The Price of Upwardness

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Abstract

Not every directed acyclic graph (DAG) whose underlying undirected graph is planar admits an upward planar drawing. We are interested in pushing the notion of upward drawings beyond planarity by considering upward k-planar drawings of DAGs in which the edges are monotonically increasing in a common direction and every edge is crossed at most k times for some integer $k \geq 1$. We show that the number of crossings per edge in a monotone drawing is in general unbounded for the class of bipartite outerplanar, cubic, or bounded pathwidth DAGs. However, it is at most two for outerpaths and it is at most quadratic in the bandwidth in general. From the computational point of view, we prove that ~~upward~~ testing upward k-planarity ~~testing~~ is NP-complete already for $k = 1$ and even for restricted instances for which upward planarity testing is polynomial. On the positive side, we can decide in linear time whether a single-source DAG admits an upward 1-planar drawing in which all vertices are incident to the outer face.

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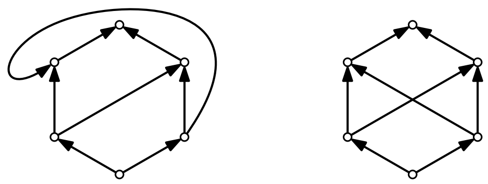
1 Introduction

Graph drawing "beyond planarity" studies the combinatorial and algorithmic questions related to representations of graphs where edges can cross but some crossing configurations are forbidden. Depending on the forbidden crossing configuration, different beyond-planar types of drawings can be defined including, for example, RAC, k-planar, fan planar, and quasi planar drawings. See [20, 34, 37] for surveys and books.

While most of the literature about beyond planar graph drawing has focused on undirected graphs (one of the few exceptions being [2, 3] which studies RAC upward drawings), we study *upward k-planar drawings* of acyclic digraphs (DAGs), i.e., drawings of DAGs where the edges monotonically increase in y-direction and each edge can be crossed at most k times. The minimum k such that a DAG admits an upward k-planar drawing is called its *upward local crossing number*. We focus on values of $k = 1, 2$ and investigate both combinatorial properties and complexity questions. Our research is motivated by the observation that well-known DAGs that are not ~~upward-planar~~upward planar, i.e., not upward 0-planar, do admit a drawing where every edge is crossed at most a constant number of times; see, e.g., Figure 1.



(a) **(b)**

Figure 1 A graph that is not upward planar but admits an upward 1-planar drawing.

Our contribution.

- A graph is an *outerpath* if it has a planar drawing in which each vertex is incident to the outer face and the internal faces induce a path in the dual graph. Papakostas [40] observed that there is a directed acyclic 8-vertex outerpath that is not upward-planar (see Figure 3a). We strengthen this observation by showing that there exists a directed acyclic fan (that is, a very specific outerpath) that has no upward-planar drawing (Proposition 1). On the other hand, we show that every directed acyclic outerpath is upward 2-planar (Theorem 9) and that the upward local crossing number is quadratic in the bandwidth (Theorem 6). However, the upward local crossing number of bipartite outerplanar DAGs (Theorem 2), bipartite DAGs with bounded pathwidth (Corollary 4), and cubic DAGs (Proposition 5) is in general unbounded.
- We show that upward 1-planarity testing is NP-complete, even for graph families where upward planarity testing can be solved in polynomial time. These include: single-source single-sink series-parallel DAGs with a fixed rotation system; single-source two-sink series-parallel DAGs where the rotation system is not fixed; and single-source single-sink

DAGs without fixed rotation system that can be obtained from a K_4 by replacing the edges with series-parallel DAGs (Theorem 11).

- Finally, following a common trend in the study of beyond planar graph representations, we consider the *outer model*, in which all vertices are required to lie on a common face while maintaining the original requirements [20,34,37]. We prove that testing whether a single-source DAG admits an upward outer-1-planar drawing can be done in linear time (Theorem 13).

Related Work. A drawing of a graph is *monotone* if all edges are drawn monotone with respect to some direction, e.g., a drawing is *y-monotone* or *upward*, if each edge intersects each horizontal line at most once. The corresponding crossing number is introduced and studied in [25,42]. Schaefer [41] mentions the upward crossing number and the local crossing number but not their combination. Schaefer [41, p. 64] also showed that a drawing with the minimum number of crossings per edge can require incident edges that cross. The edges of the provided 4-planar example graph can be oriented such that the resulting directed graph admits an upward 4-planar drawing. Thus, also an upward drawing that achieves the minimum local crossing number can require incident edges that cross. Also, the so-called strong Hanani–Tutte theorem carries over to directed graphs: Fulek et al. [25, Theorem 3.1] showed that every undirected graph that has a monotone drawing where any pair of independent edges crosses an even number of times also has a planar monotone drawing with the same vertex positions. This implies that in any upward drawing of a graph that is not upward-planar there must be a pair of independent edges that crosses an odd number of times.

Upward drawings of directed acyclic graphs have been studied in the context of (upward) book embeddings. In that model the vertices are drawn on a vertical line (a spine) following a topological order of the graph, while all edges are pointing upwards. To reduce the edge crossings, edges are partitioned into the fewest number of crossing-free subsets (pages). Studying upward book embeddings is a popular topic, which is usually centered around determining the smallest number of pages for various graph classes [23,24,31,36,39] or deciding whether a graph admits an upward drawing with a given number of pages [5,6,9,10,11]. Our model is ~~equivalent~~ related to *topological book embeddings* [29,38], which are a relaxed version of 2-page book embeddings in which edges are allowed to cross the spine. ~~To the best of our knowledge, earlier papers considered only the problem of minimizing the~~ While papers about topological book embeddings insist on planar drawings and minimize the number of spine crossings, ~~whereas we~~ we do allow crossings and want to bound the maximum number of ~~edge~~ crossings per edge ~~(ignoring the~~ (ignoring the spine).

2 Preliminaries

A *drawing* Γ of a graph G maps the vertices of G to distinct points in the plane and the edges of G to ~~Jordan arcs. An~~ open Jordan curves connecting their respective endpoints but not containing any other vertex point. A ~~*internal point*~~*crossing* ~~of an edge e in Γ is a~~ between two edges is a common point of their curves, other than a common end point. A drawing is *simple* if (a) there are no self-crossings (the Jordan curves are simple), (b) no two edge curves share more than one point~~of e which does not represent an endvertex of e. Two edges of Γ *cross* if they share exactly one internal point p and the two edges alternate around p. In what follows we only consider drawings where no edge crosses itself, no two edges cross multiple times, no three edges cross at a common point~~, and ~~no edge has an internal pointrepresenting a vertex.~~(c) no three edge curves share a common internal point.

For a vertex v of G and a drawing Γ of G, let $x_\Gamma(v)$ and $y_\Gamma(v)$ denote the x- and y-coordinates of v in Γ, respectively; when Γ is clear from the context, we may omit it and simply use the notation $x(v)$ and $y(v)$. A *face* of Γ is a region of the plane delimited by maximal uncrossed ~~arc portions~~ segments of the edges of G. The unique unbounded face of Γ is its *outer face*, the other faces are its *internal faces*. An *outer edge* is one incident to the outer face; all other edges are *inner edges*. The *rotation* of a vertex v in Γ is the counterclockwise cyclic order of the edges incident to v. The *rotation system* of Γ is the set of rotations of its vertices.

The drawing Γ is *planar* if no two of its edges cross; it is *k-planar* if each edge is crossed at most k times. A graph is *(k-)planar* if it admits a (k-)planar drawing; it is *outer (k-)planar* if it admits a (k-)planar drawing where all vertices are incident to the outer face.

A *planar embedding* $\mathcal{E}$ of a planar graph G is an equivalence class of planar drawings of G, namely those that have the same set of faces. Each face can be described as a sequence of edges and vertices of G which bound the corresponding region in the plane; each such sequence is a face of G in the embedding $\mathcal{E}$. A planar embedding $\mathcal{E}$ of a connected graph can also be described by specifying the rotation system and the outer face associated with any drawing of $\mathcal{E}$.

Let Γ be a non-planar drawing of a graph G; the *planarization* of Γ is the planar drawing Γ' of the *planarized graph* G' obtained by replacing each crossing of Γ with a dummy vertex. If Γ is 1-planar, the planarization can be obtained as follows. Let uv and wz be any two edges that cross in Γ; they are replaced in Γ' by the edges ux, xv, wx and xz, where x is the dummy vertex. Two non-planar drawings of a graph G have the same *embedding* if their planarizations have the same planar embedding. An embedding $\mathcal{E}$ of G can also be described by specifying the planarized graph G' and one of its planar embeddings. A planar graph with a given planar embedding is also called *plane graph*. An *outerplane graph* is a plane graph whose vertices are all incident to the outer face. A *fan* is a maximal outerpath that contains a vertex c that is adjacent to all other vertices; we call c the *central vertex* of the fan. A *2-tree* is a planar graph that can be reduced to an edge by iteratively removing a degree-two vertex that closes a 3-cycle. A *series-parallel graph* is a graph that can be augmented to a 2-tree by adding edges (and no vertices).

A *(simple, finite) directed graph* (*digraph* for short) G consists of a finite set $V(G)$ of *vertices* and a finite set $E(G) \subseteq \{(u,v) \mid u, v \in V(G), u \neq v\}$ of ordered pairs of vertices, which are called *edges*. A *source* (resp. *sink*) of G is a vertex with no incoming (resp. no outgoing) edges. A *single-source graph* is a digraph with a single source and, possibly, multiple sinks. A digraph G is an *st-graph* if: (i) it is acyclic and (ii) it has a single source s and a single sink t. An st-graph is a *planar st-graph* if it admits a planar embedding with s and t on the outer face. We say that a drawing of a digraph G is *upward* if every (directed) edge (u,v) of G is mapped to a y-monotone Jordan arc with $y(u) < y(v)$. Clearly, a digraph admits an upward drawing only if it does not contain a directed cycle. Therefore, we assume for the rest of the paper that the input graph is a *DAG*, a directed acyclic graph. Such a graph has a *linear extension*, i.e., a vertex order $v_1, \dots, v_n$ such that, for any directed edge (v_i, v_j), we have $i < j$. We say that a DAG is planar, outerplanar, or series-parallel if its underlying undirected graph is planar, outerplanar, or series-parallel, respectively.

Let Γ be an upward drawing of a DAG G. By the upwardness, the rotation system of Γ is such that for every vertex v of Γ the rotation of v has only one maximal subsequence of outgoing (incoming) edges. We call such a rotation system a *bimodal rotation system*. An *upward embedding* of a DAG G is an embedding of G ~~with~~ arising from an upward drawing; it naturally has a bimodal rotation system. The minimum k such that a digraph G admits an upward k-planar drawing is called its *upward local crossing number* and denoted by $\mathrm{lcr}^{\uparrow}(G)$.

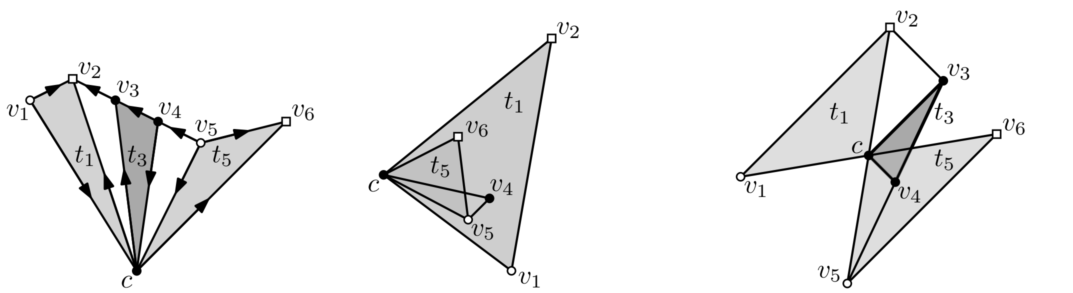
For any positive integer k, we use $[k]$ as shorthand for $\{1, 2, \ldots, k\}$. A *path-decomposition* of a graph ~~$G = (V, E)$~~ G is a sequence $P = \langle X_1, \ldots, X_\ell \rangle$ of subsets of ~~V~~ $V(G)$, called *buckets*, such that (1) for each edge~~$e \in E$~~ e of G there is a bucket that contains both end vertices of e, and (2) for every vertex v of G, the set of buckets that contain~~a particular vertex $v \in V$ forms a~~ v form a contiguous subsequence of P. The *width* of a path-decomposition is one less than the size of the largest bucket. The *pathwidth* of the graph G is the width of a path decomposition of G with the smallest width.

3 Lower Bounds

We start with a negative result that shows that even very special directed acyclic outerpaths may not admit upward-planar drawings, thus strengthening Papakostas' observation [40].

▶ **Proposition 1.** *Not every directed acyclic fan is upward-planar.*

Proof. Consider the 7-vertex fan F depicted in Figure 2a. Suppose for a contradiction that F is upward planar, that is, F admits an upward planar drawing Γ. Let c be the central vertex of F. We assume that c is placed at the origin. We say that a triangle of F is *positive* (*negative*, respectively) if the corresponding region of the plane in Γ contains the point $(\varepsilon, 0)$ $((-\varepsilon, 0)$, respectively) for a sufficiently small value $\varepsilon > 0$. The triangles that have one vertex below c and one vertex above c (namely $t_1 = \triangle cv_1v_2$, $t_3 = \triangle cv_3v_4$, and $t_5 = \triangle cv_5v_6$) are either positive or negative.



(a) the fan F **(b)** case: t_5 is contained in t_1 **(c)** a 1-planar upward drawing of F

Figure 2 A directed acyclic fan F that does not admit an upward planar drawing.

If both t_1 and t_5 are positive, then one must contain the other in Γ, say, t_1 contains t_5; see Figure 2b. But then vertices v_3 and v_4 must also lie inside t_1. If both lie inside t_5, then the edge (v_3, v_2) intersects an edge of t_5. If one of them lies inside t_5 and one does not, then the edge (v_4, v_3) intersects an edge of t_5. So both must lie outside t_5. But v_4 lies on one hand above v_5 and on the other hand below c and, thus, below v_6. So the edge (v_4, c) intersects the edge (v_5, v_6). (If t_5 ~~is contained in~~ contains t_1, the edge (c, v_3) intersects the edge (v_1, v_2).)

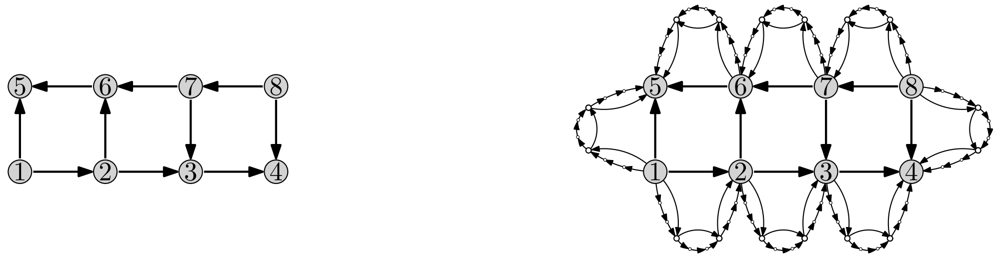
By symmetry, not both t_1 and t_5 can be negative, so exactly one of t_1 and t_5 must be negative, say, t_1; see Figure 2c. Now first assume that t_3 is positive. Due to edge (v_3, v_2), vertex v_3 must be outside t_5, so t_3 cannot be inside t_5. On the other hand, t_3 cannot contain t_5 because v_4 is above v_5. Hence t_3 intersects t_5. Finally, assume that t_3 is negative. Due to edge (v_5, v_4), vertex v_4 must be outside t_1, so t_3 cannot be inside t_1. On the other hand, t_3 cannot contain t_1 because v_3 is below v_2. Hence t_3 intersects t_1. ◀

~~the fan F case: t_5 is contained in t_1 a 1-planar upward drawing of F A directed acyclic fan F that does not admit a planar upward drawing.~~

By iteratively adding paths on every outer edge of an outerplanar but not upward-planar DAG, we can construct outerplanar DAGs with an unbounded upward local crossing number.

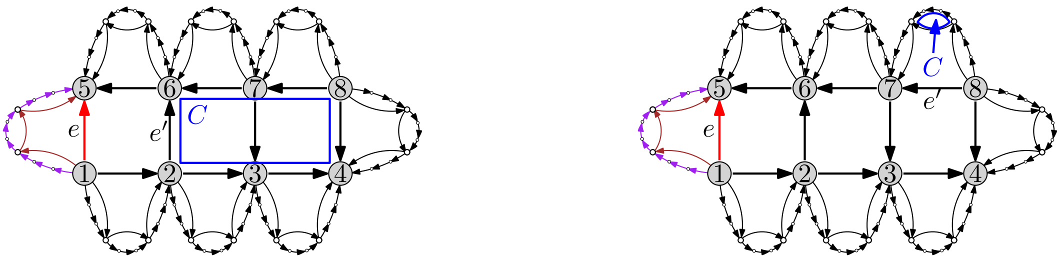
▶ **Theorem 2.** *For each $\ell \geq 0$, there is a bipartite outerplanar DAG G_ℓ with $n_\ell = 8 \cdot 3^\ell$ vertices, maximum degree $\Delta_\ell = 2\ell + 3$, and upward local crossing number greater than $\ell/6$, which is in $\Omega(\log n_\ell)$ and $\Omega(\Delta_\ell)$.*

Proof. The bipartite graph G_0 in Figure 3a is not upward planar [40]. For $\ell \geq 1$, we construct G_ℓ from $G_{\ell-1}$ by adding a 3-edge path on every outer edge of the graph. Figure 3b shows G_2. The maximum degree of G_ℓ is $\Delta_\ell = 2\ell + 3$. The number of vertices is $n_\ell = 8 + \sum_{i=1}^{\ell} 8 \cdot 3^{i-1} \cdot 2 = 8 \cdot 3^\ell$.



(a) G_0 **(b)** G_2

Figure 3 There is a family $(G_\ell)_{\ell \geq 0}$ of bipartite outerplanar graphs such that G_ℓ has n_ℓ vertices, maximum degree Δ_ℓ, and upward local crossing number in $\Omega(\Delta_\ell) \cap \Omega(\log n_\ell)$.



(a) e' inner edge of G_0 **(b)** e' outer edge of G_0

Figure 4 If e crosses e' an odd number of times then there is a cycle C of length at most 6 that is crossed by at least $\ell + 1$ edge-disjoint paths, namely the edge e and the ℓ paths added on top of e.

Consider now an upward k-planar drawing Γ of G_ℓ for some ~~k~~ $k \leq \ell$. Since G_0 is not upward planar, there must be a pair of independent edges of G_0 that crosses an odd number of times in Γ. Observe that G_0 has no upward planar drawing in which only the two inner edges cross an odd number of times, for otherwise the two cycles $\langle 1, 2, 6, 5 \rangle$ and $\langle 3, 4, 8, 7 \rangle$ would intersect an odd number of times, which is impossible. Thus, in Γ there must be an outer edge e of G_0 that is crossed by an independent edge e' of G_0 an odd number of times. We choose e' to be an outer edge of G_0, if possible.

We now determine a cycle C of G_ℓ that is crossed by e an odd number of times and does also not contain any end vertex of e. If e' is an inner edge, then we take the outer path P of G_0 that connects the ends of e' and does not contain e; this is not ~~intersected~~ crossed by e due to our choice of e'. Moreover, since e' and e are independent, it follows that P and e

do not share a vertex. Let C be the concatenation of P and e'. In this case C has length at most six. See Figure 4a.

If e' is an outer edge of G_0, we do the following: We start with the path P of length three that was added for e'. Since e is an edge of G_0 and e and e' are independent, it follows that P and e do not share a vertex. If P contains an edge that is crossed an odd number of times by e then we ~~replace e' by~~ continue with such an edge ~~and continue~~instead of e'. More precisely, let $e_1 = e'$ and initialize $i = 1$. Let ~~P_1~~ P_i be the path of length three that was added for ~~e_1~~e_i. While P_i contains an edge that is crossed an odd number of times by e, let e_{i+1} be such an edge, let P_{i+1} be the path of length three that was added for e_i, and increase i by one. Since e is crossed at most k times, this process stops at some ~~$i < k$.~~$i \leq k$. Let C be the cycle that is composed of P_i and e_i. See Figure 4b. In this case C has length four. Moreover C shares at most the end vertices of e' with G_0. Thus, since e is an edge of G_0 and e and e' are independent, it follows that C and e do not share a vertex.

Cycle C might cross itself. However, it divides the plane into cells. Since e crosses C an odd number of times, it follows that the end vertices of e must be in different cells of the plane. This means that not only e but also the ℓ edge-disjoint paths that were added on top of e have to cross C. Observe that none of these paths contains a vertex of C. But C contains at most six edges, each of which can be crossed at most k times. This is impossible if $\ell \geq 6k$. Hence, if there is an upward k-planar drawing then $\ell < 6k$, which means that $k > \ell/6$. ◀

We now show that if we expand the graph class beyond outerplanar graphs, then we get a lower bound on the upward local crossing number that is even linear in the number of vertices. The graphs in our construction have pathwidth 2, as opposed to the graphs in Theorem 2 whose pathwidth is logarithmic. Observe that a *caterpillar*, i.e., a tree that can be reduced to a path by removing all degree-1 vertices, has pathwidth 1, and that the pathwidth can increase by at most 1 if we add a vertex with some incident edges or subdivide some edges.

▶ **Theorem 3.** *For every $k \geq 1$, there exists a (planar) DAG with $\Theta(k)$ vertices, maximum degree in $\Theta(k)$, and pathwidth 2 that does not admit an upward k-planar drawing.*

Proof. Let G_k be the graph consisting of the four vertices a, b_1, b_2, and c and the following set of edges and degree-2 vertices (see also Figure 5):

- edges (a, b_1) and (a, b_2);
- for $i \in [2]$ and $j \in [3k+1]$, a *through-vertex at b_i*, i.e., a vertex $d_i^{(j)}$ and edges $(b_i, d_i^{(j)})$ and $(d_i^{(j)}, c)$;
- for $j \in [6k+1]$, a *source below a*, i.e., a vertex $s^{(j)}$ and edges $(s^{(j)}, a)$ and $(s^{(j)}, c)$;
- for $i \in [2]$ and $j \in [4k+1]$, a *sink above b_i*, i.e., a vertex $t_i^{(j)}$ and edges $(b_i, t_i^{(j)})$ and $(c, t_i^{(j)})$.

Clearly, G_k has $\mathcal{O}(k)$ vertices, and pathwidth 2, since $G - c$ is a caterpillar and has pathwidth 1.

Assume that there was an upward k-planar drawing Γ of G_k. Up to renaming, we may assume that $y(b_2) \leq y(b_1)$. Delete all but one of the through-vertices at b_1 from the drawing; in what follows, we write d_1 for the one that we keep (it does not matter which one).

Among the $3k+1$ through-vertices $d_2^{(j)}$ at b_2, there exists at least one for which the path $\langle b_2, d_2^{(j)}, c\rangle$ crosses none of the three edges in the path $\langle a, b_1, d_1, c\rangle$, for otherwise there would be an edge with more than k crossings. Delete all other through-vertices at b_2; in what follows we write d_2 for the one that we keep. Let a' be the topmost intersection point of (a, b_1) and (a, b_2) (possibly $a' = a$). Since $y(a) \leq y(a') < y(b_2) \leq y(b_1)$ the curve C_b formed by the two directed paths $\langle a', b_i, d_i, c\rangle$ (for $i \in [2]$) is drawn without crossing in Γ.

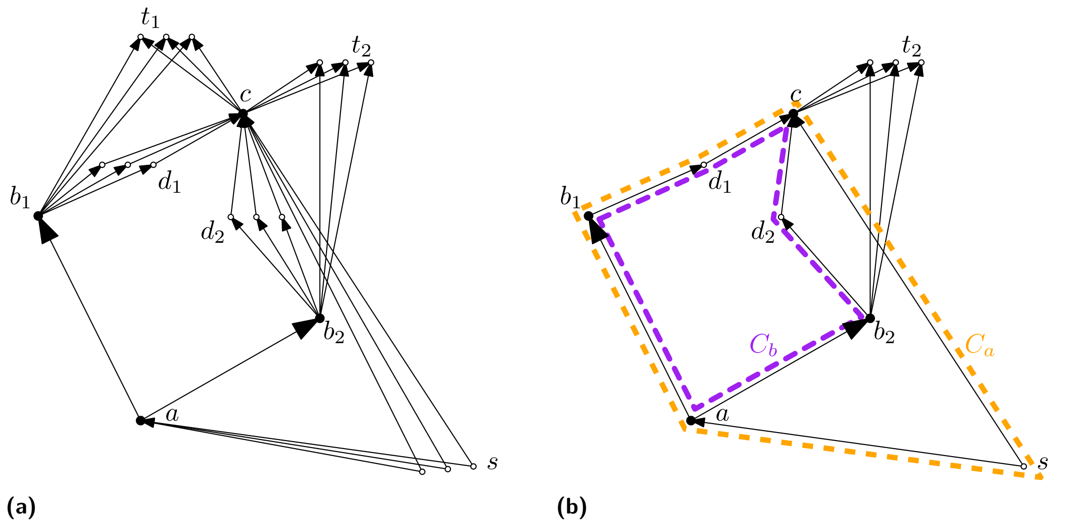


Figure 5 A graph of pathwidth 2 (drawn upward) that does not have an upward k-planar drawing. (a) We only show three of the $\Theta(k)$ vertices of each group. (b) Cycles C_a and C_b.

Curve C_b uses six edges, therefore among the $6k+1$ sources below a, there exists one, call it s, for which edge (s,c) crosses no edge of C_b. Since $y(s) < y(a)$, vertex s is outside C_b, and so the entire edge (s,c) is outside C_b, except at the endpoint c. In particular, among the three edges (d_1,c), (d_2,c), and (s,c) that are incoming at c, edge (s,c) is either leftmost or rightmost (but cannot be the middle one). We assume here that (s,c) is rightmost, the other case is symmetric. Write $\{p,q\} = \{1,2\}$ such that the left-to-right order of incoming edges at c is (d_p,c), (d_q,c), (s,c). In Figure 5, we have $p=1$ and $q=2$.

Edge (s,a) is also outside C_b, except perhaps at endpoint a, since it uses smaller y-coordinates. Let s' be the topmost intersection point of (s,a) and (s,c). Then there are no crossings in the curve C_a formed by the directed paths $\langle s', a, b_p, d_p, c\rangle$ and $\langle s', c\rangle$. By our choice of p and q, vertex d_q is *inside* C_a, and so is the entire path $\langle a', b_q, d_q, c\rangle$, except at the ends since it is part of C_b. In particular, b_q is inside C_a, whereas, for $j \in [4k+1]$, $t_q^{(j)}$ is outside C_a due to $y(c) < y(t_q^{(j)})$. It follows that one of the four edges (a,b_p), (b_p,d_p), (d_p,c) and (s,c) must be crossed at least $k+1$ times by edges from b_q to the sinks above it. Thus, the drawing was not k-planar, a contradiction. ◀

The graphs that we constructed in the proof of Theorem 3 are not bipartite, but one can make them bipartite by subdividing all edges once. This at best cuts the local crossing number in half, increases the pathwidth by at most 1, and yields the following result.

▶ **Corollary 4.** *There is a family of bipartite (planar) DAGs of constant pathwidth whose upward local crossing number is linear in the number of vertices.*

So far we needed graphs of unbounded maximum degree in order to enforce unbounded upward local crossing number. We now show that, intrinsically, this is not necessary.

▶ **Proposition 5.** *There are cubic DAGs whose upward local crossing number is at least linear in the number of vertices.*

Proof. The crossing number of a random cubic graph with n vertices is expected to be at least cn^2 for some absolute constant $c > 0$ [21], and thus there exist graphs yielding this bound. By the pigeon-hole principle, such a graph contains an edge with $\Omega(n)$ crossings among its $\Theta(n)$ edges. Impose arbitrary acyclic edge directions. ◀

4 Upper Bounds

The *bandwidth* $\mathrm{bw}(G)$ of an undirected graph G is the smallest positive integer k such that there is a labeling of the vertices by distinct numbers $1, \dots, n$ for which the labels of every pair of adjacent vertices differ by at most k.

▶ **Theorem 6.** *The upward local crossing number of a DAG G with maximum degree Δ is at most* ~~$\Delta \cdot (2\,\mathrm{bw}(G) - 2) \le 4\,\mathrm{bw}(G)(\mathrm{bw}(G) - 1)$~~$\Delta \cdot (\mathrm{bw}(G) - 2) \le 2\,\mathrm{bw}(G)(\mathrm{bw}(G) - 2)$, *so it is in* $\mathcal{O}(\Delta \cdot \mathrm{bw}(G)) \subseteq \mathcal{O}(\mathrm{bw}(G)^2)$.

Proof. Observe that the maximum degree Δ of a graph G is bounded in terms of the bandwidth of G; namely, $\Delta \le 2\,\mathrm{bw}(G)$. Consider a linear extension of G. For every vertex v of G, let $y(v)$ be its index in the extension. Now consider a labeling of G corresponding to the bandwidth. For every vertex v of G, let $x(v)$ be ~~label.~~ its label.

Construct a drawing of G by first placing every vertex v at the point $(x(v), y(v))$ and by then perturbing vertices slightly so that the points are in general position. Adjacent vertices are connected via straight-line segments.

It is easy to see that the drawing is upward since it is consistent with the linear extension. Consider an arbitrary edge (u, v) with $x(u) < x(v)$. ~~Every edge that crosses~~ The length of any edge in x-direction is bounded by $\mathrm{bw}(G)$; let $x(v) - x(u) = \ell \le \mathrm{bw}(G)$. Edge (u, v) ~~must have its left endpoint in the interval~~ ~~$[x(u) - \mathrm{bw}(G) + 1, x(v) - 1]$. Since~~ ~~$x(v) - x(u) < \mathrm{bw}(G)$~~may be crossed (a) by edges that have at least one incident vertex w with $x(u) < x(w) < x(v)$, or (b) by edges (w, w') with $x(w) < x(u) \wedge x(v) < x(w')$. Thus, there are at most ~~$2\,\mathrm{bw}(G) - 2$ such vertices distinct from u, each of which~~ $\ell - 1$ and $\mathrm{bw}(G) - \ell - 1$ possible choices for w in the two scenarios, respectively. Since each vertex is incident to at most Δ edges~~. Hence~~, (u, v) ~~has at most $\Delta \cdot (2\,\mathrm{bw}(G) - 2)$ crossings.~~ can be crossed at most $\Delta(\mathrm{bw}(G) - 2)$ times. ◀

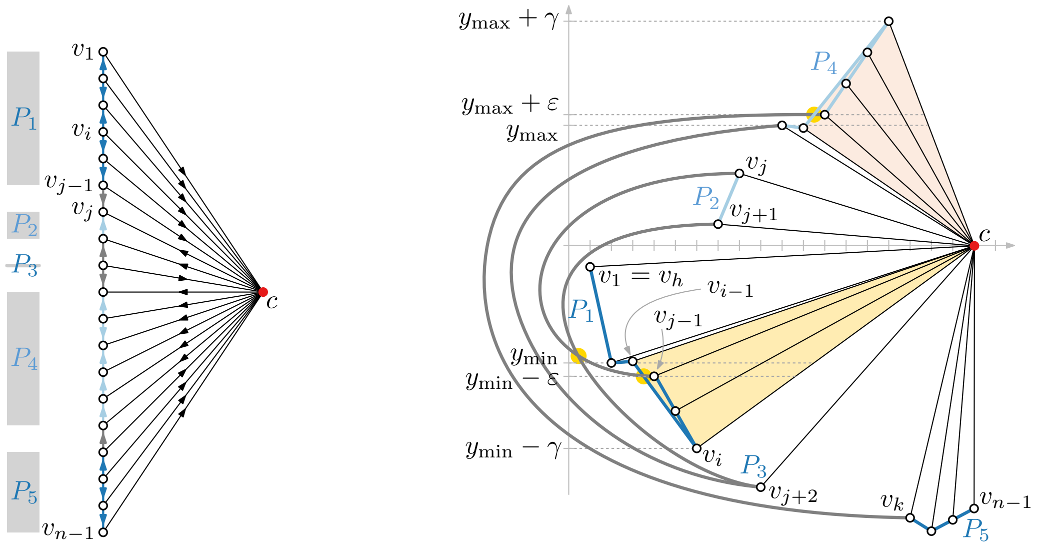
For some graphs, a sublinear bound on the bandwidth is known, see [12, 22, 43]. This gives upper bounds on the upward local crossing number of many graph classes~~(e.g., interval graphs, co-compoarability graphs, AT-free graphs, graphs of bounded treewidth); we~~. We list only a few:

▶ **Corollary 7.** *The following classes of DAGs have sublinear upward local crossing number:*

- *Square $k \times k$ grids have bandwidth $\Theta(k)$ and $\Delta = 4$, hence their upward local crossing number is in $\mathcal{O}(k) = \mathcal{O}(\sqrt{n})$.*
- ~~*Directed planar graphs with maximum degree*~~*Planar graphs of maximum degree* Δ~~*have bandwidth* $\mathcal{O}(\frac{n}{\log_\Delta n})$~~ *have bandwidth $\mathcal{O}(\frac{n}{\log_\Delta n})$ [12], hence their upward local crossing number is in $\mathcal{O}(\frac{n \cdot \Delta}{\log_\Delta n})$.*

We complement the negative result in Proposition 1 by showing that every directed acyclic outerpath allows an upward 2-planar drawing. We start with a technical lemma on fans.

▶ **Lemma 8.** *Let c be the central vertex of a directed acyclic fan G, and let $P = \langle v_1, \dots, v_{n-1} \rangle$ be the path of the remaining vertices in G. Let $P_1, \dots, P_k$ be an ordered partition of P into maximal subpaths such that, for every $i \in [k]$, the edges between P_i and c either are all directed towards c or are all directed away from c. Then there is an upward 2-planar drawing of G with the following properties:*



(a) a directed fan G

(b) upward drawing of G with at most two crossings per edge

Figure 6 ~~Upward~~ Constructing upward 2-planar drawings of fans ~~(~~according to Lemma 8~~)~~. ~~For $t \in [k]$, we add the path P_t below c (blue paths) or above c (green paths), going up and down as prescribed by the edge directions and such that no edge incident to c is crossed. We maintain the property that all vertices of P_t are on the outer face of the subgraph induced by P_t and c, except for possibly a last final part pointing upward if P_t is below c or pointing downward if P_t is above c. See the shaded areas, e.g., the final part $\langle v_i, \dots, v_{j-1} \rangle$ of P_1. The edge connecting P_t and P_{t+1} (red edges) might either cross the last edge of P_t on the outer face (e.g., the edge of P_1 between v_{i-1} and v_i) or the edge connecting P_{t-1} to P_t in order to reach the outer face. The latter may have been crossed once before (as (v_{j-1}, v_j)).~~

1. *no edge incident to c is crossed;*
2. ~~*vertex v_1 has x-coordinate 1,*~~ *the central vertex c and v_{n-1} have x-coordinate $n-1$, and the x-coordinates of* ~~$v_2, \dots, v_{n-2}$ *are*~~ $v_1, \dots, v_{n-2}$ *are pairwise distinct values within* ~~$\{2, \dots, n-2\}$;~~ $\{1, \dots, n-2\}$*;*
3. *for all edges all x-coordinates of the curves are at most $n-1$; all edges incident to c and all edges of the subpaths $P_1, \dots, P_k$ are in the vertical strip between 1 and $n-1$;*
4. *if P_1 is a directed path, then the edge between P_1 and P_2 is crossed at most once.*

Proof. We place c at $(n-1, 0)$; then we place $v_1, v_2, \dots, v_{n-1}$ above or below c depending on the direction of the edges that connect them to c; see Figure 6 for an example.

For ~~$i \in [n-2]$~~ each $\ell \in [n-2]$, we keep the invariant that, when we place ~~v_i, the leftward rayfrom v_i~~ v_ℓ, the *leftward* ray, that is, the one in direction $\binom{1}{0}$ from v_ℓ, reaches the outer face of the current drawing after crossing at most one other edge, and that this edge ~~had been crossed at most once~~ is currently crossed by at most one edge.

In order to choose appropriate y-coordinates, we maintain two values $y_{\min}$ and $y_{\max}$ indicating the minimum and maximum y-coordinate of any so far drawn vertex. Consider now a subpath $P' \in \{P_1, \dots, P_k\}$. Let v_h be the first and let v_{j-1} be the last vertex of P', i.e., $P' = \langle v_h, v_{h+1}, \dots, v_{j-1} \rangle$. We describe in detail the case that the edge from $v_h, \dots, v_{j-1}$ to c are directed towards c that is, v_h must lie below c. The other case is symmetric. We place v_h at x-coordinate h and with a y-coordinate sufficiently below $y_{\min}$. If $h = j-1$ we are done.

We now consider the ~~cases~~ case that $j = n$ or ~~$(v_{j-1}, v_{j-2}) \in E$.In that~~ $(v_{j-1}, v_{j-2}) \in E(G)$;

see, for example, path P_5 in Figure 6b. In this case, we place $v_{h+1}, \dots, v_{j-1}$ using x-coordinates $h+1, \dots, j-1$, going up and down as needed but remaining below the x-axis. The edges are drawn such that all vertices of P' remain on the outer face of the drawing. I.e., if we use straight-line edges, then, for $i \in [n-2]$, the slope of v_iv_{i+1} must be less than the slope of v_ic. Since we go towards c, we can draw P' and the edges that connect $v_1, v_2, \dots, v_{n-1}$ to c without any crossings.

If $j \neq n$ and ~~$(v_{j-2}, v_{j-1}) \in E$~~$(v_{j-2}, v_{j-1}) \in E(G)$, then let $i \in \{h, \dots, j-1\}$ be the smallest index such that the subpath $\langle v_i, v_{i+1}, \dots, v_{j-1} \rangle$ is directed. See, for example, that last part of P_1 in Figure 6b. In that case, we place $v_{h+1}, \dots, v_{i-1}$ at x-coordinates $h+1, \dots, i-1$, going slightly up and down as in the case described above. Let $y_{\min}$ be the smallest among the y-coordinates of all points placed so far.

Then we place $v_i, v_{i+1}, \dots, v_{j-1}$ in reverse order, i.e., at x-coordinates $j-1, j-2, \dots, i$.~~Set~~ For the y-coordinates, we choose $y(v_i) = y_{\min} - \gamma$ and $y(v_{j-1}) = y_{\min} - \varepsilon$ for some (large) $\gamma > 0$ and (small) $\varepsilon > 0$ ~~such that~~ with the following properties: if $i > h$ then v_{j-1} lies inside the triangle $\triangle v_{i-1}v_ic$ (pale yellow in Figure 6b) and if ~~$i > h$ and within~~ $i = h$ then v_{j-1} lies inside the triangle $\triangle ov_ic$, ~~with~~ where $o = (0,0)$~~otherwise. (Observe that in the case $i = h$, we already required that v_i is sufficiently below $y_{\min}$; this is now further specified here.)~~ . Draw $v_{i+1}, \dots, v_{j-2}$ on the segment $\overline{v_iv_{j-1}}$. ~~Now,~~ This fulfills the invariant: if $i > h$ then the vertex v_{j-1} can reach the outer face via the edge (v_i, v_{i-1}) which was not crossed so far. If $i = h$ then v_{j-1} is on the outer face if $P' = P_1$, otherwise ~~it~~ v_{j-1} can reach the current outer face by crossing the edge (v_h, v_{h-1}).~~This edge~~ Observe that, by our invariant, (v_h, v_{h-1}) might have crossed one edge ~~when it was initially drawn but so far no other edge.~~ in order to reach the outer face. While drawing P', we do not cross (v_h, v_{h-1}) again. So the potential crossing with (v_{j-1}, v_j) is at most the second crossing of (v_h, v_{h-1}), and (v_h, v_{h-1}) will not be crossed later.

Observe that when we draw the next maximal subpath, we place v_j at $(j, y_{\max}+1)$, i.e., in particular in the outer face of the current drawing. The edge from v_{j-1} to v_j must be directed towards v_j since the orientation is acylic. Thus, we can draw the edge between v_{j-1} and v_j upward with at most one crossing, causing at most a second crossing on (v_h, v_{h-1}). ◀

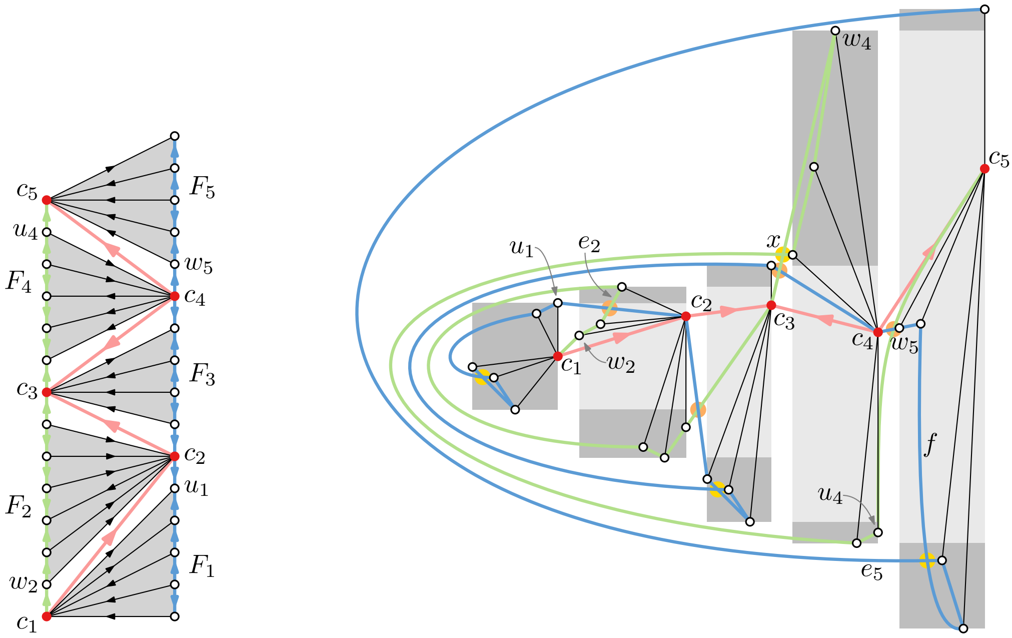
Now we describe our construction for general outerpaths; see Figure 7.

▶ **Theorem 9.** *Every directed acyclic outerpath admits an upward 2-planar drawing.*

Proof. Without loss of generality, we can assume that the given outerpath is maximal: if the outerpath has interior faces that are not triangles, we temporarily triangulate them using additional edges, which we direct such that they do not induce directed cycles and which we remove after drawing the maximal outerpath.

Let G' be such a graph; see Figure 7a. Let $c_1, c_2, \dots, c_k$ be the vertices of degree at least 4 in G' (marked red in Figure 7). These vertices form a path (light red in Figure 7); let them be numbered along this path, which we call the *backbone* of G'. We assign every vertex v that does not lie on the backbone to a neighboring backbone vertex; if v is incident to an inner edge, we assign v to the other endpoint of that edge. Otherwise v has degree 2 and is incident to a unique backbone vertex via an outer edge, and we assign v to this backbone vertex. For $i \in [k]$, backbone vertex c_i induces, together with the vertices assigned to it, a fan F_i.

We draw the backbone in an x-monotone fashion. We start by drawing F_1 with the algorithm for drawing a fan as detailed in the proof of Lemma 8; see the leftmost gray box in



(a) a directed outerpath G'

(b) upward drawing of G' with at most two crossings per edge

Figure 7 Example in- and output of our drawing algorithm (edge crossings due to Lemma 8 are highlighted in yellow; other edge crossings are highlighted in orange).

Figure 7b. Then, for $i \in \{2, \ldots, k\}$, we set $x(c_i)$ to $x(c_{i-1})$ plus the number of inner edges incident to c_i and we set $y(c_i)$ depending (i) on the y-coordinates of the two neighbors of c_i that have already been drawn (c_{i-1} and the common neighbor u_{i-1} of c_{i-1} and c_i in F_{i-1}) and (ii) on the directions of the edges that connect these vertices to c_i; see, for example, the placement of c_5 in Figure 7b. Then we draw F_i with respect to the position of c_i, again using the algorithm from the proof of Lemma 8 with the following modifications. In general, vertices in F_i that are adjacent to c_i via an edge directed towards c_i (resp. from c_i) are placed below (resp. above) all vertices in the drawings of $F_1, \ldots, F_i$; see the dark gray boxes below (resp. above) $c_2, \ldots, c_5$ in Figure 7b. If an edge of F_i connects two neighbors of c_i one of which lies above c_i and one of which lies below c_i, then we route this edge to the left of all drawings of $F_1, \ldots, F_{i-1}$.

An exception to this rule occurs if c_i and the common neighbor w_i of c_{i-1} and c_i in F_i must be both above or both below c_{i-1} due to the directions of the corresponding edges. Let u_{i-1} be the common neighbor of c_{i-1} and c_i in F_{i-1}. We assume, without loss of generality, that c_i is above c_{i-1}. Let P_1 and P_2 be the first and second maximal subpath from Lemma 8 applied to F_i, and let e_i be the edge connecting P_1 and P_2. We distinguish two subcases.

If P_1 is a directed path leaving w_i, then we draw P_1 above the edge $c_{i-1}c_i$ and we draw the edge e_i straight, without going around all drawings of $F_1, \ldots, F_{i-1}$. In this case e_i is directed from P_1 to P_2. Hence, e_i crosses the edge $u_{i-1}c_i$ if $u_{i-1}c_i$ is directed from c_i to u_{i-1}; see the situation for c_2 in Figure 7b. Note that by Property 4 of Lemma 8, e_i may receive at most a second crossing when we draw the remainder of F_i in the usual way.

Otherwise, that is, if P_1 contains an edge directed towards the ~~left~~ first endpoint w_i of P_1, let f be the first such edge. We then place the part of P_1 up to the first endpoint of f below the edge $c_{i-1}c_i$; see w_5 and f in Figure 7b. If the edge $u_{i-1}c_i$ is directed towards c_i, we draw it between w_i and the edge $c_{i-1}c_i$. Then it crosses the edge $c_{i-1}w_i$ but no other edge. We place the second endpoint of f below all vertices in $V(F_1) \cup \cdots \cup V(F_{i-1})$ and continue with the remainder of F_i as usual.

In any case, if $1 < i < k$, then the last vertex u_{i-1} of F_{i-1} is connected to c_i and c_{i-1} is connected to the first vertex w_i in F_i. These two edges may cross each other; see the crossings highlighted in orange in Figure 7b. If the edge $c_{i-1}w_i$ goes, say, up but the following outer edges go down until a vertex v_k below c_i is reached, then the edge $c_{i-1}w_i$ may be crossed a second time by the edge $v_{k-1}v_k$; see the crossing labeled x on the edge c_3w_4 in Figure 7b. ~~But~~ Observe that in this case the path P_1 is the directed path from w_i to v_{k-1}. Thus, due to Property 4 of Lemma 8, edge $v_{k-1}v_k$ had been crossed at most once within its fan. Also $c_{i-1}w_i$ cannot have a third crossing. Thus, all edges are crossed at most twice. ◀

One can argue that every maximal pathwidth-2 graph can be generated from a maximal outerpath by connecting some pairs of adjacent vertices using an arbitrary number of (new) paths of length 2. In spite of the simplicity of this operation, we cannot hope to generalize the above result to pathwidth-2 graphs; see the linear lower bound on the upward local crossing number for such graphs stated in Theorem 3.

5 Testing Upward 1-Planarity

Here, we prove that upward 1-planarity testing is NP-complete even for structurally simple DAGs, both when a bimodal rotation system is fixed and when it is not fixed.

We start with a definition. Let G_1 and G_2 be any two st–digraphs. Let s_i be the source and t_i be the sink of G_i, with $i = 1, 2$. Let G be a digraph that contains both G_1 and G_2 as induced subgraphs. Let Γ be a drawing of G and let $\Gamma_{1,2}$ be the drawing obtained by restricting Γ to the ~~nodes~~ vertices and edges of $G_1 \cup G_2$. We say that G_1 and G_2 *fully cross* in Γ if in $\Gamma_{1,2}$ every s_1t_1-path (i.e., a path directed from s_1 to t_1) crosses every s_2t_2-path. See Figures 8a and 8b for examples of st-digraph G_1 and G_2 that do not fully cross or fully cross in a drawing of $\Gamma_{1,2}$, respectively.

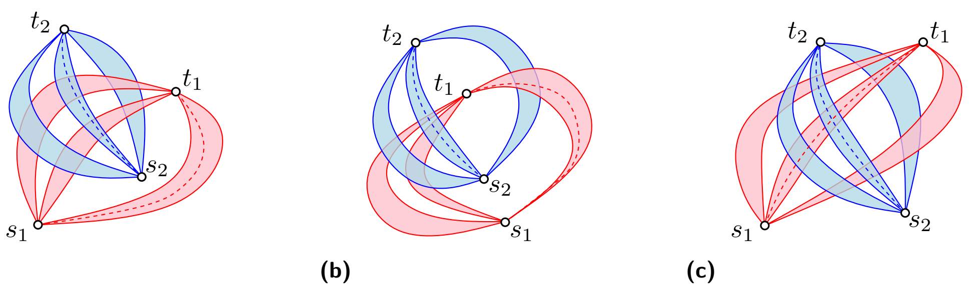
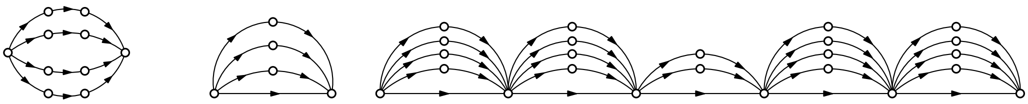


Figure 8 Illustrations for the definition of fully crossing st-subgraphs. (a) and (b) Two st-digraphs G_1 and G_2 that do not fully cross, as witnessed by the two non-crossing dashed paths. (c) Two st-digraphs G_1 and G_2 that fully cross.

We now define a few gadgets; all of them are planar st-graphs. For positive integers b and q, let a *(b, q)-parallel* be the parallel composition of b oriented paths each consisting of q edges; see Figure 9a. For a positive integer p, let a *(p)-gate* be the parallel composition of an oriented edge and a $(p - 1, 2)$-parallel; see Figure 9b. For positive integers h, q, and a, let an *(h, q, a)-chain* consist of a series of h (q)-gates, followed by exactly one (a)-gate, followed again by h (q)-gates; see Figure 9c.

An instance of 3-Partition is a multiset $I = \{a_1, a_2, a_3, \ldots, a_k\}$ of positive integers such that $b = k/3$ is an integer and $\sum_{i=1}^{k} a_i = W \cdot b$, with W integer. The 3-Partition problem



(a) a $(4, 3)$-parallel **(b)** a (4)-gate **(c)** a $(2, 5, 3)$-chain

Figure 9 Illustrations for the gadgets used in the construction of G_A and of G_B.

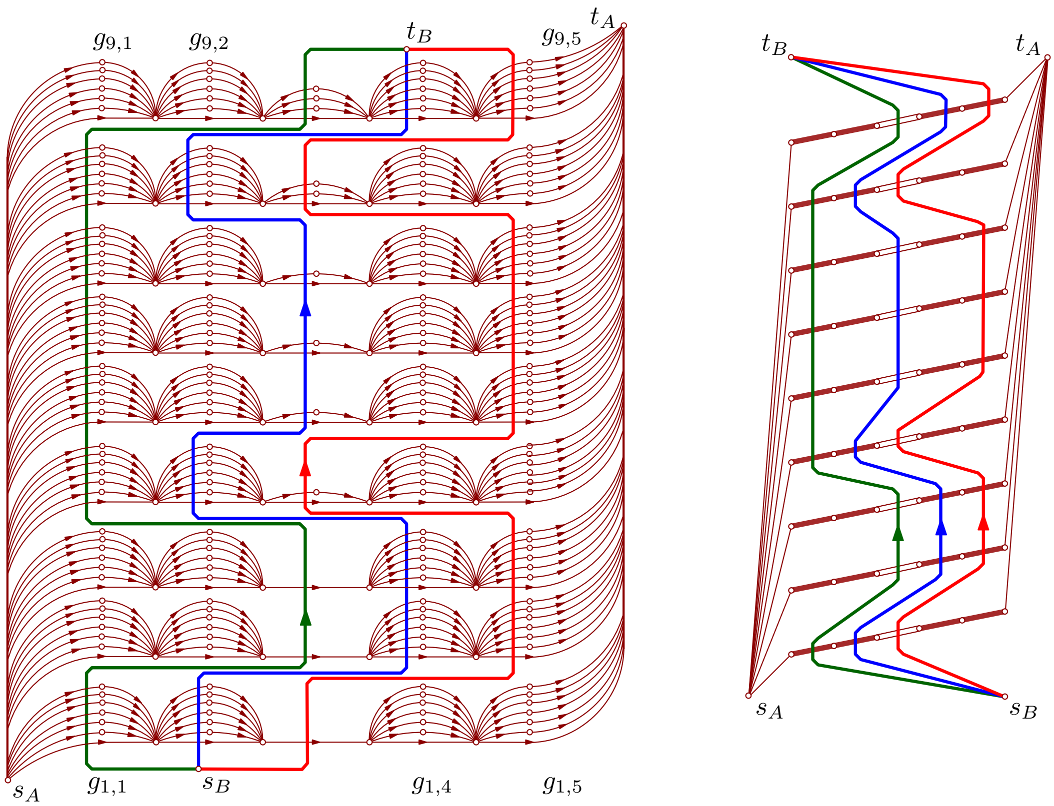
asks if there exists a partition of the set I into b 3-element subsets such that the sum of the elements in each subset is exactly W. Since 3-Partition is strongly NP-hard [26], we may assume that W is bounded by a polynomial in b.

We associate with a given instance I of 3-Partition two planar st-graphs G_A and G_B defined as follows. Digraph G_A is the parallel composition of $(b-1, W+1, a_i)$-chains, one for every $i \in \{1, \ldots, k\}$. Digraph G_B is a (b, q)-parallel, with $q = W + (k-3)(W+1)$. Note that the underlying undirected graphs of both G_A and G_B are series-parallel.

▶ **Theorem 10.** *Let I be an instance of 3-Partition and let G_A and G_B be the two planar st-graphs associated with I. Assume there exists a digraph G containing G_A and G_B as subgraphs with the following two properties: (i) if G is upward 1-planar then G_A fully crosses G_B in every upward 1-planar drawing of G; (ii) if there exists an upward 1-planar drawing of the union of G_A and G_B in which G_A and G_B fully cross, then there exists an upward 1-planar drawing of G. Then the digraph G is upward 1-planar if and only if I admits a solution.*

Proof. Assume that G is upward 1-planar and let Γ be any upward 1-planar drawing of G. We prove that Γ provides a solution of instance I of 3-Partition. By hypothesis (i), G_A and G_B fully cross in Γ. Observe that only one path among the b paths of the $\cancel{(q,b)}(b,q)$-parallel G_B can traverse the (a_i)-gate of a $(W+1, b-1, a_i)$-chain of G_A, where a_i, with $i = 1, \ldots, k$, is an element of the instance $I = \{a_1, a_2, a_3, \ldots a_k\}$ of 3-Partition, as otherwise the directed edge connecting the source and the sink of the (a_i)-gate would be traversed more than once. Since G_A and G_B fully cross in Γ, by definition every path of G_B crosses all the $(W+1, b-1, a_i)$-chains of G_A. In particular, every path of G_B must cross at least three (a_i)-gates. Indeed, if a path π of G_B crossed less than three (a_i)-gates of G_A, it ~~should~~ would cross at least $k-2$ of the $(W+1)$-gates in the $(W+1, b-1, a_i)$-chains of G_A. In order to have at most one crossing per edge, path π should have at least $(k-2)(W+1)$ edges; however, by construction, π has $W + (k-3)(W+1) = (k-2)(W+1) - 1$ edges. Also, observe that by definition of G_A, if one path of G_B crossed more than three (a_i)-gates, some other path of G_B should cross at most two (a_i)-gates. Therefore, every path π of G_B must cross exactly three (a_i)-gates and $k-3$ of the $(W+1)$-gates in Γ. Since every path π of G_B uses $(k-3)(W+1)$ of its edges to cross ~~the~~ $k-3$ of the $(W+1)$-gates in Γ, π can cross at most W edges of the (a_i)-gates. Since $\sum_{i=1}^{k} a_i = W \cdot b$, the number of crossings of each π with its three (a_i)-gates must be exactly W. It follows that if G has an upward 1-planar drawing then the instance I of 3-Partition admits a solution.

Conversely, assume that the instance I of 3-Partition admits a solution. In order to prove that G admits an upward 1-planar drawing, by hypothesis (ii) it suffices to construct an upward 1-planar drawing Γ_{AB} of G_A and G_B where G_A and G_B fully cross. We enumerate the paths of G_B as $\cancel{\pi_1, \pi_2, \cdots, \pi_b}\pi_1, \pi_2, \ldots, \pi_b$. We also enumerate the $2(b-1)$ many $(W+1)$-gates and the (a_i)-gate of a $(b-1, W+1, a_i)$-chain as $g_{i,1}, g_{i,2}, \cdots, g_{i,2(b-1)}$, where $g_{i,1}$ contains the source s_A of G_A and $g_{i,2(b-1)}$ contains the sink t_A of G_A (see also Figure 10).



(a) **(b)**

Figure 10 (a) Digraph G_A (dark red) and a schematic representation of digraph G_B where each colored curve represents a directed path with $W + (k-3)(W+1)$ edges. The corresponding instance of 3-Partition is $I = \{1, 1, 1, 2, 2, 2, 2, 3, 4\}$, with $b = 3$ and $W = 6$. The 1-planar drawing corresponds to the solution $\{1, 1, 4\}$ (green path), $\{2, 2, 2\}$ (blue path), and $\{1, 2, 3\}$ (red path). The drawing in (a) is not upward but it can be made upward by stretching it vertically as shown in (b), where thick edges represent (q)-gates and the central white-filled edges represent (a)-gates.

Consider a path π_j ~~,~~ with $1 \leq j \leq b$, and let $\{a_\chi, a_\lambda, a_\mu\}$ be the j-th bin of the solution of I. Let ν be an index such that $1 \leq \nu \leq b$.

- If ν is one of χ, λ, μ, then π_j crosses the (a_ν)-gate of the $(b-1, W+1, a_\nu)$-chain (see, for example, the red path crossing $g_{1,3}$ in Figure 10).
- If ν is not one of χ, λ, μ and the path π_h that crosses the (a_ν)-gate of the $(b-1, W+1, a_\nu)$-chain is such that $h > j$, then π_j crosses the gate $g_{\nu,j}$ of the $(W+1, b-1, a_\nu)$-chain (see, for example, the blue path crossing $g_{2,2}$ in Figure 10).
- Otherwise ($h < j$), path π_j crosses the gate $g_{\nu,b-1+j}$ of the $(b-1, W+1, a_\nu)$-chain (see, for example, the ~~green~~ red path crossing $g_{5,5}$ in Figure 10).

By the procedure above, if π_j crosses a gate $g(\nu, q)$, with $1 \leq \nu \leq k$ and $1 \leq q \leq 2(b-1)$, there is no other π_h, with $h \neq j$, such that π_h crosses $g(\nu, q)$. Also, the number of edges that π_j crosses is $a_\chi + a_\lambda + a_\mu + (k-3)(W+1) = W + (k-3)(W+1)$, which is the number of edges of π_j. Hence, it is possible to draw π_j in such a way that each of its edges is crossed exactly once and no edge of G_A is crossed more than once. This concludes the proof. ◀

▶ **Theorem 11.** *Testing upward 1-planarity is* NP*-complete even in the following restricted cases:*

1. *The bimodal rotation system is fixed, the DAG has exactly one source and exactly one sink, the underlying graph is series-parallel.*
2. *The bimodal rotation system is not fixed, the DAG has exactly one source and exactly one sink, the underlying planar graph is obtained by replacing the edges of a K_4 with series-parallel graphs.*
3. *The bimodal rotation system is not fixed, the underlying graph is series-parallel, there is one source and two sinks.*

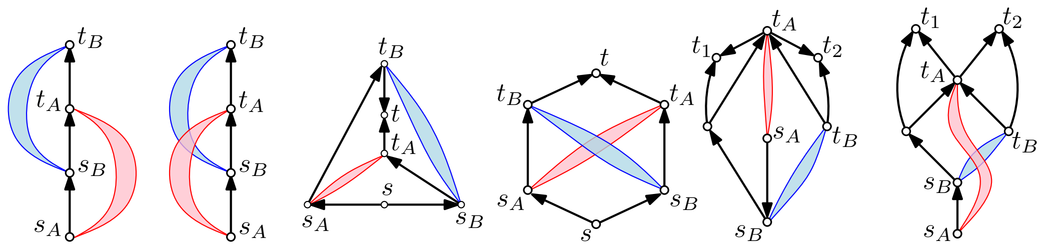
Proof. It is immediate to observe that upward 1-planarity testing is in the NP class of complexity (one can guess an upward embedding and test it in polynomial time). We show that it is NP-hard. For each case in the statement it suffices to exhibit a digraph G that contains G_A and G_B as sub-graphs and that satisfies the conditions of Theorem 10.

Let m_A and m_B be the number of edges of G_A and G_B, respectively. We ~~call~~ define a *barrier* ~~an *st*-digraph~~ to be an st-digraph consisting of a $(d, 2)$-parallel, where $d = m_A + m_B + 1$. Note that no s_At_A-path (s_Bt_B-path) can fully cross a barrier in such a way that every edge of the path is crossed at most once. This implies that neither G_A nor G_B can fully cross a barrier.

- Case 1: Refer to Figure 11a where the thick edges schematically represent barriers B_1 (from s_A to s_B), B_2 (from s_B to t_A), and B_3 (from t_A to t_B); in the figure, the red shaded shape represents G_A and the blue shaded shape represents G_B. The figure schematically represents a series-parallel digraph G with source s_A and sink t_B. As described in Figure 11b an upward 1-planar drawing where the edges of G_A precede the edges of B_1 in the left to right order of the edges exiting s_A and the edges of G_B precede the edges of B_3 in the left to right order of the edges entering t_B exists if and only if G_A crosses G_B. In fact, in any upward 1-planar drawing of G that preserves its bimodal rotation system G_A crosses G_B; conversely, starting from an upward 1-planar drawing of the union of G_A and G_B in which G_A and G_B cross, an upward 1-planar drawing of G that preserves the bimodal rotation system of Figure 11a can be obtained as shown in Figure 11b.
- Case 2: We modify a very well known small digraph that is traditionally used to exhibit a digraph that is planar but not upward planar [8, 16]. As above, in Figure 11c thick edges schematically represent barriers, the red shaded shape represents G_A, and the blue shaded shape represents G_B. Again, an upward 1-planar drawing of this digraph exists if and only if G_A fully crosses G_B (see Figure 11d). In particular, neither s_B nor t_B can be drawn in the finite region of the plane bounded by G_A since at least one s_At_A-path would fully cross a barrier. Similarly, neither s_A nor t_A can be drawn in the finite region of the plane bounded by G_B. Observe that the digraph has a single source and a single sink and its underlying planar graph is obtained by replacing the edges of a K_4 with series-parallel graphs.
- Case 3: Consider the digraph G of Figure 11e, where the thick edges schematically represent barriers; as in the previous case, the red shaded shape represents G_A and the blue shaded shape represents G_B. With analogous arguments as in the previous case, it is immediate to see that G has an upward 1-planar drawing if and only if G_A fully crosses G_B (see Figure 11f).

◀

The following corollary is a consequence of the argument used to prove the second case in the statement of Theorem 11.



(a) (b) (c) (d) (e) (f)

Figure 11 Some digraphs for the proof of Theorem 11. Thick black edges represent barriers.

▶ **Corollary 12.** *Testing upward 1-planarity is* NP*-complete for single source-single sink DAGs with a fixed bimodal rotation system, whose underlying planar graph is obtained by replacing the edges of a* K_4 *with series-parallel graphs.*

We conclude this section by remarking some differences between the complexity of upward planarity testing and upward 1-planarity testing. When the bimodal rotation system is fixed, upward planarity testing can be solved in polynomial time [7], whereas upward 1-planarity testing is NP-hard (Theorem 11). Also, when the bimodal rotation system is not fixed and the digraph has a constant number of sources and sinks, differently from upward 1-planarity testing, upward planarity testing can again be solved in polynomial time [13]. On the other hand, any digraph whose bimodal rotation system is not fixed, whose underlying graph is series-parallel, and that has only one source and only one sink is always upward planar and thus upward 1-planar. Indeed, adding an edge between any two vertices of the undirected underlying series-parallel graph yields a planar graph (see, e.g., [18]). It follows that G can be turned into a planar st-graph by connecting its source to its sink by an edge and hence it is upward planar [27]. This discussion is summarized in Table 1.

6 Testing Upward Outer-1-Planarity

Theorem 11 motivates the study of the complexity of testing upward 1-planarity with additional constraints. A common restriction in the study of beyond-planar graph drawing

Underlying planar graph	**Acyclic orientation**	**Upward planarity**: fixed embedding	**Upward planarity**: variable embedding	**Upward 1-planarity**: fixed rotation system	**Upward 1-planarity**: variable rotation system
Series-parallel	**multi-source multi-sink**	Polynomial [14, 19]	Polynomial [14, 19]	NP-complete Theorem 11 Case 1	NP-complete Theorem 11 Case 3
Series-parallel	**single-source single-sink**	Polynomial [13, 14, 19]	Polynomial [13, 14, 19]	NP-complete Theorem 11 Case 1	Trivially polynomial
General graph	**multi-source multi-sink**	Polynomial [7]	NP-complete [28]	NP-complete Corollary 12	NP-complete Theorem 11 Case 2
General graph	**single-source single-sink**	Polynomial [13]	Polynomial [13]	NP-complete Corollary 12	NP-complete Theorem 11 Case 2

Table 1 A comparison between results in the literature about the complexity of testing upward planarity and the results discussed in this paper about the complexity of testing upward 1-planarity.

problems is the one that requires that all vertices are incident to the same face. Specifically, a graph is *upward outer-1-planar* if it admits an upward 1-planar embedding in which every vertex lies on the outer face. This section is devoted to the proof of the following result.

▶ **Theorem 13.** *For single-source DAGs, upward outer-1-planarity can be tested in linear time.*

6.1 Basic Facts and Definitions

We provide the following characterization for the single-source DAGs admitting an upward outer-1-planar drawing.

▶ **Theorem 14.** *A single-source graph is upward outer-1-planar if and only if it admits an outer-1-planar embedding whose planarization is acyclic.*

Proof. Let G be a single-source graph with an upward outer-1-planar embedding Γ. Clearly, Γ is outer-1-planar. Moreover, planarizing Γ yields an upward drawing of the planarization, hence the planarization is acyclic.

Conversely, let Γ be an outer-1-planar embedding whose planarization $\Gamma^\star$ is acyclic. Observe that no crossing vertex of $\Gamma^\star$ is a source or a sink. Therefore, $\Gamma^\star$ only has a single source s and the sinks of G, which are all incident to the outer face. Let Γ^+ be obtained from $\Gamma^\star$ by adding a new super sink t, with edges from all sinks of G to t into the outer face. Clearly Γ^+ is acyclic and a planar st-graph, and therefore upward planar [17]. Removing t then yields an upward planar drawing of $\Gamma^\star$ and hence Γ is upward outer-1-planar. ◀

The following lemma follows from Theorem 14 and shows that we can decompose the problem on the biconnected components of the input graph.

▶ **Lemma 15.** *A single-source graph is upward outer-1-planar if and only if each of its blocks admits an upward outer-1-planar embedding.*

Proof. The necessity is trivial. For the sufficiency, suppose that all blocks admit an upward outer-1-planar embedding, whose planarization is thus acyclic. Combine all such planarizations at the cut-vertices so that all their vertices are incident to the outerface. Since each cycle in the graph is contained inside a block, this yields an acyclic planarization of an outer-1-planar embedding of the entire graph. This and Theorem 14 imply the statement. ◀

The SPQR-tree data structure, introduced by Di Battista and Tamassia [18], to represent the decomposition of a biconnected graph into its triconnected components is a special type of decomposition tree [35]. A *decomposition tree* of a biconnected graph G is a tree $\mathcal{T}$ whose nodes μ are associated with a biconnected multi-graph ~~$\mathrm{skel}\mu$~~ $\mathrm{skel}(\mu)$, called *skeleton*. The edges of a skeleton can be either *real* or *virtual*, and for each node μ, the virtual edges of $\mathrm{skel}(\mu)$ correspond bijectively to the edges of $\mathcal{T}$ incident to μ. The tree $\mathcal{T}$ can be inductively defined as follows. In the base case, $\mathcal{T}$ consists of a single node μ whose skeleton is the graph G consisting solely of real edges. In the inductive case, let μ be any node of $\mathcal{T}$ and let H_1 and H_2 be edge-disjoint connected subgraphs of $\mathrm{skel}(\mu)$ such that ~~$\mathrm{skel}\mu = H_1 \cup H_2$~~ $\mathrm{skel}(\mu) = H_1 \cup H_2$ and $H_1 \cap H_2 = \{u, v\}$. We obtain a new decomposition tree $\mathcal{T}'$ from $\mathcal{T}$ by splitting the node μ into two adjacent nodes ν_1 and ν_2 whose skeletons are $H_1 + uv$ and $H_2 + uv$, respectively, where uv is a new virtual edge that corresponds to the edge $\nu_1\nu_2$ of $\mathcal{T}'$; also, we replace each edge $\mu\tau$ of $\mathcal{T}$ either with the edge $\nu_1\tau$ or with the edge $\nu_2\tau$ depending on wether edge $\mu\tau$ corresponds to a virtual edge of H that belongs to H_1 or to H_2, respectively. The edges of $\mathcal{T}$ incident to μ are distributed among ν_1 and ν_2 based on whether their corresponding virtual edge belongs to H_1 or H_2. Let now $\mathcal{T}$ be an arbitrary decomposition tree of a biconnected graph. Note that each edge of G occurs as a real edge in precisely one skeleton of $\mathcal{T}$. Consider a virtual edge ~~$\{a,b\}$~~ e in the skeleton of some node μ of $\mathcal{T}$ and let ~~$e_{a,b}$~~ ε be the edge of $\mathcal{T}$ which corresponds to ~~$\{a,b\}$~~e. The *expansion graph* ~~expn($\{a,b\}$) of $\{a,b\}$~~ expn(e) of e is the graph obtained as the union of the real edges belonging to the skeletons of the subtree of $\mathcal{T}$ reachable from μ via the edge ~~$e_{a,b}$~~ε. Observe that expn($\{a,b\}$) is connected to the rest of the graph through a and b. The vertices a and b are the *poles* of expn($\{a,b\}$). For every pair of adjacent nodes μ and ν of $\mathcal{T}$, there exists a virtual edge e in skel(μ) and a virtual edge e' in skel(ν) with the same end-vertices. We say that μ (resp. ν) is the *refining* node refn(e') of e' (resp. refn(e) of e).

The SPQR-tree $\mathcal{T}$ of a biconnected graph G is a decomposition tree that has three types of nodes. The skeleton of an *S-node* is a simple cycle of length at least 3, the skeleton of a *P-node* consists of two vertices and at least three parallel edges, and the skeleton of an *R-node* is a simple 3-connected graph. Moreover, no two S-nodes and no two P-nodes are adjacent in $\mathcal{T}$. We remark that unlike the classical definition of SPQR-trees, we allow real edges in any skeleton; this avoids the need to use Q-nodes whose skeletons have two vertices and one real and one virtual edge.

The SPQR-tree of a biconnected planar graph G can be used to succinctly represent all planar embeddings of G. Specifically, any planar embedding of G uniquely defines an embedding for all the skeletons of $\mathcal{T}$. Moreover, recursively combining planar embeddings of the skeletons of $\mathcal{T}$ via 2-clique sums of such embeddings results in a planar embedding of G. The *2-clique-sum* of two embeddings $\mathcal{E}'$ and $\mathcal{E}''$ containing the virtual edge $\{u,v\}$ identifies the two copies of u and the two copies of v, removing the edge $\{u,v\}$, in such a way that edges in $\mathcal{E}'$ do not alternate with edges in $\mathcal{E}''$. Even in the presence of non-planar embeddings of skeletons, such an operation can be performed in the same fashion as long as the common virtual edge $\{u,v\}$ is crossing free in both embeddings.

Let μ be a P-node with skeleton vertices $\{u,v\}$, let e be a virtual edge in skel(μ) that is refined by an S-node λ with skeleton $(u, c_1, c_2, \dots, c_k, v, u)$. Then the *first edge* of e at u is the (virtual or real) edge $\{u, c_1\}$ of skel(λ) and the *first* ~~*edge*~~*segment* of e at v is the (virtual or real) edge $\{c_k, v\}$ of skel(λ).

6.2 Proof of Theorem 13

There are two slightly different algorithms for testing outer-1-planarity [4,32,33]. It is likely that both can be adapted to test upward outer-1-planarity. We choose to follow the approach by Auer et al. [4], which we briefly summarize in the following, as it seems slightly simpler to extend to our setting. Two key properties that both papers leverage are the facts that if G is an outer-1-planar graph and $\mathcal{T}$ is its SPQR-tree, then the skeleton of each R-node of $\mathcal{T}$ is a K_4 that contains two non-adjacent real edges and the skeleton of each P-node has at most four virtual edges and at most one real edge. In what follows, we assume that these two properties are satisfied.

6.2.1 The Outer-1-Planarity Testing Algorithm of Auer et al.

By Lemma 15, we may assume that our input graph G is biconnected and acyclic. Consider a decomposition tree $\mathcal{T}$ of G (not necessarily the SPQR-tree). Let μ be a node of $\mathcal{T}$. We call an outer-1-planar embedding of skel(μ) *good* if only real edges cross and each virtual edge is incident to the outer face.

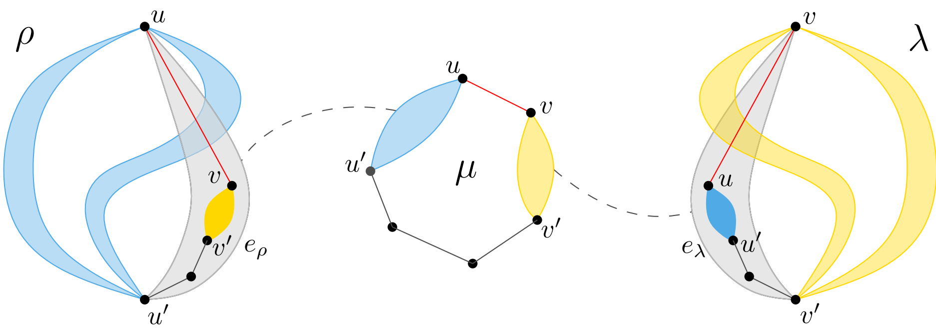


Figure 12 ~~Illustration for~~ A portion of an SPQR-tree with an S-node μ and two adjacent S-nodes ρ, λ illustrating the necessity of Condition C5. Since neither the segment of e_ρ incident to u is incident to the outer face in the embedding of skel(ρ) nor the segment of e_λ incident to v is incident to the outer face in the embedding of skel(λ), the real edge $\{u, v\}$ must be involved in at least two crossings. Namely, at least one crossing with an edge in the expansion graph of the blue virtual edges of skel(ρ) and at least one crossing with an edge in the expansion graph of the yellow virtual edges of skel(λ).

▶ **Observation 16.** *Good embeddings of all skeletons of $\mathcal{T}$ together define a unique outer-1-planar embedding of G, which is obtained by forming the 2-clique sums of the embeddings of the skeletons.*

However, if we consider the SPQR-tree, good embeddings of the skeletons may not exist even if G is outer-1-planar. Therefore, Auer et al. [4] work with a more general definition of outer-1-planar embeddings for skeletons of the SPQR-tree and show that an outer-1-planar embedding of G can be combinatorially described by embeddings of the skeletons of its SPQR-tree $\mathcal{T}$ that satisfy the following five conditions C1–C5. We note that the edges of the skeletons may cross even for P-nodes, i.e., the embeddings of the skeletons we consider are not necessarily simple drawings.

C1 Each skeleton is embedded outer-1-planar such that each virtual edge has a segment that is incident to the outer face [4, Proposition 3].

C2 Skeletons of S-nodes are embedded planarly [4, Corollary 3].

C3 Skeletons of R-nodes are embedded ~~with~~ such that virtual edges are crossing-free and precisely two non-virtual edges ~~crossing~~ cross each other [4, Corollary 2].

C4 If an edge $e = \{u, v\}$ of the skeleton of a P-node is crossed, then it is a virtual edge corresponding to an S-node. Moreover, if the segment of e incident to u (incident to v) is not incident to the outer face, then the first ~~edge~~ segment of e at u (at v) must be a real edge [4, Lemma 2].

C5 Let μ be an S-node whose skeleton contains a real edge $\{u, v\}$ and two adjacent virtual edges $\{u', u\}$ and $\{v, v'\}$ that are refined by ρ and λ, respectively. Let e_ρ, e_λ be the virtual edges that represent μ in skel(ρ) and skel(λ), respectively. If both ρ and λ are P-nodes, then the segment of e_ρ incident to u must be incident to the outer face in the embedding of skel(ρ) or the segment of e_λ incident to v must be incident to the outer face in the embedding of skel(λ).

The idea behind Conditions C4 and C5 is that crossing two virtual edges e, e' in a skeleton of a P-node μ with vertices $\{u, v\}$ in such a way that e is not incident to the outer face at u and e' is not incident to the outer face at v implies that we cross the first ~~edge~~ segment of e at u and the first ~~edge~~ segment of e' at v; see the gray-shaded regions in Figure 12 and the more formal discussion later. Condition C4 guarantees that these are real edges. On the

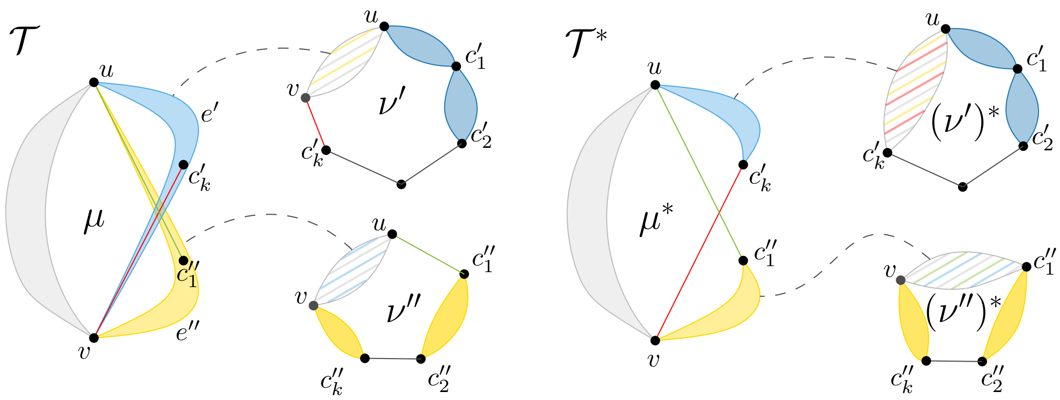


Figure 13 Illustration for the extension of a P-node μ.

other hand, Condition C5 ensures that each real edge receives at most one crossing in this way; see Figure 12 for an illustration of a case in which this is violated.

It follows from the work of Auer et al. [4] that together these conditions describe all outer-1-planar embeddings of G without unnecessary crossings (i.e., there is no outer-1-planar drawing whose crossing edge pairs form a strict subset). We summarize this ~~as follows.~~ in the following theorem. As this result does not appear explicitly in the work of Auer et al., for the sake of completeness, we briefly sketch the construction. In particular, the construction of an embedding of G from the embeddings of the skeletons is crucial for understanding our next steps.

▶ **Theorem 17.** *Let G be a biconnected graph and let $\mathcal{T}$ be its SPQR-tree. There is a bijection between the outer-1-planar embeddings of G without unnecessary crossings and the choice of an embedding for each skeleton of $\mathcal{T}$ that satisfy Conditions C1–C5. Moreover, both directions of the bijection can be computed in linear time.*

~~In the following, we briefly sketch~~

Proof. For the bijection, observe that the necessity of C1–C5 has been argued above. For the converse, we show how we obtain an embedding of G from embeddings $\{\mathcal{E}_\mu\}_{\mu \in V(\mathcal{T})}$ for each skeleton skel(μ) that satisfy C1–C5~~, as this is crucial for understanding our next steps~~. We refer to this embedding as the *combined embedding of* $\{\mathcal{E}_\mu\}_{\mu \in V(\mathcal{T})}$.

First note that, if there are no crossings that involve virtual edges in the embeddings of the skeletons, then the embeddings are good and their 2-clique sum yields an outer-1-planar embedding of G by Observation 16. If there are virtual edges that are involved in a crossing, they belong to the skeletons of P-nodes by Conditions C2 and C3. We reduce to the case of good embeddings by *extending* such P-nodes, that is, by modifying the SPQR-tree, its skeletons, and their embeddings as follows; refer to Figure 13. Let e', e'' be two crossing virtual edges in an embedding of a skeleton of a P-node μ. Let $\{u, v\}$ be the vertices of skel(μ). By Condition C4, we have that the nodes $\nu' = \text{refn}(e')$ and $\nu'' = \text{refn}(e'')$ are S-nodes with skeletons $(u, c'_1, \dots, c'_k, v, u)$ and ~~$(u, c''_1, \dots, c''_k, v, u)$~~$(u, c''_1, \dots, c''_\ell, v, u)$, respectively. Without loss of generality, we may assume that the segment of e' incident to u is incident to the outer face in the embedding of skel(μ), and hence by Condition C1 the segment of e'' incident to v is also incident to the outer face. By Condition C4, the edge $\{c'_k, v\}$ of skel(ν'), which is the first edge of e' at v, and the edge $\{u, c''_1\}$ of skel(ν''), which is the first ~~edge~~ segment of e'' at u, are real edges. We perform the following modifications. First, we shorten the skeletons of ν'

and of ν'' to $(u, c'_1, \ldots, c'_{k-1}, c'_k, u)$ and ~~$(c''_1, c''_2, \ldots, c''_k, v, c''_1)$~~$(c''_1, c''_2, \ldots, c''_\ell, v, c''_1)$, respectively, and keep their embeddings planar. Second, we subdivide the virtual edge e' (e'') with the vertex c'_k (c''_1) such that $\{v, c'_k\}$ ($\{u, c''_1\}$) is a real edge and $\{c'_k, u\}$ ($\{c''_1, v\}$) is a virtual edge that is refined by the modified S-node ν' (ν''). We embed the skeletons of such nodes so that the real edges $\{v, c'_k\}$ and $\{u, c''_1\}$ cross and the two virtual edges $\{u, c'_k\}$ and $\{v, c''_1\}$ are incident to the outer face. Note that this essentially *moves* two real edges from S-node skeletons into an adjacent P-node. See the right side of Figure 13 for the modified skeletons.

Condition C5 guarantees that no real edge needs to be moved into the skeletons of two different P-nodes. Hence we can apply this operation simultaneously and independently to all pairs of crossing virtual edges. We then arrive at a decomposition tree $\mathcal{T}^*$ of G, ~~called the *extension of* $\mathcal{T}$ *with respect to* $\{\mathcal{E}_\mu\}_{\mu \in V(\mathcal{T})}$,~~ whose skeletons have good embeddings. As explained above, we then obtain an outer-1-planar embedding of G.

Observe that this construction can be carried out in linear time. Conversely, given an outer-1-planar embedding $\mathcal{E}$ of G. The embeddings $\mathcal{E}_\mu$ of all skeletons skel(μ) can be obtained by checking the order of the incident edges for each vertex and the pairs of edges that cross. Moreover, it is then clear that applying the above construction to the $\mathcal{E}_\mu$, we reobtain $\mathcal{E}$.

◀

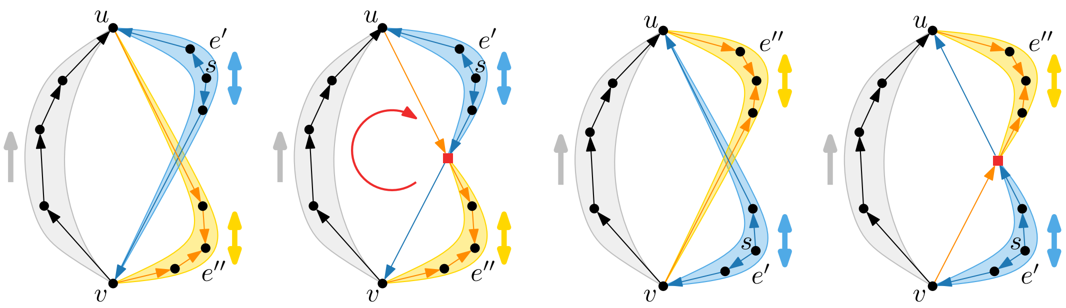
The tree $\mathcal{T}^\star$ we constructed towards the end of the proof of Theorem 17 is called the *extension of* $\mathcal{T}$ *with respect to* $\{\mathcal{E}_\mu\}_{\mu \in V(\mathcal{T})}$. Since the modification neither adds nor removes nodes of $\mathcal{T}$, there is a bijection between the nodes of $\mathcal{T}$ and $\mathcal{T}^*$. Namely, each node μ of $\mathcal{T}$ contributes a unique node μ^* to $\mathcal{T}^*$, and these are all the nodes of $\mathcal{T}^*$. The node μ^* is called the *extended node of* μ and the good embedding of skel(μ^*) obtained with the procedure described above is called the *extended embedding* of skel(μ). We further note that for P- and R-nodes μ, skel$(\mu^\star)$ only depends on the embedding $\mathcal{E}_\mu$ of skel(μ). Thus, for P-nodes and R-nodes skel$(\mu^\star)$ can also be defined if we only have an embedding of skel(μ) that satisfies C1–C4. Condition C5 is only needed if we want to combine such skeletons into the whole extension $\mathcal{T}^\star$.

6.2.2 An Algorithm to test Upward Outer-1-Planarity.

Our goal is to find an outer-1-planar embedding of G whose planarization is acyclic. Observe that it suffices to consider outer-1-planar embeddings without unnecessary crossings, since adding more crossings cannot make a cyclic planarization acyclic. Our goal is therefore to find outer-1-planar embeddings of the skeletons of the SPQR-tree $\mathcal{T}$ of G satisfying Conditions C1–C5 so that the resulting planarization of G is acyclic.

Directing virtual edges. To keep track of the existence of directed paths, we orient some edges of the skeletons of $\mathcal{T}$. Consider a skeleton skel(μ) and let $e = \{u, v\}$ be an edge of skel(μ). We orient e from u to v if either it is a real edge directed from u to v in G or if expn(e) contains a directed path from u to v. ~~Note~~

We observe some properties of the constructed orientation. First, note that a virtual edge $\{u, v\}$ is oriented in at most one direction, but it may also remain undirected in case its expansion graph contains neither a directed path from u to v nor a directed path from v to u. ~~If~~ Second, if a virtual edge ~~$\{u, v\}$~~ e with endpoints u, v is directed, say from u to v, then u is the single source of expn(e)~~. If~~, as otherwise G would contain an additional source. Finally, if e is undirected, since G has a single source s, either expn(e) contains s as a non-pole vertex or both u and v are sources of expn(e).



(a) **(b)**

Figure 14 Two embeddings of the skeleton of a P-node μ and the corresponding planarizations. The planarization in (a) contains a directed cycle, the one in (b) does not. Thick arrowed edges show the direction of the virtual edges; a double arrow indicates an undirected virtual edge.

Acyclic Embeddings. Suppose that the edges of the skeletons are directed as described above, let μ be a node with an embedding $\mathcal{E}_\mu$ that satisfies C1–C4. Recall from the remark after the proof of Theorem 17 that skel$(\mu^\star)$ and its embedding $\mathcal{E}^\star_\mu$ depend only on $\mathcal{E}_\mu$. We call $\mathcal{E}_\mu$ *acyclic* if the planarization of the extended embedding $\mathcal{E}^\star_\mu$ is acyclic; see Figure 14 for examples. It is not surprising that indeed such embeddings are necessary for the existence of an upward outer-1-planar embedding of G.

▶ **Lemma 18** (Necessity). *Let $\mathcal{E}$ be an upward outer-1-planar embedding of G without unnecessary crossings. Then the embedding of* skel(μ) *induced by $\mathcal{E}$ is acyclic for each ~~node~~P- or R-node μ of the SPQR-tree $\mathcal{T}$ of G.*

Proof. By Theorem 17,~~–~~ $\mathcal{E}$ induces embeddings $\mathcal{E}_\mu$ for each skeleton skel(μ) of $\mathcal{T}$, which satisfy the Conditions C1–C5. Let $\mathcal{T}^*$ be the extension of $\mathcal{T}$ with respect to these embeddings and recall from the proof of Theorem 17 that $\mathcal{E}$ can be obtained as the 2-clique sum of the embeddings $\mathcal{E}^\star_\mu$. Let μ be a ~~node~~ P- or R-node of $\mathcal{T}$ with embedding $\mathcal{E}_\mu$ and let μ^* be the extended node in $\mathcal{T}^*$ with embedding $\mathcal{E}^*_\mu$. As the only crossings in $\mathcal{E}^*_\mu$ are between real edges, we can form its planarization by replacing each crossing by a dummy vertex, thereby subdividing the involved directed edges into paths of length 2. Since the upward embedding $\mathcal{E}$ in particular induces an upward embedding of its planarization, it follows that the planarization of $\mathcal{E}^\star_\mu$ does not contain a cycle that consists of directed edges, i.e., $\mathcal{E}_\mu$ is acyclic. ◀

▶ **Lemma 19** (Sufficiency). *Let $\{\mathcal{E}_\mu\}_{\mu\in V(\mathcal{T})}$ be embeddings of the skeletons of $\mathcal{T}$ that satisfy Conditions C1–C5. If~~each~~ $\mathcal{E}_\mu$ is acyclic for all P- and R-nodes, then the combined embedding of $\{\mathcal{E}_\mu\}_{\mu\in V(\mathcal{T})}$ is an outer-1-planar embedding of G whose planarization is acyclic.*

Proof. Let $\mathcal{T}^\star$ be the extension of $\mathcal{T}$ with respect to the embeddings $\{\mathcal{E}_\mu\}_{\mu\in V(\mathcal{T})}$ of the skeletons and let $\mathcal{E}$ denote the combined embedding of $\{\mathcal{E}_\mu\}_{\mu\in V(\mathcal{T})}$. By Theorem 17, $\mathcal{E}$ is an outer-1-planar embedding of G. In the remainder, we show that its planarization G' is acyclic. Thus, assume for the sake of contradiction that G' contains a directed cycle C.

We will consider the *projection* of C to the planarizations of the skeletons of the nodes of $\mathcal{T}^\star$. Let $\mu^\star$ be a node of $\mathcal{T}^\star$ and consider its planarization skel$^+(\mu^\star)$. An edge e of skel$^+(\mu^\star)$ belongs to the projection of C if either (i) e is a real edge that belongs to C or (ii) e is a virtual edge whose expansion graph contains a real edge that belongs to C. Observe that for each planarized skeleton, the projection of C is either a cycle C' or a single edge. In the former case, we also say that C *projects* to the cycle C'.

Claim: There exists a ~~node~~P- or R-node $\mu^\star$ of $\mathcal{T}^\star$ where the cycle C projects to a cycle C' in the planarization $\mathrm{skel}^+(\mu^\star)$ such that the source s of G is either a vertex of $\mathrm{skel}^+(\mu^\star)$ or belongs to the expansion graph of an edge of $\mathrm{skel}^+(\mu^\star)$ that does not belong to C'.

To prove the claim, observe that the skeletons where C projects to a cycle form a subtree $\mathcal{T}_C$ of $\mathcal{T}^\star$. Similarly, the nodes whose skeletons contain s as a vertex also form a subtree $\mathcal{T}_s$ of $\mathcal{T}^\star$. We choose $\mu^\star$ either as a common node of these two subtrees (if one exists) or as the node of $\mathcal{T}_C$ that is closest to the subtree $\mathcal{T}_s$. In the former case, we have that s is a vertex of $\mathrm{skel}^+(\mu^\star)$. Note that, since s is a source, it is not contained in C and therefore it is also not contained in the projection of C to $\mathrm{skel}^+(\mu^\star)$. In particular $\mu^\star$ cannot be an S-node. In the latter case let e denote the virtual edge of $\mathrm{skel}^+(\mu^\star)$ whose expansion graph contains s. If e is contained in the projection of C to $\mathrm{skel}^+(\mu^\star)$, then $\mathrm{refn}(e)$ belongs to $\mathcal{T}_C$ and is closer to $\mathcal{T}_s$ than $\mu^\star$, which contradicts the choice of $\mu^\star$. Note that this in particular excludes the case that $\mu^\star$ is an S-node. This completes the proof of the claim.

Consider the node $\mu^\star$ from the claim and let μ be the corresponding node of $\mathcal{T}$. The projection C' of C to $\mathrm{skel}^+(\mu^\star)$ may contain both real and virtual edges. If all edges of C' are directed, the cycle is already present in $\mathrm{skel}^+(\mu^\star)$, i.e., the embedding $\mathcal{E}_\mu$ of $\mathrm{skel}(\mu)$ is not acyclic; a contradiction. Since real edges are always directed, it hence follows that C' contains an undirected virtual edge $\{u,v\}$ of $\mathrm{skel}(\mu^\star)$. Since in expanded skeletons only real edges have crossings, the edge $\{u,v\}$ is uncrossed in the embedding of $\mathrm{skel}(\mu^\star)$. Therefore $\mathrm{expn}(\{u,v\})$ is not crossed by edges outside of $\mathrm{expn}(\{u,v\})$. In particular the planarization of $\mathrm{expn}(\{u,v\})$ is ~~a split component of $\{u,v\}$~~ connected to the rest of the graph only via the vertices u and v in the planarization G' of G. Since $\{u,v\}$ belongs to C', the planarization of $\mathrm{expn}(\{u,v\})$ contains a directed path π between its two poles, say from u to v. However, since $\{u,v\}$ is undirected and does not contain s, both u and v are sources in $\mathrm{expn}(\{u,v\})$ and hence they are sources also in its planarization. This contradicts the existence of π, and hence proves the lemma. ◀

In light of Lemmas 18 and 19, it suffices to test whether the skeletons of $\mathcal{T}$ admit ~~acyclic~~ embeddings that satisfy C1–C5 and that are acyclic for P- and R-nodes. We note that C1–C4 and acyclicity are local conditions that can be checked and all solutions can be enumerated[1] locally and independently for each skeleton. On the other hand C5 is a global property, which states that for each real edge e that connects two P-nodes in a series, only one of them may put a crossing on e.

For each node μ of $\mathcal{T}$, let $\mathcal{F}_\mu$ be a subset of the embeddings of $\mathrm{skel}(\mu)$ that satisfy C1–C4. We call $\mathcal{F}_\mu$ the *feasible embeddings* of $\mathrm{skel}(\mu)$. We are interested in whether we can choose for each node μ a feasible embedding $\mathcal{E}_\mu \in \mathcal{F}_\mu$ such that together they satisfy C5. We call such a choice of embeddings *consistent*.

To find a consistent choice of embeddings, we construct an auxiliary graph H whose vertex set is $\bigcup_{\mu \in \mathcal{T}} \mathcal{F}_\mu$. We turn each set $\mathcal{F}_\mu$ into a clique and we connect two embedding choices of different P-nodes if choosing both of them simultaneously violates C5. It is then clear that H contains an independent set whose size equals the number of nodes of $\mathcal{T}$ if and only if there exists a consistent choice of embeddings. In the following we show that this can be decided in linear time by proving that H has bounded treewidth [15].

▶ **Lemma 20.** *The auxiliary graph H has bounded treewidth.*

Proof. We first consider the following construction. Let T be a tree, let I be an independent set of vertices in T, and let c be a constant. Let T' be the graph obtained by connecting

[1] Recall that P-node skeletons have at most five edges and each R-node skeleton is a K_4.

for each vertex in I its neighbors by a cycle that visits them in some arbitrary order. We say that T' is a *closure* of T with respect to I. Since I is an independent set, each block of T' is either an edge or a wheel, and hence T' has treewidth at most 3. Let T'' be obtained from T' by expanding each vertex v of T' into a clique C_v of size at most c such that each edge uv of T' is expanded into a biclique completely connecting C_u and C_v. We call T'' the *c-clique expansion* of T'. Clearly, the treewidth of T'' is at most $3c$.

We now use this to bound the treewidth of H. For each S-node λ of $\mathcal{T}$ the skeleton skel(λ) defines a circular ordering of its virtual edges. Note that embeddings of two different P-nodes μ and μ' can be connected by an edge only if they have a common S-node neighbor λ such that their corresponding virtual edges in skel(λ) are consecutive in skel(λ). Therefore the auxiliary graph H is a subgraph of the graph obtained from the SPQR-tree $\mathcal{T}$ by (i) forming the closure T' of $\mathcal{T}$ with respect to the (independent set of) S-nodes, where we connect the neighbors of the S-node in the order in which the corresponding virtual edges appear in the skeleton of the S-node and (ii) taking the c-clique-expansion of T' where c is an upper bound on the number of embeddings satisfying C1–C4 for any skeleton. Auer et al. [4] show that $c \leq 12$. Hence the treewidth of H is at most 36. ◀

▶ **Theorem 21.** *There is a linear-time algorithm for testing whether a given single-source graph admits an upward outer-1-planar embedding.*

Proof. By Lemma 15 we may assume that G is biconnected. We first compute the SPQR-tree $\mathcal{T}$ of G in linear time [30]. Next, we check that the skeleton of each R-node of $\mathcal{T}$ is a K_4, and that the skeleton of each P-node of $\mathcal{T}$ has at most four virtual edges. If this fails, we can reject the instance as it does not have an outer-1-planar embedding [4]. Next, we compute for each node μ of $\mathcal{T}$ the set $\mathcal{F}_\mu$ that contains all acyclic embeddings of skel(μ) that satisfy conditions C1–C4. Since skeletons of S-nodes have a unique planar embedding (C2) and the skeletons of P- and R-nodes have bounded size, this can be done in total linear time. It then remains to consistently choose these embeddings so that also Condition C5 is satisfied. To this end, we construct the auxiliary graph H and compute a maximum independent set, which takes linear time as well [15]. If the size of the maximum independent set is smaller than the number of nodes of $\mathcal{T}$, there is no consistent choice of embeddings and we reject the instance. Otherwise, by Theorem 17 this choice defines an outer-1-planar embedding $\mathcal{E}$. By Lemma 19 the embedding $\mathcal{E}$ has an acyclic planarization and is hence upward outer-1-planar by Theorem 14. We note that, in the positive case, the embedding $\mathcal{E}$ can also be constructed in linear time. ◀

7 Conclusion

In this paper we initiated the study of upward k-planar drawings, that is, upward drawings of directed acyclic graphs such that every edge is crossed at most k times for a given constant k. We first gave upper and lower bounds for the upward local crossing number of various graph families, i.e., the minimum k such that every graph from the respective family admits an upward k-planar drawing. We strengthen these combinatorial results by proving that testing a DAG for upward k-planarity is NP-complete even for $k = 1$. On the positive side, testing upward outer-1-planarity for single source digraphs can be done in linear time. We conclude the paper by listing some open problems that may stimulate further research.

1. Is there a directed outerpath that does not admit an upward 1-planar drawing?
2. Consider the class $\mathcal{O}_\Delta$ of outerplanar graphs (or even 2-trees) of maximum degree Δ. Is there a function f such that every graph in $\mathcal{O}_\Delta$ admits an $f(\Delta)$-planar upward drawing?

3. In light of the lower bounds in Section 3, it is natural to consider graphs with a special structure, in order to prove sublinear upper bounds on their (upward) local crossing number. For example, Wood and Telle [44, Corollary 8.3] show that every (undirected) graph of maximum degree Δ and treewidth τ admits a (straight-line) drawing in which every edge crosses $\mathcal{O}(\Delta^2\tau)$ other edges. Can the *upward* local crossing number be bounded similarly by a function in Δ and τ?
4. Do planar graphs of maximum degree Δ have upward local crossing number $\mathcal{O}(f(\Delta)n^{1-\epsilon})$ for some function f and some constant $\epsilon > 0$?
5. Can upward outer-1-planarity be efficiently tested for multi-source and multi-sink DAGs?
6. Investigate parameterized approaches to testing upward 1-planarity.

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