NP-completeness of Planar Steiner Orientation

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Abstract

We show NP-completeness for a variant of Steiner Orientation on mixed, planar graphs.

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1 Introduction

The Steiner Orientation problem is defined as follows: given a mixed graph $G = (V, E \cup A)$ with both undirected edges E and directed arcs A and a set $T \subseteq V \times V$ of k terminal pairs, is there an orientation of all edges in E, such that for every terminal pair $(s, t) \in T$ there is an s-t-path in the resulting directed graph?

In general, this problem was shown to be *NP*-complete by Arkin and Hassin [1]. Cygan et al. [3] gave an $n^{O(k)}$-time algorithm, showing it is in XP in k. Pilipczuk and Wahlström [6] improved the hardness result showing it to be $W[1]$-hard in k. For $A = \emptyset$, however, Hassin and Megiddo [5] give a polynomial time algorithm. This raises the following question: how do restrictions on G influence complexity of Steiner Orientation? These hardness proofs utilize non-planar instances. Chitnis and Feldmann [2] showed that under the Exponential Time Hypothesis, Steiner Orientation cannot be solved in $f(k) \cdot n^{o(k)}$ time, even when restricting to graphs of genus 1.

In this work, we consider the Planar Steiner Orientation problem where G is a planar graph. As a first result on computational complexity, we show the following:

▶ **Theorem 1.** Planar Steiner Orientation *is NP-complete.*

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2 Hardness Proof

To prove Theorem 1, we give a reduction from Planar Monotone 3-SAT, introduced by de Berg et al. [4] and known to be *NP*-complete. We use different gadgets for variables, clauses and edges. These are stitched together at shared undirected edges. Given a planar monotone 3-SAT formula F, we use these gadgets to create an instance of Planar Steiner Orientation resembling the incidence graph of F with $|T|$ polynomial in $|F|$. Without loss of generality we assume that every variable of F occurs both negated and unnegated.

Figure 1a shows a *flip gadget*, a building block used in other gadgets. It contains two terminal pairs (s_1, t_1) and (s_2, t_2) and two undirected (red) edges. Connecting both pairs will result in opposing directions for the two undirected edges.

For every variable x in F, we have a *variable gadget* (Figure 1b). It mimics the flip gadget, providing an undirected edge e^x_C for every positive/negative clause C containing x above/below the pairs respectively. We say that the gadget is (false) true if the undirected edges are oriented (counter-)clockwise. No other orientation allows connecting both pairs.

We use two stacked flip gadgets as an *edge gadget*: by reversing direction twice, we synchronize the outer red edges. We attach this edge to a variable and a clause gadget.

For every clause C, we have a *clause gadget* (Figure 1c). It contains a terminal pair (s, t) and has an undirected edge $\bar{e}^w_C$ for each variable w it contains. The undirected edge $\bar{e}^y_C$ in the middle is flipped to get a consistent orientation for variables set to true. The edges f and g are synchronized by two flip gadgets to ensure that at most one of them is used to connect (s, t). For the clause gadgets, we get the following Lemma:

▶ **Lemma 2.** *All pairs of clause gadgets are connected iff ≥ 1 edge $\bar{e}_C$ is directed to the right.*

Proof. In our construction all gadgets are self-contained and the terminal pairs of the flip gadgets can be connected (Appendix B). The edges f and g are both directed upwards or downwards. Hence it suffices to show the equivalence for the pair (s, t). "$\Leftarrow$": Case 1: If $\bar{e}^x_C$ is directed to the right, orient the edge f away from s. Case 2: If $\bar{e}^y_C$ is directed to the right, $\tilde{e}^y_C$ is directed to the left. Case 3: If $\bar{e}^z_C$ is directed to the right, orient the edge g pointing to t. In each of these cases s is connected to t. "$\Rightarrow$": By contraposition. As we move away from s, we can neither use the edge $\tilde{e}^y_C$ nor $\bar{e}^z_C$. Thus we have to use f which means it points away from s. To come to t we have to use one of the edges $\bar{e}^x_C$ or g. This is impossible. ◀

Provided that all terminal pairs of all variable and edge gadgets are connected (which can always be achieved), the terminal pairs of all clause gadgets can be connected if and only if the formula is satisfiable. Thus, the Planar Steiner Orientation instance has a solution if and only if the corresponding Planar Monotone 3-SAT formula is satisfiable.

Future work could involve proving $W[1]$-hardness or looking for approximation algorithms. Other graph classes – with different geometric restrictions – could also be considered.

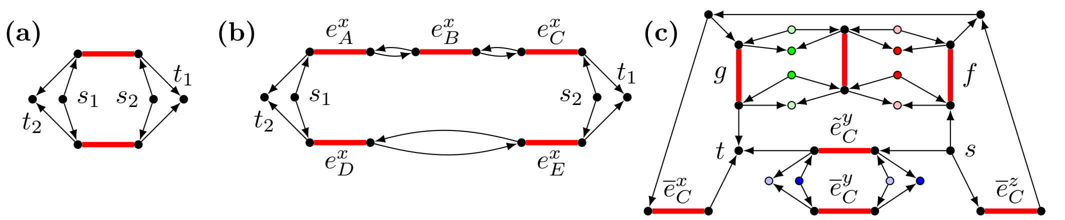

Figure 1 **(a)** The flip gadget, used to construct edge gadgets; **(b)** a variable gadget with three positive and two negative occurrences; **(c)** a clause gadget (unlabeled (s, t)-pairs color-coded).

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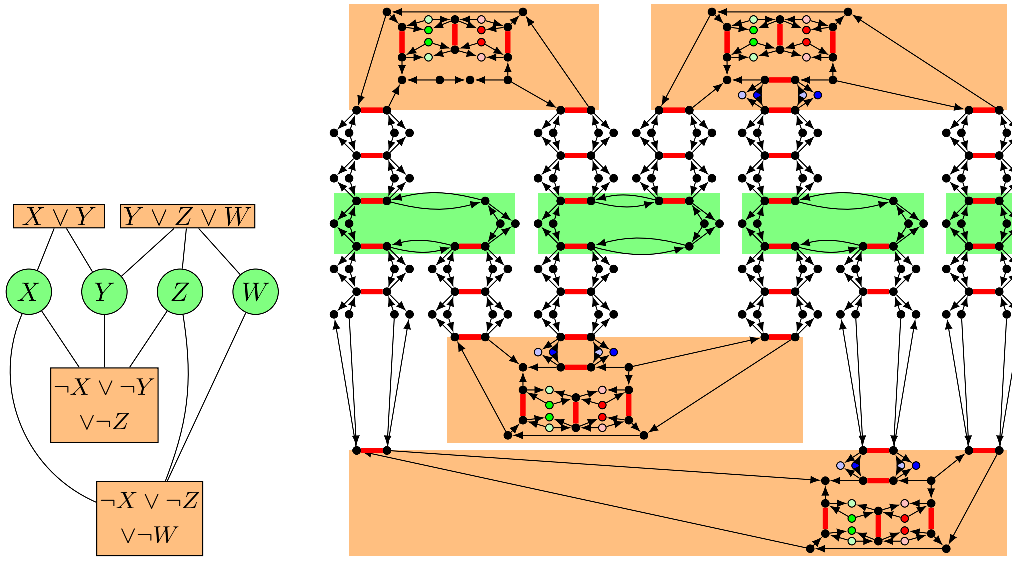
A Full Example

For better understanding, we want to provide a small but complete example. Consider the following formula:

$$F = (X \lor Y) \land (\neg X \lor \neg Z \lor \neg W) \land (Y \lor Z \lor W) \land (\neg X \lor \neg Y \lor \neg Z)$$

In Figure 2 we give the incidence graph for F together with the Planar Steiner Orientation instance corresponding to F created using the gadgets introduced above.



(a) Incidence graph of F **(b)** Corresponding Planar Steiner Orientation instance

Figure 2 A full example showing the reduction from Planar Monotone 3-SAT to Planar Steiner Orientation. Variables and variable gadgets are highlighted in green, clauses and clause gadgets in orange. Negative clause gadgets are mirrored vertically.

B Self-Containment of Gadgets

An important property of our construction is that all the gadgets that we use are *self-contained*, which means that for each gadget any simple path connecting a terminal pair of the gadget stays inside the gadget. We state the following simple observation:

▶ Observation 1. Every source s has indegree zero and every target t has outdegree zero.

Using this observation, we can show now that in our construction each clause, edge, and variable gadget is self-contained.

Clause gadgets. Assume there is a simple path connecting a terminal pair of a clause gadget, which is not fully contained within the gadget. Then there must be an edge that leaves the clause gadget and due to the structure, this edge must be part of an edge gadget. But all edges leaving a clause gadget and entering an edge gadget end in some target terminal, so by Observation 1 the path cannot re-enter the clause gadget.

Edge gadgets. Consider a simple path that leaves an edge gadget. If the leaving edge is part of a clause gadget, the path leads to a target terminal within the clause gadget or within another edge gadget, from where it cannot re-enter the original edge gadget. The case where the leaving edge is part of a variable gadget is similar.

Variable gadgets. Consider a simple path that leaves a variable gadget. Then the leaving edge is part of an edge gadget and leads to a target terminal, so the path cannot re-enter the variable gadget.

Therefore, it suffices to consider only paths within a gadget as there is a path connecting a terminal pair if and only if there is a simple path connecting this terminal pair.

C Clauses with Two Variables

For clauses with only two variables we simply replace the edge $\tilde{e}_C^y$ with an arc from left to right and omit the attached flip gadget (see Fig. 2b, clause $(X \vee Y)$). It is easy to see that this way no new possibilities for s-t-paths are created, keeping the gadget valid.