Strongly Monotone Drawings of Planar Graphs

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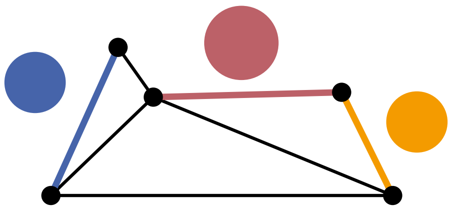
Tamara Mchedlidze	Karlsruhe Institute of Technology
Manfred Scheucher	Graz University of Technology

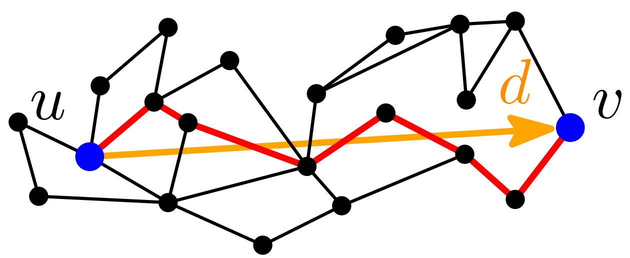


EuroCG 2016, Lugano

Strongly Monotone Drawings

We want planar straight-line graph drawings that are ...

Strongly monotone: between each vertex pair (u, v) exists a path that is monotonically increasing in direction $d = \overrightarrow{uv}$



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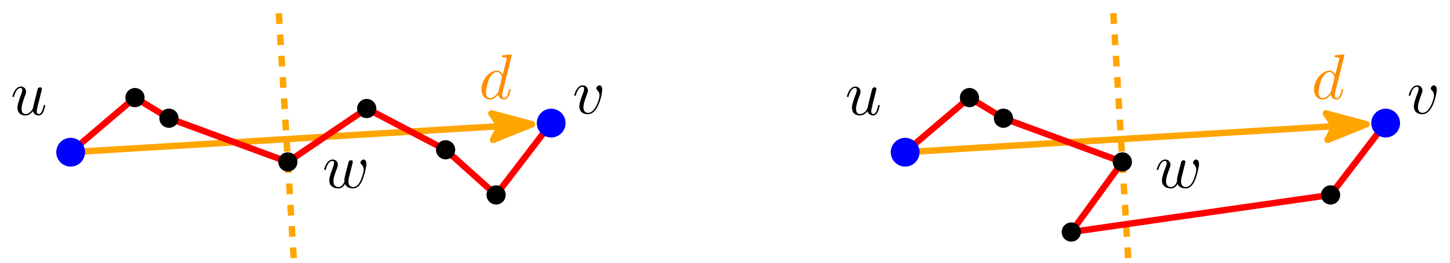

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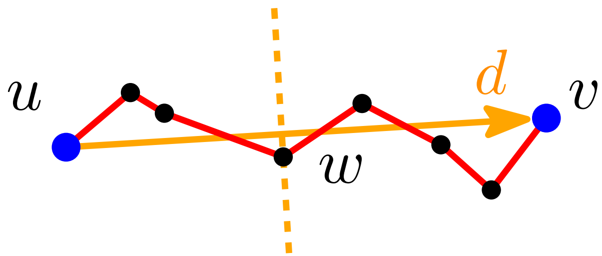

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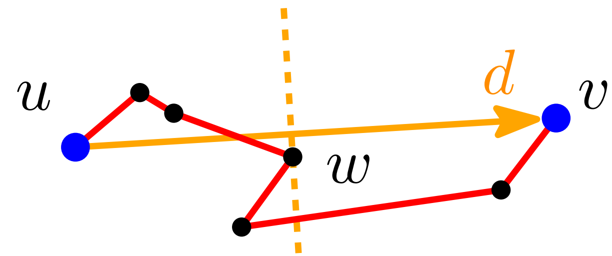

not strongly monotone

Motivation: Path-finding tasks become easy!

Related Work & Results

Strongly monotone drawings:

- do not exist for every planar graph [Kindermann et al. '14]
- exist for every 2-connected outerplanar graph [Kindermann et al. '14]
- exist for every tree [Kindermann et al. '14]
  - area required can be exponential [Nöllenburg et al. '14]

Related Work & Results

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- exist for every planar 3-connectd graph
- exist for every outerplanar graph
- exist for every 2-tree

} **our results**

3-connected Graphs

Theorem: Every 3-connected planar graph has a strongly monotone drawing.

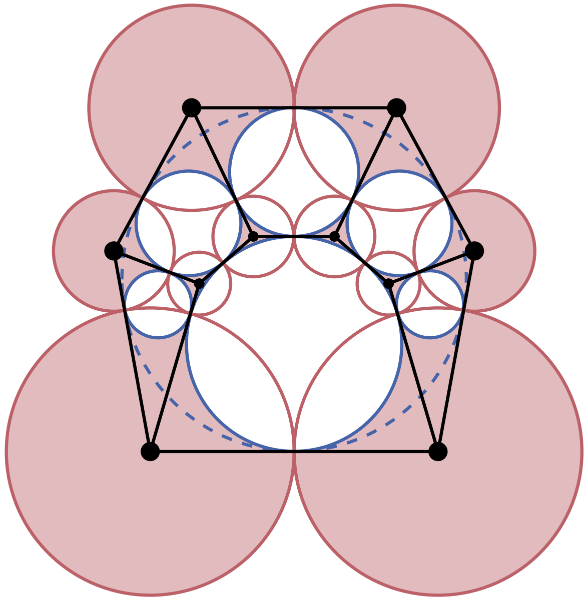
Proof idea:

Every 3-connected planar graph G admits a primal-dual circle packing $\mathcal{P}$.
[Brightwell, Scheinerman 1993]

Drawing induced by $\mathcal{P}$ is strongly monotone.

Primal Dual Circle Packings

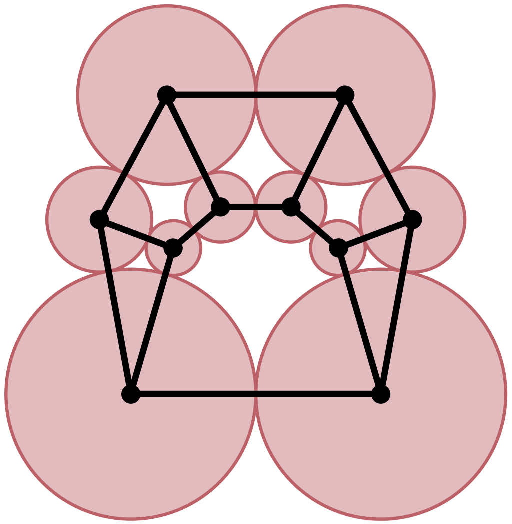
primal dual circle packing $\mathcal{P}$ of G



Primal Dual Circle Packings

circle contact representations of ...

(**primal**) graph G



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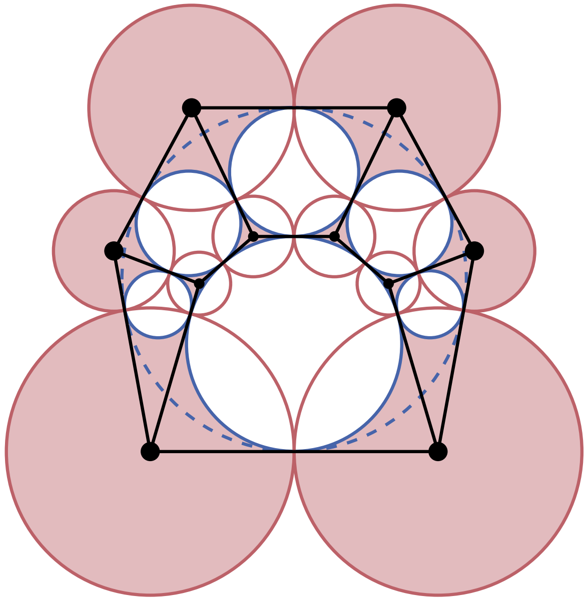

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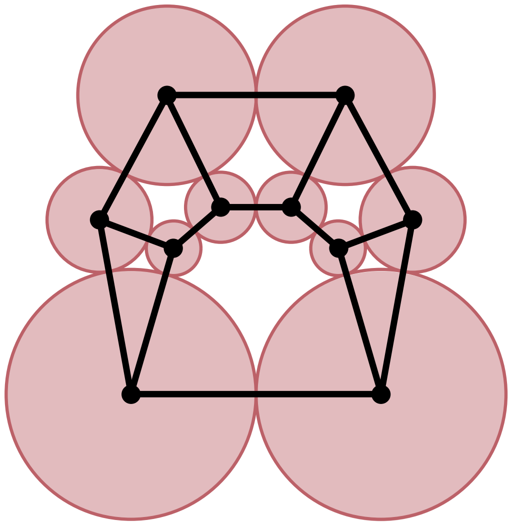
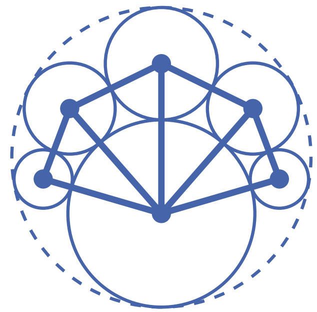


dual graph of G



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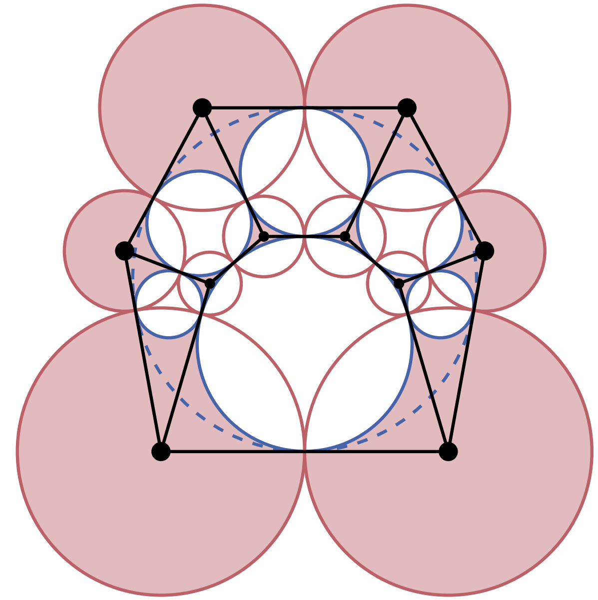

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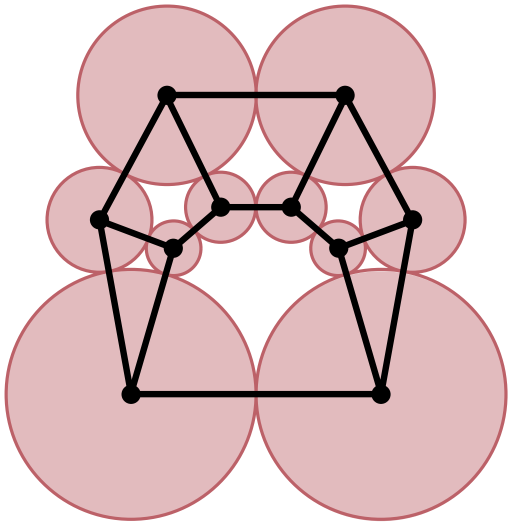

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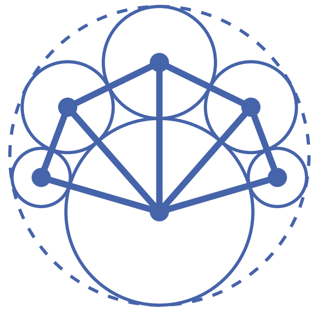
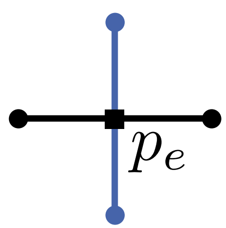


... which are **orthogonal**:

edge e **crosses**
its dual edge e^*
perpendicularly
in **edge point** p_e



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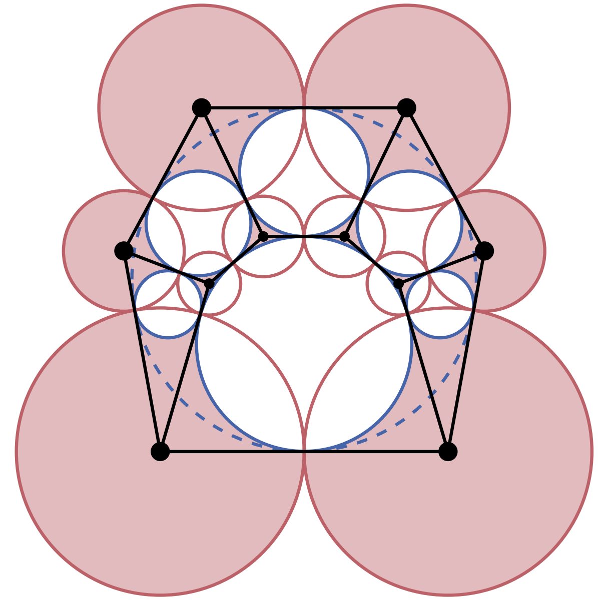

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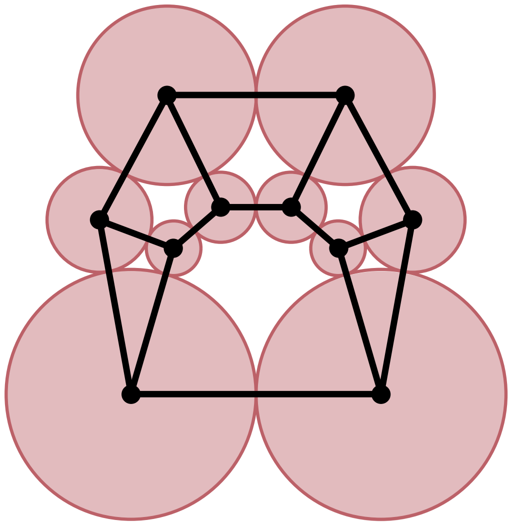

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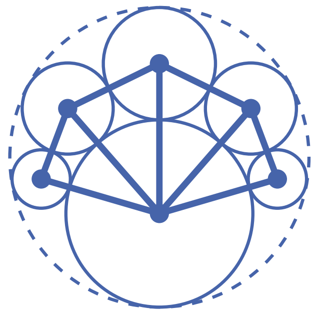

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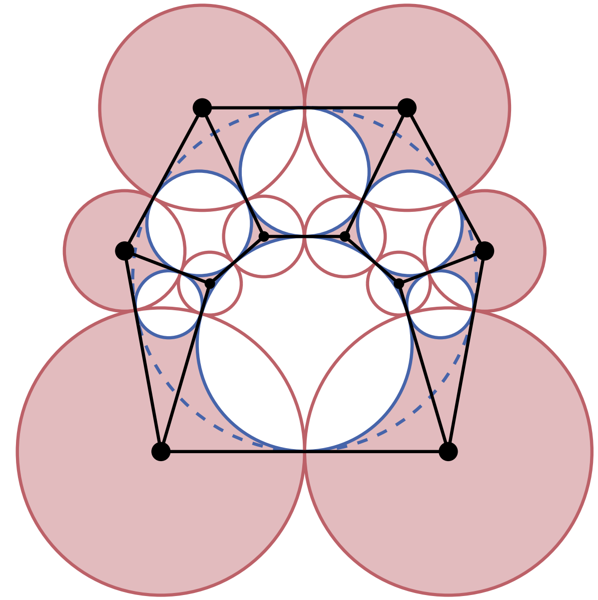

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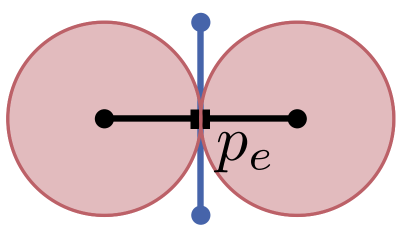
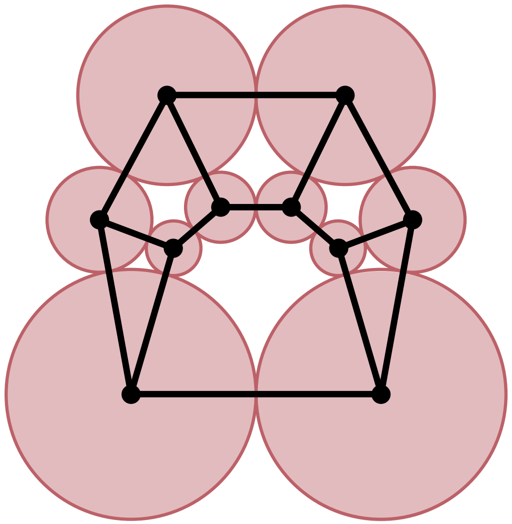
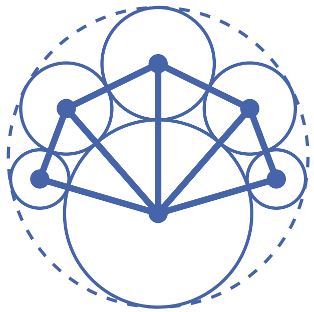


Primal Dual Circle Packings

circle contact representations of ...

(**primal**) graph G **dual** graph of G





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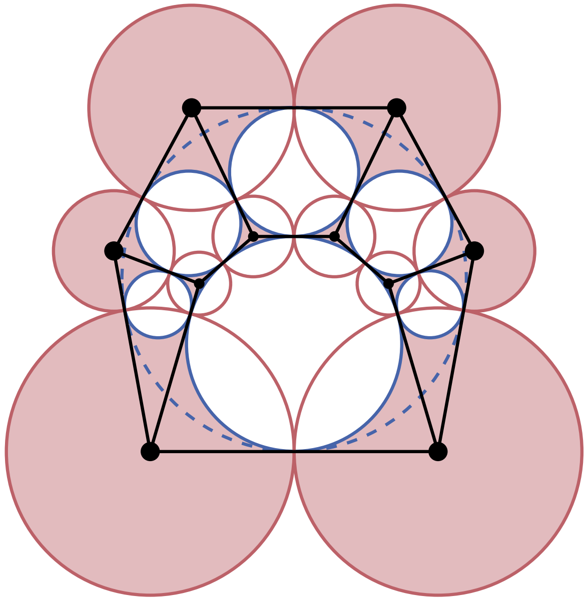

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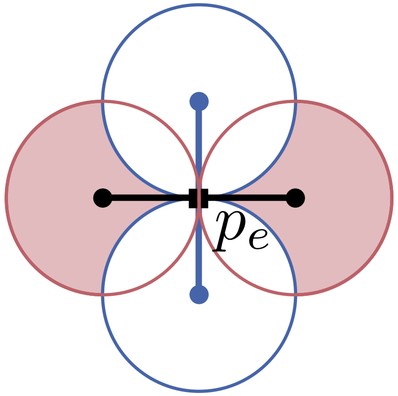

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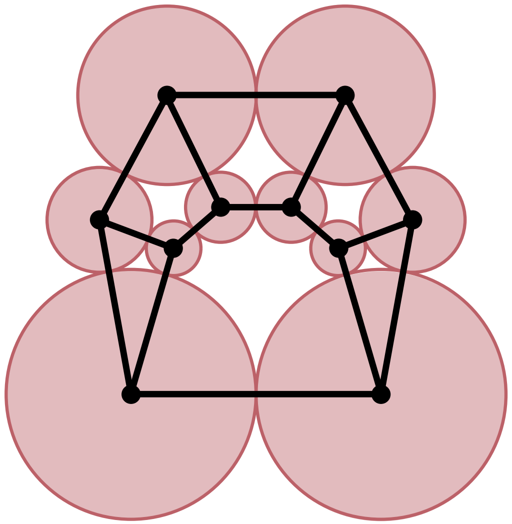

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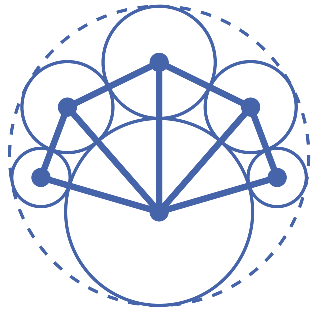

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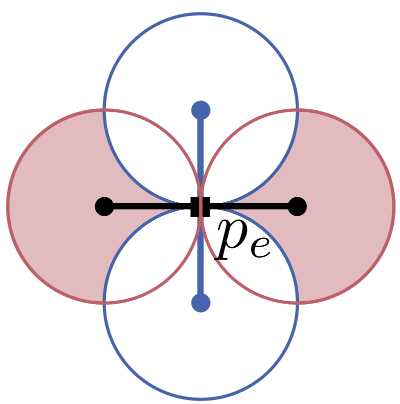
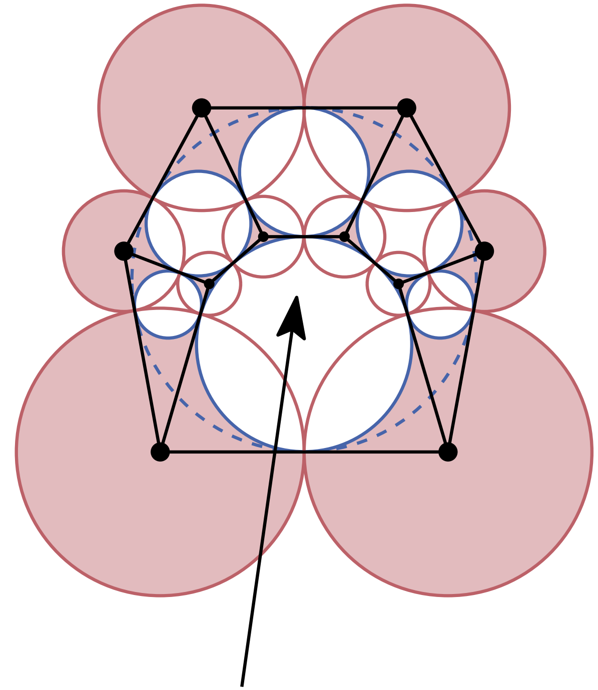


$\Rightarrow$

primal dual circle packing $\mathcal{P}$ of G



face circle = inscribed circle of face

Primal Dual Circle Packings

circle contact representations of ...

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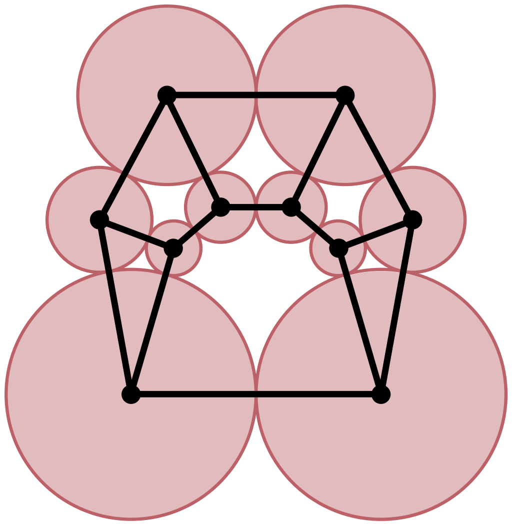

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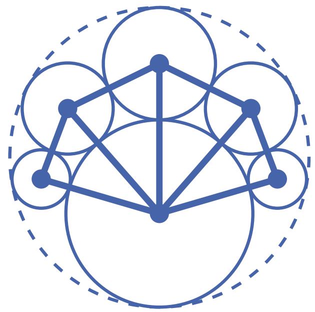

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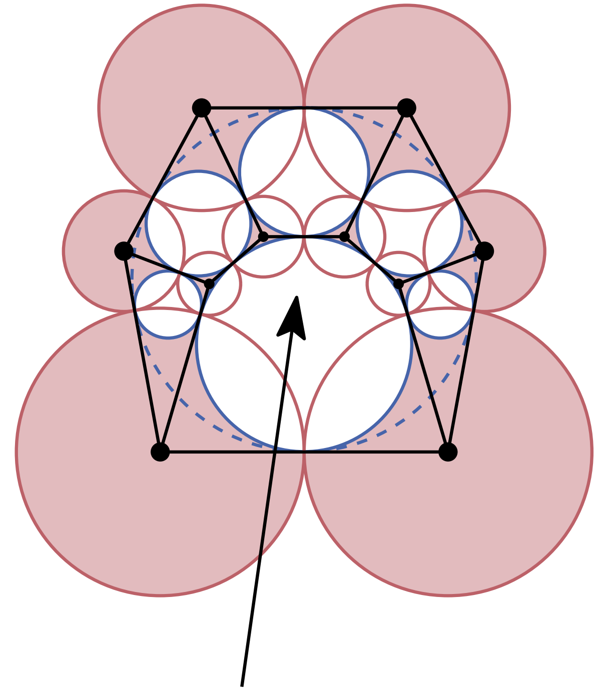

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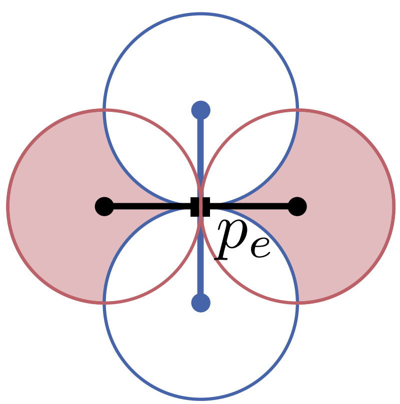
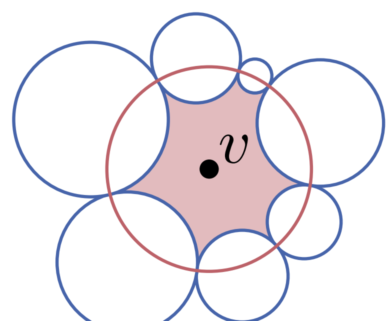


$\Rightarrow$ **face circle** $=$ inscribed circle of face

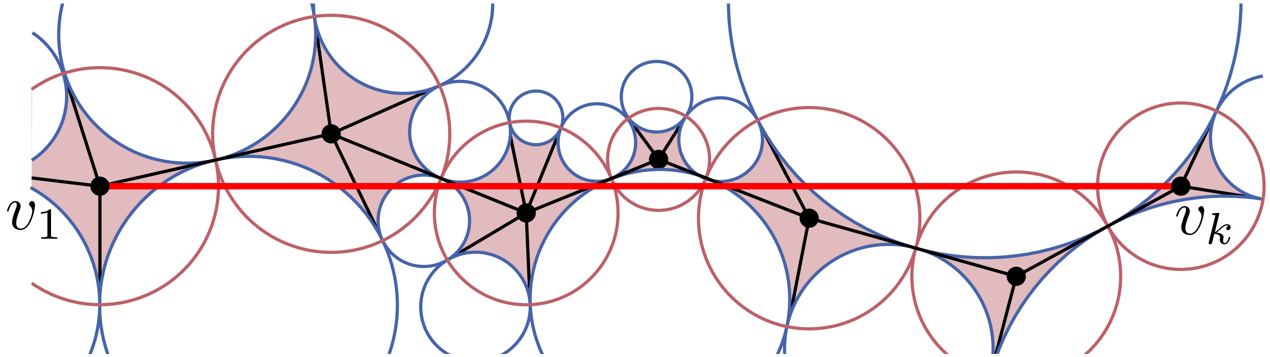
$\Rightarrow$ vertex v has 'star-shaped' **Region** R_v



v can see all of R_v!

Constructing Strongly Monotone Paths

Consider vertices v_1, v_k where w.l.o.g. $s = \overline{v_1 v_k}$ horizontal.



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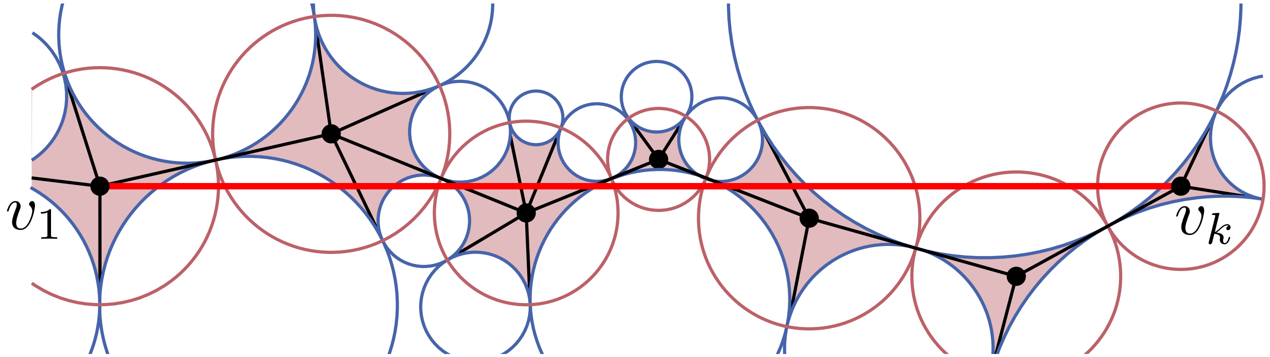
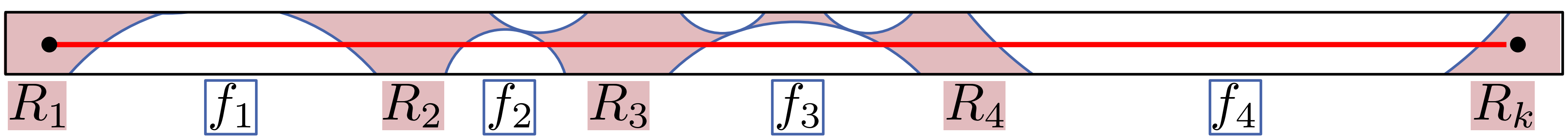


General position: s does not pass through circle centers or edge points.

$\Rightarrow$ s intersects **alternating sequence** of vertex regions and face circles:



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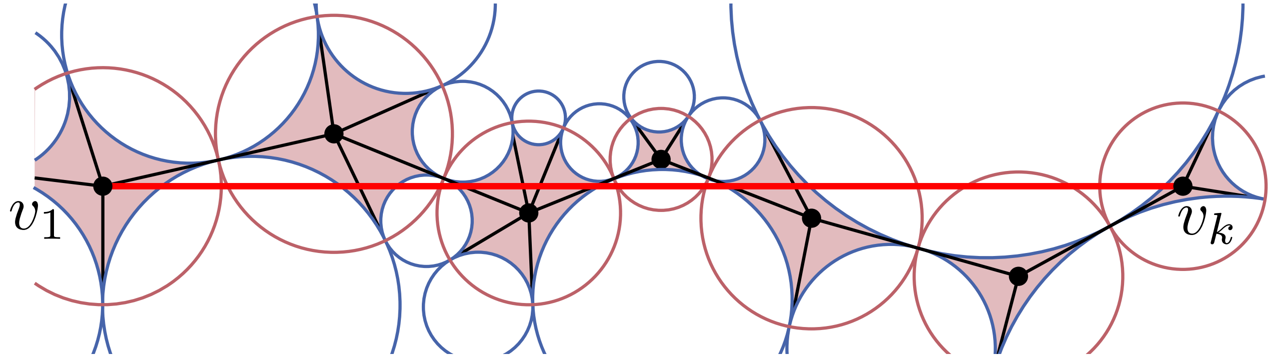

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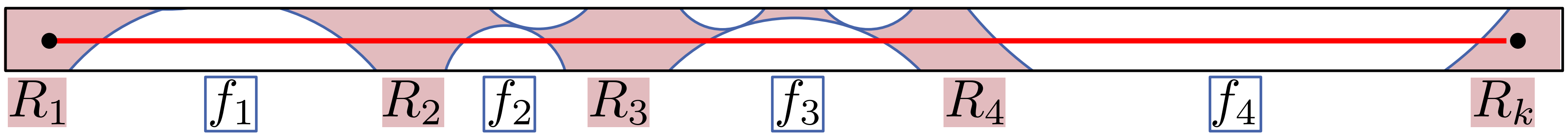
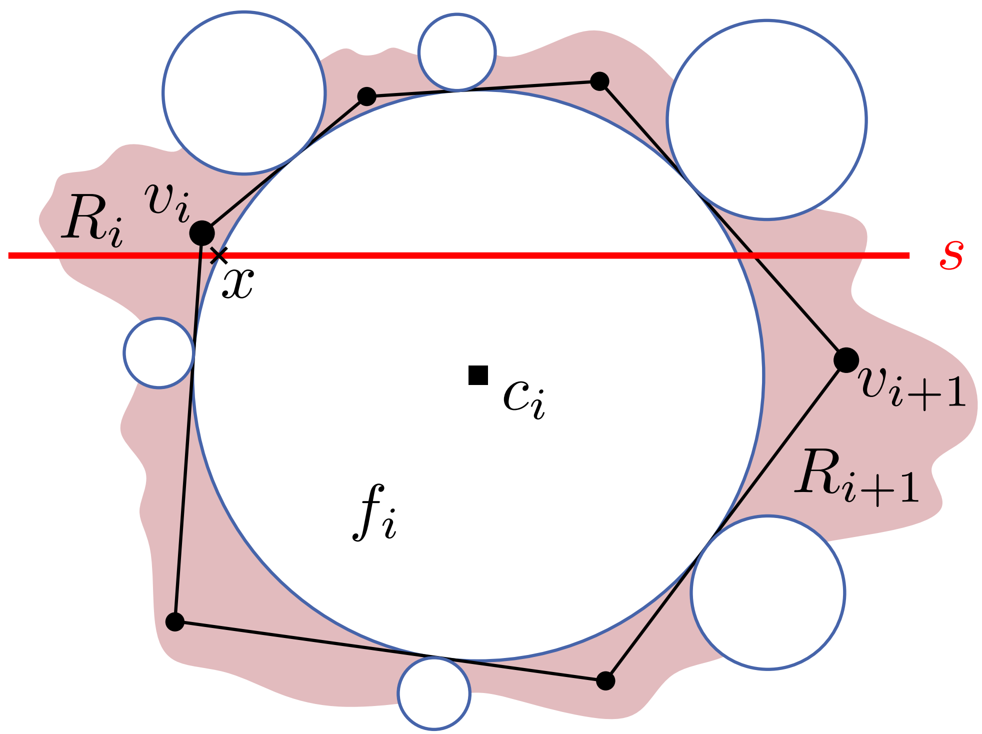


Each region R_i has some vertex v_i in its center.

Idea: Find path P_i from v_i of R_i to v_{i+1} of R_{i+1}, $1 \leq i < k$.

Concatenation yields strongly monotone (v_1, v_k)-path!

Walking around a Face Circle

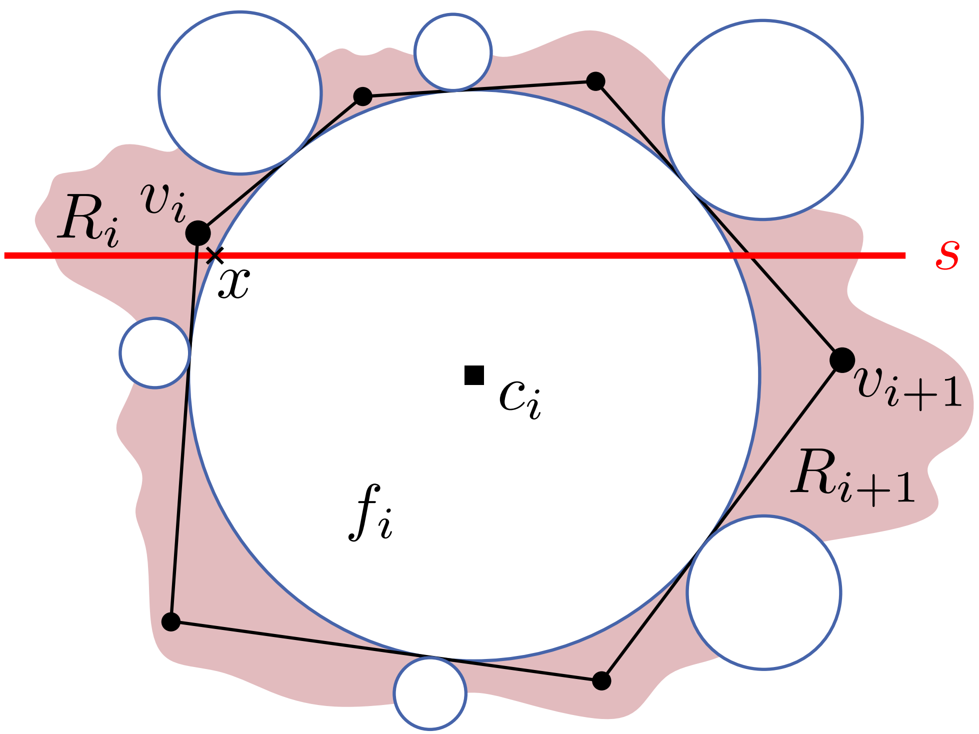


construction of $P_i = (e_1, \ldots, e_r)$

Walking around a Face Circle

Circle f_i is **inscribed circle of inner face**

$\Rightarrow$ **two paths** leading from v_i to v_{i+1}



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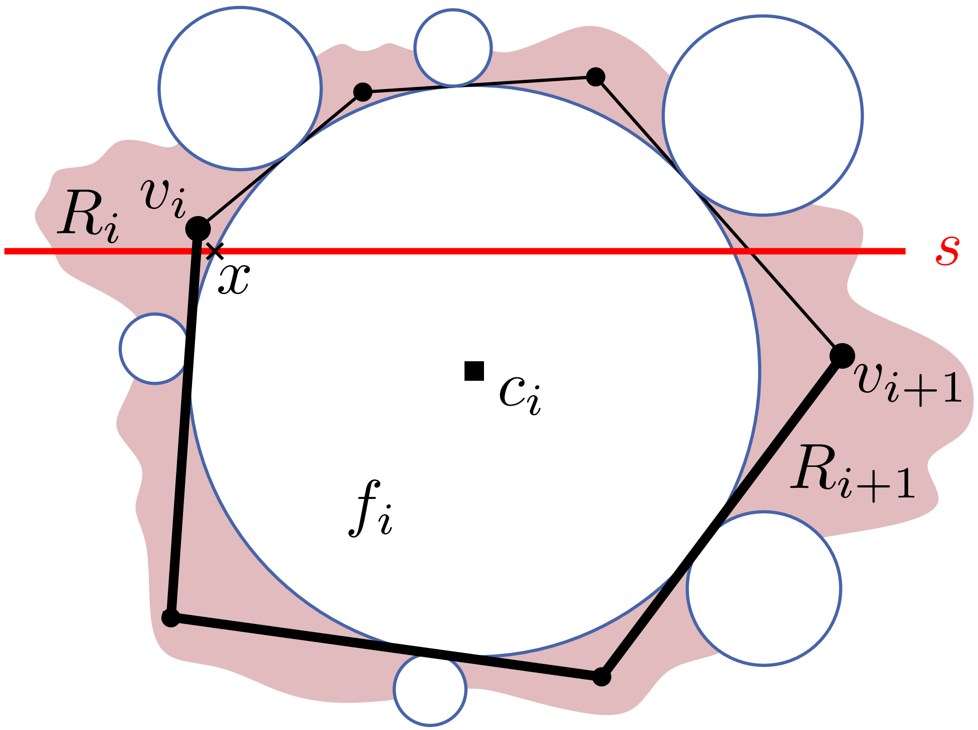

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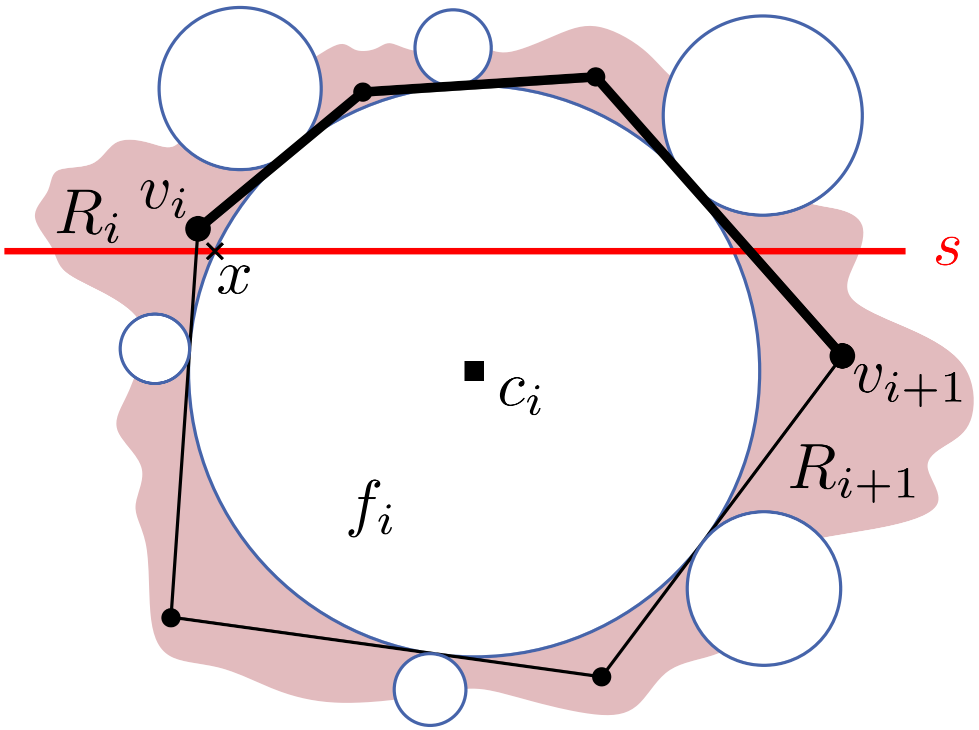

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Walking around a Face Circle

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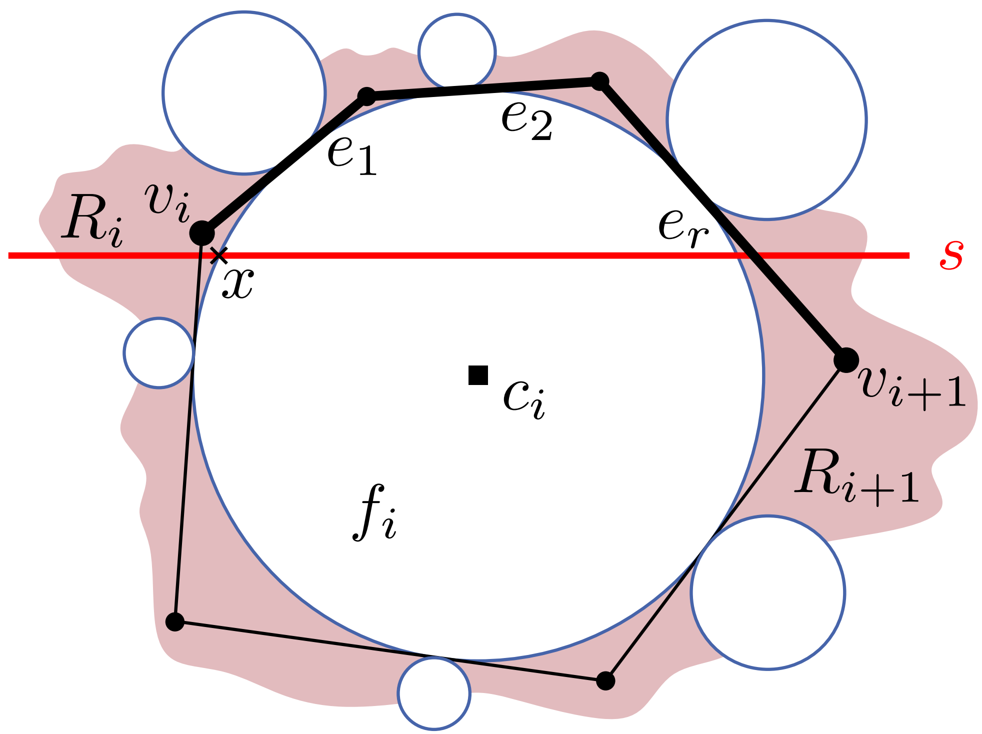
If c_i is below s, pick the upper path.
Otherwise the lower path.



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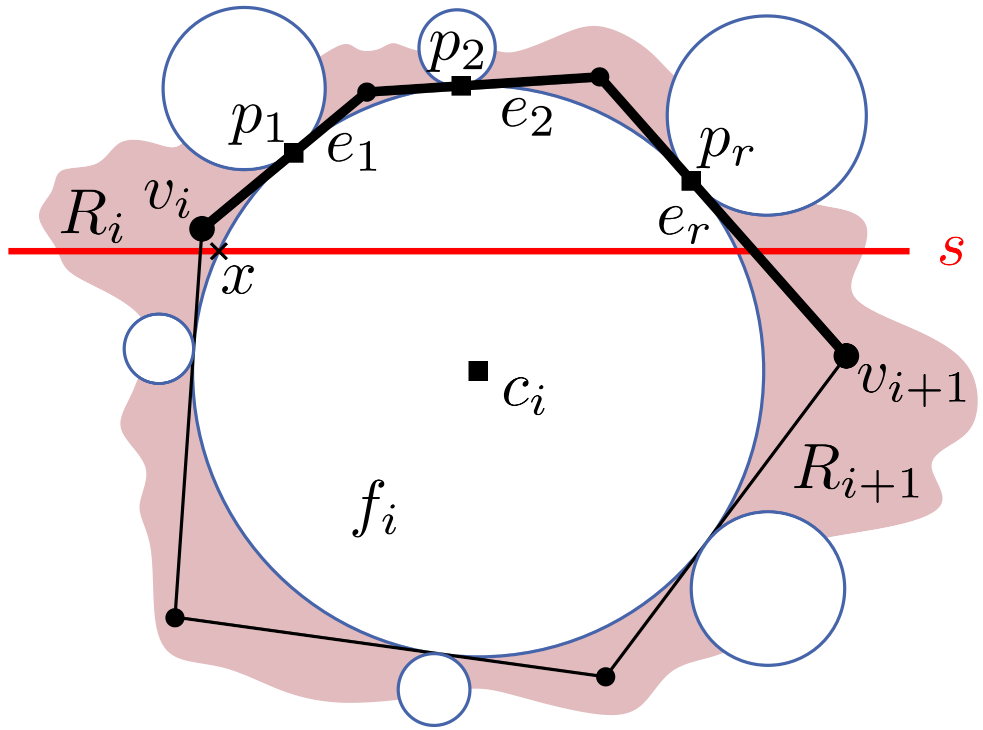
$e_1, \ldots, e_r$ tangent to f_i.

Edge points $p_1, \ldots, p_r$ **are above** s:



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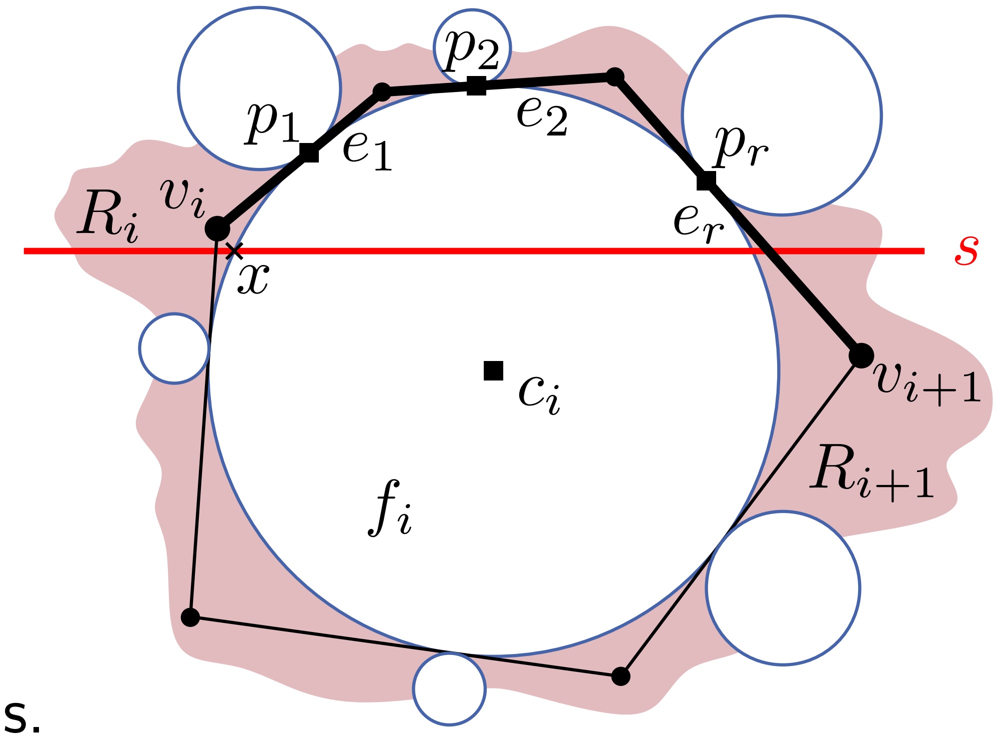
Edge points $p_1, \ldots, p_r$ **are above** s:

- Edge point of e_1 is above s
  $\Leftarrow$ v_i sees x and e_1 points upwards.



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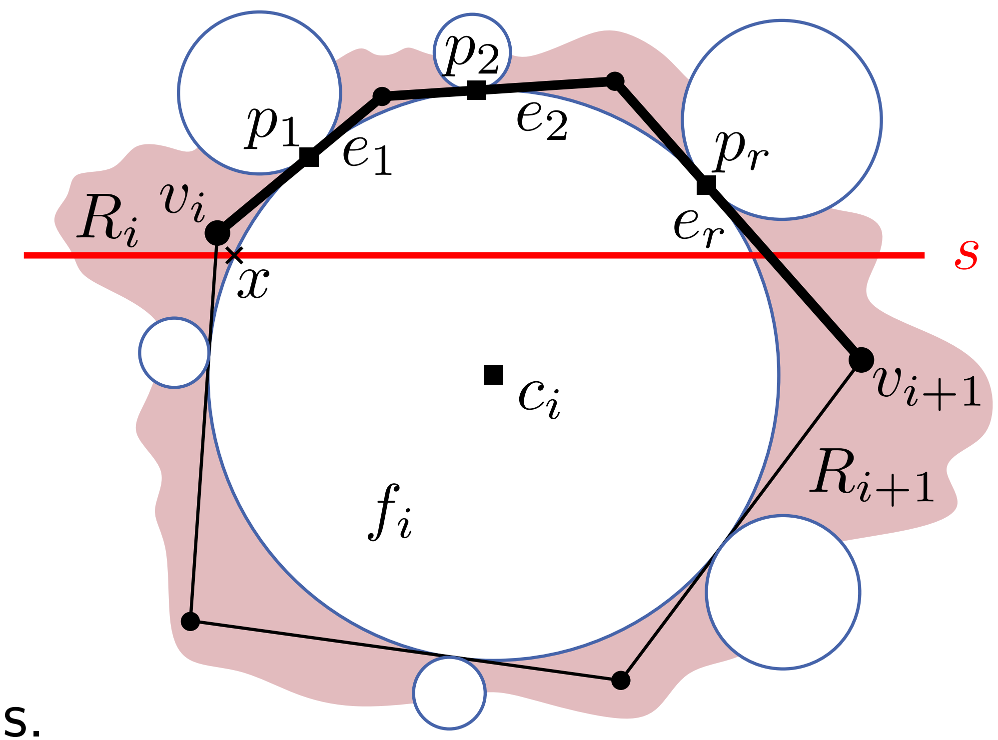

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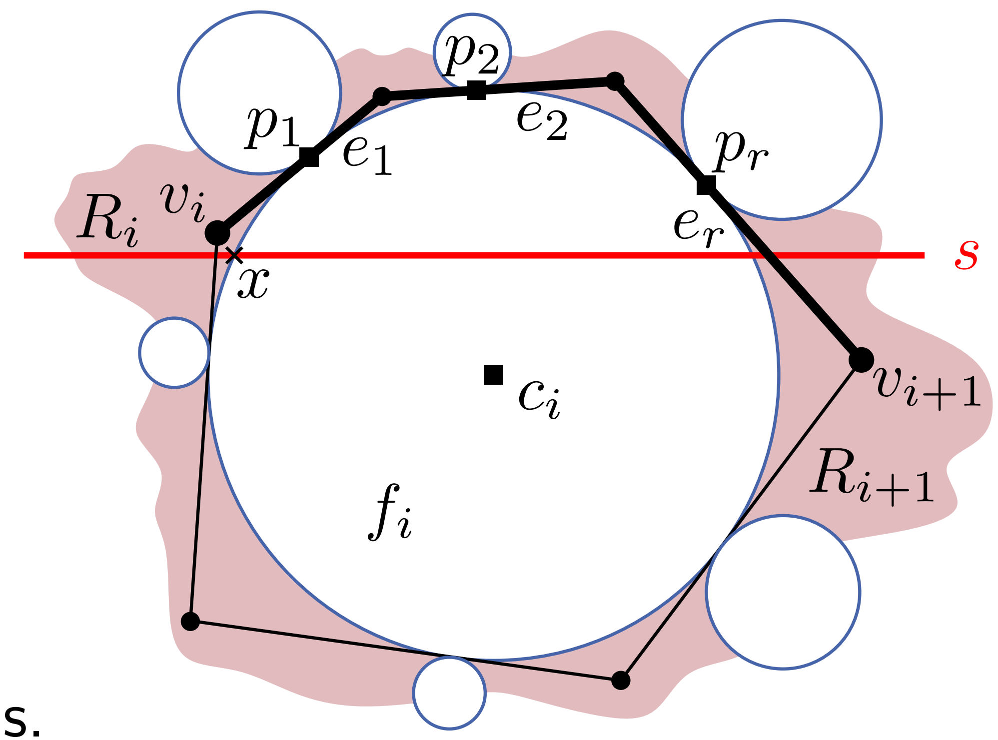

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$\Rightarrow$ all **edge points** $p_1, \ldots p_r$ **above** c_i.

$p_1, \ldots, p_r$ **in this order** around f_i.

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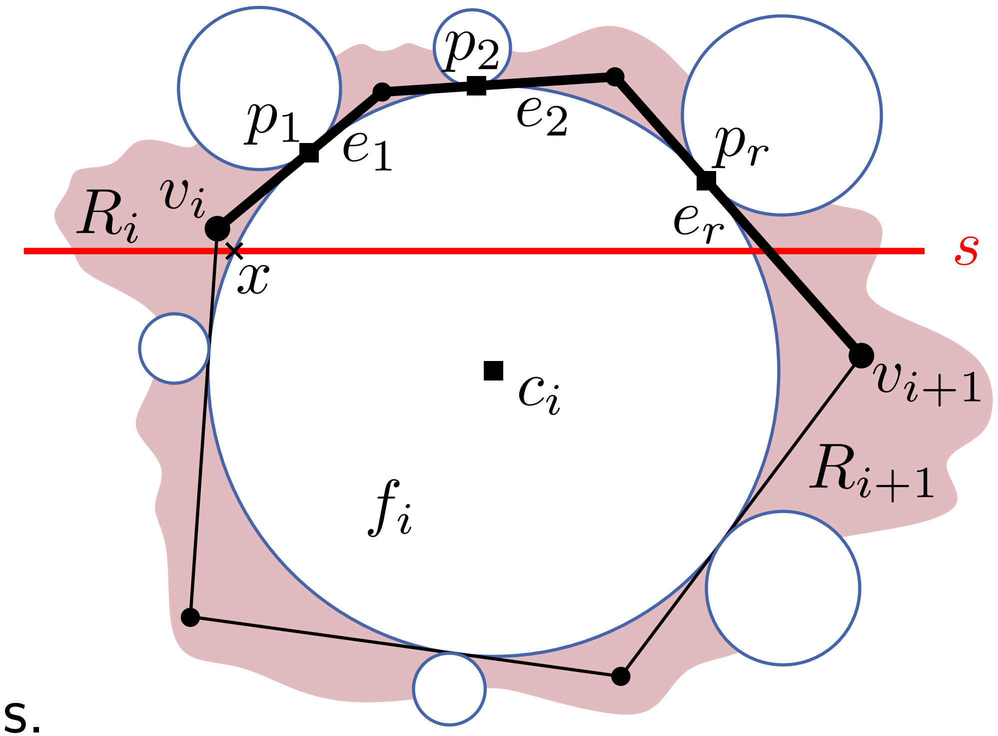


construction of $P_i = (e_1, \ldots, e_r)$

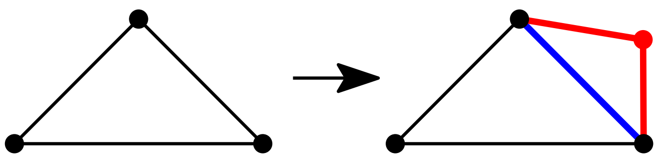
$\Rightarrow$ all **edge points** $p_1, \ldots p_r$ **above** c_i.
$p_1, \ldots, p_r$ **in this order** around f_i.

$\Rightarrow$ x-coordinates increase

$\Rightarrow$ strongly monotone

2-Trees

Graph generated from a K_3 by **stacking** vertices on edges.



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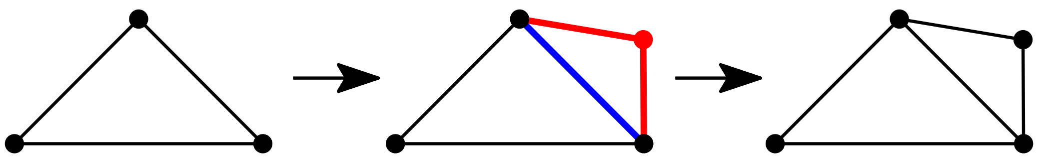

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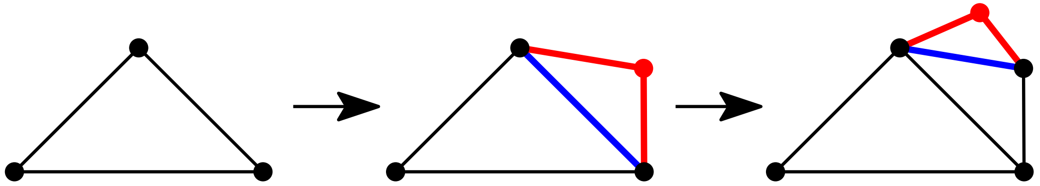

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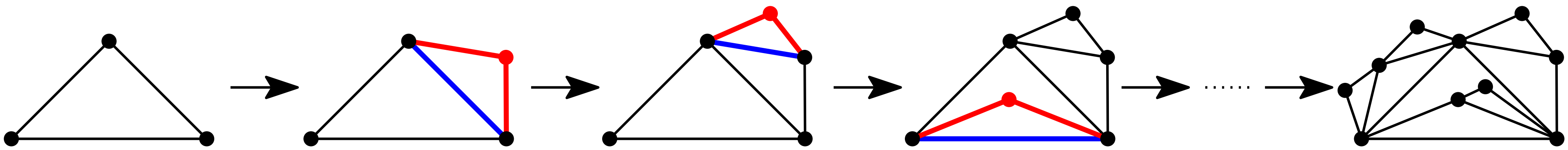

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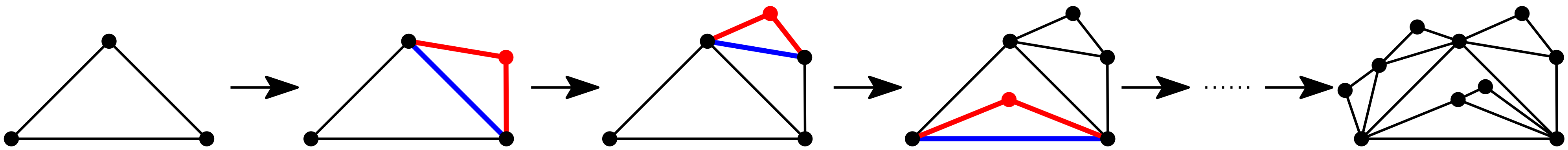
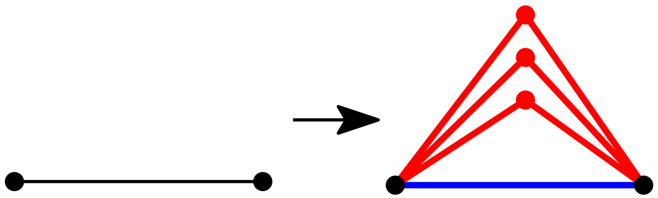


Equivalently:

Graph generated from an edge by **multistacking once per edge**.



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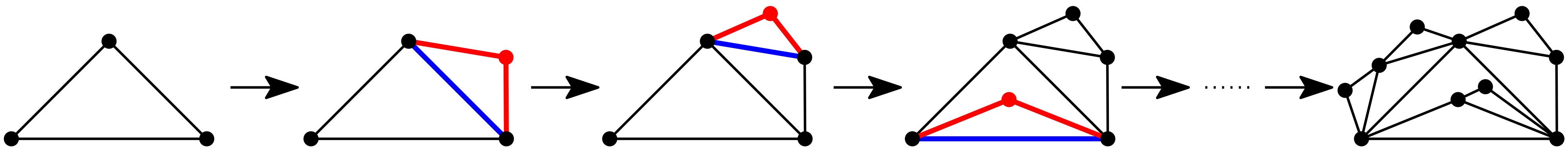

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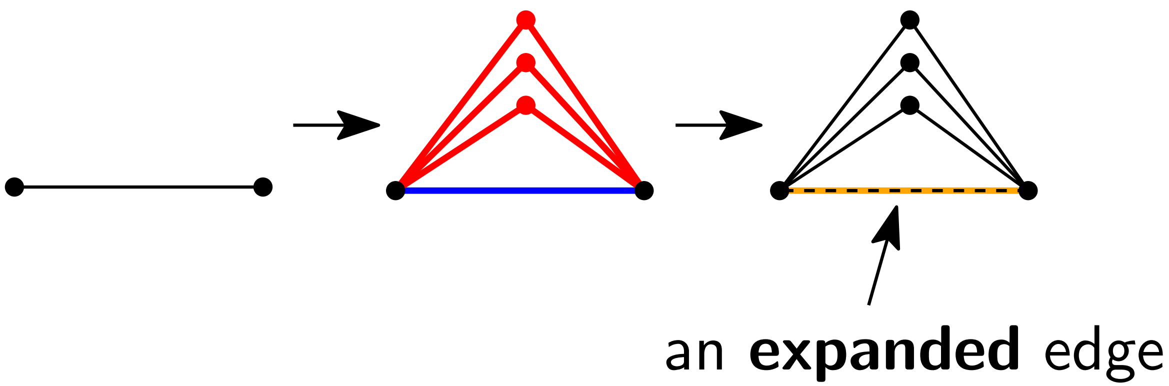

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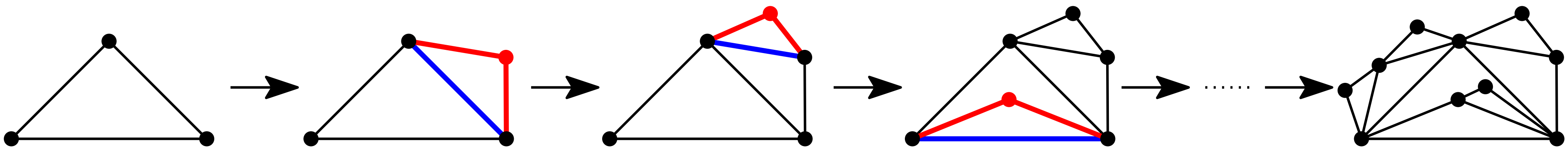

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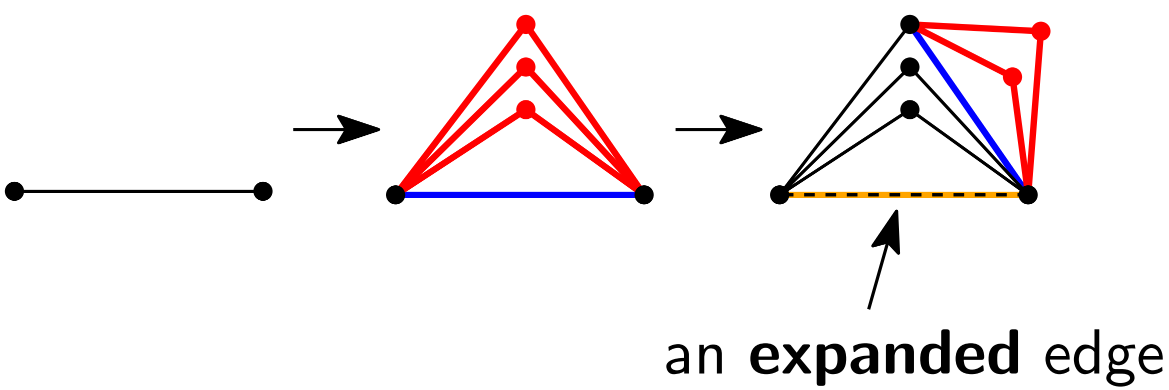

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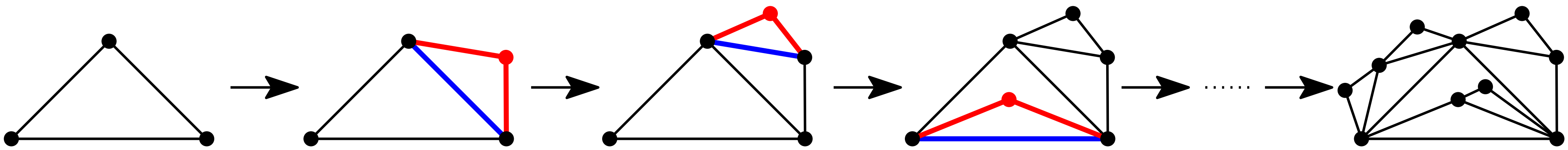

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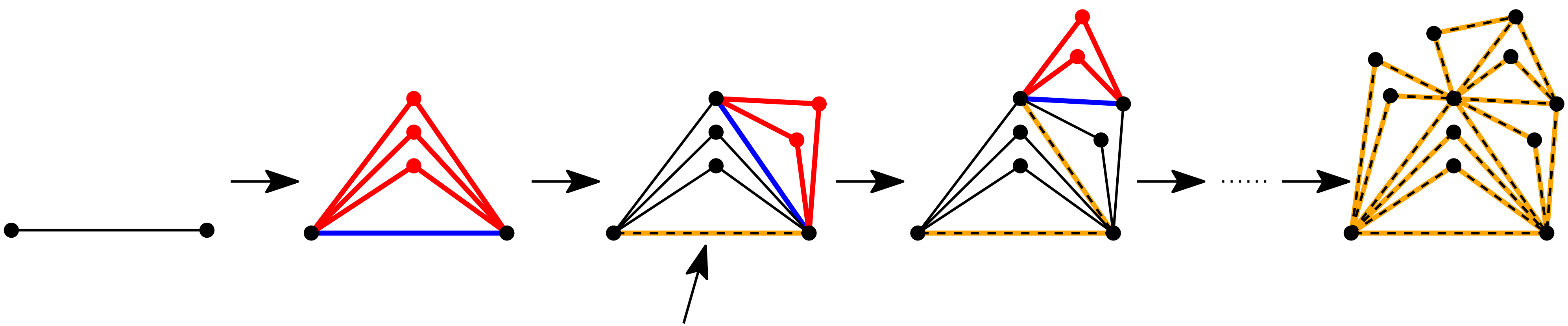

an **expanded** edge

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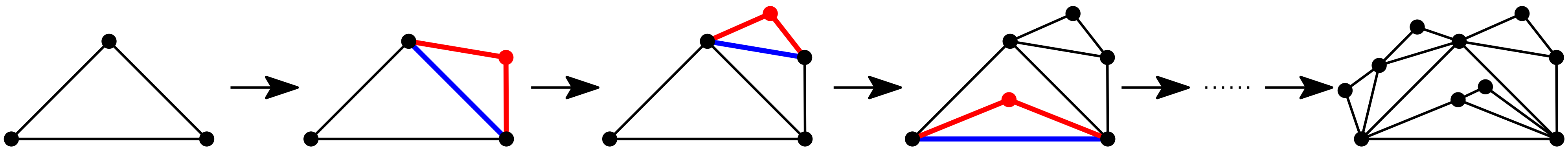

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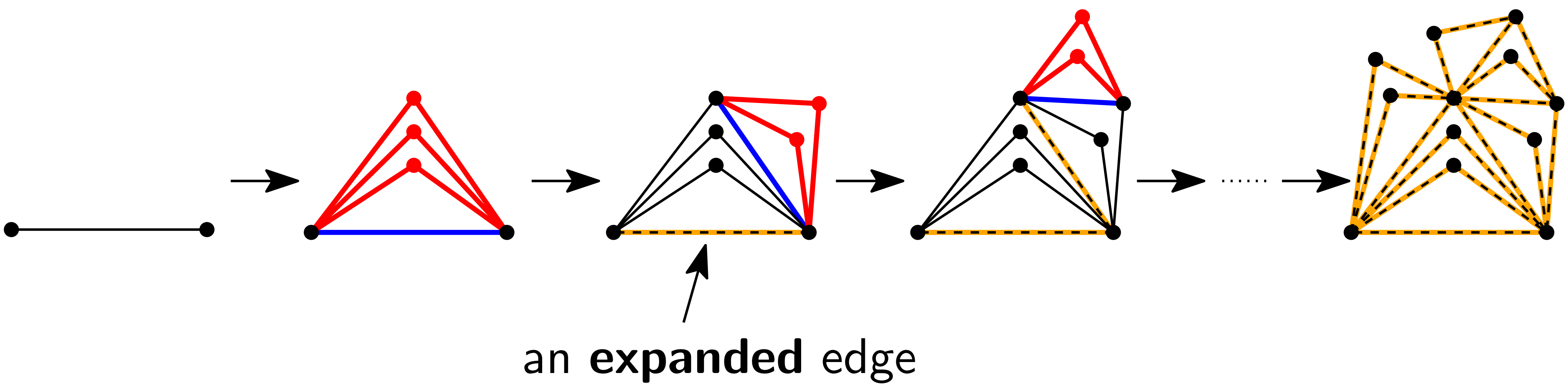
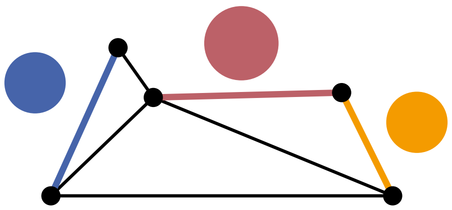


Theorem: Every 2-tree has a strongly monotone drawing.

Definitions

A **drawing with bubbles** (**DWB**) is a straight-line drawing of $G = (V, E)$ where every edge in some $E' \subseteq E$ is associated with a **bubble**.



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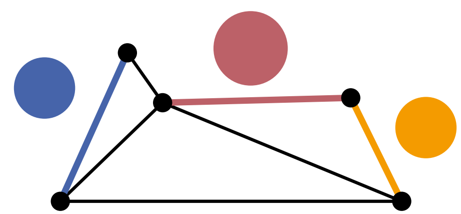
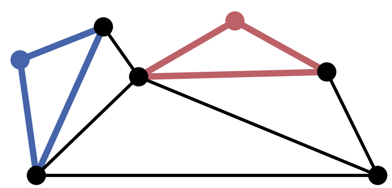


An **extension** of a DWB is a drawing obtained by stacking one vertex into each bubble of some $E'' \subseteq E'$.



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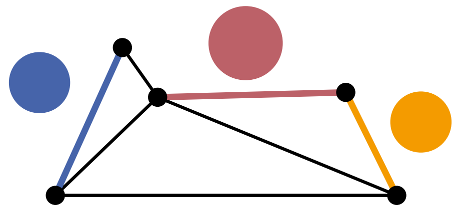

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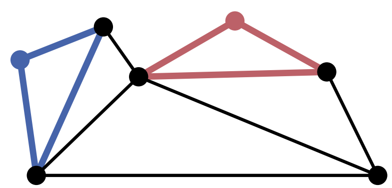

A DWB is **strongly monotone** if every extension is strongly monotone.

i.e.
- the drawing without bubbles is strongly monotone
- stacking into bubbles maintains strong monotonicty

Drawing a 2-Tree

Idea: Draw according to **multistacking** sequence.

Invariant: Before / after each multistack ...

- every non-expanded edge has a bubble
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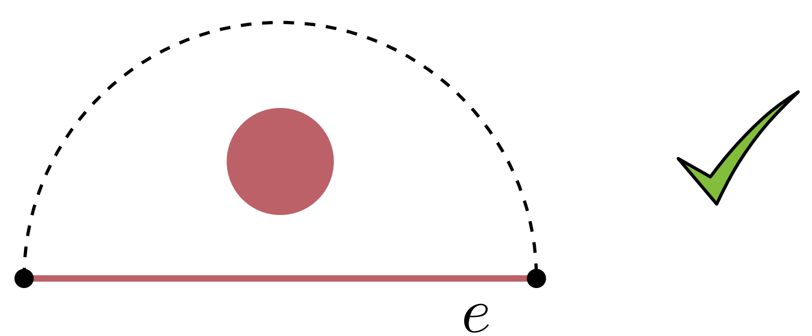
Base case:



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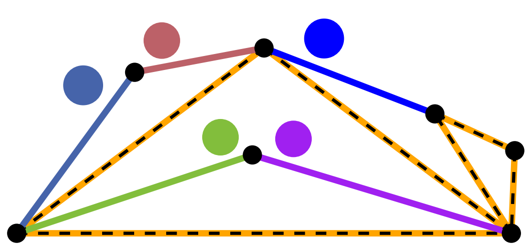

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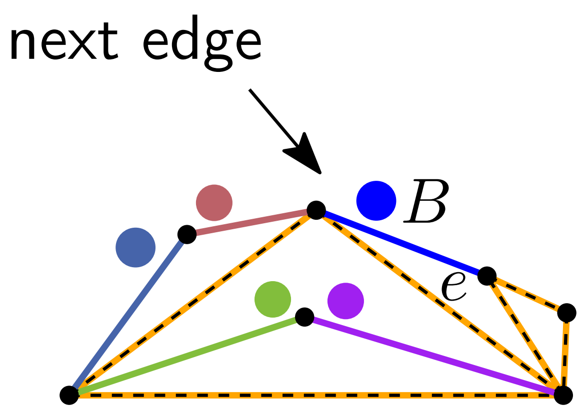

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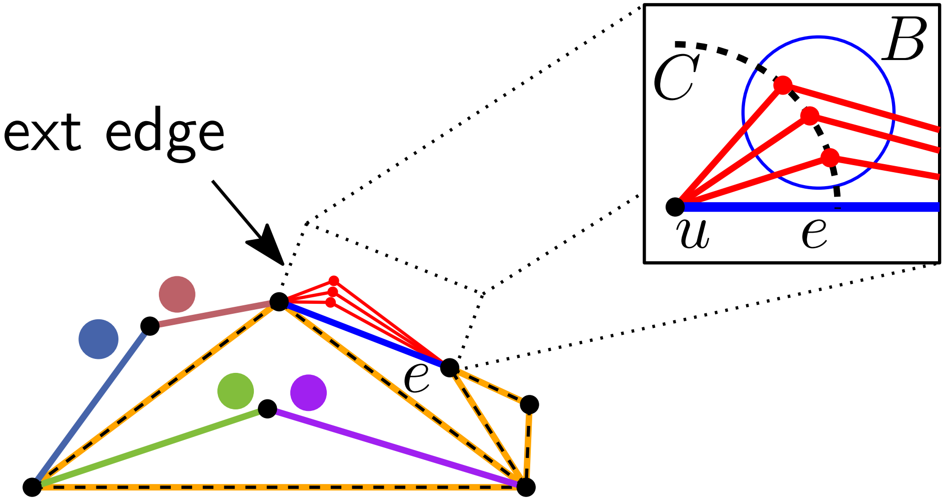
place vertices on circular arc C centered at u

next edge



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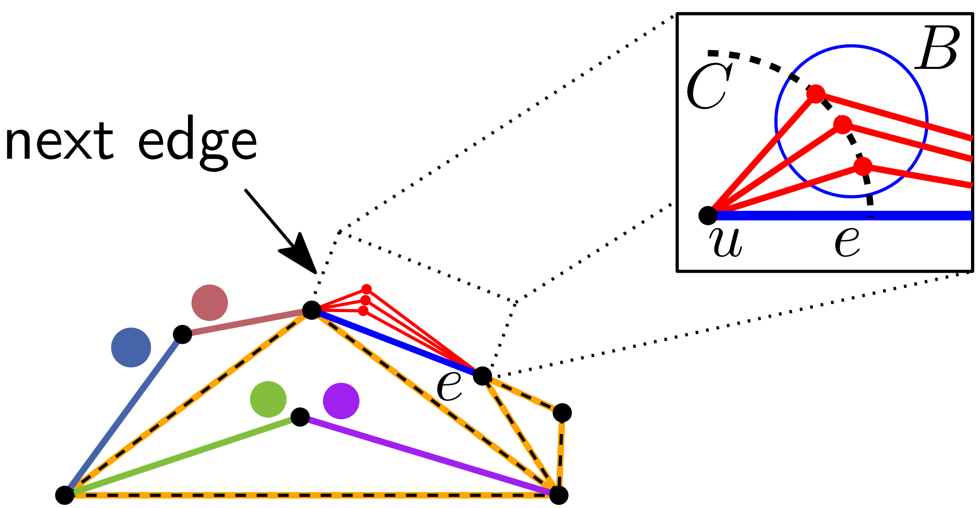

New DWB is strongly monotone:

- paths between 2 new vertices
- paths between new vertex and old vertex / bubble

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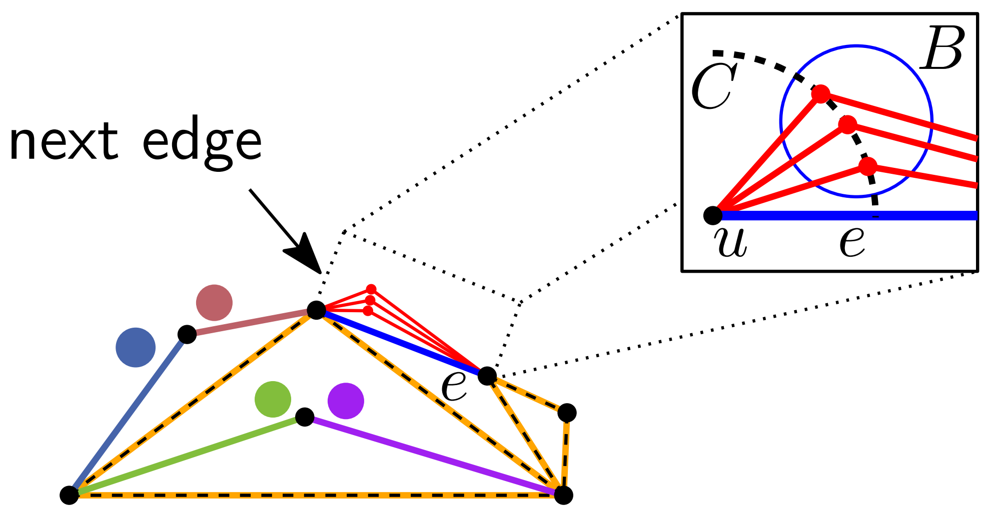

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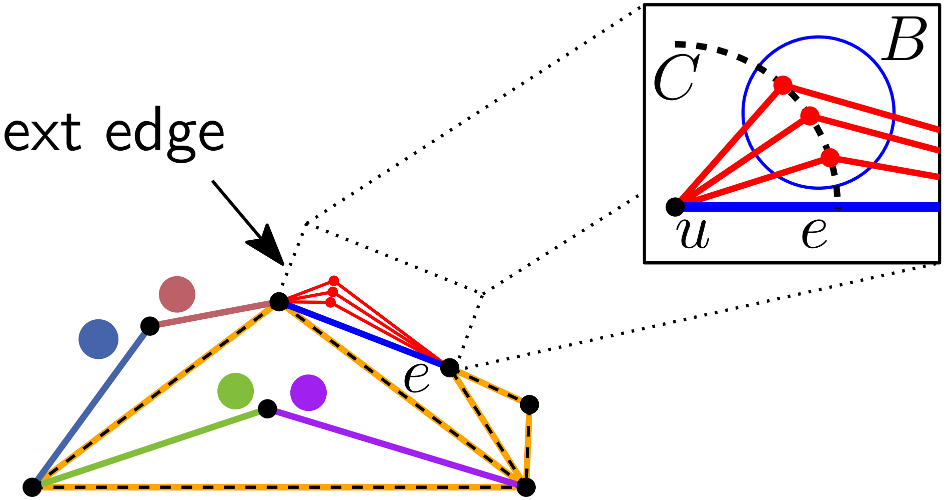

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Not done yet!

How to add bubbles for new edges?

Induction step:

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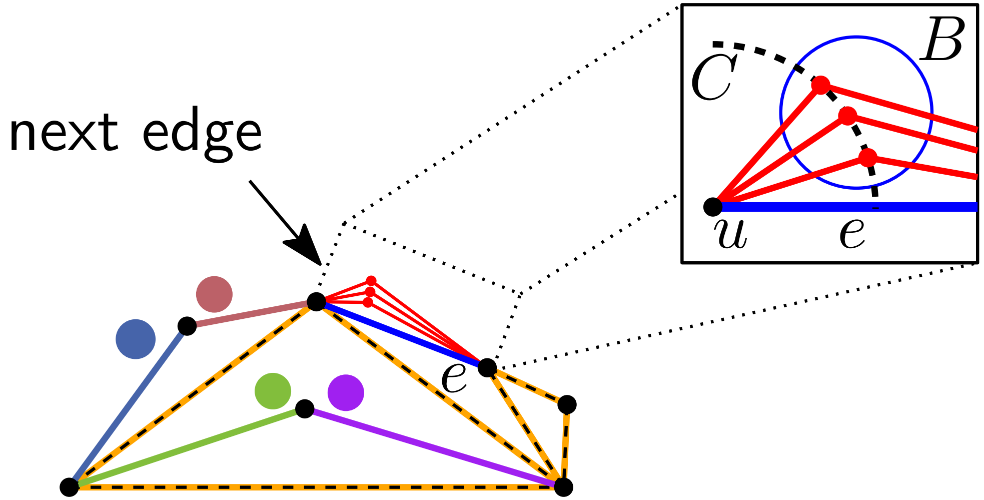

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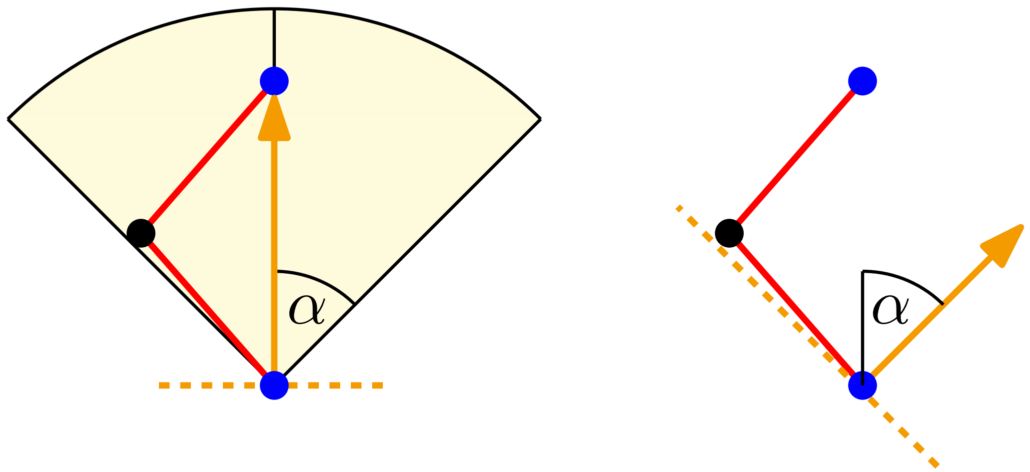
Adding Bubbles

A strongly monotone path P is α-**safe** if it remains monotone while the direction is tilted by at most α.



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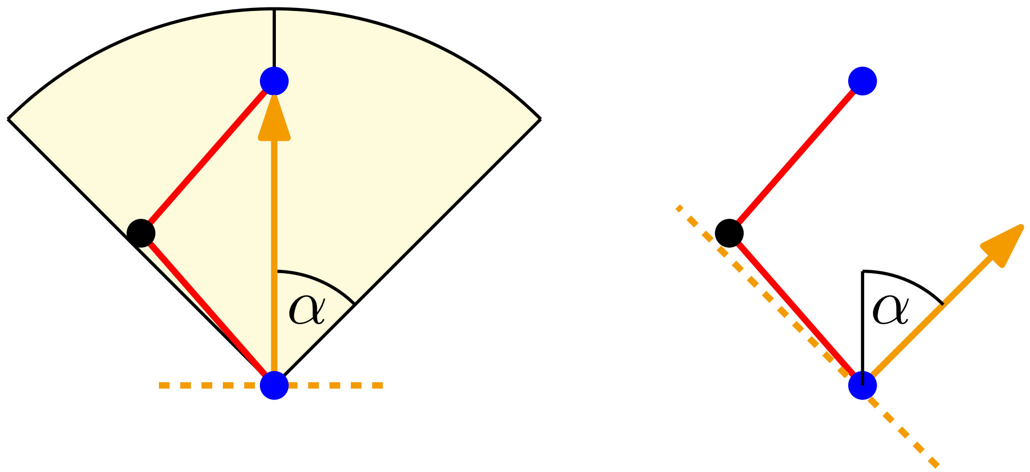

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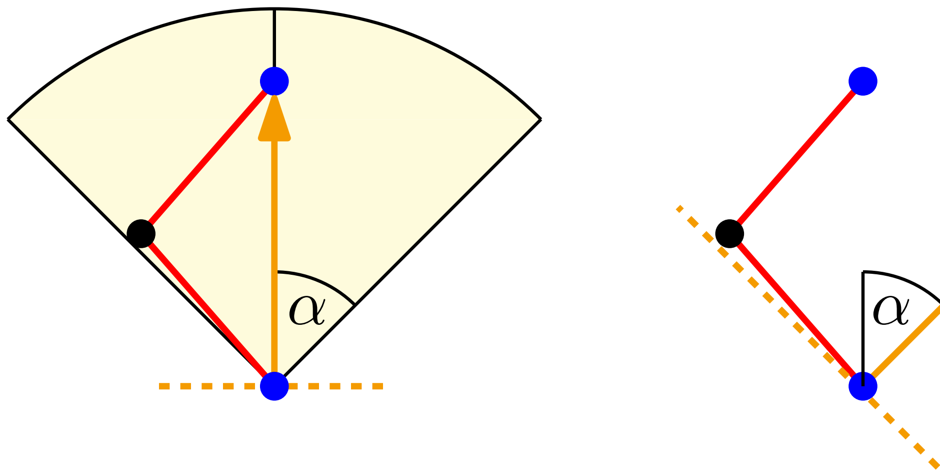

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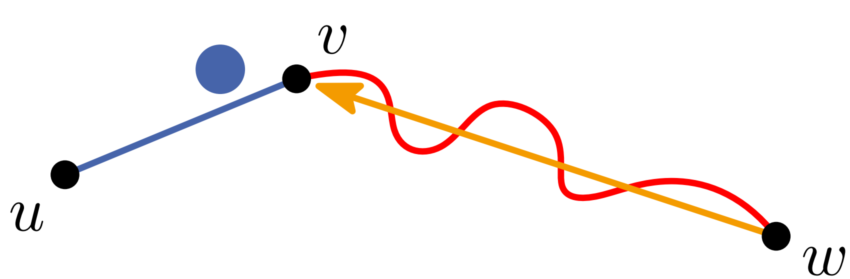

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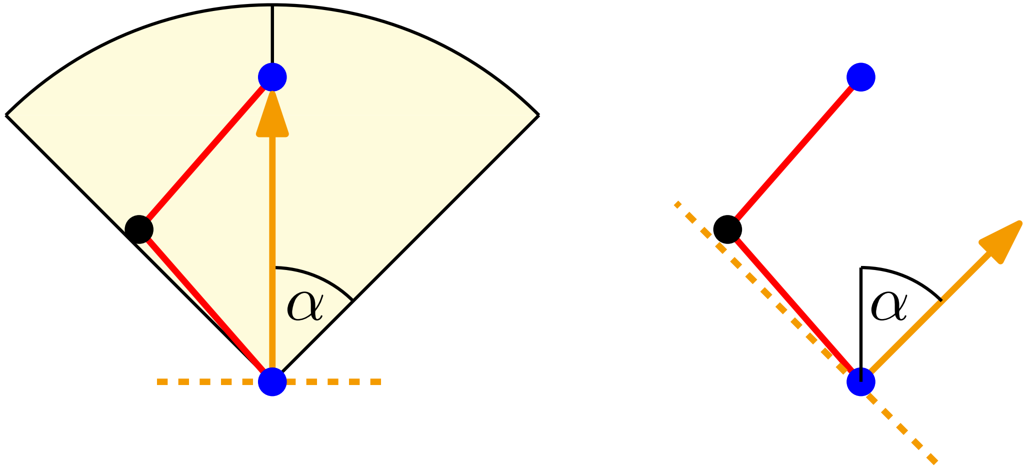

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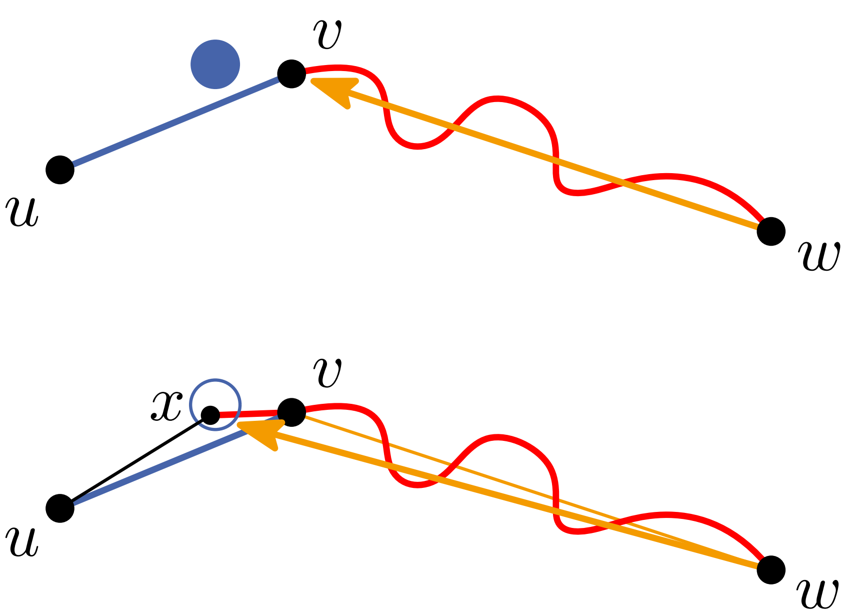

Related Work & Results & Open Problems

Strongly monotone drawings:

- do not exist for every planar graph [Kindermann et al. '14]
- exist for every 2-connected outerplanar graph [Kindermann et al. '14]
- exist for every tree [Kindermann et al. '14]
  - area required can be exponential [Nöllenburg et al. '14]

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} **our results**

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Thank you!