Computational Geometry

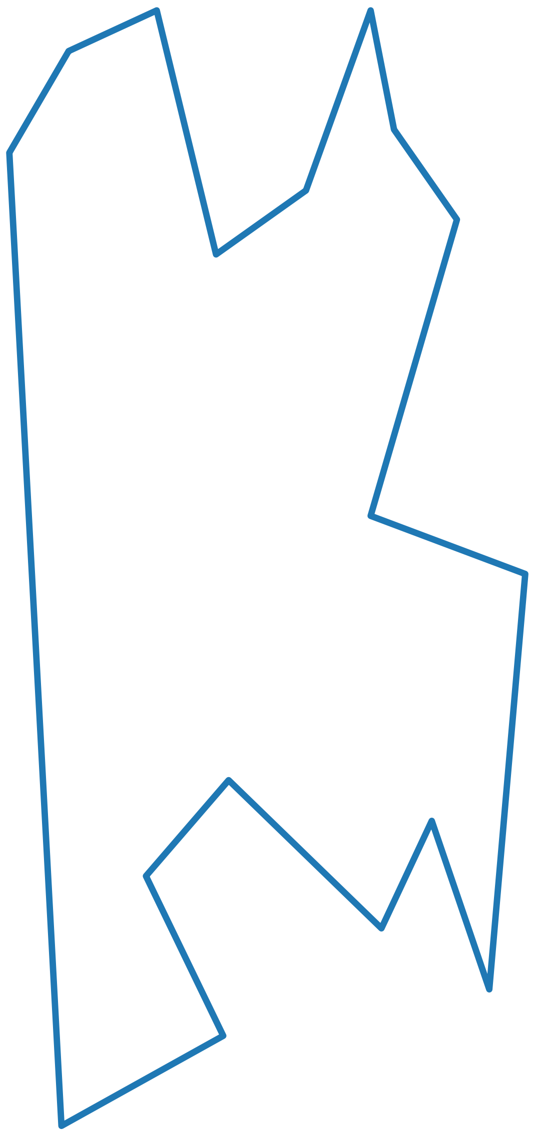
Lecture 12: Seidel's Triangulation Algorithm

Part I: General Idea

Philipp Kindermann

Winter Semester 2020

Triangulating a Polygon



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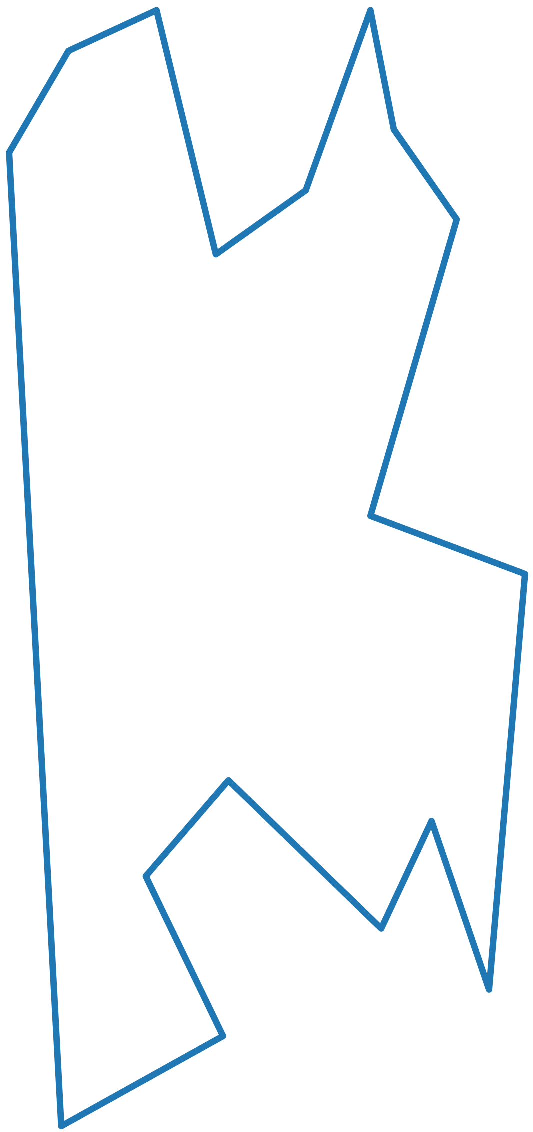

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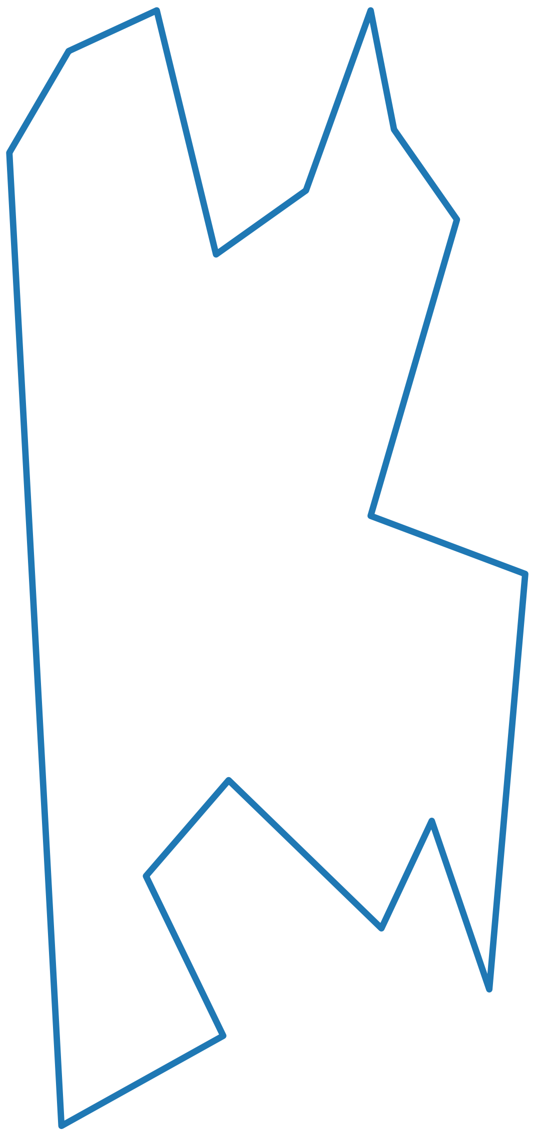

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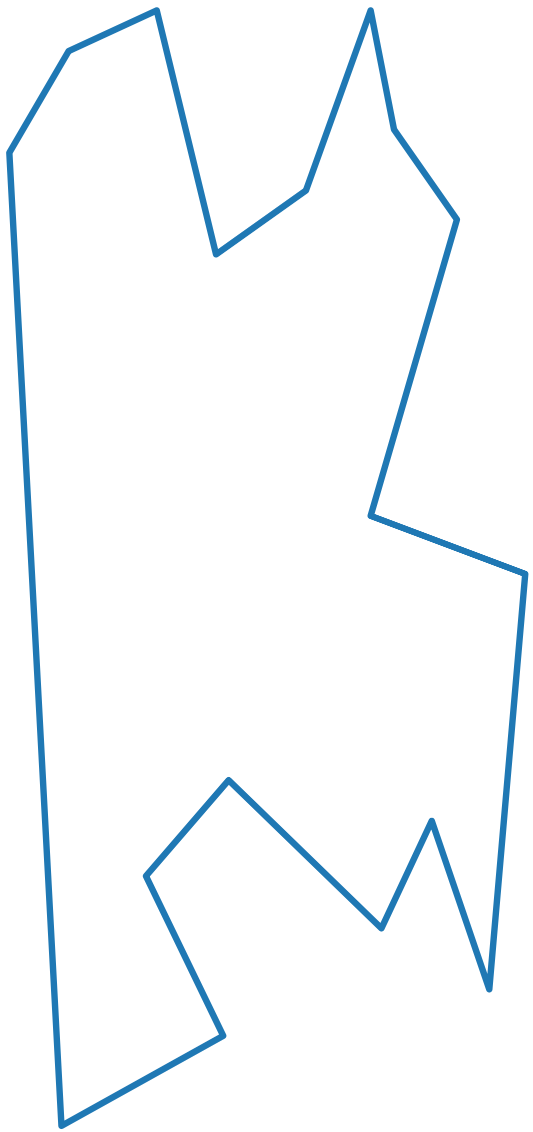

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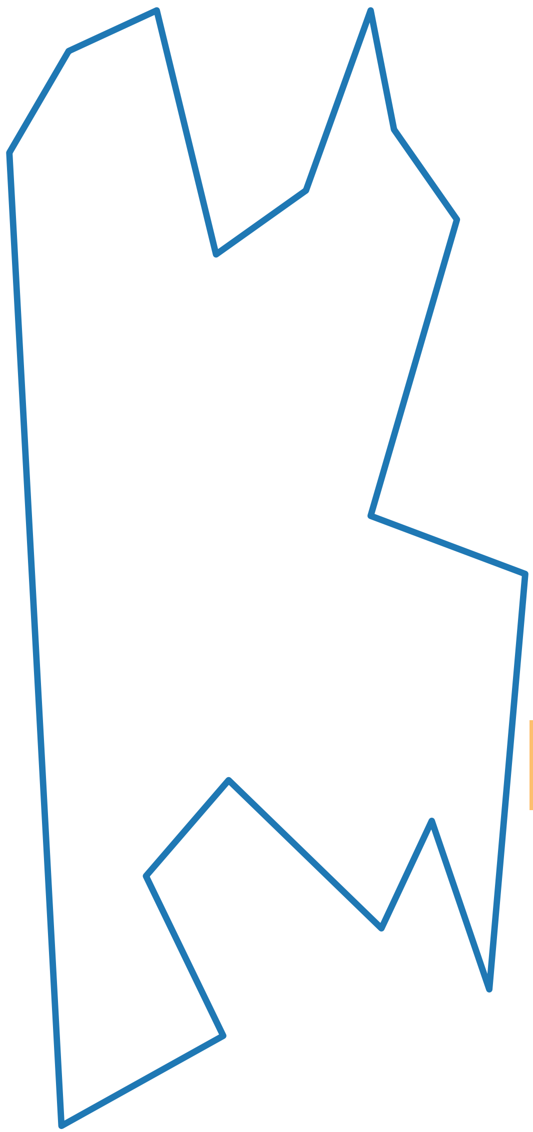

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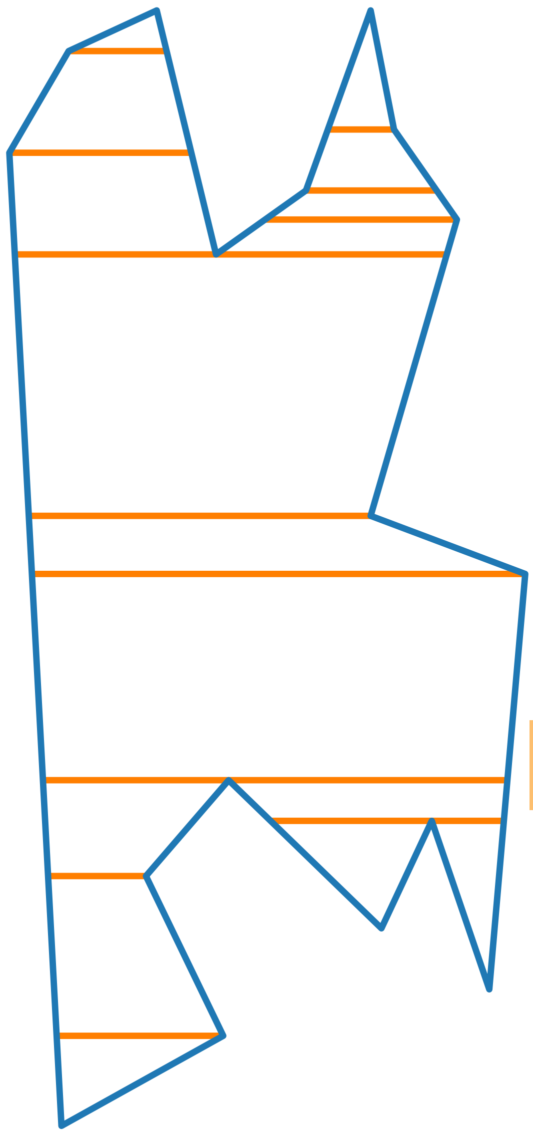

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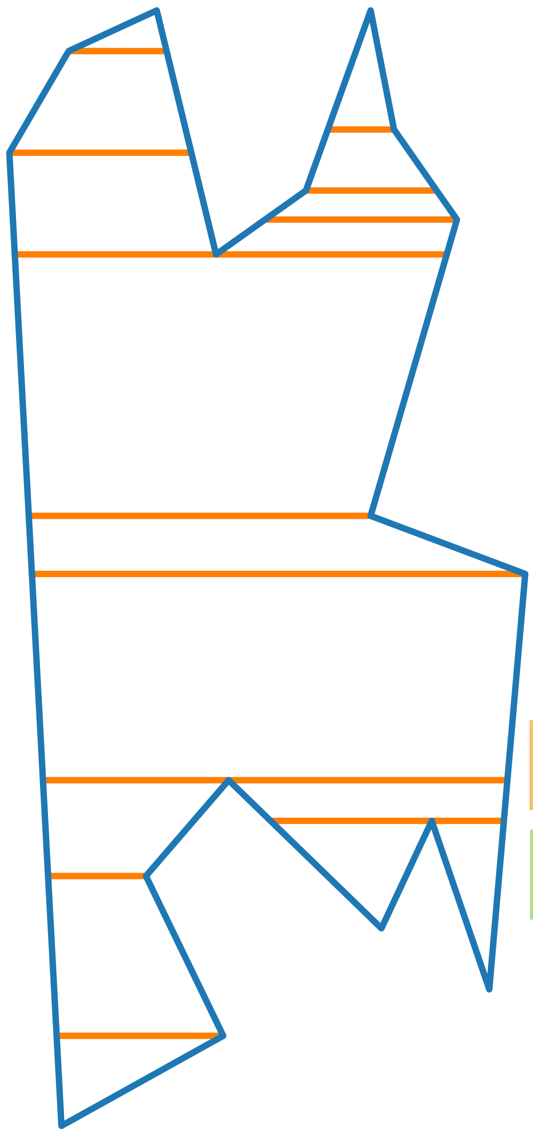

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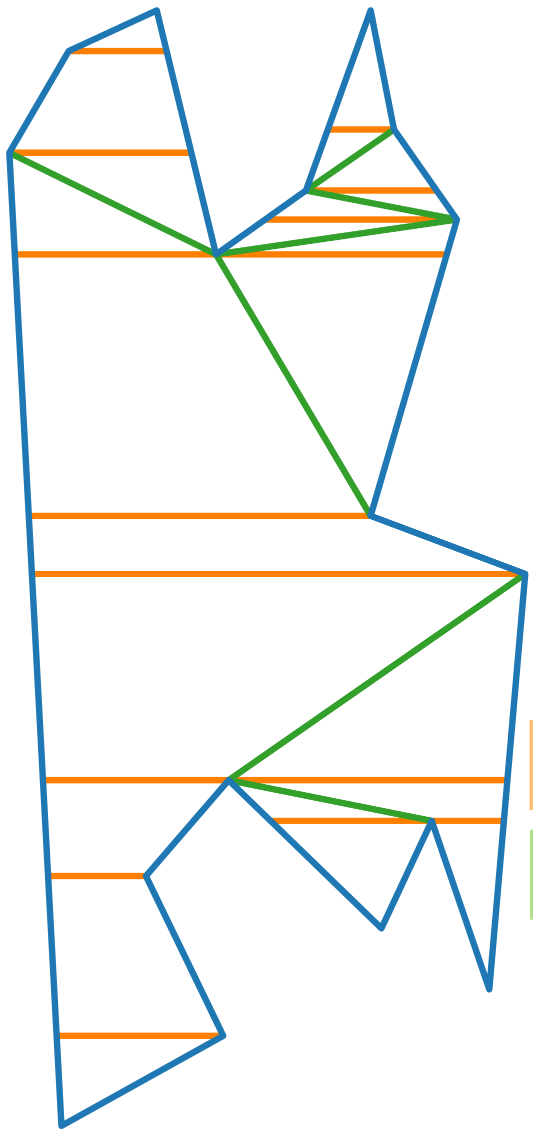

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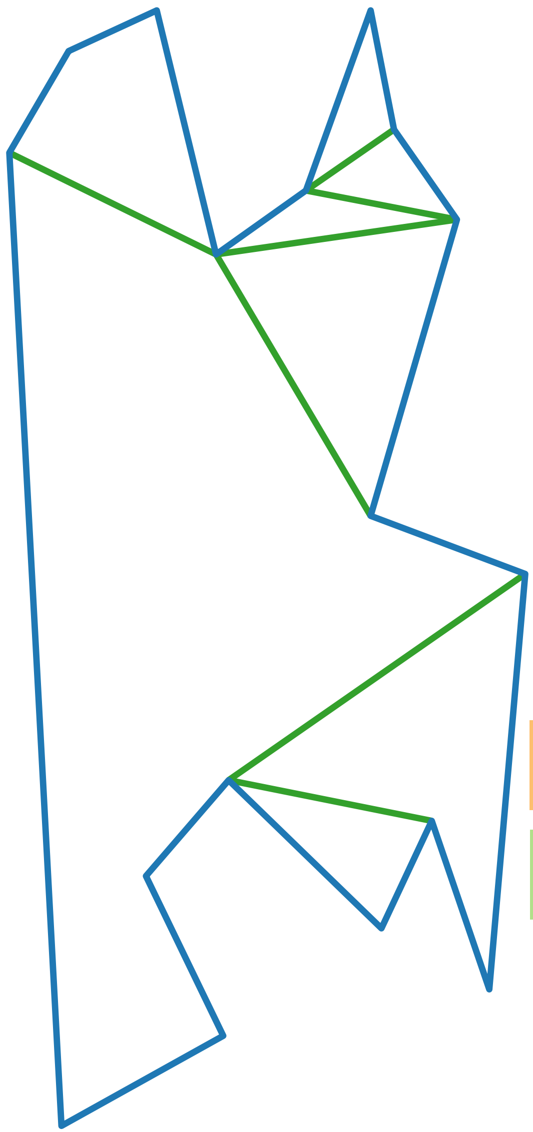

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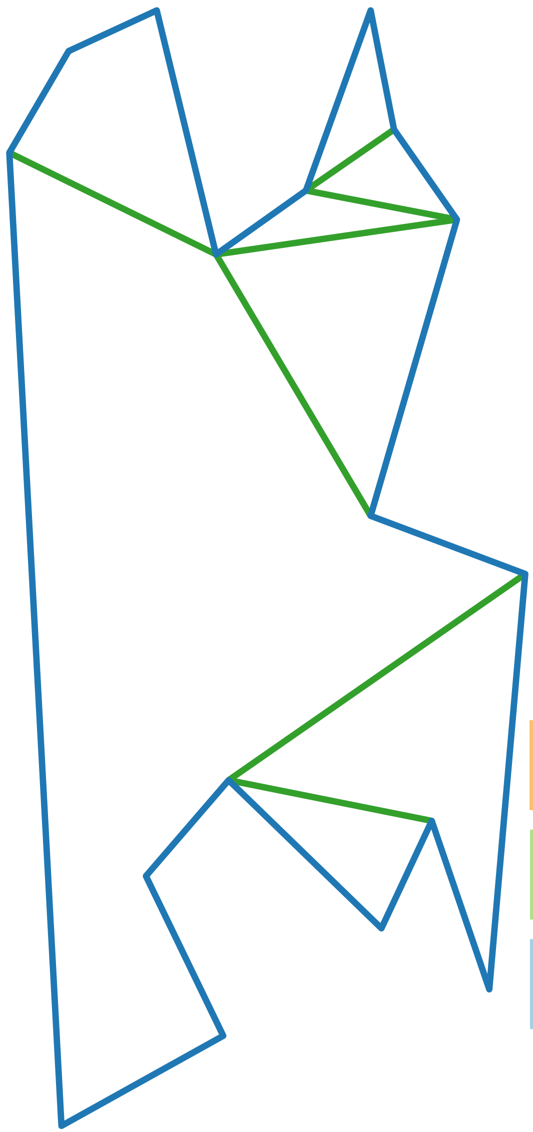

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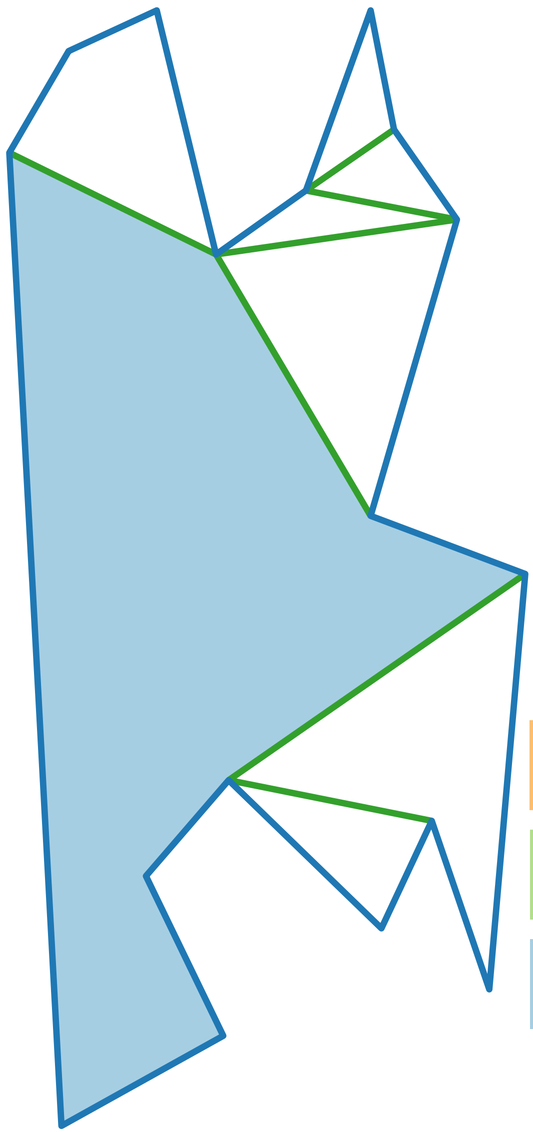

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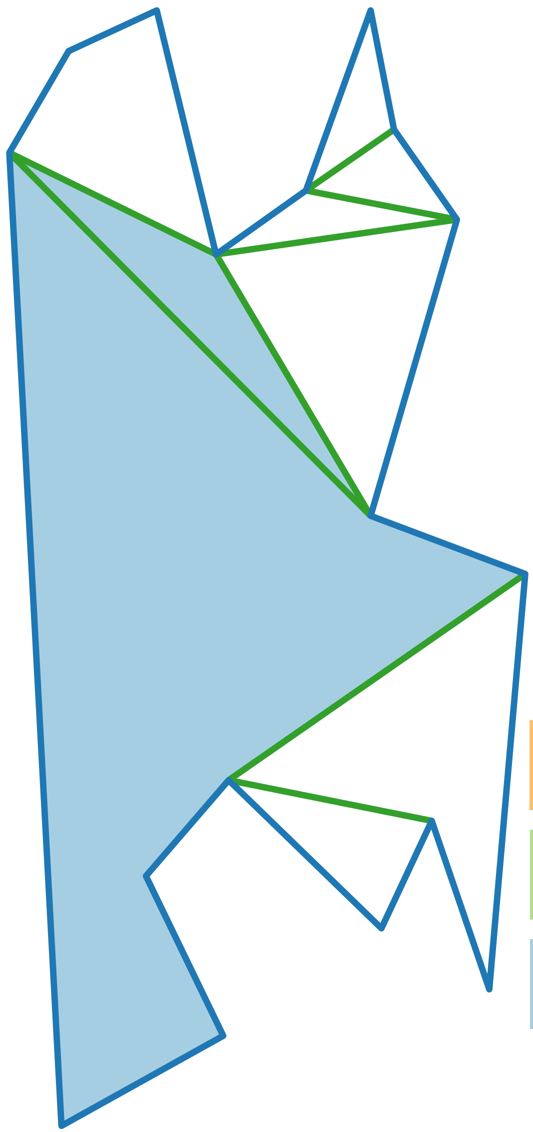

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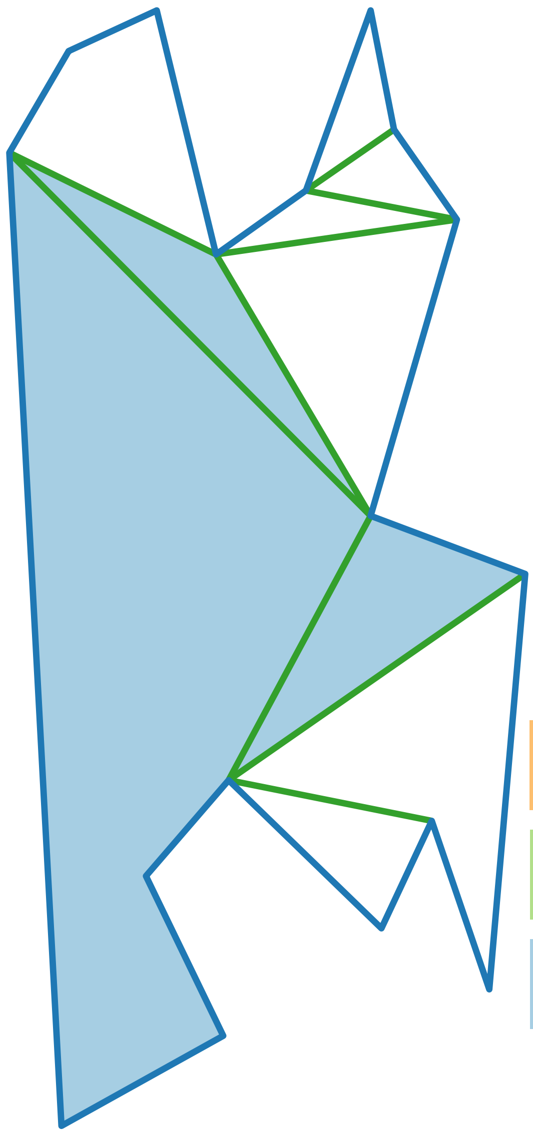

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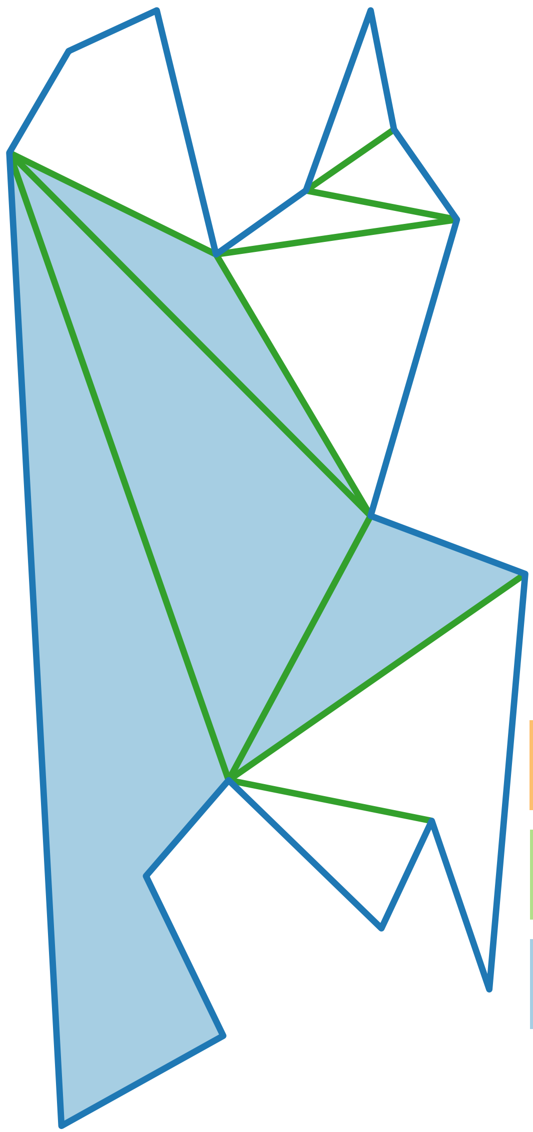

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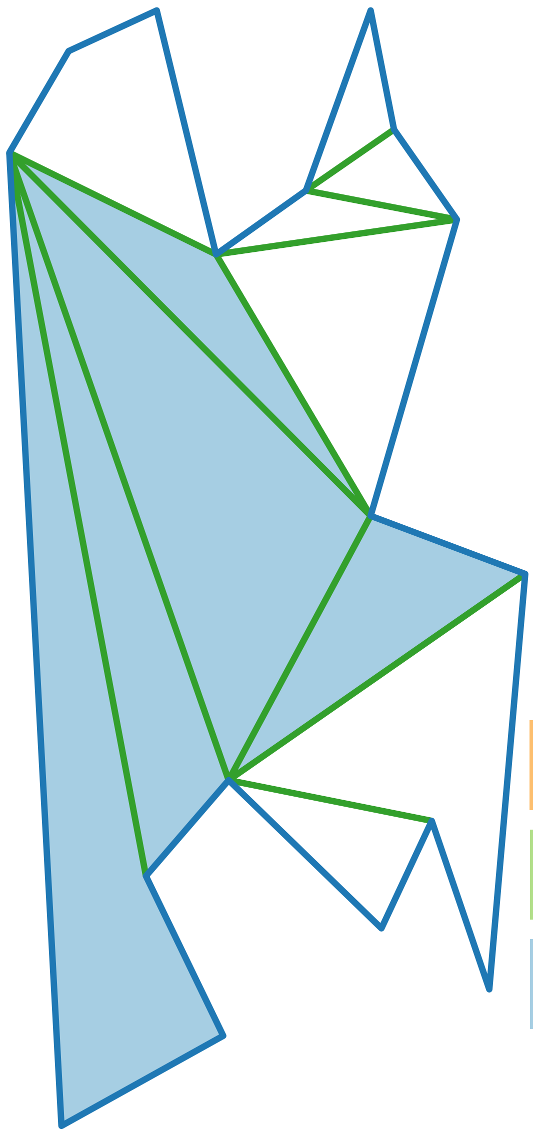

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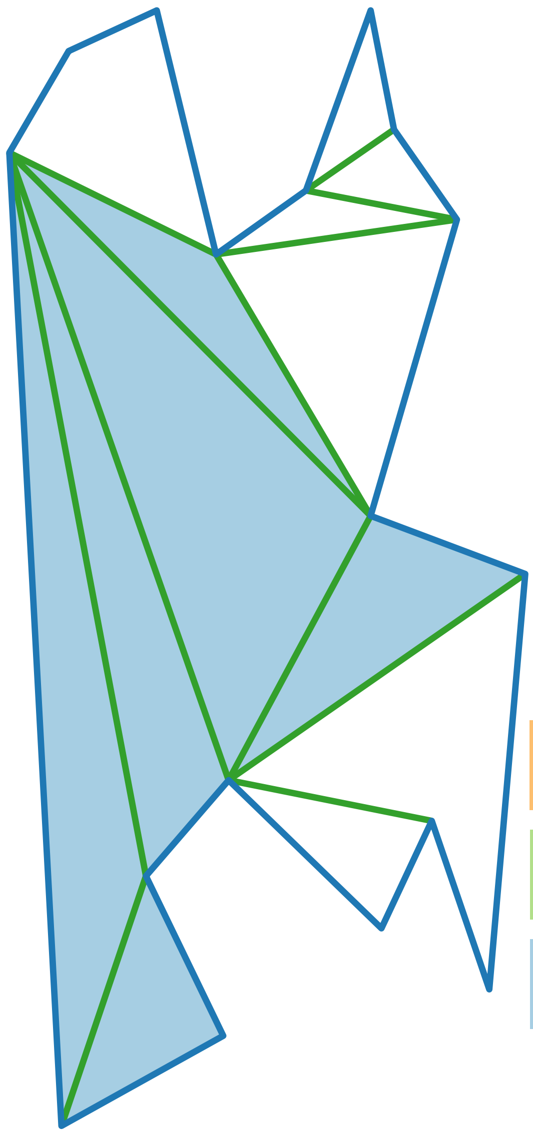

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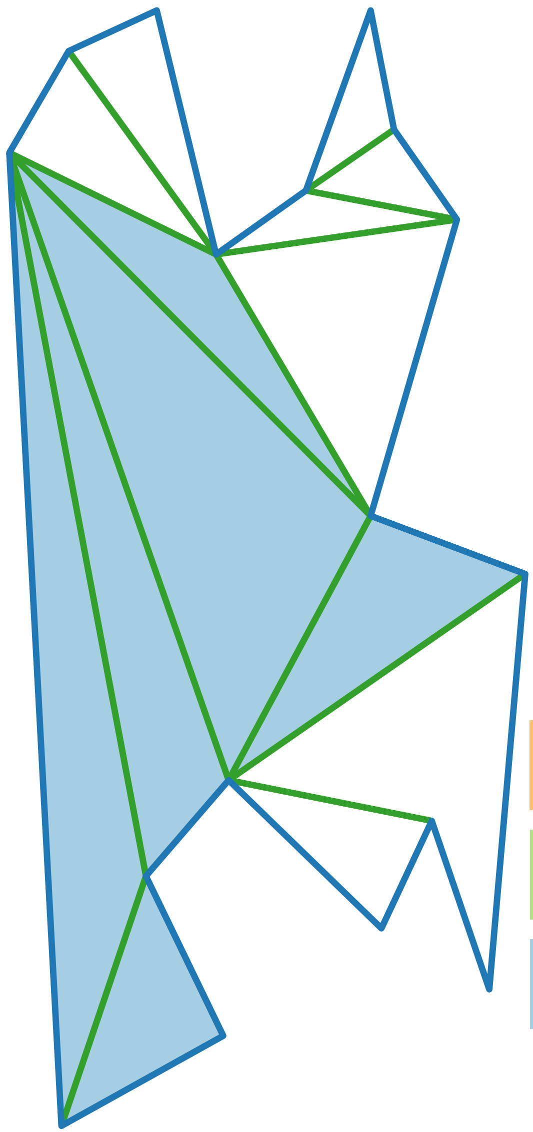

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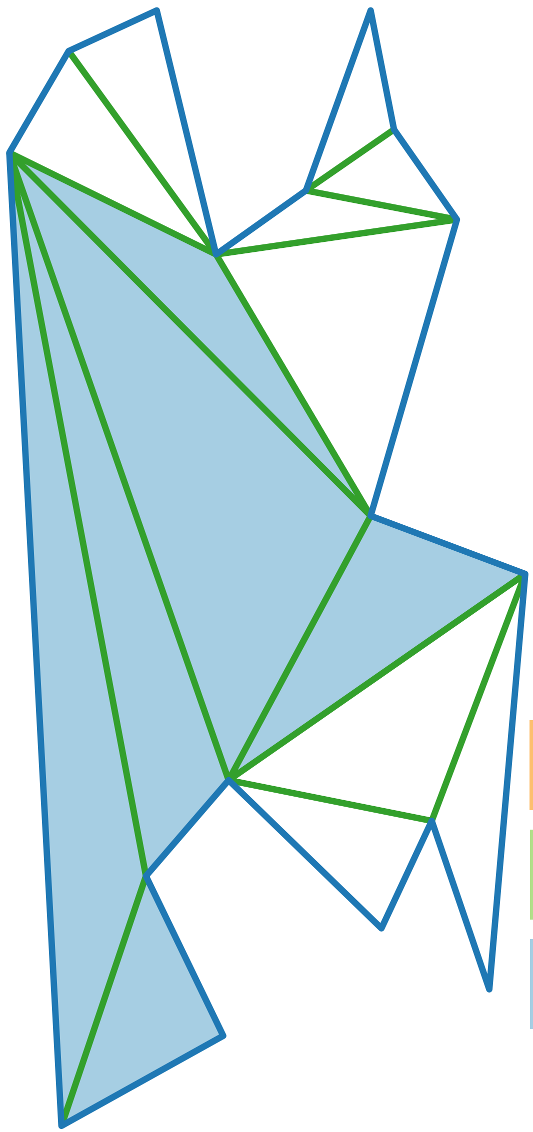

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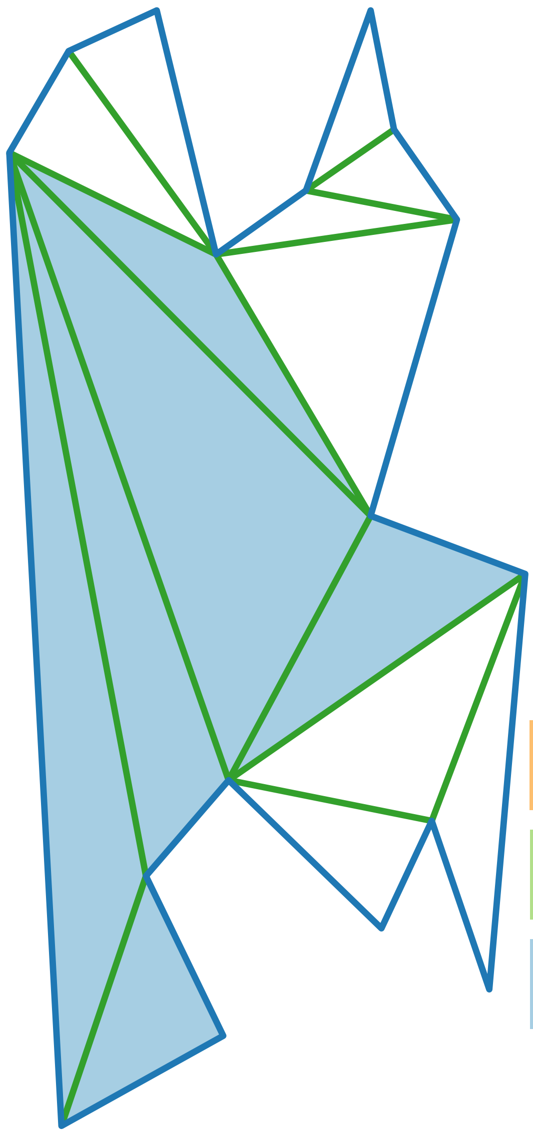

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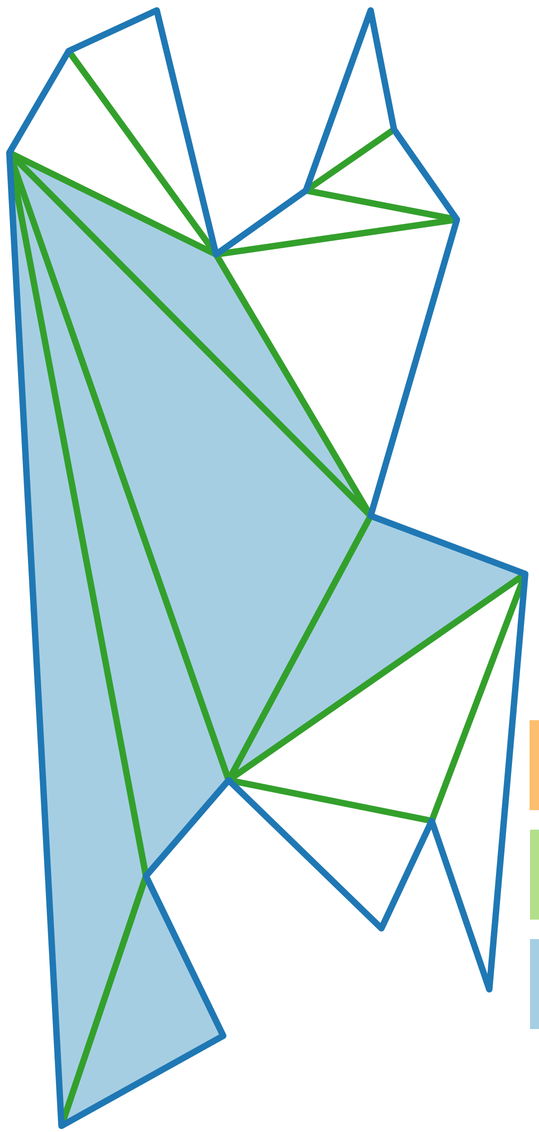

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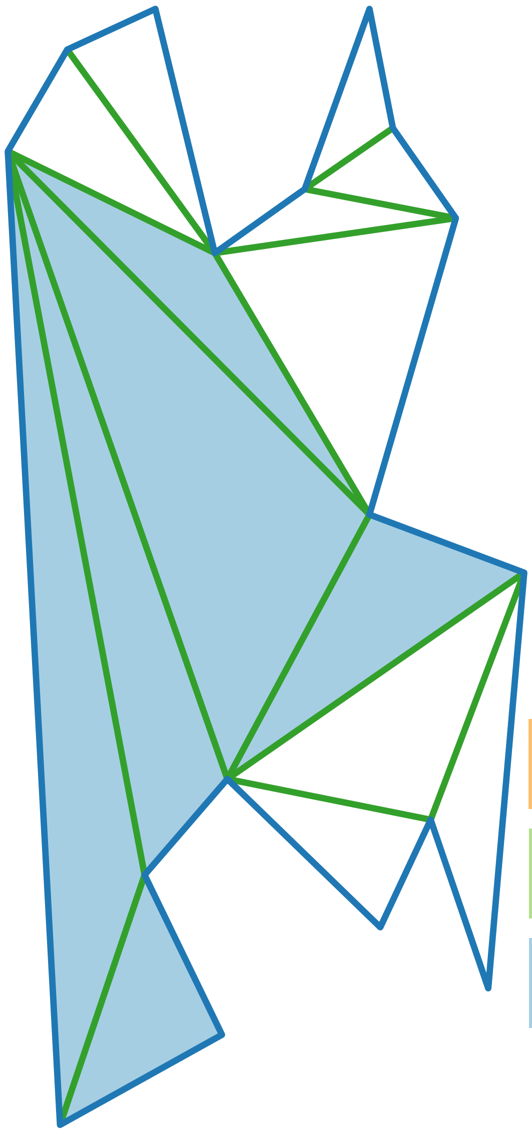

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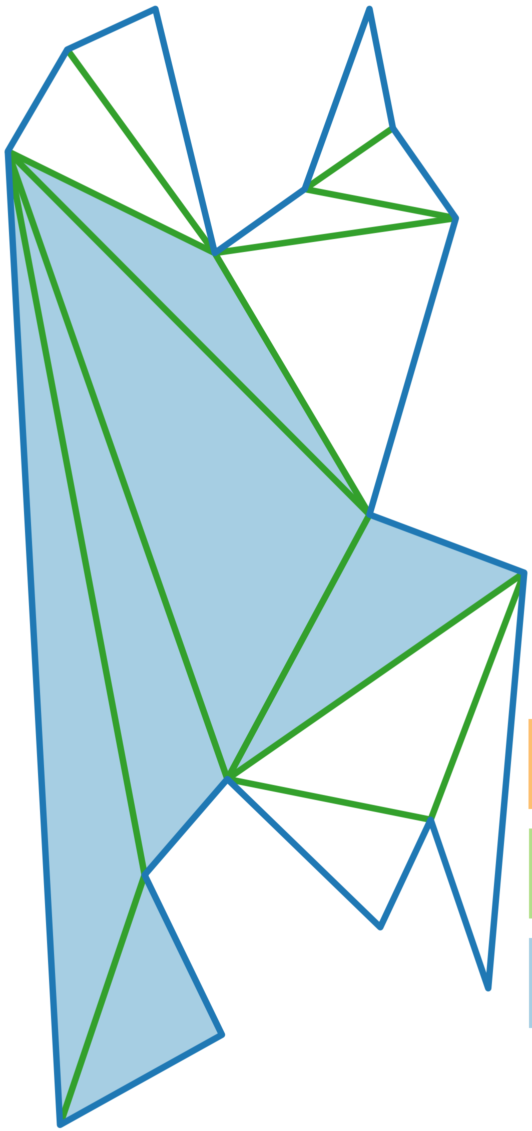

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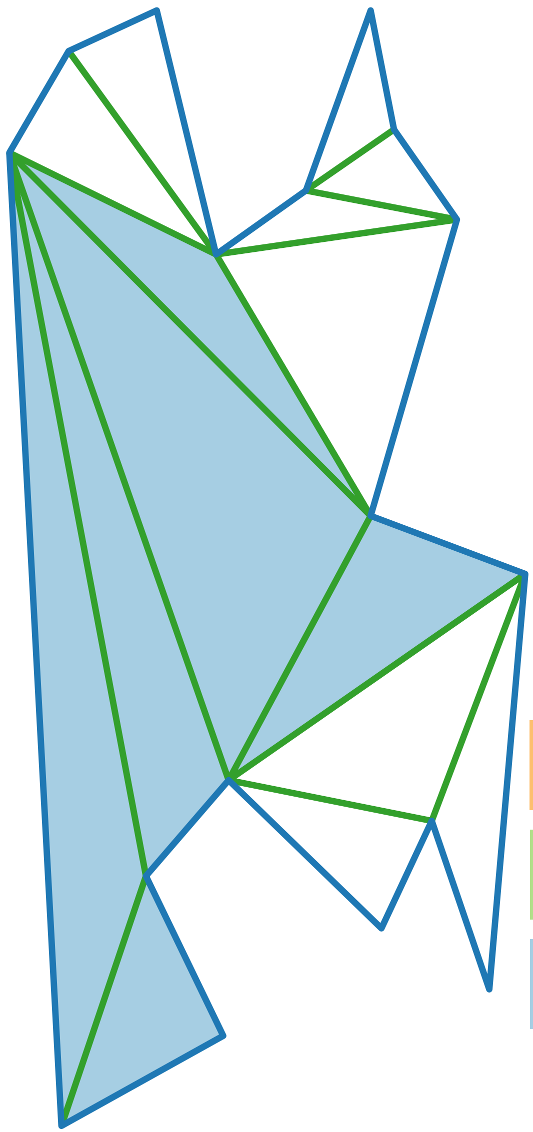

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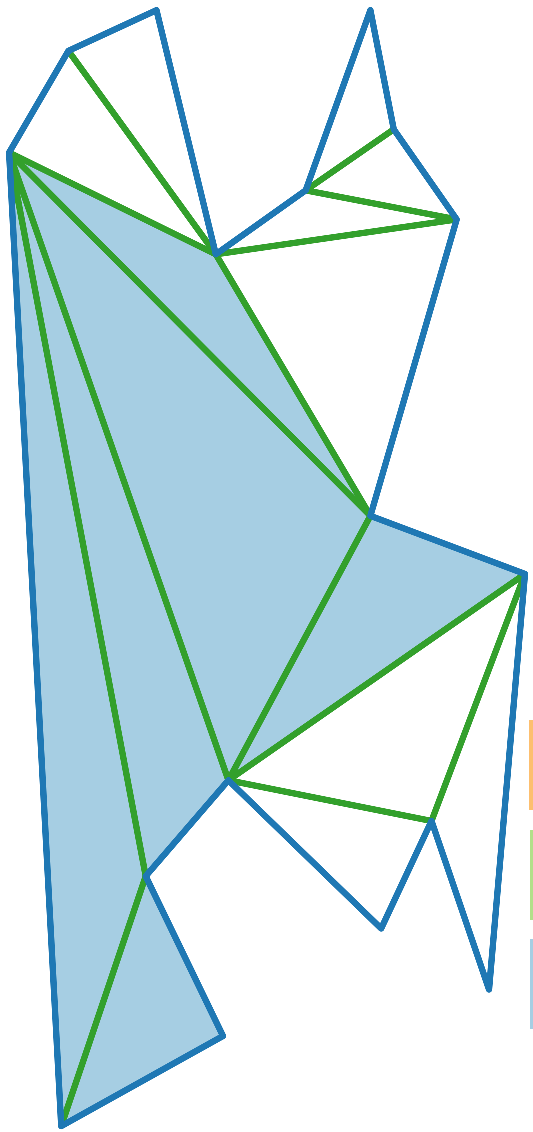

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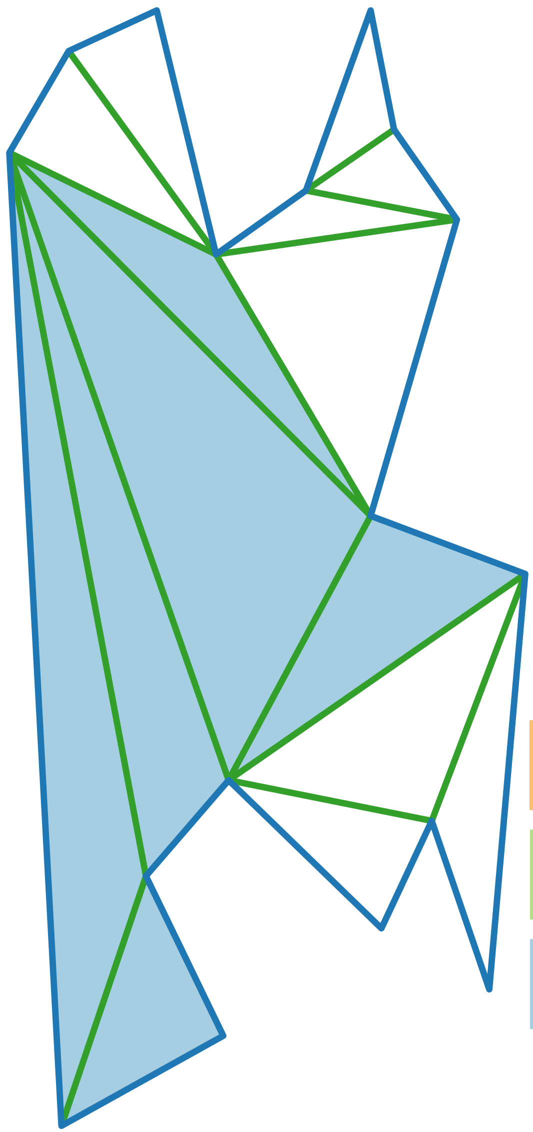

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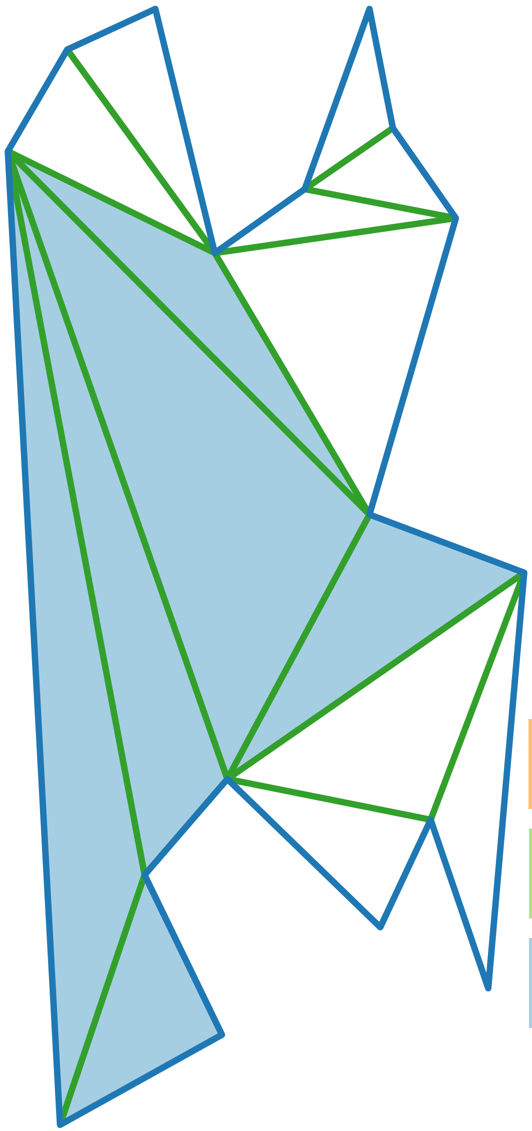

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Lemma 1. Given a trapezoidation, a polygon can be triangulated in linear time.

General Idea

Let S be a set of n non-crossing segments

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$\langle s_1, s_2, \ldots, s_n \rangle \leftarrow$ random ordering of S

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Total cost of one step: – location time
– "threading" (updating) time

Computational Geometry

Lecture 12:
Seidel's Triangulation Algorithm

Part II:
Location & Threading Time

Philipp Kindermann

Winter Semester 2020

Threading Time

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(no two points have the same y-coordinate).

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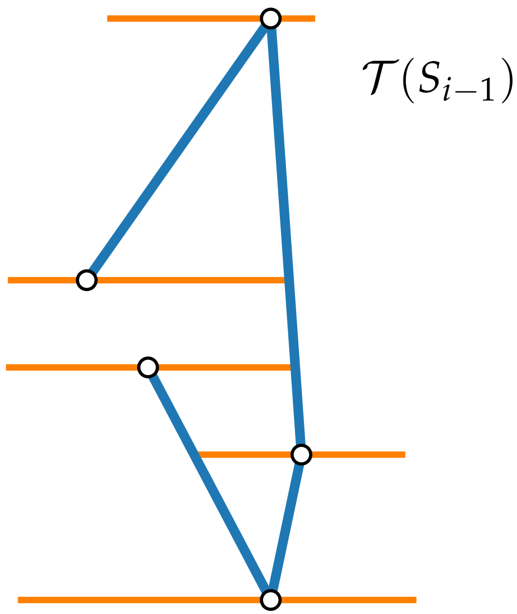

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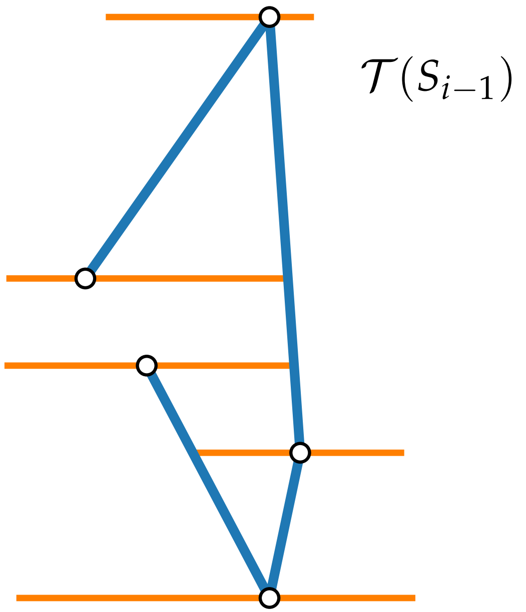

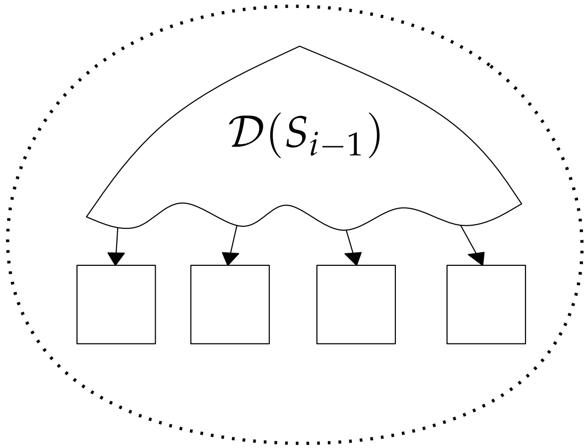

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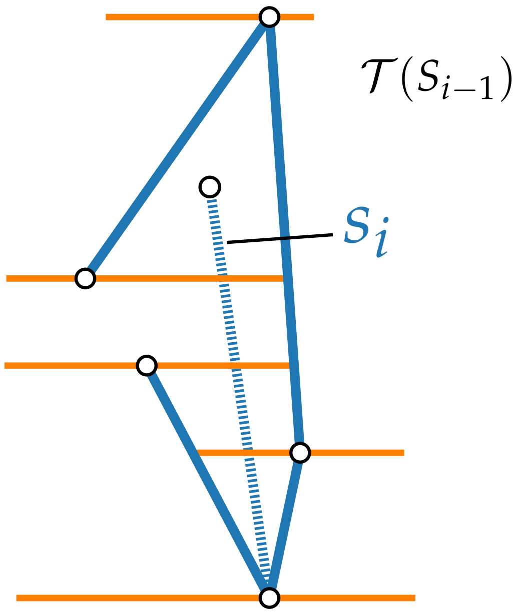

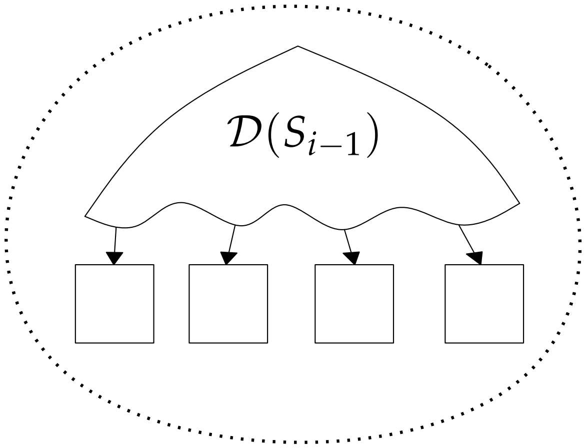

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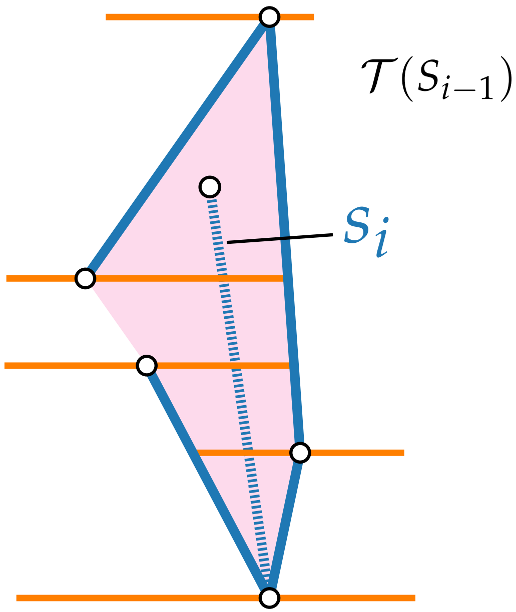

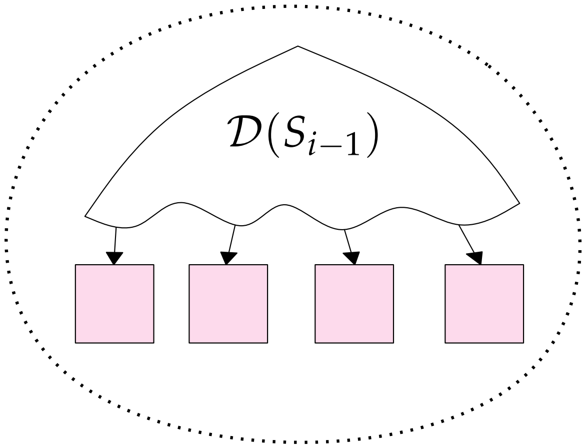

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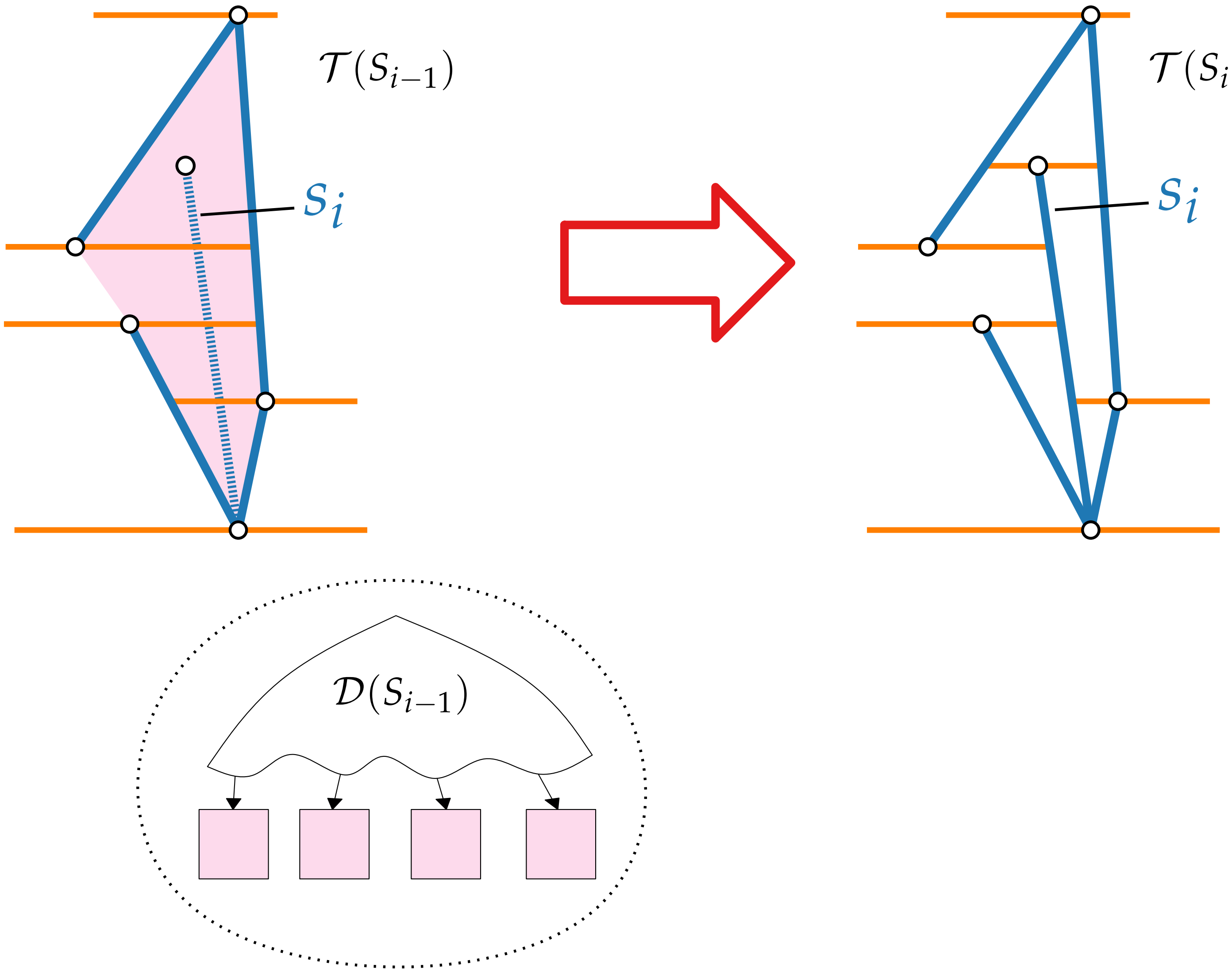

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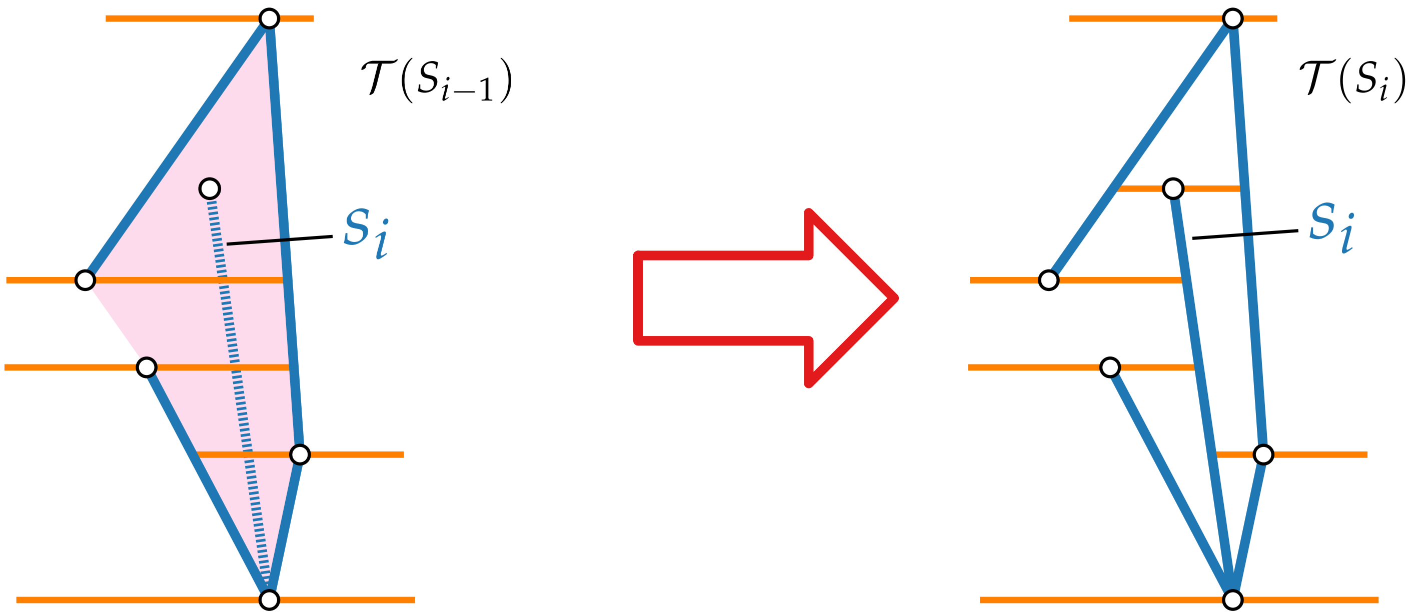

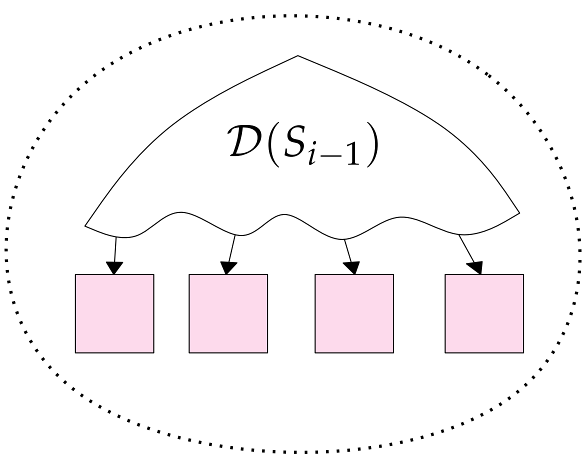

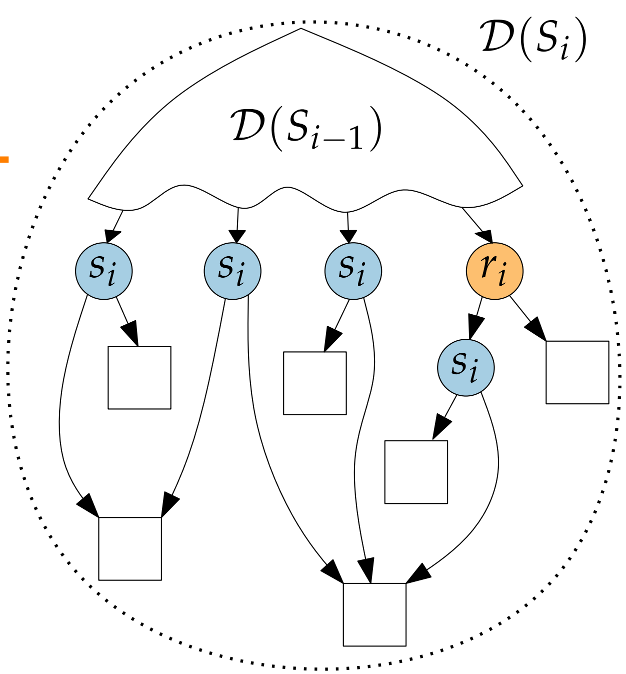

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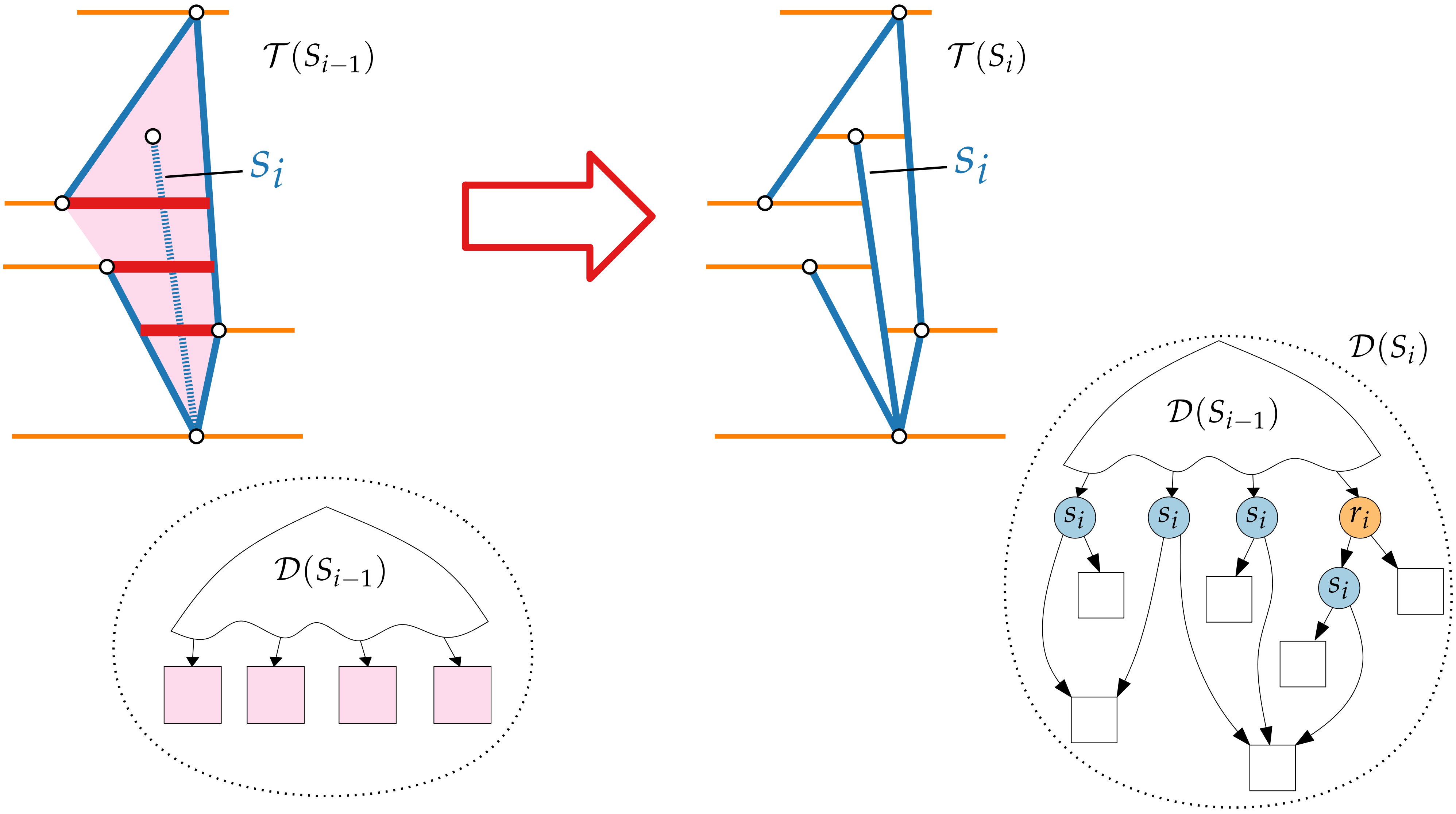

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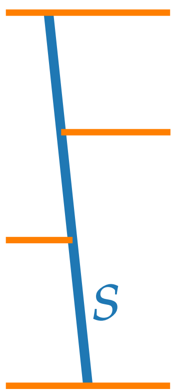

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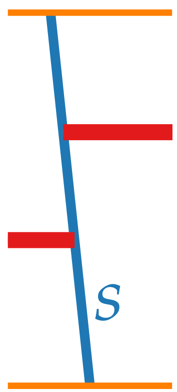

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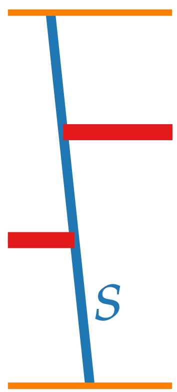

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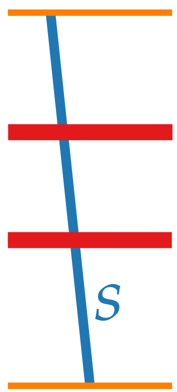

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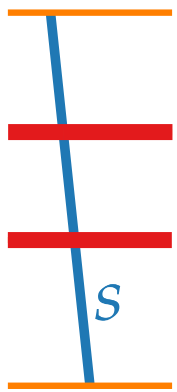

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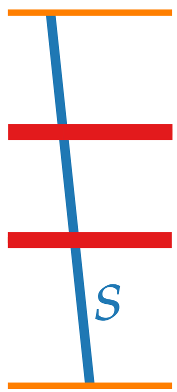

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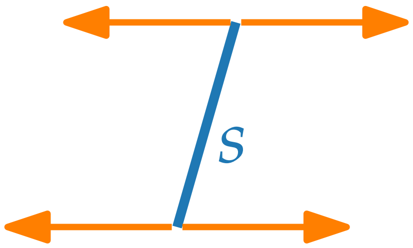

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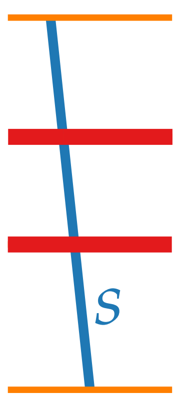

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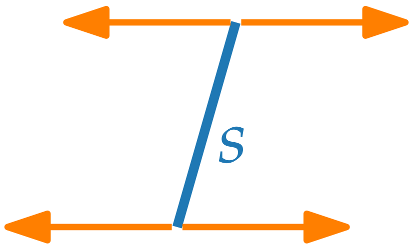

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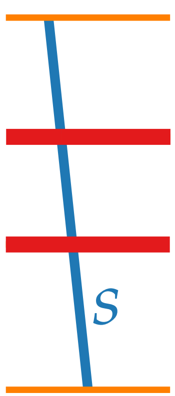

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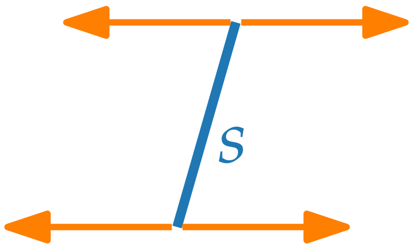

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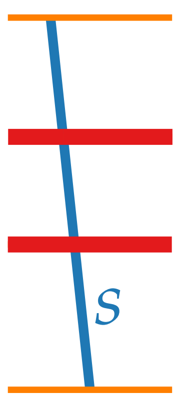

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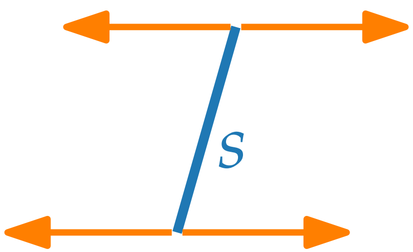

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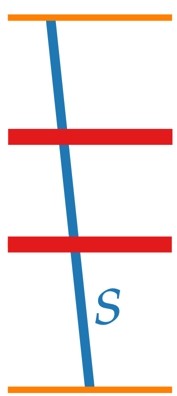

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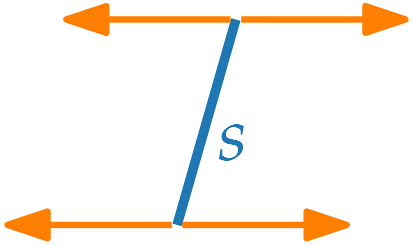

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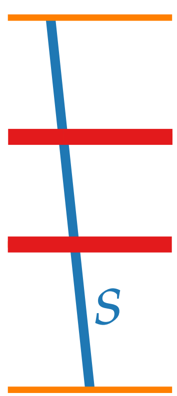

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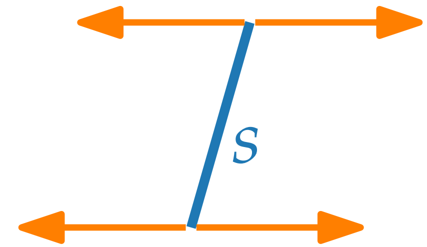

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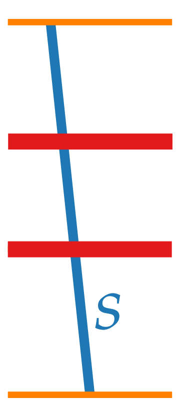

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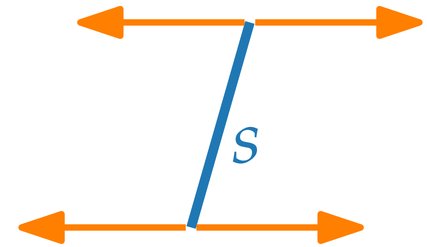

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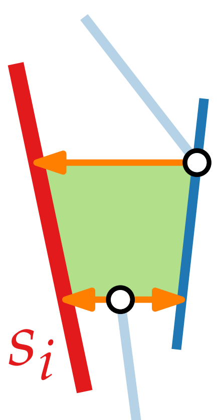

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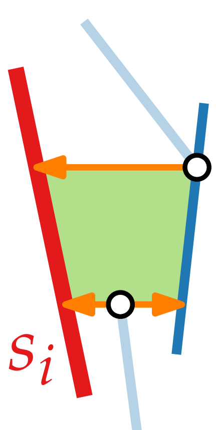

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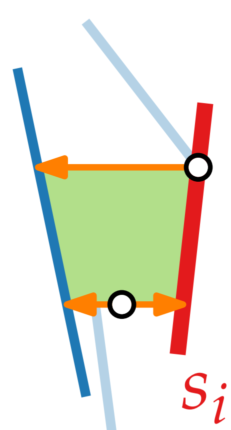

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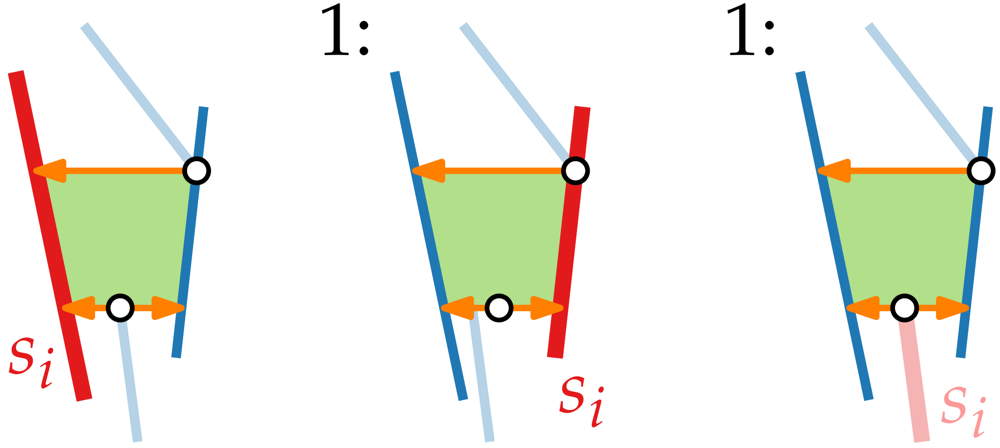

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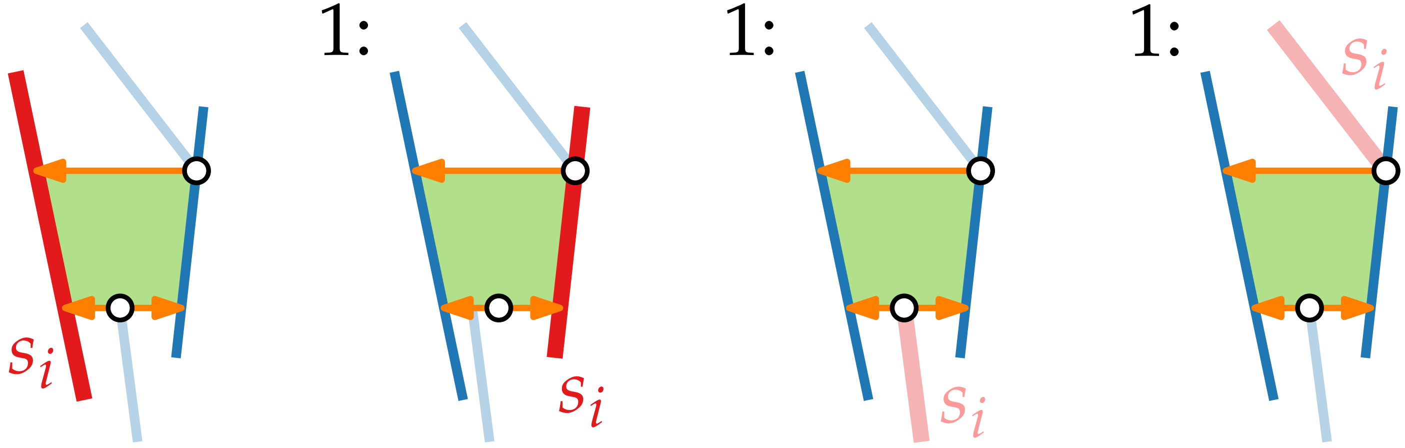

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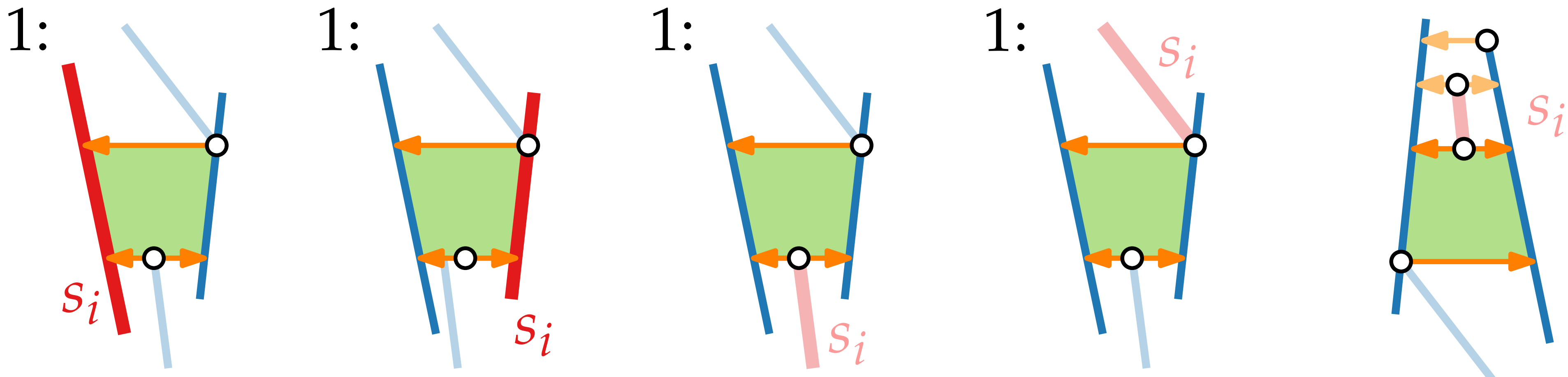

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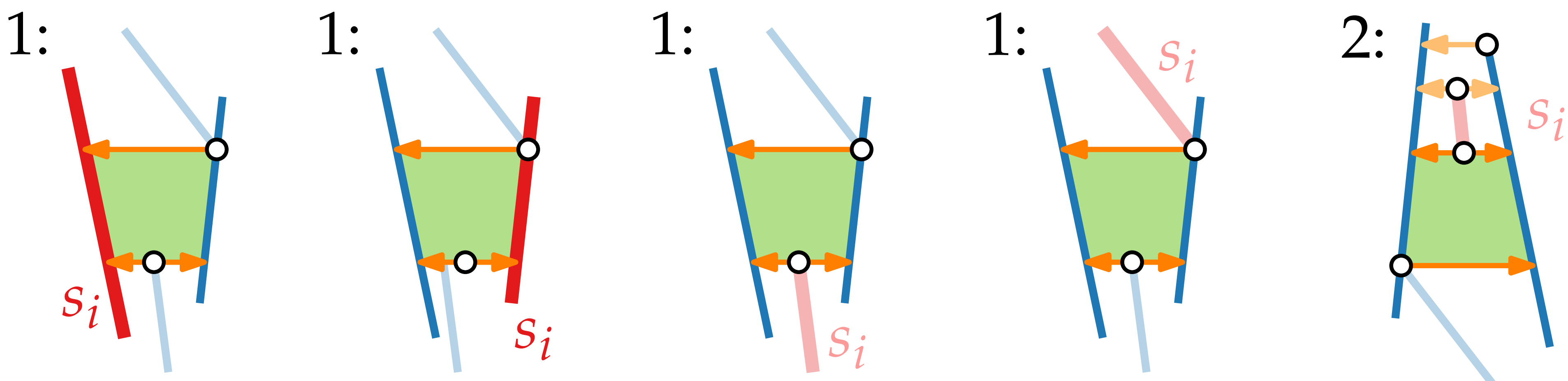

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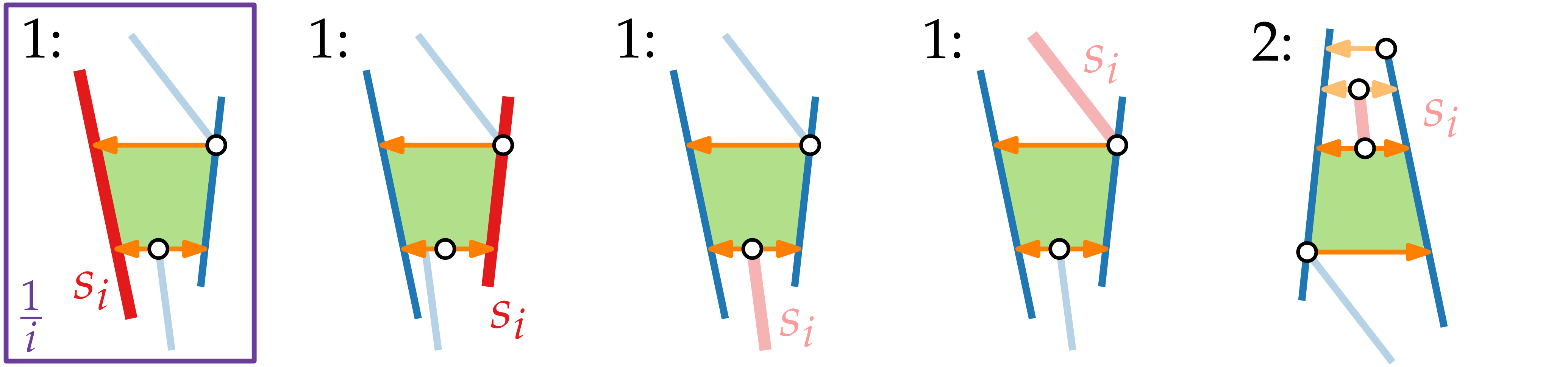

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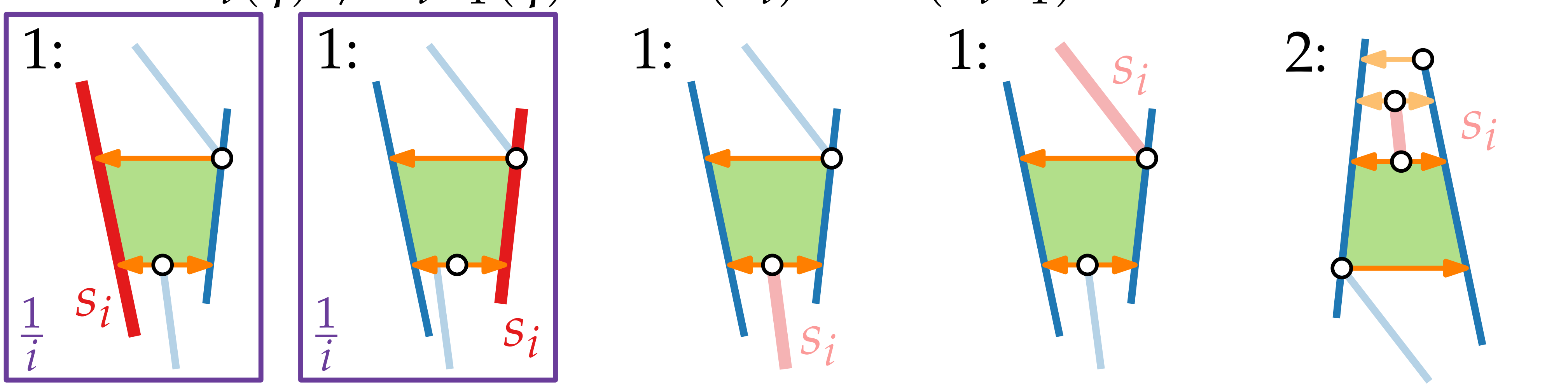

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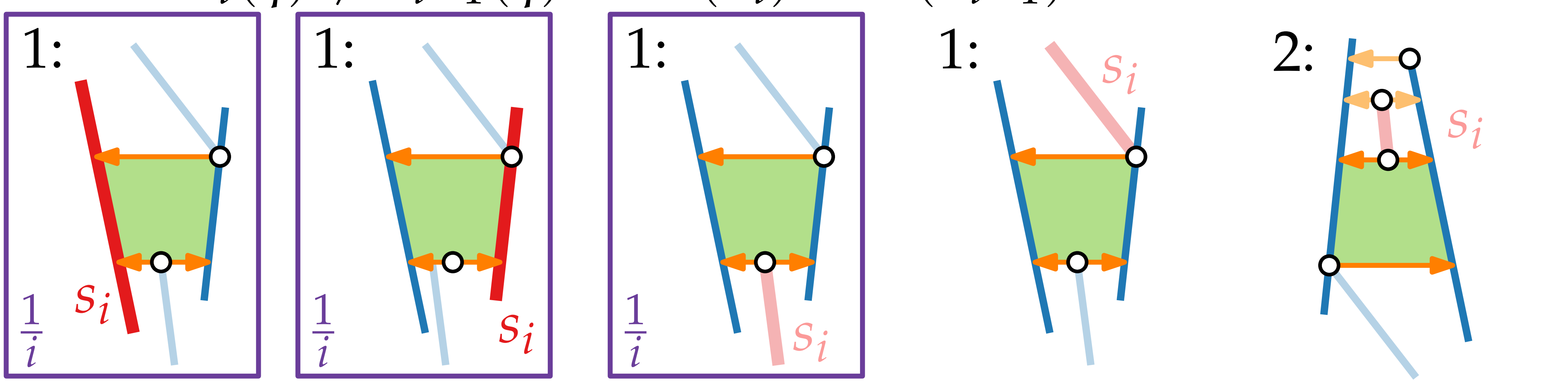

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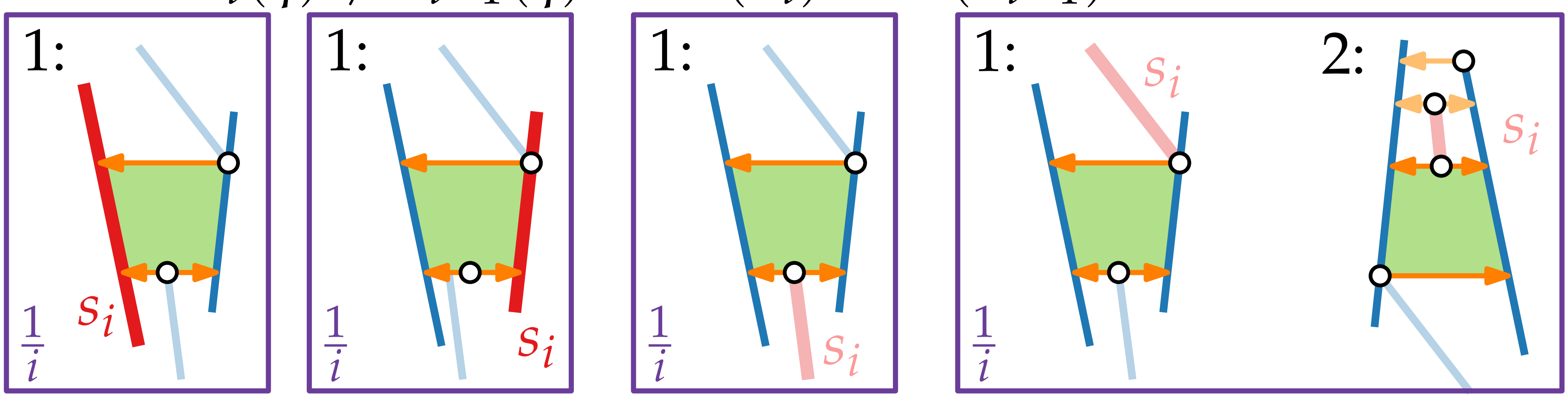

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Aim: Speed-up construction for simple polygons.

Computational Geometry

Lecture 12:
Seidel's Triangulation Algorithm

Part III:
New Approach

Philipp Kindermann

Winter Semester 2020

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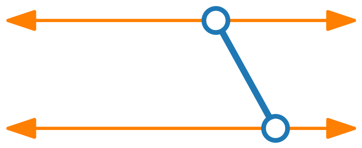

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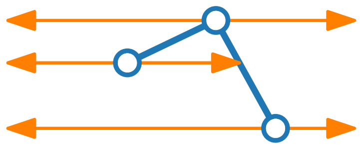

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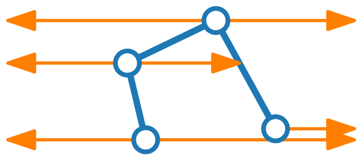

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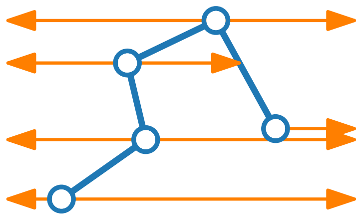

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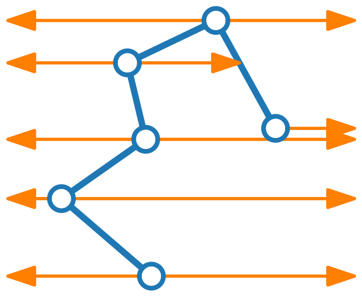

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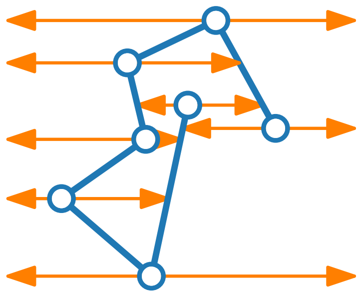

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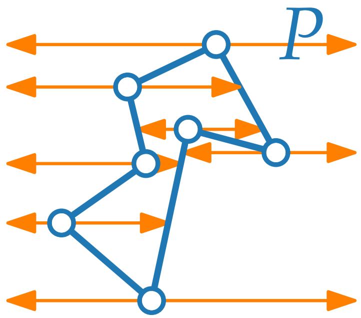

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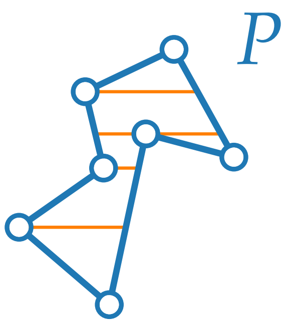

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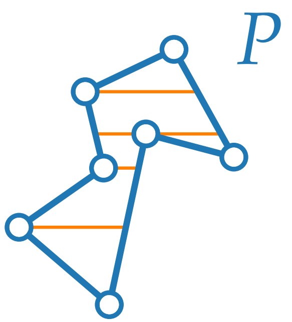

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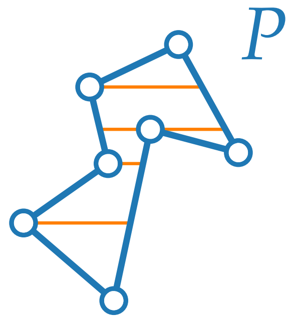

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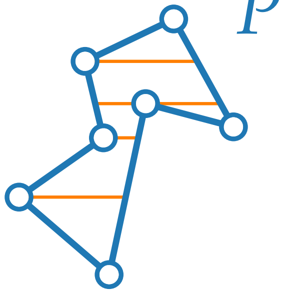

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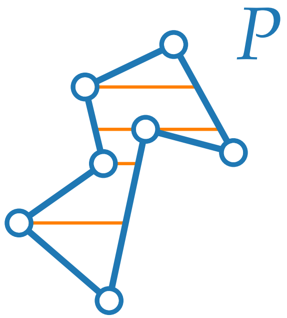

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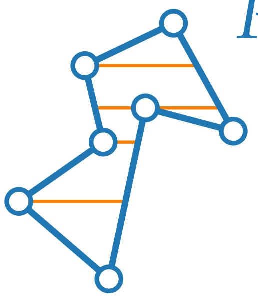

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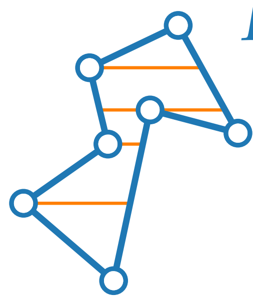

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Lemma 4. Let $1 \leq j \leq k \leq n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

Lemma 5. S as before, $R \subseteq S$ random subset, $r := |R|$. Let I be the number of intersections between rays of $\mathcal{T}(R)$ and segments in $S \setminus R$. Then $E[I] \leq 4(n - r)$, where the expectation is over all size-r subsets of S.

The Two Main New Technical Ingredients

Questions:

- How much does the intermediate location information help later?
- How expensive is it to walk along the polygon in the current trapezoidation?

Lemma 4. Let $1 \leq j \leq k \leq n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

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Computational Geometry

Lecture 12:
Seidel's Triangulation Algorithm

Part IV:
The Algorithm

Philipp Kindermann

Winter Semester 2020

Logs All Over the Place

Definition. Let the *i-th iterated logarithm* of n be defined by

$$\log^{(i)} n := \begin{cases} n & \text{if } i = 0, \\ \log_2(\log^{(i-1)} n) & \text{if } i > 0. \end{cases}$$

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Logs All Over the Place

Input:

$2^{2^{2^{2^2}}}$

Open code

Result:

2 003 529 930 406 846 464 979 072 351 560 255 750 447 825 475 569 751 419 265 016 ⋱
973 710 894 059 556 311 453 089 506 130 880 933 348 101 038 234 342 907 263 ⋱
181 822 949 382 118 812 668 869 506 364 761 547 029 165 041 871 916 351 587 ⋱
966 347 219 442 930 927 982 084 309 104 855 990 570 159 318 959 639 524 863 ⋱
372 367 203 002 916 969 592 156 108 764 948 889 254 090 805 911 457 037 675 ⋱
208 500 206 671 563 702 366 126 359 747 144 807 111 774 815 880 914 135 742 ⋱
720 967 190 151 836 282 560 618 091 458 852 699 826 141 425 030 123 391...

More digits

Decimal approximation:

$2.0035299304068464649790723515602557504478254755697 5\ldots \times 10^{19728}$

More digits

Number length:

19 729 decimal digits

n of n be defined by

if $i = 0$,

$n)$ if $i > 0$.

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65,536

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$2.0035299304068464649790723515602557504478254755 6975\ldots \times 10^{19728}$

Number length:

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Input interpretation:

estimated number of atoms in the universe

Result:

1×10^{80} atoms

$$\text{x}\{i \mid \log^{(i)} n \geq 1\}.$$

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65,536

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$2^{2^{2^{2^{2}}}}$

Open code

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2×10^{19648} per atom

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The Algorithm

PolygonTrapezoidation ((edges along) simple polygon P)

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3.1

3.2

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insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(h-1)}$)

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$\Delta \leftarrow$ the trapezoid in $\mathcal{T}_{N(h)}$ that contains v

The Algorithm

PolygonTrapezoidation ((edges along) simple polygon P)

1. $\langle s_1, s_2, \ldots, s_n \rangle :=$ random ordering of the edges of P
2. Compute $\mathcal{T}_1$ and $\mathcal{Q}_1$ for $\{s_1\}$.

foreach $v \in P$ **do** $\pi(v) \leftarrow$ ptr to the leaf of $\mathcal{Q}_1$ that contains v.

for $h = 1$ **to** $\log^\star n$ **do** // phase h

3.1 **for** $i = N(h-1)+1$ **to** $N(h)$ **do**

insert $s_i = v_i w_i$ in $\mathcal{T}_{i-1}$ using $\pi(v_i)$ (node in $\mathcal{Q}_{N(h-1)}$)

3.2 walk along P through $\mathcal{T}_{N(h)}$:

foreach vertex v **do**

$\Delta \leftarrow$ the trapezoid in $\mathcal{T}_{N(h)}$ that contains v

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return $(\mathcal{T}_n, \mathcal{Q}_n)$

Computational Geometry

Lecture 12:
Seidel's Triangulation Algorithm

Part V:
Time Complexity

Philipp Kindermann

Winter Semester 2020

Time Complexity

Step 1: Random permutation

Time Complexity

Step 1: Random permutation $O(n)$

Time Complexity

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$

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Step 3.2: Walking the polygon

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Step 3.2: Walking the polygon Lemma 5 $\Rightarrow$

Lemma 5. S as before, $R \subseteq S$ random subset, $r := |R|$. Let I be the number of intersections between rays of $\mathcal{T}(R)$ and segments in $S \setminus R$. Then $E[I] \leq 4(n - r)$, where the expectation is over all size-r subsets of S.

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– threading cost:

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$O(\log(i/N(h-1))) \subseteq$

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Step 3.1: Inserting $s_i = v_i w_i$ using $\mathcal{Q}_{N(h-1)}$

– threading cost:Lem. 2 $\Rightarrow$ expected $O(1)$ per segm.

Lemma 4. Let $1 \le j \le k \le n$ and $q \in \mathbb{R}^2$. Suppose location of q in $\mathcal{Q}(S_j)$ is known, then q can be located in $\mathcal{Q}(S_k)$ in expected time $5(H_k - H_j) \in O(\log k/j)$.

Step 4: Inserting s_i (for $N(\log^\star n) < i \le n$) using $\mathcal{Q}_{N(\log^\star n)}$

– threading cost:Lem. 2 $\Rightarrow$ $O(1)$

– locating cost: Lem. 4 $\Rightarrow$ $O(\log n / \underbrace{N(\log^\star n)}_{> n/2}) = O(1)$

$\Big\} \cdot O(n) =$

Time Complexity

$N(h) := \lceil n / \log^{(h)} n \rceil$

Step 1: Random permutation $O(n)$

Step 2: Setting up $\mathcal{T}_1$, $\mathcal{Q}_1$, and $\pi(v)$ $O(n)$

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Step 4: Inserting s_i (for $N(\log^\star n) < i \leq n$) using $\mathcal{Q}_{N(\log^\star n)}$ — $O(n)$

– threading cost: Lem. 2 $\Rightarrow O(1)$

– locating cost: Lem. 4 $\Rightarrow O(\log n / N(\log^\star n)) = O(1)$, where $N(\log^\star n) > n/2$

$\Big\} \cdot O(n) = O(n)$

Total: $O(n \log^\star n)$

The Results

Theorem. Let S be the edge set of a polygon, $|S| = n$.

- We can build $\mathcal{T}(S)$ and $\mathcal{Q}(S)$ in $O(n \log^{\star} n)$ expected time.
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Theorem. Let S be the edge set of a plane straight-line graph with k connected components, $|S| = n$.

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