Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Lehrstuhl für
INFORMATIK I
Algorithmen & Komplexität

Institut für Informatik

Coloring
Mixed Graphs

Antonio Lauerbach

05.09.2025

Basics

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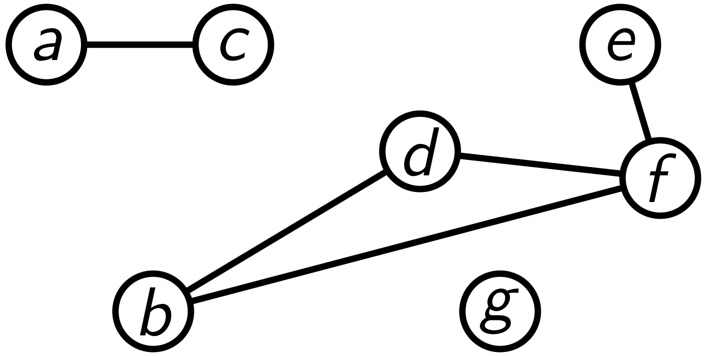
Undirected Graphs

- vertices V
- edges $E \subset \binom{V}{2}$



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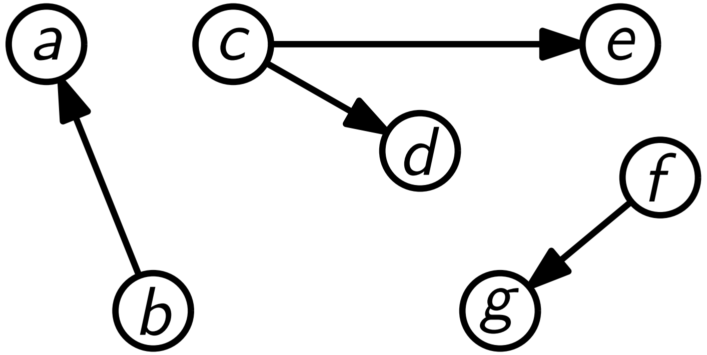

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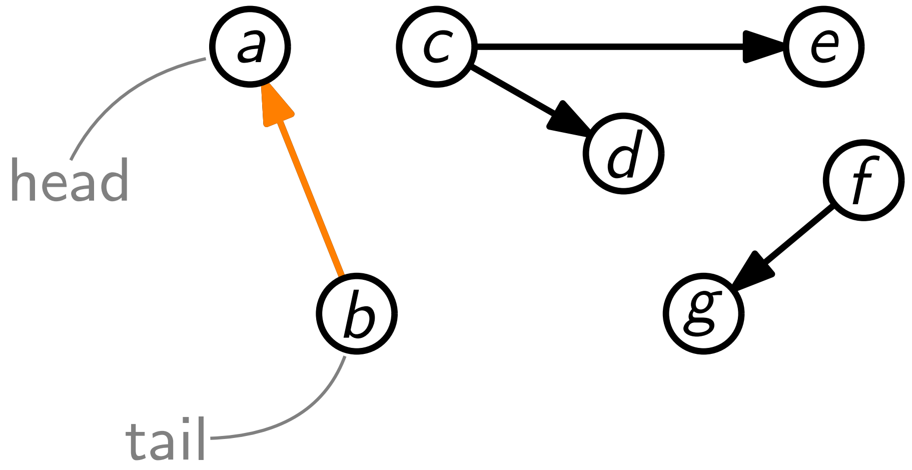

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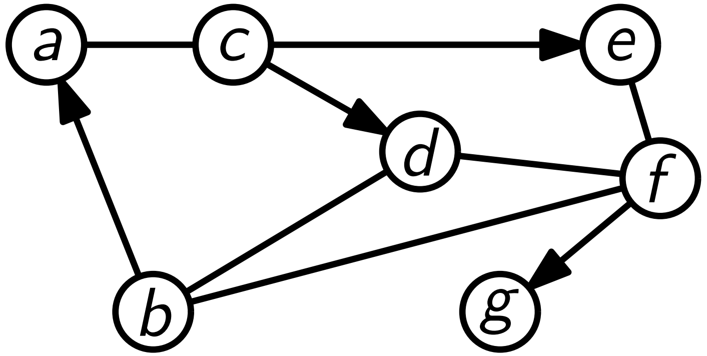

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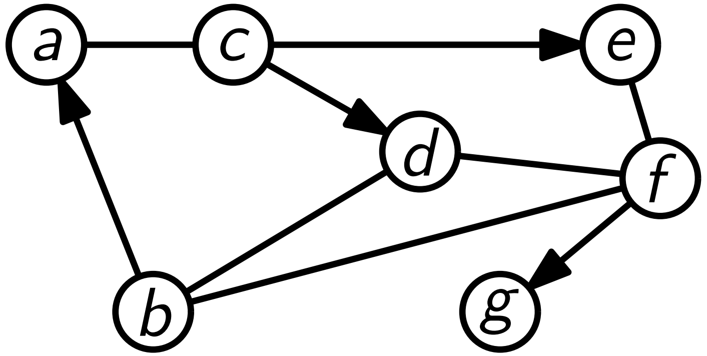

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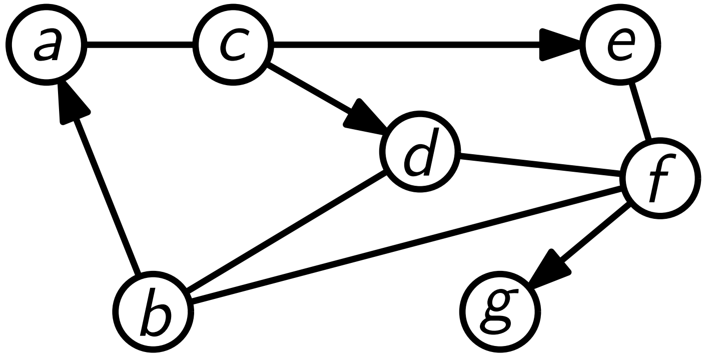

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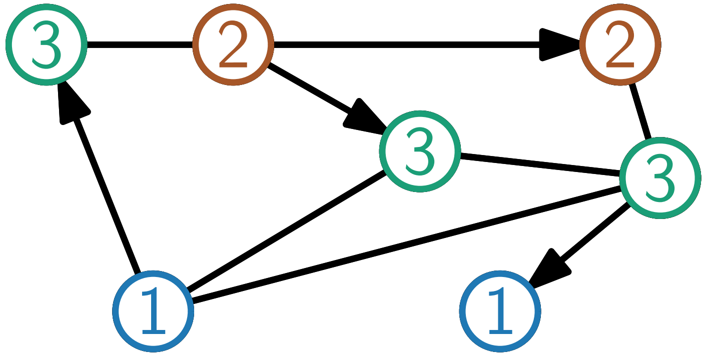

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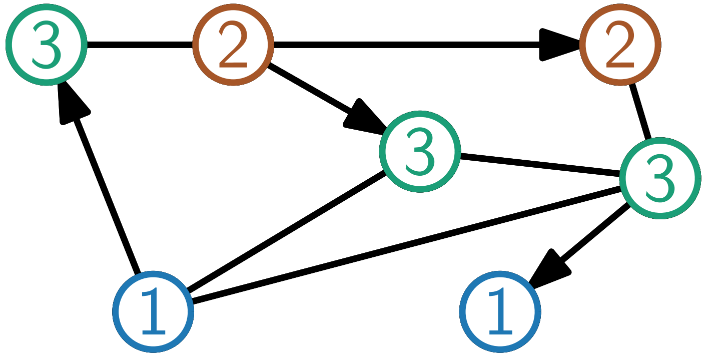

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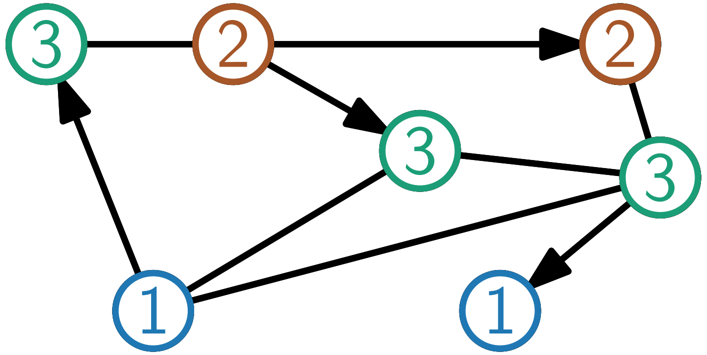

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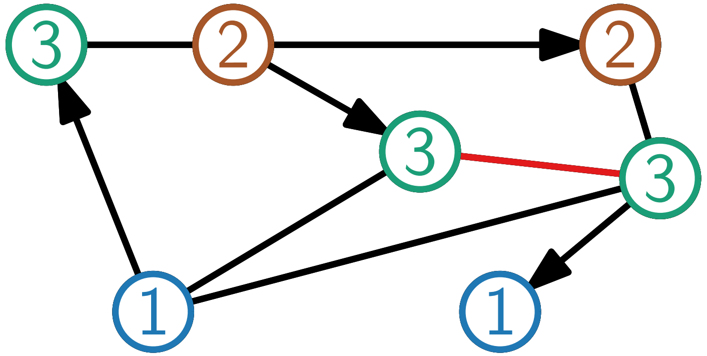

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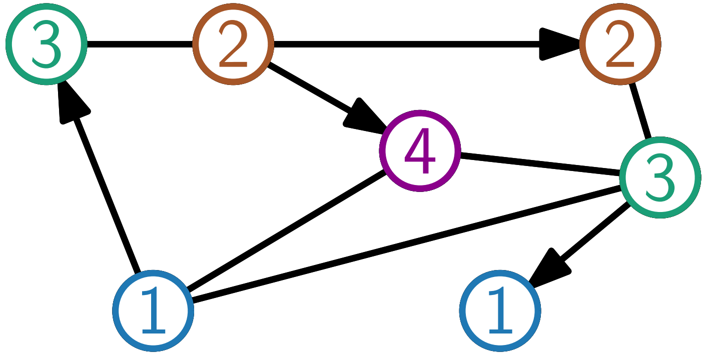

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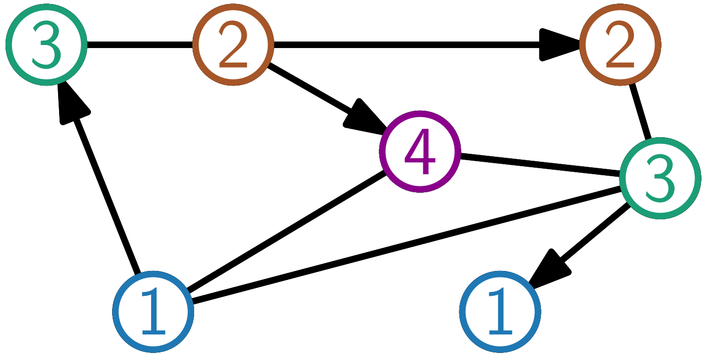

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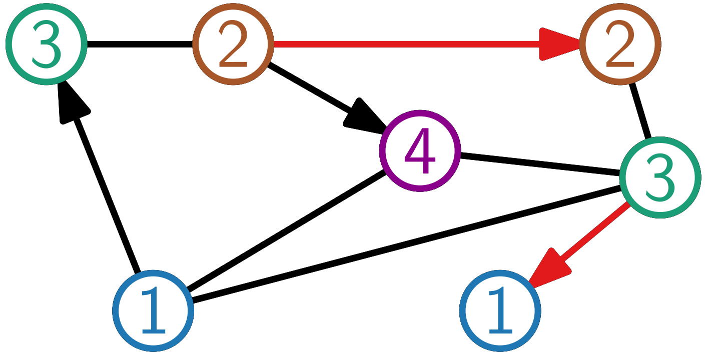

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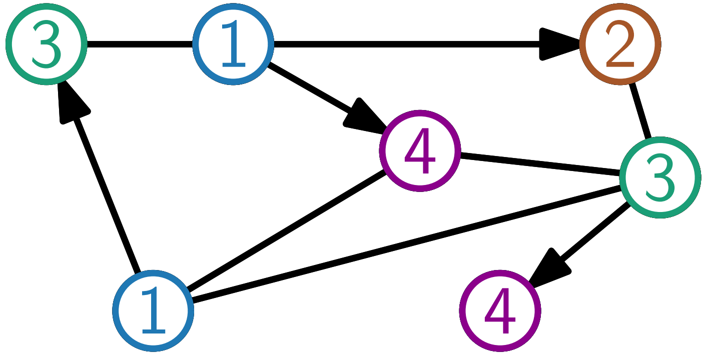

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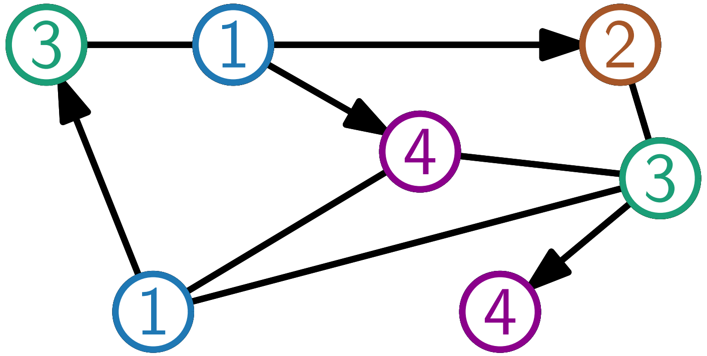

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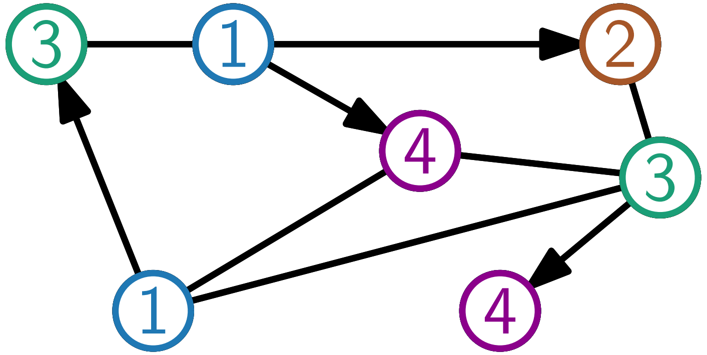

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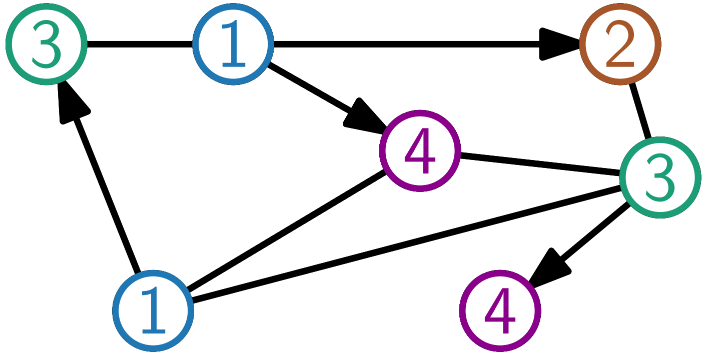

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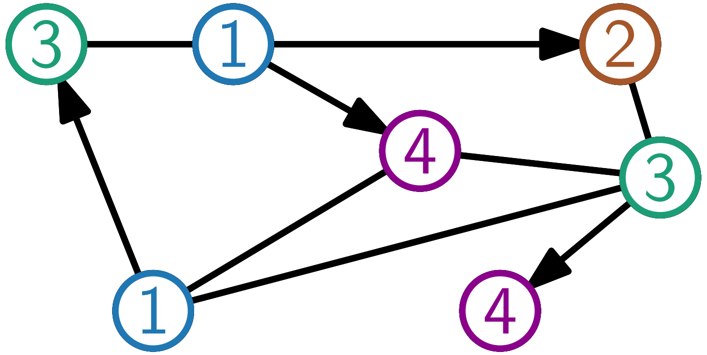

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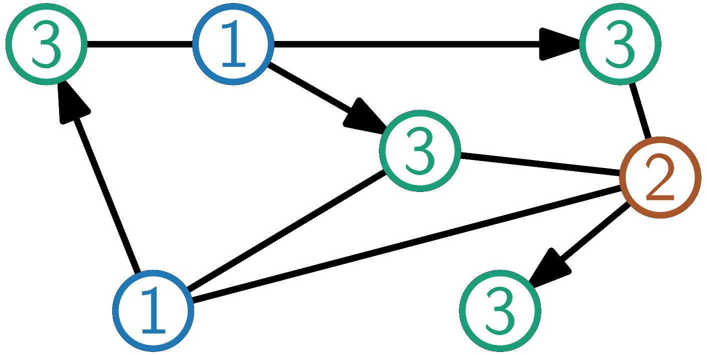

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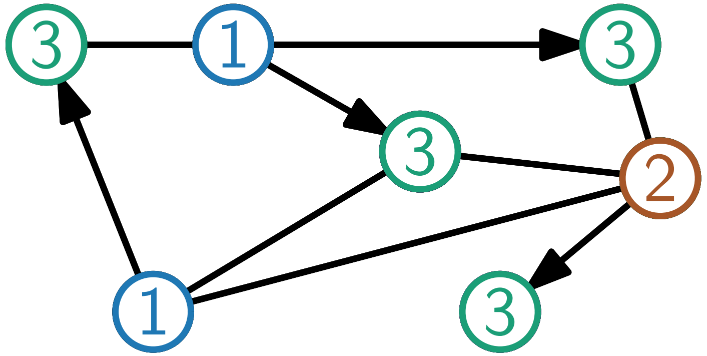

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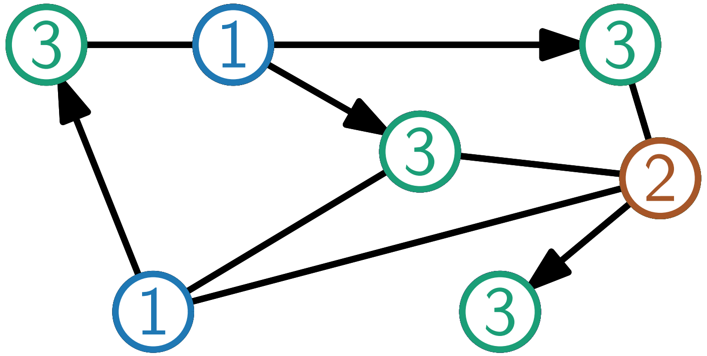

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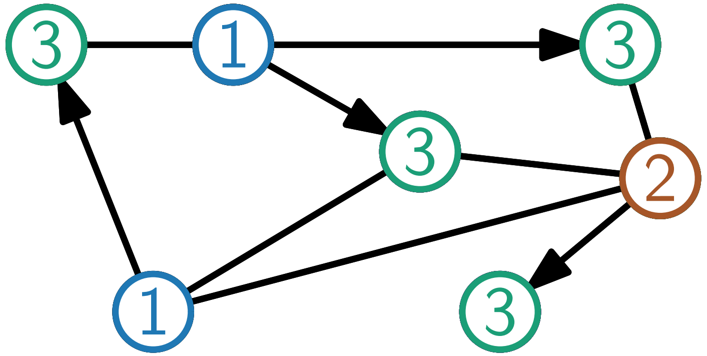

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Applications:
scheduling, frequency assignment, graph drawing

Contribution

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Related Work

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Single-Exponential-Time Algorithms

Problem	Time	Space
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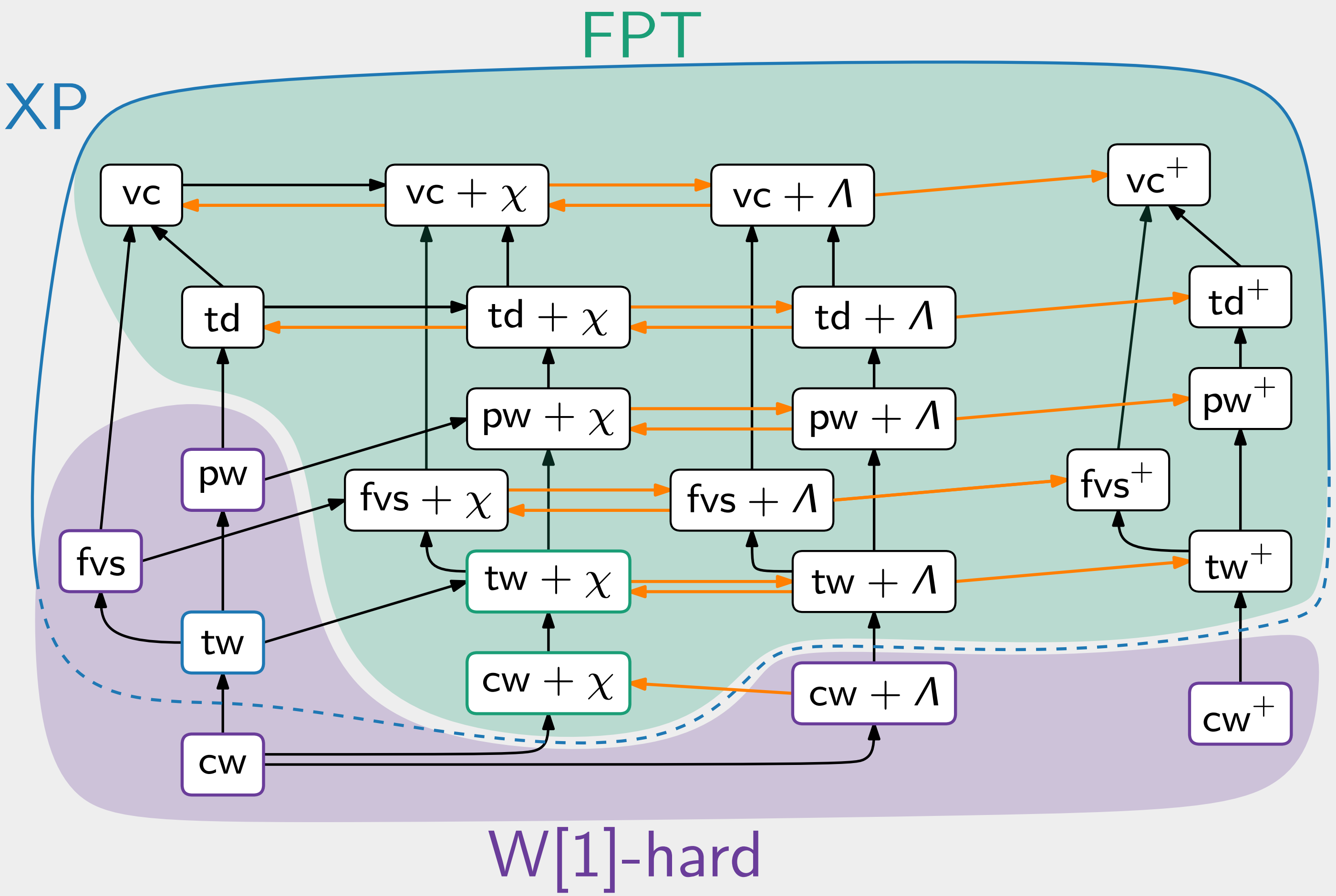

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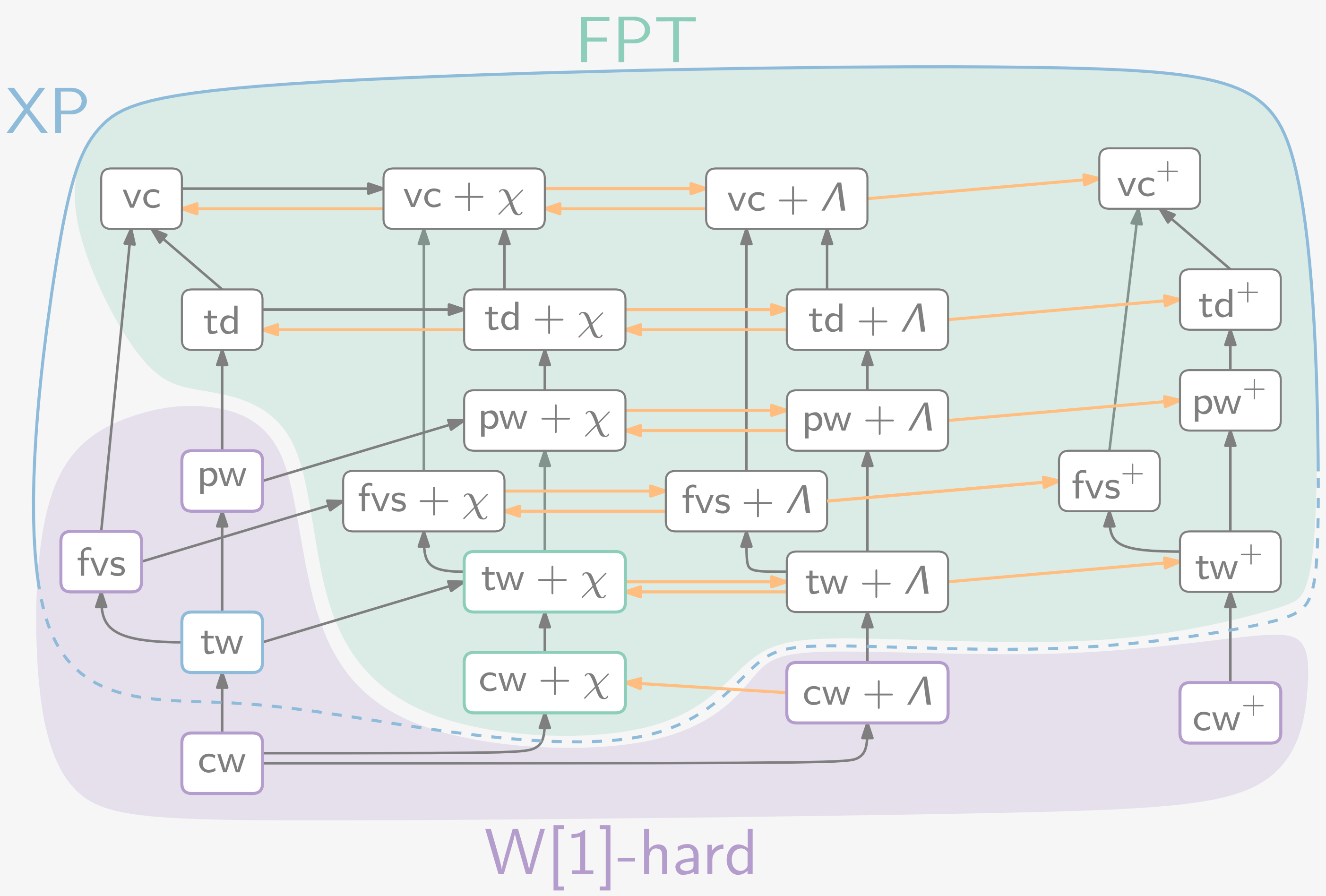

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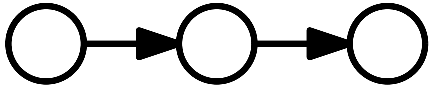

Properties – Cycles and Subgraphs

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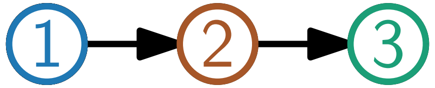

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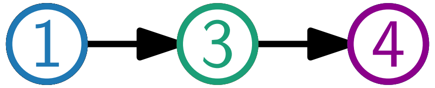

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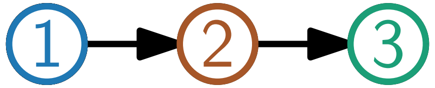

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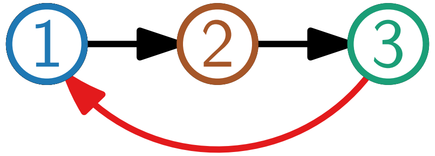

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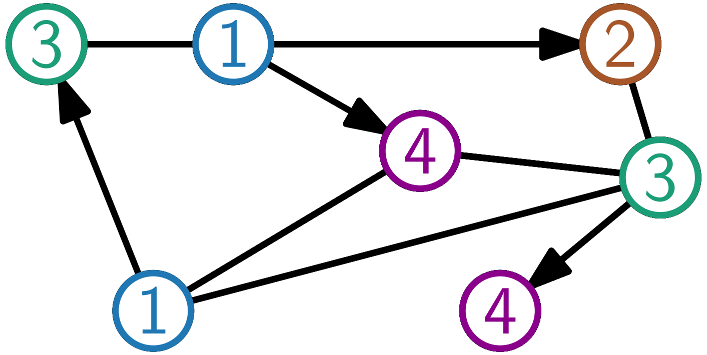

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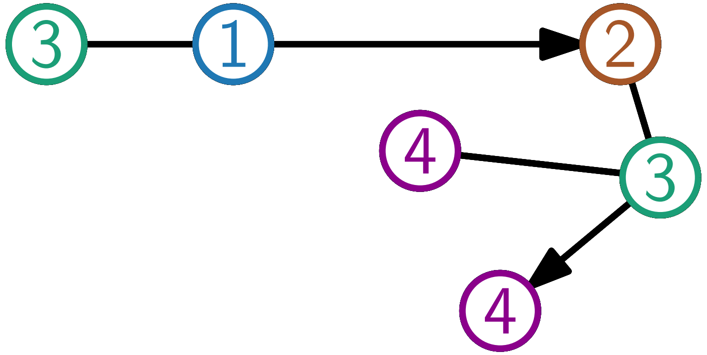

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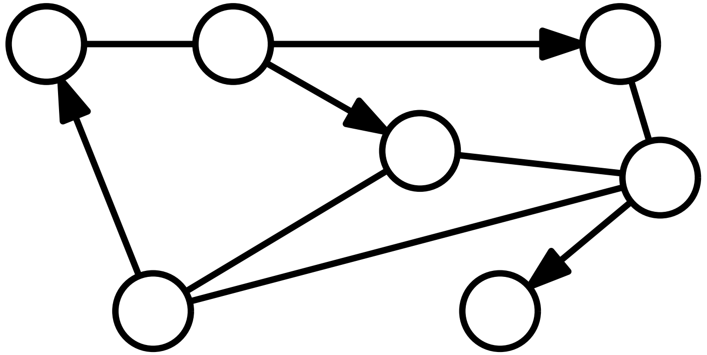

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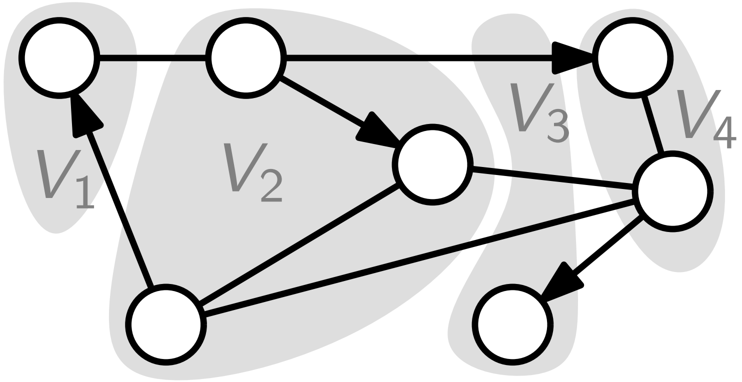

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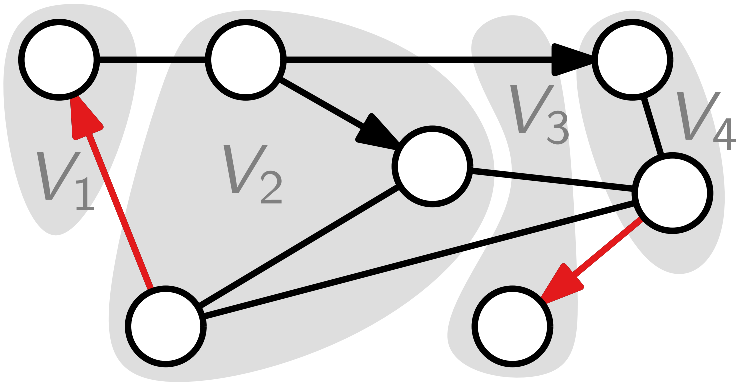

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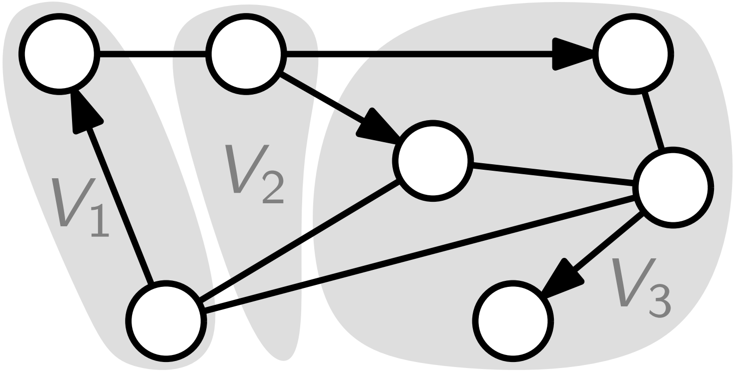

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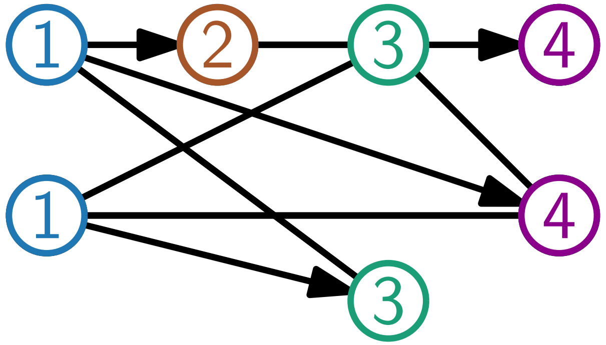

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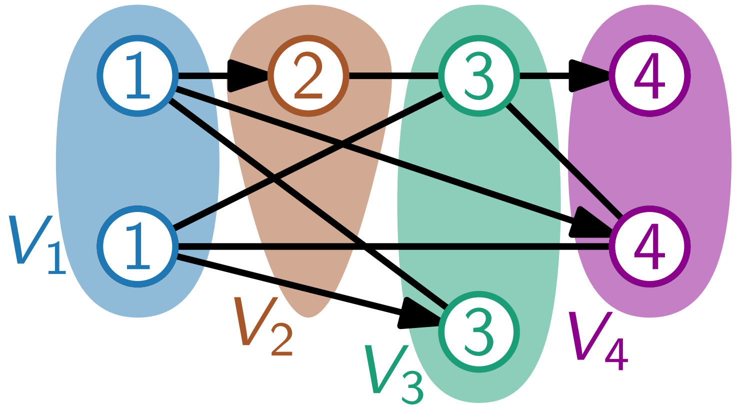

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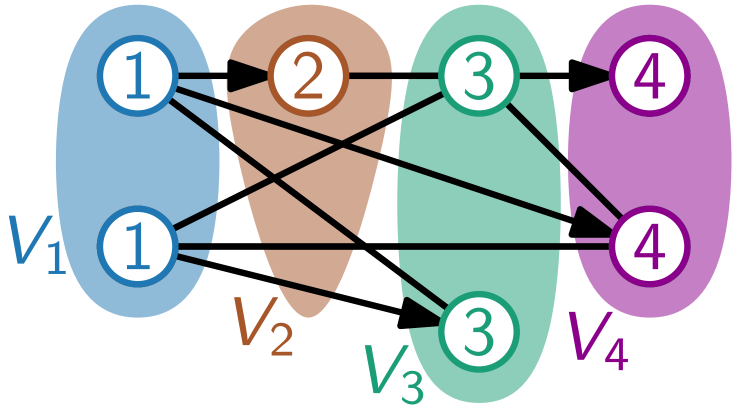

Observation:

Color classes of proper colorings are equivalent to proper partitions into independent sets

Properties – Partitions and Color Classes

Lemma

For every mixed graph G and proper partition $\langle V_1, \dots, V_k \rangle$ it holds that: $\chi(G) \leq \chi(G[V_1]) + \cdots + \chi(G[V_k])$

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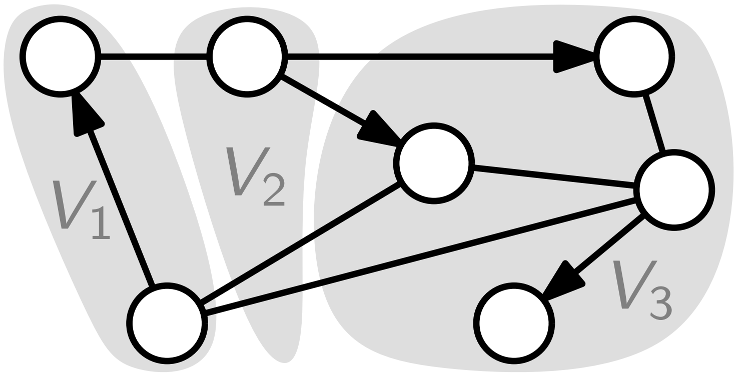

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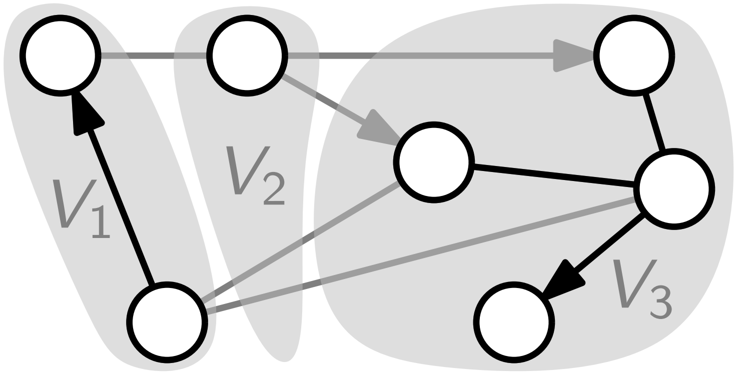

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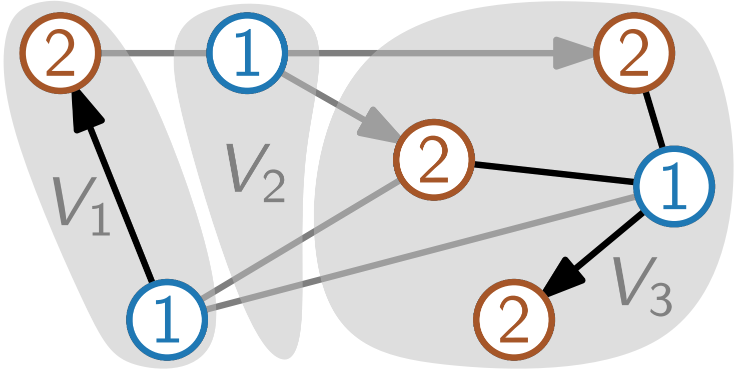

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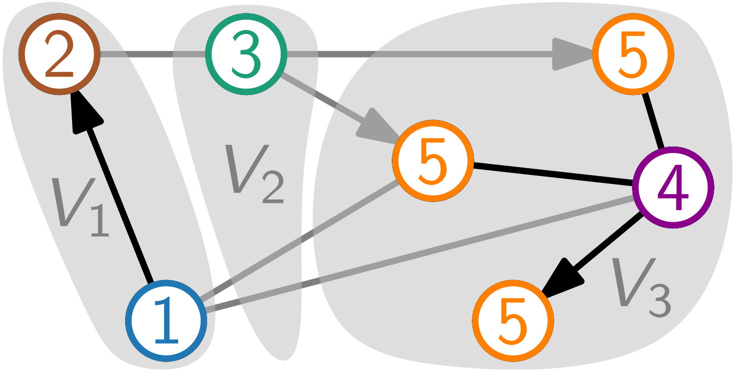

Properties – Partitions and Color Classes

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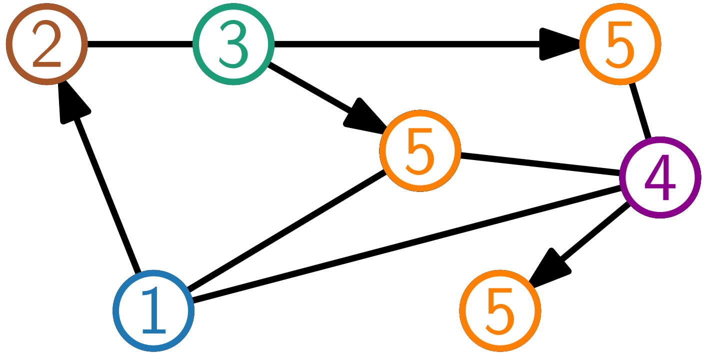
For every mixed graph G and proper partition $\langle V_1, \ldots, V_k \rangle$ it holds that: $\chi(G) \leq \chi(G[V_1]) + \cdots + \chi(G[V_k])$



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There exists an optimal coloring where the first color class is a maximal independent set

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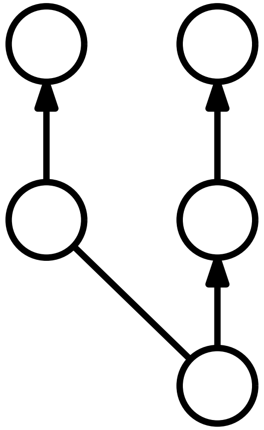

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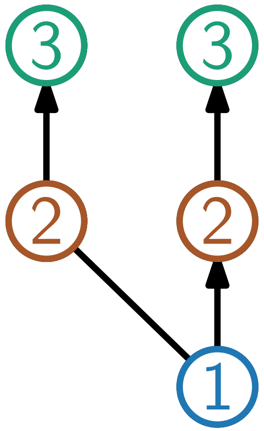

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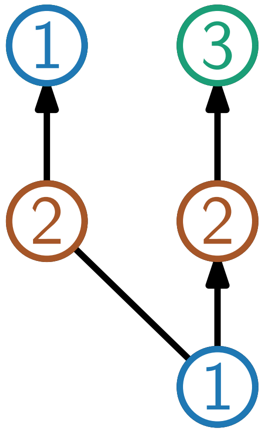

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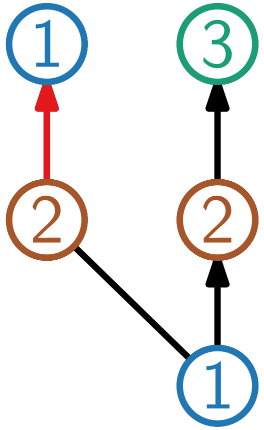

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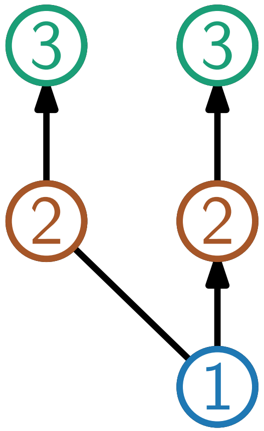

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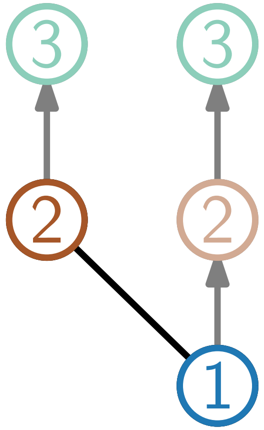

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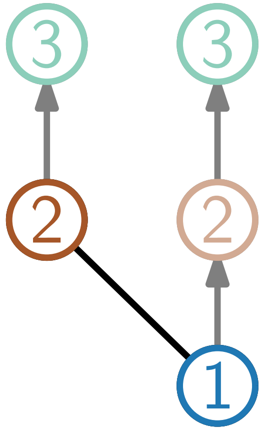

Consider $G[\mathcal{V}^\circ]$

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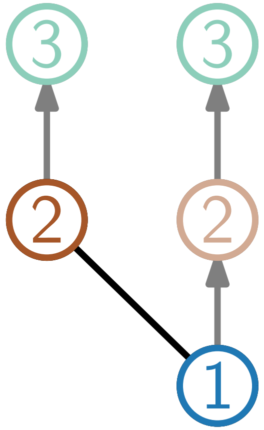

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vertices with indeg 0 (i.e. without incoming arcs)

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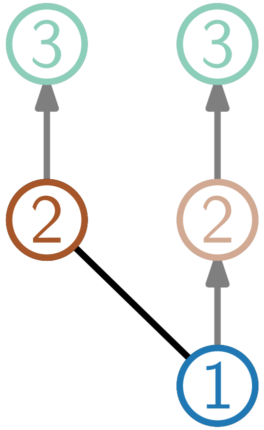

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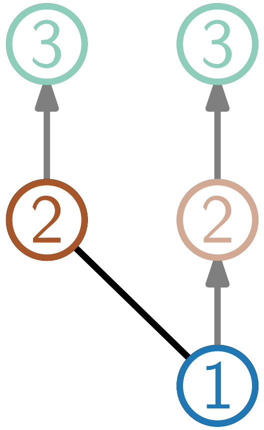

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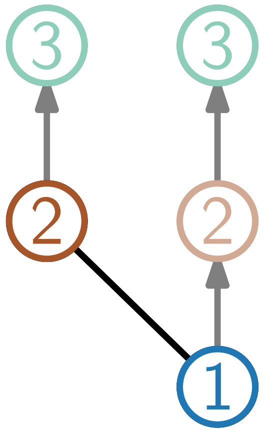

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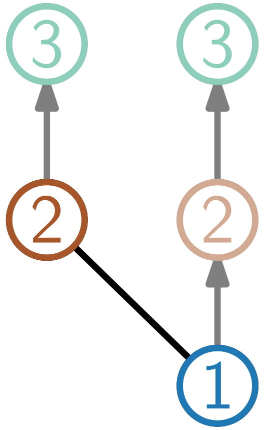

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It holds that:

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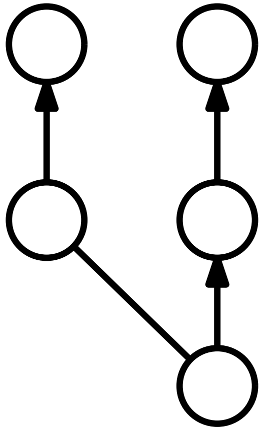

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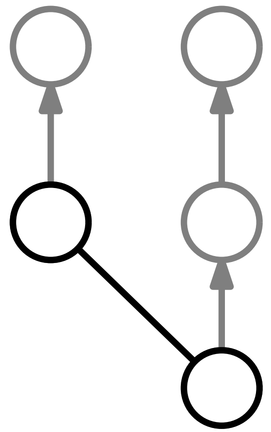

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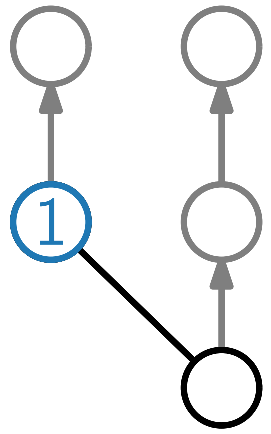

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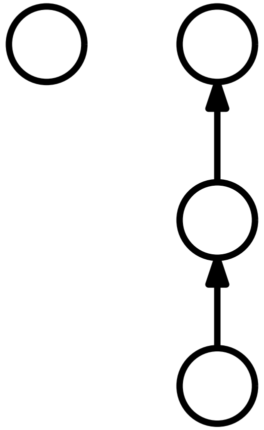

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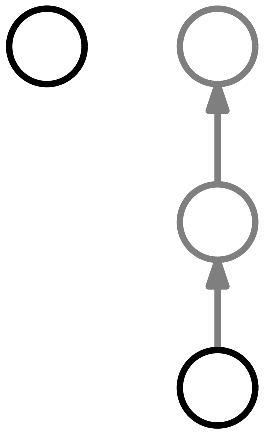

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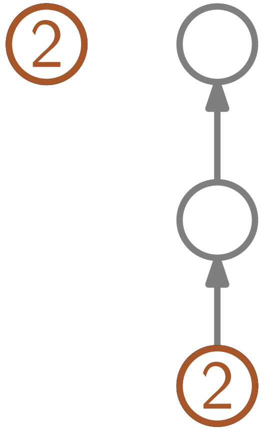

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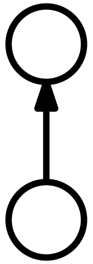

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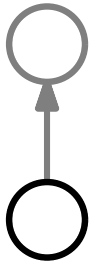

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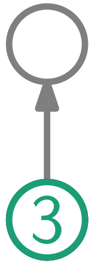

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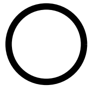

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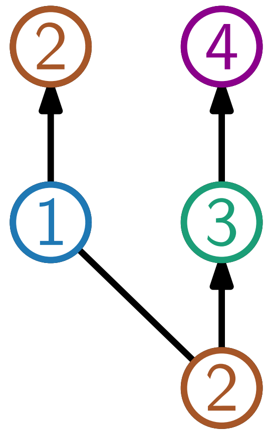

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Algo [Law76]: dynamic program to iteratively construct i-colorable subgraphs using canonical colorings

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```
LawlerColoring($G$):
    $L_0 = \{\emptyset\}$
    for $i = 1$ to $n$ do
        $L_i = \emptyset$
        foreach $S \in L_{i-1}$ do
            foreach $I \in \mathrm{MIS}(G - S)$ do
                $L_i \leftarrow L_i \cup \{S \cup I\}$
                if $S \cup I = V$ then
                    return $i$
    return 0
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Theorem

MixedColoring can be solved in $\mathcal{O}(2.44225^n)$ time and $\mathcal{O}(2^n n)$ space

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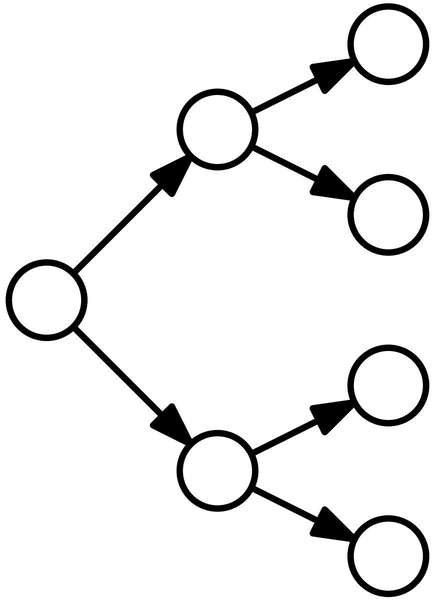

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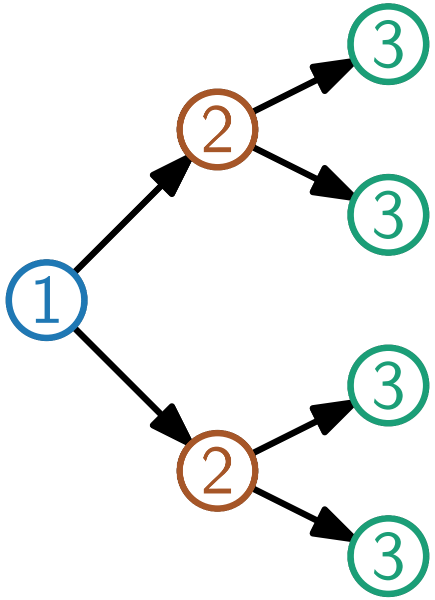

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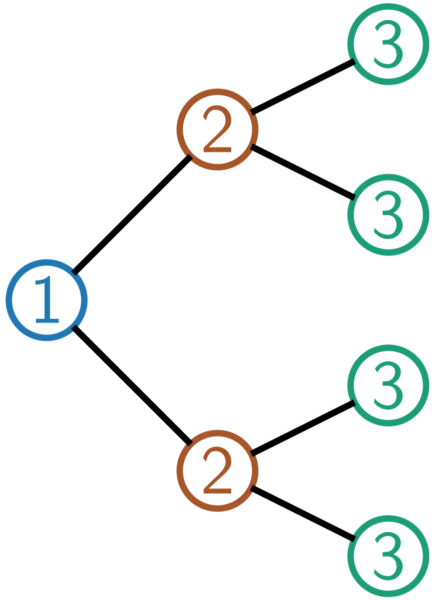

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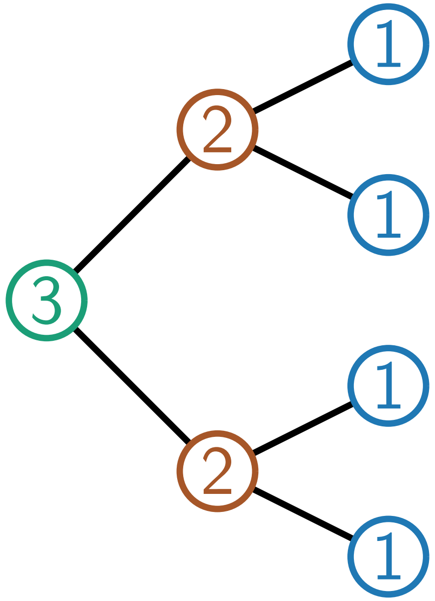

Generalization Problems – Improvements of Lawler

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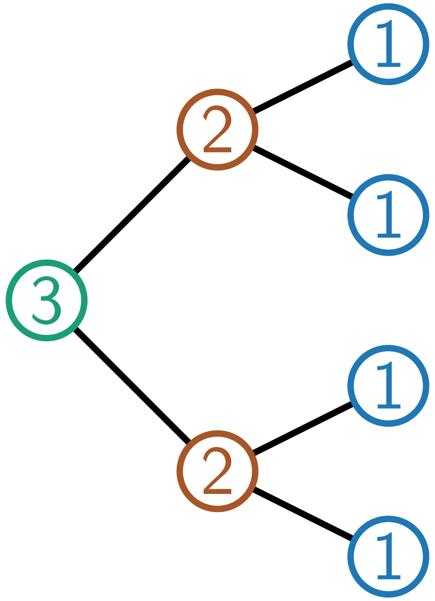

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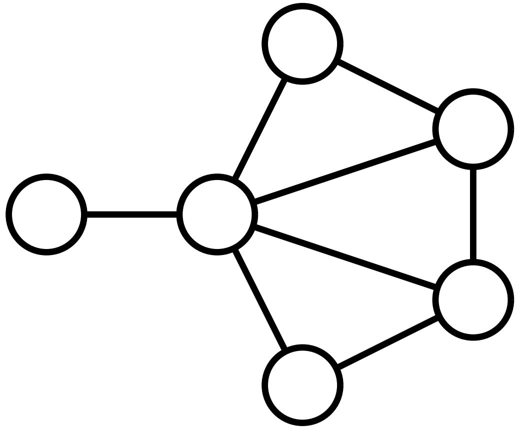

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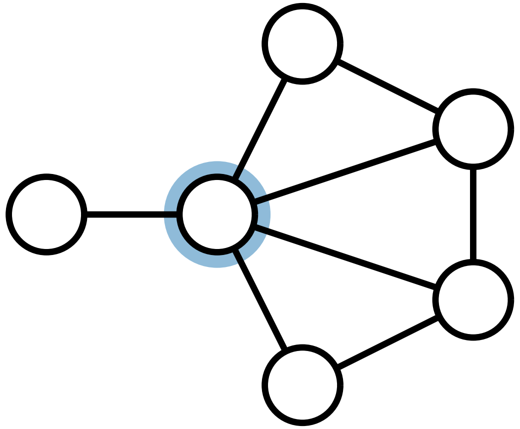

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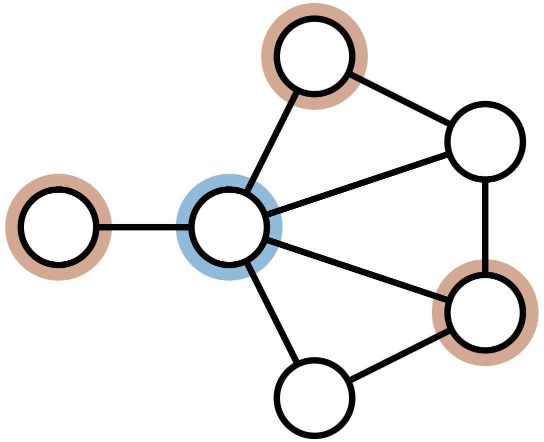

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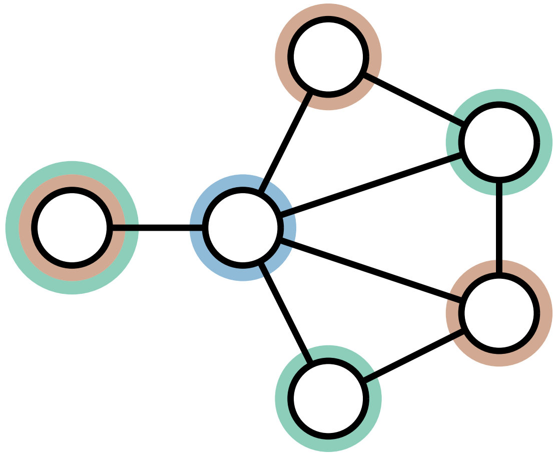

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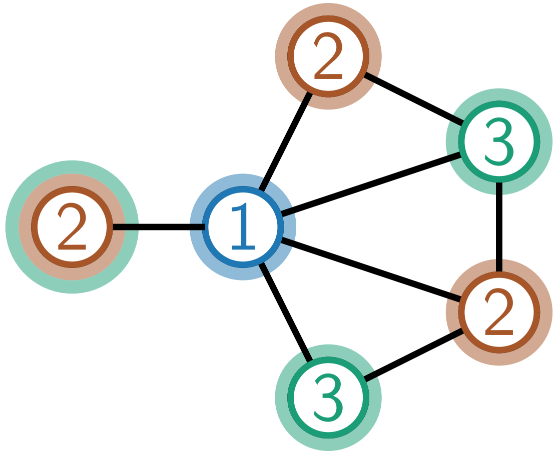

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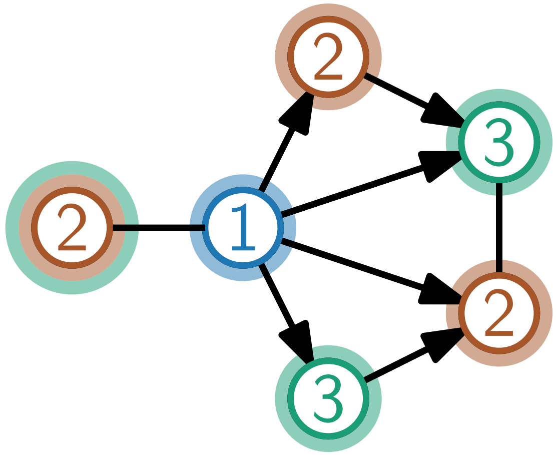

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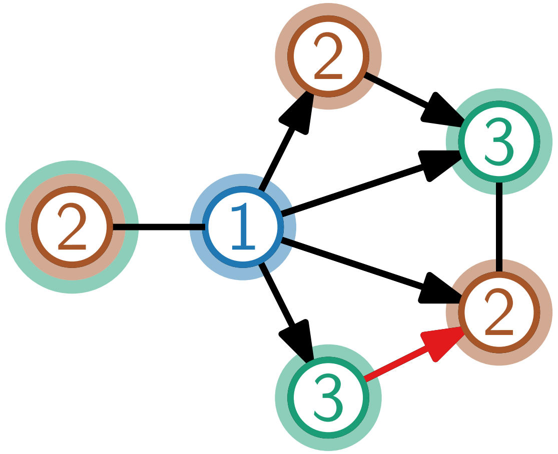

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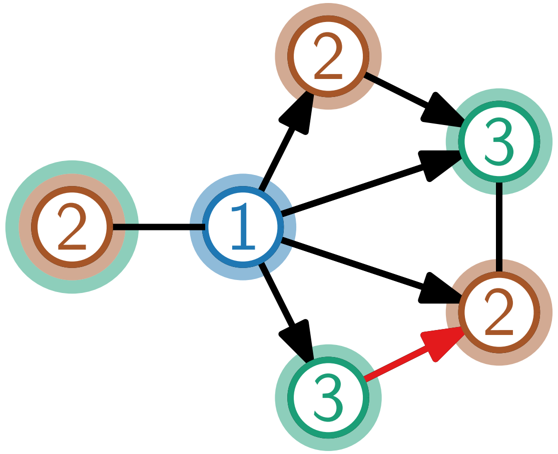

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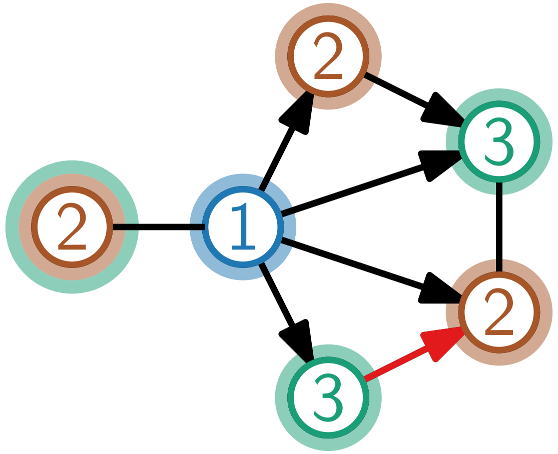

PolySpace

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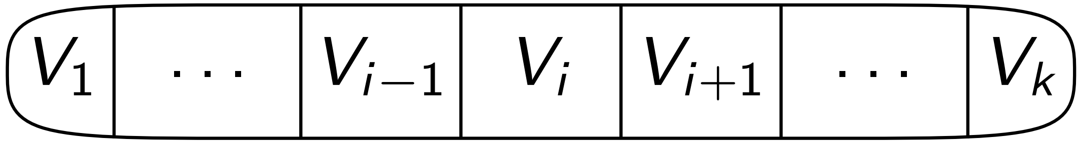

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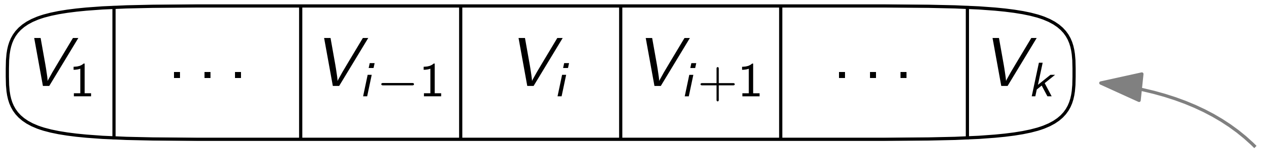

color classes of optimal coloring

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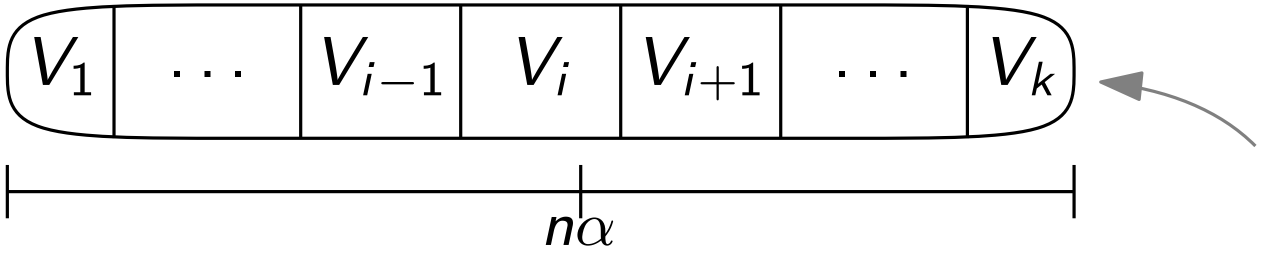

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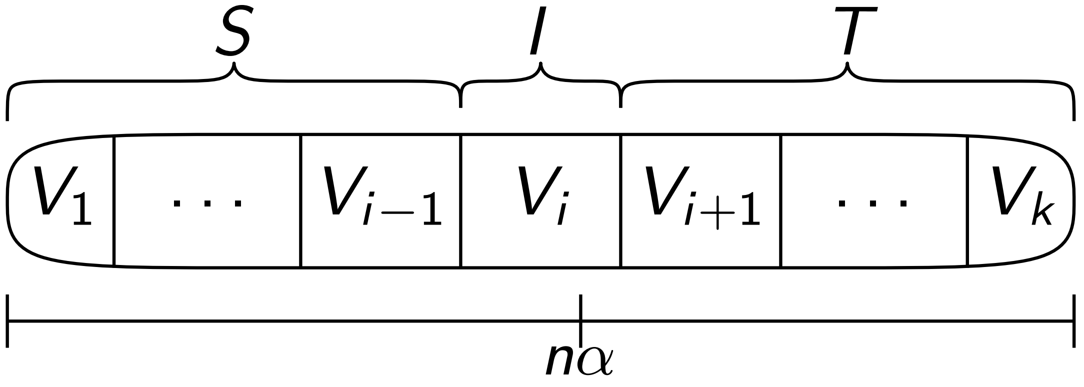

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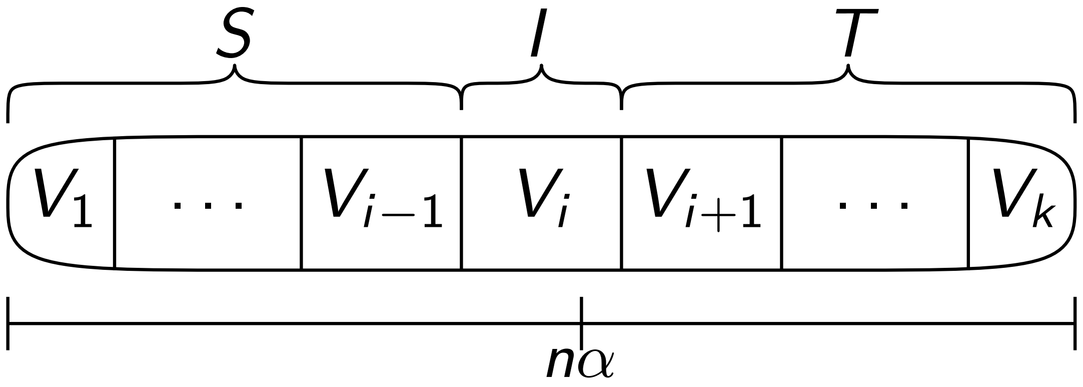

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$$\chi(G) = \min_{\substack{\text{proper partition } \langle S, I, T\rangle : \\ |S| < n\alpha,\, I \in \mathrm{IS}(G),\, |T| \leq n(1-\alpha)}} \chi(G[S]) + 1 + \chi(G[T])$$

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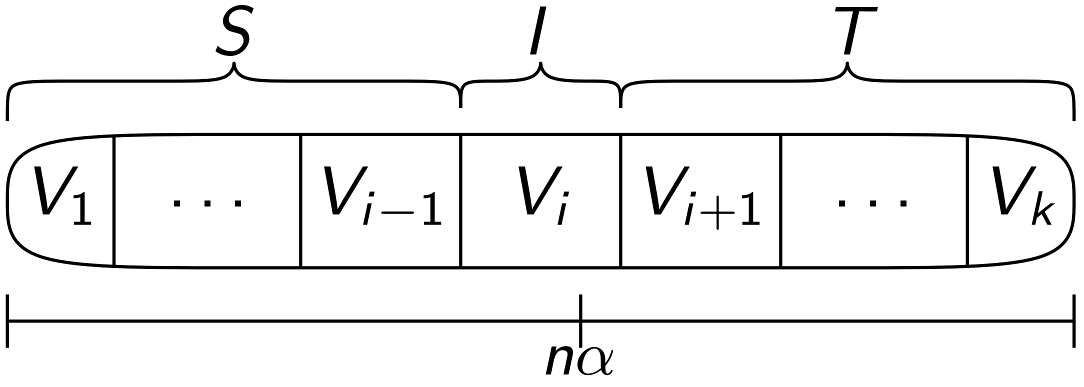

color classes of opt. canonical coloring

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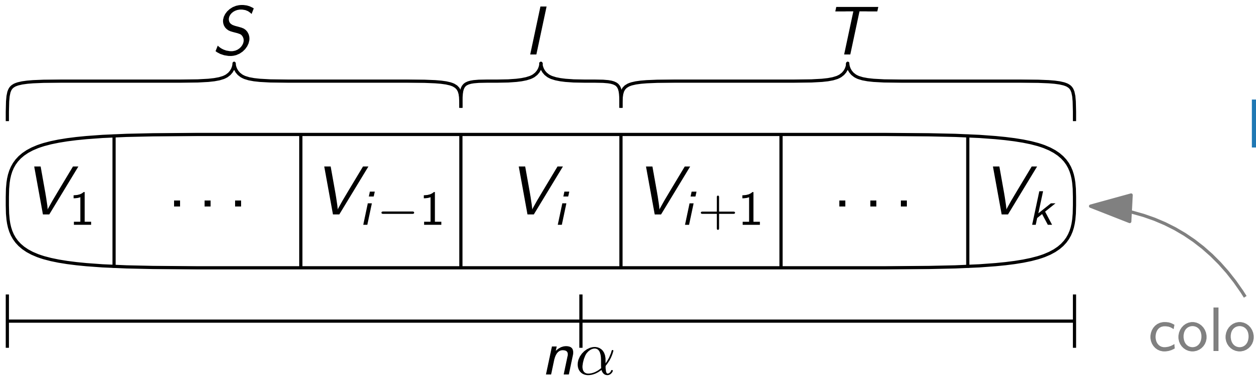

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Recall: $I = V_i \in \mathrm{MIS}^\circ(G - (V_1 \cup \ldots V_{i-1}))$

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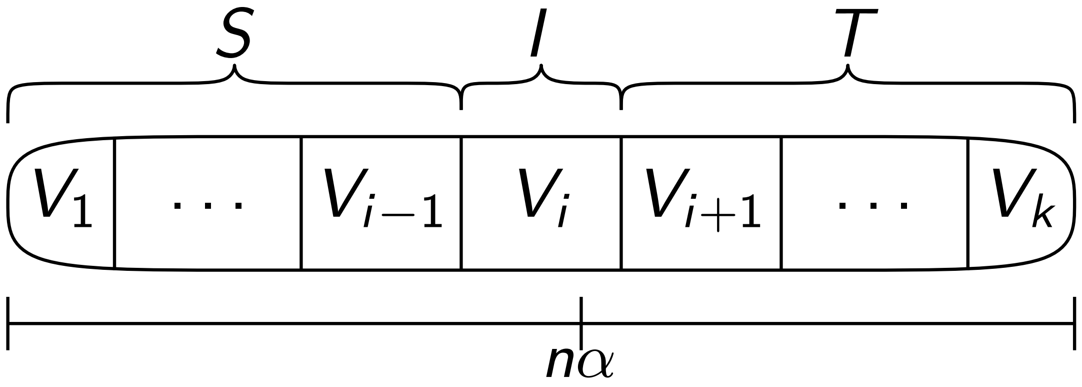

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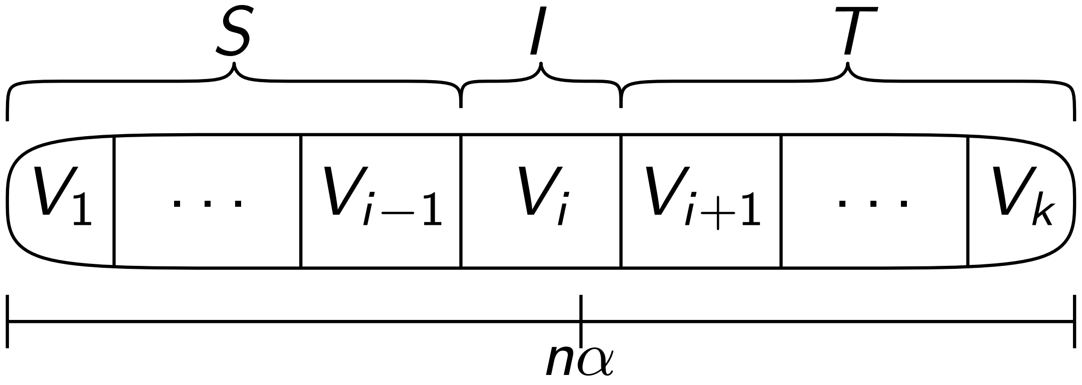

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poly via bitvector

poly via algo
from [Epp01]

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Space:

- poly for each node

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Space:

- poly for each node
- recursion depth $\mathcal{O}(\log n)$

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Space:

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- recursion depth $\mathcal{O}(\log n)$

$\Longrightarrow$ poly space overall

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Runtime:

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← poly for each step via bitvector

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poly for each step via bitvector

poly for each step via algo from [Epp01]

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poly for each step via bitvector

poly for each step via algo from [Epp01]

Runtime:

- poly in #nodes

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poly for each step via algo from [Epp01]

Runtime:

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- recursion depth $\mathcal{O}(\log n)$

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poly for each step via bitvector

poly for each step via algo from [Epp01]

Runtime:

- poly in #nodes
- recursion depth $\mathcal{O}(\log n)$

$\Longrightarrow$ poly in #leaves

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    foreach $S \subset V(G)$ with $|S| \leq \lceil n\alpha - 1 \rceil$ and $d_G^-(S) = 0$ do
        $x_1 =$ PolySpaceColor($G[S]$)
        foreach $I \in \mathrm{MIS}^\circ(G - S)$ with $|I| \geq \lceil n\alpha - |S| \rceil$ do
            $T = V(G) \setminus (S \cup I)$
            $x_2 =$ PolySpaceColor($G[T]$)
            $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

Runtime: $\mathcal{O}^*(L(n))$

PolySpace

```
PolySpaceColor($G$):
    if $G = \emptyset$ then
        return 0
    $x \leftarrow \infty$
    foreach $S \subset V(G)$ with $|S| \leq \lceil n\alpha - 1 \rceil$ and $d_G^-(S) = 0$ do
        $x_1 = $ PolySpaceColor($G[S]$)
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            $T = V(G) \setminus (S \cup I)$
            $x_2 = $ PolySpaceColor($G[T]$)
            $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

Runtime: $\mathcal{O}^*(L(n))$

max. #leaves in recursion tree of graph with n vertices

PolySpace

```
PolySpaceColor($G$):
    if $G = \emptyset$ then
        return 0
    $x \leftarrow \infty$
    foreach $S \subset V(G)$ with $|S| \leq \lceil n\alpha - 1 \rceil$ and $d_G^-(S) = 0$ do
        $x_1 =$ PolySpaceColor($G[S]$)
        foreach $I \in \mathrm{MIS}^\circ(G - S)$ with $|I| \geq \lceil n\alpha - |S| \rceil$ do
            $T = V(G) \setminus (S \cup I)$
            $x_2 =$ PolySpaceColor($G[T]$)
            $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

Runtime: $\mathcal{O}^*(L(n))$

$L(0) = 1$

PolySpace

```
PolySpaceColor($G$):
    if $G = \emptyset$ then
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    $x \leftarrow \infty$
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        $x_1 =$ PolySpaceColor($G[S]$)
        foreach $I \in \mathrm{MIS}^\circ(G - S)$ with $|I| \geq \lceil n\alpha - |S| \rceil$ do
            $T = V(G) \setminus (S \cup I)$
            $x_2 =$ PolySpaceColor($G[T]$)
            $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

Runtime: $\mathcal{O}^*(L(n))$

$L(0) = 1$

For $n \geq 1$:

PolySpace

```
PolySpaceColor($G$):
    if $G = \emptyset$ then
        return 0
    $x \leftarrow \infty$
    for $i = 0$ to $\lceil n\alpha - 1 \rceil$ do
        foreach $S \subset \binom{V(G)}{i}$ with $d_G^-(S) = 0$ do
            $x_1 = $ PolySpaceColor($G[S]$)
            for $\ell = \lceil n\alpha - i \rceil$ to $n - i$ do
                foreach $I \in \mathrm{MIS}^\circ(G - S, \ell)$ do
                    $T = V(G) \setminus (S \cup I)$
                    $x_2 = $ PolySpaceColor($G[T]$)
                    $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

PolySpace

```
PolySpaceColor($G$):
    if $G = \emptyset$ then
        return $0$
    $x \leftarrow \infty$
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            for $\ell = \lceil n\alpha - i \rceil$ to $n - i$ do
                foreach $I \in \mathrm{MIS}^\circ(G - S, \ell)$ do
                    $T = V(G) \setminus (S \cup I)$
                    $x_2 = \texttt{PolySpaceColor}(G[T])$
                    $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

$$L(n) \leq \sum_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} \cdot \left[L(i) + \sum_{\ell = \lceil n\alpha - i \rceil}^{n-i} \mu(n-i, \ell) \cdot L(n-i-\ell) \right] \text{ for } n \geq 1$$

PolySpace

```
PolySpaceColor($G$):
    if $G = \emptyset$ then
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    $x \leftarrow \infty$
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                    $x = \min\{x, x_1 + 1 + x_2\}$
    return $x$
```

max. #MIS of size ℓ in graph with $n - i$ vertices

$$L(n) \leq \sum_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} \cdot \left[L(i) + \sum_{\ell = \lceil n\alpha - i \rceil}^{n-i} \mu(n - i, \ell) \cdot L(n - i - \ell) \right] \text{ for } n \geq 1$$

PolySpace

Goal: $L(n) \leq c^n$ for $c > 1$

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

max. depth of recursion tree on graphs with n vertices

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

Base Case $(n = 0)$: $L(0) = 1 \leq c^0 \cdot (2 \cdot 0^2)^0$

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

IS $(n \geq 1)$:

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

IS $(n \geq 1)$:

$$L(n) \leq \sum_{i=0}^{\lceil n\alpha - 1 \rceil} \tbinom{n}{i} \cdot \left[L(i) + \sum_{\ell = \lceil n\alpha - i \rceil}^{n-i} \mu(n-i, \ell) \cdot L(n-i-\ell) \right]$$

PolySpace

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$$\leq n^2 \cdot \left[\max_{i=0}^{\lceil n\alpha-1\rceil} \binom{n}{i} L(i) + \max_{i=\lfloor n(1-\alpha)+1\rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} \left(\mu(i,\ell) \cdot L(i-\ell)\right) \right]$$

PolySpace

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using induction hypothesis: $L(m) \leq c^m (2m^2)^{D(m)}$ for $m < n$

PolySpace

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$$\leq n^2 \cdot \left[\max_{i=0}^{\lceil n\alpha - 1\rceil} \binom{n}{i} L(i) + \max_{i=\lfloor n(1-\alpha)+1\rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} (\mu(i,\ell) \cdot L(i-\ell)) \right]$$

using induction hypothesis: $L(m) \leq c^m (2m^2)^{D(m)}$ for $m < n$

$$\leq n^2 (2n^2)^{D(n)-1} \left[\max_{i=0}^{\lceil n\alpha - 1\rceil} \binom{n}{i} c^i + \max_{i=\lfloor n(1-\alpha)+1\rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} (\mu(i,\ell) \cdot c^{i-\ell}) \right]$$

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

IS $(n \geq 1)$:

$$L(n) \leq \sum_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} \cdot \left[L(i) + \sum_{\ell = \lceil n\alpha - i \rceil}^{n-i} \mu(n-i, \ell) \cdot L(n-i-\ell) \right]$$

$$\leq n^2 \cdot \left[\max_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} L(i) + \max_{i = \lfloor n(1-\alpha)+1 \rfloor}^{n} \binom{n}{i} \max_{\ell = \lceil i - n(1-\alpha) \rceil}^{i} (\mu(i,\ell) \cdot L(i-\ell)) \right]$$

using induction hypothesis: $L(m) \leq c^m (2m^2)^{D(m)}$ for $m < n$

$$\leq n^2 (2n^2)^{D(n)-1} \left[\underbrace{\max_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} c^i}_{\leq c^n} + \underbrace{\max_{i = \lfloor n(1-\alpha)+1 \rfloor}^{n} \binom{n}{i} \max_{\ell = \lceil i - n(1-\alpha) \rceil}^{i} (\mu(i,\ell) \cdot c^{i-\ell})}_{\leq c^n} \right]$$

Assume

PolySpace

Goal: $L(n) \leq c^n \cdot (2n^2)^{D(n)}$ for $c > 1$

IS $(n \geq 1)$:

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using induction hypothesis: $L(m) \leq c^m (2m^2)^{D(m)}$ for $m < n$

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Assume

$$\leq c^n (2n^2)^{D(n)}$$

PolySpace

$L(n) \leq c^n(2n^2)^{D(n)}$ if:

$\max_{i=0}^{\lceil n\alpha-1\rceil} \binom{n}{i} c^i \leq c^n$ and $\max_{i=\lfloor n(1-\alpha)+1\rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} \left(\mu(i,\ell)\cdot c^{i-\ell}\right) \leq c^n$

PolySpace

$L(n) \leq c^n(2n^2)^{D(n)}$ if:

$$\max_{i=0}^{\lceil n\alpha-1\rceil} \binom{n}{i} c^i \leq c^n \quad \text{and} \quad \max_{i=\lfloor n(1-\alpha)+1\rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} \left(\mu(i,\ell)\cdot c^{i-\ell}\right) \leq c^n$$

Remains to determine best c and α

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$L(n) \leq c^n (2n^2)^{D(n)}$ if:

$$\max_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} c^i \leq c^n \quad \text{and} \quad \max_{i=\lfloor n(1-\alpha)+1 \rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha) \rceil}^{i} \left(\mu(i,\ell) \cdot c^{i-\ell}\right) \leq c^n$$

Remains to determine best c and α

Consider only $\alpha \geq 0.5$

PolySpace

$L(n) \leq c^n (2n^2)^{D(n)}$ if:

$$\max_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} c^i \leq c^n \quad \text{and} \quad \max_{i=\lfloor n(1-\alpha)+1 \rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha) \rceil}^{i} \left(\mu(i,\ell) \cdot c^{i-\ell}\right) \leq c^n$$

Lemma For $\alpha \in (0,1)$ and $n \in \mathbb{N}$ it holds that:

$$\frac{1}{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)} \leq c \implies \max_{i=0}^{\lceil n\alpha - 1 \rceil} \binom{n}{i} c^i \leq c^n$$

PolySpace

$L(n) \leq c^n (2n^2)^{D(n)}$ if:

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Uses: $\binom{n}{n\beta} \leq \frac{1}{\beta^\beta (1-\beta)^{1-\beta}}$

PolySpace

$L(n) \leq c^n (2n^2)^{D(n)}$ if:

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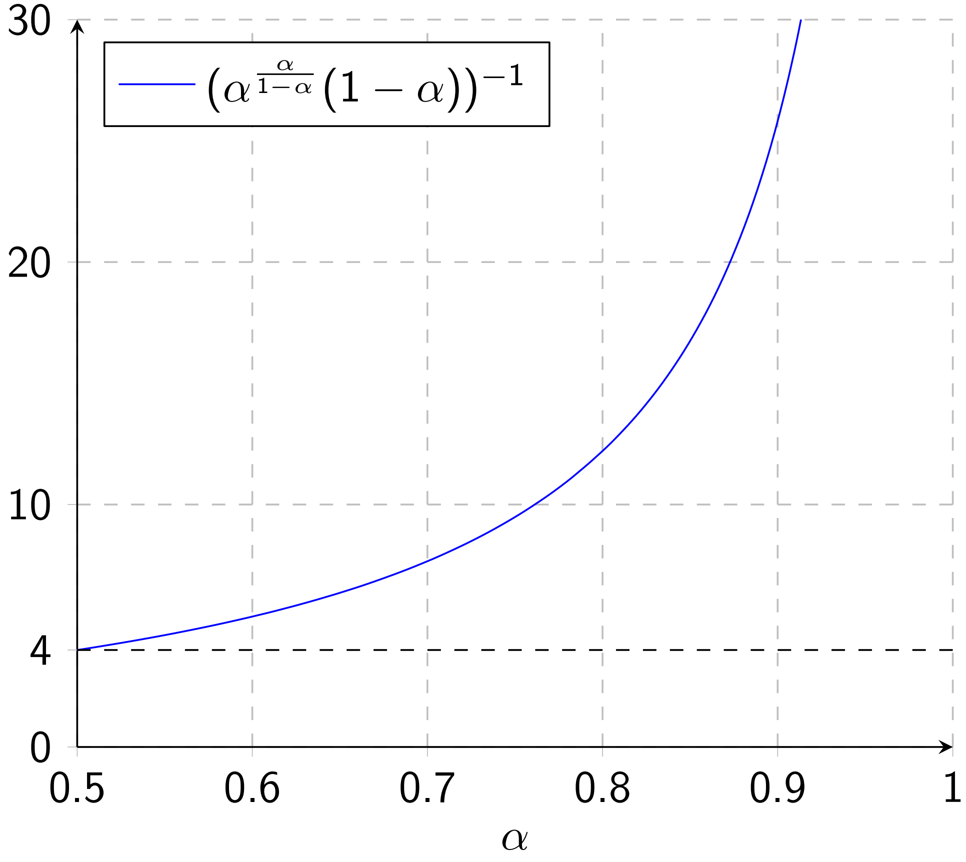

PolySpace

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Lemma For $\frac{16}{43} \leq \alpha < 1$ and $c \geq \frac{81}{64}$ it holds that:

$$\frac{81}{64}\cdot\frac{172}{81}^{\frac{1}{\alpha}} \leq c \implies \max_{i=\lfloor n(1-\alpha)+1 \rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} \left(\mu(i,\ell)\cdot c^{i-\ell}\right) \leq c^n$$

PolySpace

$L(n) \leq c^n (2n^2)^{D(n)}$ if:

$$\frac{1}{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)} \leq c \qquad \text{and} \qquad \max_{i=\lfloor n(1-\alpha)+1 \rfloor}^{n} \binom{n}{i} \max_{\ell=\lceil i-n(1-\alpha)\rceil}^{i} \left(\mu(i,\ell) \cdot c^{i-\ell}\right) \leq c^n$$

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Uses bound by [Epp01]: $\mu(i,\ell) \leq 3^{4\ell-i} 4^{i-3\ell}$

PolySpace

$L(n) \leq c^n(2n^2)^{D(n)}$ if:

$$\frac{1}{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)} \leq c \quad \text{and} \quad \frac{81}{64} \cdot \frac{172}{81}^{\frac{1}{\alpha}} \leq c$$

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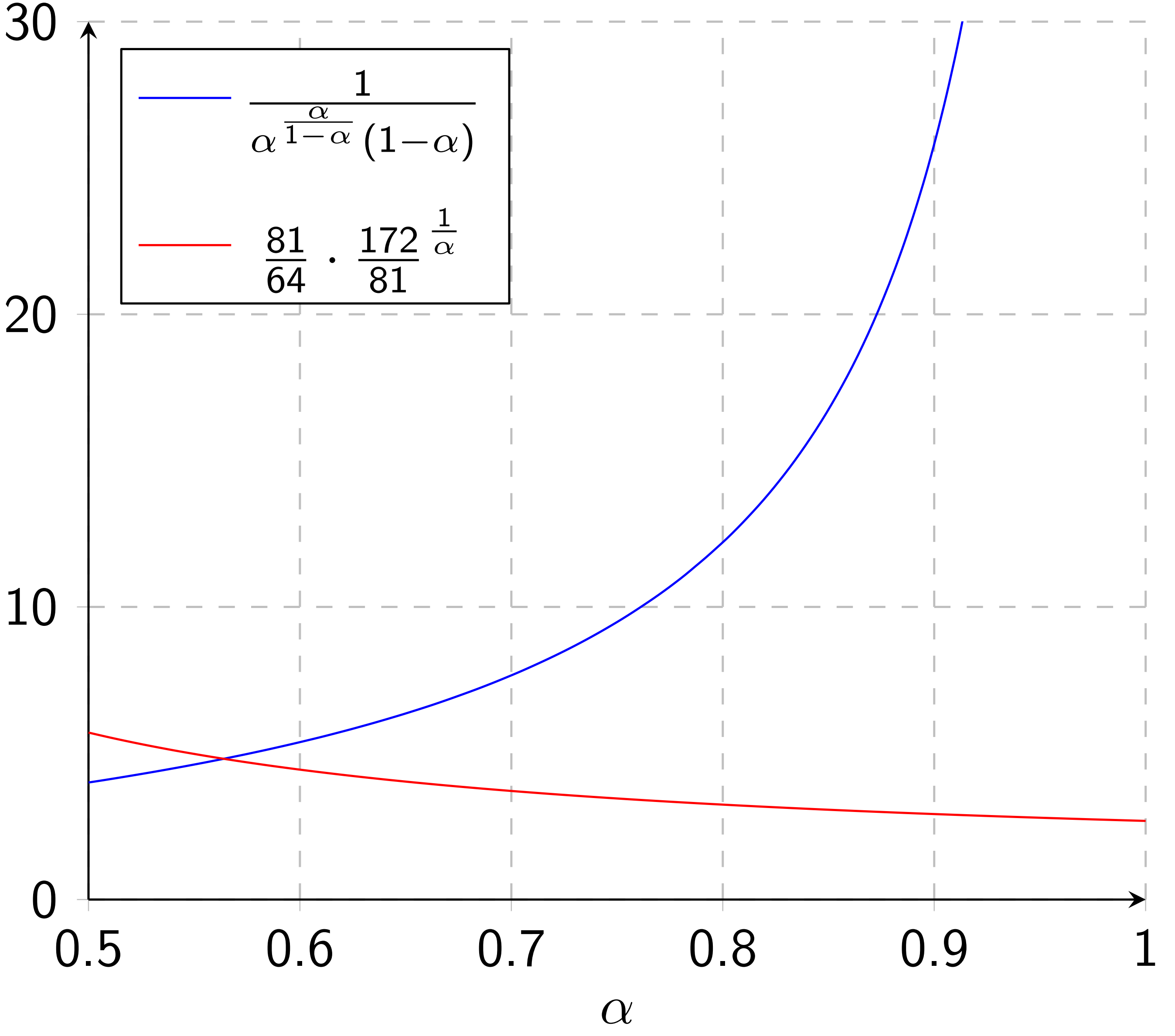

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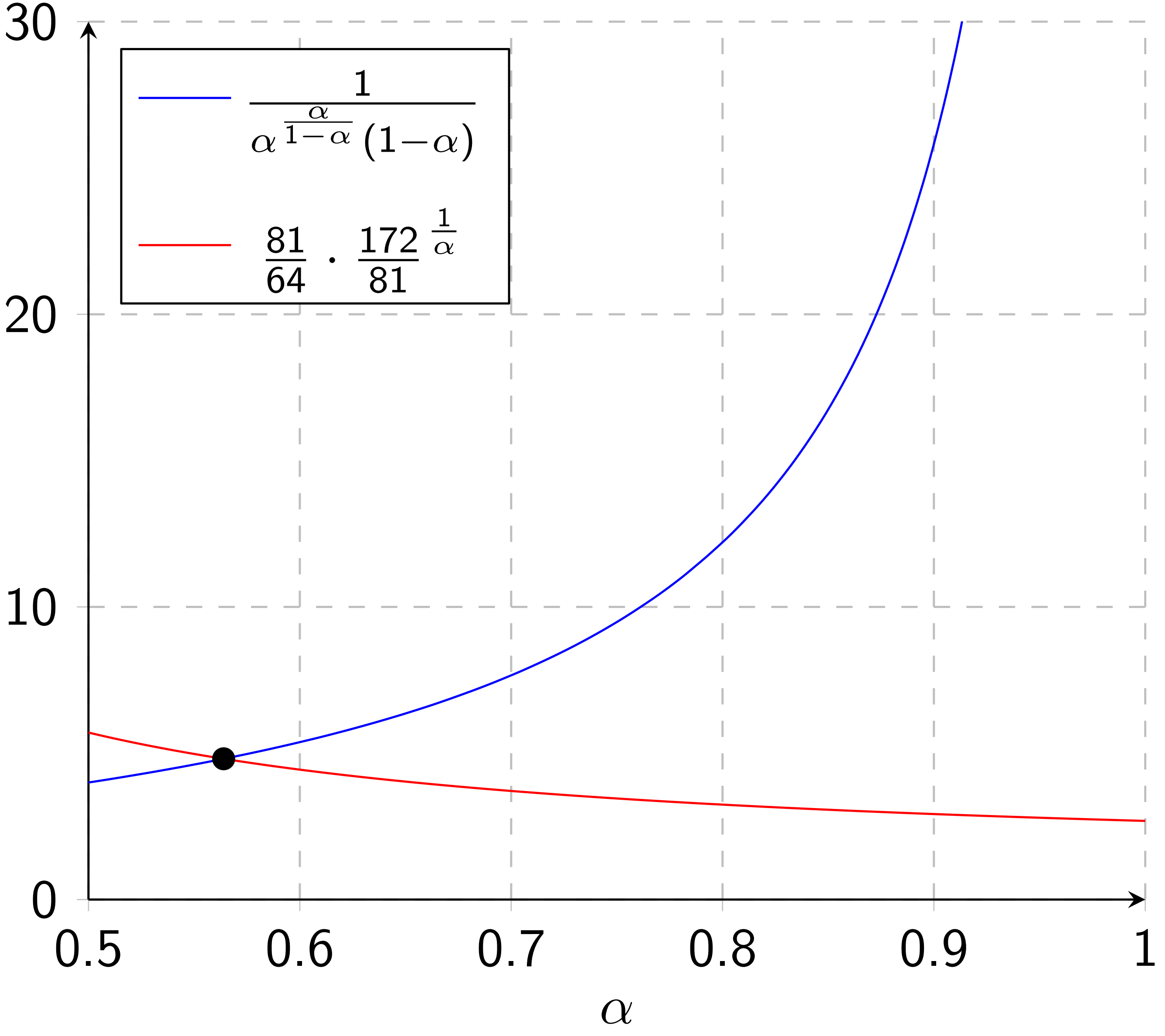
Observation:

both inequalities hold for $\alpha = 0.563964$ and $c = 4.81068$



PolySpace

$L(n) \leq c^n(2n^2)^{D(n)}$ if:

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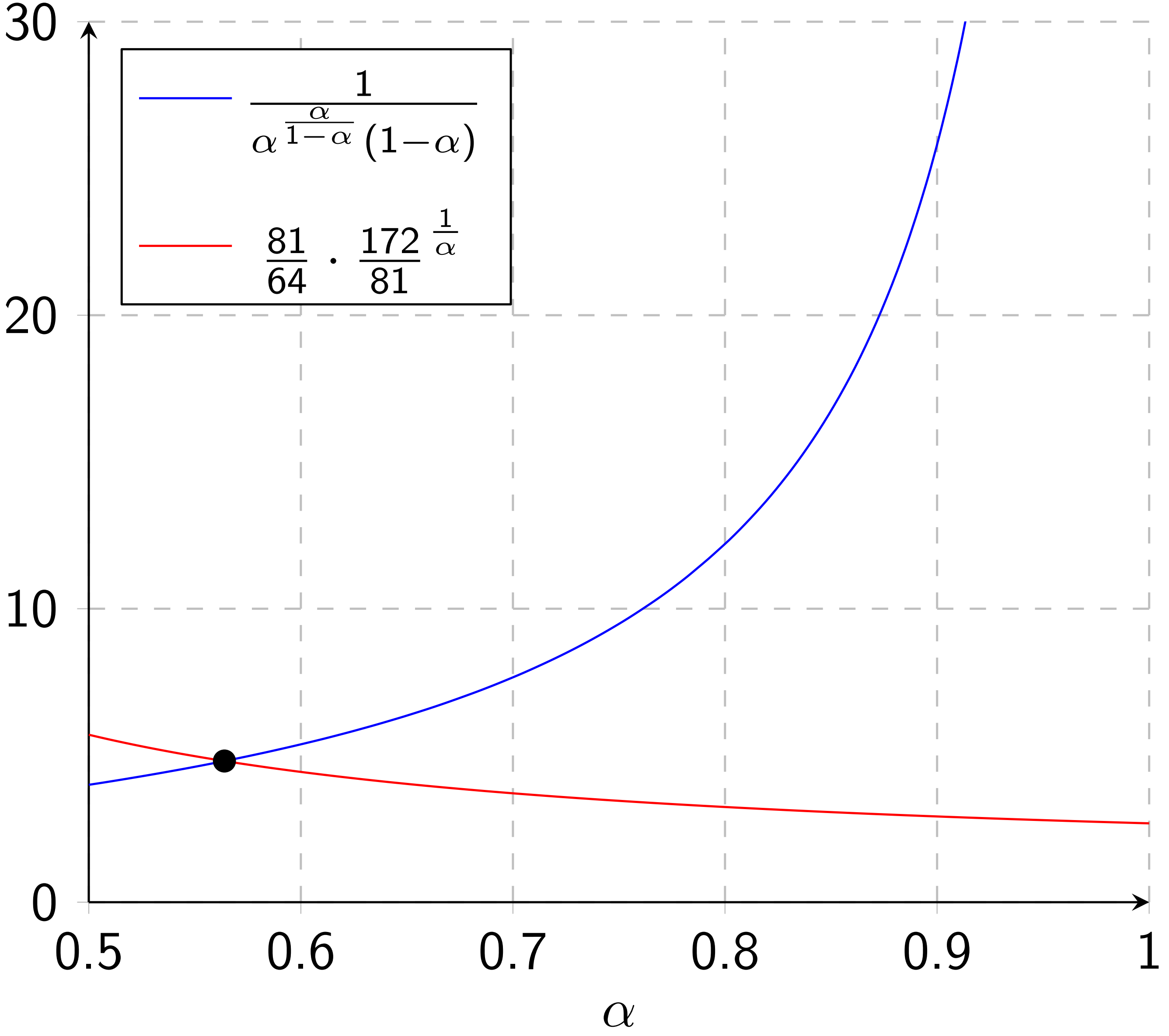
Observation:

both inequalities hold for $\alpha = 0.563964$ and $c = 4.81068$

Runtime:

$\mathcal{O}^*(4.81068^n(2n^2)^{D(n)})$



PolySpace

$L(n) \leq c^n (2n^2)^{D(n)}$ if:

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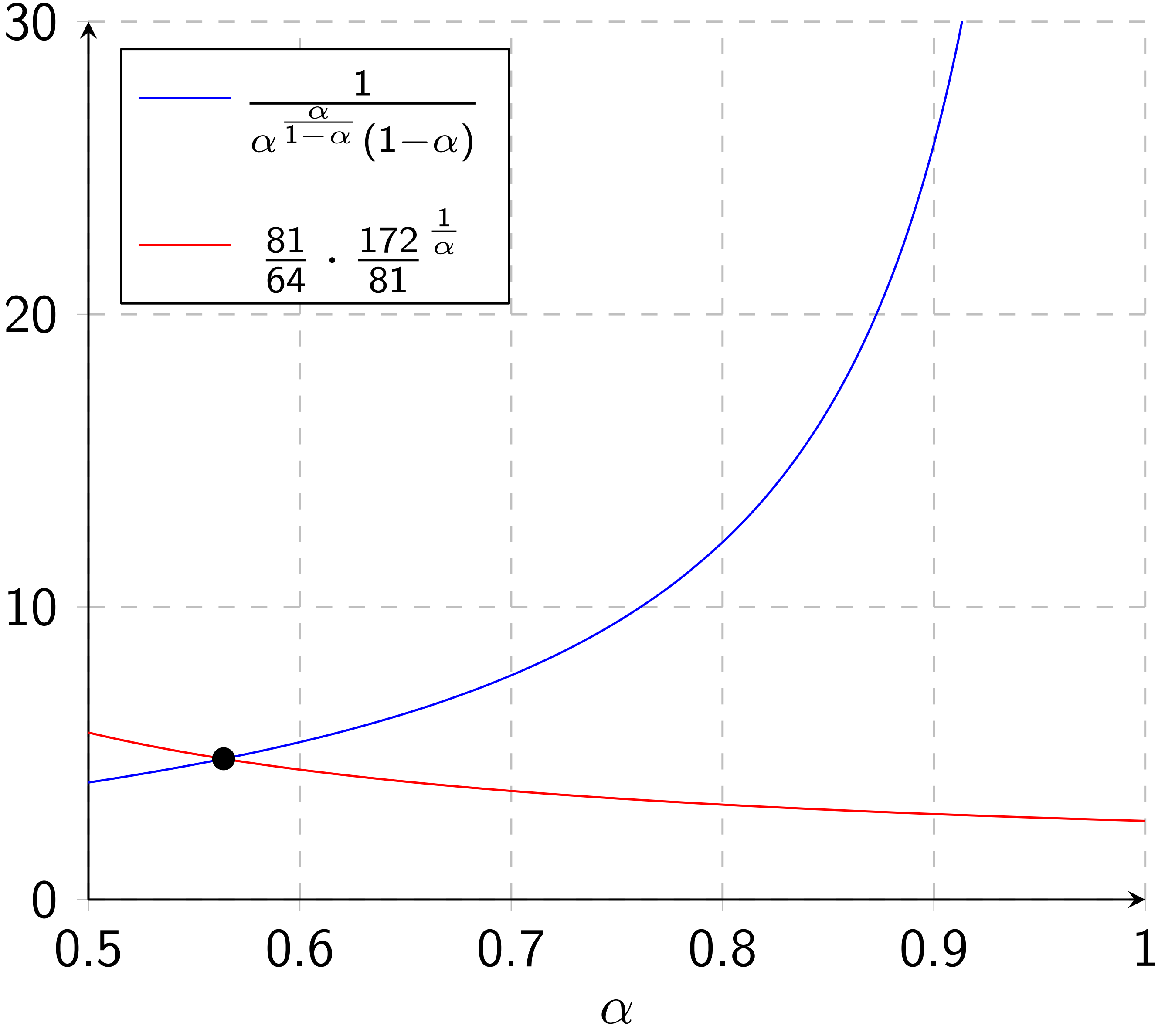
Observation:

both inequalities hold for $\alpha = 0.563964$ and $c = 4.81068$

Runtime:

$$\mathcal{O}^*(4.81068^n \underbrace{(2n^2)^{D(n)}}_{n^{\mathcal{O}(\log n)}})$$



PolySpace

$L(n) \leq c^n (2n^2)^{D(n)}$ if:

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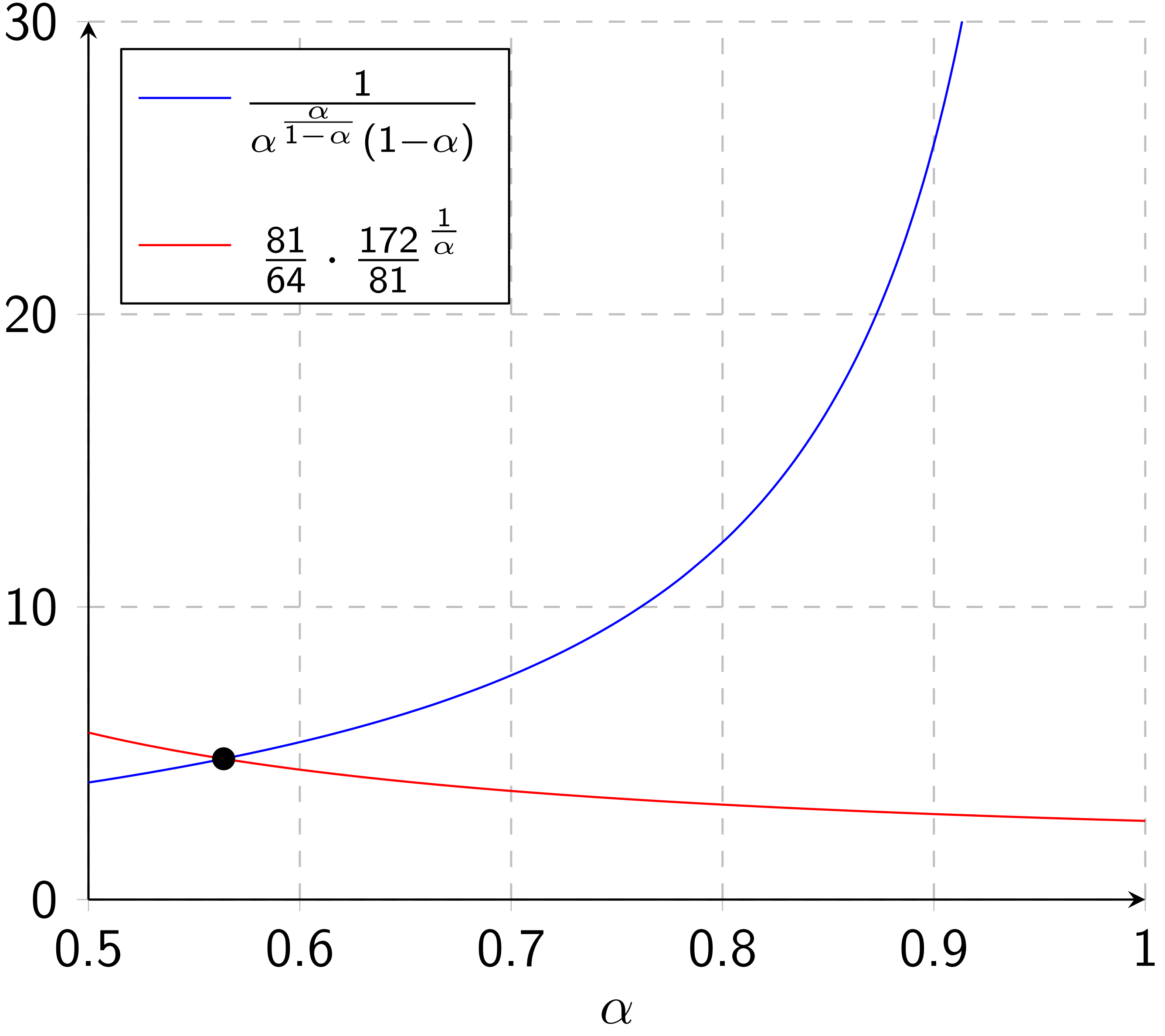
$$\mathcal{O}^*(4.81068^n \underbrace{(2n^2)^{D(n)}}_{\text{quasi-polynomial}})$$



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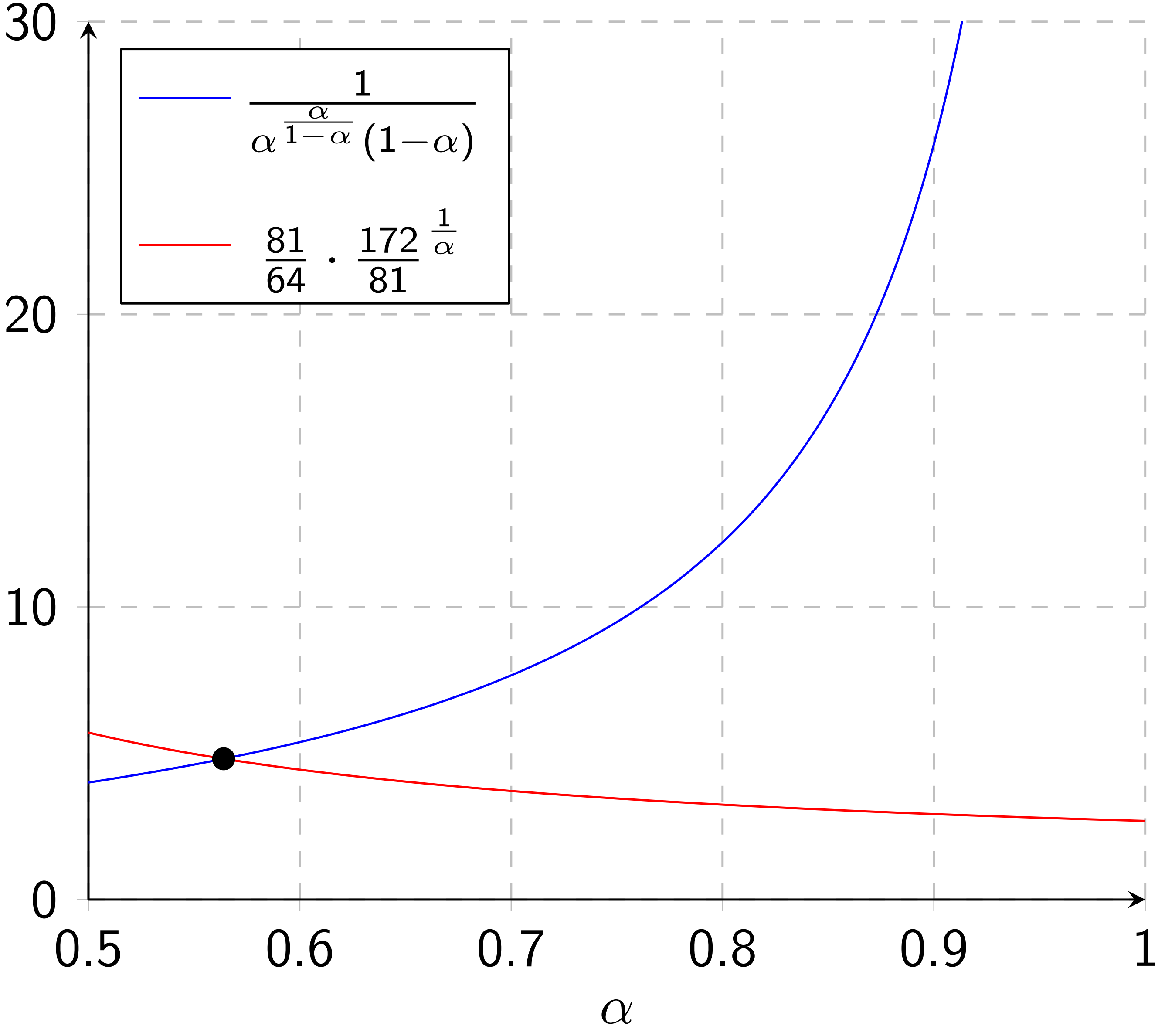
Observation:

both inequalities hold for $\alpha = 0.563964$ and $c = 4.81068$

Runtime:

$$\mathcal{O}^*(4.81068^n \underbrace{(2n^2)^{D(n)}}_{\text{quasi-polynomial}}) \subset \mathcal{O}(4.81069^n)$$



PolySpace

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Observation:

both inequalities hold for $\alpha = 0.563964$ and $c = 4.81068$

Runtime:

$\mathcal{O}^*(4.81068^n(2n^2)^{D(n)}) \subset \mathcal{O}(4.81069^n)$

Theorem

MixedColoring can be solved in $\mathcal{O}(4.81069^n)$ time and polynomial space

3-MixedColoring

3-MixedColoring

Lemma [Mei23]

3-Coloring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

3-MixedColoring

Lemma [Mei23]

3-Coloring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

- based on reduction for Constraints Satisfaction Problem
- exploits graph properties to further reduce instance size

3-MixedColoring

Lemma [Mei23]

3-Coloring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

- based on reduction for Constraints Satisfaction Problem generalizes
- exploits graph properties to further reduce instance size does not generalize

3-MixedColoring

Lemma [Mei23]

3-Coloring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

Lemma

A 3-MixedColoring instance reduces in poly time to a 3-Coloring instance with three additional vertices

3-MixedColoring

Lemma [Mei23]

3-Coloring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

Lemma

A 3-MixedColoring instance reduces in poly time to a 3-Coloring instance with three additional vertices

Idea: reduction via ListColoring

3-MixedColoring

Lemma [Mei23]

3-Coloring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

Lemma

A 3-MixedColoring instance reduces in poly time to a 3-Coloring instance with three additional vertices

Theorem

3-MixedColoring solvable in $\mathcal{O}(1.32173^n)$ time and polynomial space

k-MixedColoring

Technique 1:

k-MixedColoring

Technique 1:

Property (undirected):

There exists an optimal k-coloring where the first color class is a maximal independent set and has size at least $\lceil n/k \rceil$

k-MixedColoring

Technique 1:

Property (generalized undirected):

There exists an optimal k-coloring where the first color class is a maximal independent set with indeg 0 and has size at least $\lceil n/k \rceil$

k-MixedColoring

Technique 1:

Property (mixed):

There exists an optimal k-coloring where the first color class is a maximal independent set with indeg 0

k-MixedColoring

Technique 1:

Property (mixed):

There exists an optimal k-coloring where the first color class is a maximal independent set with indeg 0

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k-MixedColoring

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Theorem

Let $(k-1)$-MixedColoring be solvable in $\mathcal{O}^*(c^n)$ time and polynomial space for some constant $c \geq 1$. Then, k-MixedColoring is solvable with polynomial space and a runtime of, if $\frac{\lfloor ce \rfloor^{\lfloor ce+1 \rfloor}}{\lfloor ce+1 \rfloor^{\lfloor ce \rfloor} c} \leq 1$, $\mathcal{O}^*((\lceil ce \rceil^{\frac{1}{\lceil ce \rceil}} c^{1 - \frac{1}{\lceil ce \rceil}})^n)$, otherwise, $\mathcal{O}^*((\lfloor ce \rfloor^{\frac{1}{\lfloor ce \rfloor}} c^{1 - \frac{1}{\lfloor ce \rfloor}})^n)$.

k-MixedColoring

Technique 2:

k-MixedColoring

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Property (undirected):

graph k-colorable iff exists partition $\langle S, T \rangle$ where $G[S]$ is $\lceil k/2 \rceil$-colorable and $G[T]$ is $\lfloor k/2 \rfloor$-colorable

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Theorem

The k-MixedColoring problem can be solved in $\mathcal{O}^*((c+1)^n)$ time and polynomial space if $\lceil k/2 \rceil$-MixedColoring can be solved in $\mathcal{O}^*(c^n)$ time and polynomial space.

Runtimes for Poly-Space k-MixedColoring

$k \leq$	**Runtime**	**Approach**

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4	$\mathcal{O}(1.74330^n)$	T1 (3)
5	$\mathcal{O}(2.15225^n)$	T1 (4)
6	$\mathcal{O}(2.32173^n)$	T2 (3)
7	$\mathcal{O}(2.71979^n)$	T1 (6)
8	$\mathcal{O}(2.74330^n)$	T2 (4)
9	$\mathcal{O}(3.13611^n)$	T1 (8)
10	$\mathcal{O}(3.15225^n)$	T2 (5)
12	$\mathcal{O}(3.32173^n)$	T2 (6)
13	$\mathcal{O}(3.71075^n)$	T1 (12)
14	$\mathcal{O}(3.71979^n)$	T2 (7)
16	$\mathcal{O}(3.74330^n)$	T2 (8)

$k \leq$	**Runtime**	**Approach**
17	$\mathcal{O}(4.12981^n)$	T1 (16)
18	$\mathcal{O}(4.13611^n)$	T2 (9)
20	$\mathcal{O}(4.15225^n)$	T2 (10)
24	$\mathcal{O}(4.32173^n)$	T2 (12)
25	$\mathcal{O}(4.70563^n)$	T1 (24)
26	$\mathcal{O}(4.71075^n)$	T2 (13)
28	$\mathcal{O}(4.71979^n)$	T2 (14)
32	$\mathcal{O}(4.74330^n)$	T2 (16)
∞	$\mathcal{O}(4.81069^n)$	PolySpaceAlgo

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Parameterized Complexity