**Bachelor Thesis**

# 1-Planar Storyplans

Jan Sulejmani

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Advisors: Prof. Dr. Alexander Wolff
Oksana Firman, M.Sc.

Julius-Maximilians-Universität Würzburg
Lehrstuhl für Informatik I
Algorithmen und Komplexität

# Abstract

The field of dynamic graph visualization has received a lot of attention in recent years as new use cases for the exploration of complex graphs have emerged. For example, the representation of vast amounts of relational data as graphs have made a visual analysis more challenging. Storyplans have been introduced as a tool for representing such graphs in a sequential and understandable manner. While planar storyplans as well as outerplanar and forest storyplans have been studied, there has been no research that has gone into the direction of more powerful storyplans such as 1-planar storyplans. This thesis explores the construction of 1-planar storyplans and delves into the specific graph classes that admit these kinds of storyplans. We will extend the strict containment of graph classes that admit storyplans established by Fiala, Firman, Liotta, Wolff, and Zink (SOFSEM 2024) as we explore graphs that do admit 1-planar storyplans but fail to do so for planar storyplans as well as graphs that do not admit 1-planar storyplans. Furthermore, we generalize this graph containment for the graphs that admit $p$-planar storyplans. We prove that the problem of deciding whether a given graph admits a 1-planar storyplan is NP-hard and we present a parameterized algorithm attempt with respect to the vertex cover number. We conclude the thesis by stating the open problems in this field. These findings not only enhance our understanding of graph representation through storyplans but also inspire further research.

# Zusammenfassung

Das Feld der dynamischen Graphenvisualisierung hat in den letzten Jahren viel Aufmerksamkeit erhalten, da neue Anwendungsfälle für die Erkundung komplexer Graphen entstanden sind. Beispielsweise hat die Darstellung großer Mengen relationaler Daten als Graphen eine visuelle Analyse anspruchsvoller gemacht. Storypläne wurden als Werkzeug zur Darstellung solcher Graphen in einer sequenziellen und verständlichen Weise eingeführt. Während planare Storypläne sowie außenplanare und Wald-Storypläne untersucht wurden, gibt es bisher keine Forschung, die sich in Richtung mächtigerer Storypläne wie 1-planare Storypläne bewegt hat. Diese Arbeit erforscht den Aufbau von 1-planaren Storyplänen und vertieft sich in die spezifischen Graphenklassen, die solche Storypläne zulassen. Wir erweitern die strikte Hierarchie von Graphenklassen, die Storypläne zulassen, wie sie von Fiala, Firman, Liotta, Wolff und Zink (SOFSEM 2024) etabliert wurde, indem wir Graphen erkunden, die 1-planare Storypläne zulassen, dies jedoch nicht für planare Storypläne tun, sowie Graphen, die keine 1-planaren Storypläne zulassen. Darüber hinaus verallgemeinern wir diese Hierarchie um Graphenklassen, die p-planare Storypläne zulassen. Wir zeigen, dass das Problem zu entscheiden, ob ein gegebener Graph einen 1-planaren Storyplan zulässt, NP-schwer ist, und präsentieren den Versuch eines parametrisierten Algorithmus in Bezug auf die Knotenüberdeckungszahl. Wir schließen die Arbeit ab, indem wir die offenen Probleme in diesem Bereich darlegen. Diese Erkenntnisse verbessern nicht nur unser Verständnis der Graphenrepräsentation durch Storypläne, sondern inspirieren auch weitere Forschung.

# Contents

# 1 Introduction

There has been a lot of research surrounding approaches to the visual exploration of large and complex graphs. Graph drawing by force-directed layouts can be thought of as treating vertices as charged particles and edges as springs [FR91]. The layout evolves over time until the system settles into a stable state. Useful for the visualization of large-scale graphs is the hierarchical way of constructing graphs, where the visualization emphasizes parent-child relationships within the graph [STT81]. There has also been a push towards three-dimensional graph visualization techniques, which provide more depth and allow for more complex visualizations, as described by Diehl and Görg [DG02]. In the realm of dynamic graph visualization, there have emerged various time-dependent techniques [BDA+14]. These kinds of visualization techniques are used to observe the evolution of graphs over time.

One recent approach in the field of dynamic graph visualization is the use of storyplans as proposed by Binucci, Di Giacomo, Lenhart, Liotta, Montecchiani, Nöllenburg, and Symvonis [BGL+24]. Storyplans, namely outerplanar and forest storyplans, that are able to visualize graphs in a more legible way have been introduced by Fiala, Firman, Liotta, Wolff, and Zink [FFL+24]. In this thesis, we will extend the existing storyplan problem in the direction of more powerful storyplans in order to be able to construct such storyplans for larger classes of graphs. Specifically, we will investigate 1-planar storyplans, the corresponding graph classes that allow such storyplans, and the complexity of the decision problem of whether a given graph admits a 1-planar storyplan.

With an increasing number of vertices, a graph that is supposed to visualize certain relations can grow very quickly in its number of edges. In fact, the technological advances of the last twenty years have generated lots of relational data that are typically modeled as large graphs with thousands of vertices [DLM19]. Therefore, the field of graph visualization is constantly in search of beautiful and legible drawings of graphs. In a legible graph drawing, the graph should be represented in a particular way, such that the observer can read and extract the sought after information with ease, even in the largest graphs. What constitutes a beautiful drawing of a graph is controversial. However, it has been agreed upon that drawing graphs without edge crossings represents such a beauty feature. This class of graphs, which do not allow edge crossings, is called the class of *planar* graphs.

In this context, Binucci et al. [BGL+24] have specifically dealt with planar storyplans. The introduction of storyplans goes hand in hand with the desire for a method that allows us to get better at dynamically representing graphs. Intuitively, a planar storyplan can be described as a sequence of drawings, called *frames*, such that each frame is a planar subdrawing of the graph being explored. See Figure 1.1 for a planar storyplan of the

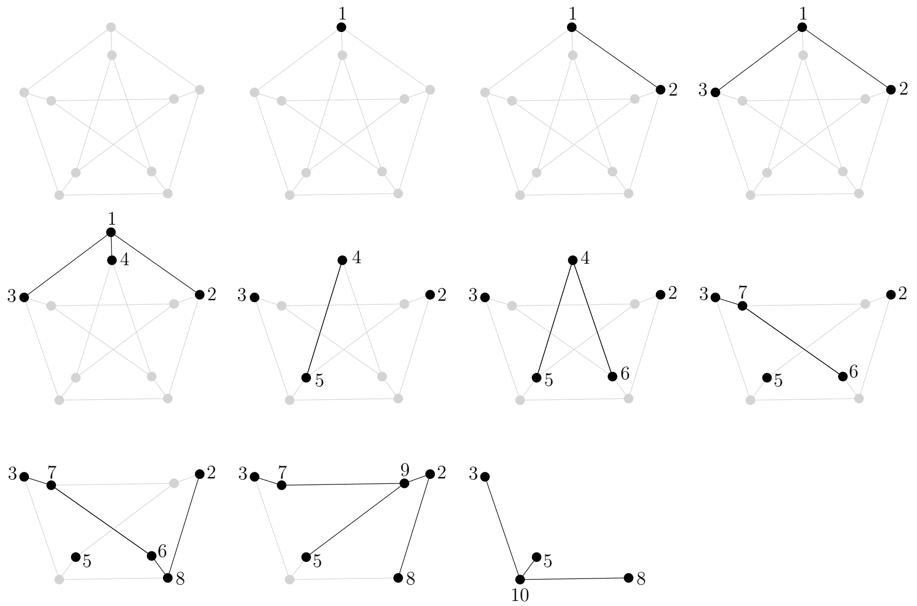



**Fig. 1.1:** A total order of the vertices, which corresponds to a planar storyplan of the Petersen graph.

Petersen graph.

A storyplan starts by *exploring* a selected vertex. This vertex remains *visible* until the neighbors of the vertex have been visited. Once all the neighbors of the vertex have been visited, the explored vertex *disappears* from the drawings of the storyplan and remains disappeared until the end of the storyplan. This implies a natural order of the explored vertices from 1 to $n$, where $n$ is the number of vertices in the graph.

Furthermore, it should be noted that there are some properties that the frames of a storyplan must fulfill. (i) A vertex should always have the same location over the sequence of these frames, i.e., once a vertex becomes visible, it should be visible at the exact same place in the next frames. (ii) An edge should always have the same location over the sequence of these frames, i.e., once an edge becomes visible, it should be visible at the exact same place in the next frames. A third property is now imposed on the frames by the restriction of planar storyplans. (iii) Each frame should show a planar drawing. This third property naturally depends on the type of storyplan being investigated. For example, Fiala et al. [FFL+24] have specifically studied *outerplanar* and *forest storyplans*. These other types of storyplans differ in their third property required of the frames. In the case of outerplanar storyplans, the third property changes to outerplanar drawings. In the case of forest storyplans, forest drawings are required.

In this thesis, we constrict property (iii) in another direction. We focus on 1-planar drawings for the frames of a storyplan. A graph is called *1-planar*, when each edge of the graph is crossed at most one time. In other words, we allow a maximum of one crossing per edge in our storyplan drawings. This naturally expands the size of the

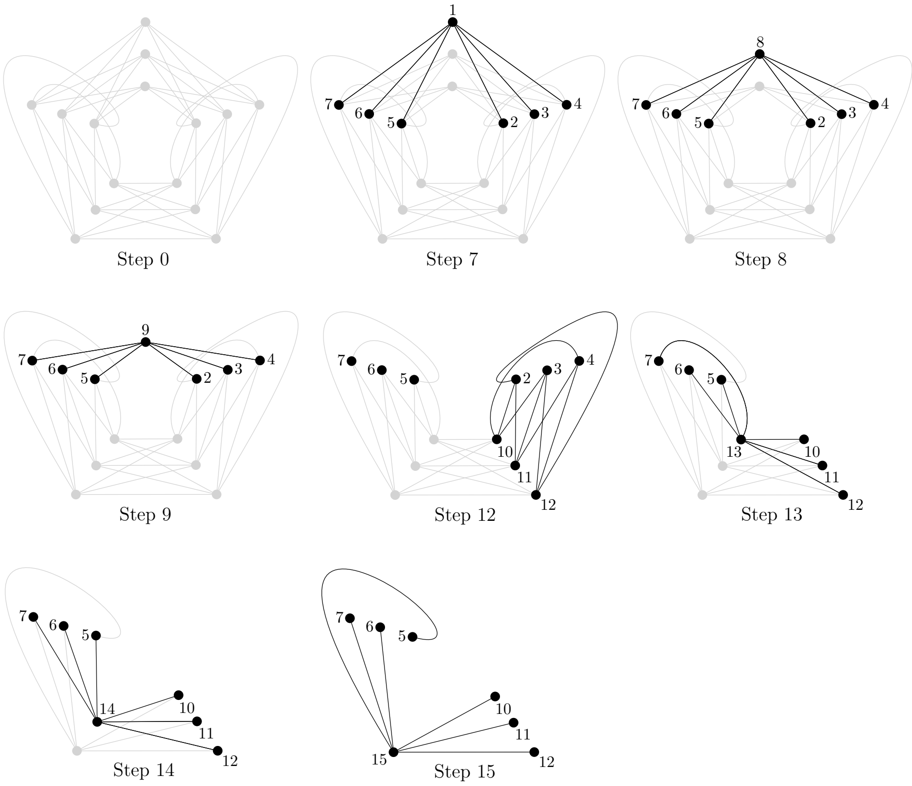



**Fig. 1.2:** A total order of the vertices that corresponds to a 1-planar storyplan of $C_{3,3,3,3,3}$. Note that the frames of steps 1-6 and steps 10-11 are not shown in this illustration.

graph universe that we can construct storyplans for. See Figure 1.2 for an example of a 1-planar storyplan. Furthermore, we study the NP-hardness and highlight a difficulty in obtaining a parameterized algorithm with respect to the vertex cover number of 1-Planar Storyplan.

**Definition 1.1.** *We call 1-Planar Storyplan the problem of deciding whether a certain graph $G$ admits a 1-planar storyplan (see Section 3.2).*

There are various models for dynamic graph visualization in the literature. Of these, the most relevant for our topic are the following ones: The works on planar storyplans [BGL+24], outerplanar or forest storyplans [FFL+24], graph stories [BLB+20], and streamed graphs [LR19].

Fiala et al. [FFL+24] have studied outerplanar and forest storyplans and have established a chain of strict containment relations

$$\mathcal{G}_{\text{forest}} \subsetneq \mathcal{G}_{\text{outerpl}} \subsetneq \mathcal{G}_{\text{planar}} \subsetneq \mathcal{G}$$

where $\mathcal{G}_{\text{forest}}$, $\mathcal{G}_{\text{outerpl}}$, and $\mathcal{G}_{\text{planar}}$ denote the classes of graphs that admit forest, outerplanar, and planar storyplans, respectively, by showing that

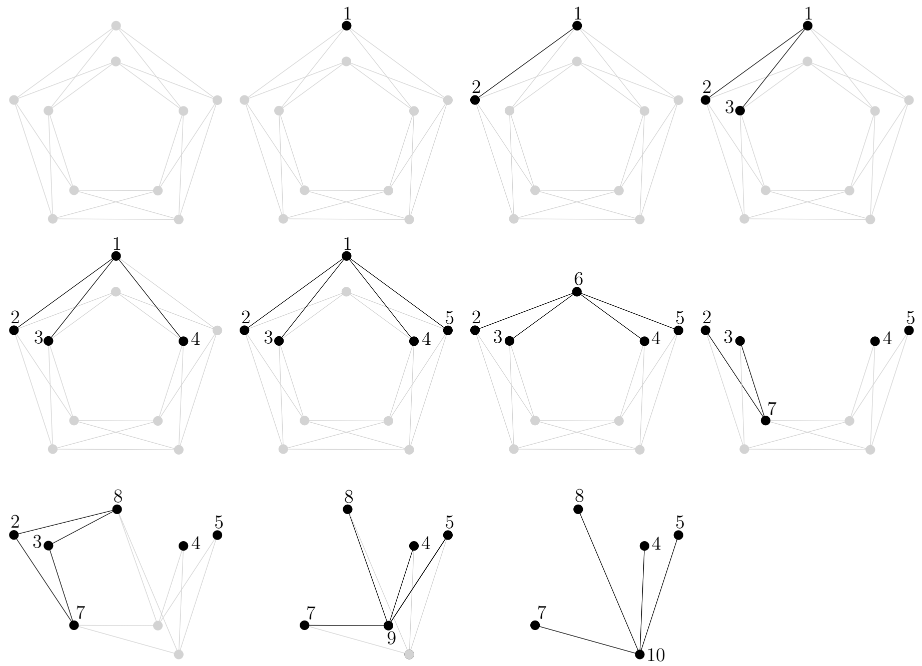


**Fig. 1.3:** A total order of the vertices, which corresponds to an outerplanar storyplan of the graph $C_{2,2,2,2,2}$. Notice, that we move the vertex 8 to another position while it is invisible.

- there is a $\triangle$-free 6-regular graph (i.e., the $C_{3,3,3,3,3}$) that does not admit a planar storyplan;
- there is a $K_4$-free 4-regular planar graph (i.e., the octahedron graph) that (trivially) admits a planar storyplan, but does not admit an outerplanar storyplan; and
- there is a $\triangle$-free 4-regular (nonplanar) graph (i.e., the $C_{2,2,2,2,2}$) that admits an outerplanar storyplan, but does not admit a forest storyplan.

We will extend this chain of strict containment relations to

$$\mathcal{G}_{\text{planar}} \subsetneq \mathcal{G}_{\text{1-planar}} \subsetneq \mathcal{G}_{\text{2-planar}} \subsetneq \mathcal{G}_{\text{3-planar}} \subsetneq \cdots \subsetneq \mathcal{G}$$

by showing that

- there is a $\triangle$-free 6-regular graph that does admit a 1-planar storyplan, but does not admit a planar storyplan;
- there is a graph that does not admit a 1-planar storyplan; and
- for every $p > 1$, the complete graph $K_n$, where $n$ is such that $K_n$ is $p$-planar (but not $(p-1)$-planar), does not admit a $(p-1)$-planar storyplan. For $p = 1$, the complete graph $K_n$, where $n$ is such that $K_n$ is $p$-planar, does not admit a planar storyplan.

Additionally, Fiala et al. [FFL+24] demonstrated that every bipartite graph admits

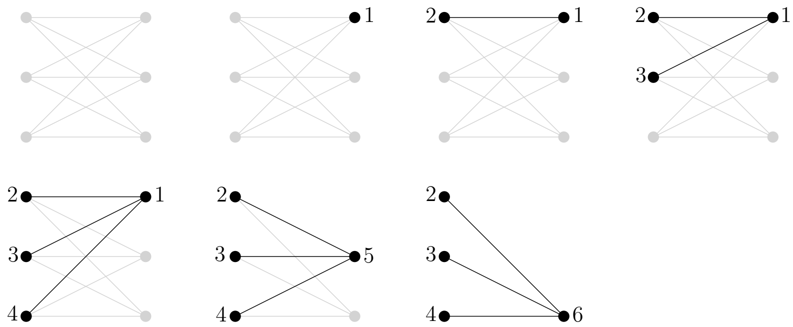



**Fig. 1.4:** A total order of the vertices, which corresponds to a forest storyplan of the bipartite graph $K_{3,3}$. Notice, that we have one partite set where we show all of the vertices and another where we show vertices one-by-one.

a forest storyplan (see Figure 1.4), the graph $C_{2,2,2,2,2}$ admits an outerplanar storyplan (see Figure 1.3), and all 2-trees admit outerplanar storyplans.

By reduction from One-In-Three 3SAT, Binucci et al. [BGL+24] have proved that solving Planar Storyplan is NP-complete. Even when the total order of vertices is given as input, as in Fixed Order Planar Storyplan, it remains NP-complete. They have complemented this hardness with two parameterized algorithms, one in the vertex cover number and one in the feedback edge set number. Furthermore, they have proven that every partial 3-tree admits a planar storyplan, which can be computed in linear time. Let $G = (V, E)$ be a graph with $n$ vertices, a vertex cover number $\kappa$ and a feedback edge set of size $\psi$. In dependency of the vertex cover number, deciding whether $G$ admits a planar storyplan, and computing one if any, can be done in $O(2^{2^{O(\kappa)}} + n^2)$ time. In dependency of the feedback edge set size, deciding whether $G$ admits a planar storyplan, and computing one if any, can be done in $O(2^{O(\psi log \psi)} + n^2)$ time.

Borrazzo et al. [BLB+20] explored the concept of graph stories. A *graph story* is formed by a graph $G$, a total order of its vertices $\tau$, and a positive integer $W$. The problem is to find a sequence of drawings $\{D_i\}, i \in [n]$ where each $D_i$ contains all vertices $v$ such that $i - W < \tau(v) \leq i$, and the position of a vertex remains constant in all drawings to which it belongs. This implies a fixed lifespan of vertices equal to $W$. A graph story $(G, \tau, W)$ is associated with a sequence $G_1, G_2, \dots, G_{n+W-1}$. For any $i \in \{1, \dots, n + W - 1\}$, the graph $G_i$ is the subgraph of $G$ induced by the set of vertices $\{v \in V : i - W < \tau(v) \leq i\}$. A *drawing story* for $(G, \tau, W)$ is a sequence $\Gamma_1, \Gamma_2, \dots, \Gamma_{n+W-1}$ of drawings such that for every $i \in \{1, \dots, n + W - 1\}$:

1. $\Gamma_i$ is a drawing of $G_i$,
2. a vertex $v$ is drawn at the same position in all the drawings $\Gamma_i$ such that $v \in V_i$, and
3. an edge $(u, v)$ is represented by the same curve in all the drawings $\Gamma_i$ such that $(u, v) \in E_i$.

Borrazzo et al. [BLB+20] proved that every graph story of a path or a tree can be drawn on a $2W \times 2W$ or a $(8W+1) \times (8W+1)$ grid, respectively, while ensuring that all drawings of the graph story are straight-line and planar. The presence of both a fixed order and a fixed lifespan distinguishes this from the approach of Binucci et al. [BGL+24]. Unlimited lifespans, in particular, allow finding storyplans where all edges are drawn in at least one step, while maintaining planarity to ensure readability even for large layouts. Besides these model differences, Binucci et al. [BGL+24] focus on the complexity of the decision problem rather than the area constraints.

Da Lozzo and Rutter [LR19] introduced streamed graphs. A *streamed graph* is a stream of edges $e_1, \ldots, e_m$ on a vertex set $V$. A streamed graph is $W$-stream planar with respect to a positive integer window size $W$ if there exists a sequence of planar drawings $\Gamma_i$ of the graphs $G_i = (V, \{e_j \mid i \leq j \leq i + W\})$ such that the common graph $G_\cap^i = G_i \cap G_{i+1}$ is drawn the same in $\Gamma_i$ and in $\Gamma_{i+1}$, for $1 \leq i < m - W$. Stream Planarity with window size $W$ asks whether a given streamed graph is $W$-stream planar. Da Lozzo and Rutter [LR19] proved that there is a constant value for $W$ such that Stream Planarity is NP-complete. They also investigated a variant where a backbone graph is given, and its edges must remain in the drawing at all times. For this variant, they proved that the problem is NP-complete for all $W \geq 2$ and can be solved in polynomial time when $W = 1$ or when the backbone graph is biconnected. The difference between Stream Planarity and the problem described by Binucci et al. [BGL+24], besides considering edges instead of vertices, is once again having a fixed order and a fixed lifespan.

# 2 Contribution and Overview

In the preceding chapter, we have introduced our thesis topic and highlighted why it is relevant to the field of dynamic graph visualization. We have presented some of the most common approaches to graph visualization and have described the relevant papers that closely relate to the field of storyplans. A focus in the preceding section has been the complexity of Planar Storyplan as shown by Binucci et al. [BGL+24], the related concepts of graph stories [BLB+20] and streamed graphs [LR19], and a discussion on the graph families that admit planar, outerplanar, or forest storyplans [FFL+24].

We establish some preliminaries in Chapter 3. These are necessary to properly understand the section on NP-hardness of 1-Planar Storyplan as well as the section on the attempt of the parameterization with respect to the vertex cover number. We introduce the reader into the broader topic and remind him or her of elementary definitions in graph theory. Furthermore, we formally define 1-planar storyplans and highlight how they are different from planar, outerplanar, and forest storyplans.

We establish a chain of strict containment relations of graph classes that admit certain types of storyplans. In particular, we extend the chain of strict containment as established by Fiala et al. [FFL+24] for forest, outerplanar, and planar storyplans (see Chapter 4). The graph class separation of $\mathcal{G}$, $\mathcal{G}_{\text{planar}}$, $\mathcal{G}_{\text{outerplanar}}$, and $\mathcal{G}_{\text{forest}}$ has already been shown. We show that there is a further distinction between $\mathcal{G}$, $\mathcal{G}_{\text{planar}}$, and $\mathcal{G}_{\text{1-planar}}$. In particular, we focus on the $\triangle$-free 6-regular graph $C_{3,3,3,3,3,3}$ and complete graphs $K_n$, where $n \geq 5$, as these have served as the examples belonging to $\mathcal{G}$ (i.e., the class of all graphs) but not to $\mathcal{G}_{\text{planar}}$ in the work of Fiala et al. [FFL+24]. As a result, we extend the already established chain of strict containment relations to

$$\mathcal{G}_{\text{planar}} \subsetneq \mathcal{G}_{\text{1-planar}} \subsetneq \mathcal{G}_{\text{2-planar}} \subsetneq \mathcal{G}_{\text{3-planar}} \subsetneq \cdots \subsetneq \mathcal{G}.$$

We prove the NP-completeness of 1-Planar Storyplan through reduction of Positive One-In-Three 3SAT in Section 5.1. We attempt to develop a parameterized algorithm for 1-Planar Storyplan in the vertex cover number in Section 5.2 (see Chapter 5). Our attempt to adapt the parameterized algorithm for Planar Storyplan as introduced by Binucci et al. [BGL+24] sheds light on a key difficulty in developing an analogous parameterization for 1-Planar Storyplan.

We conclude the thesis by discussing our findings and by stating the open problems in this field (see Chapter 6).

# 3 Preliminaries

This chapter contains the preliminaries that are necessary to understand in order to follow the proofs in this thesis. We start off with a section on elementary graph theoretical definitions (see Section 3.1). Then, we follow that up with the definition of planar storyplans while expanding that definition to include other types of storyplans, such as outerplanar, forest, and most importantly 1-planar storyplans (see Section 3.2).

## 3.1 Elementary Graph Theory

We denote a graph $G$ as $G = (V, E)$, where $V$ is the set of vertices and $E$ is the set of edges belonging to that graph, respectively. A drawing of $G$ maps each vertex in $V$ to a distinct point in the plane and each edge in $E$ to a Jordan arc between its endpoints. A graph is called *planar* if there is a *drawing* of the graph in which no edges intersect, except at a common endpoint. A *regular graph* is a graph where each vertex has the same number of neighbors. We call a regular graph with vertices of degree $k$ a *$k$-regular graph*. For example, in a 4-regular graph, every vertex has exactly four neighbors. The regions enclosed by the vertices of a graph (including the region outside the graph) are called *faces*. If the face is enclosed by three vertices, we call the face a *triangle*. Throughout this work, we denote a graph that has no triangles as $\triangle$-free. When every face of a graph is a triangle, we call the graph a *triangulation*. If a planar graph can be embedded into the plane in such a way that all its vertices lie on the boundary of the outer face, it is called *outerplanar*. A graph is called a *forest* if it is *acyclic*, i.e., it has no cycles. A cycle is a path that can be traversed to return to the starting vertex without walking the same path twice. A forest can also have isolated vertices or larger disconnected parts. This is the only difference between a forest and a *tree*. We call a graph *complete*, when it is a graph in which every vertex is connected to every other vertex. Complete graphs with $n$ vertices are denoted as $K_n$. For example, $K_5$ is the complete graph with five vertices. A graph is called *bipartite* if its vertices can be divided into two disjoint sets $A$ and $B$, such that there are no edges between vertices within the same set, but every vertex in set $A$ is connected to every vertex in set $B$. A graph is called *$p$-planar* if every edge has at most $p$ crossings. Especially, the case where $p = 1$, so called *1-planar* graphs, is of importance to our work.

## 3.2 P-Planar Storyplans

Our definition of a 1-planar storyplan is based on the definition of a planar storyplan as described by Binucci et al. [BGL+24]. Note that we use $[n]$ as a shorthand notation for the set $\{1, \dots, n\}$.

**Definition 3.1.** *A planar storyplan $S = (\tau, (D_i)_{i \in [n]})$ of $G$ is a pair defined as follows. The first element is a bijection $\tau \colon V \to [n]$ that represents a total order of the vertices of $G$. For a vertex $v \in V$, let $i_v = \tau(v)$ and let $j_v = \max_{u \in N[v]} \tau(u)$, where $N[v]$ is the set containing $v$ and its neighbors. The interval $[i_v, j_v]$ is the lifespan of $v$. We say that $v$ appears at step $i_v$, is visible at step $i$ for each $i \in [i_v, j_v]$, and disappears at step $j_v + 1$. Note that a vertex disappears only when all its neighbors have appeared. The second element of $S$ is a sequence of drawings $(D_i)_{i \in [n]}$, called frames of $S$, such that, for $i \in [n]$: (i) $D_i$ is a drawing of the graph $G_i$ induced by the vertices visible at step $i$, (ii) $D_i$ is planar, (iii) the point representing a vertex $v$ is the same over all drawings that contain $v$, and (iv) the curve representing an edge $e$ is the same over all drawings that contain $e$.*

For an outerplanar storyplan and a forest storyplan, we strengthen requirement (*ii*) to $D_i$ being outerplanar and $D_i$ being a crossing-free drawing of a forest, respectively. For a 1-planar storyplan, we weaken requirement (*ii*) to $D_i$ being 1-planar. Generally, a $p$-planar storyplan demands of all the frames in the storyplan to be $p$-planar drawings.

Note that complete bipartite graphs always admit a 1-planar storyplan. In fact, it is known that all bipartite graphs even admit forest storyplans [FFL+24]. From the fact that $K_{3,3}$ is not planar, Binucci et al. [BGL+24] obtained following property.

**Lemma 3.2.** *Let $K_{a,b} = (A \cup B, E)$ be a complete bipartite graph with $a = |A|$, $b = |B|$, and $3 \leq b \leq a$. Let $S = (\tau, \{D_i\}_{i \in [a+b]})$ be a planar storyplan of $K_{a,b}$. Exactly one of $A$ and $B$ is such that all its vertices are visible at some $i \in [a + b]$.*

We use the result of Czap and Hudák [CH12] that $K_{5,4}$ and $K_{7,3}$ are not 1-planar and show the property analogous to Lemma 3.2 for 1-planar storyplans.

**Lemma 3.3.** *Let $K_{a,b} = (A \cup B, E)$ be a complete bipartite graph with $a = |A|$, $b = |B|$, and $a \geq 5$, $b \geq 4$ (Case 1) or $a \geq 7$, $b \geq 3$ (Case 2). Let $S = (\tau, \{D_i\}_{i \in [a+b]})$ be a 1-planar storyplan of $K_{a,b}$. Exactly one of $A$ and $B$ is such that all its vertices are visible at some $i \in [a + b]$.*

*Proof.* First we show that there exists a step $i \in [a + b]$ such that all vertices of either $A$ or $B$ are visible. Recall that $i_v \in [n]$ is the step where a vertex $v$ appears and $j_v \in [n]$ with $i_v \leq j_v$ is the step where the vertex $v$ disappears. Let $i$ be such that $D_i$ contains the largest number $t$ of vertices of $A$ over all frames of $S$. If $t = a$, we are done. If $t < a$, there exist two vertices $u, v$ of $A$ such that $j_u < i_v$. Note that all vertices in $B$ are adjacent to $u$, and hence they all appear at some step smaller than or equal to $j_u$. On the other hand, since all vertices in $B$ are adjacent to $v$ as well, they cannot disappear before $i_v + 1$. It follows that all vertices of $B$ are visible at step $j_u$.

Now we show that if all vertices of one of the sets $A$ or $B$, say $A$, are all visible at some frame, then not all vertices of $B$ are visible at any step $j \in [a+b]$. Consider the interval $I = [s, t] \subseteq [a + b]$ of maximal length such that the vertices of $A$ are all visible. As we have already shown, $I$ is not empty. Let $k$ be one of the steps that contain the largest number $h$ of vertices of $B$. Observe that, since $I$ is maximal, one vertex of $A$ appears at $s$. Therefore, any vertex of $B$ visible at step smaller than $s$ is visible also at step $s$. Similarly, by the maximality of $I$, one vertex of $A$ disappears at step $t+1$, therefore any vertex of $B$ visible at a step greater than $t$ is visible also at step $t$. Consequently, we can assume that $k \in I$, and we can conclude $h \leq 3$ (Case 1) and $h \leq 2$ (Case 2), since the induced graphs $K_{5,4}$ (Case 1) and $K_{7,3}$ (Case 2) are not 1-planar and thus none of these graphs can be in any frame of a 1-planar storyplan. $\square$

In view of Lemma 3.3, we have the following definition.

**Definition 3.4.** *For a complete bipartite graph $K_{a,b}$ with $a \geq 5$, $b \geq 4$ (Case 1) or $a \geq 7$, $b \geq 3$ (Case 2) and a 1-planar storyplan $S$ of $K_{a,b}$, we call* fixed *the partite set of $K_{a,b}$ whose vertices are all visible at some step of $S$, and* flexible *the other partite set.*

# 4 Separation of Graph Classes

The graph class separation of $\mathcal{G}$, $\mathcal{G}_{\text{planar}}$, $\mathcal{G}_{\text{outerplanar}}$, and $\mathcal{G}_{\text{forest}}$ has already been shown [FFL+24]. In this chapter we show that there is a further distinction between $\mathcal{G}$, $\mathcal{G}_{\text{planar}}$, and $\mathcal{G}_{\text{1-planar}}$. More generally, we expand the class separation to $p$-planar storyplans. We specifically focus on the $\triangle$-free 6-regular graph $C_{3,3,3,3,3,3}$ and complete graphs $K_n$, where $n \geq 5$, as these have served as the examples belonging to $\mathcal{G}$ (i.e. the class of all graphs) but not to $\mathcal{G}_{\text{planar}}$ in the work of Fiala et al. [FFL+24].

**Theorem 4.1.** *For every $p > 1$, the complete graph $K_n$, where $n$ is such that $K_n$ is $p$-planar (but not $(p-1)$-planar), does not admit a $(p-1)$-planar storyplan. For $p = 1$, the complete graph $K_n$, where $n$ is such that $K_n$ is $1$-planar, does not admit a planar storyplan.*

*Proof.* This follows directly from the fact that for a complete graph $G = (V, E)$ every vertex $v \in V$ has an edge leading to every other vertex $u \neq v \in V$. In a storyplan, a vertex cannot disappear until all its neighbors have appeared. Thus, a storyplan for a complete graph will always have a frame that contains all of its vertices and edges. In a $p$-planar storyplan, all of its frames must be $p$-planar drawings. If $k < p$, a complete graph that is $p$-planar has a $p$-planar storyplan but not a $k$-planar storyplan. Otherwise, there would be a frame in its storyplan which is $p$-planar and not $k$-planar. It follows that a $p$-planar complete graph cannot admit a $p-1$-planar storyplan. □

**Theorem 4.2.** *The following statements hold:*

1. *There is a $\triangle$-free $6$-regular graph that admits a 1-planar storyplan but does not admit a planar storyplan. This establishes $\mathcal{G}_{planar} \subsetneq \mathcal{G}_{1\text{-}planar}$.*
2. *There are complete graphs that do not admit 1-planar storyplans. This establishes $\mathcal{G}_{1\text{-}planar} \subsetneq \mathcal{G}$.*
3. *There are p-planar complete graphs that do not admit $p-1$-planar storyplans (see Theorem 4.1). This establishes $\mathcal{G}_{planar} \subsetneq \mathcal{G}_{1\text{-}planar} \subsetneq \mathcal{G}_{2\text{-}planar} \subsetneq \mathcal{G}_{3\text{-}planar} \subsetneq \cdots \subsetneq \mathcal{G}$.*

*Proof.*

1. As shown by Fiala et al. [FFL+24], the graph $C_{3,3,3,3,3,3}$ does not admit a planar storyplan. We have shown that there is a total order of the vertices such that it represents a 1-planar storyplan of $C_{3,3,3,3,3,3}$ (see Figure 1.2 for an illustration).
2. The complete graph $K_n$ is planar, if $n \leq 4$. Similarly, the complete graph $K_n$ is 1-planar, if $n \leq 6$. The complete graphs $K_5$ and $K_6$ do admit 1-planar storyplans, while not admitting planar storyplans (see Figure 4.1 and Figure 4.2 for a 1-planar

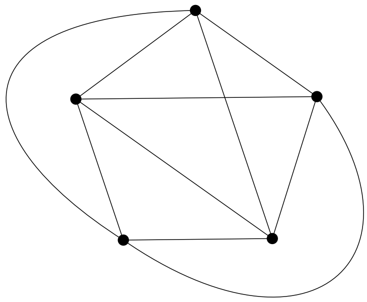


**Fig. 4.1:** A 1-planar drawing of the complete graph $K_5$.

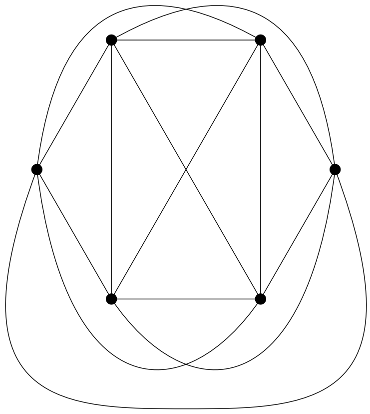


**Fig. 4.2:** A 1-planar drawing of the complete graph $K_6$.

drawing of the complete graph $K_5$ and $K_6$, respectively). According to Schumacher [Sch86], a 1-planar graph has at most $4n-8$ edges. For $n = 7$, the upper bound for the edge number is at $4 \cdot 7 - 8 = 20$, while the complete graph $K_7$ has $\sum_{i \leq 6} i = 21$ edges. Hence, the complete graph $K_7$ does not have a 1-planar drawing. It follows from Theorem 4.1 that the complete graphs $K_n$, where $n > 6$, do not admit 1-planar storyplans.

3. This follows directly from the proof of Theorem 4.1.

□

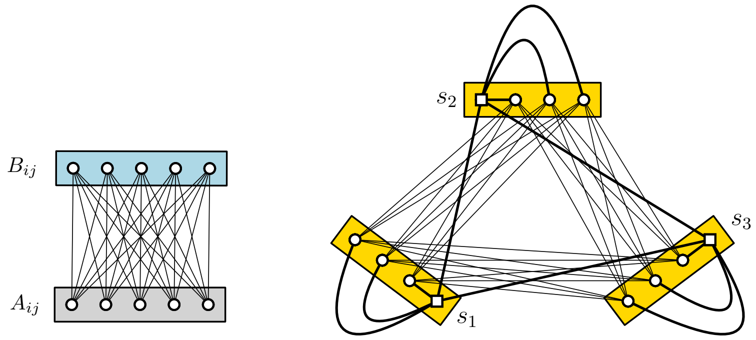
# 5 1-Planar Storyplans

In this chapter, we prove the NP-hardness of 1-Planar Storyplan (see Section 5.1) and show a parameterization attempt with respect to the vertex cover number (see Section 5.2). Our proof is based on the analogous NP-hardness proof of Planar Storyplan as described by Binucci et al. [BGL+24]. Note that we deviate from the NP-hardness proof of Planar Storyplan in several ways. First, we reduce from Positive One-In-Three 3SAT instead of One-In-Three 3SAT. This simplification improves the readability of the NP-hardness proof. Furthermore, our gadget constructs differ substantially from the ones used by Binucci et al. [BGL+24] in order to make the proof work for 1-planar storyplans. By attempting to use a parameterized algorithm for 1-Planar Storyplan that is based on the parameterized algorithm of Binucci et al. [BGL+24], we have found a difficulty in adapting some of the reduction rules. Thus, we will describe what has worked as well as discuss the main difficulty in fully adapting the parameterized algorithm for Planar Storyplan to a parameterized algorithm for 1-Planar Storyplan.

## 5.1 NP-Hardness

In this section we prove that 1-Planar Storyplan is NP-complete through reduction of Positive One-In-Three 3SAT. In Positive One-In-Three 3SAT we are given a Boolean formula in conjunctive normal form with no negative literals and at most three literals per clause. The objective is to decide whether there exists a variable assignment such that in every clause exactly one literal is assigned to be true. Note that a literal is either a positive or negative occurence of a variable. In the following, we will not distinguish between variables and literals as we only observe positive literals in our proof. One-In-Three 3SAT and the case where we allow only positive literals have been proven to be NP-complete by Thomas Jerome Schaefer [Sch78] as a special case of Schaefer's dichotomy theorem, which asserts that any problem generalizing Boolean satisfiability in a certain way is either in the class P or is NP-complete. Our NP-completeness proof is based on the NP-completeness proof of Planar Storyplan by Binucci et al. [BGL+24]. First, we describe the necessary gadgets that we need to construct the reduction. Then, we conclude the proof by showing how we can draw a 1-planar storyplan when given a solution of Positive One-In-Three 3SAT and how a solution of Positive One-In-Three 3SAT can be obtained from a 1-planar storyplan.

Let $\phi$ be a 3SAT formula over $N$ variables $\{x_i\}, i \in [N]$ and $M$ clauses $\{C_j\}, j \in [M]$. We construct an instance of 1-Planar Storyplan, i.e., a graph $G = (V, E)$, as follows




(a) Variable gadget (b) Clause gadget

**Fig. 5.1:** Illustration of the necessary gadgets.

(refer to Figure 5.1 for an illustration).

**Variable gadget.** See Figure 5.1 (a) for an illustration. Let $x_i$ be a variable occuring in a clause $C_j$. Every variable $x_i$ is represented by the complete bipartite graph $K_{5,5 \cdot p_i}$, where $p_i$ is the number of occurences of $x_i$ in $\phi$. Let $k \in [p_i]$ and let $B_{ik}$ denote the five vertices that correspond to the $k^{th}$ occurence of the variable $x_i$. We call the two sides of the variable gadget the *v-side* $A_i$ and $B_{ik}$, respectively. A true (false) assignment of $x_i$ will correspond to $A_i$ being flexible (fixed) and all $B_{ik}$ being fixed (flexible) in a possible storyplan of $G$.

**Clause gadget.** See Figure 5.1 (b) for an illustration. Each clause $C_j$ is represented by an *extended* $K_{3,3,3} = (U_1 \cup U_2 \cup U_3, F)$. The extended $K_{3,3,3}$ is obtained by drawing the $K_{3,3,3}$ and adding three vertices $s_1, s_2, s_3$, such that these three vertices are pairwise adjacent, and each $s_j$ is adjacent to all vertices in $U_j$ for $j \in [3]$. We call $s_1, s_2, s_3$ the *special vertices* of the extended $K_{3,3,3}$, while the other vertices are called *simple vertices.* We call each of the three sets of vertices $U_j \cup s_j$ a *c-side* of the clause gadget. Note that the c-sides of the clause gadgets together with the v-sides $B_{ik}$ of the variables occuring in that clause induce the graph $K_{5,4}$.

**Lemma 5.1.** *If the graph $G$ admits a 1-planar storyplan, then $\phi$ admits a satisfying assignment with exactly one true variable in each clause.*

*Proof.* Let $S = (\tau, \{D_i\}_{i \in [n]})$ be a 1-planar storyplan of $G$. For each variable gadget, we assign the value *true* (*false*) to $x_i$ if the v-side $A_i$ is flexible (fixed) in $S$. For every variable $x_i$ occuring in clause $C_j$ we have the v-side $B_{ik}$ consisting of five vertices. The v-sides $A_i$ and $B_{ik}$ together form a $K_{5,5}$, hence by Lemma 3.3 and Definition 3.4 the v-side $B_{ik}$ is fixed (flexible). Note that all $B_{ik}$, for $k \in [p_i]$, are then fixed (flexible). Similarly, the v-side $B_{ik}$ and the c-side together form a $K_{5,4}$, hence by Lemma 3.3 and Definition 3.4 the c-side is flexible (fixed). In other words, the value of $x_i$ propagates consistently throughout all its occurences. For a schematization of the gadget construction, see Figure 5.2. It remains to prove that, for any clause $C_j$ of $\phi$, precisely one variable is

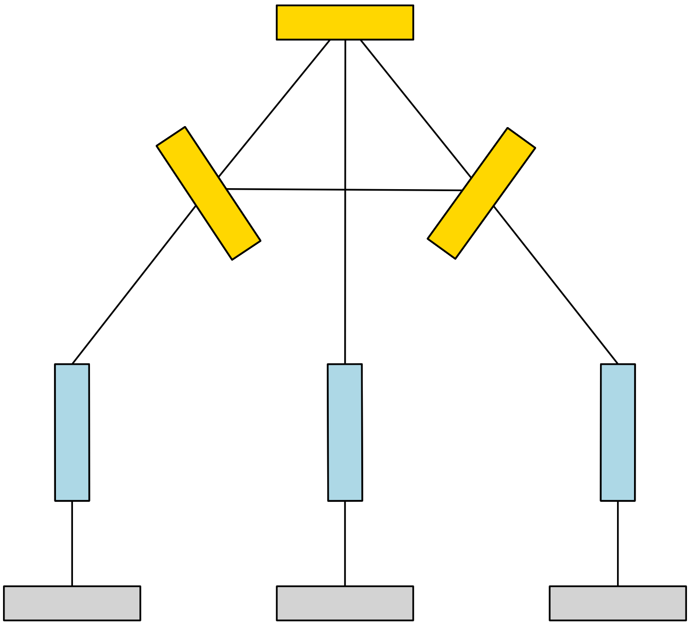


**Fig. 5.2:** Schematization of the clause $(x_1 \vee x_2 \vee x_3)$.

true for the constructed truth value assignment of $\{x_i\}, i \in [N]$. We will show this by proving that exactly one c-side of the clause gadget is flexible.

We first argue that not all c-sides can be fixed in $S$. For a contradiction, we assume that they are. Let $D_h$ be the frame of $S$ in which the last simple vertex $v$ of the clause gadget appears. Observe that all other simple vertices are also visible at step $h$. Let $u \neq v$ be a simple vertex of the clause gadget. Either $u$ is adjacent to $v$ or in the same c-side as $v$. Hence, since $i_u < h$ by assumption and since $u$ cannot disappear until $h+1$, it follows that $u$ is visible at step $h$. This implies that $D_h$ contains a drawing of $K_{3,3,3}$, which is not 1-planar [CH12]. This contradicts our assumption of a 1-planar storyplan, where every frame must be 1-planar.

Now we show that there cannot be more than one flexible c-side. If a c-side is flexible, there can be no frame containing its three simple vertices. This follows from the fact that the special vertex is adjacent to all three simple vertices on the same c-side. The three simple vertices of the same c-side cannot disappear until the special vertex of that c-side is visible. Thus, there would be a frame containing all four vertices together, which is not possible since the c-side is flexible. On the other hand, when the last simple vertex appears, its six neighbors in the other two c-sides of $K_{3,3,3}$ are all visible, and hence at least two c-sides must be fixed. Altogether, we have proved that at least two c-sides are fixed and that at least one c-side is flexible. Therefore, in each clause gadget, exactly one c-side is flexible, which corresponds to having exactly one true variable, as desired. □

**Lemma 5.2.** *If the formula $\phi$ admits a satisfying assignment with exactly one true*

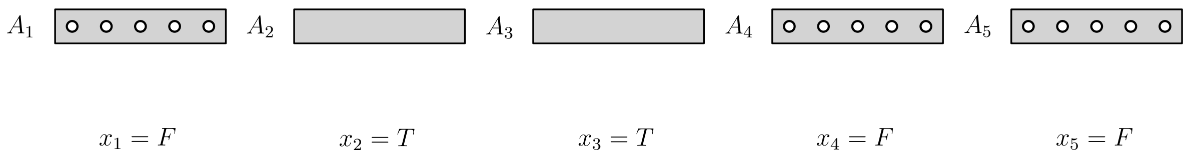



**Fig. 5.3:** Proof of Lemma 5.2: Drawing the vertices of the fixed v-sides $A_i$. The v-sides $A_i$ here correspond to following formula: $(x_1 \vee x_2 \vee x_4) \wedge (x_5 \vee x_3 \vee x_1) \wedge (x_4 \vee x_3 \vee x_1)$.

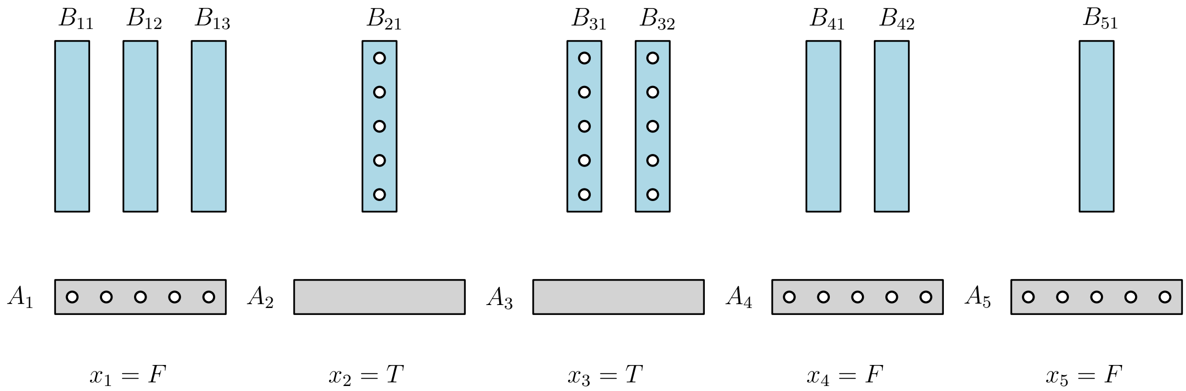



**Fig. 5.4:** Proof of Lemma 5.2: Drawing the vertices of the fixed v-sides $B_{ik}$. The v-sides $A_i$ and $B_{ik}$ here correspond to following formula: $(x_1 \vee x_2 \vee x_4) \wedge (x_5 \vee x_3 \vee x_1) \wedge (x_4 \vee x_3 \vee x_1)$.

*variable in each clause, then graph $G$ admits a 1-planar storyplan.*

*Proof.* Given a satisfying assignment of $\phi$ with one true variable per clause, we can compute a 1-planar storyplan $S = (\tau, \{D_i\}_{i \in [n]})$ of $G$. In the following, whenever we do not specify the order of a group of vertices, any relative order is valid.

Consider a single variable gadget. If $x_i$ is false in the satisfying assignment, then we let appear the five vertices of the v-side $A_i$ (i.e., the v-side $A_i$ is fixed). This procedure is repeated for all variables $x_i$ in any order (see Figure 5.3). For ease of presentation, we can imagine that all the drawn v-sides $A_i$ are horizontally aligned, as shown in Figure 5.4. Thus, for the v-sides $A_i$, it remains to draw the flexible v-sides $A_i$ that represent a true assignment to a variable $x_i$.

If $x_i$ is true, then the v-sides $B_{ik}$, for all $k \in [p_i]$, must be fixed because they form bipartite graphs $K_{5,5}$ with the v-side $A_i$, which is flexible. Therefore we let appear the five vertices of all v-sides $B_{ik}$. Similarly, if $x_i$ is false, then the v-sides $B_{ik}$, for all $k \in [p_i]$, must be flexible because they form bipartite graphs $K_{5,5}$ with the v-side $A_i$, which is fixed. Again, this procedure is repeated for all variables $x_i$ in any order. For ease of presentation, we can imagine that all the drawn v-sides $B_{ik}$ are vertically arranged along a horizontal line slightly above the v-sides $A_i$, as shown in Figure 5.4. Thus, also for v-sides $B_{ik}$, it remains to draw the flexible v-sides $B_{ik}$ (see Figure 5.5).

After having drawn the fixed v-sides, each flexible v-side can be drawn independently, by letting appear and disappear its five vertices one by one (see Figure 5.6). To conclude

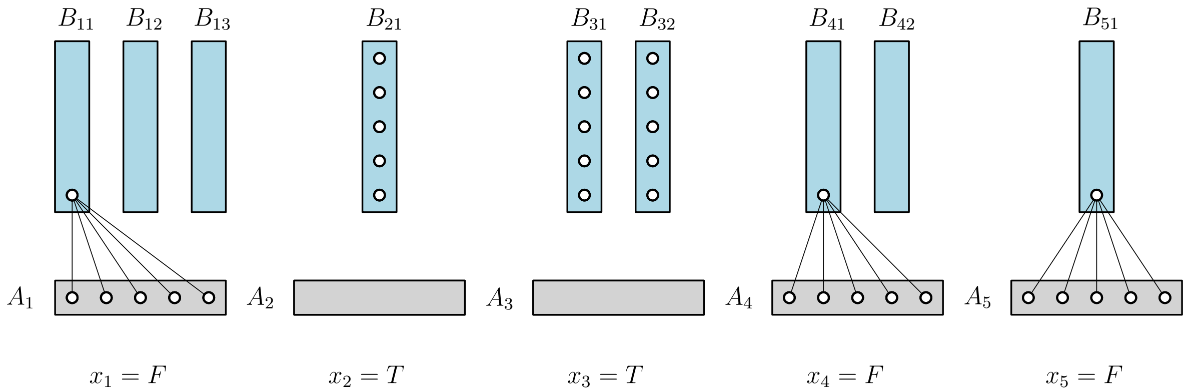



**Fig. 5.5:** Proof of Lemma 5.2: Drawing the vertices of the flexible v-sides $B_{ik}$. The v-sides $A_i$ and $B_{ik}$ here correspond to following formula: $(x_1 \vee x_2 \vee x_4) \wedge (x_5 \vee x_3 \vee x_1) \wedge (x_4 \vee x_3 \vee x_1)$.

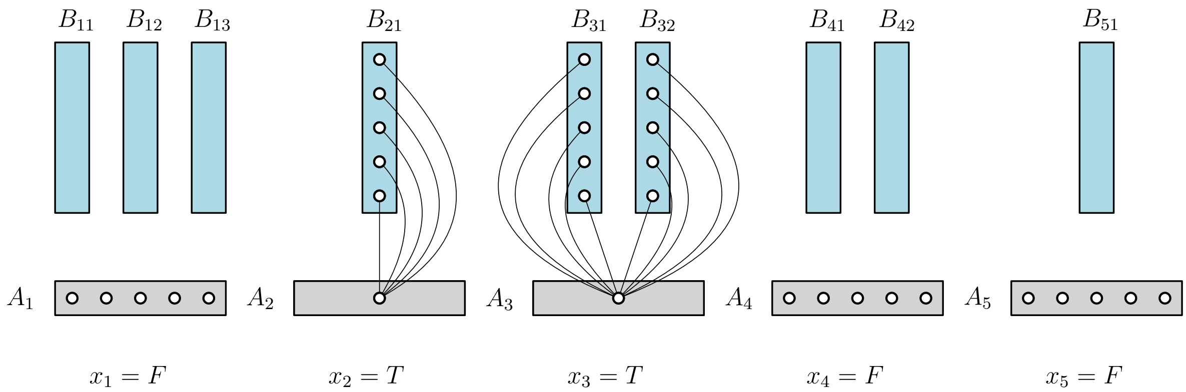



**Fig. 5.6:** Proof of Lemma 5.2: Drawing the vertices of the flexible v-sides $A_i$. The v-sides $A_i$ and $B_{ik}$ here correspond to following formula: $(x_1 \vee x_2 \vee x_4) \wedge (x_5 \vee x_3 \vee x_1) \wedge (x_4 \vee x_3 \vee x_1)$.

the proof, we first draw the clause gadgets alone, and then we show how to integrate in the storyplan the flexible v-sides $B_{ik}$ and their connections, as well as the connections of the fixed v-sides $B_{ik}$ with the corresponding c-side.

We begin by showing how to draw a single clause gadget, ignoring the connections with the linked v-sides $B_{ik}$ (see Figure 5.7). We first let appear the three special vertices of the clause gadget. Now we let appear the three simple vertices of a false variable (i.e., the fixed c-side). Right after, the special vertex connected to the three simple vertices of the fixed c-side can disappear. Next, we let appear the three simple vertices of the remaining false variable (i.e., the other fixed c-side in our clause gadget). Then, we draw the vertices of the flexible c-side one by one in a 1-planar fashion. Once the last simple vertex of the flexible c-side appears, all vertices of the clause can disappear. Hence, the last c-side hasn't appeared together but instead acted as a flexible c-side.

Figure 5.8 and Figure 5.9 illustrate the global strategy of drawing a 1-planar storyplan by showing how to draw the gadgets that represent a true variable and how to draw the

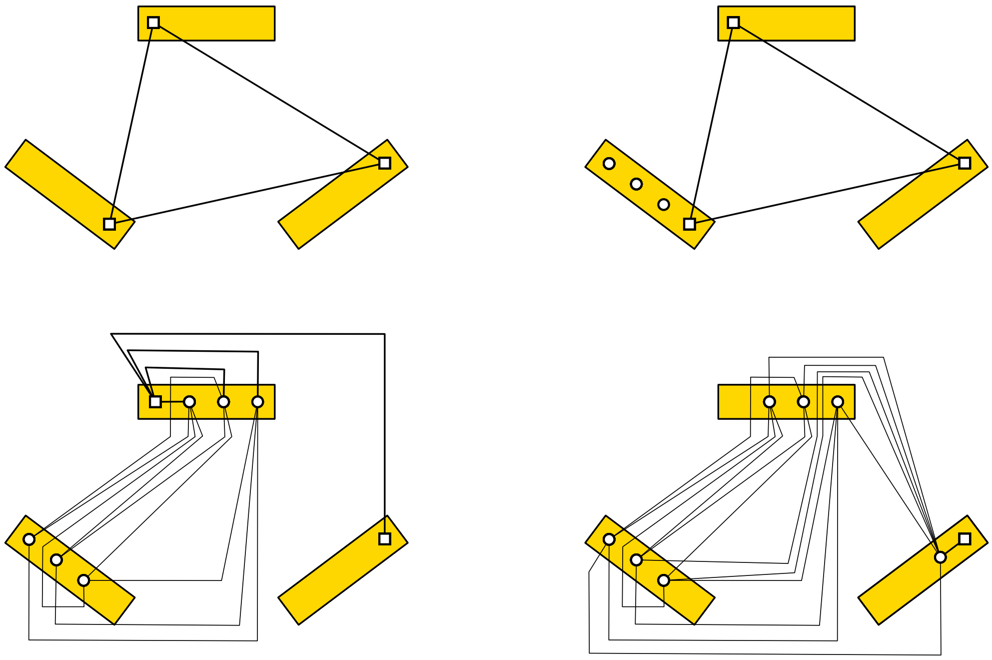


**Fig. 5.7:** Drawing a clause gadget. Note that we ignore the connections to the v-sides $B_{ik}$ and that we have only drawn one vertex of the flexible c-side.

gadgets that represent a false variable, respectively. By repeating this procedure for each clause we complete the 1-planar storyplan. □

**Theorem 5.3.** *It is NP-hard to decide whether a given graph admits a 1-planar storyplan (1-PLANAR STORYPLAN is NP-hard).*

*Proof.* Constructing the graph $G$ from the formula $\phi$ clearly takes polynomial time, and the correctness of the reduction follows from Lemma 5.1 and Lemma 5.2. This proves that 1-PLANAR STORYPLAN is NP-hard. □

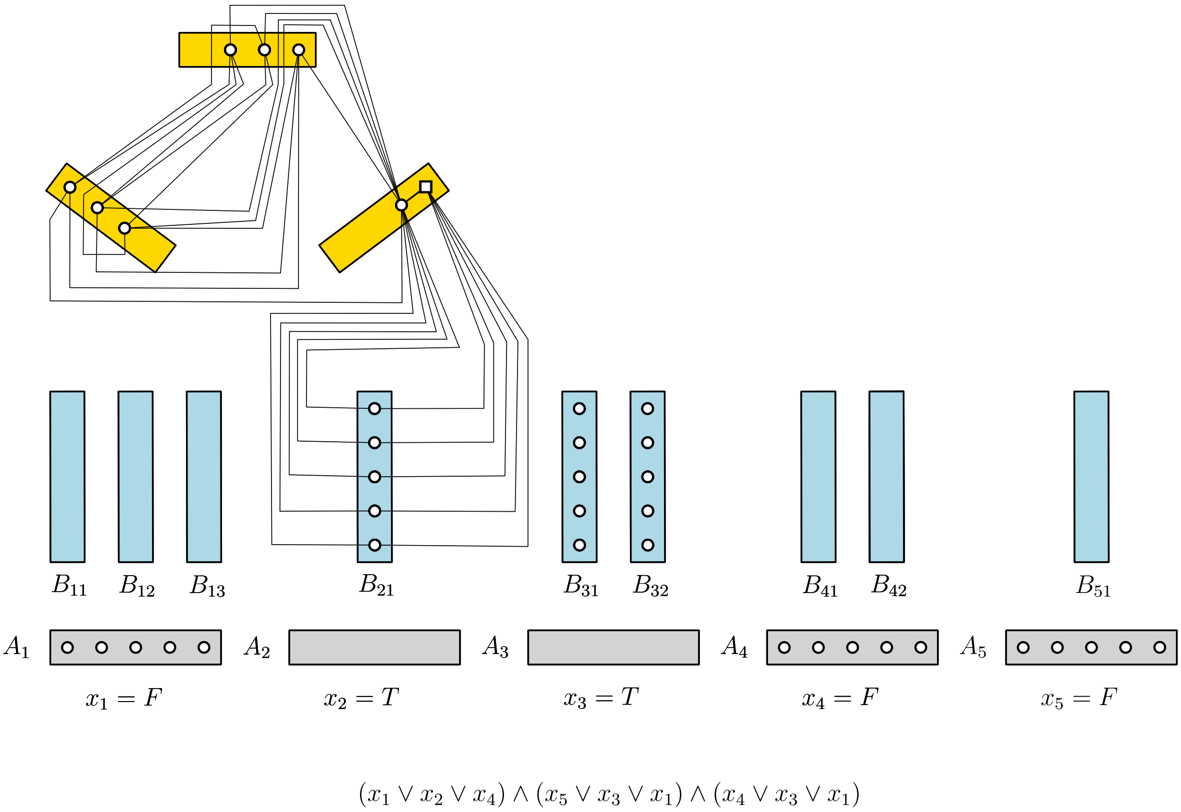
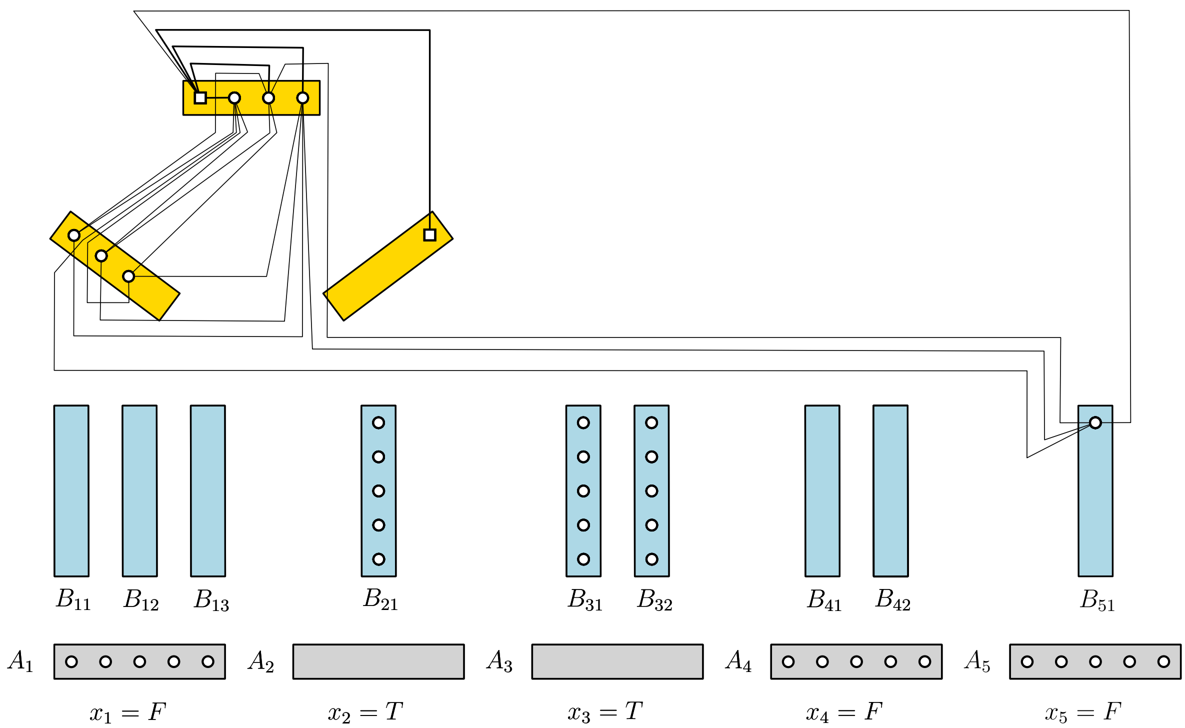



**Fig. 5.8:** Proof of Lemma 5.2: Drawing of a true variable.




$$(x_1 \vee x_2 \vee x_4) \wedge (x_5 \vee x_3 \vee x_1) \wedge (x_4 \vee x_3 \vee x_1)$$

**Fig. 5.9:** Proof of Lemma 5.2: Drawing of a false variable.

## 5.2 Parameterization Attempt

A *vertex cover* of a graph $G = (V, E)$ is a set $C \subseteq V$ such that every edge of $E$ is incident to a vertex in $C$, and the *vertex cover number* of $G$ is the minimum size of a vertex cover of $G$. Theorem 5.4 states the improvements in the time complexity that can be achieved through parameterization in the vertex cover number obtained by Binucci et al. [BGL+24]. We have attempted to adapt the parameterized algorithm to obtain an analogous theorem for 1-planar storyplans. In this section we will discuss the successful adaptations as well as the key difficulty in deriving an analogous parameterized algorithm for 1-Planar Storyplan.

**Theorem 5.4.** *Let $G = (V, E)$ be a graph with $n$ vertices and vertex cover number $\kappa$. Deciding whether $G$ admits a planar storyplan, and computing one if any, can be done in $O(2^{2^{O(\kappa)}} + n^2)$ time.*

**Algorithm Description.** Without loss of generality, we assume that the input graph $G$ does not contain isolated vertices, as they do not affect the existence of a 1-planar storyplan. Isolated vertices can become visible at any step in the storyplan and instantly disappear after that step because by definition they have no neighbors. Let $C$ be a vertex cover of size $\kappa$ of a graph $G$. For $U \subseteq C$, a vertex $v \in V \setminus C$ is of *type* $U$ if $N(v) = U$, where $N(v)$ denotes the set of neighbors of $v$ in $G$. This defines an equivalence relation on $V \setminus C$ and in particular partitions $V \setminus C$ into at most $\sum_{i=1}^{\kappa} \binom{\kappa}{i} = 2^\kappa - 1 < 2^\kappa$ distinct types. Denote by $V_U$ the set of vertices of type $U$. We define three reduction rules.

**R.1:** If there exists a type $U$ such that $|U| = 1$, then pick an arbitrary vertex $x \in V_U$ and remove it from $G$.

**R.2:** If there exists a type $U$ such that $|U| = 2$ and $|V_U| > 1$, then pick an arbitrary vertex $x \in V_U$ and remove it from $G$.

**R.3:** If there exists a type $U$ such that $|U| \geq 3$ and $|V_U| > 7$, then pick an arbitrary vertex $x \in V_U$ and remove it from $G$.

Note that reduction rules **R.1** and **R.2** are equivalent to the first two reduction rules as defined by Binucci et al. [BGL+24]. The third reduction rule has been adapted for 1-planar storyplans as the bipartite graph that is induced by $U$ and $V_U$ must not be 1-planar. In the following, we will explain why reduction rule **R.1** works and reduction rules **R.2** and **R.3** fail.

**Lemma 5.5. (Attempt)** *Let $G'$ be the graph obtained from $G$ by applying one of the reduction rules **R.1-R.3**. Then $G$ admits a 1-planar storyplan if and only if $G'$ does.*

*Proof. (Attempt)* One direction is trivial, since when a graph admits a 1-planar storyplan, then a subgraph of that graph also admits a 1-planar storyplan. Thus, we will examine the nontrivial direction. Let's assume that $G'$ admits a 1-planar storyplan $S' = (\tau', \{D'\}_{i \in [n']})$, where $n' = n - 1$. We can distinguish between three cases based

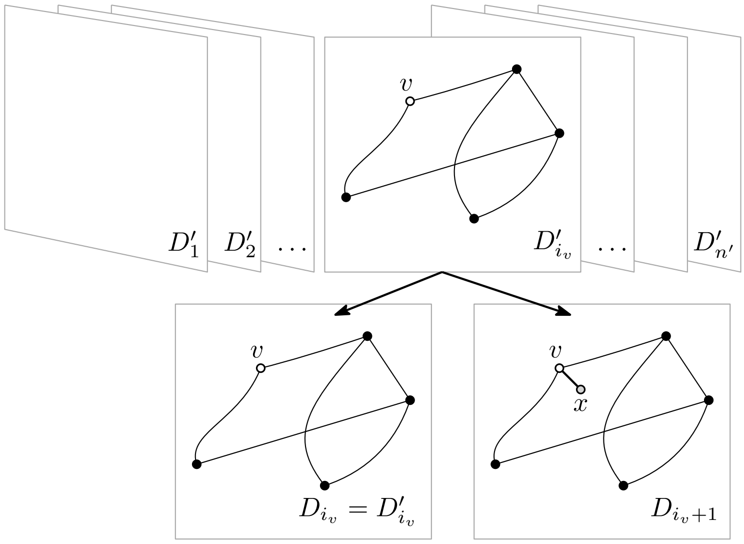



**Fig. 5.10:** Attempt of Lemma 5.5. Illustration for proof of **Case A**.

on the respective reduction rule applied to $G$. In each case, we denote by $x$ the vertex removed from $G$ to obtain $G'$.

**Case A (R.1).** See Figure 5.10 for an illustration. Let $v$ be the neighbor of $x$ in $G$, whose lifespan according to $\tau'$ is $[i_v, j_v]$. We compute $\tau$ from $\tau'$ by inserting $x$ right after $v$, consequently the lifespan of $x$ in $\tau$ is $[i_v+1, i_v+1]$. Similarly, we compute $\{D_i\}_{i\in[n]}$ from $\{D'_i\}_{i\in[n']}$ as follows. For each $i \leq i_v$, we set $D_i = D'_i$. For $i = i_v + 1$, we draw $x$ in $D'_{i_v}$ sufficiently close to $v$ such that the edge $xv$ can be drawn as a straight-line segment that does not intersect any other edge. It is not difficult to visualize that this segment can always exist in such a manner. We then set $D_i$ to be equal to the resulting drawing. Finally, for each $i > i_v + 1$, we set $D_i = D'_{i-1}$.

**Case B Attempt (R.2).** See Figure 5.12 for an illustration. By assumption $G'$ contains at least one vertex $v \neq x$ of type $U$, whose lifespan according to $\tau'$ is $[i_v, j_v]$. We compute $\tau$ from $\tau'$ by inserting $x$ right after $v$. Consequently, the lifespan of $x$ in $\tau$ is $[i_v + 1, j_v]$. The vertex $x$ will disappear after the second (i.e., the last) common neighbor $w'$ of $v$ and $x$ has appeared. For each $i \leq i_v$, we set $D_i = D'_i$. For $i = i_v + 1$, we extend $D'_i$ by drawing $x$ sufficiently close to $v$ and by drawing, for each neighbor $w$ of $x$, the edge $xw$ such that it follows the curve representing the edge $vw$. Notice, that this strategy works for planar storyplans but not for 1-planar storyplans. In the case of planar storyplans, since $vw$ is crossing-free, the same holds for $xw$. In the case of 1-planar storyplans, there is a possibility of $vw$ being crossed at most once. Thus, the edge that crosses $vw$ also crosses $xw$ except when $v$ has disappeared before $x$ appeared which we cannot guarantee. See Figure 5.11 for an illustration of the problem. We can neither guarantee that $v$ disappears before $x$ appears nor can we guarantee that the edge that crosses $vw$ disappears before we insert $x$.

**Case C Attempt (R.3).** Observe that, by assumption, the graph induced by the

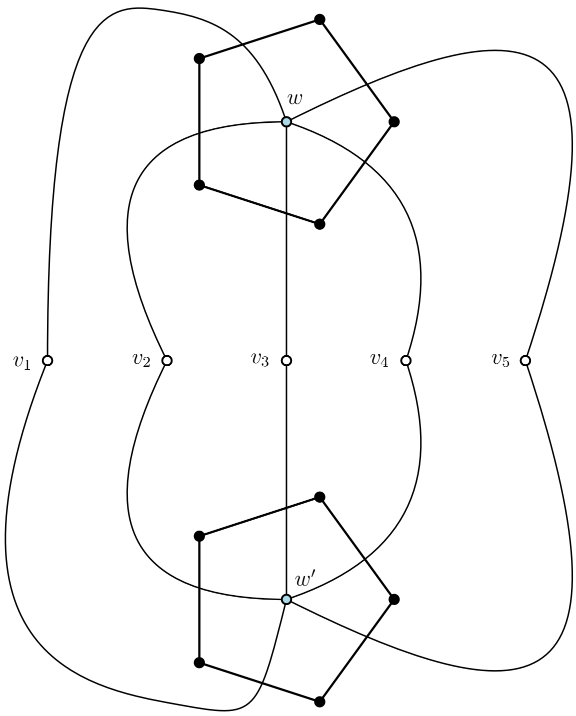



**Fig. 5.11:** Illustration of the difficulty in constructing the reduction rule as described in **Case B**. We cannot insert the vertex $x$ such that it follows the curves of one of the vertices $v_i$ for $i \in [|V_U|]$. There is no remaining edge to be crossed and we can neither guarantee that $w$ and $w'$ are still visible when $x$ is inserted nor that any $v_i$ disappears before $x$ appears

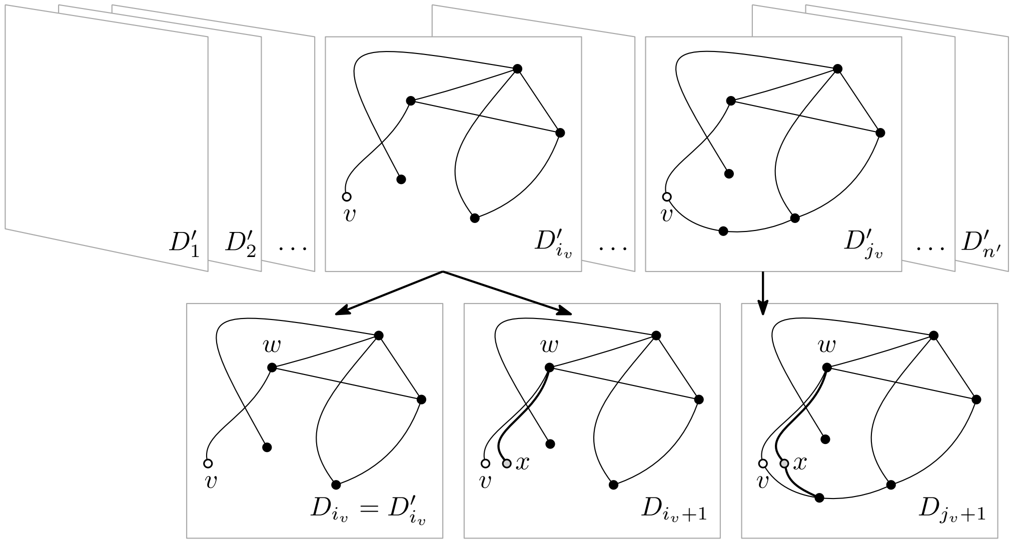



**Fig. 5.12:** Attempt of Lemma 5.5. Illustration for attempt of **Case B**.

vertices of $U \cup V_U$ contains the complete bipartite graph $K_{a,b}$ with $a \geq 3$ and $b > 7$, whose partite sets are $U$ and $V_U$. Note that these complete bipartite graphs are not 1-planar as proved by Czap and Hudák [CH12]. Thus, according to Lemma 3.3, we always have one partite set that is fixed and another that is flexible. We distinguish two subcases **Case C.1** and **Case C.2** based on whether $U$ is the fixed set or the flexible set of $K_{a,b}$. Note that **Case C.1** works as intended for 1-planar storyplans. In **Case C.2** we run into similar difficulties as described in **Case B**.

**Case C.1 (R.3).** First, suppose that $U$ is the fixed set in the bipartite graph (see Figure 5.13 for an illustration). Let $I \subseteq [n']$ be an interval in which all vertices of $U$ are visible. By assumption, $G'$ contains at least seven vertices $v_j \neq x$ of $V_U$, with $j \in [7]$. Observe that the lifespan of each vertex $v_j$ intersects $I$, therefore there is at least one vertex, say $v_1$, whose lifespan does not intersect the interval. Otherwise, there would be a frame containing $K_{3,7}$. Let $[i_1, j_1]$ be the lifespan of $v_1$. Observe that $j_1$ is in $I$. We compute $\tau$ from $\tau'$ by inserting $x$ right after $v_1$ disappears, such that its lifespan in $\tau$ is $[j_1 + 1, j_1 + 1]$. For each $i \leq j_1$, we set $D_i = D'_i$. For $i = j_1 + 1$, we take $D'_{j_1}$ and replace the drawing of $v_1$ with the drawing of $x$. Namely, we place $x$ on the same point of $v_1$ and we draw each curve $xw$ by following the curve $v_1w$. Since $v_1w$ crosses any other edge at most once and disappears before $x$ appears, the same holds for $xw$. The resulting drawing is $D_i$. Finally, for each $i > j_1 + 1$, we set $D_i = D'_{i-1}$.

**Case C.2 Attempt (R.3).** Now suppose that $V_U$ is the fixed set (see Figure 5.14 for an illustration). Let $[s, t] \subseteq [n']$ be the maximal interval in which all vertices of $V_U$ in $G'$ are visible. Let $v \neq x$ be the vertex of $V_U$ such that $\tau'(v) = s$ (i.e., the first vertex of $V_U$ that appears). We compute $\tau$ from $\tau'$ by inserting $x$ right next to $v$ such that its lifespan in $\tau$ is $[s + 1, t + 1]$. For each $i \leq s$, we set $D_i = D'_i$. For $i = s + 1$, we extend

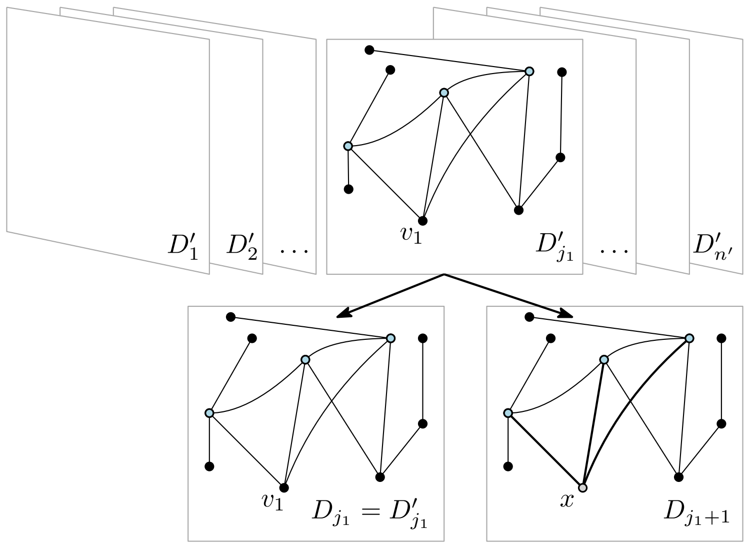



**Fig. 5.13:** Attempt of Lemma 5.5. Illustration for proof of **Case C.1** (i.e., $U$ is the fixed set).

$D'_i$ by drawing $x$ sufficiently close to $v$ and by drawing, for each neighbor $w$ of $x$, the edge $xw$ such that it follows the curve representing the edge $vw$. In the case of 1-planar storyplans, there is a possibility of $vw$ being crossed at most once. Once again, we face the same challenge as described in **Case B** and illustrated in Figure 5.11. We cannot set $D_i$ to be equal to the resulting drawing. Similarly, for each $i \in [s+2, t+1]$, we cannot extend (if needed) the frame $D'_{i-1}$ by drawing any edge $xw$ in a 1-planar manner. For each $i > t+1$, we set $D_i = D'_{i-1}$.

□

With the help of Lemma 5.5 we attempt to prove Theorem 5.4 for 1-planar storyplans.

*Proof. (Attempt)* According to Chen, Kanj, and Xia [CKX10], we can determine the vertex cover number $\kappa$ of $G$ and compute a vertex cover $C$ of size $\kappa$ in time $O(2^\kappa + \kappa \cdot n)$. To construct a kernel $G^\star$ from $G$ of size $O(2^\kappa)$, we first classify each vertex of $G$ based on its type. Then, we apply our reduction rules **R.1**, **R.2**, and **R.3** exhaustively. Thus, constructing $G^\star$ can be done in $O(2^\kappa + \kappa \cdot n)$ time, since $O(2^\kappa)$ is the number of types and $\kappa \cdot n$ is the maximum number of edges of $G$. Also, $G^\star$ contains $n^\star \leq 7 \cdot 2^\kappa < 2^{\kappa+3}$ vertices.

From Lemma 5.5 we attempt to conclude that $G$ admits a 1-planar storyplan if and only if $G^\star$ does. To establish whether $G^\star$ admits a 1-planar storyplan we proceed as follows: (1) We guess a total order $\tau^\star$ of $G^\star$; (2) For $i = 1$, we guess all 1-planar embeddings of the graph induced by the vertices visible at step $i$; (3) For each $i > 1$, we consider the embeddings computed at the previous step $i-1$, we remove from them the vertices (if any) that disappear at step $i$, we remove possible duplicates, and we try to exhaustively

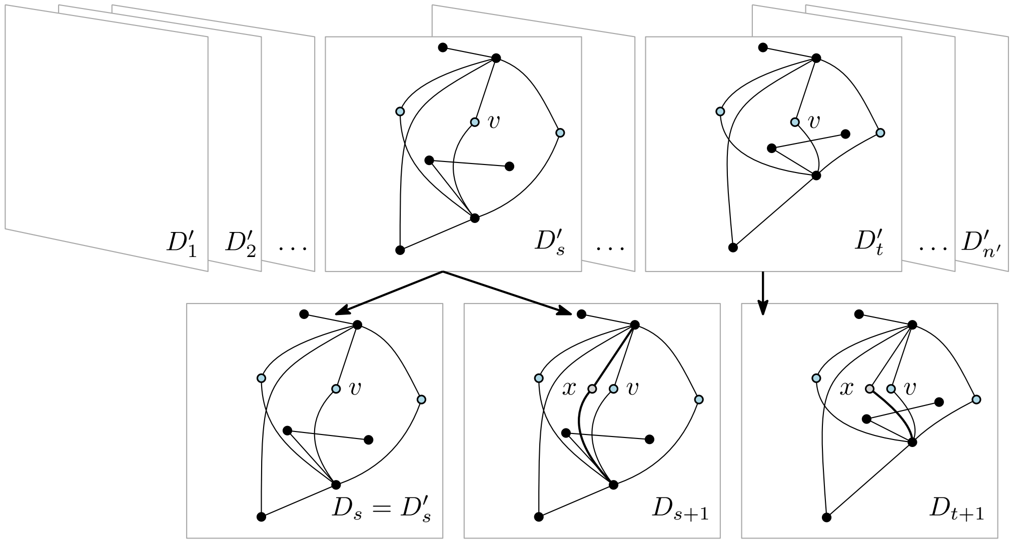



**Fig. 5.14:** Attempt of Lemma 5.5. Illustration for attempt of **Case C.2** (i.e., $V_U$ is the fixed set).

extend each of the resulting 1-planar embeddings by inserting the vertex that appears at step $i$. The algorithm halts if the set of 1-planar embeddings becomes empty. It is readily seen that $G^\star$ admits a 1-planar storyplan if and only if the algorithm terminates at step $n^\star$ with at least one 1-planar embedding. Concerning the time complexity, step (1) takes $O(2^{\kappa+3}!)$ time. Since $G^\star$ contains $O(2^{O(\kappa)})$ vertices and edges, the number of possible 1-planar embeddings are $O((2^{O(\kappa)})^{(2^{O(\kappa)})}) = O((2^{2^{O(log \kappa)} \cdot 2^{O(\kappa)}})) = O(2^{2^{O(\kappa)}})$. Hence step (2) takes $O(2^{2^{O(\kappa)}})$ time and step (3) takes $O(2^{2^{O(\kappa)}}) \cdot O(2^{2^{O(\kappa)}}) = O(2^{2^{O(\kappa)}})$ time. Starting from the 1-planar storyplan of $G^\star$, we can reinsert the missing $O(n)$ vertices each in $O(n)$ time, as detailed in Lemma 5.5. □

# 6 Conclusion

In this thesis, we have introduced 1-planar storyplans and established a distinction between the various graph classes that admit $p$-planar storyplans. This allows for a broader range of graphs to be representable by some kind of storyplan, thereby enhancing the utility of storyplans in dynamic graph visualization. The chain of strict containment relations between graph classes that allow planar, 1-planar, 2-planar, ..., and $n$-planar storyplans has been established. We devised a total order of the vertices to show that $C_{3,3,3,3,3}$ does not admit planar storyplans but does admit 1-planar storyplans. We showed that not every graph admits a 1-planar storyplan. Specifically, we have revealed that only complete graphs that are $p$-planar admit $k$-planar storyplans (where $k \geq p$). Thus, we were able to extend the strict containment of graph classes, that allow storyplans, to include $p$-planar storyplans.

Furthermore, we have formally introduced 1-PLANAR STORYPLAN and have proved the NP-hardness through reduction of POSITIVE ONE-IN-THREE 3SAT. This has built on prior work done by Binucci et al. [BGL$^+$24] and is laying the groundwork for further research in the hardness of storyplan-related problems.

Finally, we highlighted the difficulties that arise when attempting to adapt the parameterization of PLANAR STORYPLAN in the vertex cover number [BGL$^+$24] to an analogous parameterization of 1-PLANAR STORYPLAN in the vertex cover number. This result paves the way for further parameterization attempts (see Chapter 7 to get a perspective on what research can follow this thesis).

In the broader context, this research significantly contributes to the theoretical foundation of dynamic graph visualization. Our work expands the universe of graphs that can be visualized using storyplans, underlining the potential for alternative visualization techniques. By providing the NP-hardness proof for the decision problem surrounding 1-planar storyplans, this thesis not only advances our current understanding of storyplan decision problems but also points towards avenues for future research and innovation in the graph visualization domain.

# 7 Open Problems

As storyplans have only recently been introduced, the field surrounding storyplans still has a plethora of problems yet to be explored. The following are the open problems that we find are the most suitable for further studies in the domain of storyplans for graph visualization.

1. An intuitive next step in this area of research would be to develop a parameterized algorithm for 1-Planar Storyplan in the vertex cover number. This thesis has already highlighted some of the difficulties that arise when using the approach of Binucci et al. [BGL+24] for 1-Planar Storyplan. Other approaches may be more suitable for this kind of problem.
2. Further research could answer whether every 4-tree admits a 1-planar storyplan.
3. While we have shown the complexity of 1-Planar Storyplan, future work could examine the complexity of 1-Planar Storyplan Fixed Order, where the total order of the vertices $\tau\colon V \to [n]$ is already given as input of the problem.
4. In the field of storyplans, the complexity of decision problems have been studied. On the other hand, area constraint problems for the construction of certain kinds of storyplans have not been studied yet.
5. It would be interesting to attempt the parameterization of 1-Planar Storyplan in the feedback edge set number and other parameters.
6. While 1-planar storyplans have been the focus of this thesis, there is room for exploration into other types of storyplans. For instance, investigating storyplans that only allow right-angle crossings or other specific constraints can open up new possibilities.

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