Constructing Optimal Highways*

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Abstract

For two points p and q in the plane, a straight line h, called a highway, and a real $v > 1$, we define the *travel time* (also known as the *city distance*) from p and q to be the time needed to traverse a quickest path from p to q, where the distance is measured with speed v on h and with speed 1 in the underlying metric elsewhere.

Given a set S of n points in the plane and a highway speed v, we consider the problem of finding a *highway* that minimizes the maximum travel time over all pairs of points in S. If the orientation of the highway is fixed, the optimal highway can be computed in linear time, both for the L_1- and the Euclidean metric as the underlying metric. If arbitrary orientations are allowed, then the optimal highway can be computed in $O(n^2 \log n)$ time. We also consider the problem of computing an optimal pair of highways, one being horizontal, one vertical.

Keywords: Geometric facility location, min-max-min problem, city metric, time metric, optimal highways.

1 Introduction

Facility location is a branch of operations research that is concerned with placing facilities such that certain costs are minimized. For example, minimizing distances of facilities to customers helps to keep transportation costs low. Facility-location problems can be defined in unweighted or in weighted graphs where weights may or may not fulfill the triangle inequality. In this article, however, we focus on *geometric* facility location, where customers and facilities are geometric objects (such as points) in a Euclidean space. One of the classic facility-location problems is the so-called Fermat-Weber problem, where the locations of customers are given as points in the plane and the task is to locate a facility (a point) that minimizes the sum of the distances to the customers. The solution of this seemingly harmless problem, the *geometric median*, turns out to be difficult to compute exactly: in general the geometric median cannot be expressed by any formula that involves only arithmetic operations and k-th roots [12]. Luckily, there are fast approximations [8].

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The Fermat-Weber problem belongs to the class of (geometric) *min-sum* facility-location problems. Another class of facility-location problems are *min-max* problems, where the task is to place a facility such that the maximum cost incurred by any customer is minimized. For example, if the customers are points in the plane and their cost of using a facility is the Euclidean distance to the facility, then the center of the smallest enclosing circle is the location of the optimal facility, minimizing the maximum distance to the customers.

Recently, Cardinal and Langerman [10] introduced the class of *min-max-min* facility-location problems, where customers have the choice of either using or not using the facility, and their cost is the minimum cost of these two options. Transport facility location problems are typical min-max-min problems: a new railway line or highway will not be used by a customer if using it does not improve the travel time compared to existing means of transportation. Cardinal and Langerman consider three such min-max-min problems, among them the following: Given a set P of *pairs* of points, find the highway (a straight line) that minimizes the maximum travel time over all pairs in P. Here, the travel time of a pair of points (a, b) is the time taken to travel from a to b, assuming constant speed anywhere in the plane, infinite speed along the highway, and travel to and from the highway parallel to the y-axis. Cardinal and Langerman give a randomized algorithm that computes the optimal highway and whose expected running time is linear in the number of pairs. Figure 1b shows an example of an optimal highway in the case that P is the set of all pairs among the given points.

Previous work considering highways (also called *transportation networks* or *roads*) have focused on how to compute quickest paths among the customers (points) or their Voronoi diagrams under the metric induced by *given* highways. Abellanas et al. [1] started work in this area, and discussed the Voronoi diagram of a point set given a horizontal highway under the L_1-metric. They later also considered the problem under the Euclidean metric and studied shortest paths [2]. Aichholzer et al. [3] introduced the city metric induced by the L_1-metric and a highway network that consists of a number of axis-parallel line segments. They gave an efficient algorithm for constructing the Voronoi diagram and a quickest-path map for a set of points given the city metric. The running time of their algorithms was recently improved by Görke et al. [15] and Bae et al. [6]. Bae et al. [5] presented algorithms that compute the Voronoi diagram and shortest paths, using the Euclidean metric and more general highway networks whose segments can have arbitrary orientation and speed. They recently extended their approach to more general metrics including asymmetric convex distance functions [4].

In this paper, we, like Cardinal and Langerman, consider the problem of finding an optimal highway (a straight line) for a given set S of n points in the plane. We wish to place a highway such that the maximum travel time over *all* pairs of points (that is, the *travel-time diameter* of the point set) is minimized. Since there is a quadratic number of pairs, Cardinal and Langerman's algorithm takes expected quadratic time in this case. We show how to make use of the coherence between the pairs of points to get deterministic near-linear-time algorithms.

In a follow-up paper, Cardinal et al. [9] have recently also considered the version where the maximum travel time is taken over all point pairs, albeit for a different problem, the so-called *moving walkway problem*. A moving walkway is a straight-line segment $\overline{ab}$ that can only be entered and left at its endpoints. Passengers move with a given fixed speed $v > 1$ on the walkway (in either direction) and with speed 1 in the rest of the plane. Thus the travel time between two points is defined as the minimum over their Euclidean distance and the time needed via the walkway in either direction. The results of Cardinal et al. are as follows. They first consider the one-dimensional version of the problem. They show that the optimal moving walkway for a set of n points on the real line can be computed in linear (worst-case) time, but computing their travel-time diameter takes $\Theta(n \log n)$ time. In the plane, they can decide in $O(n \log n)$ time whether the travel-time diameter induced by a given walkway is bounded from above by a given value $t > 0$, and they can find the optimal horizontal walkway in $O(n \log n)$ expected time. Finally, they show how to compute in $O((n \log n) \cdot v/\varepsilon)$ expected time a $(1 + \varepsilon)$-approximation of the optimal arbitrarily oriented walkway.

In the facility-location problem that we consider we also assume that we can travel anywhere in the plane with speed 1, and that we can travel along a highway with a given speed $v > 1$. We investigate various versions of the problem. We consider both highways with infinite and finite speed. For the underlying metric we consider both the L_1-metric (as in the city Voronoi diagram) and the L_2-metric. In all cases, we show how to find the optimal highway with a *given* orientation (that is, the optimal horizontal highway). In the case of the Euclidean metric, we also consider how to find the optimal highway if we are free to choose the orientation. (We note that choosing the orientation does not make

Highway speed	Metric	Fixed orientation	Arbitrary orientation
Infinite	L_1	$O(n)$	—
	L_2	$O(n)$	$O(n \log n)$
Finite	L_1	$O(n)$	—
	L_2	$O(n)$	exact: $O(n^2 \log n)$ approx.: $O(n \log n)$

Table 1: Overview of our results for one highway.

Highway speed	Solution	Running time
Infinite	exact	$O(n \log n)$
Finite	exact	$O(n^4 \alpha(n))$
	$(1+\sqrt{2})$-approx.	$O(n \log n)$
	$(2+\varepsilon)$-approx.	$O(n \ \log(1/\varepsilon)\alpha(n)\log n)$
	$(1+\varepsilon)$-approx.	$O(n^2 \log(1/\varepsilon)\alpha(n)\log n)$

Table 2: Overview of our results for the highway cross (under the L_1-metric).

sense for the L_1-metric). Table 1 summarizes our results for all versions of the problem.

We then consider the problem of placing a *highway cross*, that is, a pair of a horizontal and a vertical highway. We can determine the optimal axis-aligned highway cross with infinite speed in $O(n \log n)$ time, see Section 5. For constant speed the problem becomes considerably harder, even under the L_1-metric. We give an exact $O(n^4\alpha(n))$-time algorithm based on computing minima of upper envelopes. We also consider approximate solutions. All our results are summarized in Table 2.

Throughout the paper we assume that the input point set S contains at least three points (if $|S| < 3$, it is trivial to find an optimal highway).

2 The optimal highway for infinite speed

As a warm-up exercise, let us consider the problem of finding the optimal placement of a horizontal highway, assuming the highway speed is infinite. In the sequel, a *strip* will always denote the (closed) region between two parallel lines.

Theorem 1 *Given n points in the plane, the middle line of the smallest enclosing* horizontal *strip is an optimal* horizontal *highway of infinite speed. It can be computed in linear time.*

The easy proof is left to the reader. What is interesting is that the optimal highway corresponds to a smallest enclosing figure—we will see this theme repeatedly in the following, see Fig. 1. Note that the result holds in any L_p-metric, as all travel to and from the highway is parallel to the y-axis.

The theorem generalizes to highways of arbitrary orientation in the Euclidean metric:

Theorem 2 *Given n points in the plane, the middle line of the smallest enclosing strip is an optimal highway of infinite speed. It can be computed in $O(n \log n)$ time.*

The algorithm used here is the rotating calipers algorithm [17]. After computing the convex hull of the point set, it runs in linear time.

(a) horizontal **(b)** arbitrary orientation

Fig. 1: Optimal infinite-speed highways (solid lines) and corresponding enclosing figures (shaded).

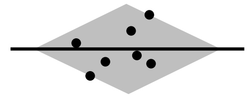
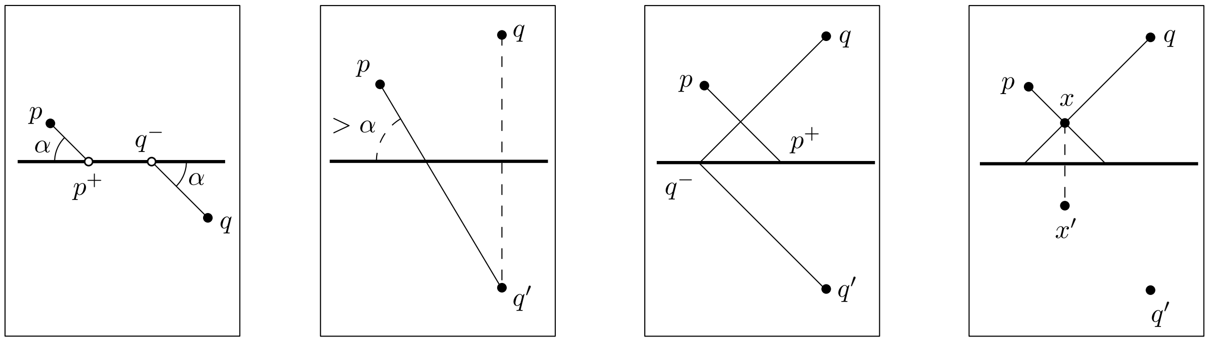


Fig. 2: Optimal constant-speed horizontal highway (solid lines) and corresponding enclosing figure (shaded).



(a) Quickest p–q path. **(b)** The line pq' forms an angle greater α with the x-axis. **(c)** The η-distance counts the time on the highway *negative*. **(d)** The path from p to q self-intersects in x.

Fig. 3: Optimal highway under the L_2-metric.

3 The optimal horizontal highway in the L_1-metric

In this section we consider horizontal highways of finite speed $v > 1$ under the L_1-metric. It will turn out that again the optimal highway corresponds to a smallest enclosing figure, namely a rhombus, that is, an equilateral quadrilateral. Let $\rho \geqslant 1$ be a real. We say that a *ρ-rhombus* is a rhombus of aspect ratio ρ, meaning that the ratio of the lengths of the longer over the shorter diagonal is ρ.

Theorem 3 *Given a set S of n points in the plane, the horizontal axis of symmetry of the smallest enclosing axis-aligned v-rhombus is an optimal horizontal highway for the L_1-metric and highway speed v. It can be computed in $O(n)$ time.*

Proof. For a pair of points $p, q \in S$, let $d_{\text{low}}(p,q) := |x_p - x_q|/v + |y_p - y_q|$. Clearly, $d_{\text{low}}(p,q)$ is a lower bound for the travel time diameter of S for any horizontal highway, and therefore $\delta := \max_{p,q\in S} d_{\text{low}}(p,q)$ is also a lower bound. We show that in fact this bound can be obtained, resulting in an optimal highway.

We observe that the point set can be enclosed in a rhombus with horizontal diagonal δv and vertical diagonal δ. For an example with $v = 2$, see Fig. 2. If we place a horizontal highway along the horizontal diagonal of this rhombus, then any point in the rhombus has travel time at most $\delta/2$ to the center of the rhombus. This implies that the travel-time diameter is at most δ.

The computation boils down to computing minimum and maximum y-axis intercepts among all lines through points of S of slope $1/v$ and $-1/v$. ◻

4 The optimal highway in the Euclidean Metric

In this section we consider optimal highways of finite speed $v > 1$ under the Euclidean metric.

Theorem 4 *Given a set S of n points in the plane, an optimal horizontal speed-v highway under the Euclidean metric can be computed in $O(n)$ time.*

Proof. The quickest path (that is, the path with shortest travel-time) between two points p and q has one of two forms [2]: It is either the segment pq; or a path consisting of three segments pp^+, p^+q^-, q^-q, where p^+ and q^- are points on the highway, and the lines pp^+ and qq^- form an angle of $\alpha = \arccos 1/v$ with the highway, see Fig. 3a.

Now let us define a norm $\eta(x, y)$ on $\mathbb{R}^2$ as

$$\eta(x, y) = |x| \cos\alpha + |y| \sin\alpha.$$

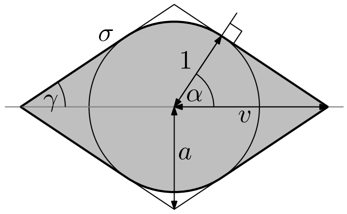


Fig. 4: Unit disk in the travel-time metric (shaded) and smallest enclosing $(\tan\alpha)$-rhombus.

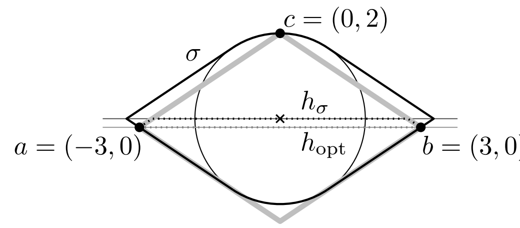


Fig. 5: The smallest shape σ enclosing $\{a, b, c\}$ does not give the optimal horizontal highway.

Since $0 < \alpha < \pi/2$, we have $\eta(x, y) > 0$ unless $(x, y) = (0, 0)$, and η is indeed a norm.

Let p and q be two points such that the highway lies in between p and q and such that the shortest path between p and q makes use of the highway. Then the travel time from p to q is $\eta(q - p)$. Indeed, let $(x, y) = q - p$, and assume $x, y \geqslant 0$. Then the travel time from p to q is $ay + (x - by)/v$, where $a = 1/\sin\alpha$ and $b = 1/\tan\alpha = a/v$. Using $1 - 1/v^2 = 1/a^2$, we get $ay + (x - by)/v = \eta(x, y)$.

When the highway cannot be used because the line pq forms an angle larger than α with the highway, then the travel time is simply the Euclidean distance $d(p, q)$, and $\eta(q - p)$ is an underestimate.

The unit circle under the norm η is a $(\sin\alpha/\cos\alpha)$-rhombus, that is, a $(\tan\alpha)$-rhombus. We can find the smallest such rhombus enclosing a given set S of n points in linear time. This means we determine the smallest factor $\delta > 0$ such that the S fits in the rhombus R with corners $(0, \delta a)$, $(0, -\delta a)$, $(\delta v, 0)$, and $(-\delta v, 0)$ (after translating the point set).

We claim that the x-axis is now an optimal highway. We already know that there is a pair of points whose η-distance is 2δ, so this is a lower bound on the diameter. We now show that for any pair of points, either their travel time (with respect to the highway at $y = 0$) is at most 2δ, or they cannot use *any* horizontal highway.

For any two points p, q in the rhombus R, we have $\eta(q - p) \leqslant 2\delta$. This means that if the highway lies in between the points, then we are already done. So assume that both p and q lie above the highway and that they *can* use a horizontal highway. Let q' be the reflection of q around the highway. Since R is symmetric with respect to $y = 0$, q' is also in R, and if p, q' can use a horizontal highway, then their travel-time distance is at most 2δ, implying that the travel time from p to q is also at most 2δ.

It remains to consider the case that p, q' cannot use a horizontal highway. This means that the line pq' forms an angle larger than α with the x-axis, see Fig. 3b. Note that $\eta(q' - p)$ still has a geometric meaning: There is a path from p to the highway, then *backwards* along the highway, then straight to q, see Fig. 3c.

The η-distance measures the whole travel time, but counting the time on the highway *negative*. Reflecting the last segment of this path back around the highway, we obtain a path from p to q with travel time $\eta(q' - p)$, still counting time spent on the highway negative. But now observe that this path self-intersects in a point x, see Fig. 3d. Let x' be the reflection of x. Then $\eta(q' - p) = d(p, x) + \eta(x' - x) + d(x', q') = d(p, x) + \eta(x' - x) + d(x, q) \geqslant d(p, x) + d(x, q) \geqslant d(p, q)$ (using $\eta \geqslant 0$). It follows $d(p, q) \leqslant \eta(q' - p) \leqslant 2\delta$. ◻

Again we found an optimal highway by computing a minimal enclosing shape. Interestingly, the shape to be minimized is not the unit circle under the travel-time metric. If the highway is the x-axis then the unit circle in the travel-time metric is the convex hull σ of the points $(-v, 0)$ and $(v, 0)$ and the Euclidean unit circle centered at the origin, see the shaded region in Fig. 4. Finding the smallest copy of σ enclosing S does not always give the optimal horizontal highway: for highway speed $v = 3/2$ the set $S = \{a, b, c\}$ depicted in Fig. 5 has travel-time diameter 4 (using highway h_{opt}), but the highway h_σ induced by σ causes the pair $\{a, b\}$ to have travel-time distance strictly greater than 4. The enclosing figure we are minimizing instead is the "rhombus approximation" of σ, see Figs. 4 and 5.

The very simple linear-time algorithm above results in a horizontal highway that minimizes the travel-time diameter of the point set, but it does not actually tell us what the travel-time diameter is. Surprisingly, it is not possible to compute the travel-time diameter within the same time bound, as the following theorem shows. Recently, Cardinal et al. [9] made a similar observation concerning the computation of the optimal walkway versus the computation of the travel-time diameter it induces. Our lemma is based on Ben-Or's algebraic computation tree model [7], which apart from the four basic

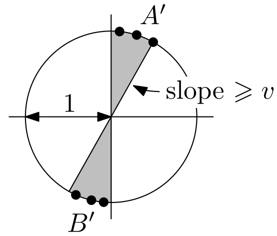


Fig. 6: Lower-bound construction for computing the travel-time diameter.

arithmetic calculations explicitly allows the taking of square roots.

Theorem 5 *In the algebraic computation-tree model, computing the travel-time diameter for a set of n points and a given highway takes $\Omega(n \log n)$ time.*

Proof. The *set disjointness problem* asks the following: given two sets A and B of n positive real numbers, is $A \cap B = \emptyset$? This problem has a lower bound of $\Omega(n \log n)$ in the algebraic computation tree model [7, Example 3]. We show how to transform this problem in linear time into a decision instance of the diameter problem.

Our instance consists of a set A' of n points and a set B' of n points, computed from A and B. All points lie on the unit circle. We first multiply all numbers in A and B by a factor $\lambda > 0$ so that they are close to zero (for instance by setting $\lambda = 1/(2v \cdot \max A \cup B)$). Then we create the point $(a, \sqrt{1-a^2})$ for each a in A and the point $(-b, -\sqrt{1-b^2})$ for each b in B, see Fig. 6. Note that since the points are close to the y-axis, no horizontal highway can be used to speed up the connection between A' and B', and so the diameter of the set is simply the Euclidean diameter. Since the points in $A' \cup B'$ lie on the unit circle, each pair of points has distance at most 2, with equality only for diametral pairs. It follows that the diameter of $A' \cup B'$ is 2 if and only if A and B contain a common number.

Given an algebraic computation tree T' that decides whether given sets A' and B' of n points have travel-time diameter at most two, we can thus construct a new algebraic computation tree T of cost $C(T) = C(T') + O(n)$ to solve the set disjointness problem. Ben-Or's lower bound $C(T) = \Omega(n \log n)$ therefore implies $C(T') = \Omega(n \log n)$. ◻

Theorem 6 *Given a set S of n points in the plane, the optimal highway with speed v can be found in $O(n^2 \log n)$ time on a Real RAM with the ability to compute trigonometric functions.*

Proof. First we compute the convex hull C of the point set S. Then we use the rotating-calipers algorithm [17] to compute the function $w : [0, \pi) \to \mathbb{R}^+$ that maps an angle ϕ to the width of the smallest strip that contains C and makes angle ϕ with the positive x-axis. The function w consists of at most n pieces, each of which is a trigonometric function that can be computed explicitly in constant time. Let $\gamma = \pi/2 - \alpha$ be the angle formed by the main diagonal and the sides of a $(\tan \alpha)$-rhombus (where $\cos \alpha = 1/v$), see Fig. 4. Then the function $u(\phi) = \max\{w(\phi - \gamma \bmod \pi), w(\phi + \gamma \bmod \pi)\}$ maps ϕ to the width of a smallest $(\tan \alpha)$-rhombus that contains C and whose main diagonal forms an angle of ϕ with the positive x-axis. In Fig. 7 we depict a set S of four points and a diagram with polygonal approximations of the functions $w(\phi - \gamma \bmod \pi)$, $w(\phi + \gamma \bmod \pi)$ (both thin black), and u (bold black) for $v = 2$. As we saw above in the proof of Theorem 4, the main diagonal of this rhombus is an optimal highway for this orientation, and the travel-time diameter $z(\phi)$ for this highway is

$$z(\phi) = \max\{u(\phi), \max_{p,q} d(p, q)\},$$

where $d(p, q)$ is the Euclidean distance of p and q and the maximum is taken over all pairs $\{p, q\}$ in S that cannot use any highway with orientation ϕ. A pair $\{p, q\}$ cannot use any highway with orientation ϕ if the line through p and q makes an angle larger than α with the highway orientation. We define now, for every pair $\{p, q\}$ in S, a function $f_{pq} : [0, \pi) \to \mathbb{R}_0^+$ with $f_{pq}(\phi) = d(p, q)$ if the angle between the line through p and q and the x-axis is between $\phi + \pi/2 - \alpha$ and $\phi + \pi/2 + \alpha$ (both modulo π), and $f_{pq}(\phi) = 0$ otherwise. (See Fig. 7 for examples.) We can then rewrite the optimal travel-time diameter for orientation ϕ as

$$z(\phi) = \max\{u(\phi), \max_{p,q \in S} f_{pq}(\phi)\}.$$

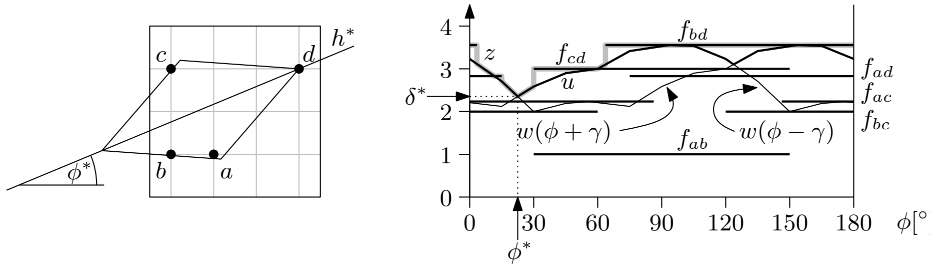


Fig. 7: The minimum of the upper envelope of u and of all functions of type f_{pq} yields the optimum speed-v highway.

The graph of z is the upper envelope of $\binom{n}{2}$ horizontal segments and of the graph of u (which has $O(n)$ breakpoints), see the bold gray polygonal chain in Fig. 7. Thus the graph of z can be computed in time $O(n^2 \log n)$ by a simple sweep, and we find an optimal orientation ϕ^* by picking a lowest point on this graph. The optimal highway h^* then goes through the main diagonal of the smallest S-enclosing $(\tan\alpha)$-rhombus with orientation ϕ^*. ◻

The function u used in the proof here can be computed in $O(n \log n)$ time, and within the same time bound we can pick an orientation ϕ for which u is minimal. This means that $u(\phi)$ is a lower bound for the travel-time diameter of *any* highway for S. Let R be the smallest enclosing $(\tan\alpha)$-rhombus with orientation ϕ. Then any point in R has travel-time distance at most $u(\phi)/(2\sin\alpha)$ from the center of R, see Fig. 4. This implies that the highway h_{app} through the main diagonal of R has a travel-time diameter of at most $u(\phi)/\sin\alpha$, which is within a factor of $1/\sin\alpha = \sqrt{v^2/(v^2-1)}$ from optimal. On the other hand, any highway yields at least the same (factor-v) approximation as building no highway at all. It is easy to see that the maximum of the function $\min\{v, \sqrt{v^2/(v^2-1)}\}$ for $v \geqslant 1$ is attained at $v = \sqrt{2}$ and has a value of $\sqrt{2}$. Thus h_{app} is in fact at least a $\sqrt{2}$-approximation of the optimal highway.

Theorem 7 *Given a set S of n points in the plane, a speed-v highway with travel-time diameter within a factor of $\min\{v, \sqrt{v^2/(v^2-1)}\} \leqslant \sqrt{2}$ from optimal can be found in $O(n \log n)$ time.*

The approximation factor of this algorithm depends on v, and since

$$\sqrt{\frac{v^2}{v^2-1}} = \sqrt{1 + \frac{1}{v^2-1}} \leqslant 1 + \frac{1}{2(v^2-1)},$$

the factor tends to 1 with growing v very quickly. For instance, for speeds $v = 2$, 3, and 10, the factors are at most 1.16, 1.06, and 1.005, respectively.

5 The optimal axis-aligned highway cross for infinite speed

Now we consider the problem of placing more than one highway. Observe that multiple parallel highways with the same speed do not reduce the maximum travel time because the quickest path using several highways can be simulated with only one highway. Instead we investigate highway crosses, that is, pairs of highways that intersect perpendicularly. We give algorithms for computing the optimal *axis-aligned* highway cross.

Definition 1 *An* enclosing cross *for a point set S is the union of a horizontal and a vertical strip of equal width containing S.*

Lemma 1 *The travel-time diameter of the optimal axis-aligned highway cross with infinite speed equals the width of the smallest enclosing cross.*

Proof. Let δ be the travel-time diameter, and let δ' be the width of a smallest enclosing cross C.

We first show that $\delta' \leqslant \delta$: Let h_1, h_2 be a pair of optimal highways. We assign each point in S to its closest highway so that S is partitioned into two subsets: one consisting of points closer to the horizontal highway and the other consisting of points closer to the vertical highway. We put around each highway the narrowest strip containing all the points assigned to the highway. Then both strips have width at

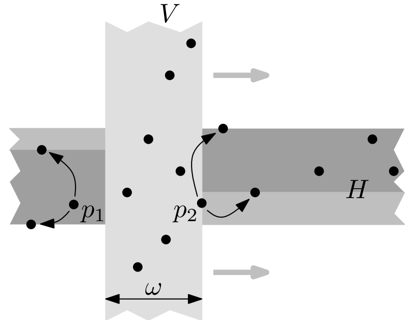


Fig. 8: Deciding whether there is an enclosing cross of width ω.

most δ, otherwise there are two points in the wider strip whose travel-time distance is larger than δ. Therefore we can obtain an enclosing cross of width δ by widening each strip until its width becomes δ. Since δ' was minimal, we have $\delta' \leqslant \delta$.

It remains to show $\delta \leqslant \delta'$: We place highways in the middle of each strip of C. This results in a pair of highways with travel-time diameter at most δ'. Since δ is optimal, we have $\delta \leqslant \delta'$. ◻

Note that once again the optimal facility corresponds to a minimal enclosing shape. This shape can be computed efficiently.

Theorem 8 *Given n points in the plane, the optimal axis-aligned highway cross for infinite speed corresponds to the smallest enclosing strip cross. It can be computed in $O(n \log n)$ time.*

Proof. The characterization follows from Lemma 1. The smallest enclosing cross of a set S of n points can be found as follows.

1. We presort the points by their x- and by their y-coordinates. For each point p, we precompute the following information: a highest and a lowest point to the left of p, and a highest and a lowest point to the right of p, see the points p_1 and p_2 and their pointers in Fig. 8. This can be done by traversing the x-sorted list of points, once from left to right and once from right to left.

2. For a given width $\omega > 0$, we can decide in linear time whether an enclosing cross of width ω exists. If this is the case, the enclosing cross can be found within the same time. Our decision algorithm is as follows. We slide a vertical strip V of width ω across the point set from left to right, see Fig. 8. We maintain a horizontal strip H of smallest width containing all the points not in V. For each point entering V from the right or leaving V from the left, we update H accordingly. This can be done in constant time per point using the precomputed information. If the width of H ever becomes ω or less, we answer "yes" and report an enclosing cross. Otherwise, we answer "no".

3. The width of the smallest enclosing cross is in the list of numbers $L = L_x \cup L_y$, where $L_x = \{x_j - x_i \mid 1 \leqslant i < j \leqslant n\}$ and $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ is the sorted sequence of x-coordinates. The list L_y is defined analogously based on the sorted sequence of y-coordinates.

4. Consider the matrix A with $A[i, j] = x_j - x_{n-i+1}$. The rows and columns of A are sorted in ascending order. Using the technique of Frederickson and Johnson [14], we can determine the k-th element of such a *sorted matrix* in $O(n)$ time without constructing A explicitly. This gives us a way to do binary search on L_x. This search consists of $O(\log n)$ steps, each of which first invokes the algorithm of Frederickson and Johnson to find the median of the remaining elements in L and then calls the decision algorithm of stage 2. Thus, the total runtime is $O(n \log n)$. Likewise we can search for the smallest value in L_y for which an enclosing cross exists. Finally we return the minimum of the two values. Notice that the algorithm in stage 2 computes not only the width of the smallest enclosing cross but also the cross itself. ◻

6 The optimal axis-aligned highway cross for finite speed

In this section we consider the optimal axis-aligned highway cross for finite speed. As underlying metric we use the L_1-metric. For this variant of the problem we have been unable to characterize the optimal

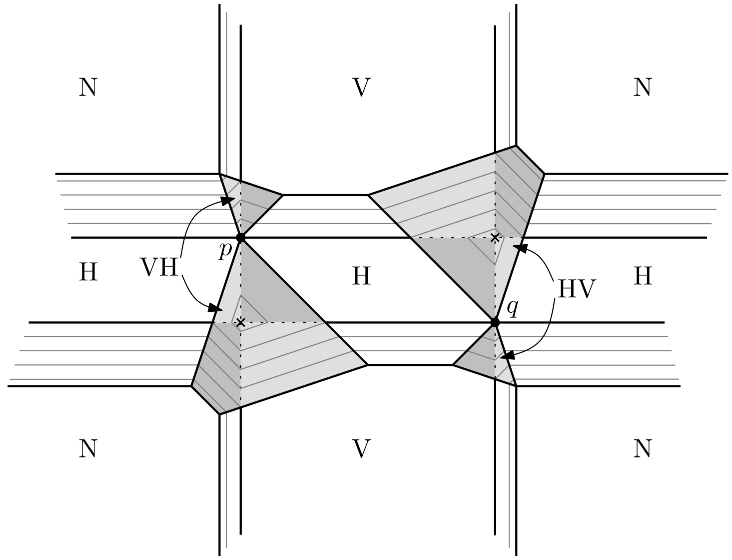


Fig. 9: Travel-time distance between two points p and q for an axis-aligned highway cross centered at (x, y). The graph of this function has 31 faces of 15 different orientations. The two points marked $\times$ are the lowest, that is, those where the corresponding highway crosses minimize the travel time from p to q. The thin dark gray lines are contour lines. If a highway cross is centered in a V- or H-region, the vertical and horizontal highway, respectively, is used by a quickest p–q path. In the VH- and HV-region both highways are used in the corresponding order. In the N-regions no highway is used.

solution by a smallest enclosing shape.

Theorem 9 *The optimal axis-aligned speed-v highway cross under the L_1-metric can be computed in $O(n^4\alpha(n))$ time.*

Proof. The optimal solution corresponds to the lowest point on the upper envelope of the pairwise distance functions, see Fig. 9. Since these functions are piecewise linear and of constant complexity, their upper envelope can be computed in $O(n^4\alpha(n))$ time [13]. ◻

6.1 The decision problem

We now present an algorithm that decides for a given $\delta > 0$ whether there is an axis-aligned speed-v highway cross such that the resulting travel-time diameter is at most δ. This will be used as a subroutine for finding approximations of the optimal highway cross, see Section 7.

Theorem 10 *Given a set P of m pairs of points in the plane, a speed $v > 1$, and a parameter $\delta > 0$, we can decide in time $O(m\alpha(m)\log m)$ whether there is an axis-aligned highway cross of speed v such that the travel-time distance of all pairs in P is at most δ. If the answer is positive, we can compute such a highway cross within the same time bound.*

Proof. For points $\sigma, p, q \in \mathbb{R}^2$, let $d_\sigma(p, q)$ denote the travel-time distance between p and q, assuming an axis-aligned highway cross with speed v has been placed (with center) at σ. We define the region

$$R(p, q) := \{\sigma \in \mathbb{R}^2 \mid d_\sigma(p, q) \leqslant \delta\}.$$

We observe that the answer to the decision problem is positive if and only if $\bigcap_{(p,q)\in P} R(p, q)$ is not empty.

The shape of the region $R(p, q)$ depends on δ. Let w, h be the horizontal and vertical distance of points p and q. If $\delta < (w+h)/v$, then $R(p, q)$ is empty. If $(w+h)/v \leqslant \delta < \min\{w+h/v, h+w/v\}$, then $R(p, q)$ consists of two convex quadrilaterals. If $\min\{w+h/v, h+w/v\} \leqslant \delta < \max\{w+h/v, h+w/v\}$, then $R(p, q)$ is infinite in one (axis-parallel) direction. If $\max\{w+h/v, h+w/v\} \leqslant \delta < w+h$, then $R(p, q)$ is infinite in both axis-parallel directions. Finally, if $\delta \geqslant w+h$, then $R(p, q) = \mathbb{R}^2$. See Fig. 10.

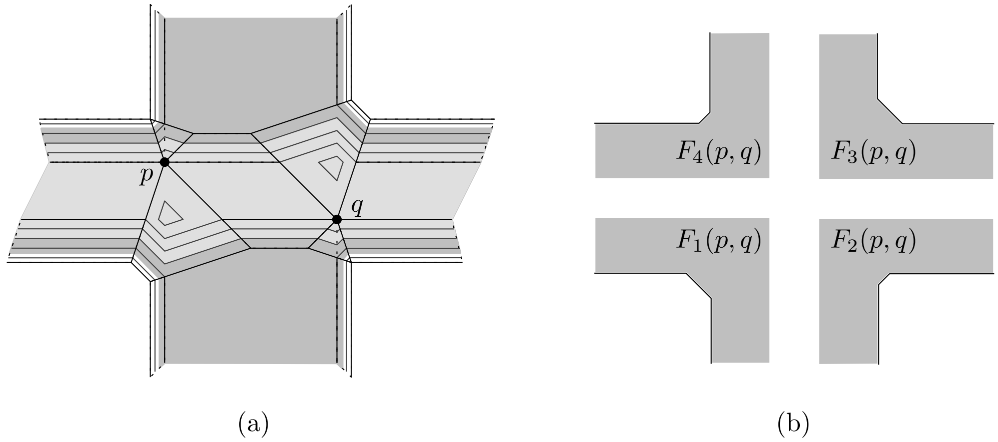


Fig. 10: (a) the regions $R(p,q)$ when $\min\{w+h/v, h+w/v\} \leqslant \delta < \max\{w+h/v, h+w/v\}$ (light gray region) and when $\max\{w+h/v, h+w/v\} \leqslant \delta < w+h$ (dark and light gray regions). (b) the dark and light gray regions can be expressed as the intersection of the four types of regions F_1, F_2, F_3, and F_4.

Let us call a planar region F (a,b)-monotone if for every point $(x,y) \in F$ and any $\lambda \geqslant 0$ the point $(x+\lambda a, y+\lambda b)$ is also in F. We observe that $R(p,q)$ can be expressed as the intersection of four regions $F_i(p,q)$, $i = 1,2,3,4$, where $F_1(p,q)$ is $(1,1)$-monotone, $F_2(p,q)$ is $(-1,1)$-monotone, $F_3(p,q)$ is $(-1,-1)$-monotone, and $F_4(p,q)$ is $(1,-1)$-monotone. Figure 10 shows an example of $R(p,q)$, which can be expressed as the intersection of F_1, F_2, F_3, and F_4. Each region is bounded by a polygonal curve of constant complexity. $F_i := \bigcap_{(p,q)\in P} F_i(p,q)$ is the lower envelope of a set of $O(m)$ line segments, has complexity $O(m\alpha(m))$ [16], and can be computed in time $O(m \log m)$. The intersections $F_1 \cap F_3$ and $F_2 \cap F_4$ can be computed by a plane sweep in time $O(m\alpha(m) \log m)$. We are left with two regions of complexity $O(m\alpha(m))$, and we need to determine whether their intersection is empty. While we do not know how to bound the complexity of this region, we can test emptiness in $O(m\alpha(m) \log m)$ time, by a simple plane sweep that stops as soon as a point in the intersection is found. Since any intersection between edges of the two regions implies that the intersection is not empty, this runs in the claimed time bound. If an intersection is found, it is the center for a highway cross with travel-time diameter at most δ. ◻

6.2 Further observations

Suppose we could characterize the travel-time diameter given the optimal highway cross to get a compact list L of candidate values as in the case of the infinite-speed highway cross. Then we could do binary search on L using the decision algorithm of Theorem 10.

Given the travel-time graph Γ_{pq} for each pair of points p and q (see Fig. 9), consider the upper envelope over all these graphs as in Theorem 9. The minimum occurs at a vertex of this upper envelope. Thus there are always at most three pairs—that is, at most six points—that define this vertex. Locally, in a neighborhood of the optimal highway, these at most six points alone have the same optimal highway cross. However, the upper envelope for these three pairs might have other minima, and the original solution might not be the global minimum. We do not know whether this is the case, and whether it could perhaps be resolved by considering a few more pairs. If that was possible, we could apply Chan's technique [11] (as Cardinal and Langerman do [10]). This would yield a randomized algorithm for the optimization problem whose expected running time is asymptotically the same as the running time of the decision algorithm.

The optimum axis-aligned highway cross for finite speed need not be contained in the strip cross for infinite speed, see Fig. 11: take the points $(-2,1)$, $(-1,2)$, $(1,2)$, $(2,1)$, $(2,-1)$, $(1,-2)$, $(-1,-2)$, $(-2,-1)$—that is, an octagon, contained in the strip cross $([-1,1]\times\mathbb{R}) \cup (\mathbb{R}\times[-1,1]))$—plus the two points $(-a,0)$ and $(a,1)$ for $a > 3v$. Then the coordinate axes are the optimum highway cross for infinite speed. It has travel-time diameter 2, so the above two strips are the optimum cover. But for any finite speed $v > 1$, the highway cross $x = a$ and $y = 0$ yields a diameter of $(2a+1)/v$, which is better than the diameter $(2a/v)+1$ caused by the coordinate axes being highways.

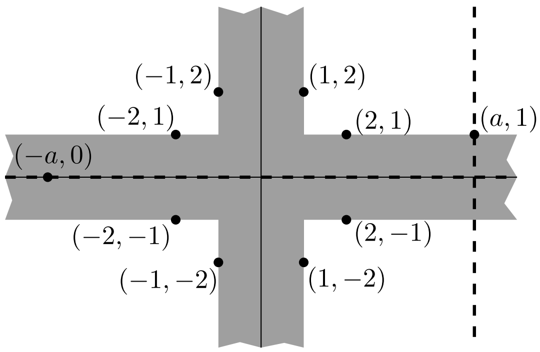


Fig. 11: Here the optimum speed-v highway cross (dashed) is not contained in the optimum speed-∞ strip cross (shaded).

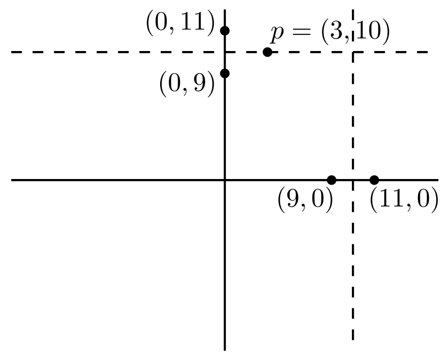


Fig. 12: After adding p, the optimal speed-∞ highway cross changes (from solid to dashed), but p does not occur in any diametral pair.

The following argument also rules out a simple incremental algorithm: Even for infinite speed there are point sets such that the addition of one point changes the diameter, and the new point does not occur in any diametral pair. An example (see Fig. 12) for infinite speed is given by the points $(0, 9)$, $(0, 11)$, $(9, 0)$, $(11, 0)$ since now the coordinate axes are an optimal highway cross—with a diameter of 0. If we add the point $(3, 10)$, the optimal highway cross is centered at $(10, 10)$ and has a diameter of 2.

7 Approximations for the optimal axis-aligned highway cross

Consider a set S of n points. Let C be the smallest enclosing cross for S, and let h_1, h_2 be the middle line of each strip of C. We call h_1, h_2 the *median highways* for S.

Lemma 2 *The travel-time diameter δ_{med} of the median highways (with speed v) is at most $2+1/v$ times the travel-time diameter δ_{opt} of an optimal axis-aligned speed-v highway cross for S. There are point sets S where for $v \geqslant \sqrt{3}$ the travel-time diameter of the median highways is at least $2 - 1/(v+2)$ times the optimum.*

Proof. We can scale S such that its L_1-diameter is 2—this does not change the travel-time ratio. Let w be the width of C after scaling. Observe that $\delta_{\text{opt}} \geqslant 2/v$, as there are points at L_1-distance 2. Furthermore, we have $\delta_{\text{opt}} \geqslant w$, since using the optimal highways at infinite speed cannot achieve diameter less than w.

On the other hand, $\delta_{\text{med}} \leqslant w + (2 + w)/v$, since any point can reach a point on the highways at distance at most $w/2$, and the maximum distance of such points on the highways is at most $2 + w$. This implies $\delta_{\text{med}} \leqslant (1 + 1/v)w + 2/v \leqslant (1 + 1/v)\delta_{\text{opt}} + \delta_{\text{opt}} = (2 + 1/v)\delta_{\text{opt}}$.

For the lower-bound example, let speed $v > 1$ be given, and set parameter $\omega = 1/(v + 2)$. We will construct a point set S such that the smallest enclosing cross has width 2ω, the median highway has travel-time diameter $4/v - 2/(v(v+2))$, and the optimal highway cross has travel-time diameter at most $2/v$, implying the lower bound.

Let $\varepsilon > 0$ be very small. Our point set S consists of the points $(1, 0)$, $(-1, 0)$, $(0, 1)$, $(0, -1)$, $(-2\omega, \omega + \varepsilon)$, $(-2\omega, -\omega - \varepsilon)$, $(\varepsilon, -2\omega)$, as in Fig. 13. We claim that S has a unique smallest enclosing strip of width 2ω, centered around the lines $x = -\omega$ and $y = -\omega$. Indeed, the line $x = 0$ must be in the vertical strip (otherwise the horizontal strip would have width at least 2), while the line $y = 0$ must be in the horizontal strip. Similarly, the line $x = -2\omega$ must be in the vertical strip as well, and this now fixes the vertical strip of width 2ω around the line $x = -\omega$. It follows that the remaining point $(\varepsilon, -2\omega)$ is in the horizontal strip, fixing that strip around $y = -\omega$.

The median highway cross has travel-time diameter $2\omega + (2+2\omega)/v = 4/v - 2/(v(v+2))$ if the diameter is less than or equal to the L_1-distance of these points (note that the diameter is determined by $(1, 0)$ and $(0, 1)$.) That is, for $v \geqslant \sqrt{3}$ the median highway cross has travel-time diameter $4/v - 2/(v(v + 2))$. Consider now a highway cross with center at the origin. The four outer points and the point $(\varepsilon, -2\omega)$ can be reached from the origin within travel time $1/v$. The remaining two points can be reached with travel time $\omega(1 + 2/v)$ (ignoring all ε-terms). The travel-time distance between these two points is 2ω,

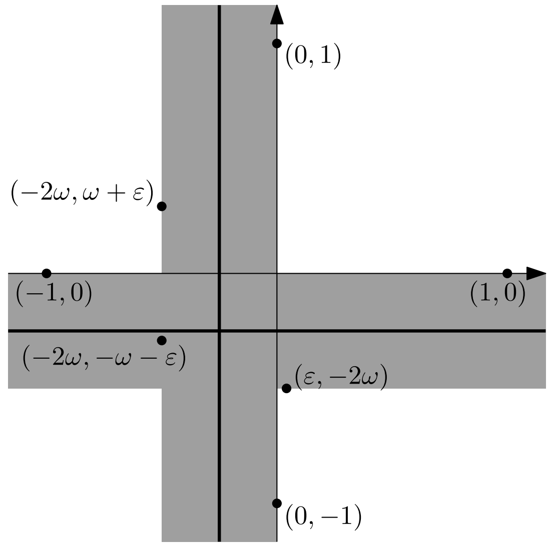


Fig. 13: The median highway for $v = 2$ (bold lines).

and so the travel-time diameter is bounded by

$$\max\{2/v, 1/v + \omega(1 + 2/v), 2\omega\} = 2/v. \qquad \square$$

We can improve the result in Lemma 2 by a simple observation.

Theorem 11 *Given a set S of n points we can compute in $O(n \log n)$ time an axis-aligned highway cross whose travel-time diameter is at most $1 + \sqrt{2}$ times the travel-time diameter of an optimal axis-aligned speed-v highway cross for S.*

Proof. According to Theorem 8 the median highways can be computed in $O(n \log n)$ time. According to Lemma 2 they yield a factor-$(2 + 1/v)$ approximation for the optimal travel-time diameter. Note that the approximation factor tends to 3 when the speed goes to 1. Clearly *not* building a highway cross is a factor-v approximation. Balancing out the two terms yields $\min\{2 + 1/v, v\} \leqslant 1 + \sqrt{2}$. $\square$

The following lemma will allow us to do better.

Lemma 3 *Let s be any point in S. Let H_s be the highway cross that minimizes the maximum travel time to s. The travel-time diameter of H_s is at most twice the travel-time diameter δ_{opt} of an optimal axis-aligned speed-v highway cross for S.*

Proof. Let $\{p, q\}$ be a pair of points in S and let $s' \in S$ be a point of maximum travel-time distance from s given H_s. We denote by d_s the metric induced by H_s and by d_{opt} the metric induced by the optimal highway cross. Then $d_s(s, p) \leqslant d_s(s, s') \leqslant d_{\text{opt}}(s, s') \leqslant \delta_{\text{opt}}$ and, by symmetry, $d_s(s, q) \leqslant \delta_{\text{opt}}$. This yields $d_s(p, q) \leqslant d_s(p, s) + d_s(s, q) \leqslant 2\delta_{\text{opt}}$. $\square$

Note that H_s is usually *not* centered at s (consider, for instance, the set $S = \{(0, 1), (1, 0)\}$ whose optimal highway cross is centered at the origin).

Based on the constant-factor approximation from Theorem 11 we can use binary search and the decision procedure of Theorem 10 to get the following.

Theorem 12 *Given a set S of n points in the plane, we can compute in $O(\log(1/\varepsilon)\alpha(n) n \log n)$ time a $(2 + \varepsilon)$-approximation for the optimal axis-aligned speed-v highway cross for S.*

Proof. Let s be any point in S and H_s be the highway cross that minimizes the maximum travel time to s.

According to Theorem 3 the travel-time diameter δ_s given H_s is at most twice the travel-time diameter δ_{opt} given an optimal axis-aligned speed-v highway cross for S, that is, $\delta_s \leqslant 2\delta_{\text{opt}}$, so a $(1 + \varepsilon/2)$-approximation δ_ε to δ_s is a $(2 + \varepsilon)$-approximation for δ_{opt}.

We now describe how to compute a $(1+\varepsilon)$-approximation for δ_s by binary search. Recall that the median highways yield a travel-time diameter of $\delta_{\mathrm{med}} \leqslant (1+\sqrt{2})\delta_{\mathrm{opt}} \leqslant 3\delta_{\mathrm{opt}} \leqslant 3\delta_s$ of δ_s, see Theorem 11. The median highways can be computed in $O(n \log n)$ time according to Theorem 8.

Now we conceptually subdivide the interval $I = [0, 2\delta_{\mathrm{med}}]$ into at most $N = 6/\varepsilon$ pieces of length $\delta_{\mathrm{med}} \cdot \varepsilon/3$, and denote the increasing sequence of interval endpoints by $\Delta = (\delta_1, \ldots, \delta_N)$. Since $\delta_s \leqslant 2\delta_{\mathrm{opt}} \leqslant 2\delta_{\mathrm{med}}$, we know that δ_s lies in I. Hence there is an index $i \in \{1, \ldots, N\}$ such that $\delta_i < \delta_s \leqslant \delta_{i+1}$.

Setting $\delta_\varepsilon = \delta_{i+1}$ we find that $\delta_s \geqslant \delta_i = \delta_\varepsilon - \delta_{\mathrm{med}} \cdot \varepsilon/3$. This yields $\delta_\varepsilon \leqslant \delta_s + \delta_{\mathrm{med}} \cdot \varepsilon/3 \leqslant \delta_s + 3\delta_s \cdot \varepsilon/3 \leqslant (1+\varepsilon)\delta_s$. Thus δ_ε is indeed a $(1+\varepsilon)$-approximation of δ_s.

For a given $\delta > 0$ we run the decision algorithm of Theorem 10, using the set of $n-1$ pairs $P = \{(s,q) \mid q \in S \setminus \{s\}\}$. Each such test takes $O(\alpha(n) n \log n)$ time. Using $O(\log(1/\varepsilon))$ calls to this decision procedure, we can determine δ_ε by binary search on Δ. We return the highway cross computed by the decision procedure for the largest $\delta_i \leqslant \delta_s$. □

If we are willing to invest more time, we can even get a $(1+\varepsilon)$-approximation of the optimal travel-time diameter δ_{opt}.

Theorem 13 *Given a set S of n points in the plane, we can compute in $O(\log(1/\varepsilon)\alpha(n)n^2 \log n)$ time a $(1+\varepsilon)$-approximation for the travel-time diameter of S under the optimal axis-aligned speed-v highway cross.*

Proof. We again first compute the median highways to get an upper bound δ_{med} for the optimal travel-time diameter δ_{opt} and then do binary search. We can now use the interval $[0, \delta_{\mathrm{med}}]$, which contains δ_{opt}. We stop when the interval size is sufficiently small, that is, at most $\delta_{\mathrm{med}} \cdot \varepsilon/3$. This time we use the decision algorithm of Theorem 10 with the set P of all $\binom{n}{2}$ pairs of points in S. □

8 Concluding remarks

There are many ways how this problem can be extended. First, can we compute an optimal highway with arbitrary orientation under the Euclidean metric in $o(n^2 \log n)$ (worst-case) time? Second, consider highways with different speeds, different slopes, or bounded lengths. Third, suppose an existing network of (axis-parallel) highways and a real $\ell > 0$ is given. Where should a new (axis-parallel) highway segment of length ℓ be placed in order to minimize the travel-time diameter of the resulting network?

Our algorithm for computing the optimal axis-aligned highway cross with finite speed runs in $O(n^4\alpha(n))$ time. In order to find a faster algorithm, it would be helpful to gain a better understanding of the upper envelope $\mathcal{E}$ over the travel-time graphs Γ_{pq} for all pairs of points p and q (see Fig. 9). If we could show that there is always a constant number of point pairs with the property that the upper envelope of their travel-time graphs has the same global minimum as $\mathcal{E}$, then we could apply Chan's technique [11]. This would yield a randomized algorithm for the optimization problem whose expected running time is asymptotically the same as the running time of the corresponding decision algorithm, that is, $O(n^2\alpha(n)\log n)$.

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