such that the set of these 2/3-disks is pairwise disjoint. For each of the remaining unmatched disks D we pack one 2/3-disk in D. Lemma 2 ensures that these disks are disjoint to the disks that have been packed for the matched pairs. Let $\mathcal{B}_{2/3}$ be the set of all disks packed as above. Its cardinality is at least $\frac{1}{6} \cdot \frac{3}{2} \cdot m' + \frac{5}{6} \cdot m' = \frac{13}{12} \cdot m'$. Since the cardinality of $\mathcal{B}$ is at least $\frac{12}{13} \cdot m$, the set $\mathcal{B}_{2/3}$ contains at least m 2/3-disks and we can conclude with the following theorem:

Theorem 4 *Algorithm 1 is a 2/3-approximation for the* ApproxSize *problem. Its running time is $O(n^{625})$.*

6 Conclusion

Naturally, our result is purely of theoretic interest. The bottleneck for the running time is the application of Hochbaum and Maas' PTAS. To obtain an algorithm with better running time, it seems to be unavoidable to use a completely different approach. For future work it would also be desirable to narrow the gap between the known approximation (2/3) and the inapproximability result (13/14). We conjecture that, unless $P = NP$, the lower bound of 2/3 is indeed tight.

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