Languages Defined by Recurrent Circuits

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Outline

Examples of Regular Expression

- $L_2 \stackrel{\text{df}}{=} (aa \cup b)^*$
  (words containing *a*-blocks of even length)
- $L_3 \stackrel{\text{df}}{=} (aaa \cup b)^*$
  (. . . of length multiple of three)
- $L_6 \stackrel{\text{df}}{=} L_2 \cap L_3 = (aa \cup b)^* \cap (aaa \cup b)^*$
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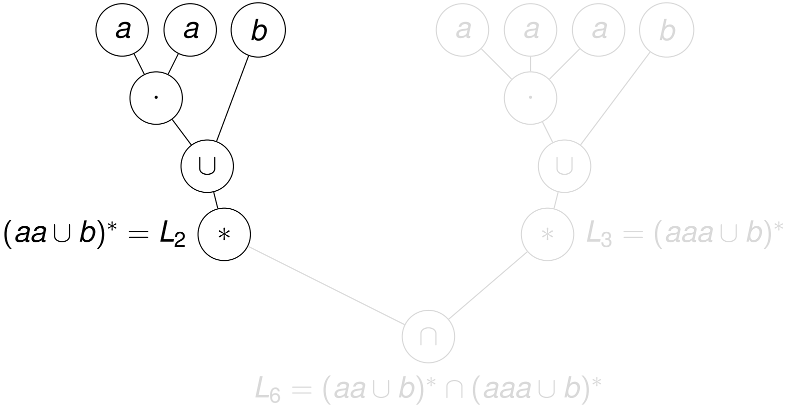

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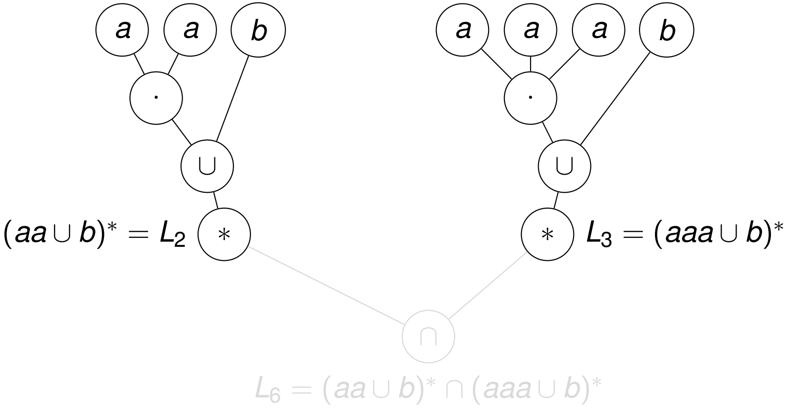

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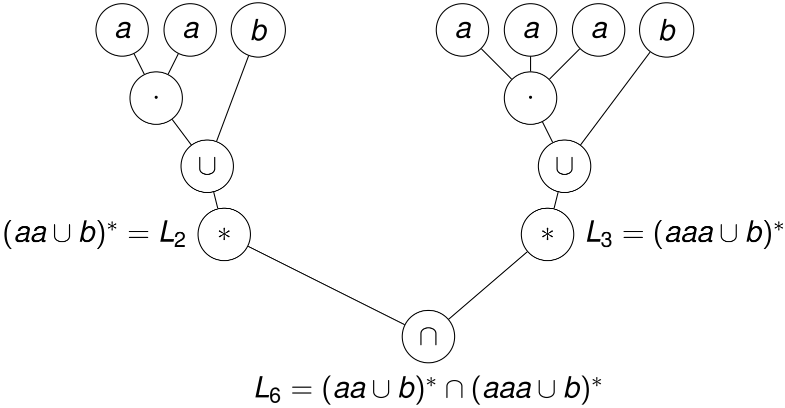

From Regular Expression to Recurrent Circuits

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Definition of the Syntax

For $\mathcal{O} \subseteq \{\cup, \cap, \cdot\}$ a recurrent $\mathcal{O}$-circuit C over the alphabet Σ is

- a directed graph (V, E) with ordered edges
- where to each gate in V is assigned:
  - one operation from $\mathcal{O}$ and
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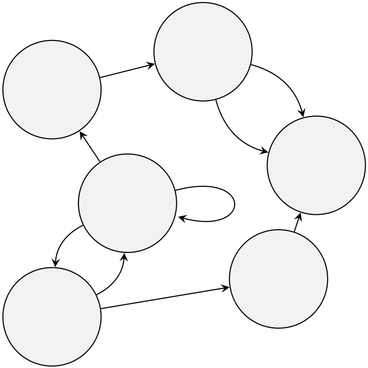

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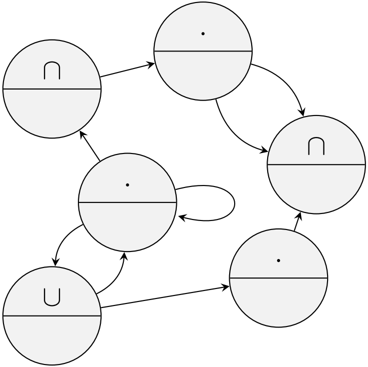

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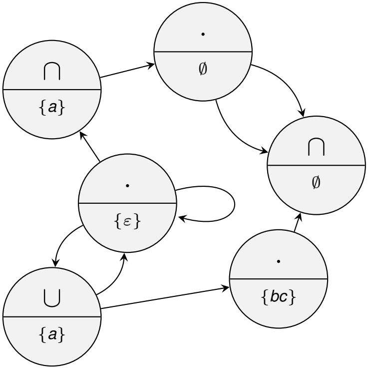

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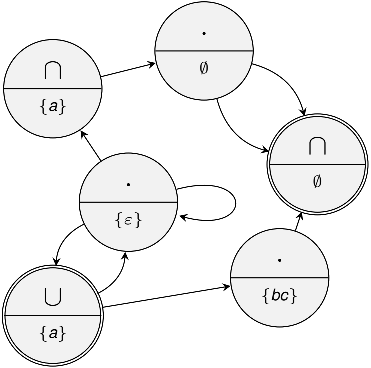

Definition of the Semantics

Let C be a recurrent circuit. For each node v we define:

- $C(v, 0) \stackrel{\text{df}}{=}$ initial set of v
- If $u_1, u_2, \ldots, u_n$ are the ordered predecessors of v and o is the operation of v then

$$C(v, t+1) \stackrel{\text{df}}{=} C(v, t) \cup \begin{cases} \bigcup_{i=1}^{n} C(u_i, t) & \text{if } o = \cup \\ \bigcap_{i=1}^{n} C(u_i, t) & \text{if } o = \cap \\ C(u_1, t) \cdot \ldots \cdot C(u_n, t) & \text{if } o = \cdot \end{cases}$$

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Definition of Recurrent Circuit Classes

Definition

- For a recurrent circuit C with output gates V' we define $L(C) \overset{\mathrm{df}}{=} \bigcup_{v \in V'} C(v)$.
- For any $\mathcal{O} \subseteq \{\cup, \cap, \cdot\}$ we define $\mathrm{RC}(\mathcal{O}) \overset{\mathrm{df}}{=} \{L \mid L = L(C)$ for some recurrent $\mathcal{O}$-circuit $C\}$.

Overview of the Classes

$$\mathrm{CSL}$$

$$\mathrm{RC}(\cup,\cap,\cdot) = \mathrm{CCFL}$$

$$\mathrm{RC}(\cap,\cdot) \qquad \mathrm{RC}(\cup,\cdot) = \mathrm{CFL}$$

$$\mathrm{RC}(\cdot)$$

$$\mathrm{REG}$$

$$\mathrm{RC}(\cap) = \mathrm{RC}(\cup) = \mathrm{RC}(\cup,\cap)$$

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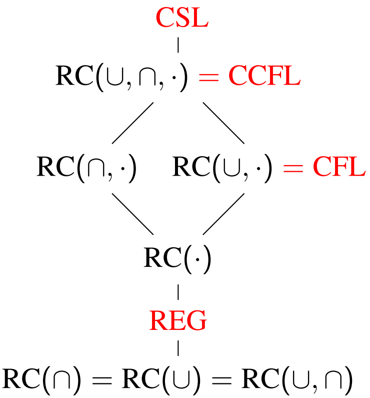

First Observations

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For every gate v of the circuit C with operation o and predecessors $u_1, u_2, \ldots, u_n$:

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Equivalence to Context Free Languages

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$\mathrm{RC}(\cup, \cdot) = \mathrm{CFL}$

Sketch of Construction.

Gates and nonterminals correspond in a natural way:
Concatenation gate A with predecessors $A_1, A_2, \ldots, A_n$ corresponds to production $A \to A_1 A_2 \ldots A_n$.
Union gate B with predecessors $B_1, B_2, \ldots, B_n$ corresponds to production $B \to B_1 \mid B_2 \mid \cdots \mid B_n$. □

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Conjunctive Context Free Languages (CCFL)

- Defined by Okhotin in 2001.
- Context free grammars with intersection.
- Contain productions of the form $A \to w_1 \& w_2 \& \ldots \& w_n$ with $w_i \in (N \cup \Sigma)^*$.

Definition

$x \in (N \cup \Sigma)^*$ can be derived from $w_1 \& w_2 \& \ldots \& w_n$ iff it can be derived from all w_i, $i = 1, 2, \ldots, n$.

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Facts about CCFL by Okhotin

- Efficient parsing algorithm: Adaption of CYK, also $O(n^3)$.
- This implies $\mathrm{CCFL} \subsetneq \mathrm{P}$.
- There is a P-complete language in CCFL.
- $\Gamma_\cap(\mathrm{CFL}) \subsetneq \mathrm{CCFL} \subseteq \mathrm{CSL}$
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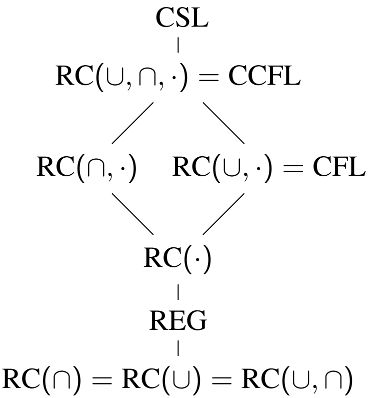

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Nice (Non-)Closure Proofs for $\mathrm{RC}(\mathcal{O})$

Theorem

$\mathrm{RC}(\cdot)$ *is not closed under intersection with regular languages.*

Sketch of Proof.

Theorem of Chomsky and Schützenberger:
$L \in \mathrm{CFL} \iff L = h(D_n^* \cap R)$, for some homomorphism h, $n \in \mathbb{N}$ and $R \in \mathrm{REG}$
(D_n^* is the (one-sided) Dyck language with n bracket-types).
Since $D_n^* \in \mathrm{RC}(\cdot)$, this would imply $\mathrm{RC}(\cdot) = \mathrm{CFL}$, but $\mathrm{RC}(\cdot) \subsetneq \mathrm{CFL}$. □

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Thus $\mathrm{CFL} \subseteq \Gamma_{h,\cap}(\mathrm{RC}(\cdot)) \Rightarrow \Gamma_{h,\cap}(\mathrm{CFL}) \subseteq \Gamma_{h,\cap}(\mathrm{RC}(\cdot))$
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