Bachelor Thesis

Property-Preserving Reductions and Hardness for Variants of Level Planarity

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Abstract

The problem Level Planarity asks for a planar drawing of a graph where the y-coordinates of vertices are fixed. It can be generalized to the problem Constrained Level Planarity where the left-to-right order of the vertices on a horizontal line is constrained by a partial order. The problem Ordered Level Planarity is a special case of Constrained Level Planarity where the orders on the vertices are total.

In this thesis, we investigate reductions from Constrained Level Planarity to Ordered Level Planarity. Adapting a reduction by Sieper [Sie22] we show that there are reductions from Constrained Level Planarity to Ordered Level Planarity which preserve various graph properties such as outerplanarity, chordality, perfectness, pathwidth, treedepth, maximum degree, and cycle graphs. We also provide a reduction that, under certain conditions, maintains k-connectivity for arbitrary k.

By modifying a $\mathcal{NP}$-hardness proof from Brückner and Rutter [BR17] that reduces Planar Monotone 3-Satisfiability to Constrained Level Planarity we show that Constrained Level Planarity is $\mathcal{NP}$-hard when restricted to cycle graphs as well as 5-connected graphs. Using the reductions from this thesis we then show that Ordered Level Planarity is also $\mathcal{NP}$-hard when restricted to cycle as well as 5-connected graphs.

Zusammenfassung

Das Problem Level Planarity besteht darin, für einen gegeben Graphen zu entscheiden, ob es eine planar Zeichnung gibt, in der sich die Knoten auf zuvor festgelegten horizontalen Linien befinden. Es kann zum Problem Constrained Level Planarity verallgemeinert werden, bei dem die Reihenfolge der Knoten von links nach rechts durch eine partielle Ordnung beschränkt ist. Das Problem Ordered Level Planarity ist ein Spezialfall des Problems Constrained Level Planarity, bei dem die Ordnungen auf den Knoten total sind.

In dieser Arbeit untersuchen wir Reduktionen von Constrained Level Planarity auf Ordered Level Planarity. Ausgehend von einer Reduktion von Sieper [Sie22] zeigen wir, dass es Reduktionen von Constrained Level Planarity auf Ordered Level Planarity gibt, die unterschiedliche Grapheigenschaften, wie Außenplanarität, Chordalität, Perfektheit, Pfadweite, Baumtiefe, Maximalgrad und Kreisgraphen, erhalten. Außerdem präsentieren wir eine Reduktion, die, unter gewissen Bedingungen, k-fach Zusammenhang für beliebige k aufrechterhält.

Durch Modifikation eines $\mathcal{NP}$-Schwere Beweises von Brückner und Rutter [BR17], in dem Planar Monotone 3-Satisfiability auf Constrained Level Planarity reduziert wird, zeigen wir, dass Constrained Level Planarity auch für Kreisgraphen sowie 5-fach zusammenhängende Graphen $\mathcal{NP}$-schwer ist. Mithilfe der Reduktionen aus dieser Arbeit folgern wir anschließend, dass Ordered Level Planarity ebenfalls für Kreisgraphen sowie 5-fach zusammenhängende Graphen $\mathcal{NP}$-schwer ist.

Contents

1 Introduction

When visualizing hierarchical data, such as networks or corporate structures, a top-down approach is often used to enhance clarity and improve understanding of the data. Due to crossings diminishing visual clarity and readability, it is desirable to obtain drawings of a graph that are free of crossings. Additionally, as it is often useful to group the data on horizontal lines, the question arises as to whether a given graph can be drawn planar with the vertices on prescribed horizontal lines, known as levels. This problem is known as Level Planarity. Since it may also be desired to have the vertices placed in a grid pattern, this leads to the problem of Ordered Level Planarity. Alternatively, a less strict version of this problem would be to require that the vertices on a level be placed according to a partial order, a problem known as Constrained Level Planarity.

1.1 Related Work

The problem of Level Planarity has been studied for a long time with Di Battista and Nardelli [BN88] showing in 1988 that Level Planarity can be solved in polynomial time for graphs with a single source. This result was later generalized by Jünger et al. [JLM98] to a linear-time recognition algorithm. Further improvements by Jünger and Leipert [JL99] led to an algorithm that also determines an embedding in linear time. As this algorithm is quite complicated, simpler, but asymptotically slower, algorithms were developed, such as a quadratic-time algorithm by Fulek et al. [FPSŠ13]. Another quadratic-time algorithm was given by Harrigan and Healy [HH08] which also allows for constraints on the order of incident edges around vertices.

Over the years, many variations of level planarity have been studied. For example, the problems Radial Level Planarity, Cyclic Level Planarity and Torus Level Planarity draw the graph on the surface of a standing cylinder, rolling cylinder, and torus, respectively, instead of a plane. Bachmaier et al. [BBF05] provided a linear-time recognition and embedding algorithm for Radial Level Planarity. For strongly-connected cyclic level graphs, a linear-time testing and embedding algorithm was provided by Bachmaier and Brunner [BB08]. Angelini et al. [ALB+16] later showed that Radial Level Planarity and Cyclic Level Planarity reduce in linear time to Torus Level Planarity and provided a general polynomial-time testing algorithm for these problems. Forster and Bachmaier [FB04] introduced the problem of Clustered Level Planarity that allows for the visualization of vertex clusterings and provided an efficient algorithm for the proper case where all edges connect vertices on adjacent levels. Another variation, T-Level Planarity, where there are restrictions on which vertices can appear consecutively on a level, was introduced by Wotzlaw et al. [WSP12]. They provided a quadratic-time algorithm for proper instances with a constant level-width, i.e. a constant number of vertices per level. Angelini et al. [ALB+15] managed to generalize the algorithms for Clustered Level Planarity and T-Level Planarity to proper graphs and also showed that these problems are $\mathcal{NP}$-complete in the general case. Brückner and Rutter [BR17] introduced the problem of Constrained Level Planarity, where a partial order is set on the vertices of each level, as well as Partial Level Planarity, where a drawing of a subgraph is already fixed. They provided an efficient algorithm for graphs with only one source and showed that both problems are $\mathcal{NP}$-hard in the general case. The problem Ordered Level Planarity, where both the x and y-coordinate of vertices are prescribed, was introduced by Klemz and Rote [KR19]. They showed that Ordered Level Planarity is a special case of several graph drawing problems, such as T-Level Planarity, Clustered Level Planarity and Constrained Level Planarity. As the problem Constrained Level Planarity is $\mathcal{NP}$-hard in the general case there has been research into $\mathcal{FPT}$-algorithms parametrized by certain graph properties. Recently, Sieper [Sie22] showed that there are no $\mathcal{FPT}$-algorithms for Constrained Level Planarity parametrized by the number of levels, pathwidth, or the maximum number of vertices per level as these cases are $\mathcal{NP}$-hard. In doing so, Sieper also provided a reduction from Constrained Level Planarity to Ordered Level Planarity.

1.2 Contribution

Seeing as Ordered Level Planarity is a special case of Constrained Level Planarity and there are reductions from Constrained Level Planarity to Ordered Level Planarity, we further analyze the connection between these two problems. We start by formalizing the problems and properties which we will use throughout this thesis in Chapter 2. Adapting the reduction from Sieper [Sie22, Chapter 5], we will show in Chapter 3 that Constrained Level Planarity can be reduced in polynomial time to Ordered Level Planarity while preserving several graph properties, such as outerplanarity, chordality, perfectness, pathwidth, treedepth, maximum degree, and cycle graphs. We also show in Section 3.3 that k-connectivity can be maintained for arbitrary k under certain conditions.

In Chapter 4 we adapt the $\mathcal{NP}$-hardness proof from Brückner and Rutter [BR17] reducing Planar Monotone 3-Satisfiability to Constrained Level Planarity in order to show that Constrained Level Planarity is $\mathcal{NP}$-hard even when restricted to cycle graphs as well as 5-connected graphs. Using the reductions from this thesis we then show that Ordered Level Planarity is also $\mathcal{NP}$-hard when restricted to cycle as well as 5-connected graphs. Lastly, we conclude our work and outline questions for future work in Chapter 5.

2 Preliminaries

We will now define the problems which this work focuses on. An example instance for each problem can be seen in Figure 2.1. The terminology is based on the one used by Klemz and Rote [KR19].

Definition 2.1 (Level Planarity)**.** *An* h-level graph $\mathcal{G} = (G, \gamma)$ *is a directed graph* $G = (V, E)$ *together with a* level assignment $\gamma : V \to [h] := \{1, 2, \ldots, h\}$ *such that* $\gamma(u) < \gamma(v)$ *for each edge* $(u, v) \in E$*. The value* h *is therefore the* height *of* $\mathcal{G}$*. The set* $V_\ell(\mathcal{G}) := \{v \in V \mid \gamma(v) = \ell\}$ *is the set of vertices on the* ℓ*-th* level *of* $\mathcal{G}$ *with* width $\lambda_\ell(\mathcal{G}) := |V_\ell(\mathcal{G})|$*. The* level-width $\lambda(\mathcal{G})$ *of* $\mathcal{G}$ *is the maximum width of any level in* $\mathcal{G}$*. A* level drawing *of* $\mathcal{G}$ *is a drawing where the* y*-coordinate of each vertex* v *is* $\gamma(v)$ *and each edge* $e = (u, v)$ *is a* y*-monotone arc. A crossing-free level drawing of* $\mathcal{G}$ *is a* level planar drawing *and* $\mathcal{G}$ *is* level planar *if and only if it admits a level planar drawing. The problem* Level Planarity *asks whether a given level graph is level planar.*

As the y-coordinates prescribed by γ act merely as a way to encode a total preorder of the vertices, Level Planarity is equivalent in terms of realizability to the generalization where γ maps to h distinct and arbitrary real numbers. We can also generalize Level Planarity by enabling the restriction of the order in which vertices are drawn on each level which yields the following problem.

Definition 2.2 (Constrained Level Planarity)**.** *A* constrained h-level graph $\mathcal{G}$ *is a triple* $(G, \gamma, (\prec_\ell)_{1 \leq \ell \leq h})$ *where* (G, γ) *is an* h*-level graph and the vertex order on each level* $\ell \in [h]$ *is constrained by the partial order* $\prec_\ell$*. A* constrained level planar drawing *of* $\mathcal{G}$ *is a level planar drawing compatible with the constraints, i.e. with the left-to-right order of the vertices in* $V_\ell(\mathcal{G})$ *being a linear extension of* $\prec_\ell$ *for each level* $\ell \in [h]$*. For a pair of vertices* $u, v \in V_\ell(\mathcal{G})$ *we refer to* $u \prec_\ell v$ *as a* constraint *on* u *and* v*. The graph* $\mathcal{G}$ *is* constrained level planar *if and only if it admits a constrained level planar drawing. The problem* Constrained Level Planarity *asks whether a given graph* $\mathcal{G}$ *is constrained level planar.*

We define Ordered Level Planarity as a special case of Constrained Level Planarity with total instead of partial orders. It remains equivalent in terms of realizability to the definition by Klemz and Rote [KR19] as prescribing the x-coordinate of each vertex acts merely as a way to encode a total order of the vertices on each level

Definition 2.3 (Ordered Level Planarity)**.** *The problem* Ordered Level Planarity *corresponds to a special case of* Constrained Level Planarity *where the order of the vertices on each level is total. An instance* $\mathcal{G}$ *of* Ordered Level Planarity *is an* ordered level graph*. A constrained level planar drawing of* $\mathcal{G}$ *is referred to as* ordered level planar drawing*, and we say that* $\mathcal{G}$ *is* ordered level planar *when it is constrained level planar.*

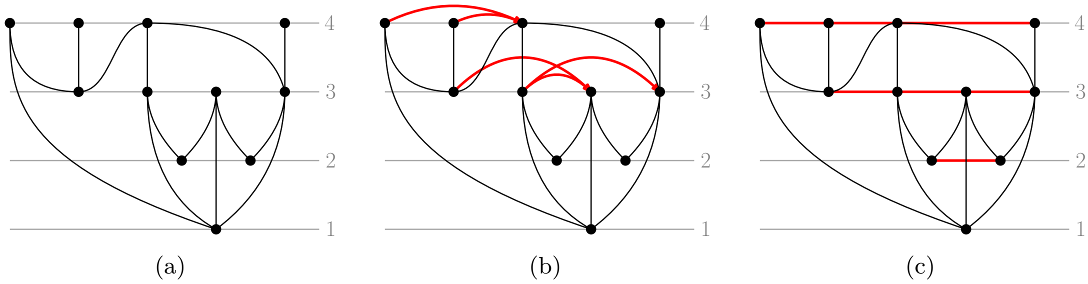


Fig. 2.1: (a) A 4-level graph. (b) The same level graph as a constrained level graph. The constraints are drawn as (red) arrows. (c) The same level graph as an ordered level graph. The constraints on the vertices of a level are drawn as a single (red) line for visual clarity since the order of the vertices is total.

Before we can proceed to the main part of this work we will define some additional terms and conventions. We say that two level graphs $\mathcal{G}$ and $\mathcal{G}'$ are equivalent (in terms of level planarity) if and only if, $\mathcal{G}$ is level planar if and only if $\mathcal{G}'$ is level planar. We say the same for constrained and ordered level graphs.

For a level graph $\mathcal{G} = (G, \gamma)$ with $G = (V, E)$ we define $V(\mathcal{G}) := V$ as the set of vertices, $n(\mathcal{G}) := |V|$ as the number of vertices, $E(\mathcal{G}) := E$ as the set of edges and $m(\mathcal{G}) := |E|$ as the number of edges. We also set $c(\mathcal{G})$ as the number of constraints in $\mathcal{G}$.

The size of an instance $\mathcal{G}$ of Constrained Level Planarity depends on the number of vertices, edges, and constraints and is therefore in $O(n(\mathcal{G})+c(\mathcal{G})+m(\mathcal{G})) \subset O(n^2(\mathcal{G}))$. If $\mathcal{G}$ is planar the number of edges must be in $O(n(\mathcal{G}))$ and therefore the size of $\mathcal{G}$ is in $O(n(\mathcal{G}) + c(\mathcal{G}))$.

For a level graph $\mathcal{G}$ and each vertex $v \in V(\mathcal{G})$ we denote the set of *outgoing edges* as $E^+_{\mathcal{G}}(v) := \{(v, u) \in E(\mathcal{G})\}$. The outgoing edges connect v with its *upward neighbours* $N^+_{\mathcal{G}}(v) := \{u \in V \mid (v, u) \in E(\mathcal{G})\}$. The set $E^-_{\mathcal{G}}(v) := \{(u, v) \in E \mid \gamma(u) < \gamma(v)\}$ are the *incoming edges* which connect to *downward neighbours* $N^-_{\mathcal{G}}(v) := \{u \in V \mid (u, v) \in E^-_{\mathcal{G}}\}$. Combining these sets yields the *incident edges* $E_{\mathcal{G}}(v) := E^+_{\mathcal{G}}(v) \cup E^-_{\mathcal{G}}(v)$ and *adjacent vertices* $N_{\mathcal{G}}(v) := N^+_{\mathcal{G}}(v) \cup N^-_{\mathcal{G}}(v)$ of a vertex $v \in V(\mathcal{G})$. For an edge $e \in E(\mathcal{G})$ with $e = (u, v)$ we define its *upper endpoint* as $e^+ := v$ and its *lower endpoint* as $e^- := u$. A vertex without neighbours is an *isolated vertex*. The *outdegree* of a vertex $v \in V(\mathcal{G})$ is the number of outgoing edges $\deg^+_{\mathcal{G}}(v) := |E^+_{\mathcal{G}}(v)|$ and the *indegree* is the number of incoming edges $\deg^-_{\mathcal{G}}(v) := |E^-_{\mathcal{G}}(v)|$. The *degree* of a vertex $v \in V$ is the number of incident edges $\deg_{\mathcal{G}}(v) := |E_{\mathcal{G}}|$. We denote the *maximum degree* of a graph $\mathcal{G}$ as $\Delta(\mathcal{G}) := \max\{\deg_{\mathcal{G}}(v) \mid v \in V\}$.

In the case that an h-level graph $\mathcal{G}$ has no isolated vertices we can partition the set of vertices $V(\mathcal{G})$ into *sources* $Q(\mathcal{G}) := \{v \in V(\mathcal{G}) \mid \deg^-_{\mathcal{G}}(v) = 0\}$ with only outgoing edges, *sinks* $S(\mathcal{G}) := \{v \in V(\mathcal{G}) \mid \deg^+_{\mathcal{G}}(v) = 0\}$ with only incoming edges, and *intermediate vertices* $R(\mathcal{G}) := \{v \in V(\mathcal{G}) \mid \deg^-_{\mathcal{G}}(v) \neq 0 \neq \deg^+_{\mathcal{G}}(v)\}$ which have both incoming and outgoing edges, as seen in Figure 2.2a. This partition also applies to each level $\ell \in [h]$ with $V_\ell(\mathcal{G})$ being partitioned into $Q_\ell(\mathcal{G}) := Q(\mathcal{G}) \cap V_\ell(\mathcal{G})$, $R_\ell(\mathcal{G}) := R(\mathcal{G}) \cap V_\ell(\mathcal{G})$ and $S_\ell(\mathcal{G}) := S(\mathcal{G}) \cap V_\ell(\mathcal{G})$.

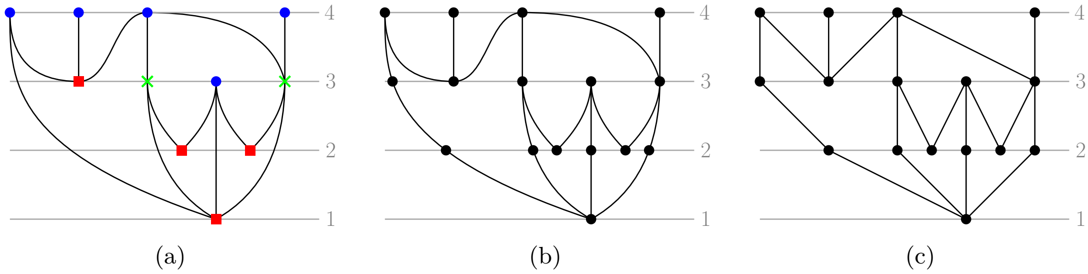


Fig. 2.2: (a) A 4-level graph $\mathcal{G}$ with its vertices partitioned into sources $Q(\mathcal{G})$ drawn as (red) squares, sinks $S(\mathcal{G})$ as (blue) circles, and intermediate vertices $R(\mathcal{G})$ as (green) crosses. (b) The proper version $\mathcal{G}^P$ of the graph $\mathcal{G}$ where the edges spanning multiple levels have been subdivided. (c) The proper graph can also be drawn straight.

For a constrained level graph $\mathcal{G} = ((V,E),\gamma,(\prec_\ell)_\ell)$ we say that a constrained level graph $\mathcal{G}' = ((V',E'),\gamma',(\prec'_\ell)_\ell)$ is a *subgraph* of $\mathcal{G}$, in symbols $\mathcal{G}' \subset \mathcal{G}$ if and only if $V' \subset V$, $E' \subset E$, $\gamma'(v) = \gamma(v)$ for all $v \in V'$ and for each constraint $v \prec'_\ell u$ in $\mathcal{G}'$ there is the constraint $v \prec_\ell u$ in $\mathcal{G}$. Note that if $\mathcal{G}$ is constrained level planar all its subgraphs are also constrained level planar. For a subset of vertices $U \subset V$ the *induced subgraph* of $\mathcal{G}$ is $\mathcal{G}[U] := ((U,\bar{E}),\bar{\gamma},(\bar{\prec}_\ell)_\ell)$ where $\bar{E}$, $\bar{\gamma}$ and $\bar{\prec}$ are restricted to the vertices in U.

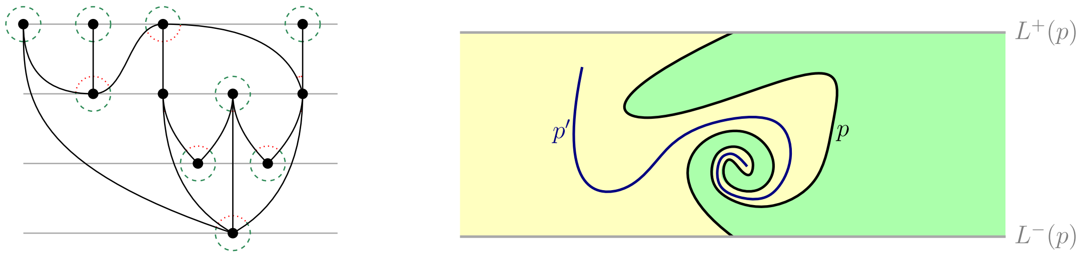
Since the direction of the edges of a level graph is clear from context, we will treat them for the sake of simplicity as undirected from now on.

A *walk* $p = \langle v_0,\ldots,v_k\rangle$ of *length* $k \geq 0$ is a finite sequence of vertices in $\mathcal{G}$ such that $\{v_{i-1},v_i\} \in E(\mathcal{G})$ for all $i \in [k]$. If $v_0 = v_k$ we say that the walk p is *closed*. If no two vertices in p are the same, except for the case that $v_0 = v_k$, we say that p is a *path*. We call a path that is also closed a *cycle*. A graph that consists solely of a cycle is a *cycle graph*. If $\gamma(v_{i-1}) < \gamma(v_i)$ for all $i \in [k]$ or $\gamma(v_{i-1}) > \gamma(v_i)$ for all $i \in [k]$ we say that the path p is *(y-)monotone*. We define $L^+(p) := \max_{v\in p}\gamma(v)$ and $L^-(p) := \min_{v\in p}\gamma(v)$ as the lowest and highest levels of a path p and $L(p) := \{L^-(p),\ldots,L^+(p)\}$ as the set of levels which p *touches*. Further, we say that a path p *crosses* a level ℓ at a vertex $v \in V_\ell(\mathcal{G})$ with $v \in p$ if and only if v is not an endpoint of p. We say that a path $p = \langle v_0,\ldots,v_k\rangle$ is *bounded* (by its endpoints) if and only if they are the highest and lowest points of the path, i.e. $L^-(p) = \gamma(v_0)$ and $L^+(p) = \gamma(v_k)$ or $L^+(p) = \gamma(v_0)$ and $L^-(p) = \gamma(v_k)$.

A level graph $\mathcal{G}$ is *proper* if and only if for every edge $e \in E(\mathcal{G})$ the incident vertices lie on consecutive levels, i.e. $\gamma(e^+) = \gamma(e^-) + 1$. Every h-level graph $\mathcal{G}$ can be transformed into a proper h-level graph $\mathcal{G}^P$ by subdividing all edges spanning multiple levels, as seen in Figure 2.2b. A level drawing Γ of $\mathcal{G}$ can be similarly transformed into a level drawing Γ^P of $\mathcal{G}^P$. Note that $\mathcal{G}$ and $\mathcal{G}^P$ are equivalent. Assuming that $\mathcal{G}$ is planar and $h \leq n(\mathcal{G})$ the graph $\mathcal{G}^P$ has $O(n^2(\mathcal{G}))$ vertices as $\mathcal{G}$ has $O(n(\mathcal{G}))$ edges.

A level drawing Γ of a proper h-level graph $\mathcal{G}$ defines total linear orders $\prec^\Gamma_\ell$ on the vertices $V_\ell(\mathcal{G})$ of each level $\ell \in [h]$, given by their left-to-right order in the drawing. We call this family of orders a *level embedding*. The embedding derived from a level planar drawing is a *level planar embedding*. Jünger et al. [JLM98] argued that a level embedding Γ of a proper h-level graph $\mathcal{G}$ is level planar if and only if for every pair of edges $\{u_1, v_1\}, \{u_2, v_2\} \in E(\mathcal{G})$ such that $\gamma(u_1) = \gamma(u_2) = \ell$ and $\gamma(v_1) = \gamma(v_2) = \ell + 1$ as well as $u_1 \neq u_2$ and $v_1 \neq v_2$ it follows from $u_1 \prec_\ell^\Gamma u_2$ that $v_1 \prec_{\ell+1}^\Gamma v_2$. As we can obtain a level planar drawing of $\mathcal{G}$ from a given level planar embedding, by realizing all edges with straight lines and positioning the vertices on each level according to the order from the embedding, see for example Figure 2.2c, we will from now on mostly use the level embedding. Jünger and Leipert [JL99] showed that a level planar embedding can be obtained in linear time. However, since we want all relations between vertices explicitly encoded in the embedding our embedding can have up to quadratic size compared to the graph $\mathcal{G}$. We therefore also need quadratic time instead of linear to obtain a level planar embedding. Since a y-monotone path p has exactly one vertex on each level in $L(p)$ we can compare it to another y-monotone path p' on a level $\ell \in L(p) \cap L(p')$ by setting $p \prec_\ell^\Gamma p'$ if $v \prec_\ell^\Gamma v'$ and $p' \prec_\ell^\Gamma p$ if $v' \prec_\ell^\Gamma v$ where v and v' are the respective vertices of the paths on the level ℓ. This order is defined on all levels where the paths do not share a point. As we can treat vertices as paths of length 0 we can therefore also compare paths to vertices. This means that all vertices and edges on a level ℓ can be compared to one another except for edges with the same endpoint. We therefore denote for a vertex $v \in V(\mathcal{G})$ the total linear orders on the outgoing and incoming edges as $\prec_{v^+}^\Gamma$ and $\prec_{v^-}^\Gamma$, respectively. When given a level embedding Γ these orders can be derived from the order of the edges on the adjacent levels since the edges must have an order on those as we do not allow multi-edges. These orders are then also included in the order $\prec_\ell^\Gamma$ where $\ell = \gamma(v)$. This enables us to rewrite the planarity criterion as $\mathcal{G}$ is level planar if and only if for every pair of edges $e_1, e_2 \in E(\mathcal{G})$ and levels $\ell^-, \ell^+ \in L(e_1) \cap L(e_2)$ with $\ell^- < \ell^+$ as well as $e_1^- \neq e_2^-$ if $\ell^- = L^-(e_1) = L^-(e_2)$ and $e_1^+ \neq e_2^+$ if $\ell^+ = L^+(e_1) = L^+(e_2)$ it follows from $e_1 \prec_{\ell^-}^\Gamma e_2$ that $e_1 \prec_{\ell^+}^\Gamma e_2$. Therefore, if the embedding is planar, the order of two vertex-disjoint monotone paths must be the same on all levels that both paths touch. We can also compare a bounded path p to each path p' with $L(p') \subset L(p)$ that is vertex-disjoint to p. For this, we look at the area between the levels $L^-(p)$ and $L^+(p)$. This area is divided by the path p into two halves, as seen in Figure 2.3b. The path p' has to be completely in one half as it cannot be crossing p since they are vertex-disjoint, and it also cannot pass above or below p since p' does not go that high or low. Therefore, we can say that p' is to the right of p setting $p \prec_\ell^\Gamma p'$ for every $\ell \in L(p')$ if p' is in the right half. Otherwise, p' is to the left of p, and we set $p' \prec_\ell^\Gamma p$. Note that this definition is consistent with the one for monotone paths.

As we can transform a level drawing of an h-level graph $\mathcal{G}$ to a level drawing of its proper version $\mathcal{G}^P$ and vice versa we can generalize the level embedding to non-proper level graphs. We therefore compare the order of edges and even paths in $\mathcal{G}$ by comparing their subdivided equivalents in $\mathcal{G}^P$. Since the subdivision does not affect y-monotonicity, the path p^P in $\mathcal{G}^P$ corresponding to a y-monotone path p in $\mathcal{G}$ is also y-monotone. This means that the path in $\mathcal{G}^P$ corresponding to an edge in $\mathcal{G}$ is always y-monotone. We set $E_\ell(\mathcal{G}) := \{e \in E(\mathcal{G}) \mid e \text{ crosses } \ell\}$ for each $\ell \in [h]$ as the set of edges crossing the level ℓ. Using this we can equivalently define a level embedding of a (non-proper) level graph $\mathcal{G}$ as the family of total orders over the vertices and crossing edges $\mathcal{E}_\ell(\mathcal{G}) := V_\ell(\mathcal{G}) \cup E_\ell(\mathcal{G})$ on each level $\ell \in [h]$. The planarity criterion remains the same for these embeddings.



(a) (b)

Fig. 2.3: (a) A graph with the small angles highlighted as dotted (red) arcs and the big angles highlighted as dashed (green) arcs.
(b) The path p splits the area into a (yellow) left and a (green) right half. As p' is in the left half we say that p' is to the left of p.

We will now define the *combinatorial embedding* where the *cyclic order* of the incident edges of each vertex is given. Let U be a set and C a ternary relation on U, consisting of triples (x, y, z) with $x, y, z \in U$ pairwise unequal. Then C is a cyclic order if and only if the following properties are fulfilled:

1. C is cyclic: $(x, y, z) \in C \implies (y, z, x) \in C$
2. C is asymmetric: $(x, y, z) \in C \implies (z, y, x) \notin C$
3. C is transitive: $(x, y, z), (x, z, w) \in C \implies (x, y, w) \in C$
4. C is total: $x, y, z \in U, x \neq y \neq z \neq x \implies (x, y, z) \in C \vee (z, y, x) \in C$

Novák [Nov84] showed that given a cyclic order C and an element $x \in U$ we can obtain two linear orders $\prec_x$, $\prec^x$ with x as either the smallest or biggest element. According to Huntington [Hun35] the order on the elements in $U \setminus \{x\}$ is the same in $\prec_x$ and $\prec^x$. We can derive a cyclic order Λ_v of the edges around each vertex $v \in V(\mathcal{G})$ from a level planar embedding Γ by first joining the linear orders $\prec^{\Gamma}_{v^+}$ and $\prec^{\Gamma}_{v^-}$, setting the incoming edges to be before the outgoing edges, and then turning this into a cyclic order. The family of all the cyclic orders of the vertices in $\mathcal{G}$ is a combinatorial embedding Λ of $\mathcal{G}$. When given a cyclic order Λ_v of the incident edges of a source $v \in V(\mathcal{G})$ and an edge $e \in E_{\mathcal{G}}(v)$ the orders $\prec_e$ and $\prec^e$ correspond to linear orders of the incident edges of v with e being the left and rightmost edge, respectively.

For a level planar drawing Γ of $\mathcal{G}$, we set $F(\Gamma)$ as the set of faces in Γ. Given a face $f \in F(\Gamma)$ we set $\partial f := \langle v_0, v_1, \ldots, v_0 \rangle$ as the boundary of f, i.e. the closed walk surrounding f. For a vertex v we say, following [GT01], that v is a *local source* with respect to f if v has two outgoing edges on ∂f. If v has two incoming edges on ∂f it is a *local sink*. The angle α between those two edges is *large* if $\alpha > \pi$ and otherwise small, as seen exemplarily in Figure 2.3a. Note that each vertex can have at most one large angle.

We will now specify a situation that plays an important role in the reductions. Given a constrained level planar embedding Γ of a constrained level graph $\mathcal{G}$ we say that a vertex $u \in V(\mathcal{G})$ is *nested* in a vertex $v \in V(\mathcal{G})$ if and only if there are two bounded paths $p_L = \langle v, \dots, v_L \rangle$ and $p_R = \langle v, \dots, v_R \rangle$ such that p_L and p_R are vertex-disjoint and $p_L \prec^\Gamma_{\gamma(u)} u \prec^\Gamma_{\gamma(u)} p_R$. For a nested vertex, there cannot be a monotone path $p_u = \langle u, \dots, u' \rangle$ with $\gamma(v) \in L(p_u)$ that does not contain v and is vertex-disjoint to p_L and p_R as then either $v \prec^\Gamma_{\gamma(v)} p_u$ or $p_u \prec^\Gamma_{\gamma(v)} v$. This contradicts Γ being constrained level planar because of $u \prec_{\gamma(u)} p_R$ and $u \prec_{\gamma(u)} p_L$ which forces p_u to cross either p_L or p_R. For a vertex u nested in a vertex v we say that u is *nested below* if $\gamma(u) < \gamma(v)$ and *nested above* if $\gamma(u) > \gamma(v)$. We also say that a vertex u *can be nested* in a vertex v if there is a constrained level planar embedding of $\mathcal{G}$ where u is nested in v.

Properties We will now introduce some properties of graphs, which we will preserve in the reductions. First off, a graph $\mathcal{G}$ is *connected* if there is a path between any two vertices $u, v \in V(\mathcal{G})$ otherwise it is *disconnected*. A graph is *k-connected* if the graph remains connected when removing any $k-1$ vertices.

For two vertices $u, v \in V(\mathcal{G})$ of a graph $\mathcal{G}$ the distance $\mathrm{d}_\mathcal{G}(u, v)$ between u and v corresponds to the length of the shortest path between u and v in $\mathcal{G}$. The *diameter* $\mathrm{diam}(\mathcal{G}) := \max\{\mathrm{d}_\mathcal{G}(u, v) \mid u, v \in V(\mathcal{G})\}$ of a graph $\mathcal{G}$ is the longest distance between two vertices.

A *k-coloring* of $\mathcal{G}$ is a function $f\colon V(\mathcal{G}) \to [k]$ such that $f(v) \neq f(u)$ for each edge $\{v, u\} \in E(\mathcal{G})$. The *chromatic number* $\chi(\mathcal{G})$ is the smallest k such that there is a k-coloring of $\mathcal{G}$. A set $U \subset V(\mathcal{G})$ is a *clique* if and only if all vertices in U are adjacent to each other, i.e. $\{v, u\} \in E(\mathcal{G})$ for every $u, v \in U$ with $u \neq v$. The *clique number* $\omega(\mathcal{G})$ is the size of the biggest clique in $\mathcal{G}$. We say that a graph is *perfect* if and only if for each induced subgraph $\mathcal{G}'$ of $\mathcal{G}$ it is $\omega(\mathcal{G}') = \chi(\mathcal{G}')$.

A *chordal* graph is one where each cycle with a length greater than 3 has a *chord*, an edge connecting two vertices of the cycle that is itself not a part of the cycle. A graph $\mathcal{G}$ for which exists a planar drawing where all vertices are incident to the outer face is level planar is *outerplanar*.

A *tree-decomposition* of a graph $\mathcal{G}$, as defined by Robertson and Seymour [RS86], is a tree $T = (V_T, E_T)$ where each node $i \in V_T$ belongs to a bag $X_i \subset V(\mathcal{G})$ with the following properties:

1. $\bigcup_{i \in V_T} X_i = V(\mathcal{G})$
2. $\forall \{u, v\} \in E(\mathcal{G}) \exists i \in V_T\colon u, v \in X_i$
3. $\forall i, j, k \in V_i\colon j$ lies on a path of T from i to $k \implies X_i \cap X_k \subseteq X_j$

The last property is equivalent to requiring that the nodes corresponding to bags containing a vertex v induce a connected subgraph of T. The *width* of the tree-decomposition T is $\mathrm{w}(T) := \max\{|X_i| - 1 \mid i \in V_T\}$. The *treewidth* $\mathrm{tw}(\mathcal{G})$ of $\mathcal{G}$ is the minimum width of a tree-decomposition of $\mathcal{G}$. A *path-decomposition* is a tree-decomposition T where T is a path and the *pathwidth* $\mathrm{pw}(\mathcal{G})$ of $\mathcal{G}$ is the minimum width of a path-decomposition of $\mathcal{G}$. Therefore, the treewidth is bounded by the pathwidth.

The *treedepth* $\mathrm{td}(\mathcal{G})$ of a graph $\mathcal{G}$, as defined by Nešetřil and Ossona de Mendez [NO06], is the minimum height of a rooted forest F such that for every edge $\{u, v\} \in E(\mathcal{G})$ there is an ancestor-descendant relationship between u and v in F.

3 Reductions Preserving Graph Properties

In this chapter, we will present reductions from Constrained Level Planarity to Ordered Level Planarity that preserve outerplanarity, chordality, perfectness, pathwidth, treedepth, maximum degree, cycle graphs, and, under certain conditions, k-connectivity. We will begin by providing reductions from Constrained Level Planarity to Ordered Level Planarity for some trivial cases.

Lemma 3.1. *Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph. In the following cases we can transform the graph $\mathcal{G}$ in quadratic time to an equivalent ordered level graph $\mathcal{G}' = (G, \gamma, (\prec'_\ell)_\ell)$ by only adding constraints:*

1. *The graph $\mathcal{G}$ contains only isolated vertices.*
2. *The graph $\mathcal{G}$ is not level planar.*

Proof. A constrained level graph that only contains isolated vertices is always constrained level planar, regardless of the constraints, as an embedding that is on each level a linear extension of the constraints on the vertices of that level respects the constraints and is trivially planar as there are no edges. We therefore can set the total order $\prec'_\ell$ for each level $\ell \in [h]$ to be an arbitrary linear extension of the partial order $\prec_\ell$ thereby obtaining an ordered level planar graph $\mathcal{G}'$.

If the graph $\mathcal{G}$ is not level planar it also is not constrained level planar. We therefore can set the total order $\prec'_\ell$ for each level $\ell \in [h]$ to be an arbitrary linear extension of the partial order $\prec_\ell$ thereby obtaining an ordered level planar graph $\mathcal{G}'$. The graph $\mathcal{G}'$ is equivalent to $\mathcal{G}$, as there cannot be an ordered level planar embedding of $\mathcal{G}'$ since this would also be a constrained level planar embedding of $\mathcal{G}$ which contradicts $\mathcal{G}$ not being level planar.

As during the reductions we only add constraints and there are at most $O(n^2)$ constraints for a graph with n vertices, adding these constraints takes at most quadratic time. □

As we can test if a graph is level planar and contains at least one edge in linear time we will assume from now on that the graphs are level planar and contain edges. We will now show that we can further assume that the graph does not contain any isolated vertices.

Lemma 3.2. *Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph and $U \subseteq V(\mathcal{G})$ the isolated vertices in $\mathcal{G}$. The graph $\mathcal{G}$ can be transformed in linear time to the induced subgraph $\mathcal{G}' = \mathcal{G}[V(\mathcal{G}) \setminus U]$ which does not contain any isolated vertices. Further, the graphs $\mathcal{G}$ and $\mathcal{G}'$ are equivalent.*

Proof. The transformation can be performed in linear time as isolated vertices can be found in linear time and removing those vertices and their incident edges as well as the constraints on these vertices can also be done in linear time.

We will now show that the two graphs are equivalent. If $\mathcal{G}$ is constrained level planar so is $\mathcal{G}'$ as it is a subgraph of $\mathcal{G}$. It remains to show that if $\mathcal{G}'$ is constrained level planar with embedding Γ' there is a constrained level planar embedding Γ for $\mathcal{G}$. To achieve this we transform Γ' to Γ by adding the isolated vertices into the orders of Γ'. Let $u \in U$ be an isolated vertex and let $\ell = \gamma(u)$. If there is no constraint of the form $v \prec_\ell u$ in $\mathcal{G}$ we can place u at the left of the level ℓ. Similarly, if there is no constraint $u \prec_\ell v$ in $\mathcal{G}$ we place u to the right of all vertices on ℓ. Otherwise, there are constraints $v_L \prec_\ell u$ and $u \prec_\ell v_R$ in $\mathcal{G}$. We can assume without loss of generality that v_L is the rightmost and v_R is the leftmost vertex with such a constraint. Therefore, it must also be $v_L \prec_\ell v_R$, and we can insert the vertex u between v_L and v_R. If we apply this iteratively to each vertex $u \in U$ we obtain a constrained level planar embedding Γ for $\mathcal{G}$. □

We therefore now have to deal only with graphs that are level planar and free of isolated vertices. For those, the reductions consist mostly of two parts. The first is to move the constraints of each level to separate levels, a process we call constraint expansion. The second part is the so-called vertex expansion where the vertices from each level, that do not have constraints between them, are moved to distinct levels such that in the end there is a total order on the vertices of each level. For simplicity, we will treat the levels as rational numbers instead of natural ones during the reductions, as we can convert them to natural numbers afterwards in linear time. This especially allows us to insert new levels between the existing levels without having to reassign the levels of all vertices above the inserted levels.

We will begin by defining the *vertex expansion.* For a constrained h-level graph $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ we say that a level $\tilde{\ell} \in [h]$ has been *vertex expanded* if and only if the graph $\mathcal{G}$ has been transformed to a constrained level graph $\mathcal{G}' = (G, \gamma', (\prec'_\ell)_\ell)$ such that in place of the level $\tilde{\ell}$ there are consecutive new levels $\tilde{\ell}_1, \ldots, \tilde{\ell}_k$ with the vertices from $\tilde{\ell}$ distributed between them, i.e. $V_{\tilde{\ell}}(\mathcal{G}) = V_{\tilde{\ell}_1}(\mathcal{G}') \dot{\cup} \ldots \dot{\cup} V_{\tilde{\ell}_k}(\mathcal{G}')$. Further, for each constraint $v \prec_{\tilde{\ell}} u$ in $\mathcal{G}$ there must be the constraint $v \prec'_{\gamma'(v)} u$ in $\mathcal{G}'$. With this, we can now formulate the following lemma.

Lemma 3.3. *Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph and $\mathcal{G}' = (G, \gamma', (\prec'_\ell)_\ell)$ a constrained level graph that differs from $\mathcal{G}$ solely in that a level $\tilde{\ell} \in [h]$ from $\mathcal{G}$ has been vertex expanded in $\mathcal{G}'$. If in every constrained level planar embedding of $\mathcal{G}'$ there is no nesting between any two vertices $v, u \in V_{\tilde{\ell}}(\mathcal{G})$, the graphs $\mathcal{G}$ and $\mathcal{G}'$ are equivalent.*

Proof. In order to prove that $\mathcal{G}$ and $\mathcal{G}'$ are equivalent we will first show that we can obtain a constrained level planar embedding Γ' of $\mathcal{G}'$ from a constrained level planar embedding Γ of $\mathcal{G}$. The idea is to keep the order from Γ as $\mathcal{G}'$ simply changed the level assignments of the vertices. For each level $\ell \in [h] \setminus \{\tilde{\ell}\}$ we set $\prec_\ell^{\Gamma'}$ to be the same as $\prec_\ell^{\Gamma}$ since these levels contain the same vertices and edges. For the new levels in $\mathcal{G}'$ we set the order according to the order of the elements on the level $\tilde{\ell}$ in Γ. This means that for two elements $a, b \in \mathcal{E}_\ell(\mathcal{G}')$ on a level $\ell = \tilde{\ell}_i$ with $i \in [k]$ we set $a \prec_\ell^{\Gamma'} b$ if $a \prec_{\tilde{\ell}}^{\Gamma} b$. We will now show that the obtained embedding Γ' is constrained level planar. First of all, all constraints are respected in Γ'. Indeed, for a level $\ell \in [h] \setminus \{\tilde{\ell}\}$ in $\mathcal{G}$ and a constraint $v \prec'_\ell u$ it is also $v \prec_\ell u$ since we kept the vertices and the constraints on this level. As we also kept the order from the embedding Γ on the level ℓ it is also $v \prec^{\Gamma'}_\ell u$ due to $v \prec^\Gamma_\ell u$ as Γ respects the constraint $v \prec_\ell u$. For a constraint $v \prec'_\ell u$ on a new level $\ell = \tilde{\ell}_i$ for $i \in [k]$ there must be the constraint $v \prec_{\tilde{\ell}} u$ in $\mathcal{G}$. This indicates that it must also be $v \prec^\Gamma_{\tilde{\ell}} u$. Since we copied the order $\prec^{\Gamma'}_\ell$ from $\prec^\Gamma_{\tilde{\ell}}$ it is also $v \prec^{\Gamma'}_\ell u$. It remains to show that the embedding is planar. In order for Γ' to be planar, we have to show that for two edges $e_1, e_2 \in E(\mathcal{G}')$ and levels $\ell^-, \ell^+ \in L(e_1) \cap L(e_2)$ with $\ell^- < \ell^+$ as well as $e_1^- \neq e_2^-$ if $\ell^- = L^-(e_1) = L^-(e_2)$ and $e_1^+ \neq e_2^+$ if $\ell^+ = L^+(e_1) = L^+(e_2)$ it follows from $e_1 \prec^{\Gamma'}_{\ell^-} e_2$ that $e_1 \prec^{\Gamma'}_{\ell^+} e_2$. Given such edges and levels with $e_1 \prec^{\Gamma'}_{\ell^-} e_2$ we will now show that it also is $e_1 \prec^{\Gamma'}_{\ell^+} e_2$. If $\ell^- \neq \tilde{\ell}_i$ and $\ell^+ \neq \tilde{\ell}_j$ for $i, j \in [k]$, we have $e_1 \prec^\Gamma_{\ell^-} e_2$ and therefore $e_1 \prec^\Gamma_{\ell^+} e_2$ since these edges appear on the corresponding levels in $\mathcal{G}$ and we copied their order in Γ'. It follows that $e_1 \prec^{\Gamma'}_{\ell^+} e_2$. If instead $\ell^+ = \tilde{\ell}_i$ for an $i \in [k]$ we have $e_1 \prec^\Gamma_{\ell^-} e_2$, and therefore it must also be $e_1 \prec^\Gamma_{\tilde{\ell}} e_2$ since an edge that touches the level $\tilde{\ell}_i$ in $\mathcal{G}'$ touches the level $\tilde{\ell}$ in $\mathcal{G}$. We therefore have $e_1 \prec^{\Gamma'}_{\ell^+} e_2$ as for those levels we copied the order from $\prec^\Gamma_{\tilde{\ell}}$. This applies analog to the case where $\ell^- = \tilde{\ell}_i$ and $\ell^+ \neq \tilde{\ell}_j$ for $i, j \in [k]$. If $\ell^- = \tilde{\ell}_i$ and $\ell^+ = \tilde{\ell}_j$ for $i, j \in [k]$ we must have $e_1 \prec^\Gamma_{\tilde{\ell}} e_2$ since $\prec^{\Gamma'}_{\ell^-}$ was copied from $\prec^\Gamma_{\tilde{\ell}}$. Therefore, it must also be $e_1 \prec^{\Gamma'}_{\ell^+} e_2$ as $\prec^{\Gamma'}_{\ell^+}$ was also copied from $\prec^\Gamma_{\tilde{\ell}}$.

It now only remains to show that if $\mathcal{G}'$ is constrained level planar with embedding Γ', we can obtain a constrained level planar embedding Γ for $\mathcal{G}$. This is done by mostly copying the order from Γ'. For each level $\ell \in [h] \setminus \{\tilde{\ell}\}$ we set $\prec^\Gamma_\ell$ to be the same as $\prec^{\Gamma'}_\ell$ since these levels contain the same vertices and edges. For the level $\tilde{\ell}$ we cannot directly copy the order from one of the new levels as the vertices in $V_{\tilde{\ell}}(\mathcal{G})$ are not all on the same level in $\mathcal{G}'$. We therefore use the order of the incident edges instead, to set the order of two vertices. For two elements $a, b \in \mathcal{E}_{\tilde{\ell}}(\mathcal{G})$ we set $a \prec^\Gamma_{\tilde{\ell}} b$ if there is an $i \in [k]$ such that for $\ell = \tilde{\ell}_i$ we have $a' \prec^{\Gamma'}_\ell b'$ where a' (b') is either the same element or, in the case that a' (b') is a vertex, an incident edge. This order is unambiguous. Indeed, this is clear for edges due to them being y-monotone and therefore having the same order on all levels which they touch. It remains to show that given two vertices $v, u \in V_{\tilde{\ell}}(\mathcal{G})$ their order in Γ is also unambiguous. Suppose that this were not the case with there being $v \prec^\Gamma_{\tilde{\ell}} u$ and $u \prec^\Gamma_{\tilde{\ell}} v$. Then there would be edges $e^v_L, e^v_R \in E_{\mathcal{G}'}(v)$ and $e^u_L, e^u_R \in E_{\mathcal{G}'}(u)$ such that $e^v_L \prec^{\Gamma'}_\ell e^u_R$ and $e^u_L \prec^{\Gamma'}_{\ell'} e^v_R$ for some levels ℓ and ℓ'. Assume without loss of generality that $\gamma'(v) < \gamma'(u)$. Then if e^v_L and e^v_R are outgoing edges of v the vertex u is nested in v due to $e^v_L \prec^{\Gamma'}_{\gamma'(u)} u \prec^{\Gamma'}_{\gamma'(u)} e^v_R$ which contradicts the fact that there is no nesting between any vertices $v, u \in V_{\tilde{\ell}}(\mathcal{G})$ in $\mathcal{G}'$. If, on the other hand, e^v_L and e^v_R are incoming edges, the edges e^u_L and e^u_R must also be incoming edges which leads to $e^u_L \prec^{\Gamma'}_{\gamma'(v)} v \prec^{\Gamma'}_{\gamma'(v)} e^u_R$ and therefore v being nested in u, again, a contradiction. Otherwise, let without loss of generality e^v_L be an incoming and e^v_R an outgoing edge. Connecting them forms therefore a monotone path p^v. It is then $e^u_L \prec^{\Gamma'}_{\ell'} p^v$ and $p^v \prec^{\Gamma'}_\ell e^u_R$ which leads to the contradiction that $u \prec^{\Gamma'}_{\gamma'(u)} p^v \prec^{\Gamma'}_{\gamma'(u)} u$. Therefore, the order defined by Γ is unambiguous. Currently, the order set on the level $\tilde{\ell}$ in Γ is not necessarily total. This is because for a source

$v \in Q_{\tilde{\ell}}(\mathcal{G})$ and a sink $u \in S_{\tilde{\ell}}(\mathcal{G})$ with $\gamma'(v) > \gamma'(u)$ there does not exist a level in $\mathcal{G}'$ which is shared by an incident edge of u and an incident edge of v. We therefore choose an arbitrary linear extension of the current order and use this to set the order between the remaining elements. It remains to show that Γ is constrained level planar. First of all, the constraints are respected in Γ. Indeed, for a level $\ell \in [h] \setminus \{\tilde{\ell}\}$ we kept the order on the vertices of the level ℓ which has the same constraints in $\mathcal{G}'$. For a constraint $c = (v \prec_{\tilde{\ell}} u)$ on the level $\tilde{\ell}$ in $\mathcal{G}$ there must be a level $\ell = \tilde{\ell}_i$ for an $i \in [k]$ in $\mathcal{G}'$ such that $v \prec'_\ell u$. Therefore, it must be $v \prec^{\Gamma'}_\ell u$ which means that we set $v \prec^{\Gamma}_{\tilde{\ell}} u$ thereby respecting the constraint. In order for Γ to be planar we have to show that for two edges $e_1, e_2 \in E(\mathcal{G})$ and levels $\ell^-, \ell^+ \in L(e_1) \cap L(e_2)$ with $\ell^- < \ell^+$ as well as $e_1^- \neq e_2^-$ if $\ell^- = L^-(e_1) = L^-(e_2)$ and $e_1^+ \neq e_2^+$ if $\ell^+ = L^+(e_1) = L^+(e_2)$ it follows from $e_1 \prec^{\Gamma}_{\ell^-} e_2$ that $e_1 \prec^{\Gamma}_{\ell^+} e_2$. If $\ell^- \neq \tilde{\ell} \neq \ell^+$ we have $e_1 \prec^{\Gamma'}_{\ell^-} e_2$ and therefore $e_1 \prec^{\Gamma'}_{\ell^+} e_2$ since these edges appear on the corresponding levels in $\mathcal{G}'$. It follows that $e_1 \prec^{\Gamma}_{\ell^+} e_2$. If $\ell^+ = \tilde{\ell}$ then it must be $e_1 \prec^{\Gamma'}_{\ell^-} e_2$ and $e_1 \prec^{\Gamma'}_{\ell} e_2$ for $\ell = \tilde{\ell}_1$ since these edges touch the level $\tilde{\ell}$ in $\mathcal{G}$ and therefore must touch the level $\tilde{\ell}_1$ in $\mathcal{G}'$ as they cannot end on an earlier level. Therefore, it is $e_1 \prec^{\Gamma}_{\ell^+} e_2$. This is analog for $\ell^- = \tilde{\ell}$ with both edges touching the level $\tilde{\ell}_k$. This shows that Γ is a constrained level planar embedding of $\mathcal{G}$. Therefore, the graphs $\mathcal{G}$ and $\mathcal{G}'$ are equivalent. □

Fig. 3.1: (a) A constrained level graph $\mathcal{G}$ that is not constrained level planar. (b) The graph $\mathcal{G}'$ obtained from $\mathcal{G}$ by vertex expanding the level $\tilde{\ell}$ is constrained level planar due to u being able to be nested in v.

If we did not require in Lemma 3.3 that there be no nesting between vertices from the level $\tilde{\ell}$ in constrained level planar embeddings of $\mathcal{G}'$ we would have the problem that $\mathcal{G}'$ may be constrained level planar even if $\mathcal{G}$ is not, as demonstrated in Figure 3.1. We will therefore now show a few cases in which nesting between vertices of the vertex-expanded level is not possible after the expansion. This leads to some concrete methods for vertex expansion which transform a graph $\mathcal{G}$ to an equivalent graph $\mathcal{G}'$.

Lemma 3.4. *Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph and $\tilde{\ell} \in [h]$ a level without constraints. Then the following transformations are possible $O(\lambda_{\tilde{\ell}}(\mathcal{G}))$ time:*

1. *The graph $\mathcal{G}$ can be transformed to a constrained level graph $\mathcal{G}' = (G, \gamma', (\prec'_\ell)_\ell)$, where the level $\tilde{\ell}$ has been vertex expanded into at most three consecutive levels $\tilde{\ell}_q \leq \tilde{\ell}_r \leq \tilde{\ell}_s$, such that the sources $Q_{\tilde{\ell}}(\mathcal{G})$ are on $\tilde{\ell}_q$, the sinks $S_{\tilde{\ell}}(\mathcal{G})$ on $\tilde{\ell}_s$ and the intermediate vertices $R_{\tilde{\ell}}(\mathcal{G})$ on $\tilde{\ell}_r$. The graphs $\mathcal{G}$ and $\mathcal{G}'$ are equivalent.*

2. *The graph $\mathcal{G}$ can be transformed to a constrained level graph $\mathcal{G}' = (G, \gamma', (\prec'_\ell)_\ell)$, where the level $\tilde{\ell}$ has been vertex expanded into consecutive levels such that each vertex $v \in V_{\tilde{\ell}}(\mathcal{G})$ is on a distinct new level. The graphs $\mathcal{G}$ and $\mathcal{G}'$ are equivalent if one of the following conditions apply:*
   a) *All vertices in $V_{\tilde{\ell}}(\mathcal{G})$ are intermediate vertices.*
   b) *All vertices in $V_{\tilde{\ell}}(\mathcal{G})$ have degree* 1.

Proof. For the first transformation, we expand the level $\tilde{\ell}$ by replacing it with at most three new consecutive levels $\tilde{\ell}_q \leq \tilde{\ell}_r \leq \tilde{\ell}_s$. We assign all sources $Q_{\tilde{\ell}}(\mathcal{G})$ to the level $\tilde{\ell}_q$ all intermediate vertices $R_{\tilde{\ell}}(\mathcal{G})$ to $\tilde{\ell}_r$ and all sinks $S_{\tilde{\ell}}(\mathcal{G})$ to $\tilde{\ell}_s$, which yields the graph $\mathcal{G}'$, as illustrated in Figure 3.2. As we only reassign the levels of vertices in $V_{\tilde{\ell}}(\mathcal{G})$ this can be done $O(\lambda_{\tilde{\ell}}(\mathcal{G}))$ time. In order for $\mathcal{G}'$ to be equivalent to $\mathcal{G}$ we will now show that there cannot be nesting between vertices from $V_{\tilde{\ell}}(\mathcal{G})$ in $\mathcal{G}'$. A source on the level $\tilde{\ell}_q$ cannot be nested in an intermediate vertex on $\tilde{\ell}_r$ or a sink on $\tilde{\ell}_s$, as it has an edge going upwards past $\tilde{\ell}_s$. This is symmetric for the sinks, as they have an edge going downwards past $\tilde{\ell}_q$. An intermediate vertex also cannot be nested in a source or a sink as it has edges going upwards past $\tilde{\ell}_s$ and downwards past $\tilde{\ell}_q$. Therefore, the graph $\mathcal{G}'$ is equivalent to $\mathcal{G}$ according to Lemma 3.3.

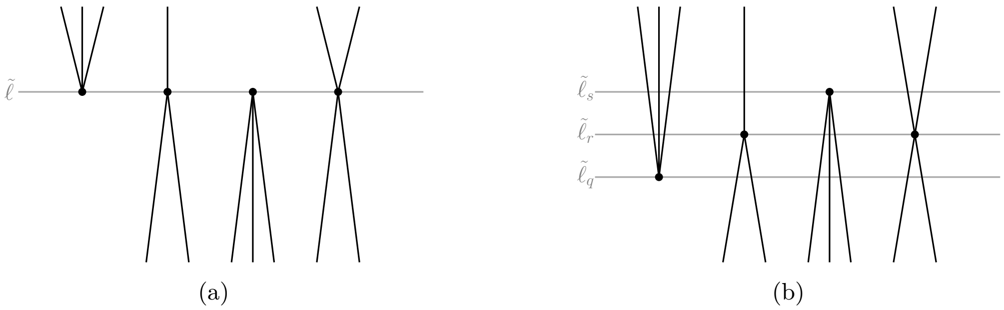


Fig. 3.2: A level without constraints before (a) and after (b) the vertex expansion with the first method.

For the second transformation, we expand the level $\tilde{\ell}$ by replacing it with consecutive new levels $\tilde{\ell}_v$ for each $v \in V_{\tilde{\ell}}(\mathcal{G})$, setting $\gamma'(v) = \tilde{\ell}_v$ for each $v \in V_{\tilde{\ell}}(\mathcal{G})$. As this transformation only reassigns the levels of vertices in $V_{\tilde{\ell}}(\mathcal{G})$ and adds $\lambda_{\tilde{\ell}}(\mathcal{G})$ levels the transformation can be performed in $O(\lambda_{\tilde{\ell}}(\mathcal{G}))$ time. In order for the graph $\mathcal{G}'$ to be equivalent to $\mathcal{G}$ according to Lemma 3.3, we have to show that there can be no nesting between vertices from $V_{\tilde{\ell}}(\mathcal{G})$ in $\mathcal{G}'$. We will first show that this is the case if all vertices in $V_{\tilde{\ell}}(\mathcal{G})$ are intermediate vertices. Given two intermediate vertices $u, v \in V_{\tilde{\ell}}(\mathcal{G})$ they cannot be nested in each other in $\mathcal{G}'$ as u has an incident edge that crosses $\gamma'(v)$ and v has an incident edge crossing $\gamma'(u)$. In the second case, where all vertices in $V_{\tilde{\ell}}(\mathcal{G})$ have degree 1 there also cannot be any nesting, as in order for a vertex u to be nested in a vertex v there must be two vertex-disjoint paths starting in v. This cannot be the case when v has only one incident edge. □

Now we will define the *constraint expansion*. We say that for a constrained h-level graph $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ a level $\tilde{\ell} \in [h]$ has been *constraint expanded* if and only if the graph $\mathcal{G}$ has been transformed to a constrained level graph $\mathcal{G}'$ in such a way that each constraint from the level $\tilde{\ell}$ is on its own level in $\mathcal{G}'$. Depending on which constraint expansion is used in a reduction from Constrained Level Planarity to Ordered Level Planarity a different set of properties is preserved. We will therefore present three distinct ways to constraint expand a level of a constrained level graph such that the resulting graph is equivalent.

Lemma 3.5. *Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph and $\tilde{\ell} \in [h]$. The graph $\mathcal{G}$ can be transformed to an equivalent constrained level graph $\mathcal{G}' = (G', \gamma', (\prec'_\ell)_\ell)$ where the level $\tilde{\ell} \in [h]$ has been constraint expanded. This can be achieved in one of the following ways, illustrated exemplarily in Figure 3.3:*

1. *For each constraint $c = (v \prec_{\tilde{\ell}} u)$ there is a new* constraint level *$\tilde{\ell}_c$ added directly below $\tilde{\ell}$. On this level, there are only the two* constraint vertices *v_c and u_c with the constraint $v_c \prec'_{\tilde{\ell}_c} u_c$. Further, the* constraint edges *$\{v_c, v\}$ and $\{u_c, u\}$ are added. The constraint c is then removed from $\prec'_{\tilde{\ell}}$. This transformation can be performed in time linear to the number of vertices and constraints on $\tilde{\ell}$ in $\mathcal{G}$.*

2. *For each constraint $c = (v \prec_{\tilde{\ell}} u)$ there is a new* constraint level *$\tilde{\ell}_c$ added directly below $\tilde{\ell}$. On this level, there are only the two* constraint vertices *v_c and u_c with the constraint $v_c \prec'_{\tilde{\ell}_c} u_c$. Further, there are additional constraint vertices $\tilde{v}_c$ and $\tilde{u}_c$ added to the level $\tilde{\ell}$ with constraints $v \prec'_{\tilde{\ell}} \tilde{v}_c$ and $\tilde{u}_c \prec'_{\tilde{\ell}} u$ as well as* constraint edges *$\{v_c, \tilde{v}_c\}$ and $\{u_c, \tilde{u}_c\}$. The constraint c is then removed from $\prec'_{\tilde{\ell}}$. Moreover, a total order is set on the* bundle *$b(v) := \{v\} \cup \{\tilde{v}_c \mid c$ is a constraint on v in $\mathcal{G}\}$, consisting of v and its constraint vertices on the level $\tilde{\ell}$, of each vertex $v \in V_{\tilde{l}}(\mathcal{G})$. This transformation can be performed in time quadratic to the number of vertices and constraints on $\tilde{\ell}$ in $\mathcal{G}$.*

3. *The level $\tilde{\ell}$ is replaced with two new levels $\tilde{\ell}_b$ and $\tilde{\ell}_s$ with $\tilde{\ell}_b < \tilde{\ell}_s$. The sources and intermediate vertices $Q_{\tilde{\ell}}(\mathcal{G}) \cup R_{\tilde{\ell}}(\mathcal{G})$ are assigned to the level $\tilde{\ell}_b$ and the sinks $S_{\tilde{\ell}}(\mathcal{G})$ to the level $\tilde{\ell}_s$. For each intermediate vertex and source $v \in Q_{\tilde{\ell}}(\mathcal{G}) \cup R_{\tilde{\ell}}(\mathcal{G})$ one outgoing edge is designated as the* constraint path *p_v. The same is done for each sink $v \in S_{\tilde{\ell}}(\mathcal{G})$ where one incoming edge is designated as constraint path p_v. For each constraint $c = (v \prec_{\tilde{\ell}} u)$ there is a* constraint level *$\tilde{\ell}_c$ added between $\tilde{\ell}_b$ and $\tilde{\ell}_s$. On this constraint level are two* constraint vertices *v_c and u_c which subdivide the constraint paths p_v and p_u, respectively, and have the constraint $v_c \prec'_{\tilde{\ell}_c} u_c$. This transformation can be performed in time linear to the number of vertices and constraints on $\tilde{\ell}$.*

Proof. We will provide a separate proof for each constraint expansion method.

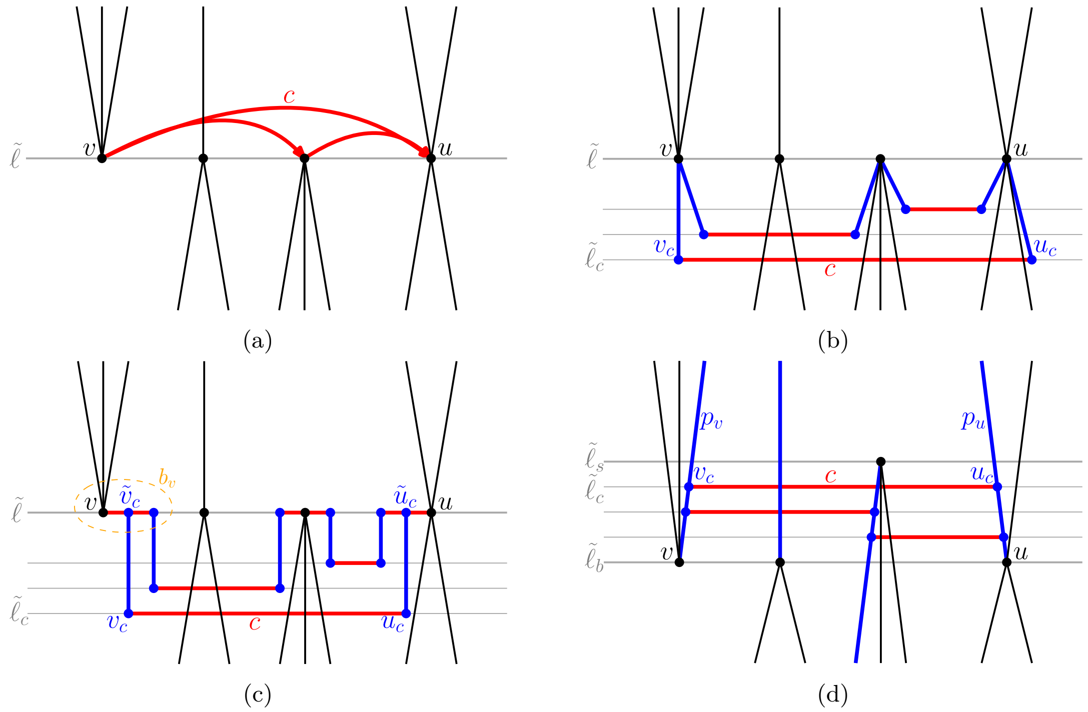


Fig. 3.3: A level $\tilde{\ell}$ (a) of a constrained level graph is constraint expanded with the first (b), second (c) and third (d) method.

1. For the first method, we begin by showing that if $\mathcal{G}$ is constrained level planar so is $\mathcal{G}'$. Let Γ be a constrained level planar embedding of $\mathcal{G}$. We can obtain a constrained level planar embedding Γ' for $\mathcal{G}'$ by copying the order on the elements of each level $\ell \in [h]$ since these levels remained unchanged in $\mathcal{G}'$. In order to also have an order on the added constraint edges we set for each vertex $v \in V_{\tilde{\ell}}(\mathcal{G})$ the order $\prec_{v^-}^{\Gamma'}$ on the incoming edges, which includes the constraint edges, to be a linear extension of $\prec_{v^-}^{\Gamma}$. We then set for a constraint level $\tilde{\ell}_c$ the order on the edges touching that level to be the same as on $\tilde{\ell}$, since all edges that touch the level $\tilde{\ell}_c$ also touch $\tilde{\ell}$ and therefore can be compared using $\prec_{\tilde{\ell}}^{\Gamma'}$. We now have to show that the embedding Γ' respects the constraints and is level planar. For a constraint $v \prec_{\ell}' u$ with $\ell \neq \tilde{\ell}$ it is also $v \prec_{\ell} u$ and therefore $v \prec_{\ell}^{\Gamma} u$ and $v \prec_{\ell}^{\Gamma'} u$ since we copied the order on the vertices of each level $\ell \in [h]$ from Γ. Assume that for a constraint $c = (v \prec_{\tilde{\ell}} u)$ the corresponding constraint $v_c \prec_{\tilde{\ell}_c}' u_c$ in $\mathcal{G}'$ is not respected by Γ'. Then it must be $u_c \prec_{\tilde{\ell}_c}^{\Gamma'} v_c$ as well as $v \prec_{\tilde{\ell}}^{\Gamma} u$ since Γ respects the constraint c. This leads to a contradiction as it forces the constraint edges $\{v_c, v\}$ and $\{u_c, u\}$ to cross. Therefore, the constraints on the constraint levels are also respected in Γ' which means that Γ' respects the constraints. The embedding is also planar. Indeed, due to the order on each constraint level being the same as on $\tilde{\ell}$ and Γ' having the same order as Γ on each level $\ell \in [h]$ all edges run in parallel.

It remains to show that $\mathcal{G}$ is constrained level planar if $\mathcal{G}'$ is constrained level planar. If $\mathcal{G}'$ is constrained level planar with embedding Γ' we can obtain a constrained level planar embedding Γ for $\mathcal{G}$ by setting $v \prec_\ell^\Gamma u$ if $v \prec_\ell^{\Gamma'} u$ for each pair of vertices $u, v \in V_\ell(\mathcal{G})$ and level $\ell \in [h]$. This embedding respects the constraints, as for each constraint $v \prec_\ell u$ with $\ell \neq \tilde{\ell}$ it is also $v \prec'_\ell u$ which is respected by Γ', and therefore it is $v \prec_\ell^\Gamma u$. For a constraint $c = (v \prec_{\tilde{\ell}} u)$ it is $v_c \prec'_{\tilde{\ell}_c} u_c$ and therefore $v_c \prec_{\tilde{\ell}_c}^{\Gamma'} u_c$. Because of the constraint edges $\{v_c, v\}$ and $\{u_c, u\}$ it must also be $v \prec_{\tilde{\ell}}^{\Gamma'} u$. The embedding is also planar as all edges in $\mathcal{G}$ are also in $\mathcal{G}'$ and their order was directly copied. Therefore, the first method transforms $\mathcal{G}$ to an equivalent constrained level graph $\mathcal{G}'$ where the level $\tilde{\ell}$ has been constraint expanded.

This transformation can be performed in time linear to the number of vertices and constraints on $\tilde{\ell}$ in $\mathcal{G}$ as for each constraint on $\tilde{\ell}$ we add a new level, two vertices, two edges and one constraint.

2. For the second method, we begin by showing that if $\mathcal{G}$ is constrained level planar so is $\mathcal{G}'$. Let $\mathcal{G}$ be constrained level planar with embedding Γ. We can obtain a constrained level planar embedding Γ' for $\mathcal{G}'$ by first copying the order on all levels $\ell \in [h]$ from Γ. The order on the vertices of the level $\tilde{\ell}$ in $\mathcal{G}'$ is not yet total as we added constraint vertices to this level. For a constraint vertex $\tilde{v}_c$ we therefore set its order to be the same order as that of the vertex v in Γ. As this does not establish an order between the vertices of the bundle $b(v)$ of each vertex $v \in V_{\tilde{\ell}}(\mathcal{G})$ we set the order on the vertices of $b(v)$ according to the constraints in $\mathcal{G}'$ which form a total order on these vertices. This order on the elements of the level $\tilde{\ell}$ is then copied to the constraint levels. We now have to show that Γ' is a constrained level planar embedding of $\mathcal{G}'$. First off, the embedding Γ' respects the constraints in $\mathcal{G}'$, as for a constraint $v \prec'_\ell u$ with $\ell \neq \tilde{\ell}$ it is also $v \prec_\ell u$ and therefore $v \prec_\ell^\Gamma u$ and $v \prec_\ell^{\Gamma'} u$ since we copied the order on those vertices from Γ. A constraint $v' \prec'_{\tilde{\ell}} v''$ is also respected as there are only constraints between vertices in the bundle $b(v)$ of a vertex $v \in V_{\tilde{\ell}}(\mathcal{G})$ in $\mathcal{G}'$, and we set the order in Γ' according to these constraints. Suppose that for a constraint $c = (v \prec_{\tilde{\ell}} u)$ in $\mathcal{G}$ the corresponding constraint $v_c \prec_{\tilde{\ell}_c} u_c$ is not respected by Γ'. Then it must be $u_c \prec_{\tilde{\ell}_c}^{\Gamma'} v_c$ as well as $v \prec_{\tilde{\ell}}^\Gamma u$ since Γ respects the constraint c. This leads to a contradiction as it forces the constraint edges $\{v_c, \tilde{v}_c\}$ and $\{u_c, \tilde{u}_c\}$ to cross. Therefore, all constraints are respected in Γ'. The embedding is also planar, as we set the order on each constraint level to be the same as on $\tilde{\ell}$ and Γ' has the same order as Γ on each level $\ell \in [h]$ which leads to all edges running in parallel.

It remains to show that $\mathcal{G}$ is constrained level planar if $\mathcal{G}'$ is constrained level planar. If $\mathcal{G}'$ is constrained level planar with embedding Γ' we can obtain a constrained level planar embedding Γ for $\mathcal{G}$ by setting $v \prec_\ell^\Gamma u$ if $v \prec_\ell^{\Gamma'} u$ for each $u, v \in V_\ell(\mathcal{G})$ and $\ell \in [h]$. This embedding respects the constraints, as for each $v \prec_\ell u$ with $\ell \neq \tilde{\ell}$ it is also $v \prec'_\ell u$ and therefore $v \prec_\ell^{\Gamma'} u$. For a constraint $c = (v \prec_{\tilde{\ell}} u)$ it is $v_c \prec'_{\tilde{\ell}_c} u_c$ and therefore $v_c \prec_{\tilde{\ell}_c}^{\Gamma'} u_c$. As there are constraint edges $\{v_c, \tilde{v}_c\}$ and $\{u_c, \tilde{u}_c\}$ this means that it also must be $\tilde{v}_c \prec_{\tilde{\ell}}^{\Gamma'} \tilde{u}_c$ and because of the constraints $v \prec'_{\tilde{\ell}} \tilde{v}_c$ and $\tilde{u}_c \prec'_{\tilde{\ell}} u$ it must be $v \prec_{\tilde{\ell}}^{\Gamma'} \tilde{v}_c \prec_{\tilde{\ell}}^{\Gamma'} \tilde{u}_c \prec_{\tilde{\ell}}^{\Gamma'} u$ and therefore $v \prec_{\tilde{\ell}}^\Gamma u$. The embedding is also planar as all edges in $\mathcal{G}$ are also in $\mathcal{G}'$ and their order was copied from Γ'. Therefore, the second method transforms $\mathcal{G}$ to an equivalent constrained level graph $\mathcal{G}'$ where the level $\tilde{\ell}$ has been constraint expanded.

Regarding the time of this transformation, we add for each constraint a new constraint level with two vertices, two edges and one constraint. However, since we also add two new vertices on $\tilde{\ell}$ which have a constraint with each vertex in their respective bundles, the number of constraints added in total is up to quadratic compared to the number of constraints on $\tilde{\ell}$ in $\mathcal{G}$. Therefore, this transformation takes up to quadratic time regarding the number of vertices and constraints on $\tilde{\ell}$ in $\mathcal{G}$.

3. For the third method, we will begin by showing that if $\mathcal{G}$ is constrained level planar so is $\mathcal{G}'$. Given a constrained level planar embedding Γ of $\mathcal{G}$ we can obtain a constrained level planar embedding Γ' of $\mathcal{G}'$ by maintaining and extending the orders from Γ. For each level $\ell \in [h] \setminus \{\tilde{\ell}\}$ we set $\prec_\ell^{\Gamma'}$ to be the same order as $\prec_\ell^{\Gamma}$ as these levels contain the same edges and vertices. For the levels $\tilde{\ell}_b$ and $\tilde{\ell}_s$ as well as the constraint levels we copy the orders from $\prec_{\tilde{\ell}}^{\Gamma}$. We thereby set the order for the constraint paths according to the order of the original edge in Γ. This embedding respects all constraints in $\mathcal{G}'$ since for constraints on levels other than $\tilde{\ell}$ the constraints remained the same, and we copied the order of Γ. For a constraint $v_c \prec'_{\tilde{\ell}_c} u_c$ there is a constraint $c = (v \prec_{\tilde{\ell}} u)$ in $\mathcal{G}$. Therefore, it must be $v \prec_{\tilde{\ell}}^{\Gamma} u$. This means that it is $p_v \prec_{\tilde{\ell}_c}^{\Gamma'} p_u$ thereby respecting the constraint. The embedding Γ' is also planar since we copied the orders on all levels other than $\tilde{\ell}$ and for the constraint levels as well as $\tilde{\ell}_b$ and $\tilde{\ell}_s$ we copied the order from the level $\tilde{\ell}$, therefore maintaining the edges parallel.

It remains to show that $\mathcal{G}$ is constrained level planar if $\mathcal{G}'$ is constrained level planar. Let $\mathcal{G}'$ be constrained level planar with embedding Γ'. We can construct a constrained level planar embedding Γ for $\mathcal{G}$ by first copying the order on all levels other than $\tilde{\ell}$. For two elements $a, b \in \mathcal{E}_{\tilde{\ell}}(\mathcal{G})$ we set $a \prec_{\tilde{\ell}}^{\Gamma} b$ if $a' \prec_{\tilde{\ell}_b}^{\Gamma} b'$ where a' (b') is either a (b) or p_a (p_b) if a (b) is a sink, as a (b) is then on $\tilde{\ell}_s$ in $\mathcal{G}'$. This embedding respects the constraints on all levels since on levels other than $\tilde{\ell}$ the order was copied from Γ' and there are the same constraints in $\mathcal{G}'$. Suppose that Γ does not respect a constraint $c = (v \prec_{\tilde{\ell}} u)$. Then it is $u \prec_{\tilde{\ell}}^{\Gamma} v$, and therefore it must be $p_u \prec_{\tilde{\ell}_b}^{\Gamma'} p_v$. This indicates that it is also $p_u \prec_{\tilde{\ell}_c}^{\Gamma'} p_v$ which contradicts Γ' being constraint level planar because of $v_c \prec'_{\tilde{\ell}_c} u_c$. The embedding Γ is planar as for a pair of edges that do not touch the level $\tilde{\ell}$ in $\mathcal{G}$ the order in Γ was copied for each level from Γ' where they run in parallel. For edges that touch $\tilde{\ell}$ in $\mathcal{G}$, the order on $\tilde{\ell}$ in Γ was copied from $\tilde{\ell}_b$ which all edges touching $\tilde{\ell}$ in $\mathcal{G}$ touch in $\mathcal{G}'$. The only exceptions to that are the edges that were chosen as constraint paths. However, since the constraint paths are monotone these edges also run in parallel. Therefore, the third method transforms $\mathcal{G}$ to an equivalent constrained level graph $\mathcal{G}'$ where the level $\tilde{\ell}$ has been constraint expanded.

This transformation takes time linear in the number of vertices and constraints on $\tilde{\ell}$ in $\mathcal{G}$ as for each constraint on $\tilde{\ell}$ two paths are subdivided with two new vertices which have a single constraint. Further, the levels of the vertices on $\tilde{\ell}$ are reassigned which takes $O(\lambda_{\tilde{\ell}}(\mathcal{G}))$ time. □

With this, we now have all the necessary components to reduce Constrained Level Planarity to Ordered Level Planarity while preserving various graph properties.

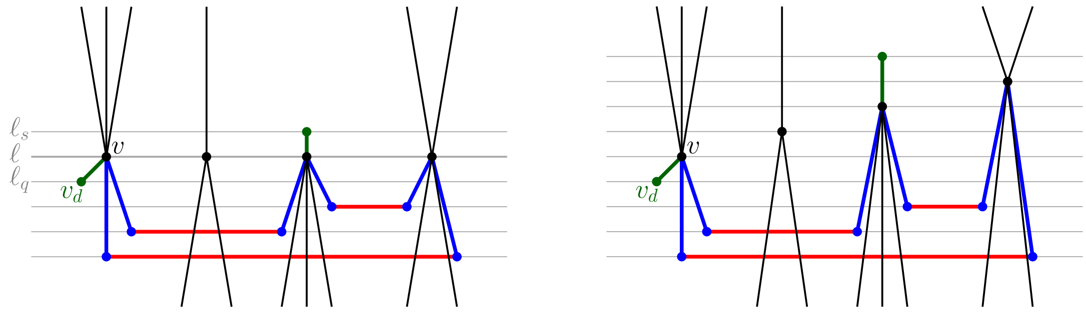
3.1 Outerplanarity, Chordality, Perfectness, Diameter, Pathwidth & Treedepth

We will start by presenting a simple reduction that preserves various properties such as outerplanarity, chordality, and perfectness. It also increases the diameter by at most 2.

Theorem 3.6. *An instance $\mathcal{G}$ of* Constrained Level Planarity *can be reduced in linear time to an instance of* Ordered Level Planarity *with level-width* 2 *and height in $O(n(\mathcal{G}) + c(\mathcal{G}))$ while preserving outerplanarity, chordality, clique number, chromatic number and perfectness. It also increases the diameter by at most* 2*, the pathwidth and treedepth by at most* 1 *and keeps a connected graph connected.*

Proof. Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph. In order to transform $\mathcal{G}$ to an equivalent ordered level graph we will begin by using the first constraint expansion method from Lemma 3.5 iteratively on each level $\ell \in [h]$ to obtain the graph $\tilde{\mathcal{G}}$. The constraint levels in $\tilde{\mathcal{G}}$ have already a total order on their vertices since they only consist of two vertices with a constraint. It therefore only remains to vertex expand the original levels to obtain an ordered level graph. To be able to use Lemma 3.4 we will transform all vertices on each level $\ell \in [h]$ into intermediate vertices. This is achieved by adding two new levels ℓ_q and ℓ_s for each level $\ell \in [h]$ with the levels $\ell_q < \ell < \ell_s$ being consecutive. For each source $v \in Q_\ell(\tilde{\mathcal{G}})$ we add a vertex v_d on ℓ_q and an edge $\{v_d, v\}$. For each sink $v \in S_\ell(\tilde{\mathcal{G}})$ we similarly add a vertex v_d on ℓ_s and an edge $\{v, v_d\}$, as seen exemplarily in Figure 3.4a. Adding these edges maintains constrained level planarity as we can always draw them directly below respective above the corresponding vertex. The level ℓ can then be vertex expanded according to Lemma 3.4 without modifying constrained level planarity, as seen for example in Figure 3.4b, as it only contains intermediate vertices. The levels ℓ_q and ℓ_s can also be expanded according to the same lemma as they only contain vertices with degree 1. Let $\mathcal{G}' = (G', \gamma', (\prec_\ell)_\ell)$ be the graph obtained after iteratively expanding all levels. It is an ordered level graph as all levels without constraints contain only one vertex and there is a total order on the two vertices of each constraint level. Therefore, its level-width is 2. As we added a new level for each vertex $v \in V(\mathcal{G})$ as well as new levels for the vertices v_d for each source and sink v and a level for each constraint the height of $\mathcal{G}'$ is in $O(n(\mathcal{G}) + c(\mathcal{G}))$. The transformation can be performed in linear time as the constraint and vertex expansions take in total linear time. The graph $\mathcal{G}'$ is also equivalent to $\mathcal{G}$ as the constraint expansion and the vertex expansion did not modify constrained level planarity making the reduction valid.

Properties We will now show that this reduction preserves the aforementioned properties, starting with the clique and chromatic number. Assume without loss of generality that $\omega(\mathcal{G}) \geq 2$ and $\chi(\mathcal{G}) \geq 2$. Otherwise, $\mathcal{G}$ consists only of isolated vertices and can be reduced to an ordered level planar graph by only adding constraints, therefore maintaining the clique number and chromatic number, according to Lemma 3.1. Since $\mathcal{G}$ is a subgraph of $\mathcal{G}'$ all cliques from $\mathcal{G}$ are in $\mathcal{G}'$. Further, the added cliques have all a size of 2 as they consist of a vertex $v \in V(\mathcal{G})$ and either a constraint vertex v_c or, for sources and sinks, the



(a) (b) The fully expanded level

Fig. 3.4: A level with the added (green) edges (a) to prevent nesting after the level is vertex expanded (b).

vertex v_d. Therefore, the clique number remains the same. As an added vertex v_c or v_d is only adjacent to the vertex v it can be colored with a color not used by v. As $\chi(\mathcal{G}) \geq 2$ we do not need any additional colors. The chromatic number is therefore also preserved.

We will now show that perfectness is preserved. Assume $\mathcal{G}$ is perfect and let $\mathcal{G}'[U']$ be an induced subgraph of $\mathcal{G}'$ for an arbitrary $U' \subset V(\mathcal{G}')$. In order for $\mathcal{G}'$ to be perfect we have to show that $\omega(\mathcal{G}'[U]) = \chi(\mathcal{G}'[U])$. We know that since $\mathcal{G}$ is perfect it must be $\omega(\mathcal{G}[U]) = \chi(\mathcal{G}[U])$ where $U = U' \cap V(\mathcal{G})$. If $\omega(\mathcal{G}[U]) = \chi(\mathcal{G}[U]) \geq 2$, it is also $\omega(\mathcal{G}[U]) = \omega(\mathcal{G}'[U']) = \chi(\mathcal{G}'[U']) = \chi(\mathcal{G}[U])$ as $\mathcal{G}[U] = \mathcal{G}'[U] \subset \mathcal{G}[U']$ and $U' \setminus U$ only contains vertices added during the reduction. For those we already argued that when adding them the clique number and chromatic number are maintained if they are at least 2. If $\omega(\mathcal{G}[U]) = \chi(\mathcal{G}[U]) < 2$ and $\mathcal{G}'[U']$ contains only isolated vertices, it is $\omega(\mathcal{G}'[U']) = \chi(\mathcal{G}'[U'])$. Otherwise, we have a clique of size 2 and also need 2 colors for $\mathcal{G}'[U']$ which leads to $\omega(\mathcal{G}'[U']) = 2 = \chi(\mathcal{G}'[U'])$. Therefore, perfectness is preserved by the reduction.

If $\mathcal{G}$ is chordal then $\mathcal{G}'$ is also chordal, as we retained the chords of the cycles in $\mathcal{G}$ since G is a subgraph of G', and we added no new cycles since we only added vertices with a single incident edge.

The reduction also preserves outerplanarity as for an outerplanar graph $\mathcal{G}$ the added vertices can always be drawn in the outer face of a planar drawing of $\mathcal{G}$ as they each have only one edge connecting them to an original vertex.

If $\mathcal{G}$ is connected so is $\mathcal{G}'$ as all added vertices are connected to a vertex $v \in V(\mathcal{G})$. Since they are directly connected to v the diameter only increases by at most 2. Let $v', u' \in V(\mathcal{G}') \setminus V(\mathcal{G})$ be two added vertices that are adjacent to $v, u \in V(\mathcal{G})$, respectively. Then it is $\mathrm{d}(v', v) = 1$ and $\mathrm{d}(u, u') = 1$ which means that the distance between v' and u' is $\mathrm{d}(v', u') = \mathrm{d}(v, u) + 2 \leq \mathrm{diam}(\mathcal{G}) + 2$. Therefore, it is $\mathrm{diam}(\mathcal{G}') \leq \mathrm{diam}(\mathcal{G}) + 2$.

It remains to show that the pathwidth and treedepth increase by at most 1. Let $T = (V_T, E_T)$ be a minimal path-decomposition of $\mathcal{G}$. We will now construct a path-decomposition for $\mathcal{G}'$ with an increase in width of at most 1. For each vertex $v \in V(\mathcal{G})$ let $i \in V_T$ be a node whose bag X_i contains v. Let further without loss of generality i_L and i_R be the nodes adjacent to i. In the case that v has no new adjacent vertices in $\mathcal{G}'$

we do not change anything. Otherwise, let $\{v_1, \ldots, v_k\} = N_{\mathcal{G}'}(v) \setminus N_{\mathcal{G}}(v)$ be the set of newly adjacent vertices of v. We replace the node i with k new nodes $i_1, \ldots, i_k$ and set the new bags to be $X_{i_j} = X_i \cup \{v_j\}$ for $j \in [k]$. Then, for each incident edge of v there is a bag containing said edge. In the tree T we replace the edges $\{i_L, i\}$ and $\{i, i_R\}$ with the edges $\{i_L, i_1\}, \{i_1, i_2\}, \ldots, \{i_k, i_R\}$. This yields a valid path-decomposition as the tree is still a path, and we further added all vertices to bags. Also, every edge is inside a bag and since $X_i \subset X_{i_j}$ the bags containing a vertex u still induce a connected subgraph. As the added bags contain one element more than the original bag the pathwidth increases by at most 1.

We will now show that the treedepth increases by at most 1. Let F be a minimum height rooted forest such that for every edge $\{u, v\} \in E(\mathcal{G})$ there is an ancestor-descendant relationship between u and v in F. We modify this forest to a forest F' that is a valid minimum height rooted forest for $\mathcal{G}'$. Since in $\mathcal{G}'$ we only added edges incident to vertices $v \in V(\mathcal{G})$ we only have to add relationships from each vertex $v \in V(\mathcal{G})$ to its newly adjacent vertices. This can be achieved, by making each newly adjacent vertex $v' \in N_{\mathcal{G}'}(v) \setminus N_{\mathcal{G}}(v)$ of v the child of v in F'. Since in F' we only added children to vertices of F the height of F' is at most one greater than the height of F. Therefore, the treedepth of $\mathcal{G}'$ increased by at most 1 compared to $\mathcal{G}$. □

By using the second constraint expansion method from Lemma 3.5, we can preserve pathwidth and treedepth at the cost of losing connectivity and an increased level-width. Due to how we represent the graph it also takes quadratic time instead of linear.

Theorem 3.7. *An instance $\mathcal{G}$ of* Constrained Level Planarity *can be reduced in quadratic time to an instance of* Ordered Level Planarity *with level-width in $O(n(\mathcal{G}))$ and height in $O(n(\mathcal{G}) + c(\mathcal{G}))$ while preserving pathwidth and treedepth.*

Proof. Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph. In order to transform $\mathcal{G}$ to an equivalent ordered level graph we will perform the following steps iteratively for each level $\tilde{\ell} \in [h]$: First, we use the second constraint expansion method from Lemma 3.5 to obtain the equivalent graph $\bar{\mathcal{G}}_{\tilde{\ell}}$. As the added constraint levels have already total orders it therefore remains to vertex expand the level $\tilde{\ell}$. When vertex expanding the level $\tilde{\ell}$ we have to keep the vertices in the bundle $b(v)$ of each vertex $v \in V_{\tilde{\ell}}(\mathcal{G})$ together on the same level, as there is a total order set on them through constraints. In order for a vertex expansion of the level $\tilde{\ell}$ to not modify constrained level planarity according to the Lemma 3.3 we have to prevent nesting between the vertices of $V_{\tilde{\ell}}(\bar{\mathcal{G}})$ after the vertex expansion. For this, we add so-called claws to each sink and source in $V_{\tilde{\ell}}(\bar{\mathcal{G}})$, as seen in Figure 3.6a, obtaining the graph $\check{\mathcal{G}}_{\tilde{\ell}} = (\check{G}, \check{\gamma}, (\check{\prec}_\ell)_\ell)$. A *claw* is a star consisting of the four vertices v_M, v_L, v_R and v_T with v_M being the center vertex, as illustrated in Figure 3.5. A claw added to a sink $v \in S_{\tilde{\ell}}(\bar{\mathcal{G}})$ has the level assignments $\check{\gamma}(v_L) = \check{\gamma}(v_R) = \check{\gamma}(v)$, $\check{\gamma}(v_M) = \tilde{\ell}_{sm}$, $\check{\gamma}(v_T) = \tilde{\ell}_{st}$ where $\tilde{\ell}_{sm}$ and $\tilde{\ell}_{st}$ are new consecutive levels directly above $\tilde{\ell}$ with $\tilde{\ell}_{sm} < \tilde{\ell}_{st}$. The claw further adds constraints $v_L \check{\prec}_\ell v \check{\prec}_\ell v_R$, forcing the vertex v to be nested in v_M. We therefore say that the claw *holds* the vertex v. Further, we add constraints $v_L \check{\prec}_\ell u$ and $v_R \check{\prec}_\ell u$ for each constraint $v \check{\prec}_\ell u$ and analog constraints for each $u \check{\prec}_\ell v$. A claw for a source $v \in Q_{\tilde{\ell}}(\bar{\mathcal{G}})$ is similar with the level assignments $\check{\gamma}(v_M) = \tilde{\ell}_{qm}$ and

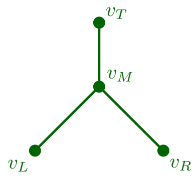


Fig. 3.5: A claw

$\check{\gamma}(v_T) = \tilde{\ell}_{qt}$ where $\tilde{\ell}_{qm}$ and $\tilde{\ell}_{qt}$ are consecutive levels directly below $\tilde{\ell}$ with $\tilde{\ell}_{qm} > \tilde{\ell}_{qt}$. As adding these claws maintains constrained level planarity since the claws can be drawn directly above or below their respective vertices, the graph $\check{\mathcal{G}}_{\tilde{\ell}}$ is equivalent to the graph $\bar{\mathcal{G}}_{\tilde{\ell}}$. We can now vertex expand the level $\tilde{\ell}$ into levels $\tilde{\ell}_v$ for each $v \in V_{\tilde{\ell}}(\mathcal{G})$ with the vertices on $\tilde{\ell}_v$ being the vertices from the bundle $b(v)$ plus the endpoints of the claws holding vertices of said bundle, as seen in Figure 3.6b. Therefore, such a level $\tilde{\ell}_v$ has a width in $O(n(\mathcal{G}))$. Let $\tilde{\mathcal{G}}_{\tilde{\ell}} = (\tilde{G}, \tilde{\gamma}, (\tilde{\prec}_\ell)_\ell)$ be the graph obtained after vertex expanding the level $\tilde{\ell}$, which, as we will show, is equivalent to $\check{\mathcal{G}}_{\tilde{\ell}}$. We then vertex expand the levels $\tilde{\ell}_{qt}$, $\tilde{\ell}_{qm}$, $\tilde{\ell}_{sm}$ and $\tilde{\ell}_{st}$ which can be done according to Lemma 3.4 without modifying constrained level planarity as $\tilde{\ell}_{qm}$ and $\tilde{\ell}_{sm}$ have only intermediate vertices and $\tilde{\ell}_{qt}$ and $\tilde{\ell}_{st}$ only have vertices with degree 1. This leads to all the added levels having total orders.

Let $\mathcal{G}' = (G', \gamma', (\prec'_\ell)_\ell)$ be the ordered level graph obtained after performing these steps iteratively for each level $\tilde{\ell} \in [h]$. The reduction takes quadratic time due to the second constraint expansion taking quadratic time in total. As we added a new level for each constraint and up to three levels for each vertex the height of $\mathcal{G}'$ is in $O(n(\mathcal{G}) + c(\mathcal{G}))$.

In order for $\mathcal{G}'$ to be equivalent to $\mathcal{G}$ it remains to show that for each level $\tilde{\ell} \in [h]$ the graph $\tilde{\mathcal{G}}_{\tilde{\ell}}$ is equivalent to $\check{\mathcal{G}}_{\tilde{\ell}}$. For this, we show that there is no nesting between the vertices $V_{\tilde{\ell}}(\check{\mathcal{G}}_{\tilde{\ell}})$ in a constrained level planar embedding $\tilde{\Gamma}$ of $\tilde{\mathcal{G}}_{\tilde{\ell}}$. First off, an intermediate vertex $u \in R_{\tilde{\ell}}(\check{\mathcal{G}})$ cannot be nested in another vertex $v \in V_{\tilde{\ell}}(\check{\mathcal{G}})$, as u has edges going above $\tilde{\ell}_{st}$ and below $\tilde{\ell}_{qt}$ in $\tilde{\mathcal{G}}_{\tilde{\ell}}$. Having an incident edge that extends past $\tilde{\ell}_{st}$ is also the reason why a source in $Q_{\tilde{\ell}}(\check{\mathcal{G}})$ cannot be nested in sinks or below intermediate vertices from $V_{\tilde{\ell}}(\check{\mathcal{G}})$ in $\tilde{\mathcal{G}}_{\tilde{\ell}}$. For sinks in $S_{\tilde{\ell}}(\check{\mathcal{G}})$ it is an edge going down past $\tilde{\ell}_{qt}$ that prevents them from being nested in sources or above intermediate vertices from $V_{\tilde{\ell}}(\check{\mathcal{G}})$ in the graph $\tilde{\mathcal{G}}_{\tilde{\ell}}$. It therefore only remains to show that a source cannot be nested in another source or above an intermediate vertex and that a sink cannot be nested in another sink or below an intermediate vertex. A vertex cannot be nested in a claw endpoint or a constraint vertex as these have a degree of 1. If a source v is an endpoint of a claw, the claw must be holding a sink v'. This implies that if v is nested in a source or above an intermediate vertex u, the rest of the claw must also be nested in u since all outgoing edges of u go above $\tilde{\ell}_{st}$. Therefore, the sink v' must also be nested above u which, as we already argued, is not possible. If the source v is not an endpoint of a claw then it must have a claw holding it. Let e_L and e_R be the edges incident to the vertex u between which v is nested and v_L, v_R, v_M, v_T the vertices of the claw holding v. If u is an intermediate vertex it must have an edge e_d going down past $\tilde{\ell}_{qm}$. Assume without loss of generality that it is $e_d \prec^{\tilde{\Gamma}}_{\tilde{\ell}_{qm}} v_M$. This contradicts $\tilde{\Gamma}$ being constrained level planar because of $v_L \prec^{\tilde{\Gamma}}_{\tilde{\gamma}(v)} v \prec^{\tilde{\Gamma}}_{\tilde{\gamma}(v)} e_R$ which forces

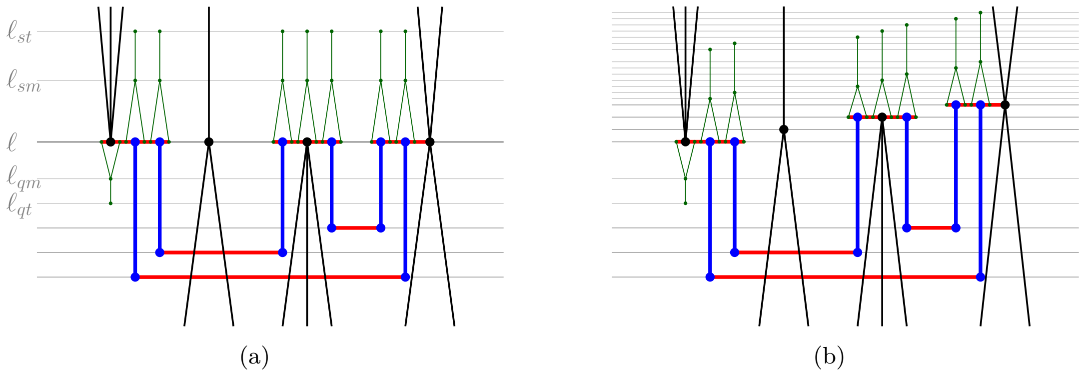


Fig. 3.6: A level with the added (green) claws (a) to prevent nesting after the levels are vertex expanded (b).

a crossing between $e_{vl} = \{v_M, v_L\}$ and e_R or e_d. If instead, u is a source, it must also have a claw holding it. Let u_L, u_R, u_M, u_T be the vertices of the claw holding u. Then it is without loss of generality $v_M \prec^{\tilde{\Gamma}}_{\tilde{\ell}_{qm}} u_M$ as well as $e_L \prec^{\tilde{\Gamma}}_{\tilde{\gamma}(v)} v \prec^{\tilde{\Gamma}}_{\tilde{\gamma}(v)} v_R$ and therefore $u \prec^{\tilde{\Gamma}}_{\tilde{\gamma}(u)} e_{vr}$ with $e_{vr} = \{v_M, v_R\}$. Together with $u_L \prec^{\tilde{\Gamma}}_{\tilde{\gamma}(u)} u$ this forces the edges e_{vr} and $e_{ul} = \{u_M, u_L\}$ to cross thereby contradicting $\tilde{\Gamma}$ being constrained level planar. In the case that u is a sink the situation is analog for u being nested in a sink or below an intermediate vertex. Therefore, there is no nesting between the vertices $V_{\tilde{\ell}}(\check{\mathcal{G}}_{\tilde{\ell}})$ in the graph $\tilde{\mathcal{G}}_{\tilde{\ell}}$.

Properties It remains to show that this reduction preserves pathwidth and treedepth. If $\mathcal{G}$ has only isolated vertices it has pathwidth 0 and a treedepth smaller than 2, and we can reduce it to an equivalent ordered level graph with the same pathwidth and treedepth according to Lemma 3.1. Otherwise, the pathwidth is at least 1 and the reduction preserves pathwidth as we only added claws, which have a pathwidth of 1, that are not connected to the original graph. As $\mathcal{G}$ has a treedepth of at least 2 and since the added claws have treedepth 2, the treedepth of the resulting graph $\mathcal{G}'$ is the same as the treedepth of $\mathcal{G}$. This is because we can use a separate tree for each claw thereby not increasing the height of the forest. □

3.2 Maximum Degree & Cycle Graphs

We will now provide a method to reduce an instance of Constrained Level Planarity to an instance of Ordered Level Planarity by subdividing edges. This can be used in reductions that maintain the maximum degree of an instance, or to transform a constrained level cycle graph into an ordered level cycle graph.

Theorem 3.8. *An instance $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ of Constrained Level Planarity can be reduced in $O(n(\mathcal{G}) + c(\mathcal{G}) + \Delta^3(\mathcal{G}) \cdot n(\mathcal{G}))$ time to an instance $\mathcal{G}' = (G', \gamma', (\prec'_\ell)_\ell)$ of Ordered Level Planarity with level-width $\lambda(\mathcal{G}') \leq 3$ and G' being a subdivision of G. The height of $\mathcal{G}$ is in $O(n(\mathcal{G}) + c(\mathcal{G}) + \Delta^3(\mathcal{G}) \cdot n(\mathcal{G}))$.*

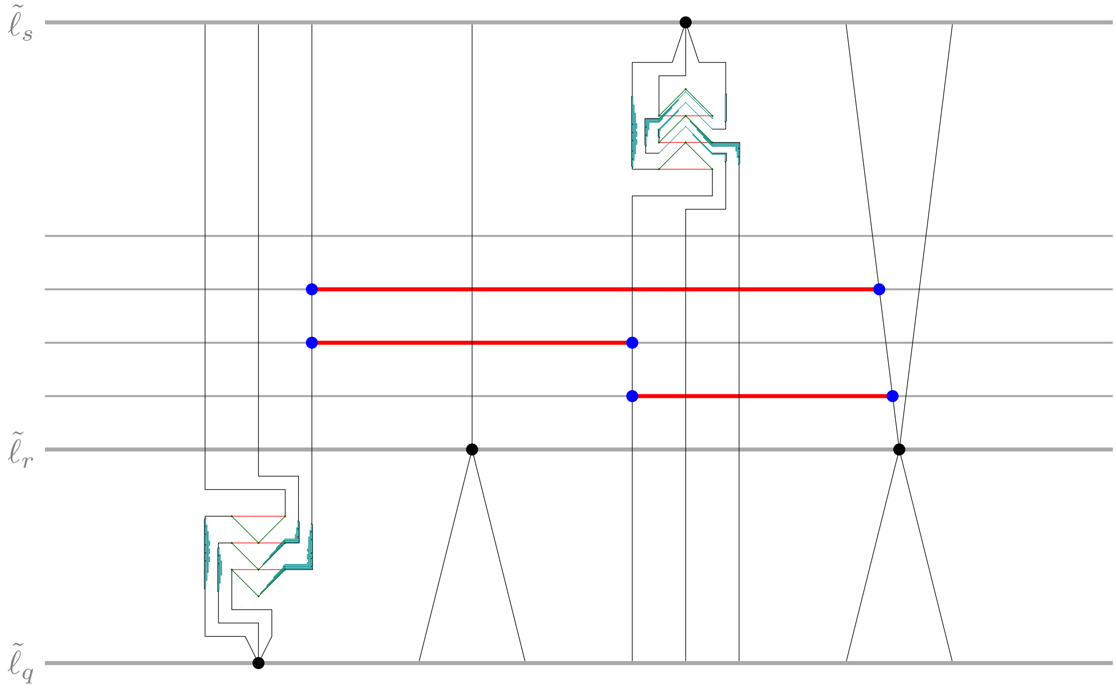
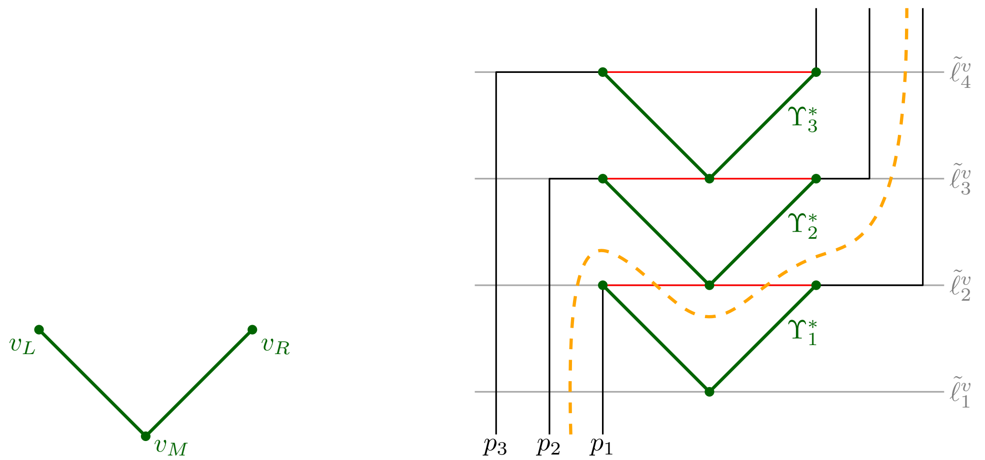


Fig. 3.7: A constraint expanded level with the added wedge-chains.

Proof. Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph. We will transform $\mathcal{G}$ to an equivalent ordered level graph by doing the following steps iteratively for each level $\tilde{\ell} \in [h]$: First, we use the third constraint expansion method from Lemma 3.5 on $\tilde{\ell}$ to obtain the graph $\bar{\mathcal{G}}_{\tilde{\ell}} = (\bar{G}, \bar{\gamma}, (\bar{\prec}_l)_l)$. Next, we will vertex expand the new levels $\tilde{\ell}_b$ and $\tilde{\ell}_s$. First, we vertex expand the level $\tilde{\ell}_b$ into the two levels $\tilde{\ell}_q < \tilde{\ell}_r$ with the sources on $\tilde{\ell}_q$ and the intermediate vertices on $\tilde{\ell}_r$ without modifying constrained level planarity according to Lemma 3.4. Before we can vertex expand the levels $\tilde{\ell}_q$ and $\tilde{\ell}_s$ we have to prevent nesting in the expanded graph. For this, we add a so-called wedge-chain to each source $v \in Q_{\tilde{\ell}}(\mathcal{G})$ which subdivides each incident edge with so-called wedges. This gadget prevents nesting between the vertices after the level $\tilde{\ell}_q$ is expanded and is also used upside down for each sink in $S_{\tilde{\ell}}(\mathcal{G})$, as seen in Figure 3.7. Let $\check{\mathcal{G}}_{\tilde{\ell}} = (\check{G}, \check{\gamma}, (\check{\prec}_\ell)_\ell)$ be the graph obtained after adding the wedge-chains. As the wedge-chains do not impose any restrictions on the orders of the incident edges of v in level planar embeddings, the graph $\check{\mathcal{G}}_{\tilde{\ell}}$ is equivalent to $\bar{\mathcal{G}}_{\tilde{\ell}}$. We then vertex expand the levels $\tilde{\ell}_q$ and $\tilde{\ell}_s$ into consecutive levels $\tilde{\ell}_v$ for each $v \in V_{\tilde{\ell}_q}(\check{\mathcal{G}})$ and consecutive levels $\tilde{\ell}_v$ for each $v \in V_{\tilde{\ell}_s}(\check{\mathcal{G}})$ while not modifying constrained level planarity according to Lemma 3.3, as the wedge-chains prohibit nesting between sources $V_{\tilde{\ell}_q}(\check{\mathcal{G}})$ as well as sinks $V_{\tilde{\ell}_s}(\check{\mathcal{G}})$ in the resulting graph. The level $\tilde{\ell}_r$ is also vertex expanded without modifying constrained level planarity according to Lemma 3.4 as it only contains intermediate vertices.

Let $\mathcal{G}'$ be the graph obtained after performing these steps iteratively for each level $\ell \in [h]$. Then $\mathcal{G}'$ is an ordered level graph and equivalent to $\mathcal{G}$. As each wedge-chain adds at most $\Delta^3(\mathcal{G})$ levels, and we add one level for each constraint as well as one for each vertex $v \in V(\mathcal{G})$, the height of $\mathcal{G}'$ is in $O(n(\mathcal{G}) + c(\mathcal{G}) + \Delta^3(\mathcal{G}) \cdot n(\mathcal{G}))$. The reduction can therefore be performed in $O(n(\mathcal{G}) + c(\mathcal{G}) + \Delta^3(\mathcal{G}) \cdot n(\mathcal{G}))$ time.

Technical Details In order to create the *wedge-chain* for a source $v \in Q_{\tilde{\ell}}(\mathcal{G})$ we first add the consecutive levels $\tilde{\ell}^v_1, \dots, \tilde{\ell}^v_{\deg(v)+1}$ directly above $\tilde{\ell}_q$. Let $\{e_1, \dots, e_{\deg(v)}\} = E_{\mathcal{G}}(v)$ be the incident edges of the vertex v. For each edge e_i we add a *wedge* Υ^*_i, see Figure 3.8a, consisting of the vertices v_L, v_M, v_R with the level assignments $\check{\gamma}(v_L) = \check{\gamma}(v_R) = \tilde{\ell}^v_{i+1}$ and $\check{\gamma}(v_M) = \tilde{\ell}^v_i$. The edge $e_i = \{v, u\}$ is then replaced by the path $p_i = \langle v, v_L, v_M, v_R, u \rangle$. We then chain those wedges together for each level $\ell = \tilde{\ell}_{i+1}$ with $i \in [\deg(v) - 1]$ by setting the order $v_L \check{\prec}_\ell v'_M \check{\prec}_\ell v_R$ where v_L and v_R are the left and right vertices from the wedge Υ^*_i and v'_M is the middle vertex from the wedge Υ^*_{i+1}. As there is no wedge above the wedge $\Upsilon^*_{\deg(v)}$ we only set the order $v_L \check{\prec}_\ell v_R$ for the two vertices on $\ell = \tilde{\ell}_{\deg(v)+1}$. With this, the wedge-chain already prevents nesting between different sources in a constrained level planar drawing Γ' of the graph $\mathcal{G}'$ where the level $\tilde{\ell}_q$ has been vertex expanded. This is because a source $u \in Q_{\tilde{\ell}}(\mathcal{G})$ with $\gamma'(u) > \gamma'(v)$ that is nested between two paths p_i and p_j of v would have at least one outgoing edge e_u. Since $\gamma'(u)$ is below the levels of the wedge-chain of v, the edge e_u would need to pass between the wedges Υ^*_i and Υ^*_j which is not possible, as seen exemplarily in Figure 3.8b. Indeed, let without loss of generality $j = i + 1$ such that the vertex u is nested between the paths p_i and p_j of v, where p_i and p_j correspond to the now subdivided edges e_i and e_j. Then the edge e_u is to the right of p_j and the left of p_i and therefore needs to pass directly between the two wedges Υ^*_i and Υ^*_j. This requires the edge e_u to go above v_L and below v'_M where v_L is the left vertex of Υ^*_i and v'_M the middle vertex of Υ^*_j. Since $\gamma'(v_L) = \gamma'(v'_M)$ it follows that e_u is not y-monotone, a contradiction to Γ' being a constrained level planar embedding.



(a) (b)

Fig. 3.8: A wedge (a) is used to construct a wedge-chain (b). Any path located between the paths of the wedge-chain, such as the dashed (orange) path, must pass between the wedges and cannot be y-monotone.

Since an edge cannot pass between two wedges, the order of the incident edges of v is currently prescribed by the levels of the wedges, with the left-to-right order of the incident edges corresponding to the top-to-bottom order of the wedges. In order to avoid restricting the order of the incident edges of v we therefore add additional wedges which allow the now subdivided edges to pass between the already present wedges. To achieve this we add between every consecutive pair of levels $\tilde{\ell}_i^v, \tilde{\ell}_{i+1}^v$ for $i \in [\deg(v)]$ in total $\deg^2(v)$ new consecutive levels, $\deg(v)$ for each edge incident to v. We set these levels to be $\tilde{\ell}_{i,1,1}^v < \cdots < \tilde{\ell}_{i,1,\deg(v)}^v < \tilde{\ell}_{i,2,1}^v < \cdots < \tilde{\ell}_{i,\deg(v),\deg(v)}^v$. For each $k \in [\deg(v)]$ and each $j \in [\deg(v)]$ we subdivide the path p_k that correspond to the edge e_k and insert a wedge $\Upsilon_k^{i,j}$ consisting of the vertices v_L, v_R and v_M with the level assignments $\check{\gamma}(v_L) = \check{\gamma}(v_R) = \tilde{\ell}_{i+1,j,k}^v$ and $\check{\gamma}(v_M) = \tilde{\ell}_{i,j,k}^v$. As these wedges allow paths to switch sides by passing between wedges we call them *switch-wedges* to separate them from the *fixed-wedges* that were added beforehand. With this, the wedge-chain no longer restricts the order of the incident edges of v, as exemplified in Figure 3.9. Indeed, let Γ be a constrained level planar embedding of $\mathcal{G}$ and $i \in [\deg(v) - 1]$. Let $e_{i+1} \prec_{v^+}^{\Gamma} e_i$ and $e^2, \ldots, e^d$ be the edges between e_i and e_{i+1} in the order from Γ, such that $e_{i+1} \prec_{v^+}^{\Gamma} e^d \prec_{v^+}^{\Gamma} \cdots \prec_{v^+}^{\Gamma} e^2 \prec_{v^+}^{\Gamma} e_i$. Then the paths $p^2, \ldots, p^d$ that correspond to the edges $e^2, \ldots, e^d$ must be to the left of the vertex v_L corresponding to the wedge Υ_i^* when they first touch the level $\tilde{\ell}_i^v$ but to the right of the wedge Υ_{i+1}^* when they first touch the level $\tilde{\ell}_{i+1}^v$. Therefore, they must pass between the wedges Υ_i^* and Υ_{i+1}^*. The path p^j uses its wedge $\Upsilon_k^{i,j}$, with k being the index of the edge e_k corresponding to the path p^j, to switch sides. If instead $e_i \prec_{v^+}^{\Gamma} e_{i+1}$, let similarly $e^1, \ldots, e^{d+1}$ be the edges from e_i to e_{i+1} including e_i and e_{i+1} such that $e_i = e^1 \prec_{v^+}^{\Gamma} e^2 \prec_{v^+}^{\Gamma} \cdots \prec_{v^+}^{\Gamma} e^{d+1} = e_{i+1}$. The paths $p^1, \ldots, p^{d+1}$ corresponding to these edges must now pass between the wedges Υ_i^* and Υ_{i+1}^* from right-to-left. The path p^j therefore uses its wedge $\Upsilon_k^{i,j}$ with k being again the index of the corresponding edge e_k. This works even though when a path p^j uses its wedge $\Upsilon_k^{i,j}$ to switch sides, all paths from p_1 to p_j now must also pass between Υ_i^* and $\Upsilon_k^{i,j}$. This is due to the fact that the paths $p^{j'}$ in between, with $1 \leq j' < j$, use the wedge $\Upsilon_{k'}^{i,j'}$, with k' being the index of the corresponding edge $e_{k'}$, which is located between Υ_i^* and $\Upsilon_k^{i,j}$. As paths corresponding to edges that are incident to other sources $u \in Q_{\tilde{\ell}}(\mathcal{G})$ have no subdivisions on these levels they still cannot pass between the wedges of the wedge-chain, which means that nesting is still prevented in every level planar drawing of the vertex-expanded graph. In order for the wedge-chain to have total orders on each level, it remains to vertex expand the levels of the wedge-chain that do not contain constraints, which are precisely the levels of the switch-wedges. This can be done using the first method from the Lemma 3.4 as each level $\tilde{\ell}_{i,j,k}^v$ contains at most one source v'_M from the wedge $\Upsilon_k^{i,j}$ as well as at most one intermediate vertex v_R and one sinks v_L from the lower wedge $\Upsilon_k^{i-1,j}$. The levels $\tilde{\ell}_i^v$ with fixed-wedges already have a total order on their up to three vertices. Therefore, there are at most three vertices on a level of the wedge-chain. We have thus shown that wedge-chains prevent nesting between different sources in the expanded graph and inserting them maintains constrained level planarity. □

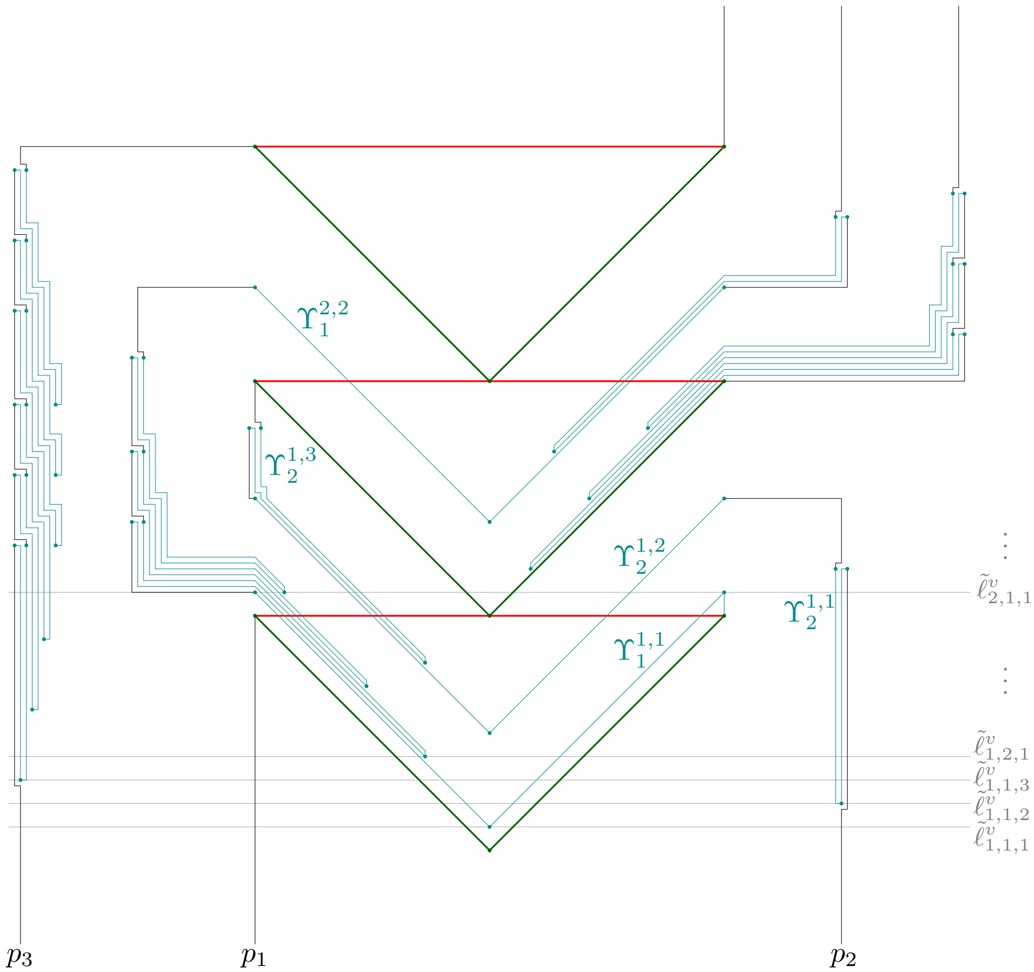


Fig. 3.9: A wedge-chain with the (blue) switch-wedges added. As the order of the paths does not match the order of the wedges the paths p_1 and p_2 have to switch sides.

We can now use this as part of a reduction from Constrained Level Planarity to Ordered Level Planarity that maintains the maximum degree.

Theorem 3.9. *Constrained Level Planarity can be reduced in polynomial time to Ordered Level Planarity while maintaining the maximum degree.*

Proof. Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a constrained h-level graph. We can easily determine the maximum degree of $\mathcal{G}$ in linear time. If the graph has a maximum degree greater than 1 we can use Theorem 3.8 to transform it in polynomial time to an equivalent ordered level graph by only subdividing edges. Since the degree of the original vertices does not change and the added vertices have degree 2 the maximum degree is maintained. If $\mathcal{G}$ has instead a maximum degree of 0 it only consists of isolated vertices and can therefore be transformed to an equivalent ordered level graph in quadratic time by only adding constraints, thereby keeping the maximum degree at 0, according to Lemma 3.1.

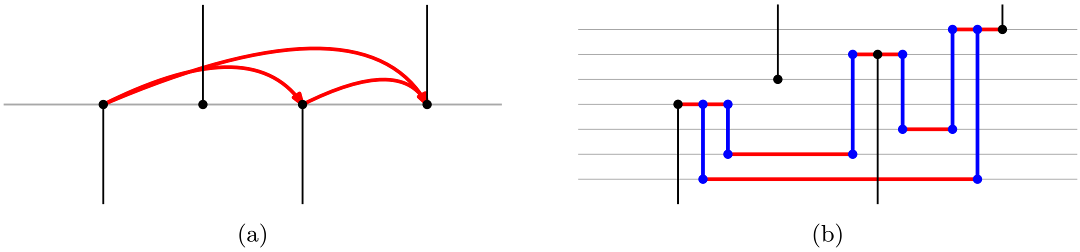


Fig. 3.10: (a) A level of a constrained level graph with maximum degree 1. (b) The level after being constraint and vertex expanded.

If instead $\mathcal{G}$ has maximum degree 1 we can construct an equivalent ordered level graph $\mathcal{G}' = (G', \gamma', (\prec'_\ell)_\ell)$ by doing the following: First we use the second constraint expansion method from Lemma 3.5 iteratively on every level $\ell \in [h]$. Let $\tilde{\mathcal{G}}$ be the resulting graph. As in $\tilde{\mathcal{G}}$ the constraint levels have total orders it only remains to vertex expand the original levels. We do this by replacing each level $\ell \in [h]$ with vertex levels ℓ_v for each vertex $v \in V_\ell(\mathcal{G})$ and assigning all vertices in the bundle $b(v)$ to the level ℓ_v as seen in Figure 3.10. We also transfer the total order on the vertices of the bundle $b(v)$ to the level ℓ_v. Let $\mathcal{G}'$ be the graph obtained by iteratively vertex expanding all original levels in $\tilde{\mathcal{G}}$. The graph $\mathcal{G}'$ has total orders on each level, as the constraint levels have two vertices with a constraint and the vertex levels contain the bundle $b(v)$ of an original vertex $v \in V_\ell(\mathcal{G})$ with a total order set as well. Thus, $\mathcal{G}'$ is an ordered level graph. It was obtained in polynomial time due to the constraint and vertex expansion taking polynomial time. There also cannot be any nesting between vertices in a level planar embedding of $\mathcal{G}'$ since they all have degree 1. This means that the graph $\mathcal{G}'$ is equivalent to the graph $\tilde{\mathcal{G}}$ according to Lemma 3.3. Therefore, $\mathcal{G}'$ is also equivalent to $\mathcal{G}$. As a result, an instance of Constrained Level Planarity can be reduced in polynomial time to an instance of Ordered Level Planarity while maintaining the maximum degree. □

The reduction from Theorem 3.8 can also be used to transform a constrained level cycle graph to an equivalent ordered level cycle graph.

Corollary 3.10. *An instance of Constrained Level Planarity that is a cycle graph can be reduced in linear time to an instance of Ordered Level Planarity that is also a cycle graph.*

Proof. Let $\mathcal{G}$ be a constrained level cycle graph. We can transform it according to Theorem 3.8 to an equivalent ordered level graph $\mathcal{G}'$ in $O(n(\mathcal{G}) + c(\mathcal{G}) + \Delta^3(\mathcal{G}) \cdot n(\mathcal{G}))$ time. As $\mathcal{G}$ is a cycle it is therefore $\Delta(\mathcal{G}) = 2$ which means that the transformation takes place in $O(n(\mathcal{G}) + c(\mathcal{G}))$ and therefore linear time. Since the graph $\mathcal{G}'$ is obtained from $\mathcal{G}$ by subdividing edges it is also a cycle, as subdividing a cycle yields a cycle. □

3.3 Connectivity

In this section, we will show that a k-connected instance of Constrained Level Planarity can be reduced to a k-connected instance of Ordered Level Planarity for arbitrary k as long as nesting is prevented in the resulting graph. The basic idea of the reduction is to use the third constraint expansion method from Lemma 3.5. However, as this subdivides edges we lose k-connectivity for $k \geq 3$. We remedy this by using a gadget, see Definition 3.13, which enables us to increase the connectivity of the graph. Since its use heavily restricts the level planar drawings we will first show that for a 3-connected graph the level planar drawings are already fairly restricted.

Lemma 3.11. *Let $\mathcal{G}$ be a 3-connected level planar graph. In every level planar drawing of $\mathcal{G}$, the set of faces is the same. Further, there are at most two faces that appear as an outer face in level planar drawings of $\mathcal{G}$. Moreover, if two distinct faces appear as an outer face in level planar drawings of $\mathcal{G}$ they must be incident to the edge from the lowest to the highest vertex.*

Proof. The first point, that in every level planar drawing of $\mathcal{G}$ there is the same set of faces, follows from the fact that 3-connected planar graphs have a unique combinatorial embedding, up to reflection, as shown by Whitney [Whi32] and Fleischner [Fle73].

We will now show that there are at most two faces that appear as an outer face in level planar drawings of $\mathcal{G}$. Let v_B and v_T be the lowest and highest vertex of $\mathcal{G}$. Since in level planar drawings, all edges must be drawn y-monotone, there cannot be an edge drawn above v_T or below v_B. Therefore, both of these vertices are incident to the outer face and the outer face is bounded by two vertex-disjoint paths from v_B to v_T, as each crossing point of the paths would be a cut vertex. Let f_1 and f_2 be two faces that appear as outer faces in level planar drawings of $\mathcal{G}$. Let Γ be a level planar drawing where f_1 is the outer face and let $V_L^{f_2}$ be the set of vertices drawn to the left of f_2 and $V_R^{f_2}$ the set of vertices drawn to the right of f_2, as illustrated in Figure 3.11a. Since v_B and v_T are incident to the face f_2, it splits the graph in two. This forces all paths between vertices

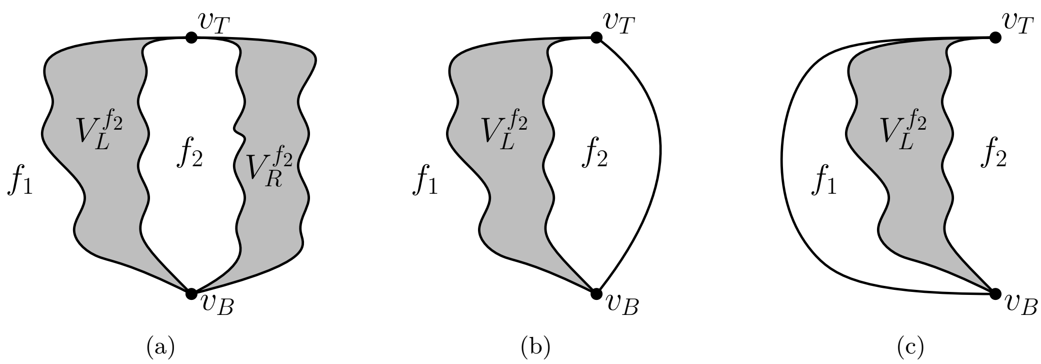


Fig. 3.11: (a) A level planar drawing where f_1 is the outer face and the sets $V_L^{f_2}$ and $V_R^{f_2}$ contain the vertices to the left and right of f_2, respectively. (b) The set $V_R^{f_2}$ is empty as in its place is the edge $\{v_B, v_T\}$. (c) A level planar drawing with f_2 as the outer face.

in $V_L^{f_2}$ and $V_R^{f_2}$ to go through either v_T or v_B. Therefore, one of those sets must be empty as otherwise v_T and v_B form a separation pair which contradicts 3-connectivity. This implies that one of the paths from v_B to v_T incident to f_2 must be the edge $\{v_B, v_T\}$, as shown in Figure 3.11b. As this argument also applies to f_1 in a level planar drawing Γ' where f_2 is the outer face, see Figure 3.11c, both faces f_1 and f_2 must be incident to the edge $\{v_B, v_T\}$. Since an edge can only be incident to two faces, there are at most two faces that appear as an outer face in level planar drawings of $\mathcal{G}$. □

We will now show that not only the outer face of a 3-connected level graph is restricted in level planar drawings, but also the order on the incident edges of a vertex.

Lemma 3.12. *Let $\mathcal{G} = (G, \gamma, (\prec_\ell)_\ell)$ be a 3-connected constrained level graph that is level planar. The following holds:*

1. *For each intermediate vertex $v \in R(\mathcal{G})$ the linear orders on the incoming edges and on the outgoing edges of v are unique, up to reflection.*
2. *Let $v \in Q(\mathcal{G})$ ($v \in S(\mathcal{G})$) be a source (sink) in $\mathcal{G}$. There is an edge $e \in E^+_{\mathcal{G}}(v)$ ($e \in E^-_{\mathcal{G}}(v)$) incident to v such that it is in any level planar embedding of $\mathcal{G}$ the left- or rightmost incident edge of v. We call such an edge e an* outer edge *of v.*

We can obtain the unique linear orders for all intermediate vertices and the outer edges for all sources and sinks in polynomial time.

Proof. Since $\mathcal{G}$ is 3-connected it has a unique combinatorial embedding Λ in the plane, up to reflection, as shown by Whitney [Whi32] and Fleischner [Fle73]. Since all edges are drawn as y-monotone arcs, for an intermediate vertex $v \in R(\mathcal{G})$ all outgoing edges are drawn above the level $\gamma(v)$ while all incoming edges are drawn below. This indicates that in the cyclic order Λ_v of the incident edges of v, all outgoing edges must appear consecutively just as all the incoming edges must appear consecutively. This establishes unique linear orders, up to reflection, on the incoming edges and on the outgoing edges of v. As we can obtain a level planar embedding of $\mathcal{G}$ in quadratic time we can obtain the order of the incoming and the outgoing edges of an intermediate vertex in polynomial time.

It therefore only remains to show that for each source and sink, there exists an outer edge and that we can obtain such an edge in polynomial time. We will show this only for sources as it is analogous for sinks. If v is the lowest vertex, and the outer face is the same in all level planar embeddings, the order of the incident edges of v is always the same. This is because the incident edges of v that are part of the boundary of the outer face must be the left and rightmost edges since the outer face has a big angle at v as v is the lowest vertex. Therefore, the leftmost incident edge would be an outer edge. If there are level planar embeddings of $\mathcal{G}$ with distinct outer faces, the graph $\mathcal{G}$ must contain the edge $e = \{v, v_T\}$, where v_T is the highest vertex, according to the Lemma 3.11. As each face that appears as an outer face in a level planar embedding of $\mathcal{G}$ must be incident to the edge e, the edge e must be the left or rightmost edge in every level planar embedding of $\mathcal{G}$. This makes e an outer edge of v.

If v is not the lowest vertex and the order on the incident edges of v is the same in every level planar embedding, the leftmost edge is an outer edge. If, on the other hand, there are several level planar embeddings with distinct linear orders on the incident edges of v, we will show that there is also an outer edge. To do this, we will list some conditions that must be satisfied when there are several level planar embeddings with distinct linear orders on the incident edges of v. In order to do that, we must first introduce the setting.

Let Γ be a level planar embedding of $\mathcal{G}$ and let e_L and e_R be the left and rightmost edges of v in Γ, respectively. We can characterize the linear orders on the incident edges of v by the face which has a big angle at v, as the left and rightmost incident edges of v must be incident to this face. Let therefore f be the face with which v has a big angle in Γ, as illustrated in Figure 3.12a. Let $\partial f = \langle v, u, v_3, \dots, v_k, v \rangle$ be the directed boundary of f. Let u_t be the vertex with the highest level in ∂f before the first vertex with a level below or equal to $\gamma(v)$ called u_ℓ. Let v_B be the vertex with the lowest level, which must be below $\gamma(v)$ since v is not the lowest vertex. Further, let w_t be the vertex with the highest level after the last vertex with a level below or equal to $\gamma(v)$ called w_ℓ. Assume without loss of generality that $\gamma(u_t) \leq \gamma(w_t)$ (otherwise we can look at the horizontally reflected situation where the roles are switched). Because $\mathcal{G}$ is 3-connected there has to be a path p' from v to v_B that is vertex-disjoint to the paths $\langle v, u, \dots, v_B \rangle$ and $\langle v, \dots, w_t, \dots, v_B \rangle$ along the boundary of f. Let v' be the first vertex on this path p' above $\gamma(u_t)$ and let p_v be the path from v to v' along p', as seen in Figure 3.12a. Due to how we chose v' the path p_v does not go below $\gamma(v)$. A similar argument can be made for each vertex x nested in v between the paths $p_L = \langle v, \dots, u_t \rangle$ and $p_R = \langle v, \dots, w_t \rangle$. It must also have a path to a vertex x' with $\gamma(x') > \gamma(u_t)$ that does not go below $\gamma(v)$ nor includes v and is vertex-disjoint to the paths $\langle v, u, \dots, v_B \rangle$ and $\langle v, \dots, w_t, \dots, v_B \rangle$ along the boundary of f. If such a path did not exist the vertices v and u_t or v and w_t would be a separation pair depending on if x is on the left or right of the path p_v which separates the area between $\gamma(v)$ and $\gamma(u_t)$.

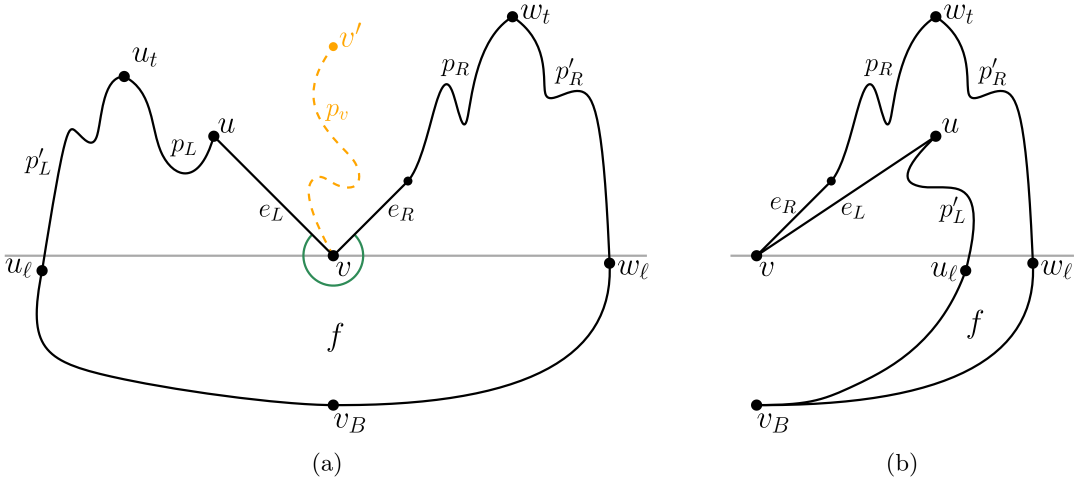


Fig. 3.12: Two level planar embeddings Γ (a) and Γ' (b) with distinct orders on the incident edges of v. For Γ' to be level planar it must be $u = u_t$ as well as $\gamma(u) < \gamma(w_t)$.

Let Γ' be a level planar embedding of $\mathcal{G}$ with a different order on the incident edges of v than Γ. As e_L and e_R are the outermost edges in Γ this means that in Γ' the edge e_L is to the right of e_R. If $\gamma(u_t) = \gamma(w_t)$ this is not possible. Indeed, if e_L is to the right of e_R the path $p'_R = \langle w_t, \dots, w_\ell \rangle$ has to be to the left of the path p_R as otherwise it would lead to a contradiction. This is because, it would be $p_R \prec^{\Gamma'}_{\gamma(v)} p_L \prec^{\Gamma'}_{\gamma(v)} p'_R$ if p'_R is to the right of p_R and therefore $p_R \prec^{\Gamma'}_{\gamma(w_t)} p_L \prec^{\Gamma'}_{\gamma(w_t)} p'_R$ which is equivalent to the impossible order $w_t \prec^{\Gamma'}_{\gamma(w_t)} u_t \prec^{\Gamma'}_{\gamma(w_t)} w_t$. If the path p'_R is to the left of p_R, all the other outgoing edges of v must have either their upper endpoint nested below w_t, since p'_R goes below $\gamma(v)$, or be to the right of p_R with their upper endpoint nested below u_t, as we will soon show. This leads to a contradiction, as there is a vertex v' with $\gamma(v') > \gamma(u_t) = \gamma(w_t)$ which is connected via a bounded path p_v to v and would therefore also have to be nested in u_t or v_t since it cannot get out due to p_v not going below $\gamma(v)$. The vertices to the right of p_R have to be nested in u_t, as otherwise the path $p'_L = \langle u_t, \dots, u_\ell \rangle$ would have to be to the left of the path p_L. This would, similarly to the path p'_R being on the left of p_R, force the bounded path p_R to be on the right of the path p_L which means that the order of the incident edges of v is as before, a contradiction to there being a distinct order in Γ'. This shows that there cannot be several distinct orders on the incident edges of v in level planar embeddings if $\gamma(u_t) = \gamma(w_t)$.

We will now show that unless $u_t = u$ we also have a contradiction. Take a look at the situation in Γ' where the edge e_L is to the right of e_R. Suppose $u_t \neq u$. This means that u is nested in v in Γ. As shown previously there must be a path p_u from u to a vertex u' with $\gamma(u') > \gamma(u_t)$ that does not go below $\gamma(v)$ nor through v. The vertex u' therefore cannot be nested below u_t in Γ'. As all paths that start in u must be drawn between the paths p_L and p'_L in Γ', due to p'_L being to the right of p_L, the path from u to u' would need to go below $\gamma(v)$ or cross another edge to reach u', a contradiction to Γ' being constrained level planar. The path p'_L must be to the right of p_L in Γ', as otherwise that would, similarly to before, establish the order from Γ instead. Therefore, when there are several level planar embeddings with distinct orders on the incident edges of v it must be $u = u_t$ and $\gamma(u) < \gamma(w_t)$.

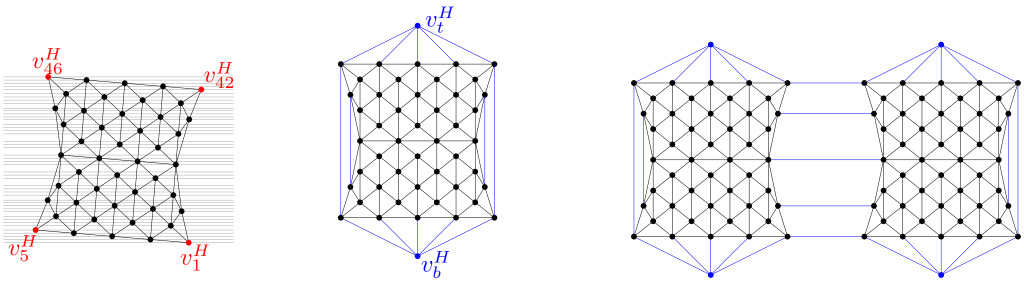
We will now show that the orders for Γ and Γ' can only differ by one edge. Suppose that the orders from Γ and Γ' differ by more than one edge. Let $e_{LR} = \{v, u_{LR}\}$ be the edge directly to the right of e_L in Γ. Then e_{LR} must also be to the right of e_R in Γ' as the order from Γ' differs by more than one edge from the order in Γ. Further, it is either $\gamma(u_{LR}) > \gamma(u) = \gamma(u_t)$ or there is a path from u_{LR} to a vertex u'_{LR} above $\gamma(u)$ that does not go below $\gamma(v)$ nor through v and is vertex-disjoint to the paths $\langle v, u, \dots, v_B \rangle$ and $\langle v, \dots, w_t, \dots, v_B \rangle$. As we have shown that if p_L is to the right of p_R the path p'_L cannot be to the left of p_L, it therefore follows that the path $\langle v, u_{LR}, \dots, u'_{LR} \rangle$ must cross either p_L or p'_L thereby contradicting Γ' being constrained level planar. This shows that the different orders can only differ by one edge and that therefore if $u = u_t$ and $\gamma(u) < \gamma(w_t)$ the edge e_L is an outer edge of v since it is the rightmost edge in Γ', as illustrated in Figure 3.12b.

We can therefore obtain an outer edge for each source v by checking if the above requirements for there being more than one order on the incident edges of v, $u = u_t$ and

$\gamma(u) < \gamma(w_t)$, apply. If so we directly obtain the outer edge $\{v, u\}$. Otherwise, the order is always the same, and we choose the leftmost edge as it is an outer edge. For sinks, we perform a similar procedure as the situation is symmetric. As we can obtain a level planar embedding in quadratic time, we can choose all outer edges in polynomial time. □

We will now present the gadget which we will use the increase the connectivity of a graph after it was constraint expanded.

Definition 3.13 (Hourglass)**.** *An* hourglass H *consists of* 46 *vertices* $v_1^H, \dots, v_{46}^H$*, with each vertex* v_i^H *being on a distinct level* ℓ_i*, as illustrated in Figure 3.13a. On its own, an hourglass is only* 3*-connected as the four corner vertices* $v_1^H, v_5^H, v_{42}^H, v_{46}^H$ *have a degree of* 3*. An hourglass can be made* 5*-connected by adding two edges to the left and right side as well as a vertex* v_b^H *below that is connected to the bottom side and a vertex* v_t^H *above that is connected to the top side, as shown in Figure 3.13b.*



(a) (b) (c)

Fig. 3.13: (a) An hourglass. Due to the (red) corner vertices, it is only 3-connected. (b) An hourglass with additional (blue) edges making it 5-connected. For the sake of visual clarity, we usually draw the hourglass upright. (c) Two hourglasses connected by their sides form a 5-connected graph.

In order to build a 5-connected graph out of hourglasses, we can connect two hourglasses by running five parallel edges between two of their sides, as seen exemplarily in Figure 3.13c. Note that if we have two hourglasses that share the same levels, i.e. $\gamma(v_i^{H_1}) = \gamma(v_i^{H_2}) = \ell_i$ for all $i \in [46]$ as well as $\gamma(v_t^{H_1}) = \gamma(v_t^{H_2}) = \ell_t$ and $\gamma(v_b^{H_1}) = \gamma(v_b^{H_2}) = \ell_b$, there cannot be nesting between their vertices when we vertex expand these levels. We are now equipped to show how to reduce Constrained Level Planarity to Ordered Level Planarity while maintaining k-connectivity for 3, 4, and 5-connected graphs.

Theorem 3.14. *A* k*-connected instance* $\mathcal{G}$ *of* Constrained Level Planarity *with* $3 \leq k \leq 5$ *and height* h *reduces in polynomial time to a* k*-connected instance* $\mathcal{G}'$ *of* Ordered Level Planarity *with level-width* $\lambda(\mathcal{G}') \leq 2$ *and height in* $O(n^2(\mathcal{G}))$*, as long as for each level* $\ell \in [h]$ *there is no nesting between its sources as well as between its sinks in any level planar embedding of a graph where the level* ℓ *has been vertex expanded.*

Proof. If the graph $\mathcal{G}$ is not level planar, which we can test in linear time, we can reduce $\mathcal{G}$ to an equivalent k-connected ordered level graph according to Lemma 3.1 as this reduction only adds constraints. Otherwise, $\mathcal{G}$ is level planar, and we can transform $\mathcal{G}$ to an equivalent ordered level graph $\mathcal{G}'$ by doing the following: First, we determine the order of the incident edges for each intermediate vertex as well as the outer edge for each source and sink according to Lemma 3.12. Subsequently, we perform the third constraint expansion from Lemma 3.5 for each level $\ell \in [h]$ and make sure that the constraint path is not the selected outer edge of its vertex. This can be done without a problem, as the constraint path must merely be an incident edge of the vertex and each vertex has at least three incident edges. This implies that there are at least two edges other than the outer edge that can be used as a constraint path. We then vertex expand for each $\ell \in [h]$ the level ℓ_b that was added during the constraint expansion, into the two levels ℓ_q and ℓ_r with the sources being on ℓ_q and the intermediate vertices on ℓ_r without modifying constrained level planarity according to Lemma 3.4.

To obtain an ordered level graph it only remains to vertex expand the levels ℓ_q, ℓ_r and ℓ_s for each $\ell \in [h]$. We can use the level expansion from Lemma 3.4 to vertex expand the level ℓ_r as it only contains intermediate vertices. We can also vertex expand the level ℓ_q into separate levels ℓ_v for each source v on ℓ_q, as exemplified in Figure 3.14a. The resulting graph is equivalent to the previous according to Lemma 3.3 as we required that there be no nesting between the sources from the level ℓ in any level planar embedding of a graph where the level ℓ has been vertex expanded. This applies analog to the sinks on the level ℓ_s. After expanding all the levels iteratively we therefore obtain an equivalent ordered level graph.

This graph is not necessarily k-connected as we subdivided the constraint paths when performing the constraint expansion. In order to obtain a k-connected graph, we replace each vertex with its so-called wide version yielding an equivalent k-connected ordered level graph, as seen for example in Figure 3.14b. As the constraint and vertex expansion as well as the replacement of the vertices with their wide counterparts did not modify constrained level planarity, the resulting ordered level graph $\mathcal{G}'$ is equivalent to the graph $\mathcal{G}$. The height of $\mathcal{G}'$ is in $O(n^2(\mathcal{G}))$ as for each constraint, edge, and vertex a constant number of levels were added. Regarding the time of the reduction, the constraint and vertex expansion take polynomial time, as does determining the orders of the incident edges of intermediate vertices and the outer edges of sources and sinks. Since replacing the vertices with their wide versions can be done in polynomial time, the reduction takes place in polynomial time.

Technical Details To convert a vertex v to a *wide* vertex we use hourglasses. If v is an intermediate vertex, the linear order of the incoming as well as the outgoing edges of v is unique, up to reflection, according to Lemma 3.12. Let $e_{t,1}, \dots, e_{t,k^+}$ be the outgoing edges of v from left-to-right and $e_{b,1}, \dots, e_{b,k^-}$ be the incoming edges from left-to-right. We then substitute v with $k = \max(k^+, k^-)$ hourglasses $H_1^v, \dots, H_k^v$ where H_i^v has a connection from its left side to the right side of H_{i+1}^v for all $i \in [k-1]$. We further replace each edge $e_{t,i}$ with five edges originating from the top of H_i^v, and we replace each edge $e_{b,i}$ with five edges connecting to the bottom of H_i^v. These edges run either in parallel to another hourglass, in case the other endpoint of the edge was also replaced with an hourglass, or otherwise converge on the singular endpoint.

If v is a source we use the same process as with intermediate vertices on all edges except the outer edge e_o. We then add a vertex v_o below the hourglasses and connect it with five edges to the underside of H_1^v (since v has at least three incident edges there are at least two hourglasses added for v). We then replace the edge e_o with one or multiple edges starting at v_o, depending on if the other endpoint got converted into an hourglass. The resulting vertex is internally 5-connected. For sinks, we proceed similarly.

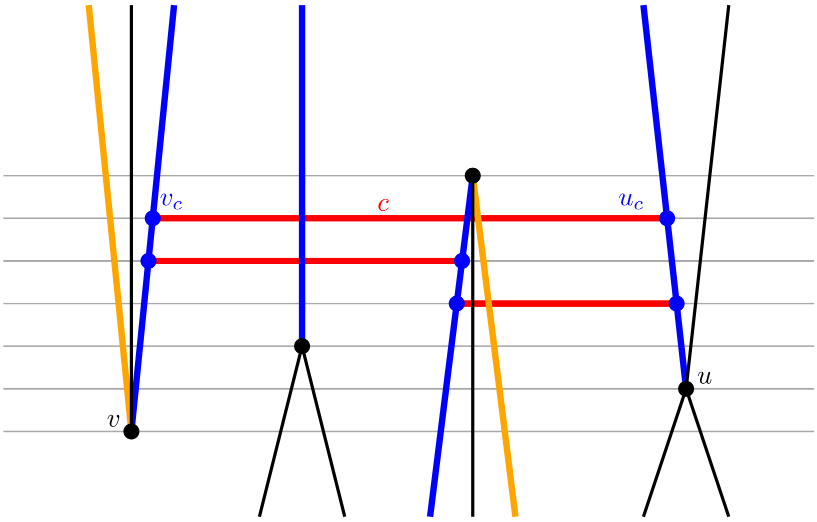
We also replace each constraint vertex v_c with an hourglass H_c^v and use five parallel edges between the constraint vertices on the same constraint path. As at least one end of the constraint path has an hourglass, since we added hourglasses at the endpoint of each edge that is not an outer edge, the constraint vertices are connected through five vertex-disjoint paths with the rest of the graph, as seen in Figure 3.14b. For a constraint $v_c \prec_{\ell_c} u_c$ we set constraints between the corresponding vertices of the hourglasses H_c^v and H_c^u instead. This process does not modify constrained level planarity, as two hourglasses cannot be nested in each other, and therefore the situation remains equivalent to there being single vertices.

In order for the wide vertices to have total orders on their levels, as the hourglasses used when converting a vertex to a wide vertex share the same levels, we vertex expand these hourglasses without modifying constrained level planarity according to Lemma 3.3 as we noted that there is no nesting between the vertices of these hourglasses after vertex expanding the levels shared by them. Therefore, there are at most two vertices per level and the level-width is $\lambda(\mathcal{G}') \leq 2$. □

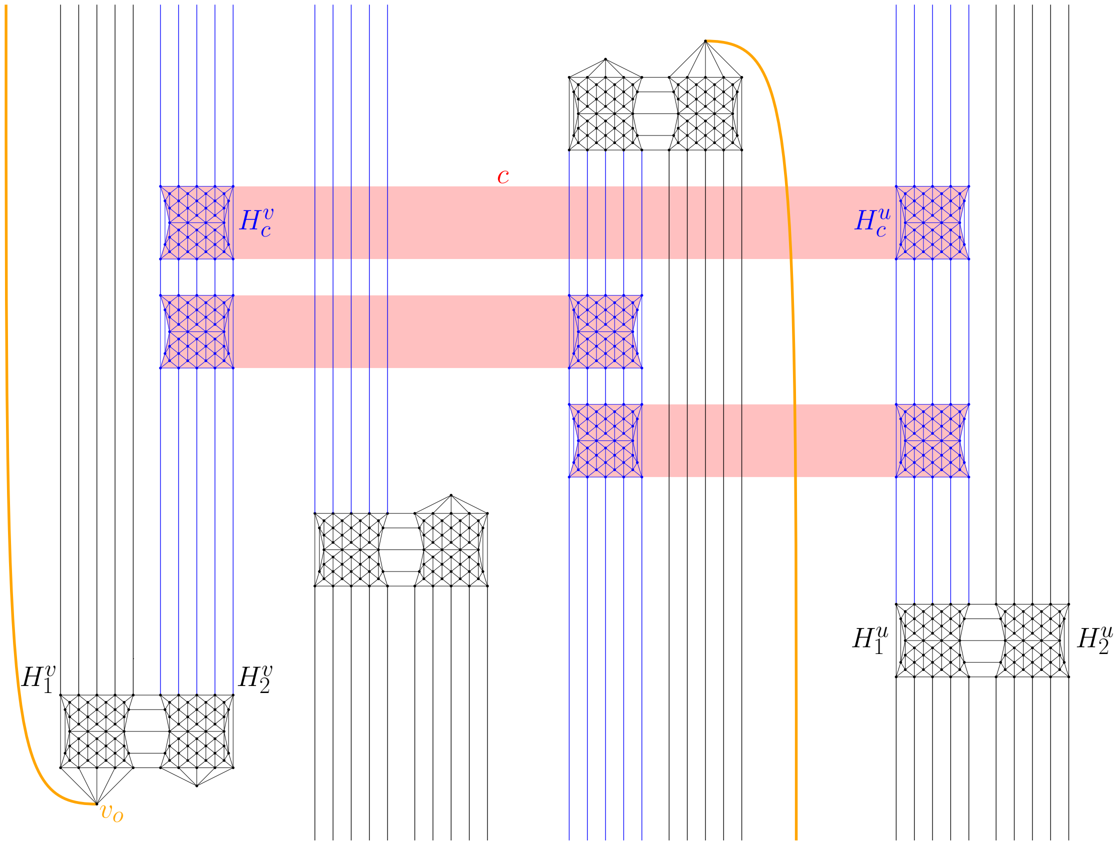
The restriction in the previous theorem could be removed if a way is found to prevent nesting between vertices that were on the same level after that level is vertex expanded. Lifting this restriction would also greatly improve the following result.

Corollary 3.15. *An instance $\mathcal{G}$ of* Constrained Level Planarity *with height h reduces in polynomial time to an instance $\mathcal{G}'$ of* Ordered Level Planarity *while maintaining k-connectivity for all $k \in \mathbb{N}_0$, with the restriction that if $3 \leq k \leq 5$ there must not for any level $\ell \in [h]$ be any nesting between its sources as well as between its sinks in any level planar embedding of a graph where the level ℓ has been vertex expanded.*

Proof. It is well known, see for example [Ski20, Chapter 18.8], that for a given graph the connectivity k can be computed in polynomial time. If $k \leq 2$ we can use Theorem 3.8 to reduce $\mathcal{G}$ to an equivalent ordered level graph $\mathcal{G}'$ in polynomial time by subdividing edges. As when subdividing multiple vertex-disjoint paths they remain vertex-disjoint, the connectivity remains the same for $k \leq 2$. Note that for $k \geq 6$ the graph $\mathcal{G}$ cannot be planar, and we can therefore reduce it to an equivalent k-connected ordered level graph in polynomial time according to Lemma 3.1. Otherwise, the graph $\mathcal{G}$ is k-connected with $3 \leq k \leq 5$ which means that it can be reduced to an equivalent k-connected ordered level graph in polynomial time according to Theorem 3.14. □



(a)

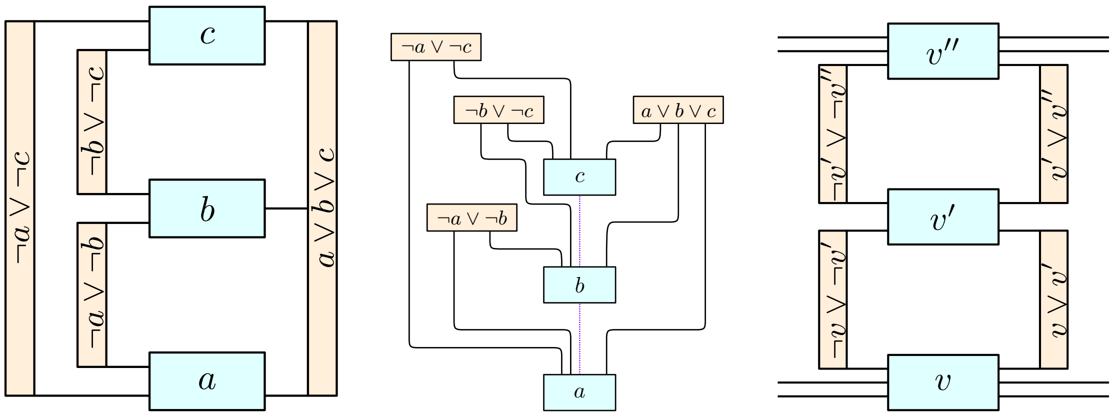


(b)

Fig. 3.14: (a) A level after constraint and vertex expansion, where the (orange) outer edges have not been used as constraint paths. The vertices are then replaced by hourglasses (b). For the sake of visual clarity, the constraints between two hourglasses have not been drawn individually but are drawn instead as a (red) band.

4 Hardness of Cycle and 5-Connected Graphs

We will now show that Constrained Level Planarity is $\mathcal{NP}$-hard for cycle and 5-connected graphs by adapting the proof of Brückner and Rutter [BR17] which reduces Planar Monotone 3-Satisfiability to Constrained Level Planarity. A *monotone* 3-Satisfiability formula $\varphi = (\mathcal{V}, \mathcal{C})$, where $\mathcal{V}$ is a set of variables and $\mathcal{C}$ a set of clauses, is a formula that only contains *positive* or *negative* clauses. This means that each clause either only contains positive literals or only negative literals. A *planar* 3-Satisfiability formula $\varphi = (\mathcal{V}, \mathcal{C})$ is one where the *variable-clause graph* $G_\varphi = (\mathcal{V} \dot\cup \mathcal{C}, E)$ is planar. This graph only contains edges between variables and clauses $\{v, c\} \in E$ if the clause $c \in \mathcal{C}$ contains a literal v or $\neg v$ of the variable $v \in \mathcal{V}$. The problem Planar Monotone 3-Satisfiability asks whether a given planar monotone 3-Satisfiability formula is satisfiable. It was shown to be $\mathcal{NP}$-complete by De Berg and Khosravi [DBK12].



(a) (b) (c)

Fig. 4.1: The variable-clause graph (a) and a different drawing of the variable-clause graph with y-monotone edges (b). To ensure that the graph is connected additional edges can be inserted between the variables, visualized as dotted (purple) lines.
(c) The construction to have each literal appear at most three times in clauses.

Given an instance of Planar Monotone 3-Satisfiability it is possible to draw the variable-clause graph such that all variables are drawn as boxes on a single vertical line, positive clauses are drawn as rectangles to the right of the variables and negative clauses are drawn as rectangles to the left of the variables, as seen in Figure 4.1a. The drawing can be modified such that the edges are y-monotone arcs that connect from the bottom of the clause rectangles to the top of the variable boxes, as seen in Figure 4.1b.

We can assume that each literal appears in at most three clauses. Otherwise, we can split a variable into two equivalent variables. This is achieved by adding for a variable v, whose literal appears more than three times, the variables v' and v'' as well as the clauses $(v \vee v')$, $(\neg v \vee \neg v')$, $(v' \vee v'')$ and $(\neg v' \vee \neg v'')$. As then v is equivalent to v'' we can use the literals of v'' instead of the literals of v. As this procedure used each literal from v once but added two literals of v'' which can be used instead, we can repeat this process until each literal appears at most three times in clauses.

In order to convert an instance of Planar Monotone 3-Satisfiability to an instance of Constrained Level Planarity we will replace each variable with a so-called variable gadget, each clause with a clause gadget and each edge with a pipe gadget.

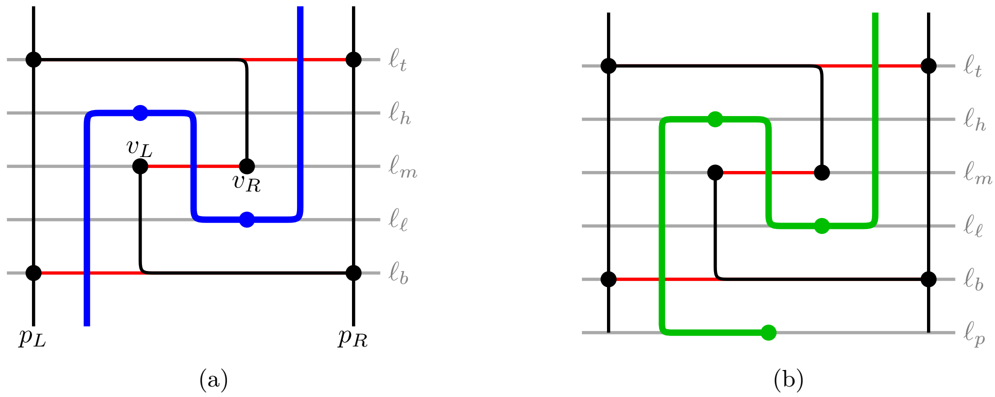
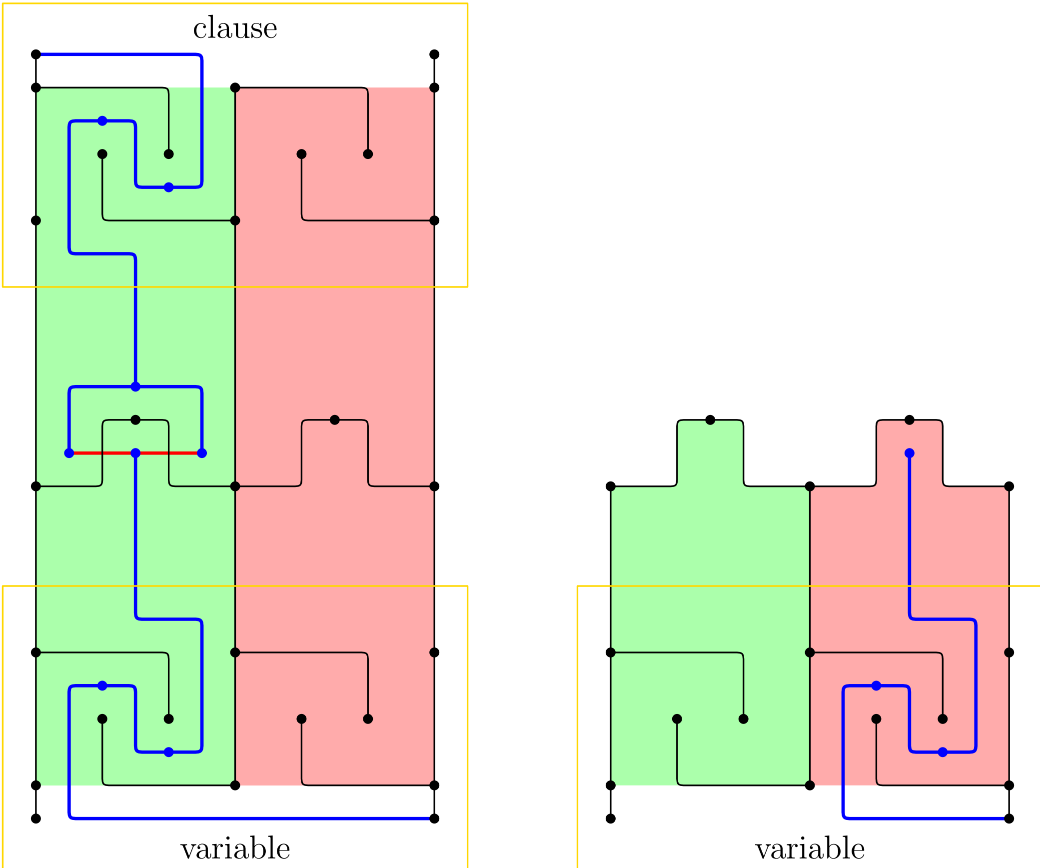


Fig. 4.2: (a) A (black) gate with a (blue) path passing through it in a zigzag manner. (b) A gate that is blocked by a (green) plug.

These gadgets all consist of gates. A *gate*, as seen in Figure 4.2a, consists of two paths p_L and p_R running in parallel that have their order fixed through constraints on the levels ℓ_t and ℓ_b. A gate restricts the number of paths that can be drawn between p_L and p_R to one, by having two vertices v_L and v_R on the level ℓ_m that connect to p_R and p_L, respectively. There is also the constraint $v_L \prec_{\ell_m} v_R$ fixing v_L to the left of v_R. With this, a path can only pass through the gate by changing direction at the levels ℓ_h and ℓ_ℓ. This is the reason that only one path can pass through a gate, as a second path that also changes directions at ℓ_h and ℓ_ℓ would cross the first path. Therefore, we can use *plugs*, paths that only pass through a gate and end directly after, as in Figure 4.2b, to make a gate impassable for other paths.

A *pipe gadget*, illustrated in Figure 4.3a, consists of two *channels* which are divided by a path in the middle, and one *conductor* which starts on the lower right side of the pipe and *flows* through either the left or right channel to the upper left. To connect the middle path to the outer walls of the pipe, the channels are divided into a lower and upper part with the conductor being split into a claw on top and a *tip* below. We force the two parts of the conductor to be drawn in the same channel by putting constraints on the claw and the tip such that the tip has to be drawn inside the claw. On each end of a channel, there is a gate. We can therefore use these pipes to pass the values of literals, by saying that the passed value is true if the conductor uses the left channel and otherwise false.



(a) (b)

Fig. 4.3: (a) The pipe gadget with the (blue) conductor flowing through the left channel transmitting the value true. (b) The lower half of a pipe that is used when a literal occurs less than three times. The (black) parts of the pipe, that are not the conductor, are fixed with constraints that are omitted in these drawings for the sake of visual clarity.

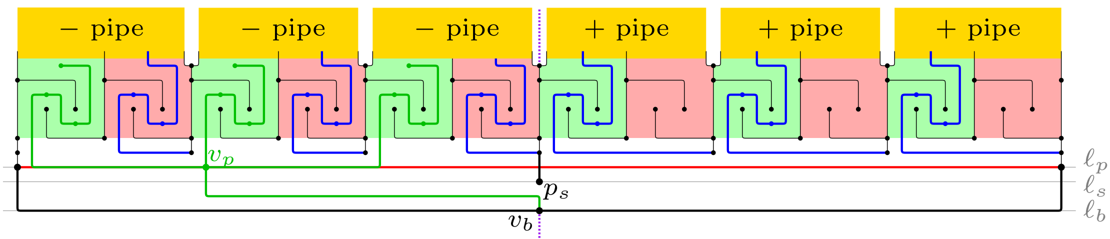


Fig. 4.4: A variable gadget that is configured as true with the (green) plugs on the left side of p_s. It can be connected to the other variable gadgets via the dotted (purple) lines. The constraints on the gates are omitted in the drawing.

The *variable gadget*, illustrated in Figure 4.4, consists of the merged lower ends of six pipes with the pipes of positive literals on the right being separated from the pipes of negative literals on the left by the path p_s going down to the level ℓ_s. In case a literal appears less than three times we still add a pipe but only its lower half, shown in Figure 4.3b. Each gadget then has three plugs, connected at the vertex v_p at the level ℓ_p which in turn is connected to the lowest vertex v_b of the gadget on the level ℓ_b. The vertex v_b also connects to the outermost pipes. Since the plugs are bounded paths with their lowest vertex being their common vertex v_p at the level ℓ_p, they must all be either completely to the left or completely to the right of the separating path p_s. Further, each plug has to plug exactly one gate since they cannot be drawn outside the variable gadget, as there are constraints on v_p forcing it to be drawn inside. All plugs are also forced into the left channels of the pipes since if a plug is inside the right channel of a pipe the conductor of said pipe cannot be drawn planar. Therefore, we can say that the variable gadget is configured as true or false depending on if the plugs are on the left or right side of the gadget. This is because all conductors on that side are forced into the right channels, therefore transmitting the value false for those literals, while the conductors on the other side can flow through the left channels and transmit the value true. To make sure that the resulting graph is connected regardless of the clauses, we further add edges between the vertex gadgets, as illustrated schematically in Figure 4.1b.

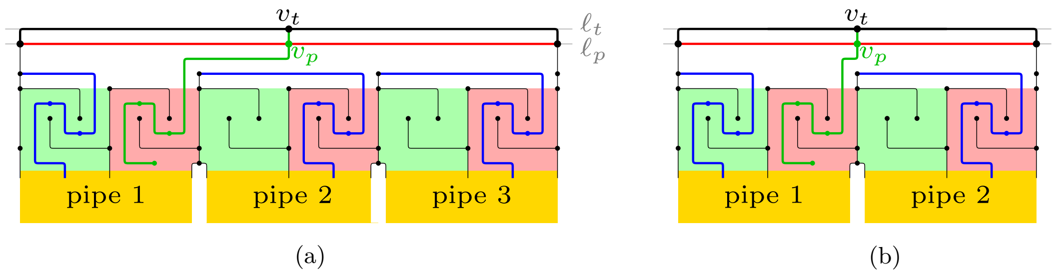


Fig. 4.5: The clause gadgets for a clause containing three (a) and two (b) literals. The constraints on the gates are omitted for the sake of visual clarity.

The *clause gadget*, illustrated in Figure 4.5, consists of one upper pipe end for each literal as well as a single plug. The outermost paths of the pipes connect to the highest vertex v_t of the gadget on the level ℓ_t. The plug is forced to be drawn inside the gadget by putting constraints on its vertex v_p on the level ℓ_p below the level ℓ_t. Therefore, the plug must go into the right channel of a pipe forcing the conductor of said pipe to flow through the left channel. This means that at least one pipe that ends in the clause gadget must transmit the value true in a level planar drawing.

Given an instance φ of Planar Monotone 3-Satisfiability we can obtain the corresponding Constrained Level Planarity instance $\mathcal{G}$ in polynomial time. We will now show that φ is satisfiable if and only if $\mathcal{G}$ is constrained level planar. If there is a constrained level planar drawing of $\mathcal{G}$, then for each negative clause at least one pipe must transmit the value true with its conductor flowing through the left channel. Therefore, the variable on the other end of that pipe must be configured as false. Otherwise, the conductor would be forced to flow through the right channel leading to a contradiction. This applies similarly to positive clauses. Therefore, φ must be satisfiable. If φ is satisfiable, we can configure all variables according to a satisfying assignment. Then for a negative clause, at least one of its literals must be true and therefore one of the adjacent variables is configured as false. This means that the conductor connecting the two can flow through the left channel and therefore the plug of the clause can go into the right channel without violating constrained level planarity. The situation is analogous for positive clauses. This shows that φ is satisfiable if and only if there is a constrained level planar drawing of $\mathcal{G}$. Therefore, Planar Monotone 3-Satisfiability reduces in polynomial time to Constrained Level Planarity which is therefore $\mathcal{NP}$-hard in the general case.

By replacing the edges with a cycle, as seen in Figure 4.6, we can reduce an instance of Planar Monotone 3-Satisfiability in polynomial time to an instance of Constrained Level Planarity that is a cycle. We therefore obtain the following theorem.

Theorem 4.1. *The problem Constrained Level Planarity is $\mathcal{NP}$-hard even when restricted to cycle graphs.*

As we have previously shown in Corollary 3.10 that we can transform a constrained level cycle graph into an ordered level cycle graph in polynomial time it follows that Ordered Level Planarity is also $\mathcal{NP}$-hard for cycle graphs.

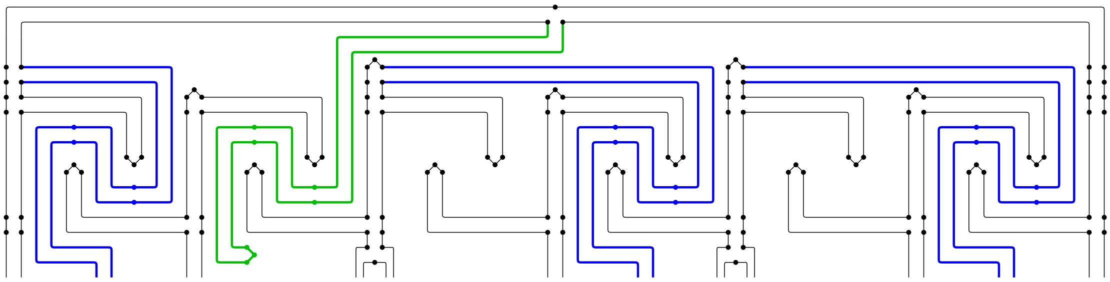
Corollary 4.2. *The problem Ordered Level Planarity is $\mathcal{NP}$-hard even when restricted to cycle graphs.*

If we instead replace the vertices in the variable, pipe, and clause gadgets with hourglasses and each edge with five parallel edges, as seen in Figure 4.7, the resulting graph is 5-connected.

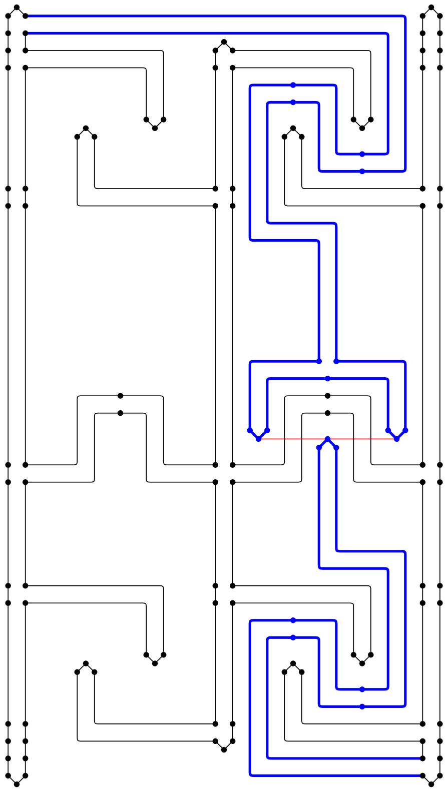
Theorem 4.3. *The problem Constrained Level Planarity is $\mathcal{NP}$-hard even when restricted to 5-connected graphs.*

Using the fact that there is no nesting between the vertices of different hourglasses that share the same levels in a graph where these levels were vertex expanded, we can reduce the 5-connected constrained level graph, obtained from reducing an instance of Planar Monotone 3-Satisfiability in polynomial time to an instance of Constrained Level Planarity, to an instance of Ordered Level Planarity in polynomial time while preserving 5-connectivity.

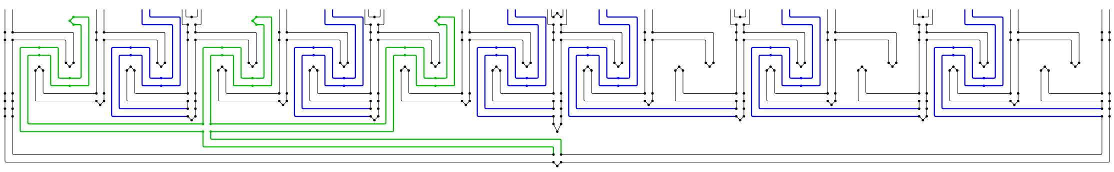
Corollary 4.4. *The problem Ordered Level Planarity is $\mathcal{NP}$-hard even when restricted to 5-connected graphs.*



(a)



(b)



(c)

Fig. 4.6: The clause (a), pipe (b) and variable (c) gadget for a cycle graph. The pipe gadget transmits the value false while the variable gadget is configured as true. The constraints on the gates are omitted in the drawings.

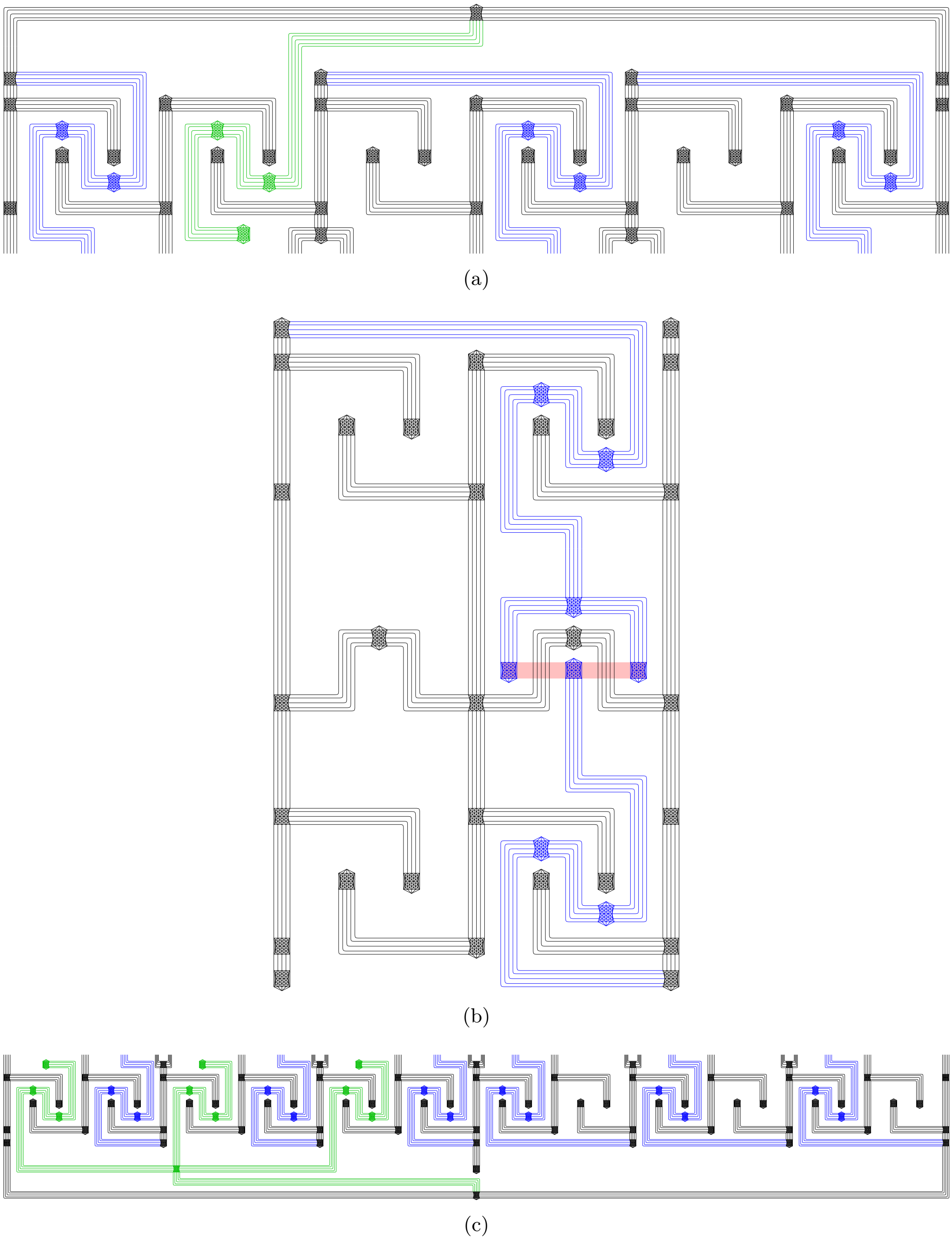


Fig. 4.7: The clause (a), pipe (b) and variable (c) gadget for a 5-connected graph. The pipe gadget transmits the value false while the variable gadget is configured as true. The constraints on the gates are omitted in the drawings, while the constraints between the claw and the tip are visualized as a (red) band.

5 Conclusion

In this thesis, we investigated reductions from Constrained Level Planarity to Ordered Level Planarity that preserve several properties of graphs. We were able to preserve outerplanarity, chordality, and perfectness as well as pathwidth, treedepth, maximum degree, and cycle graphs. We also showed that k-connectivity can be preserved under certain conditions. It would therefore be an intriguing subject for future work to try and find a reduction that maintains k-connectivity unconditionally. Another interesting aspect for a future investigation would be to see if the diameter can be maintained, as we only were able to provide a reduction that increases the diameter by at most 2. It would also be interesting to see if there are reductions that are able to preserve even more properties at once, thereby bringing Constrained Level Planarity and Ordered Level Planarity more closely together. Seeing as Ordered Level Planarity is a special case of several other graph drawing problems, such as Clustered Level Planarity and T-Level Planarity, it would be interesting to see how Constrained Level Planarity relates to them. There is also room to improve the reductions. For example, the reduction that maintains the maximum degree currently greatly increases the size of the graph for a non-constant maximum degree. The reductions could also be used to extend $\mathcal{FPT}$-algorithms for Ordered Level Planarity that are parametrized by one of the maintained properties to Constrained Level Planarity.

We also modified a proof by Brückner and Rutter [BR17] reducing Planar Monotone 3-Satisfiability to Constrained Level Planarity in order to show $\mathcal{NP}$-hardness of Constrained Level Planarity even when restricted to cycle as well as 5-connected graphs. We then used the reductions from this thesis to show that Ordered Level Planarity is also $\mathcal{NP}$-hard when restricted to cycle as well as 5-connected graphs. It remains to see if Constrained Level Planarity and Ordered Level Planarity are $\mathcal{NP}$-hard for other special cases, such as a constant treedepth.

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Würzburg, den 14. Juli 2023

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Antonio Lauerbach