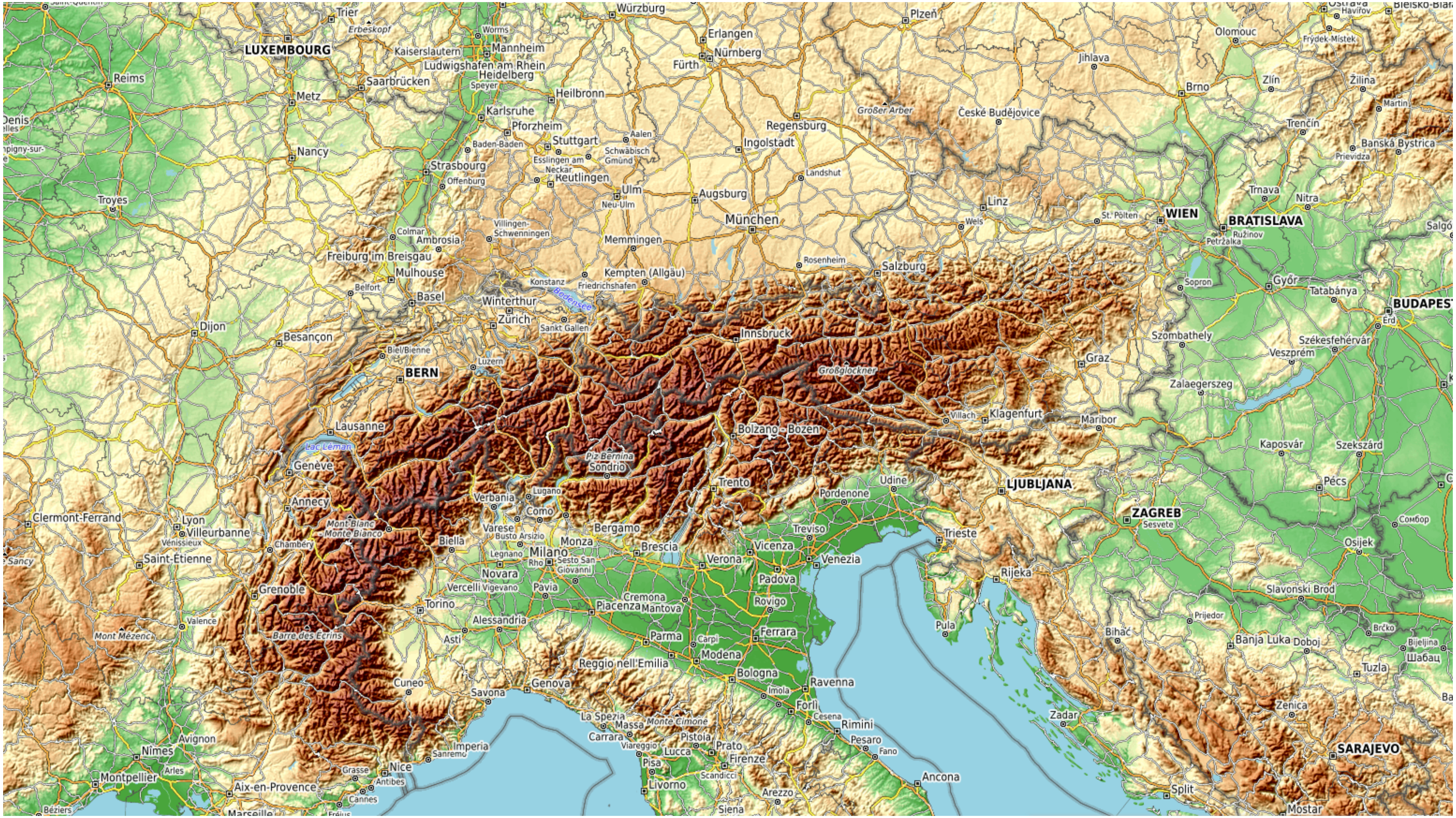
Computational Geometry

Lecture 8:
Delaunay Triangulations
or
Height Interpolation

Part I:
Height Interpolation

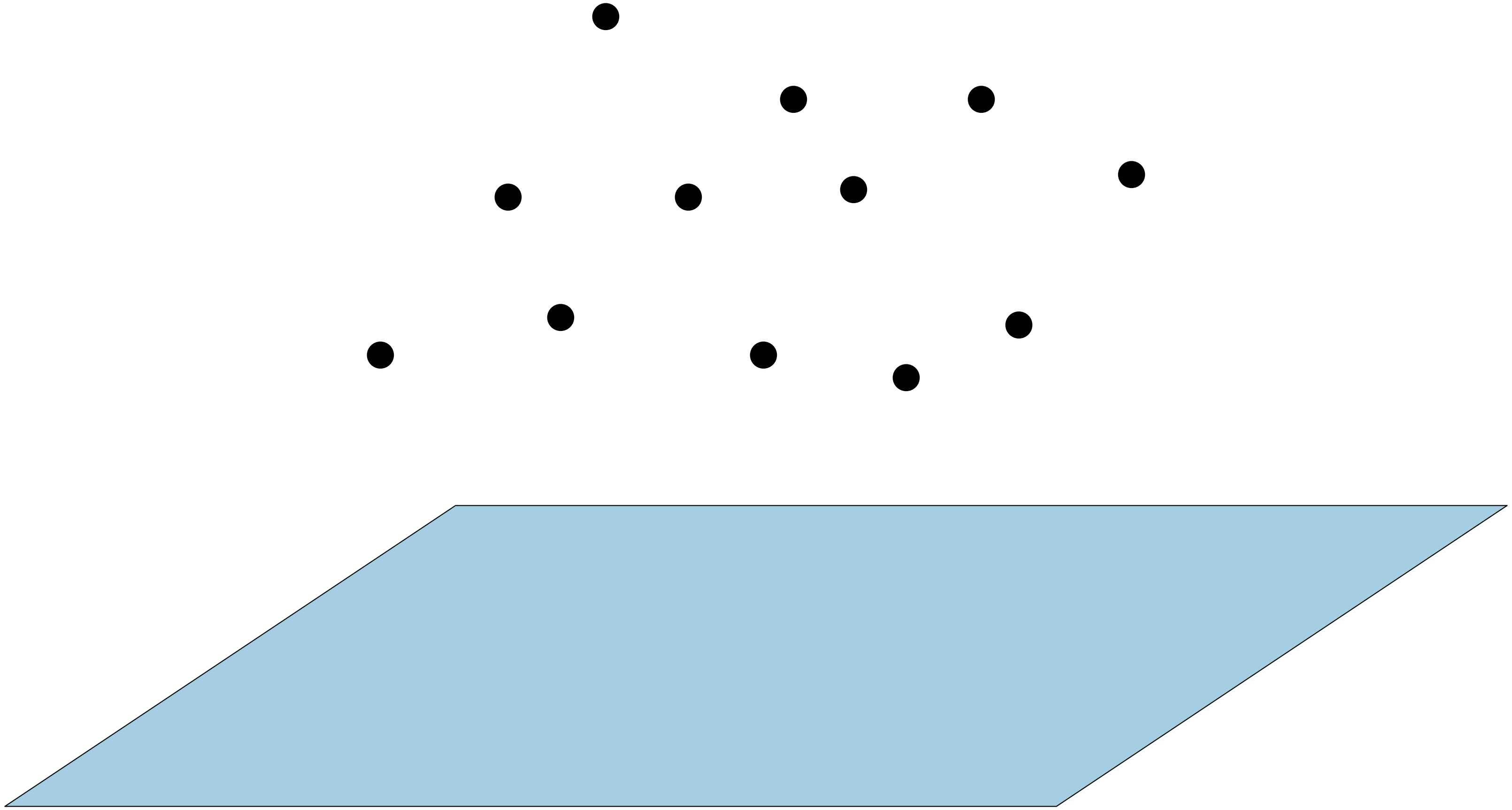
Philipp Kindermann

Summer Semester 2020



[opentopomap.org]

Height Interpolation



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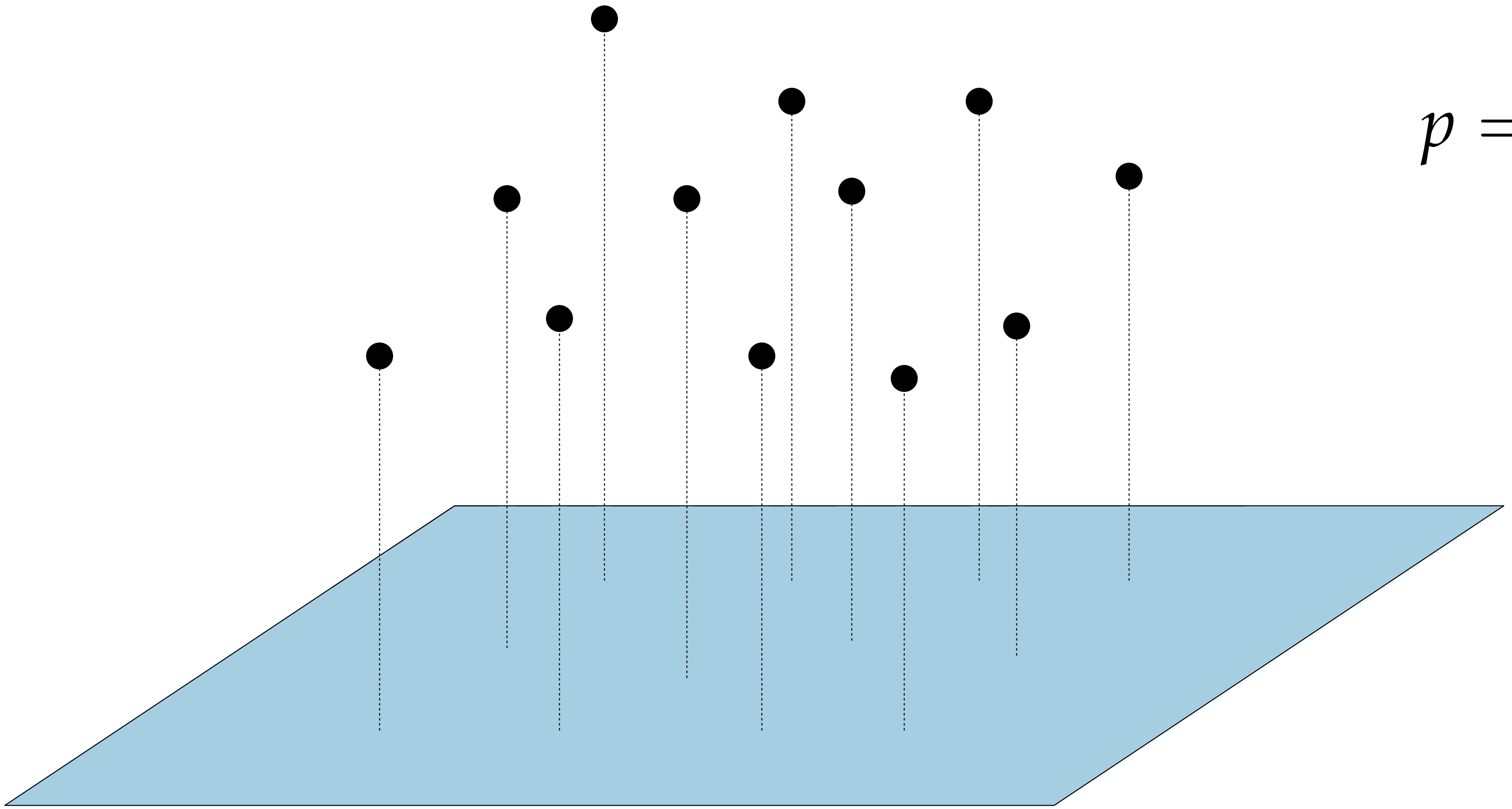

$$p = (x_p, y_p, z_p)$$

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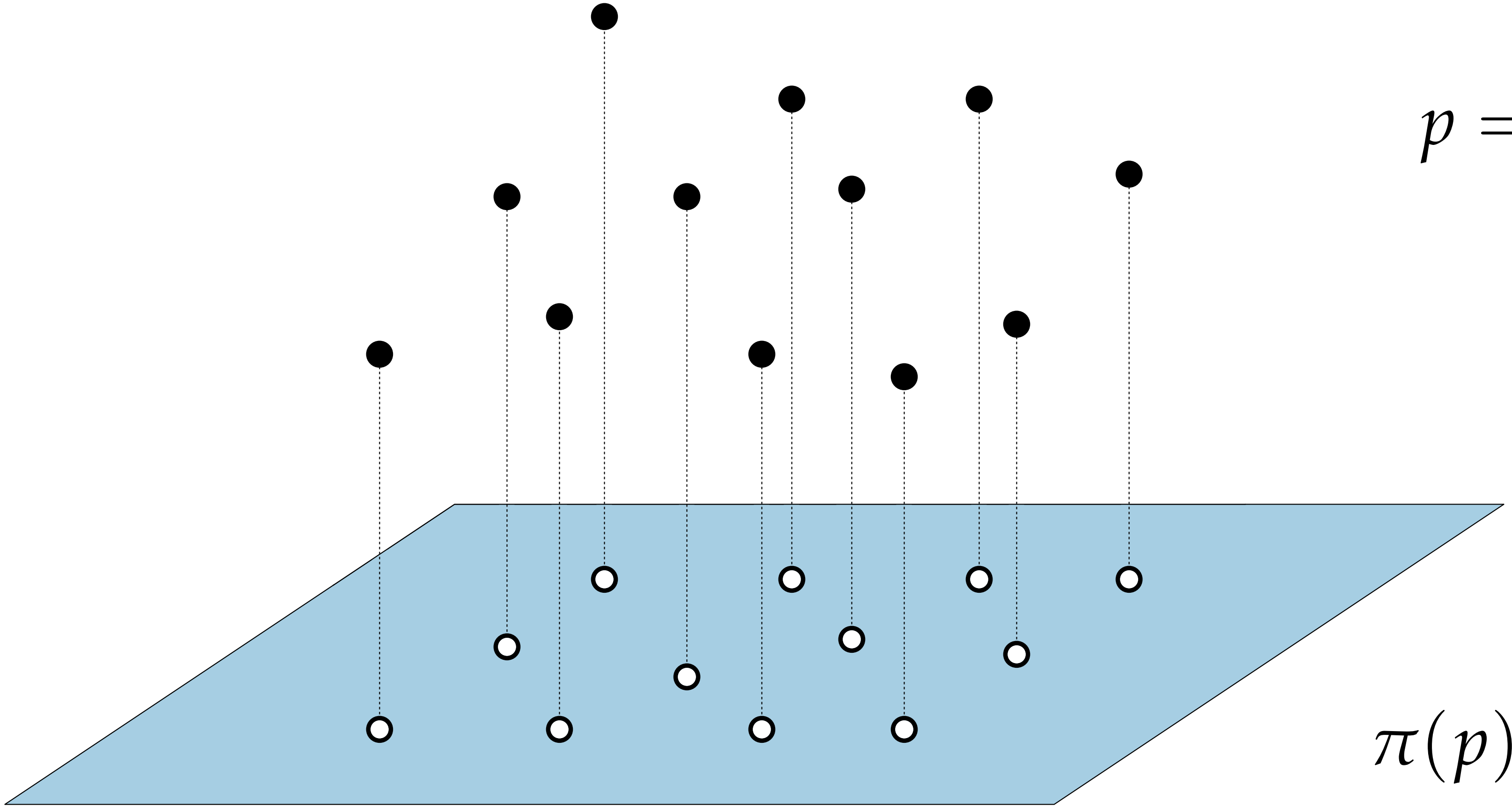

$$p = (x_p, y_p, z_p)$$

$$\pi(p) = (x_p, y_p, 0)$$

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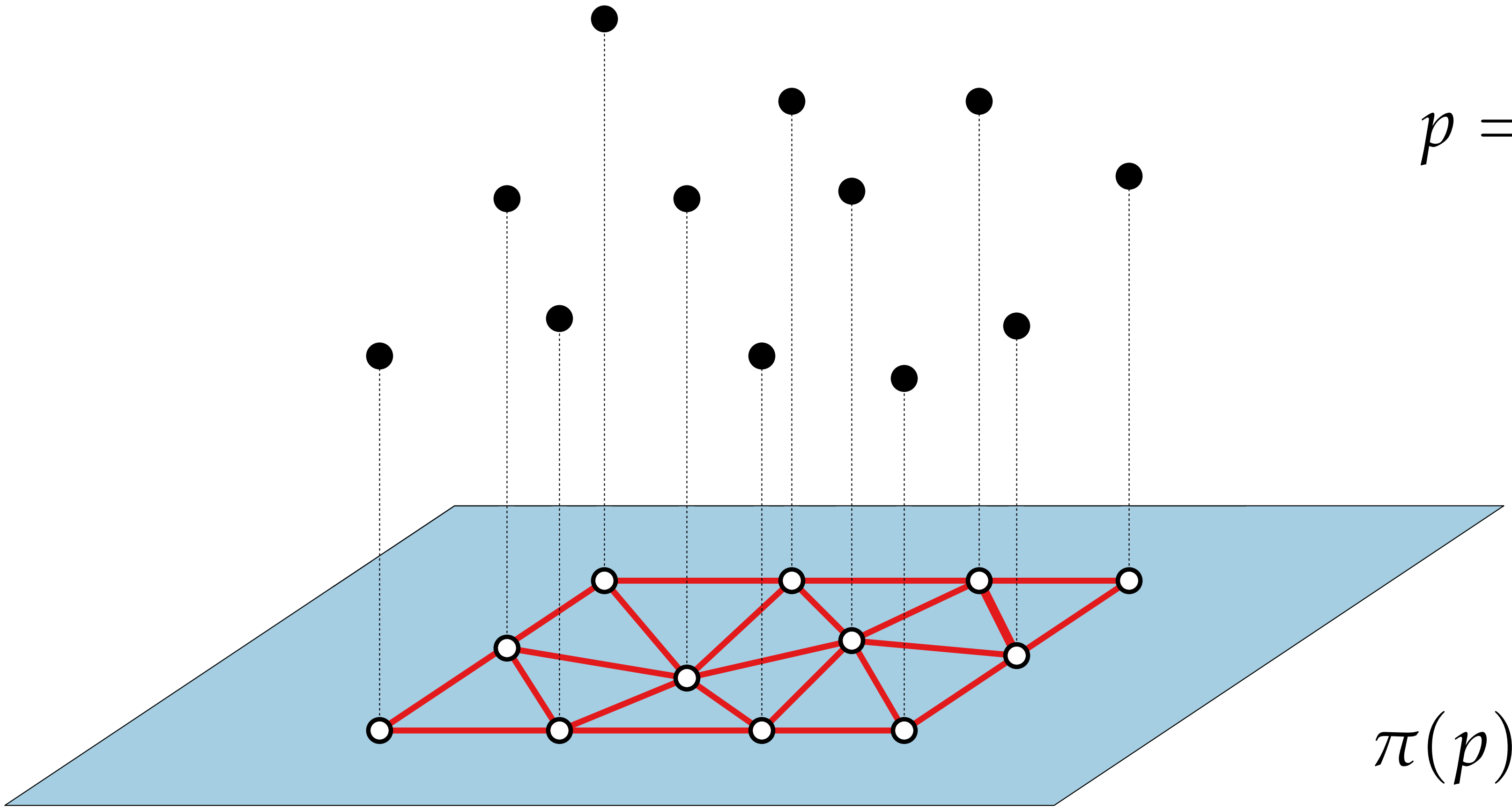

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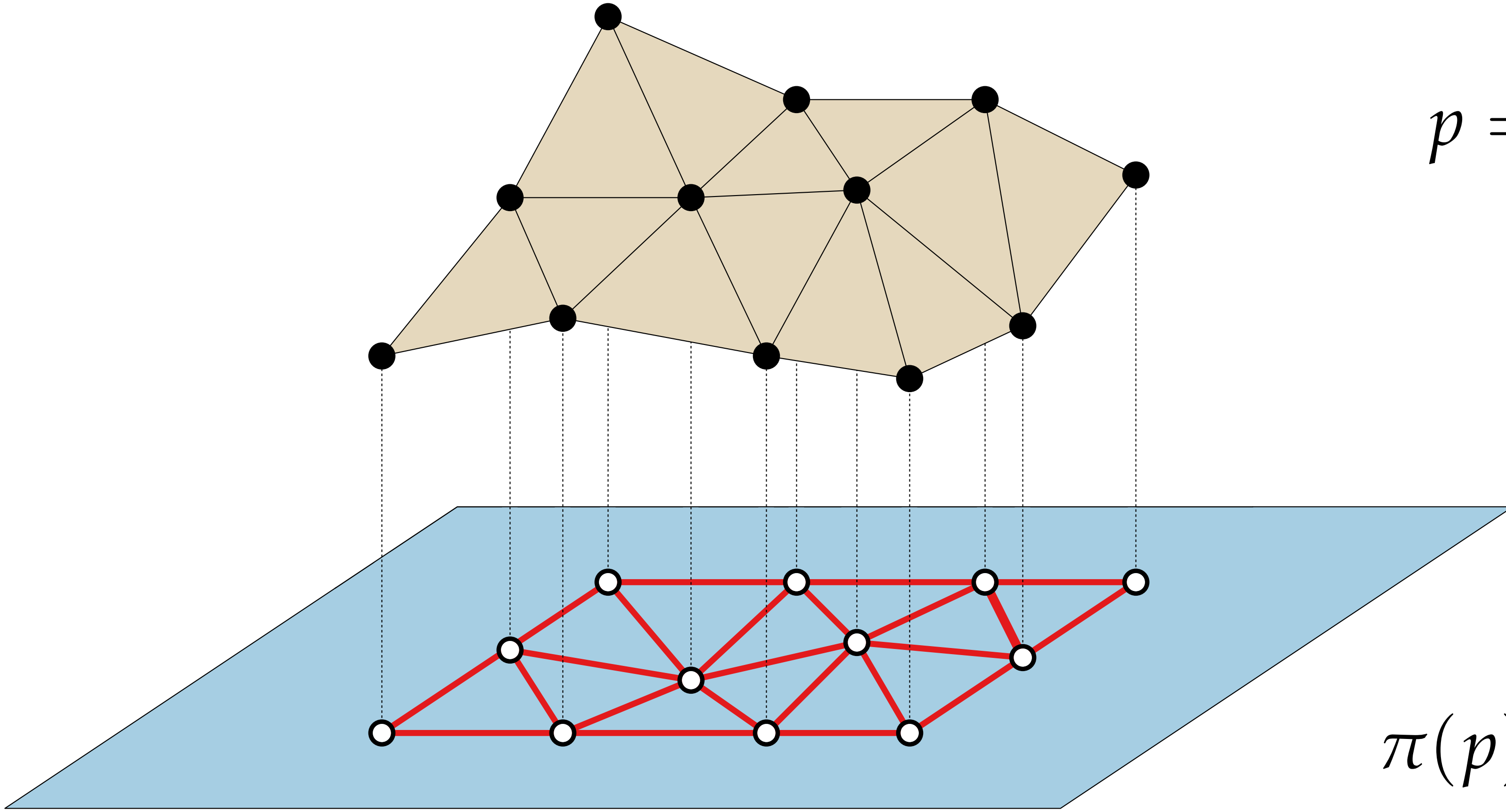

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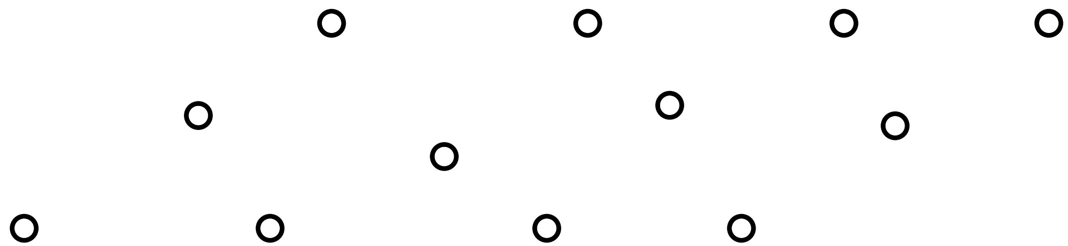
Triangulation of Planar Point Sets

Definition. Given $P \subset \mathbb{R}^2$, a *triangulation* of P is a maximal planar subdivision with vtx set P, that is, no edge can be added without crossing.



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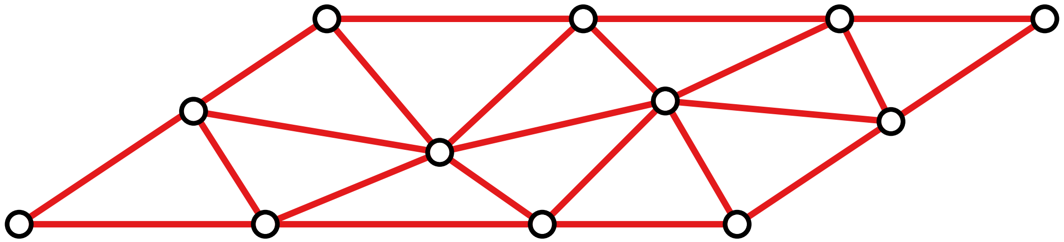

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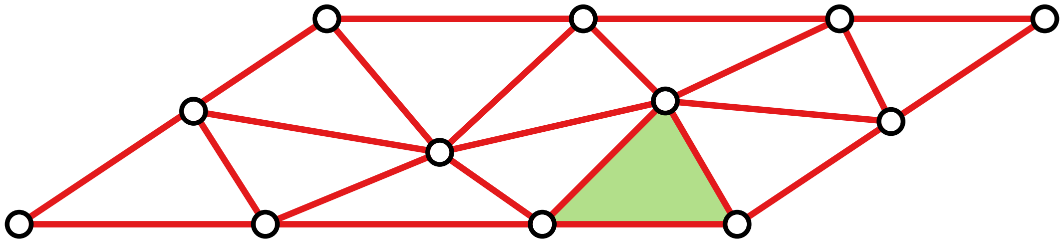

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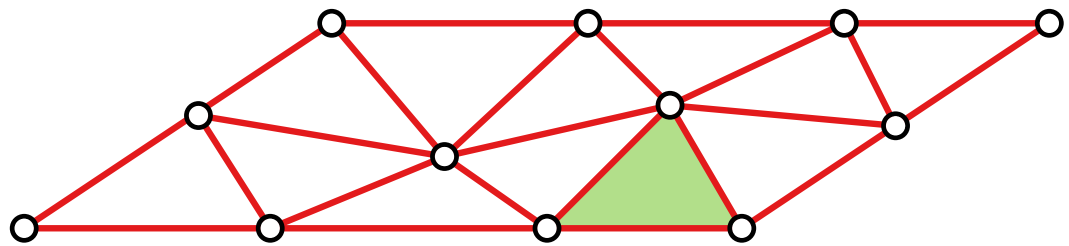

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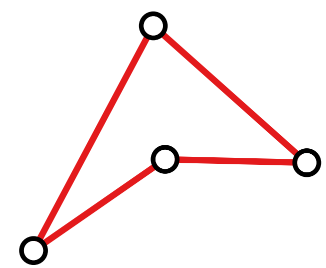

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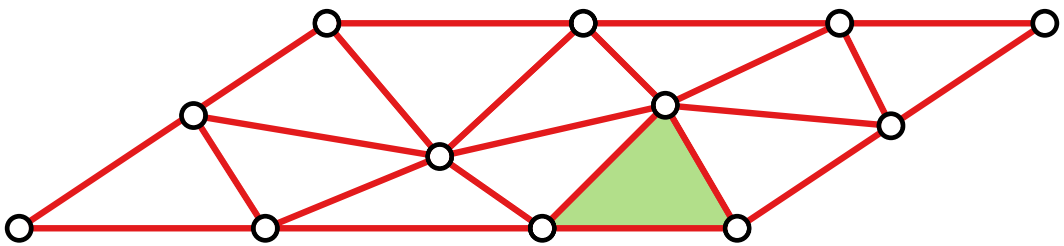

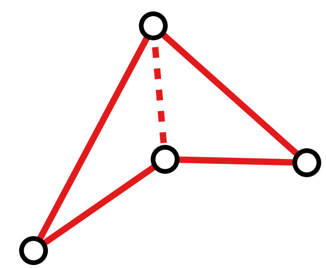

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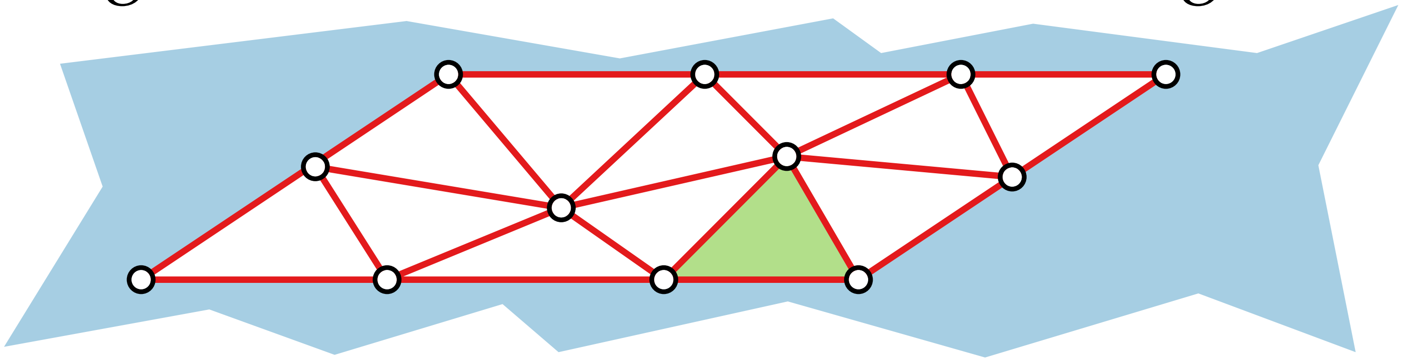

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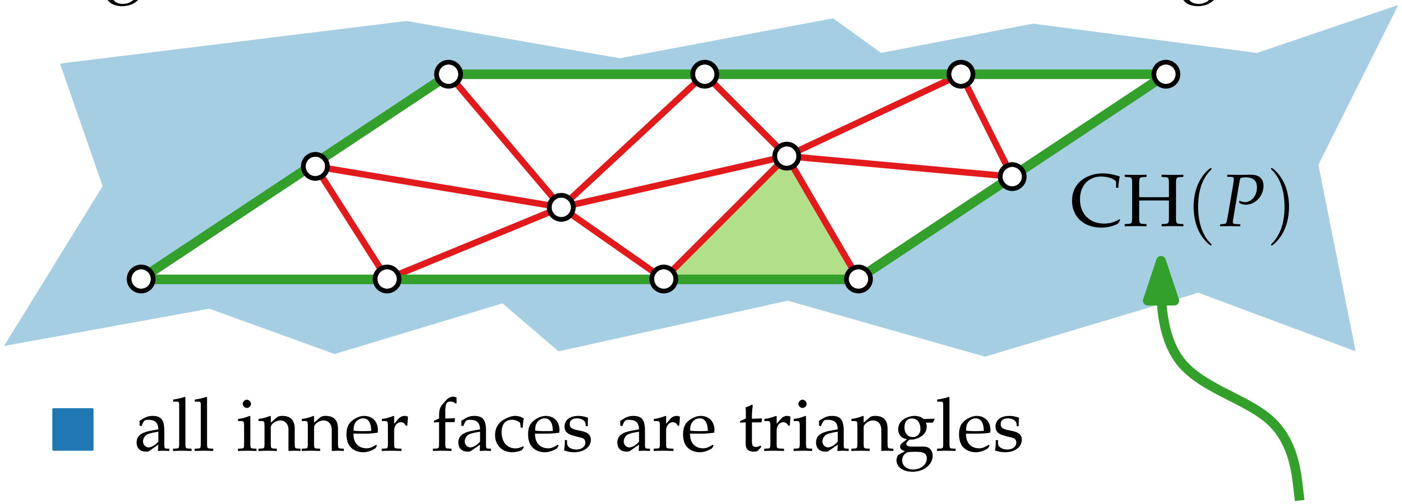

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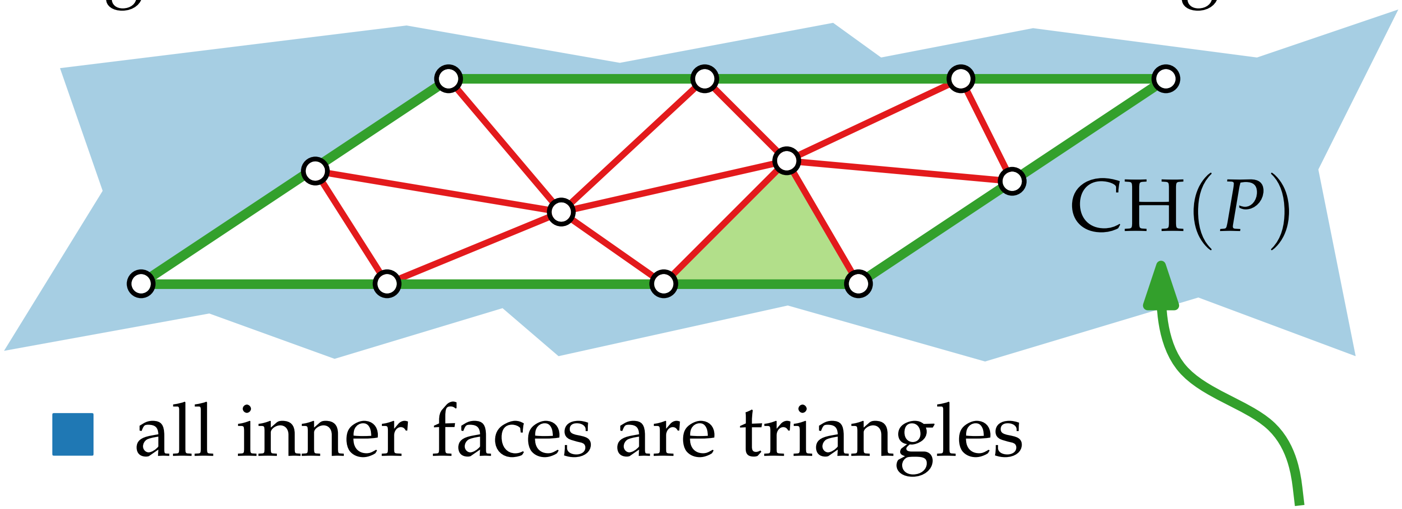

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Theorem. Let $P \subset \mathbb{R}^2$ be a set of n sites, not all collinear, and let h be the number of sites on $\partial \mathrm{CH}(P)$.

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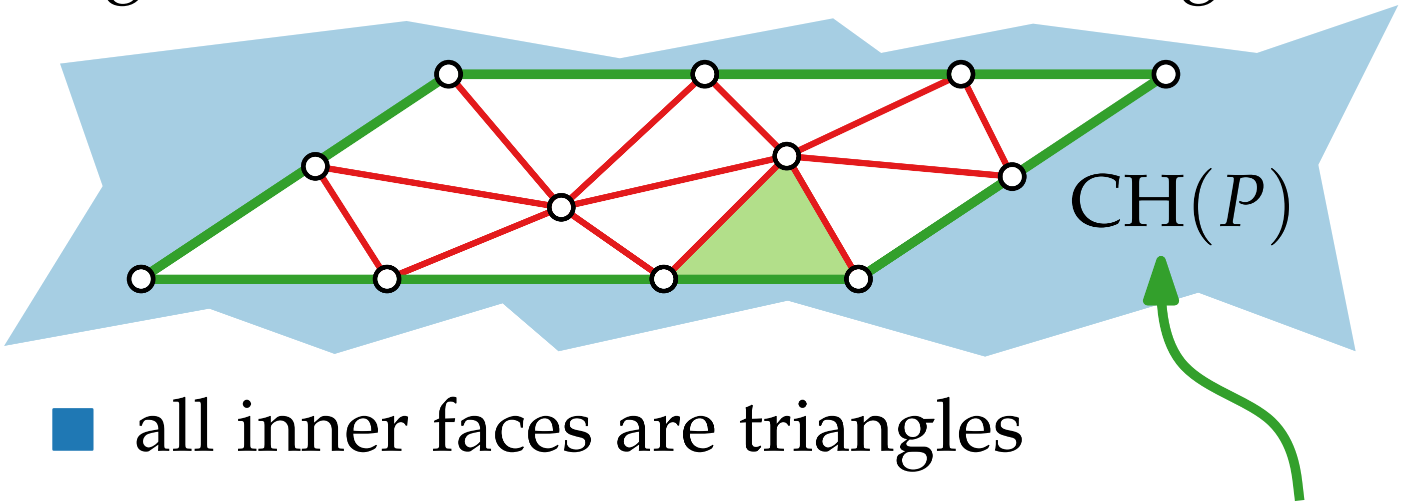

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Then *any* triangulation of P has

? triangles and ? edges.

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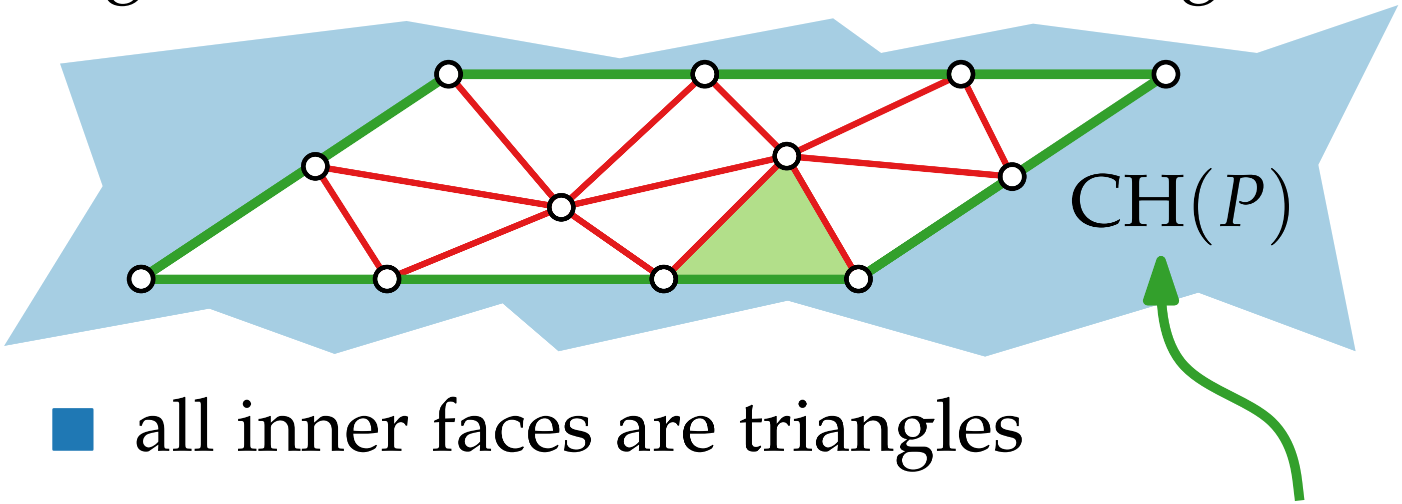

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Theorem. Let $P \subset \mathbb{R}^2$ be a set of n sites, not all collinear, and let h be the number of sites on $\partial\mathrm{CH}(P)$.

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? triangles and $3n - 3 - h$ edges.

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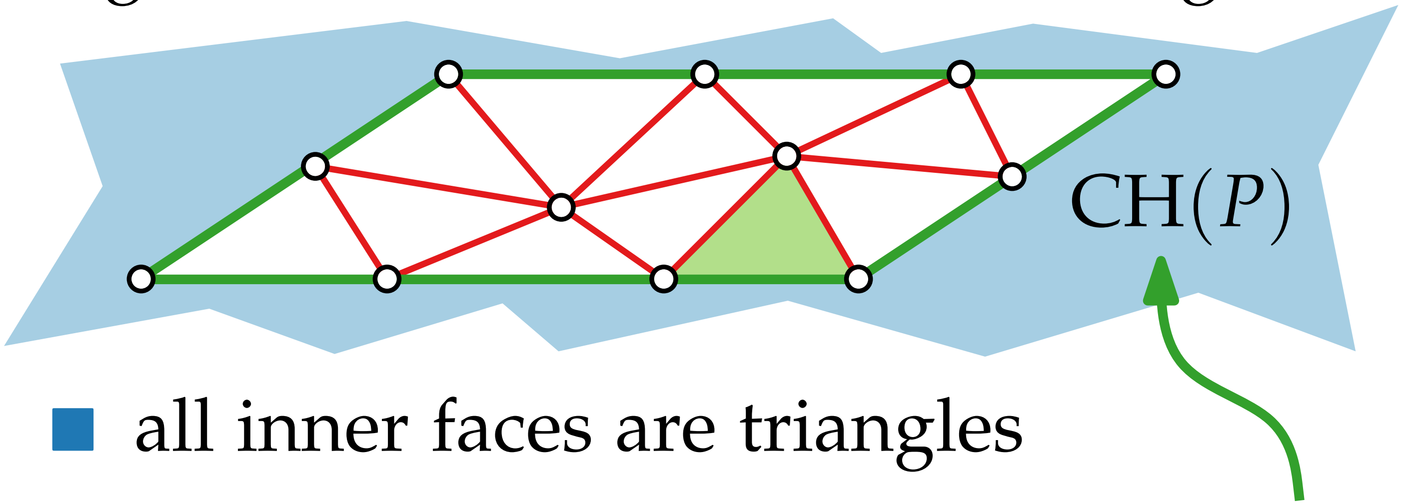

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Theorem. Let $P \subset \mathbb{R}^2$ be a set of n sites, not all collinear, and let h be the number of sites on $\partial\mathrm{CH}(P)$.

Then *any* triangulation of P has $2n - 2 - h$ triangles and $3n - 3 - h$ edges.

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Part II:
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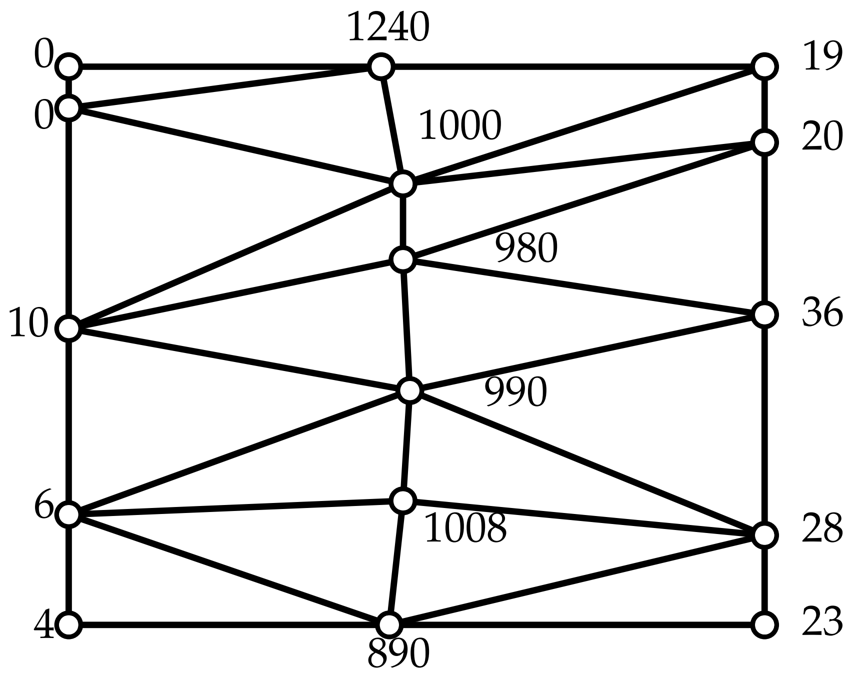
Back to Height Interpolation



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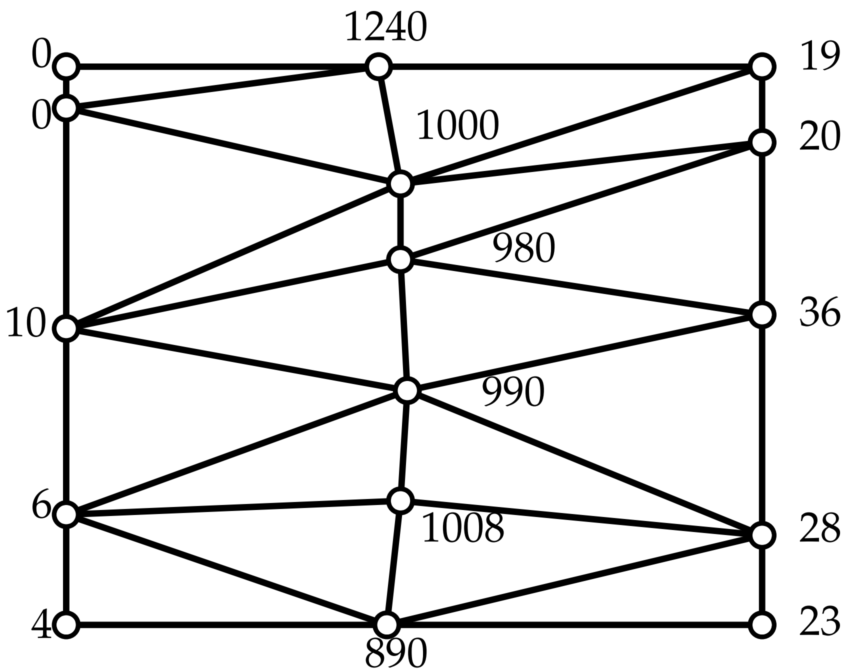

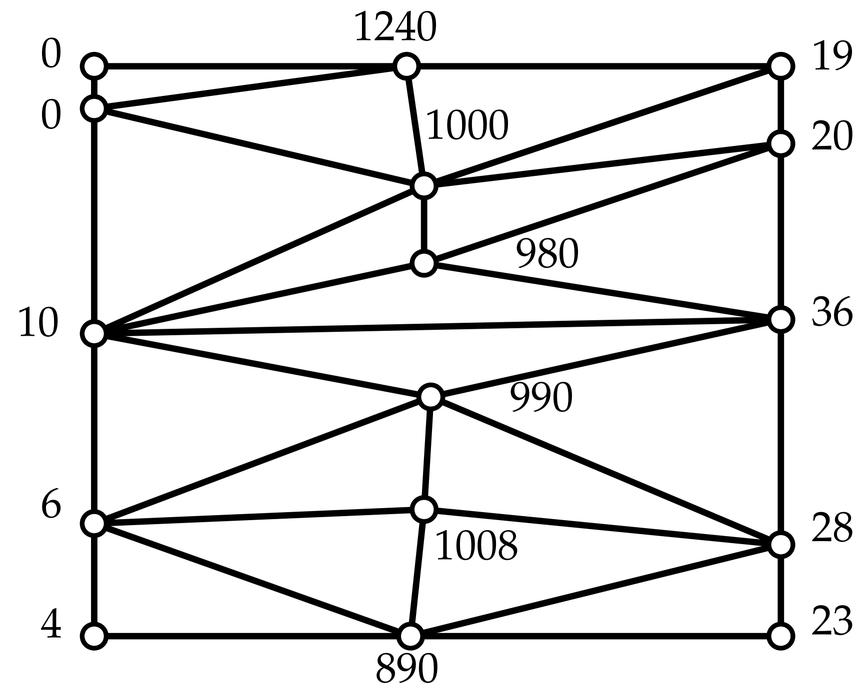

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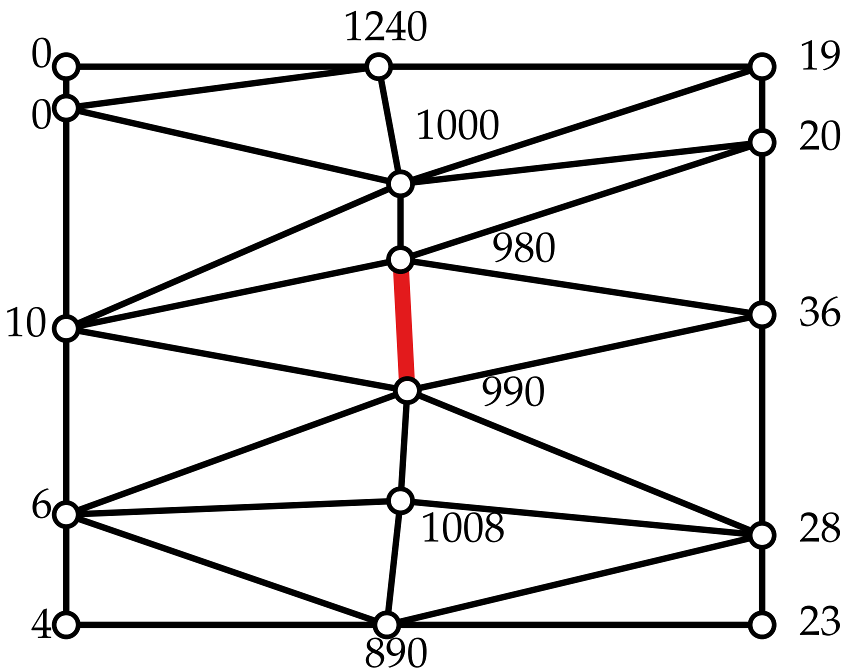

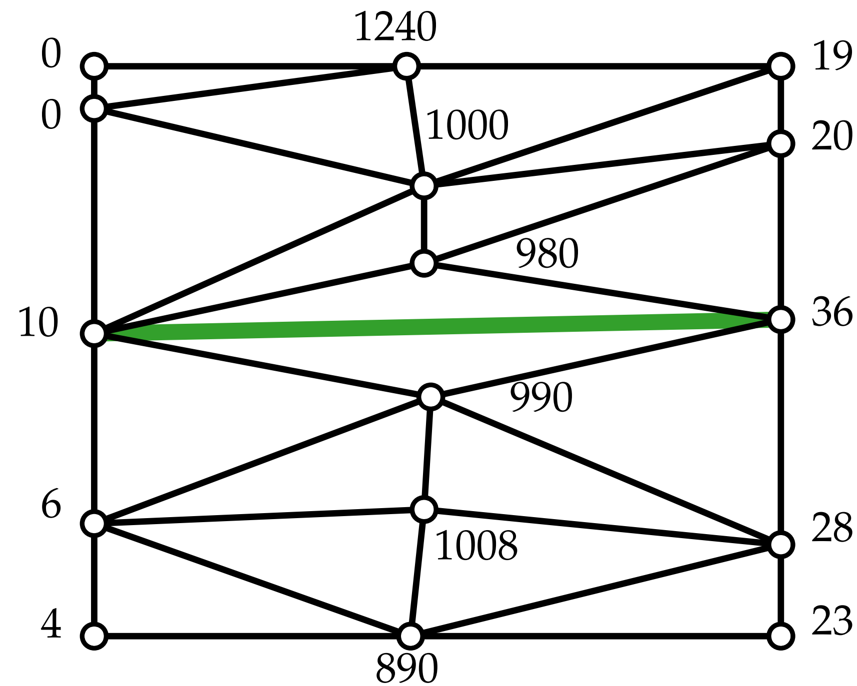

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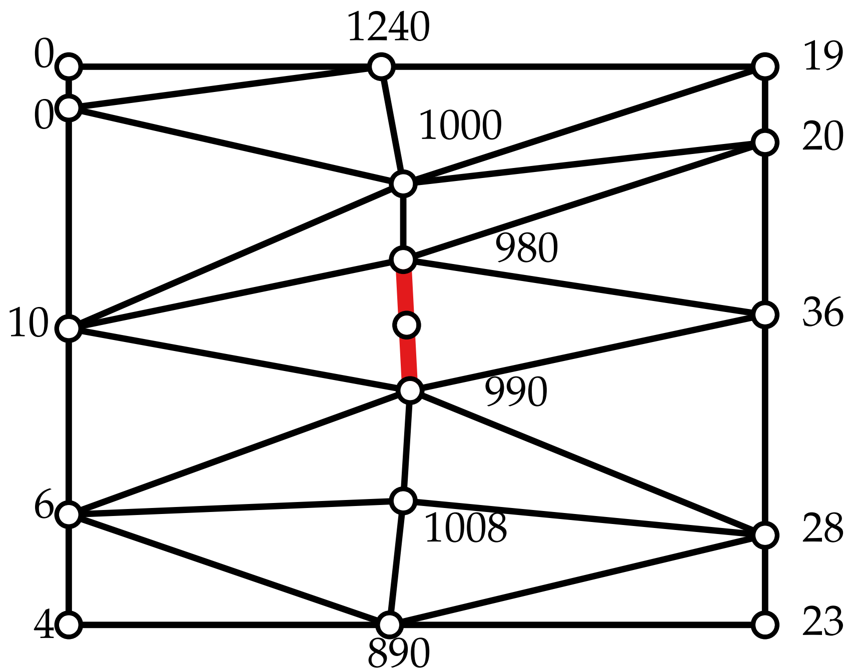

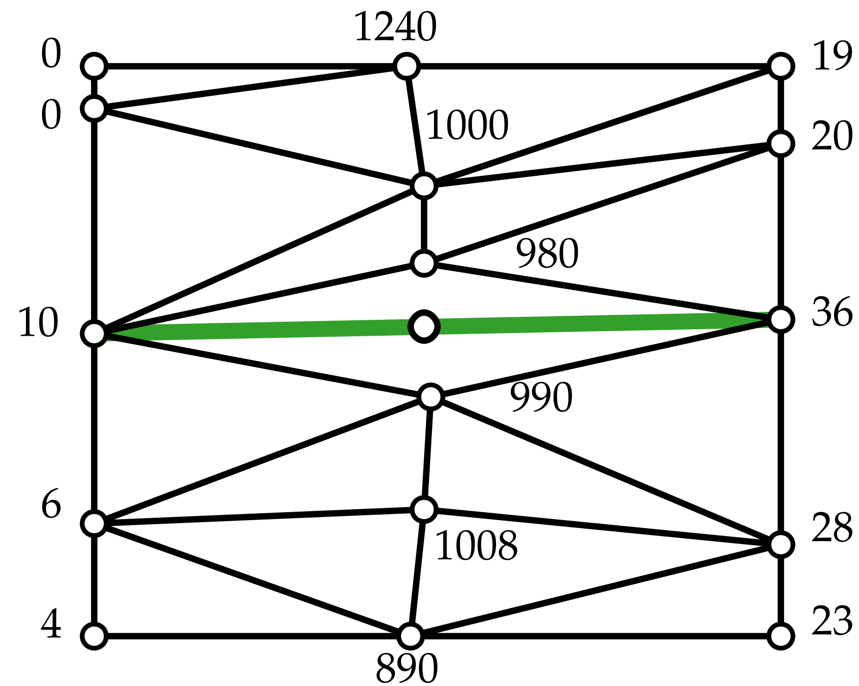

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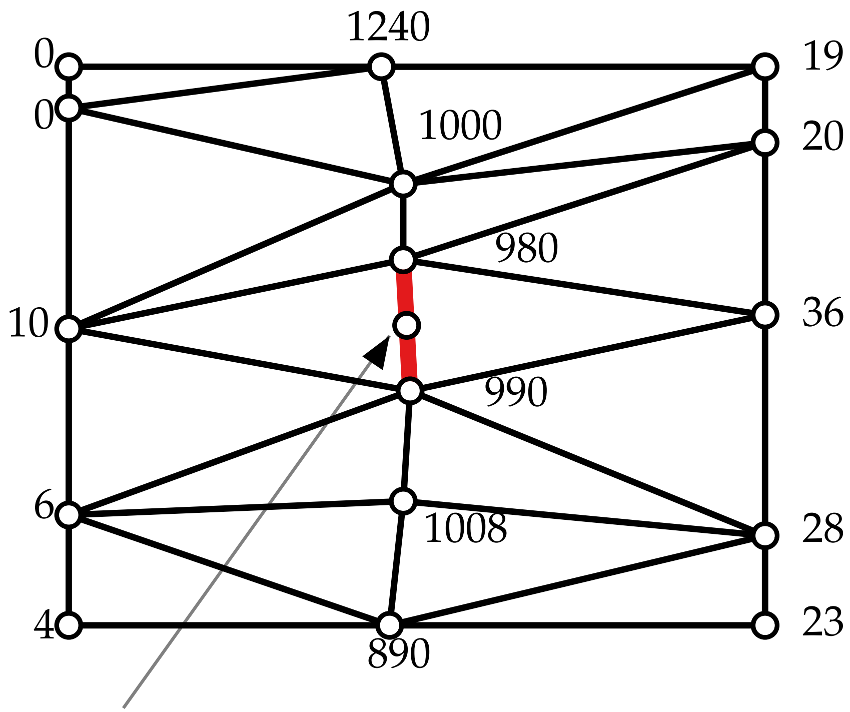
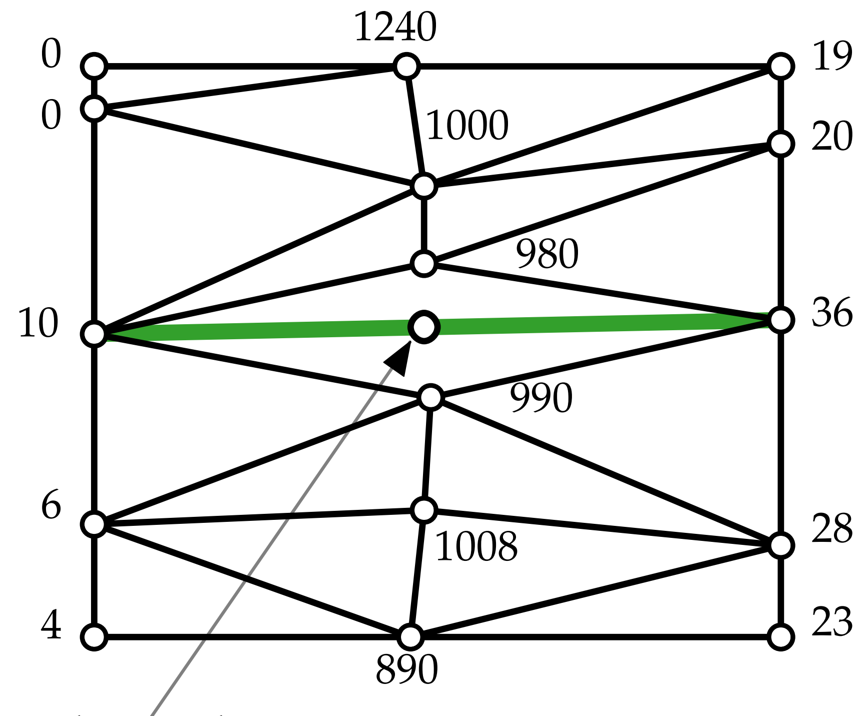
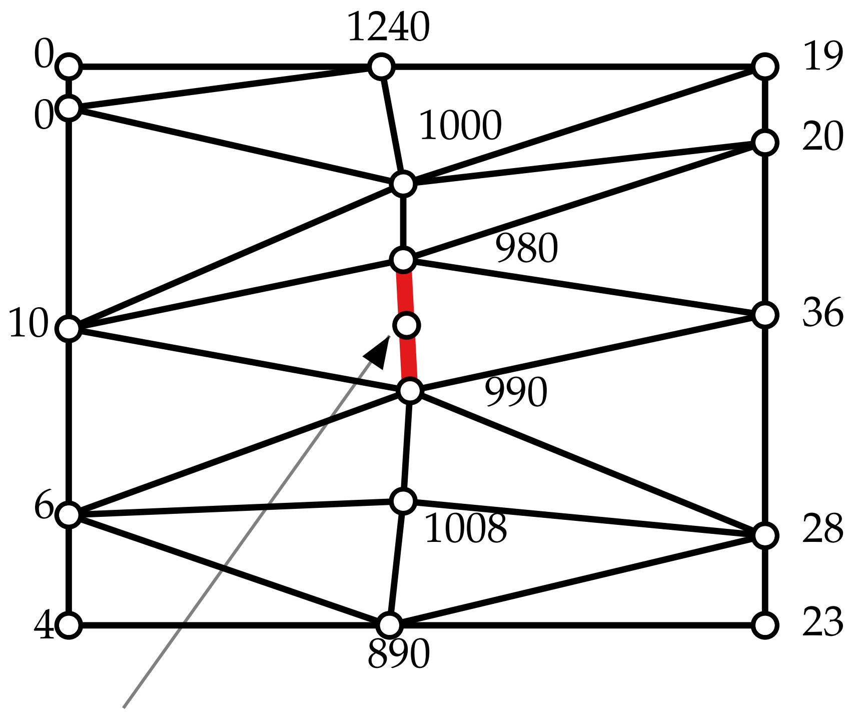




height = 985

height = 23

Back to Height Interpolation



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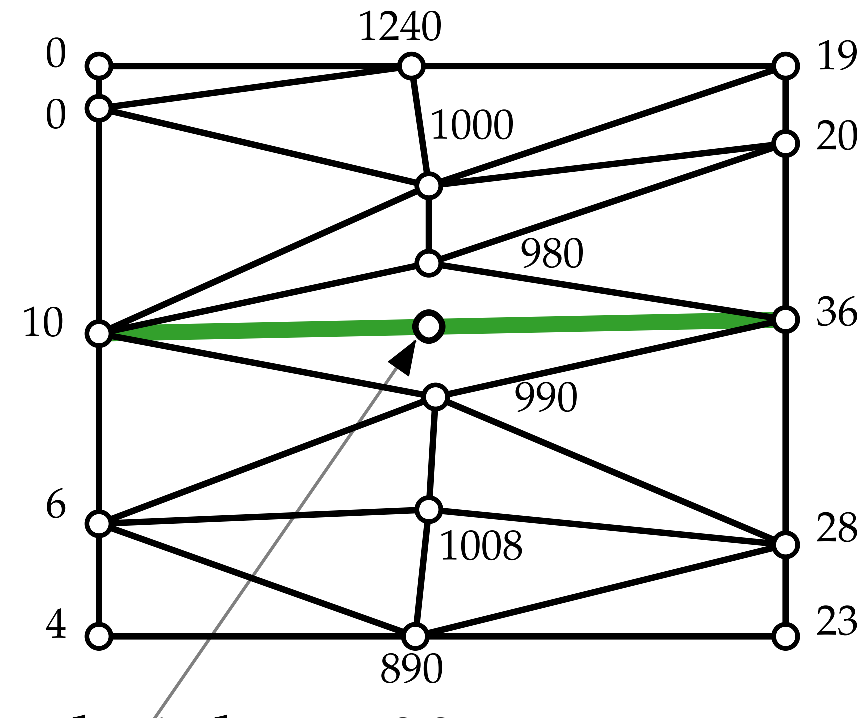

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Intuition. Avoid "skinny" triangles!

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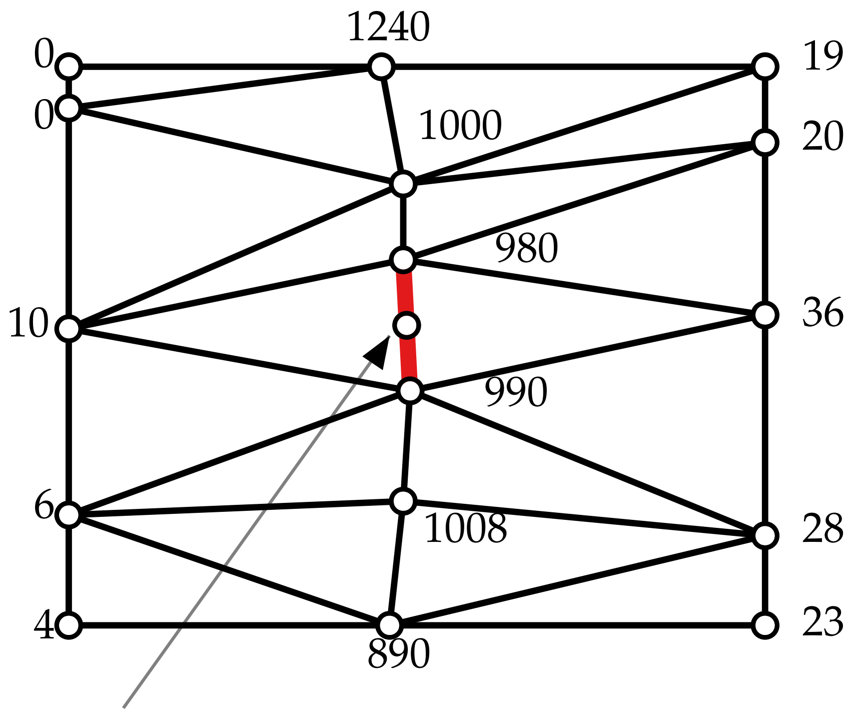

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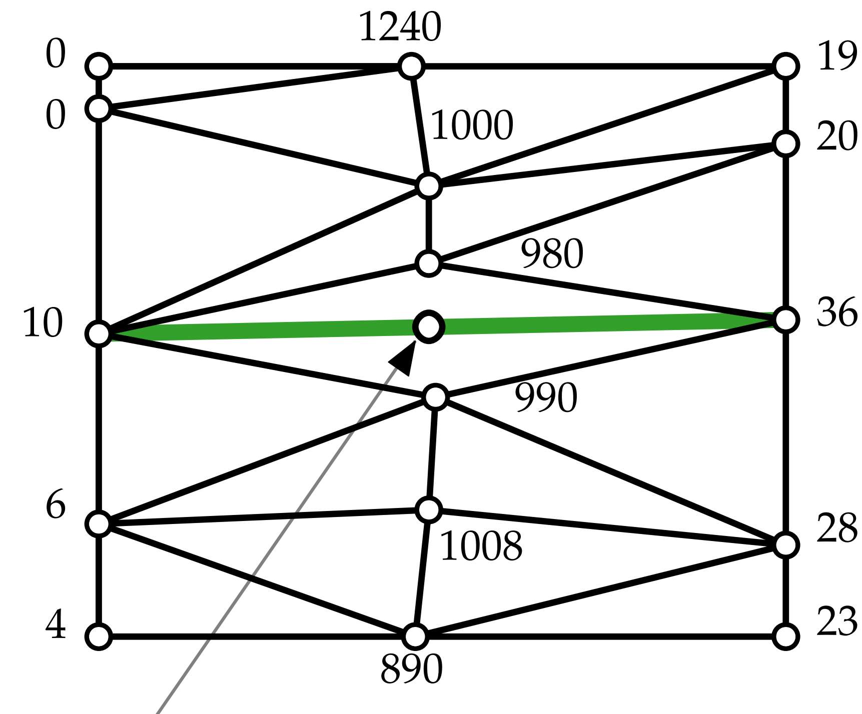

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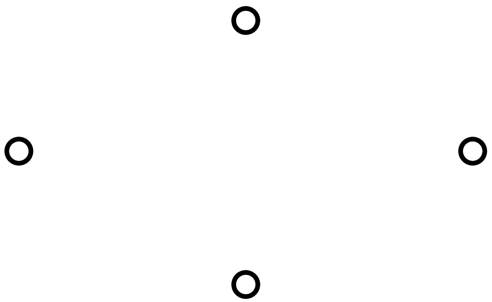
Intuition. Avoid "skinny" triangles!

In other words: avoid small angles!

Angle-Optimal Triangulations

Definition. Given a set $P \subset \mathbb{R}^2$



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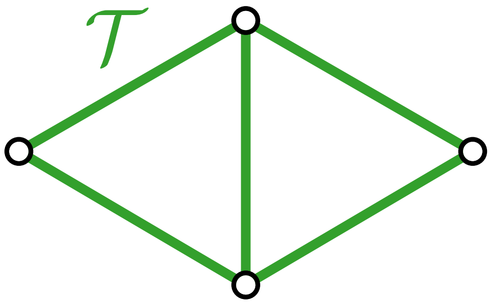
Definition. Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of P,



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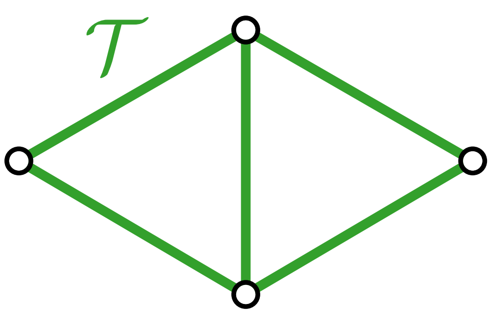

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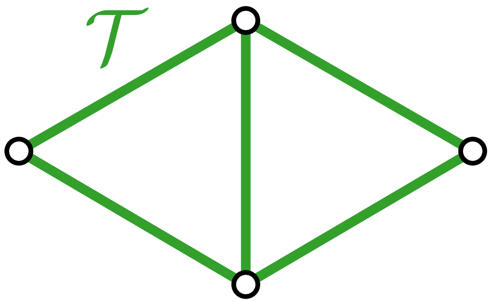
Definition. Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of P, let m be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the *angle vector* of $\mathcal{T}$



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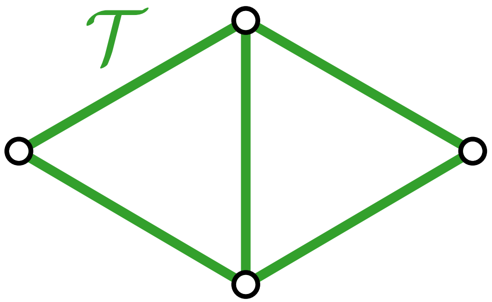

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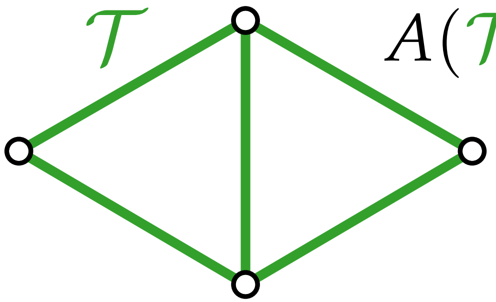

$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

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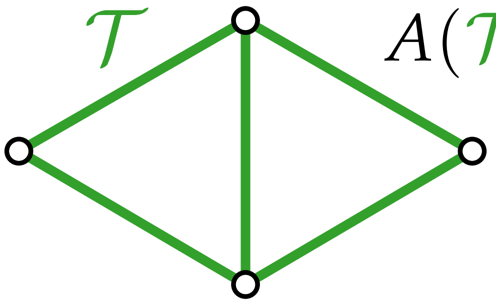
We say $A(\mathcal{T}) > A(\mathcal{T}')$

$\mathcal{T}$



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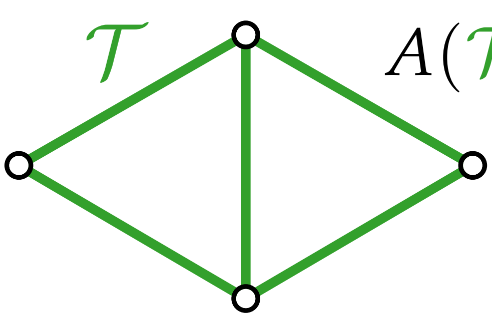

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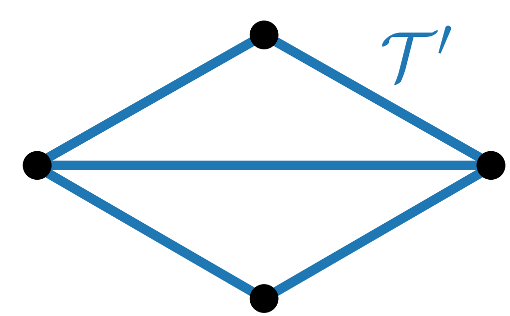

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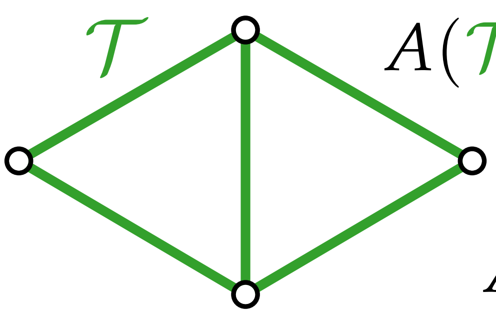

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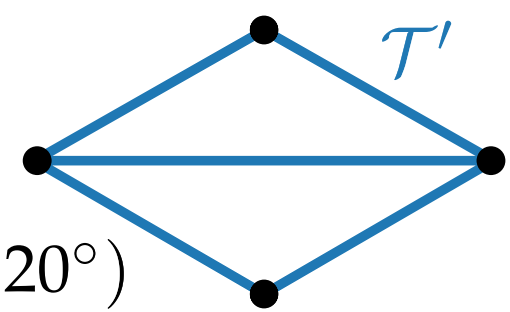

$$A(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$$

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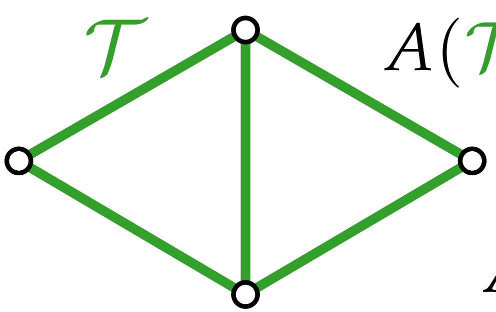

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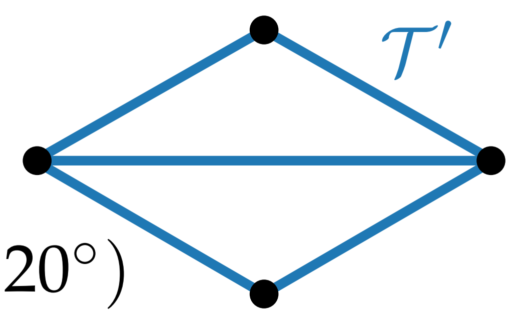

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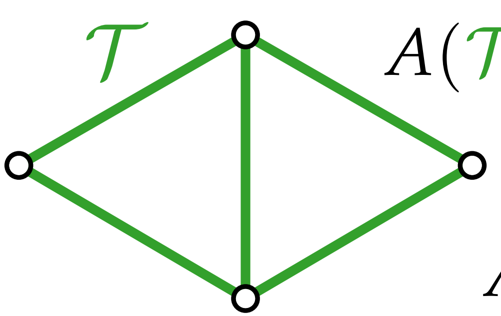

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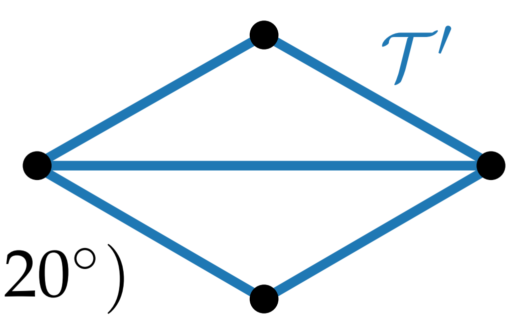

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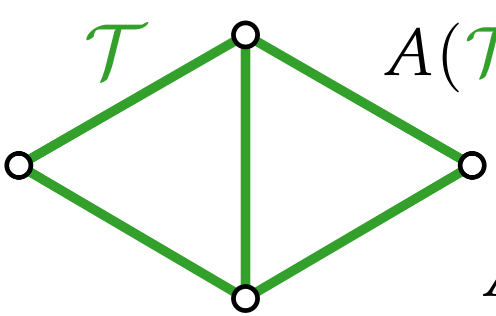

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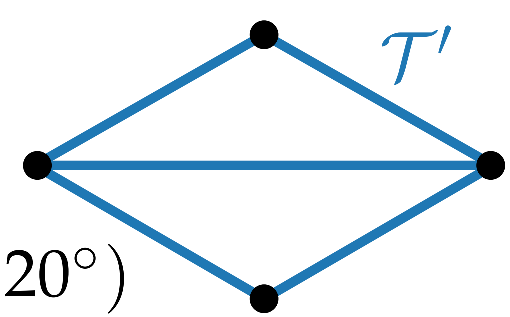

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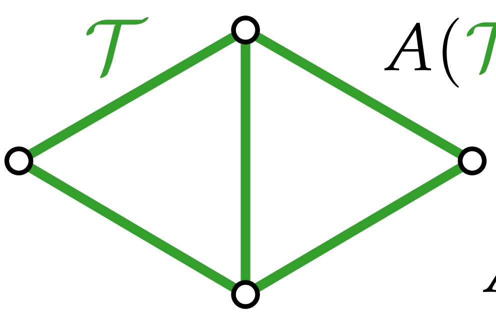
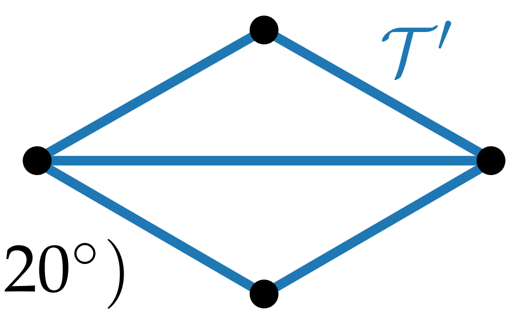
Angle-Optimal Triangulations

Definition. Given a set $P \subset \mathbb{R}^2$ and a triangulation $\mathcal{T}$ of P, let m be the number of triangles in $\mathcal{T}$ and let $A(\mathcal{T}) = (\alpha_1, \ldots, \alpha_{3m})$ be the *angle vector* of $\mathcal{T}$, where $\alpha_1 \leq \cdots \leq \alpha_{3m}$ are the angles in the triangles of $\mathcal{T}$.

We say $A(\mathcal{T}) > A(\mathcal{T}')$
if $\exists i \in \{1, ..., 3m\} : \alpha_i > \alpha'_i$ and $\forall j < i : \alpha_j = \alpha'_j$.



$A(\mathcal{T}) = (60°, 60°, 60°, 60°, 60°, 60°)$



$A(\mathcal{T}') = (30°, 30°, 30°, 30°, 120°, 120°)$

Angle-Optimal Triangulations

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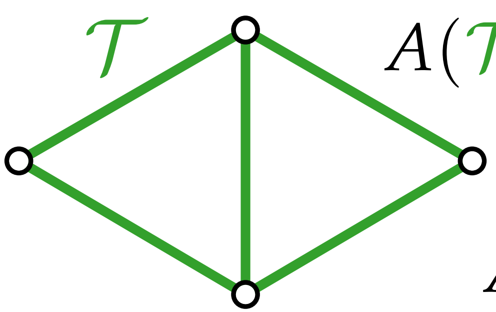

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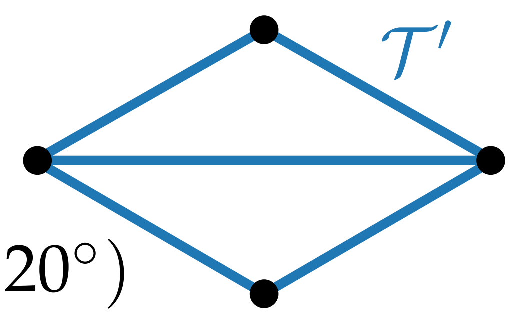

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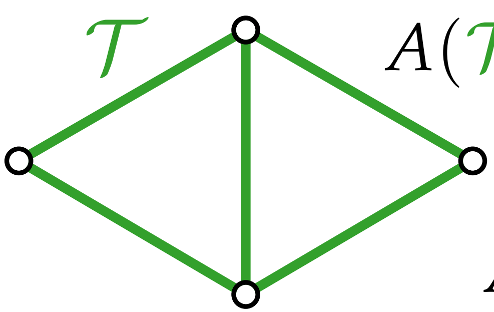
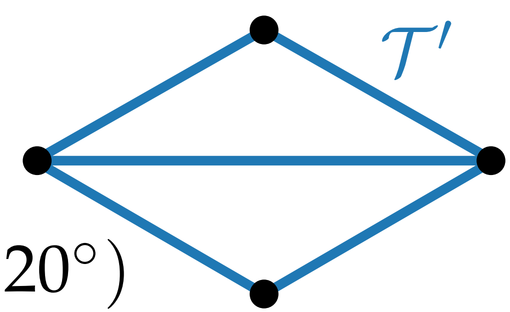
$\mathcal{T}$ is *angle-optimal* if
$A(\mathcal{T}) \geq A(\mathcal{T}')$ for all triangulations $\mathcal{T}'$ of P.

$\mathcal{T}$



$A(\mathcal{T}) = (60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ, 60^\circ)$

$\mathcal{T}'$



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Angle-Optimal Triangulations

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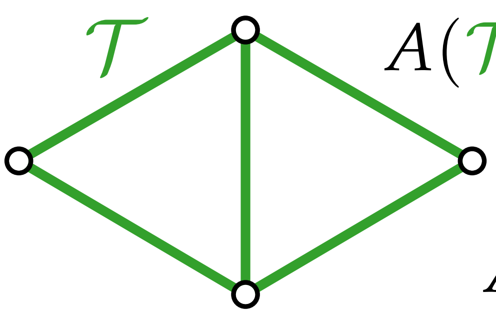

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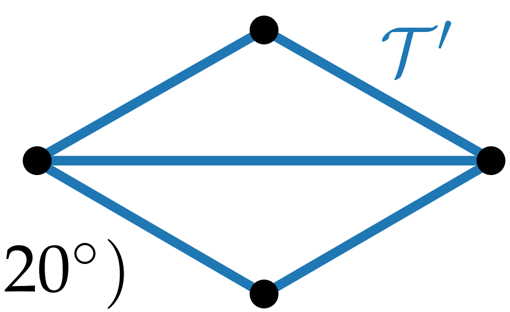


Computational Geometry

Lecture 8:
Delaunay Triangulations
or
Height Interpolation

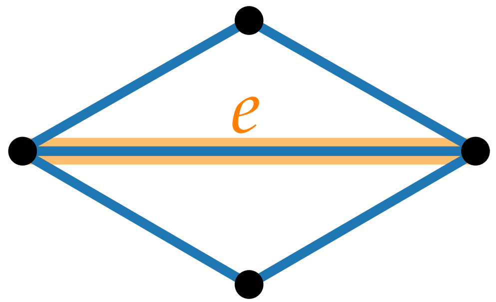
Part III:
Edge Flips & Legal Triangulations

Philipp Kindermann

Summer Semester 2020

Edge Flips

Definition. Let $\mathcal{T}$ be a triangulation. An edge e of $\mathcal{T}$ is *illegal* if the minimum angle in the two triangles adjacent to e increases when flipping.



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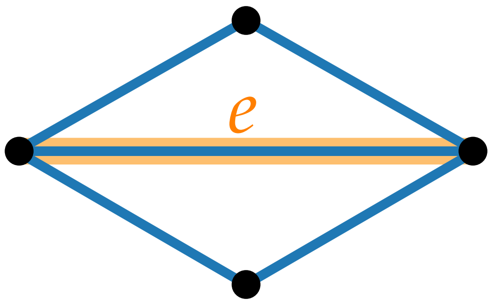
$\min_i \alpha_i = 30^\circ$



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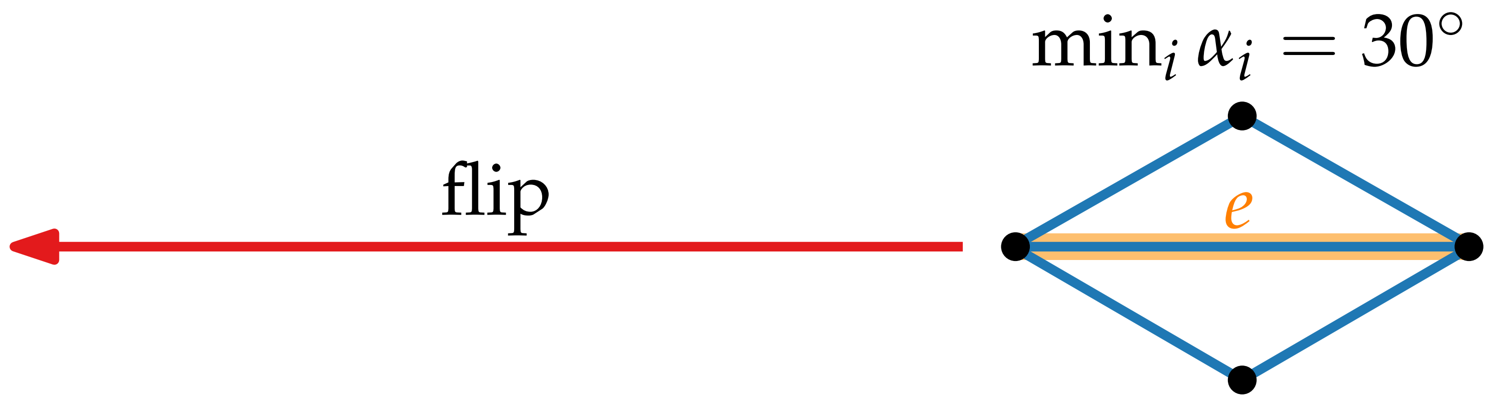

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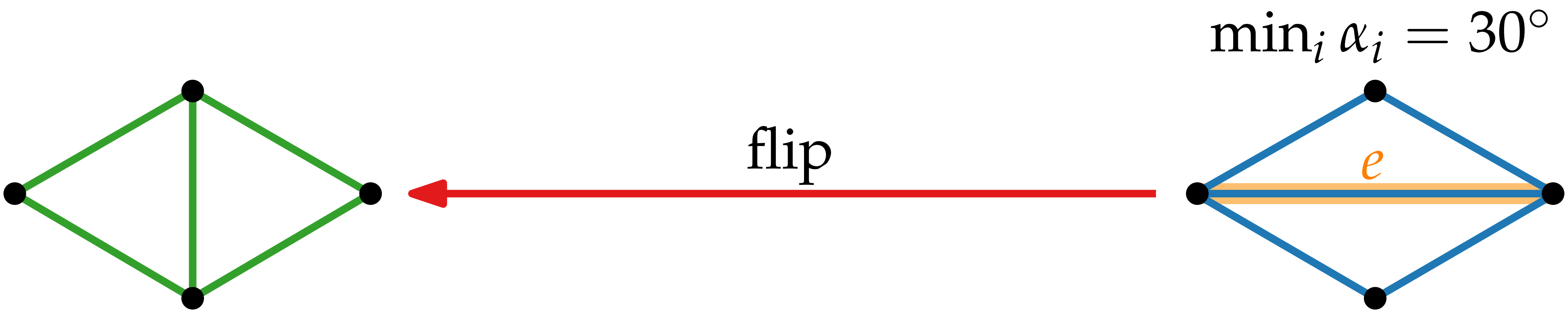

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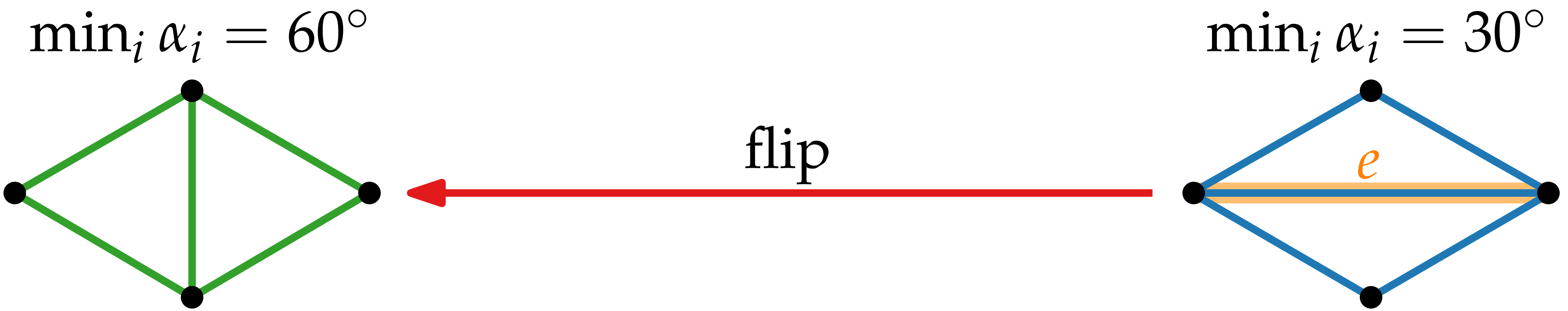

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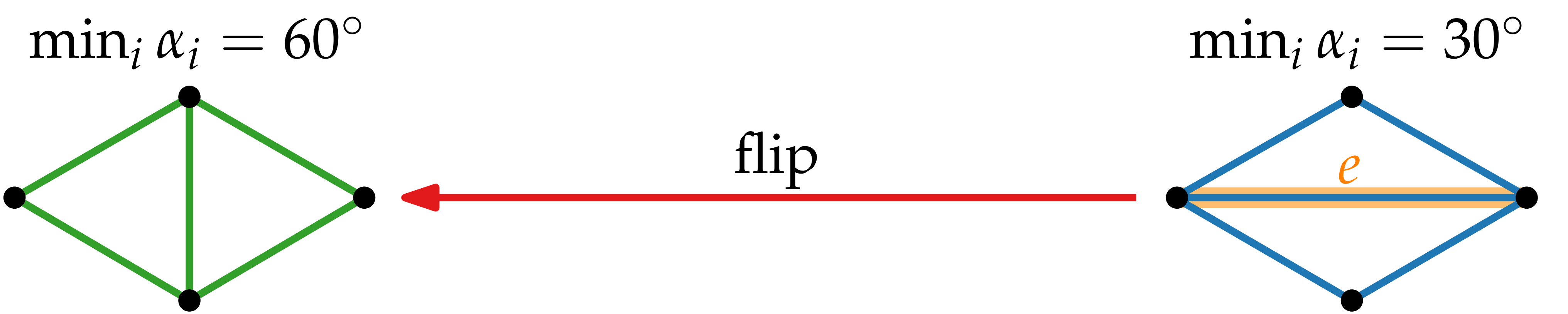
Observe. Let e be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$.



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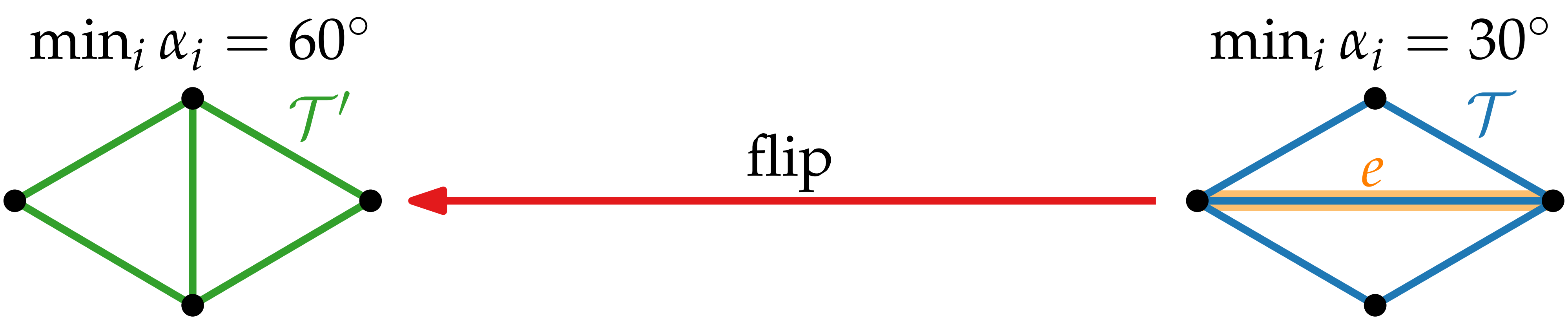

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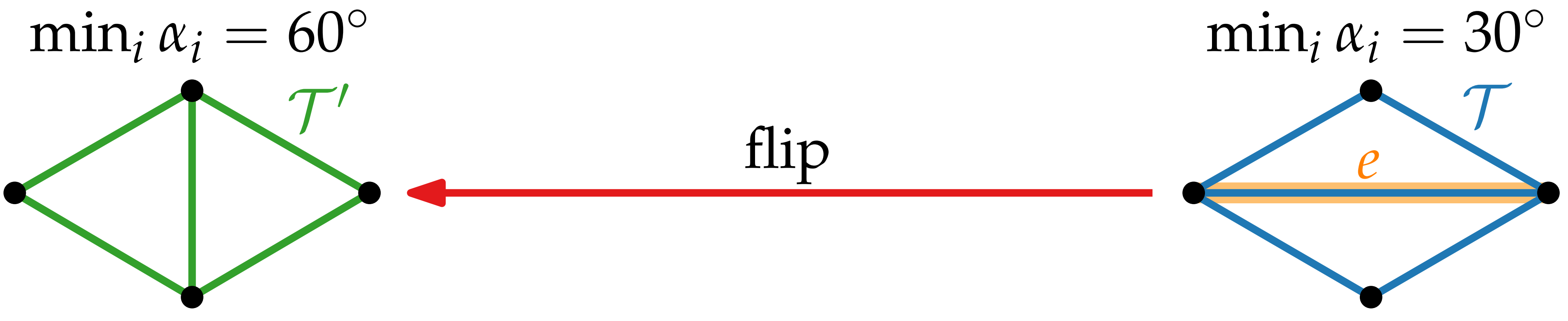
Observe. Let e be an illegal edge of $\mathcal{T}$, and $\mathcal{T}' = \text{flip}(\mathcal{T}, e)$. Then $A(\mathcal{T}') > A(\mathcal{T})$.



This is all Greek to me...

Theorem. [Thales]

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The diameter of a circle always subtends a right angle to any point on the circle.

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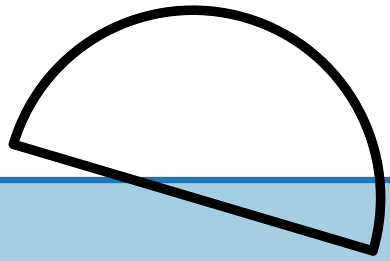

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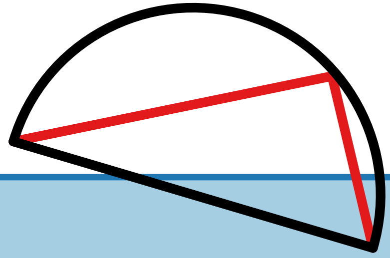

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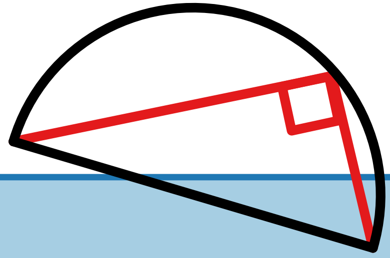

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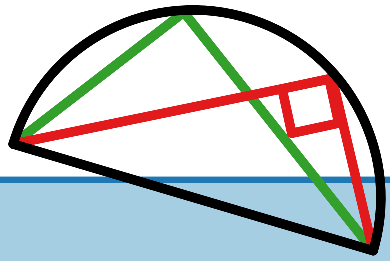

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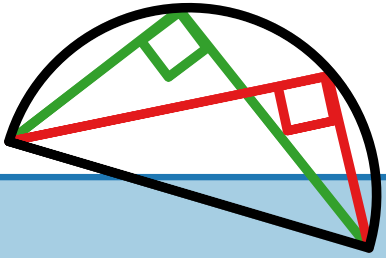

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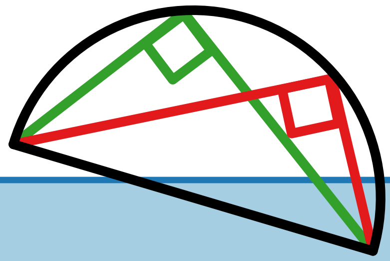

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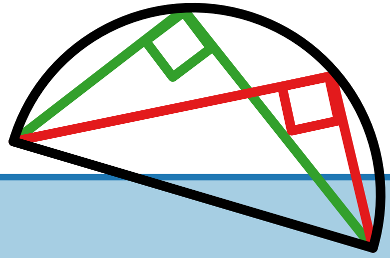

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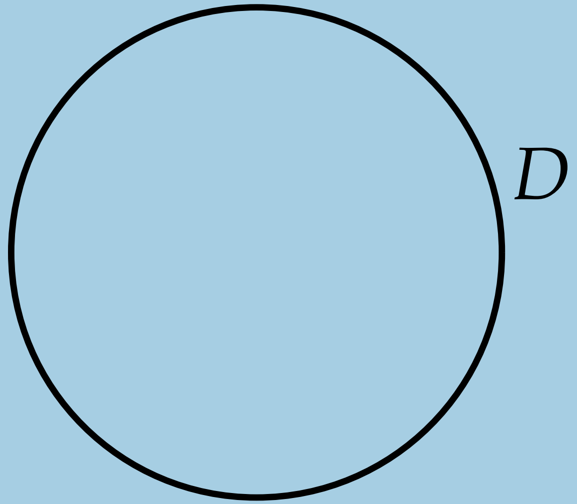

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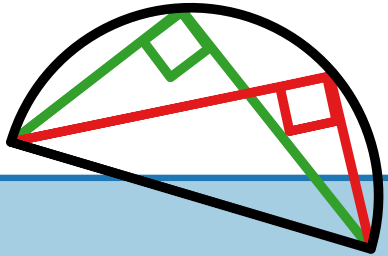

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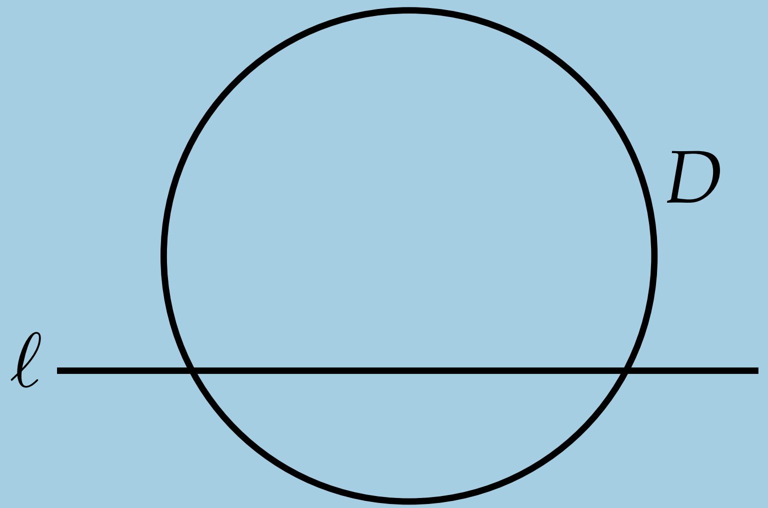

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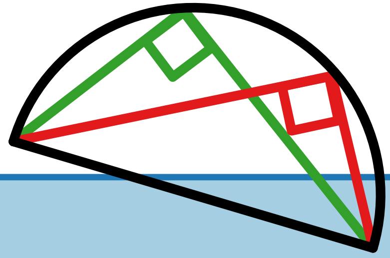

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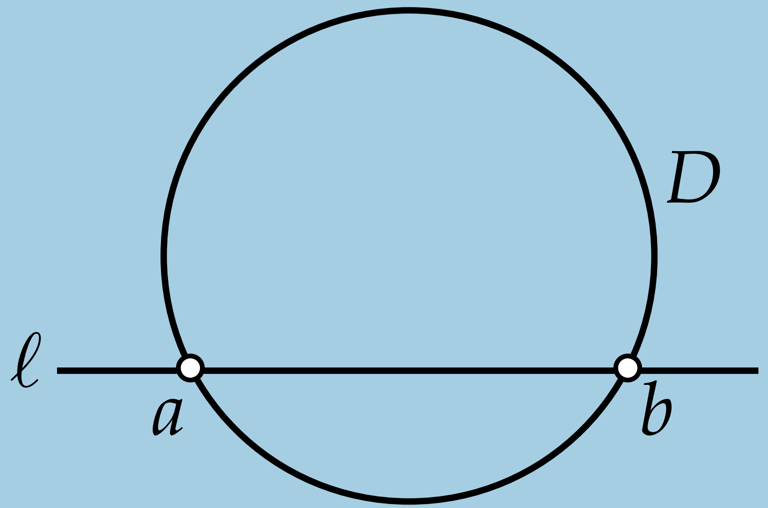

$$\{a, b\} := \ell \cap \partial D \ (a \neq b)$$

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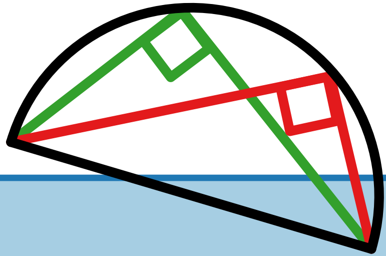

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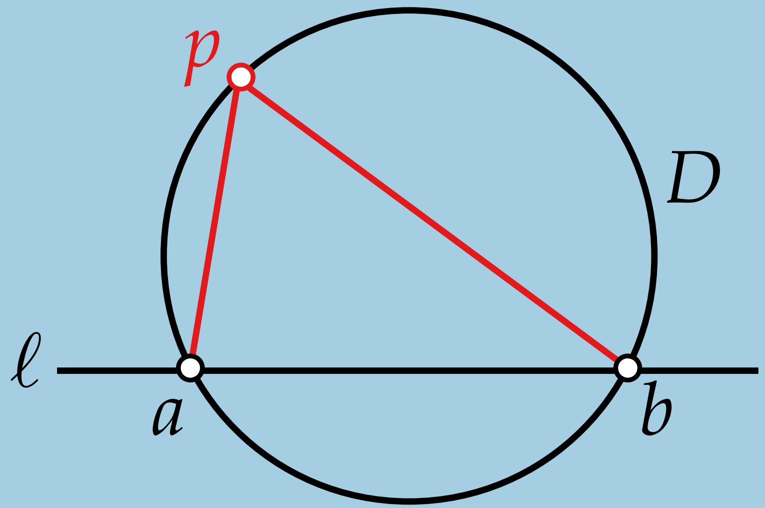

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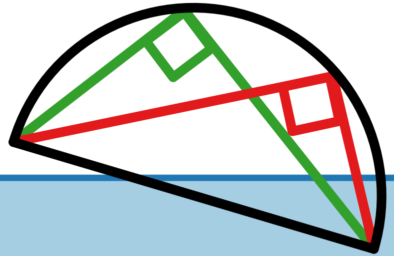

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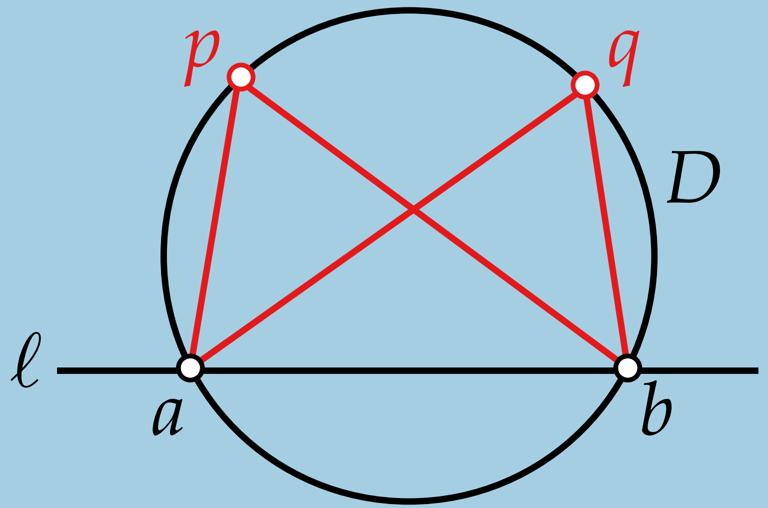

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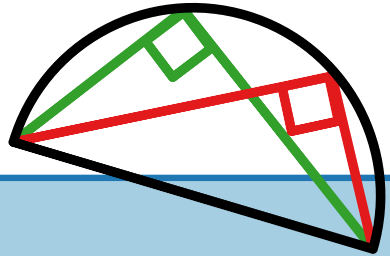

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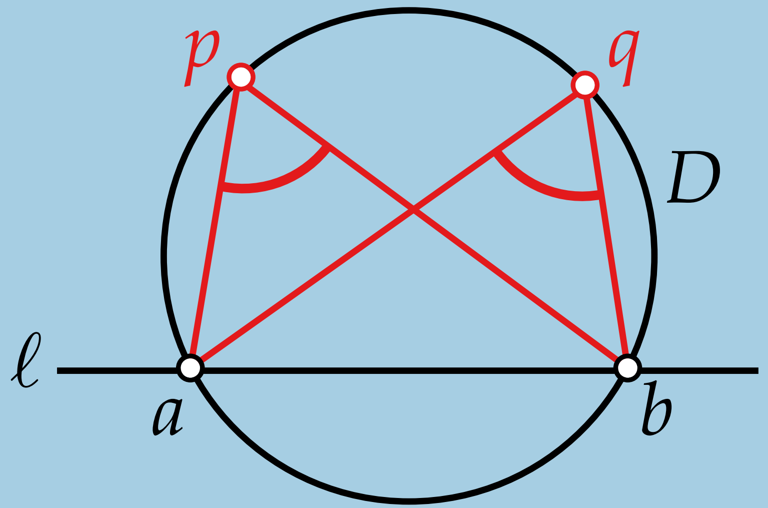

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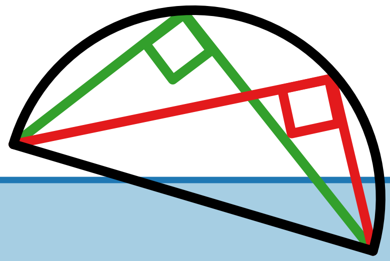

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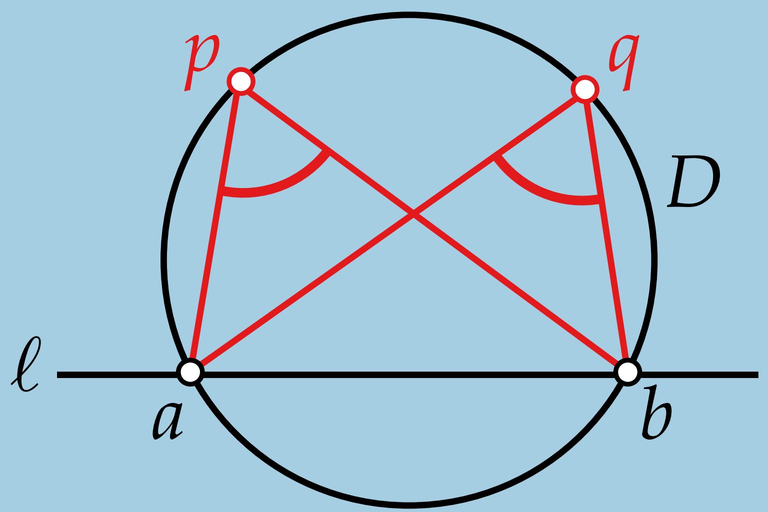

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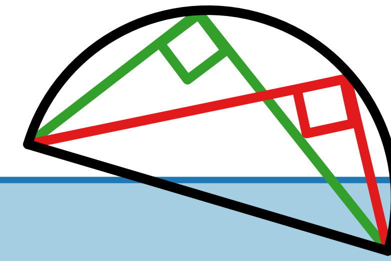

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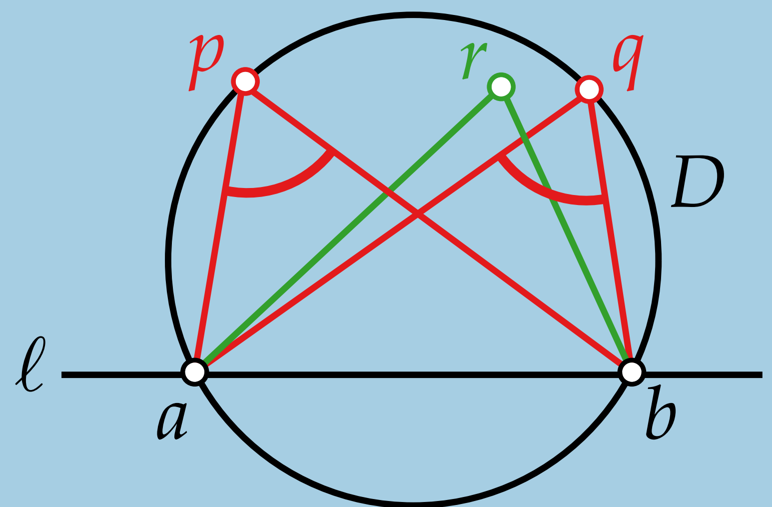

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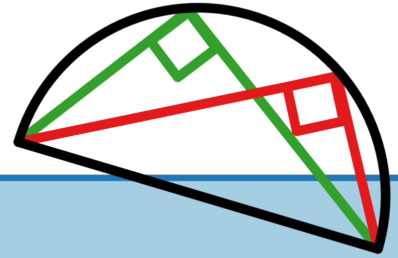

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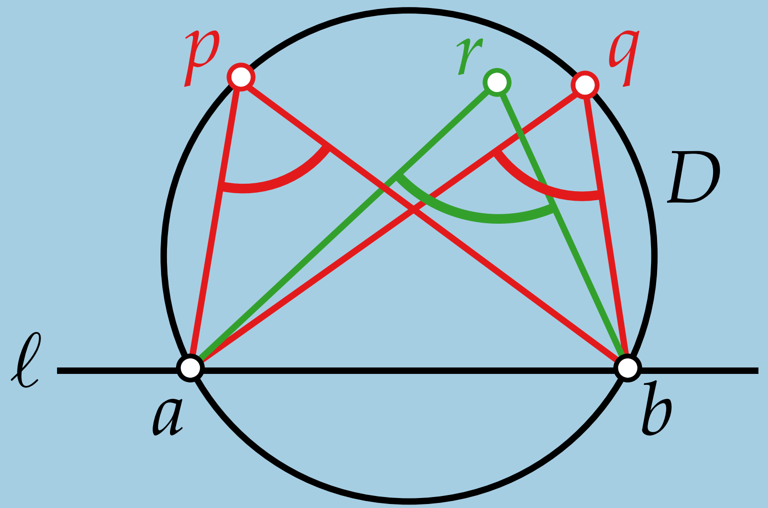

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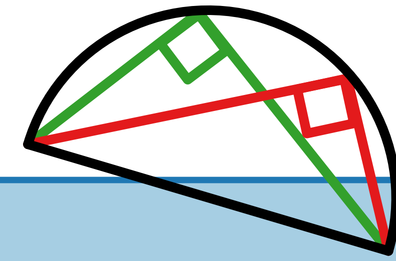

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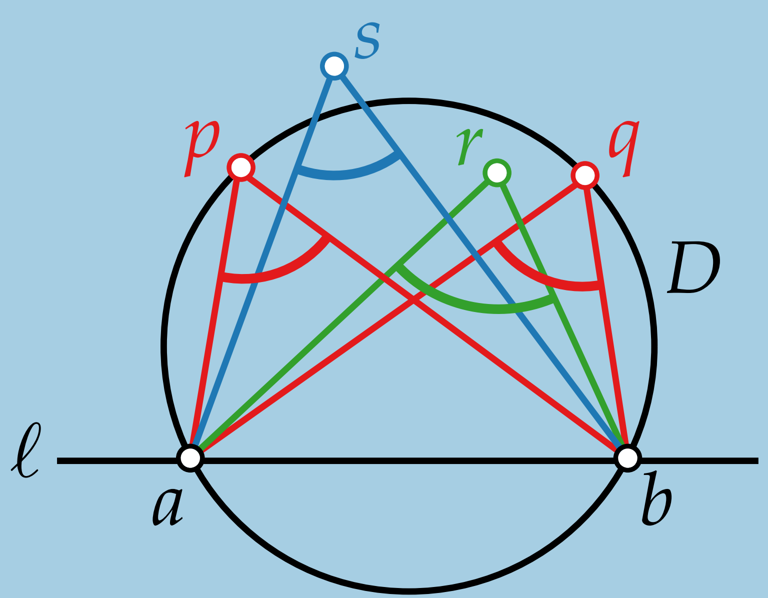

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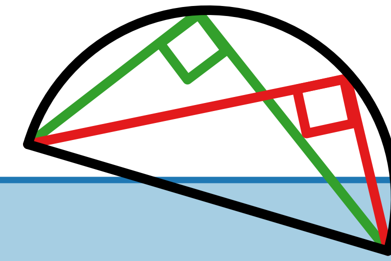

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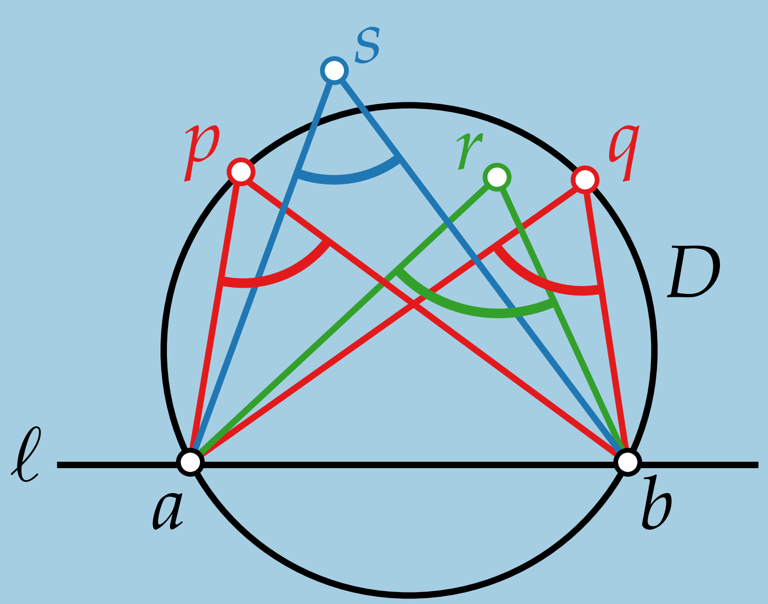

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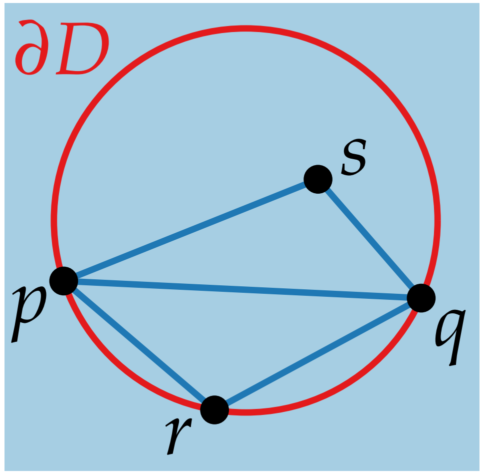
$s \notin D$

$$\angle asb < \angle apb = \angle aqb < \angle arb$$

Legal Triangulations

Lemma. Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$.
Then edge pq is illegal iff $s \in \text{int}(D)$.



Legal Triangulations

Lemma. Let $\Delta prq, \Delta pqs \in \mathcal{T}$ and $p, q, r \in \partial D$.
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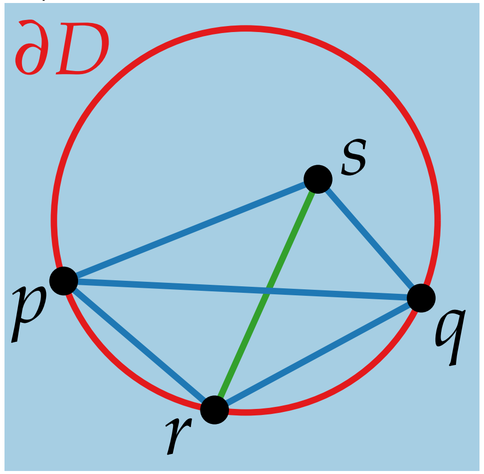
If p, q, r, s in convex position and $s \notin \partial D$,
then either pq or rs is illegal.



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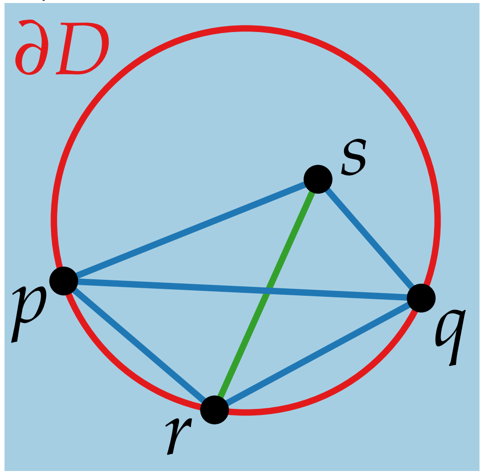

Proof.

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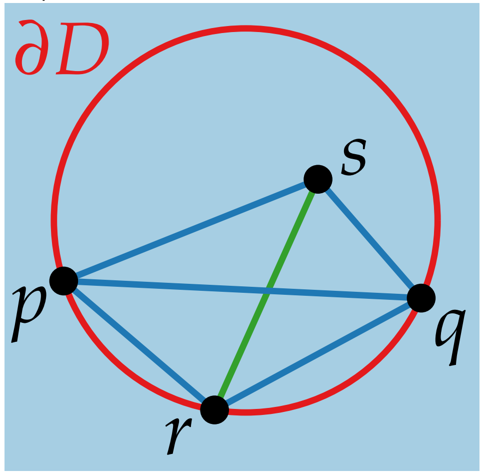

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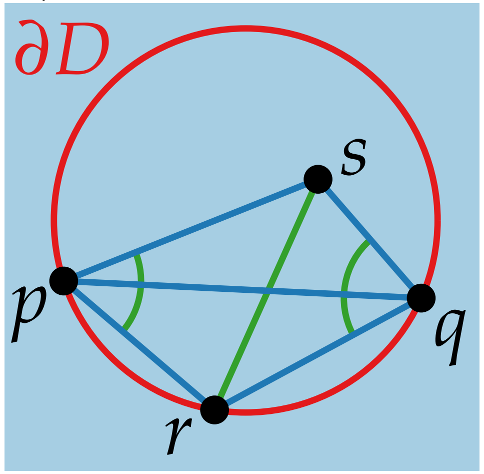

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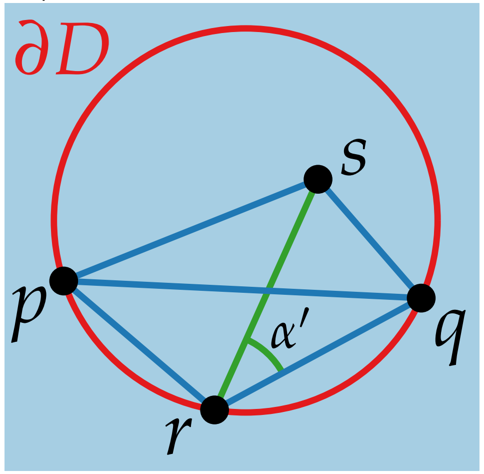

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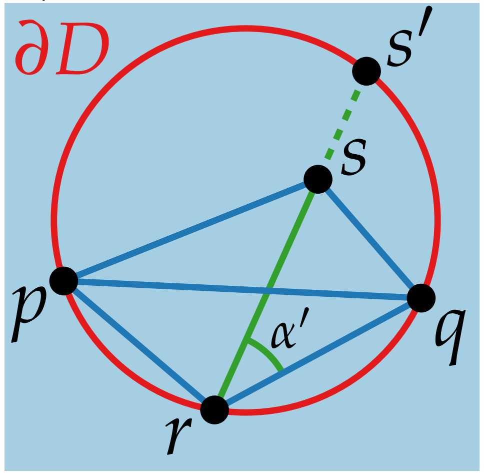

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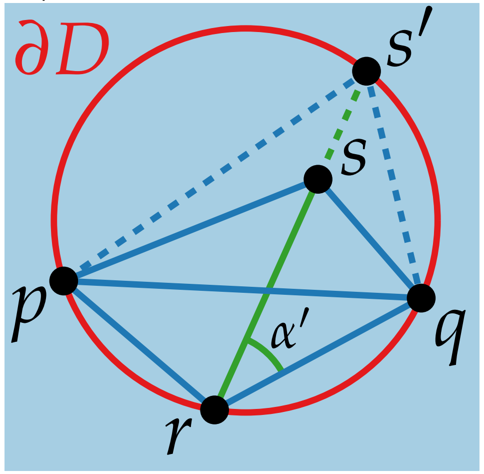

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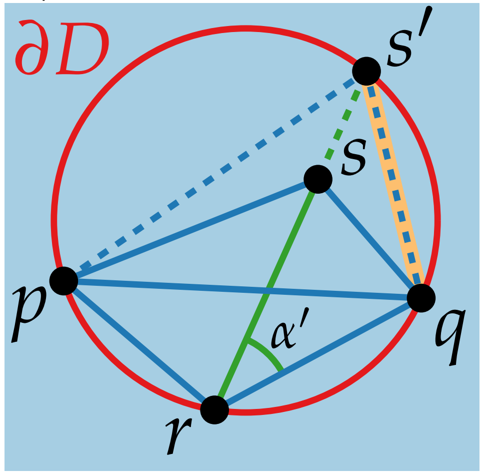

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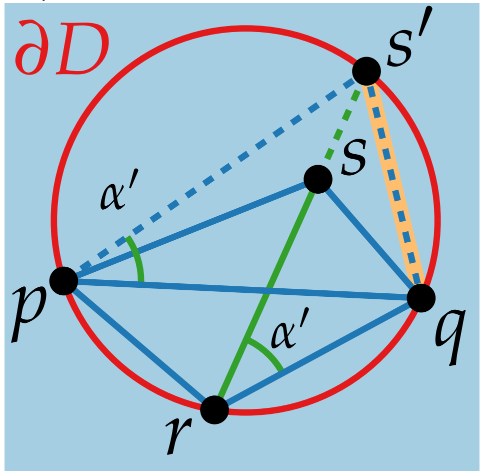

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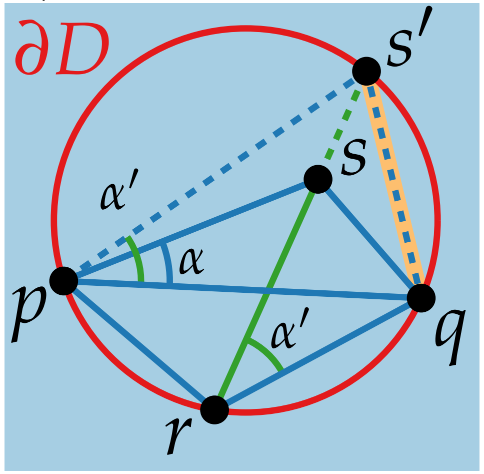

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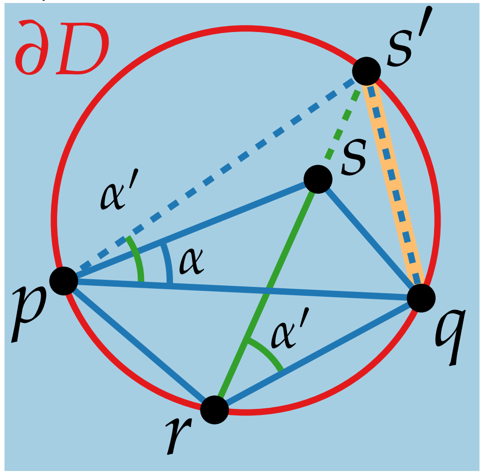

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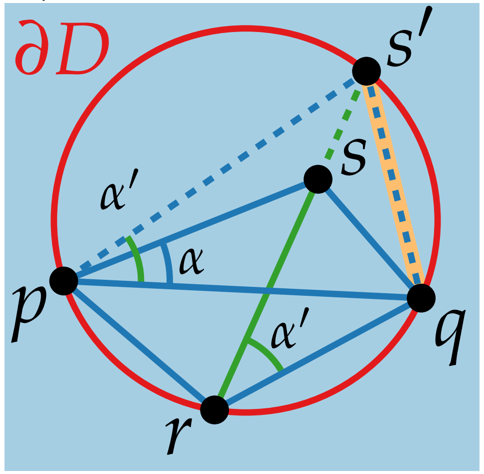

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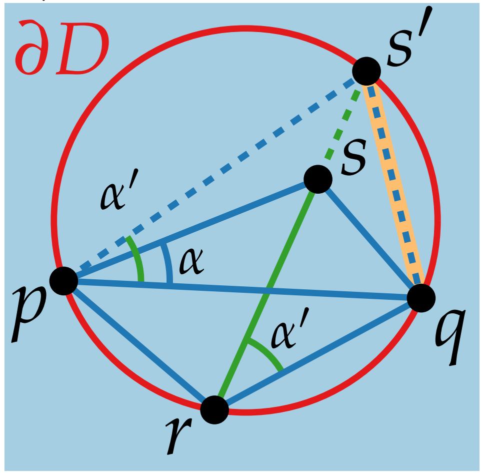

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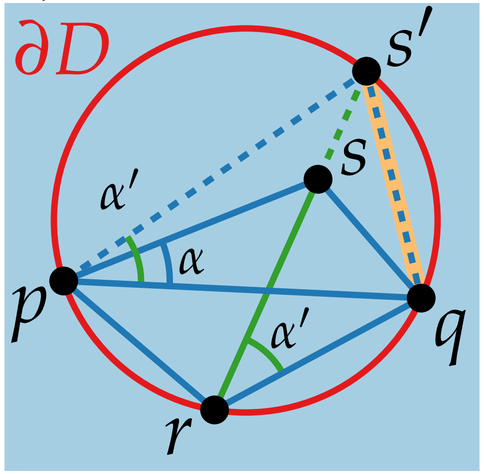

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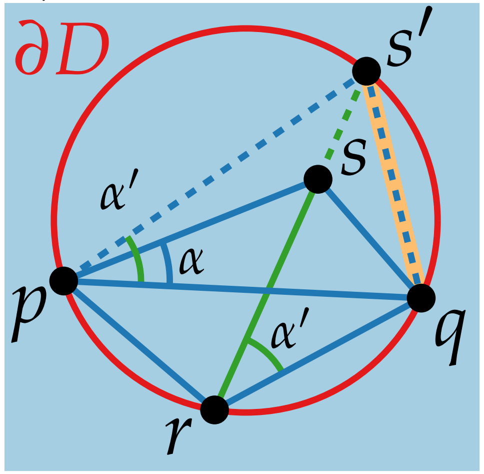

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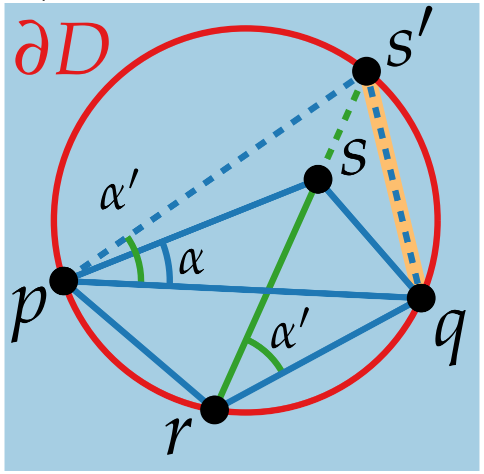

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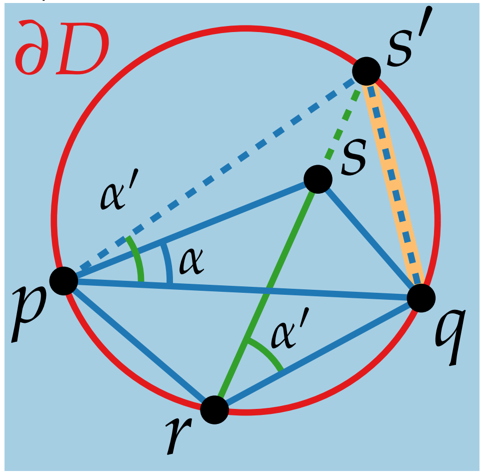

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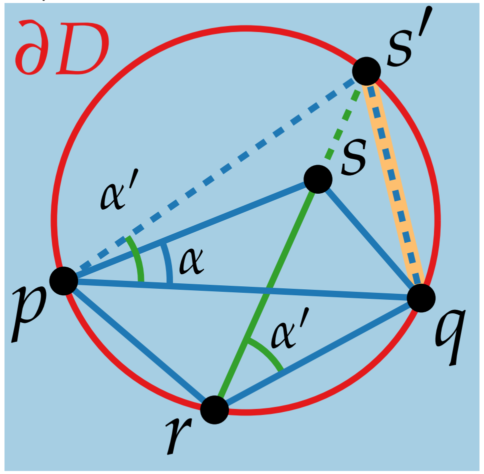

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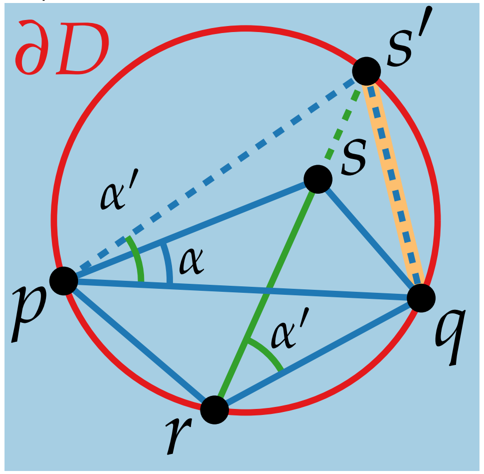

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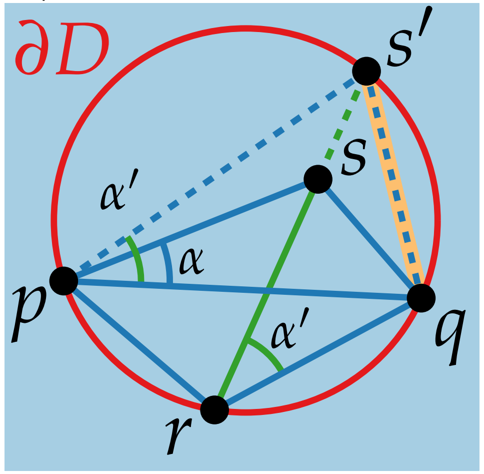

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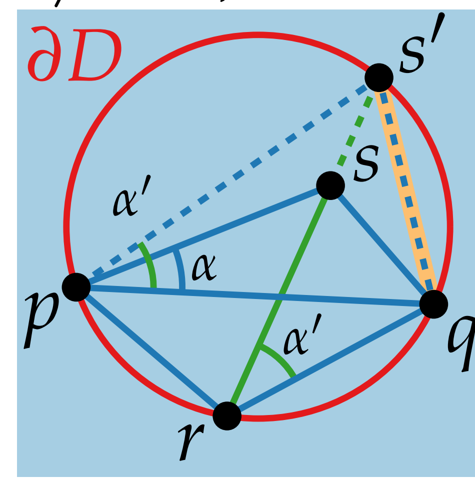

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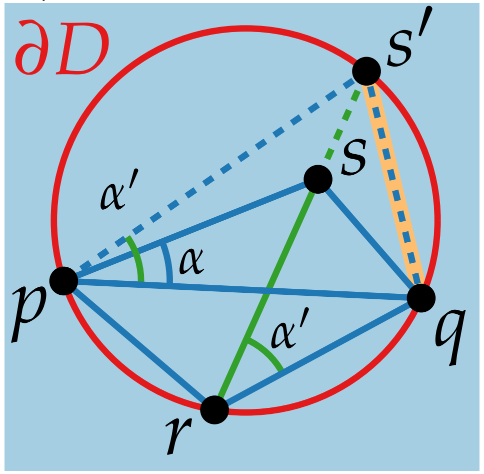

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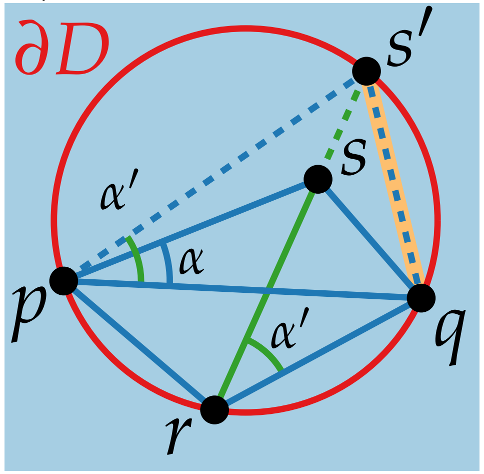

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To clarify things, we'll introduce yet another type of triangulation. . .

Computational Geometry

Lecture 8:
Delaunay Triangulations
or
Height Interpolation

Part IV:
Delaunay Triangulation

Philipp Kindermann

Summer Semester 2020

Voronoi & Delaunay

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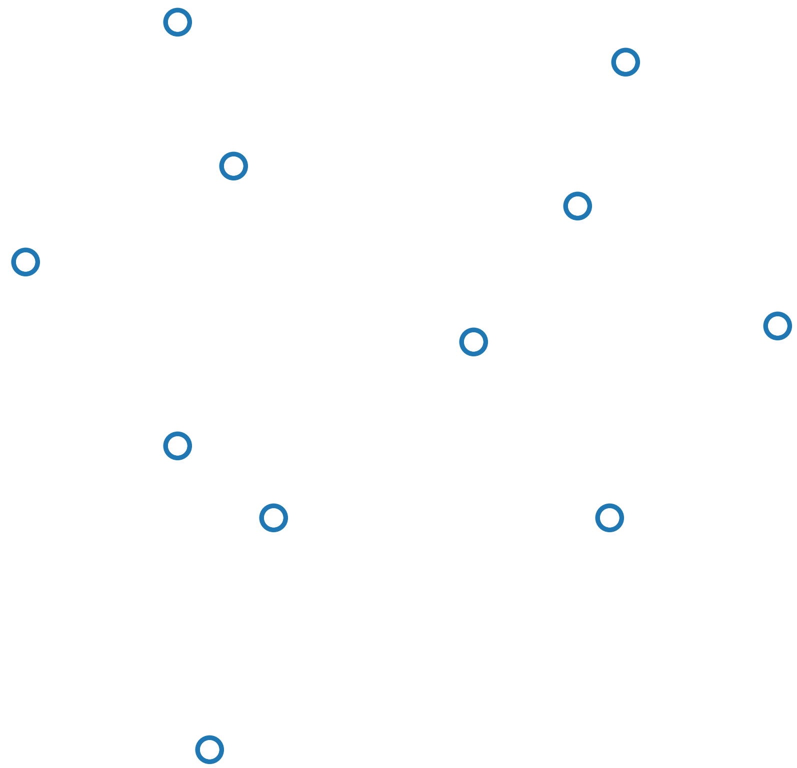
Definition: The *Delaunay graph* $\mathcal{DG}(P)$ is the straight-line drawing of $\mathcal{G}$.

From Voronoi to Delaunay

$P \subset \mathbb{R}^2$



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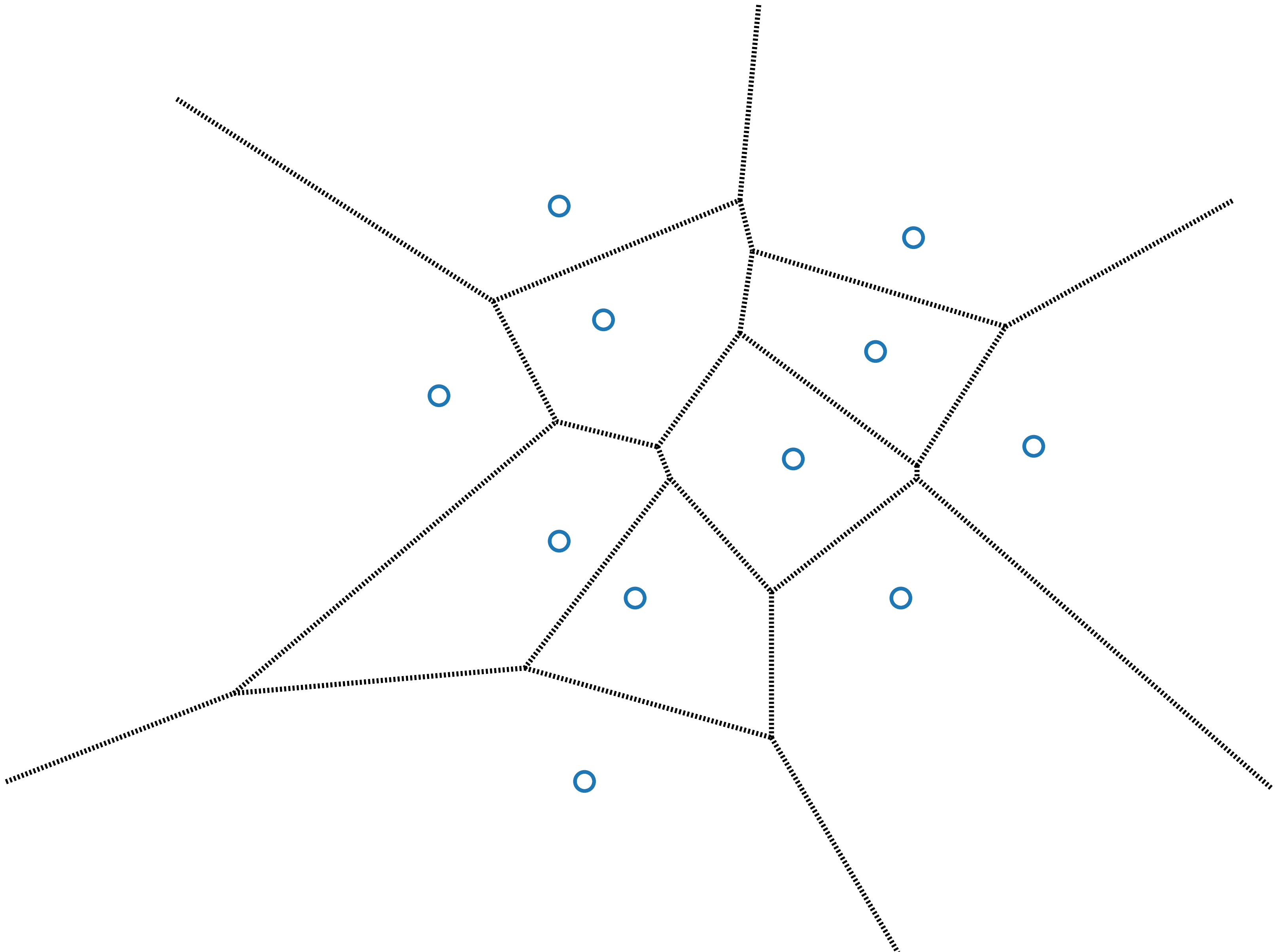

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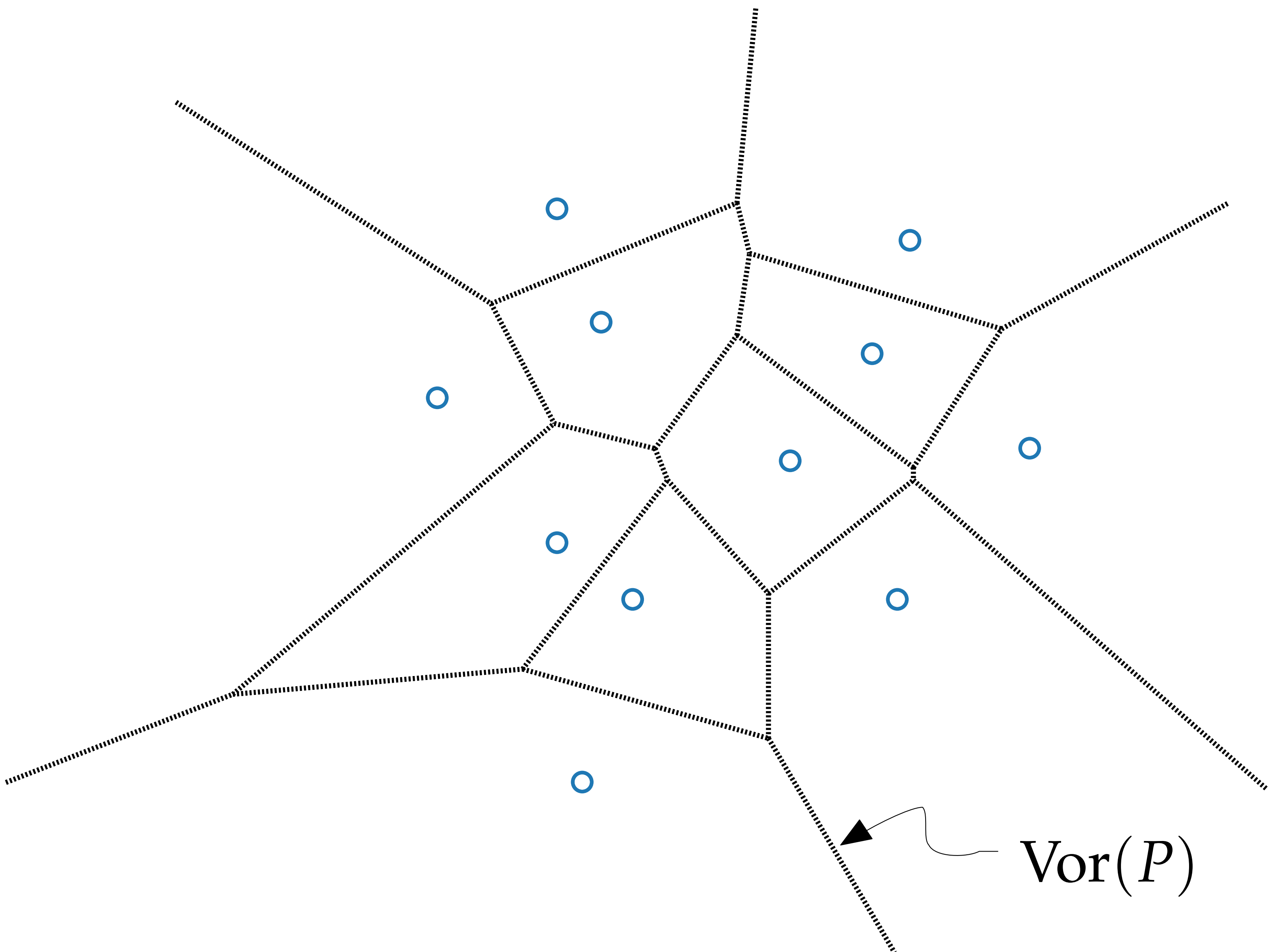

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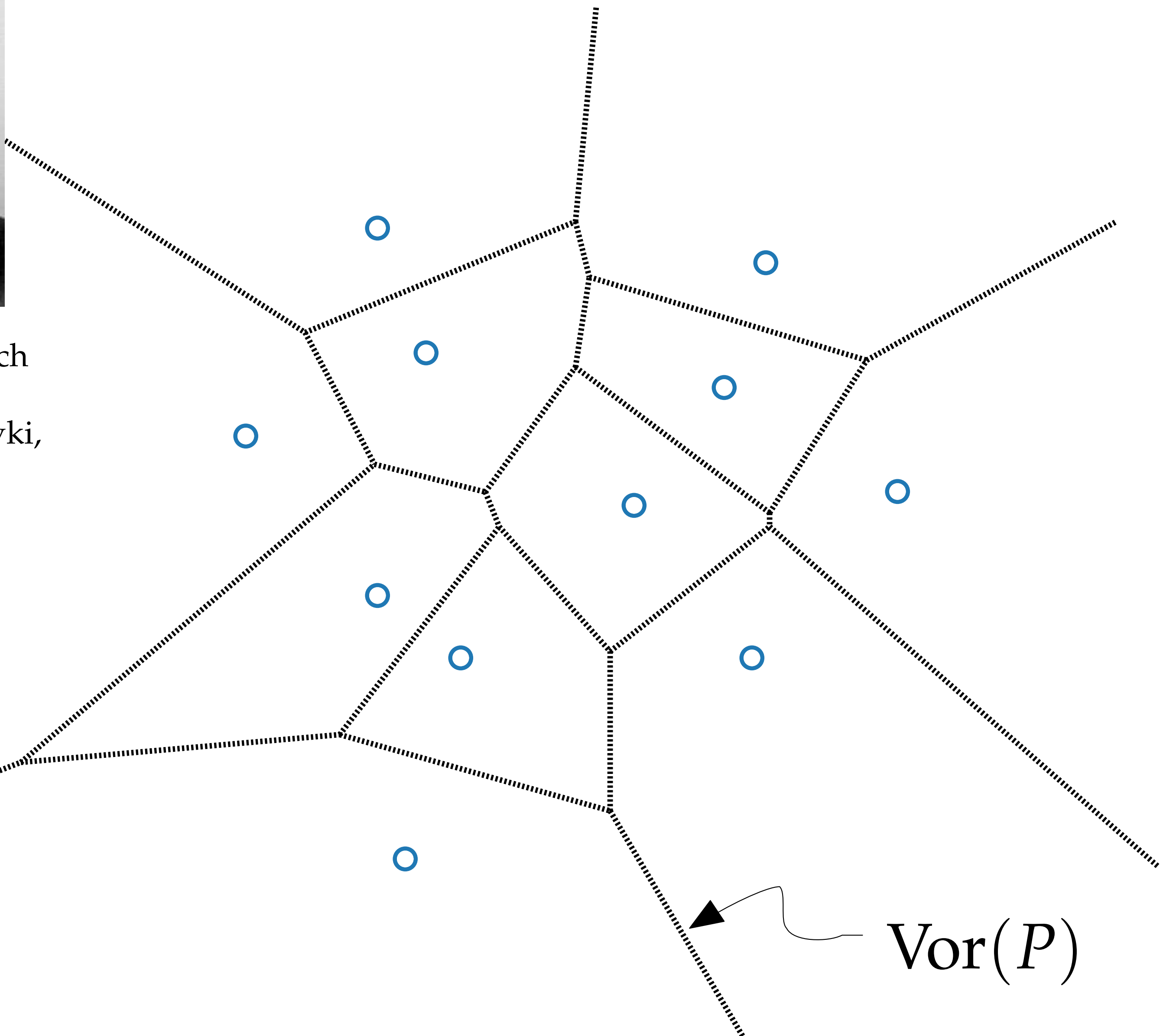

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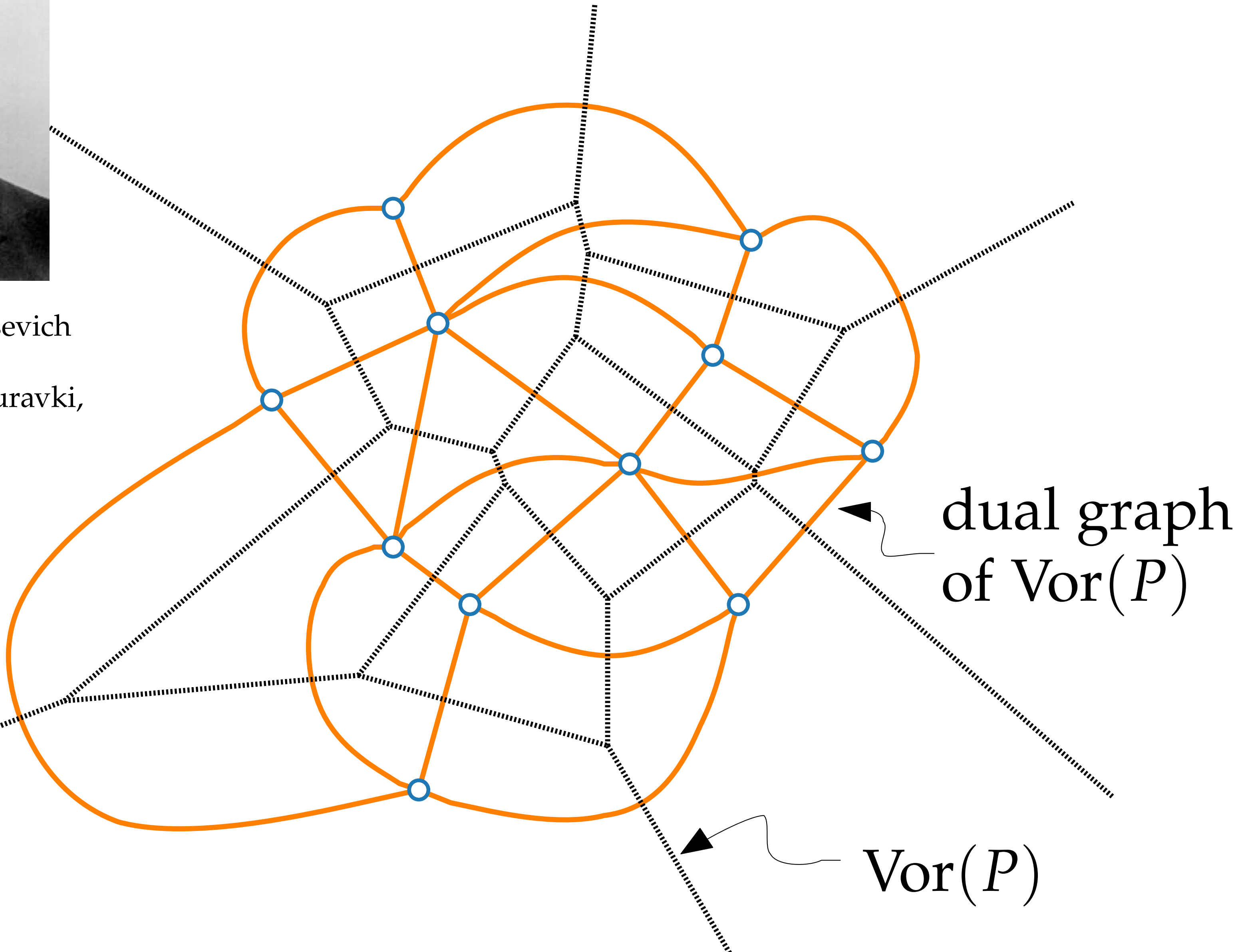

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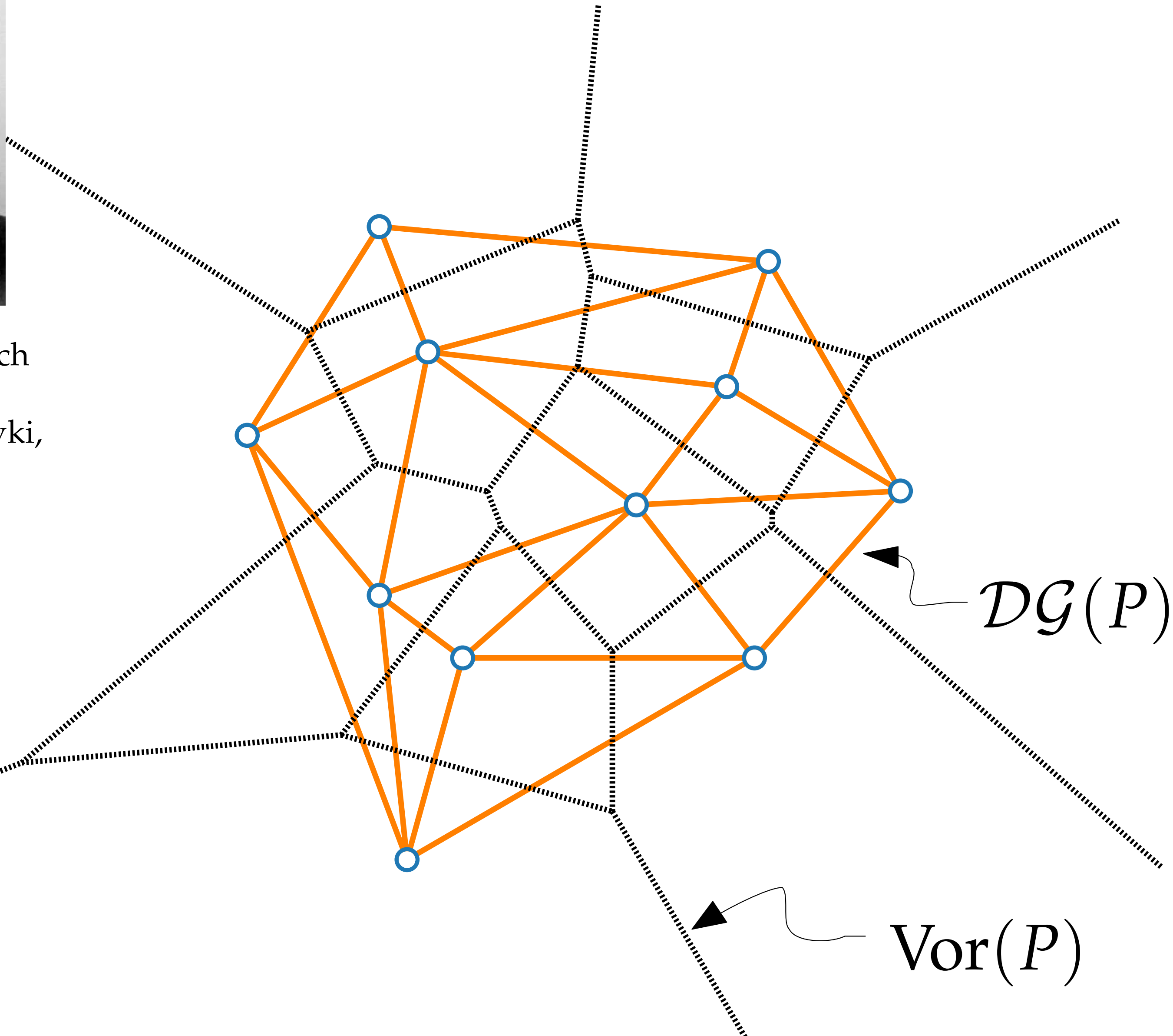

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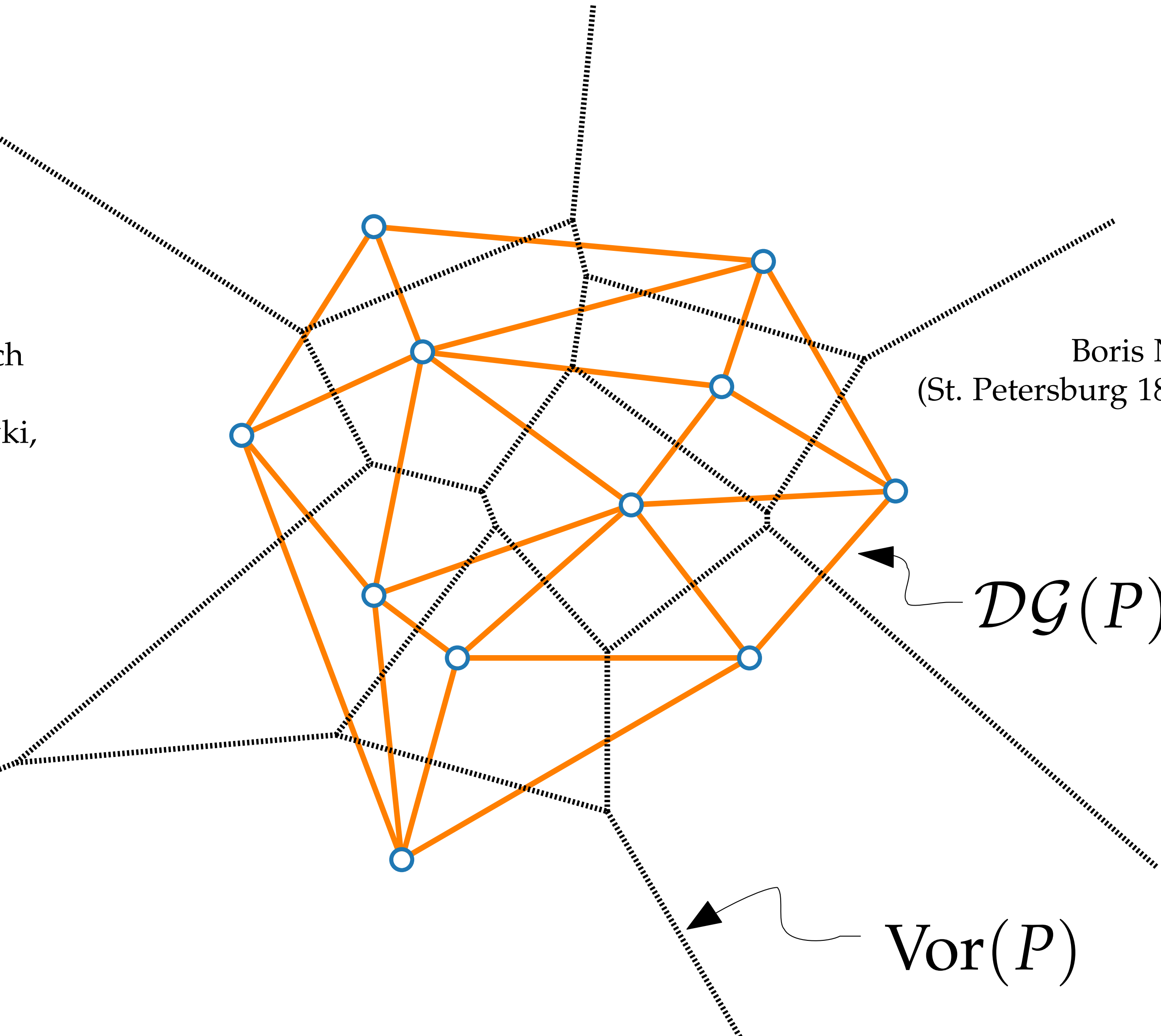

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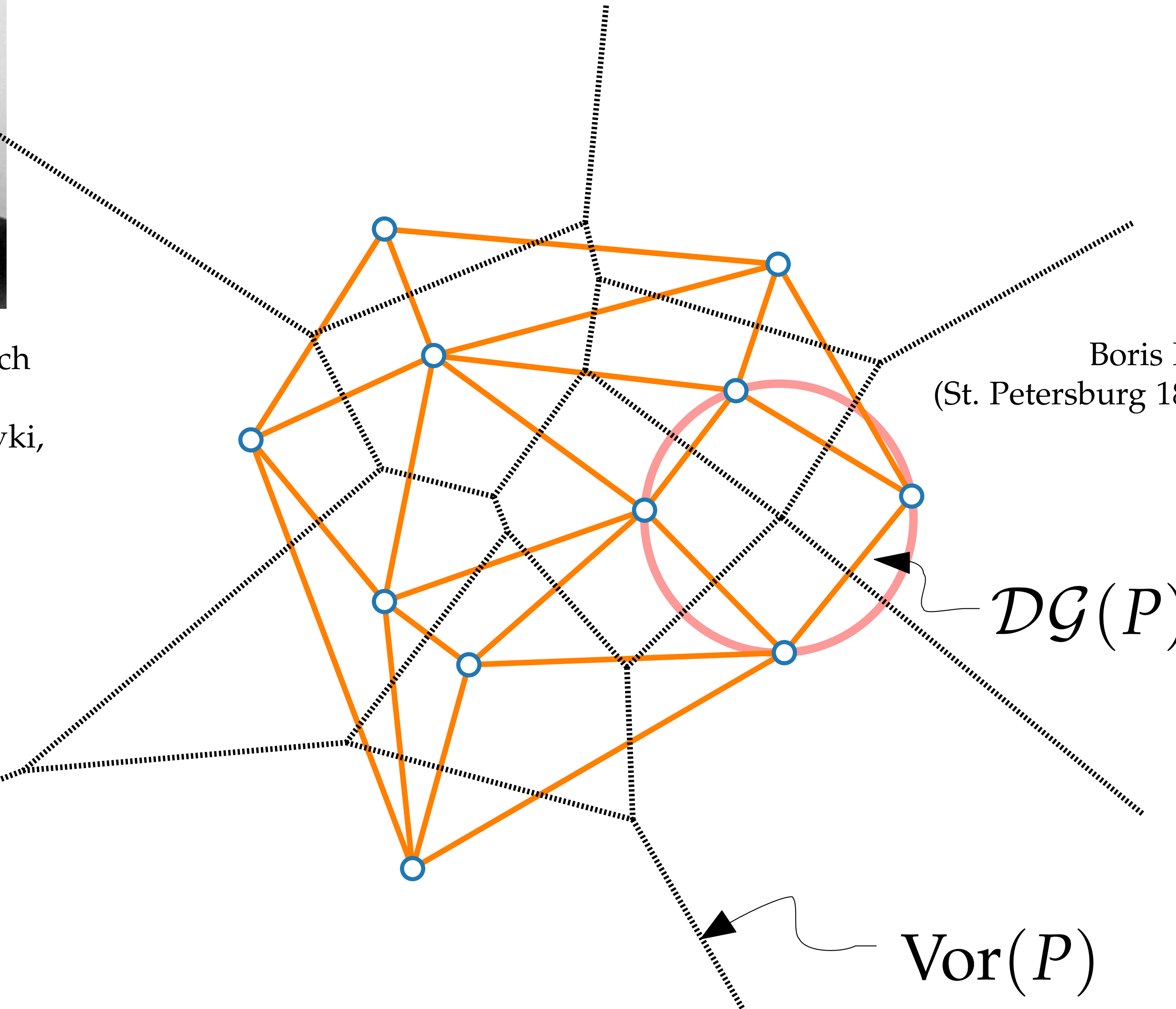

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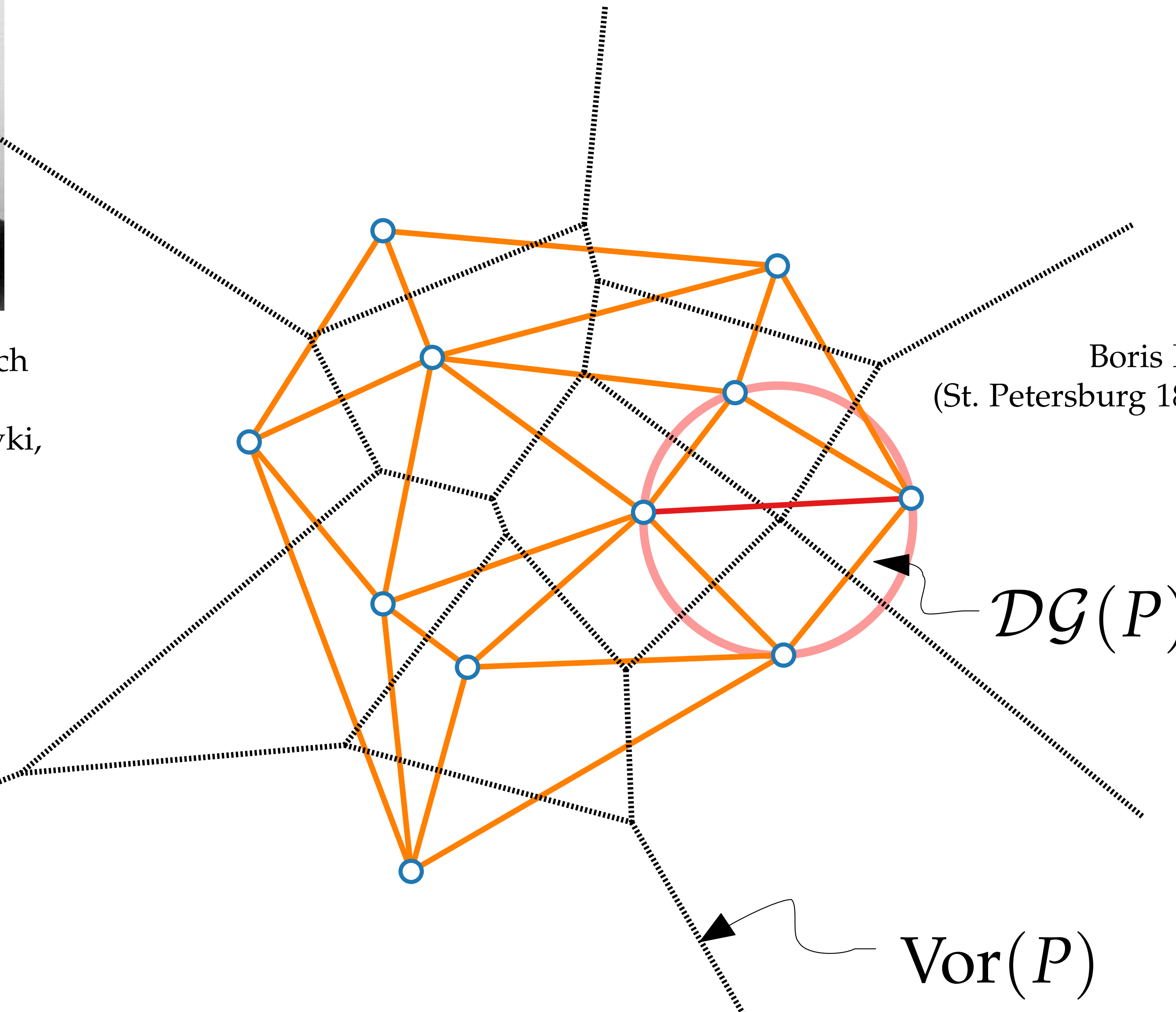
Georgy Feodosevich Voronoy (1868–1908 Zhuravki, now Ukraine)



Boris Nikolaevich Delone (St. Petersburg 1890–1980 Moscow)

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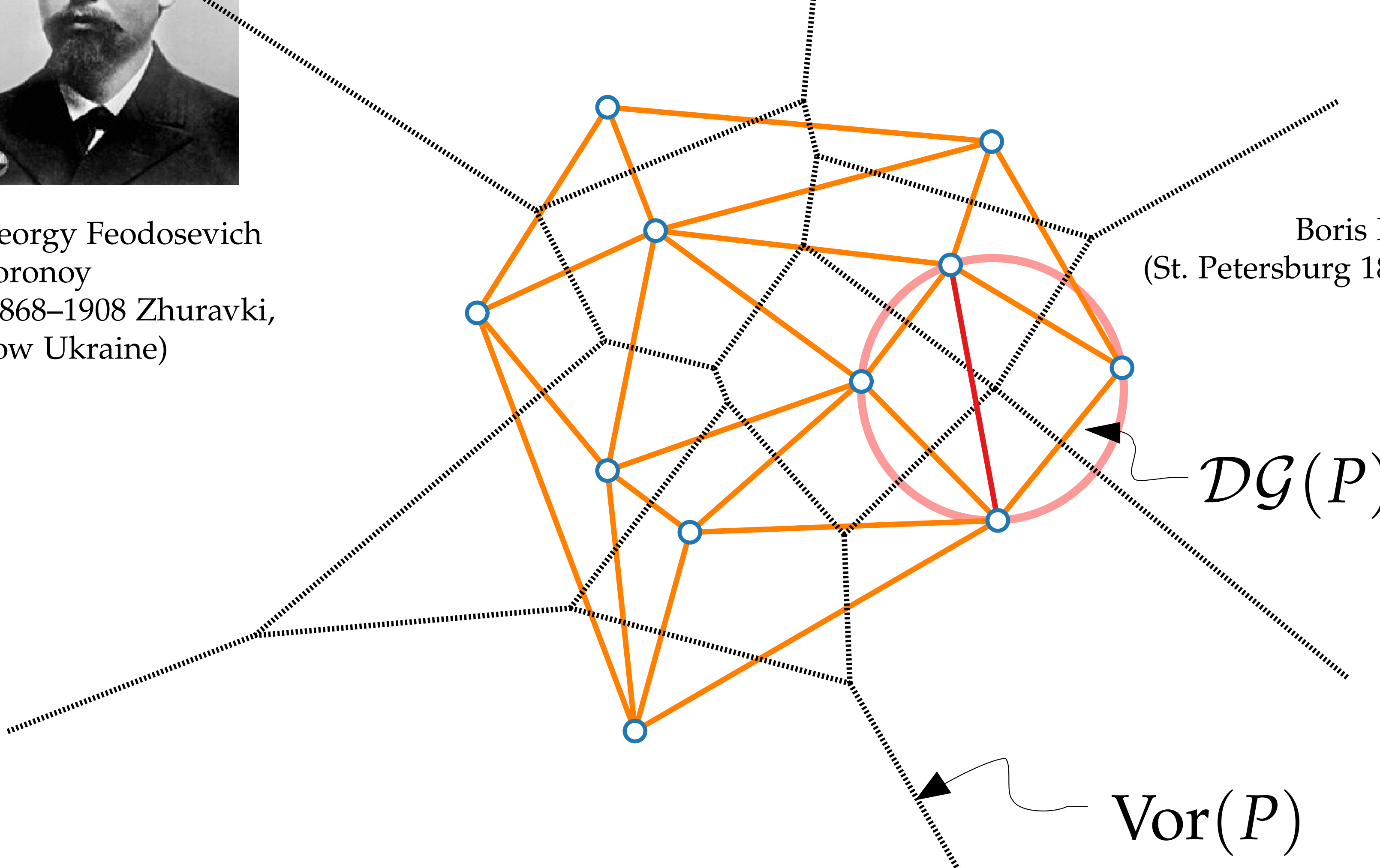

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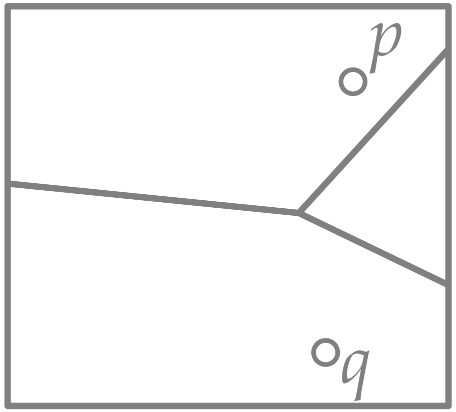

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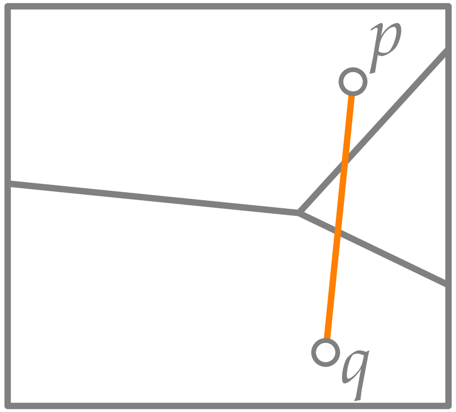

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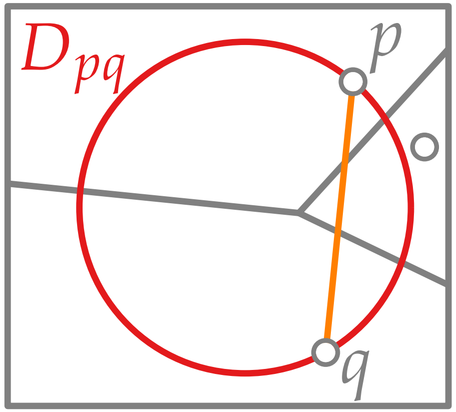

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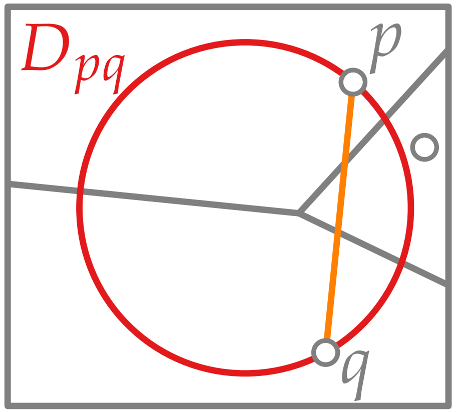

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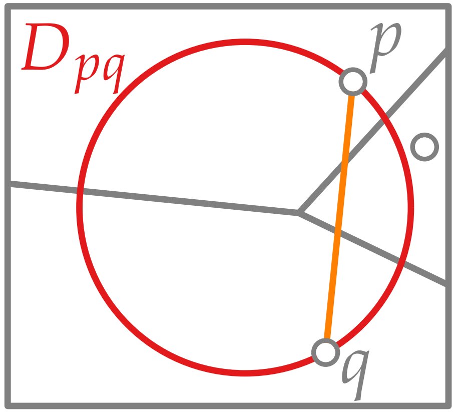

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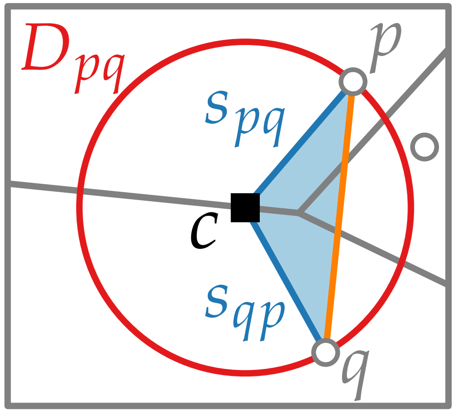

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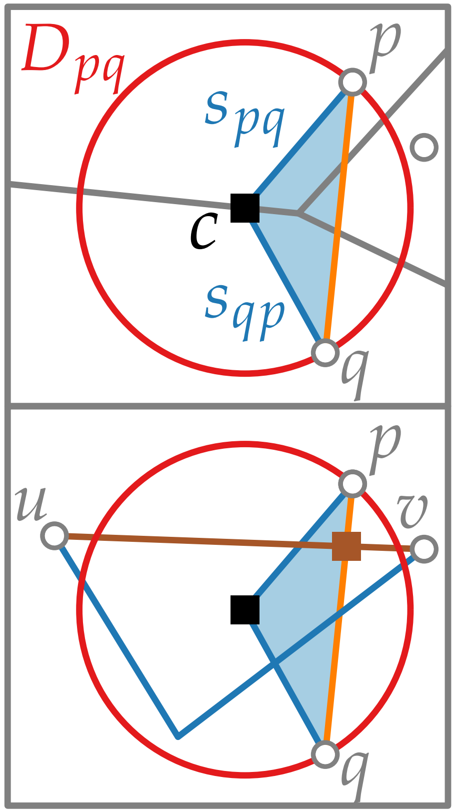

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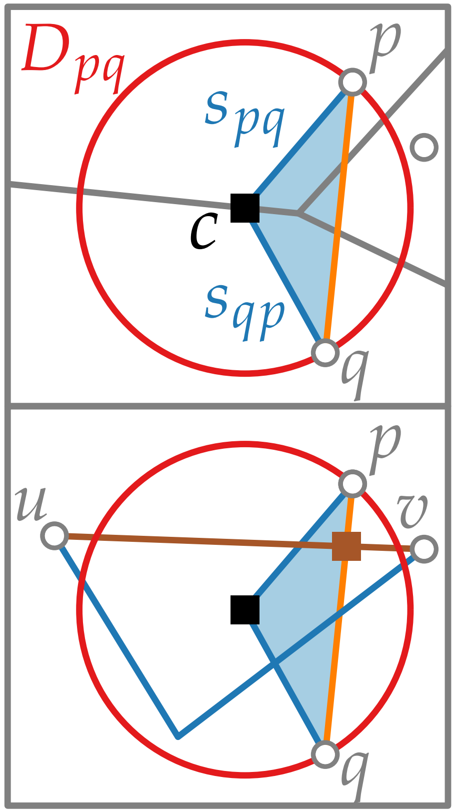

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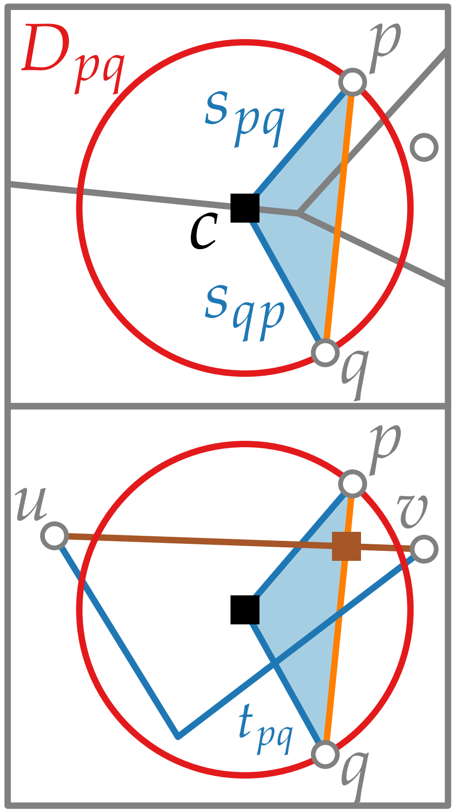

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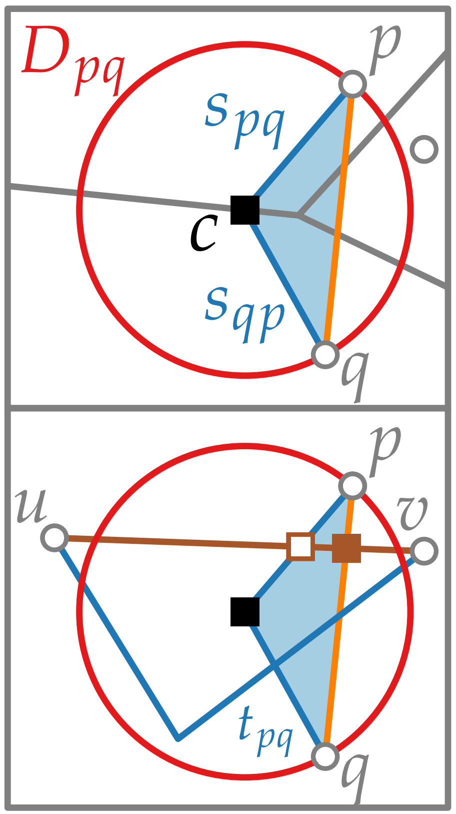

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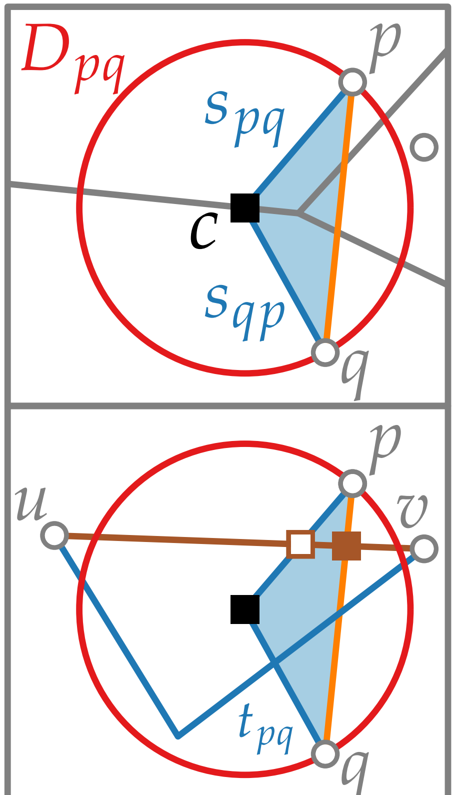

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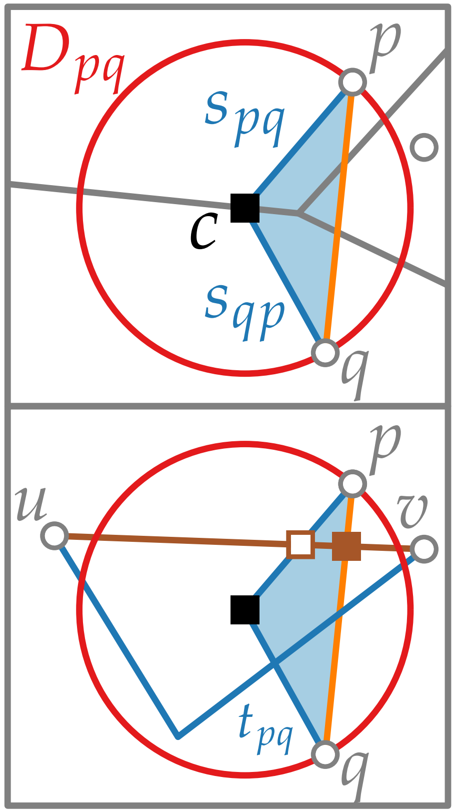

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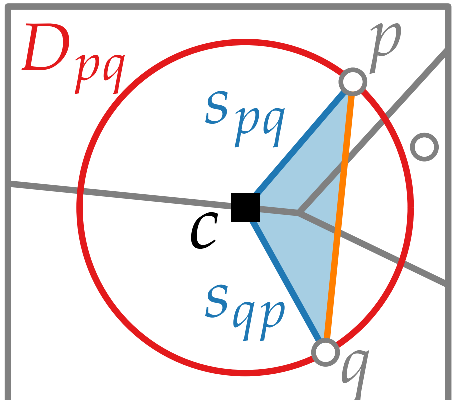

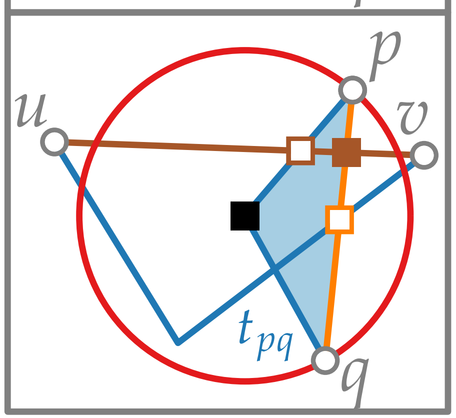

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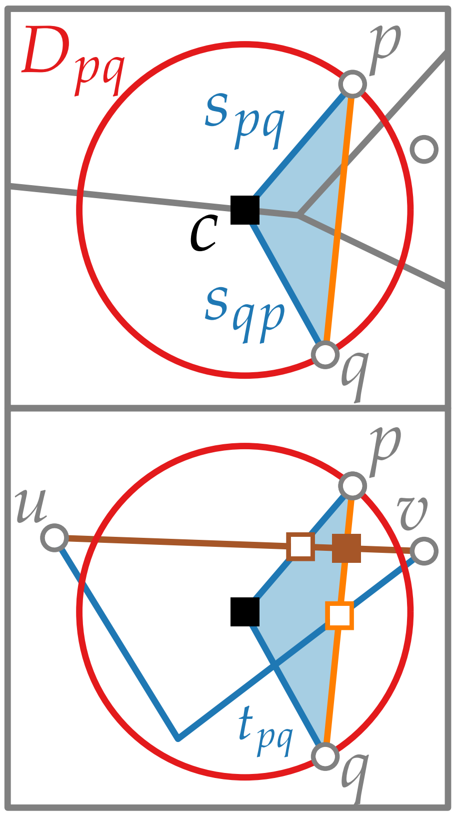

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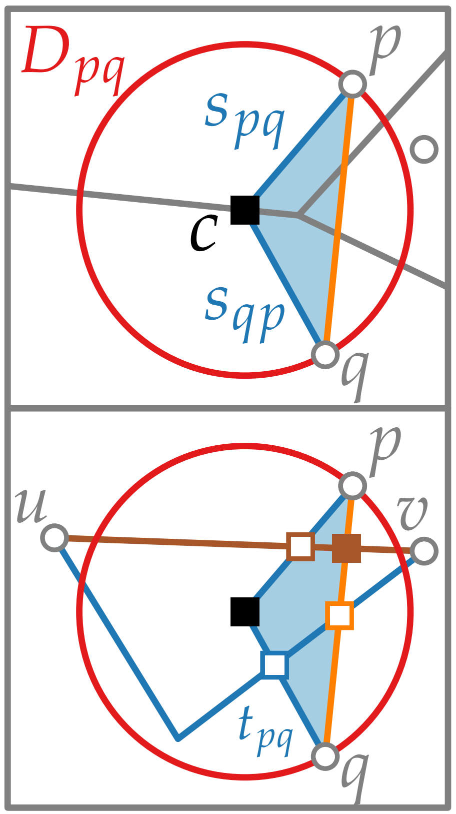

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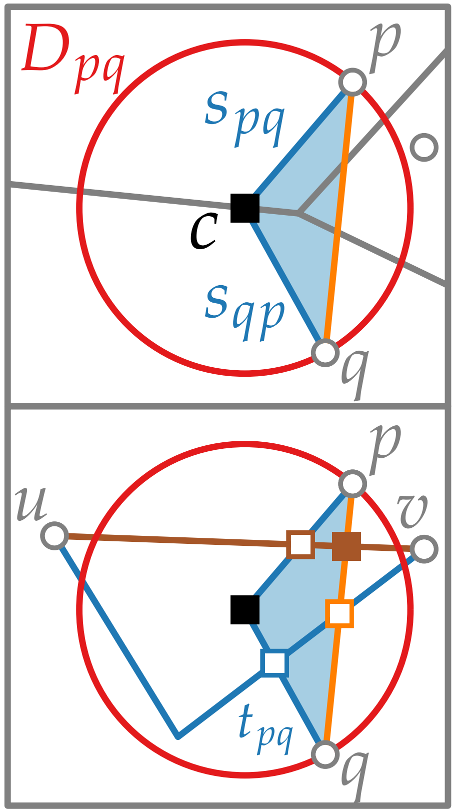

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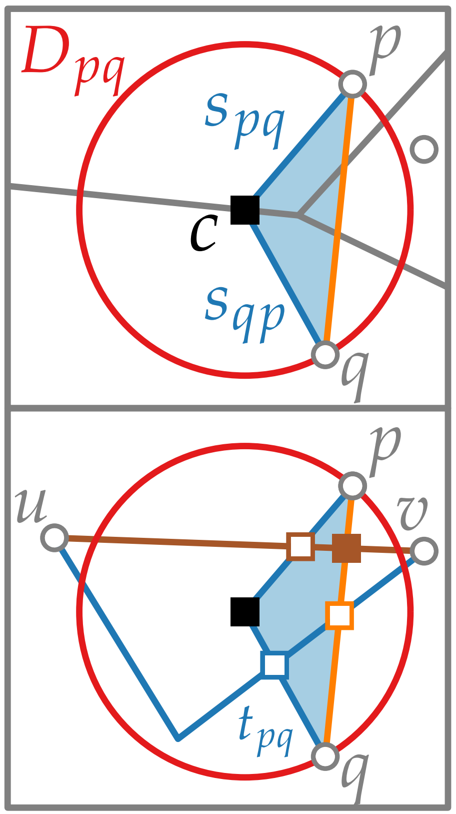

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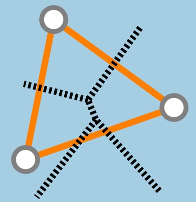

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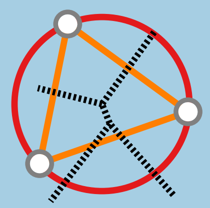

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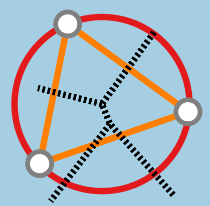

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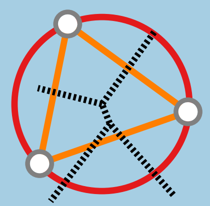

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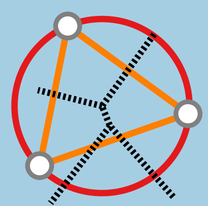

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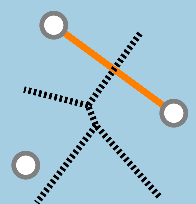

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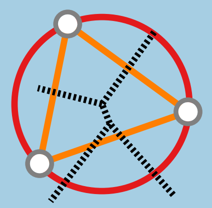

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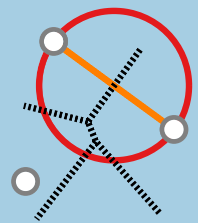

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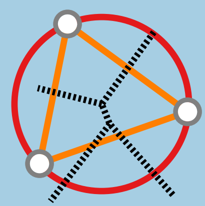

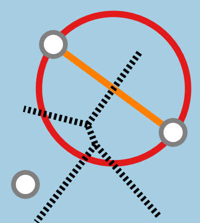

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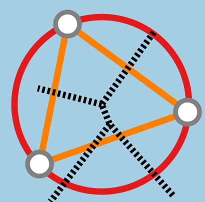

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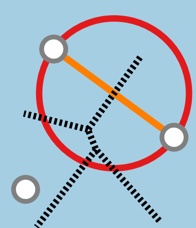

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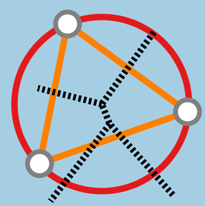

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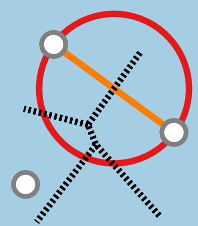

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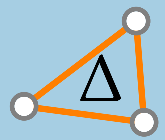

$\mathcal{T}$ Delaunay $\Leftrightarrow$ for each triangle Δ of $\mathcal{T}$:

Characterization

Characterization of Voronoi vertices and Voronoi edges $\Rightarrow$

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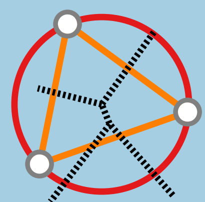

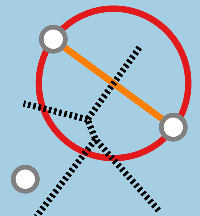

(i) Three pts $p, q, r \in P$ are vertices of the same face of $\mathcal{DG}(P) \Leftrightarrow \mathrm{int}(C(p,q,r)) \cap P = \emptyset$

(ii) Two pts $p, q \in P$ form an edge of $\mathcal{DG}(P) \Leftrightarrow$ there is a disk D with
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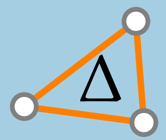

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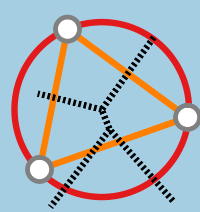

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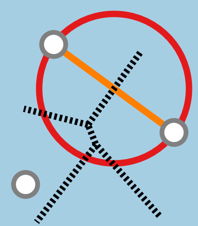

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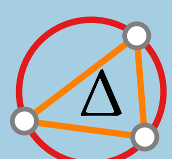

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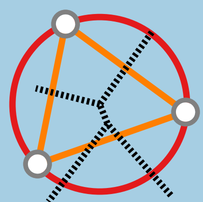

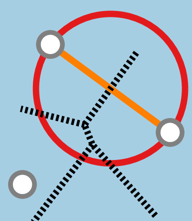

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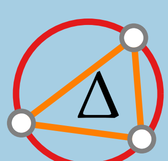

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("empty-circumcircle property")

Computational Geometry

Lecture 8:
Delaunay Triangulations
or
Height Interpolation

Part V:
Correctness & Computation

Philipp Kindermann

Summer Semester 2020

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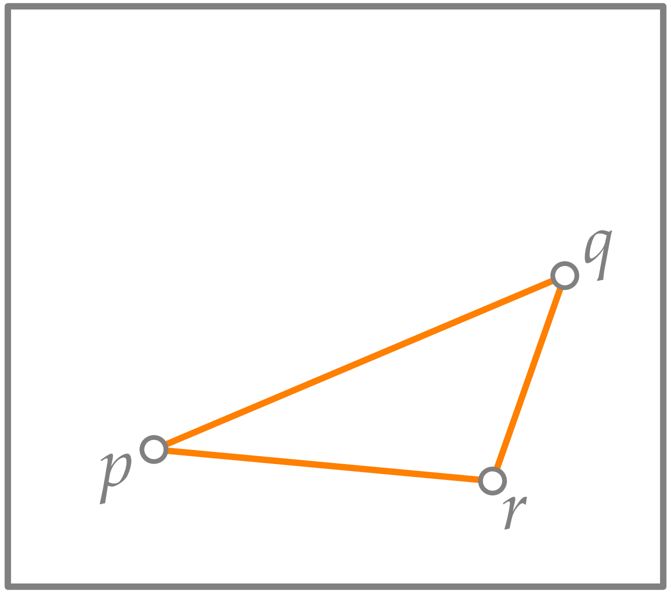

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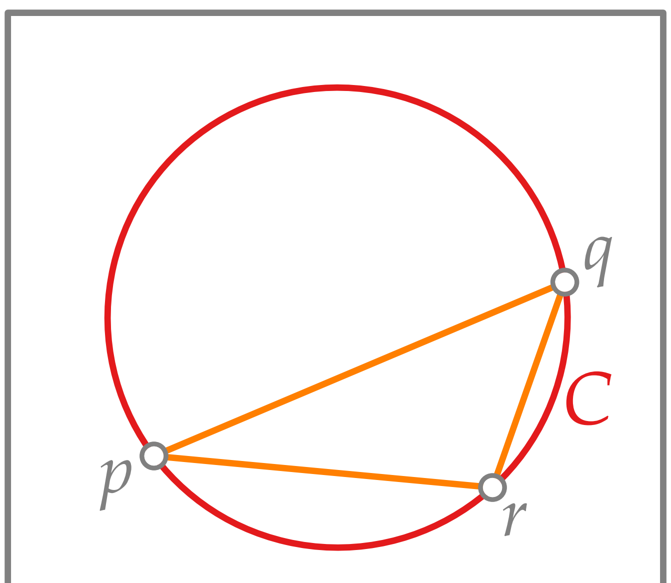

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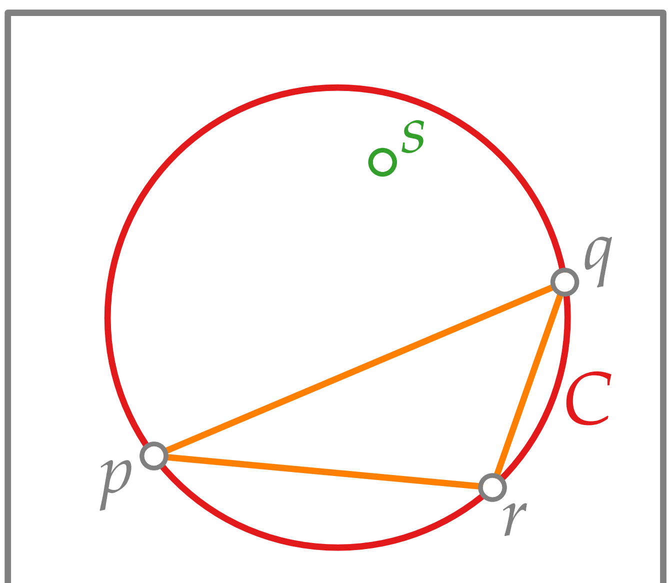

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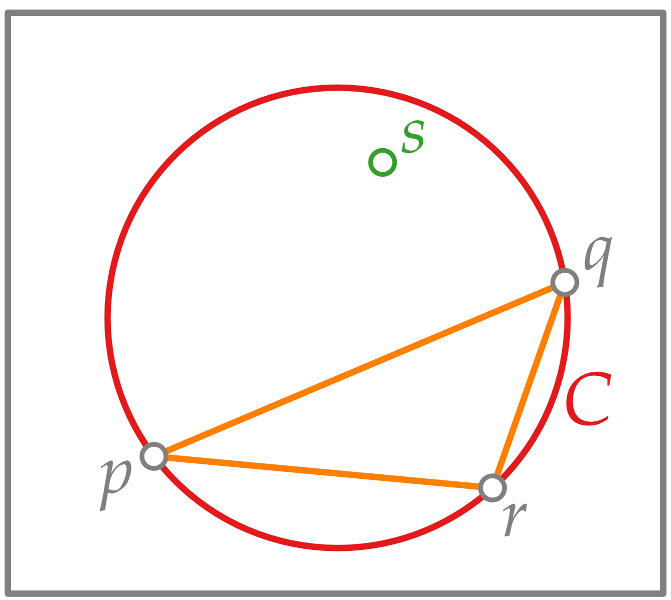

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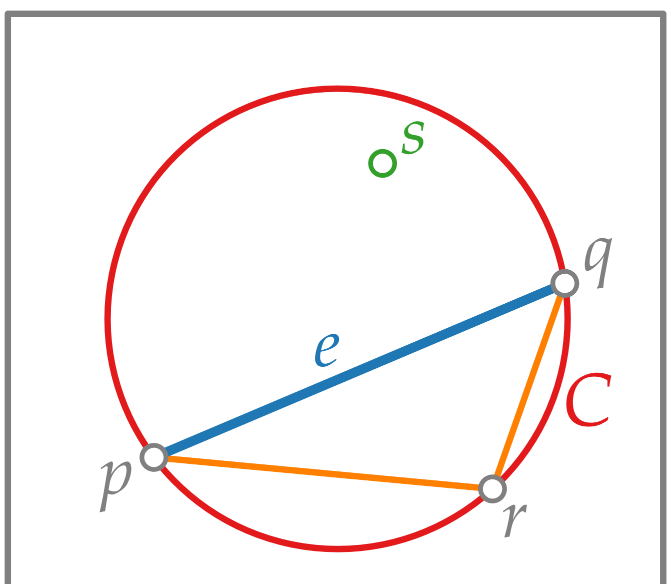

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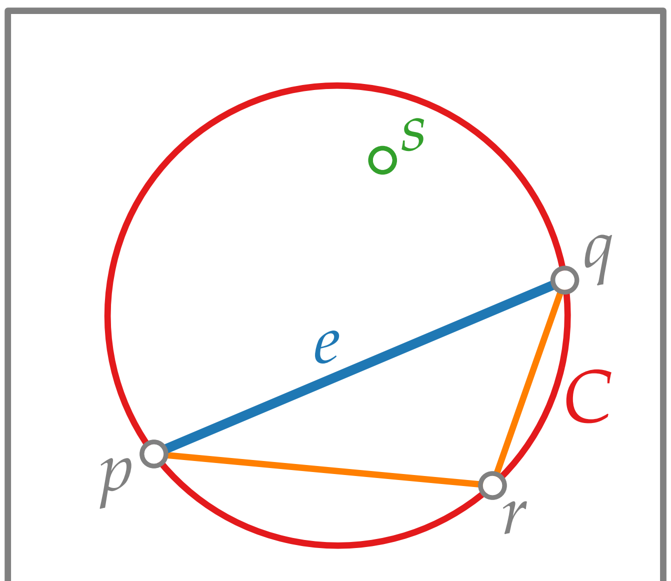

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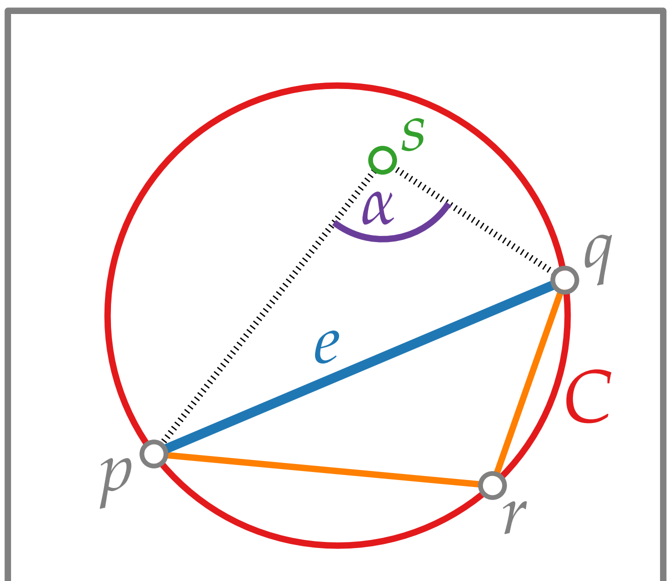

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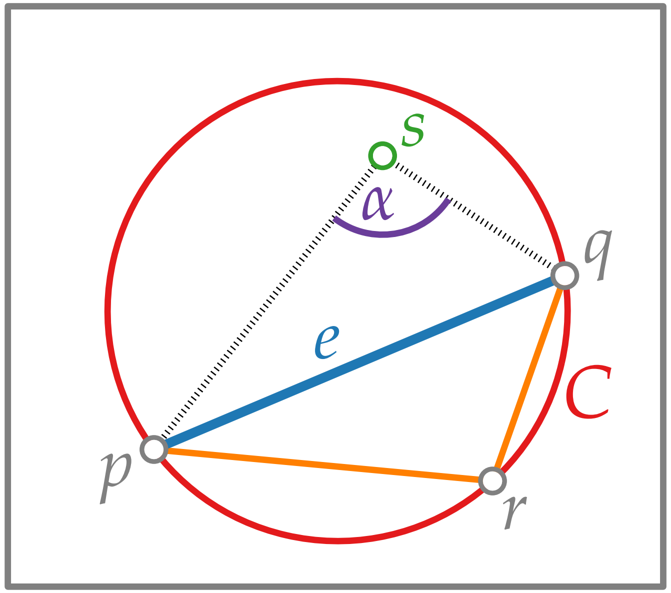

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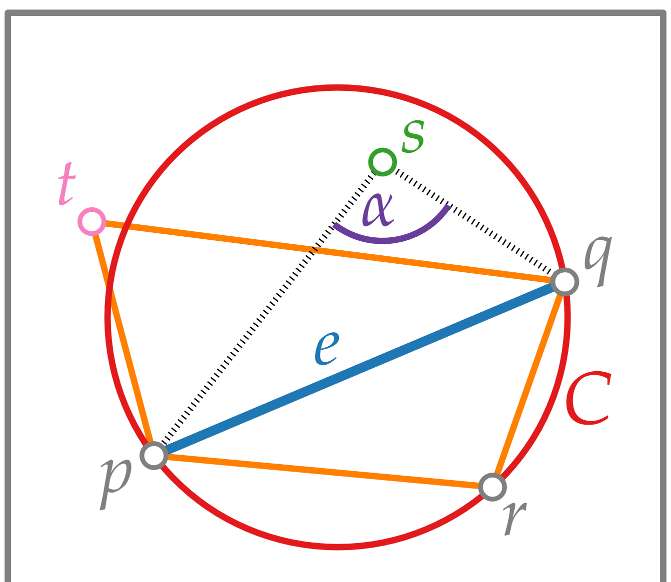

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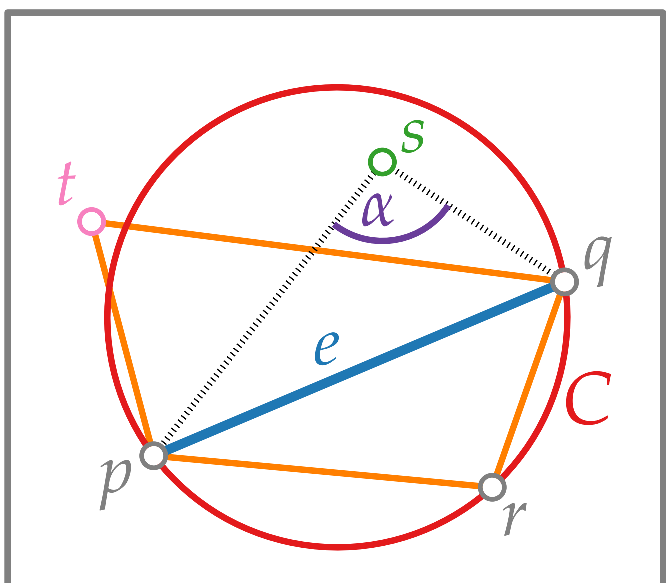

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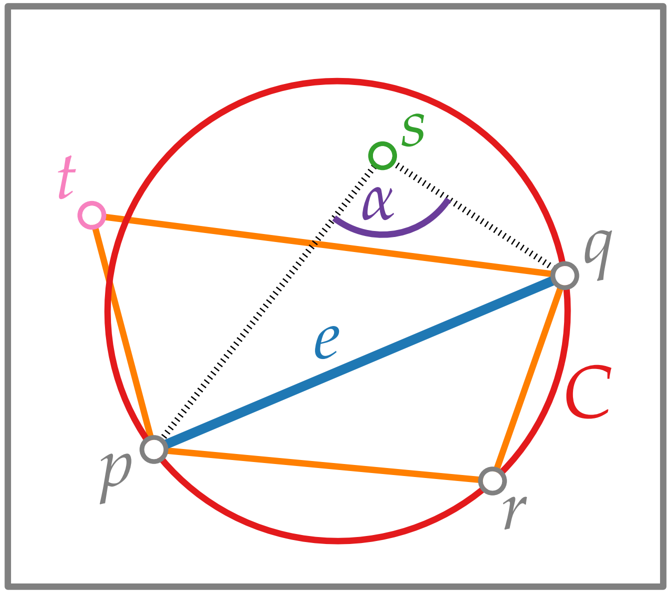

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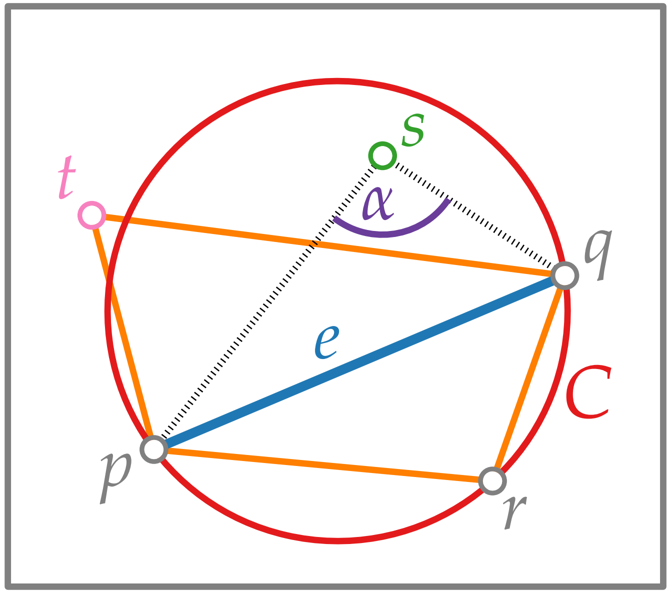

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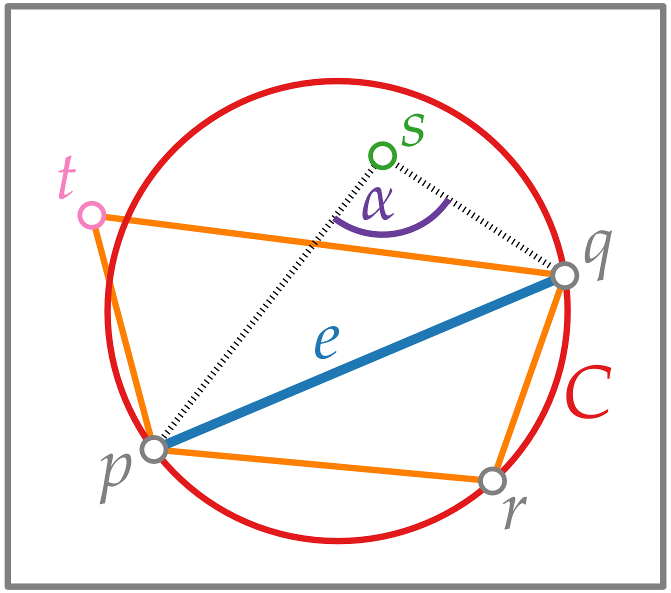

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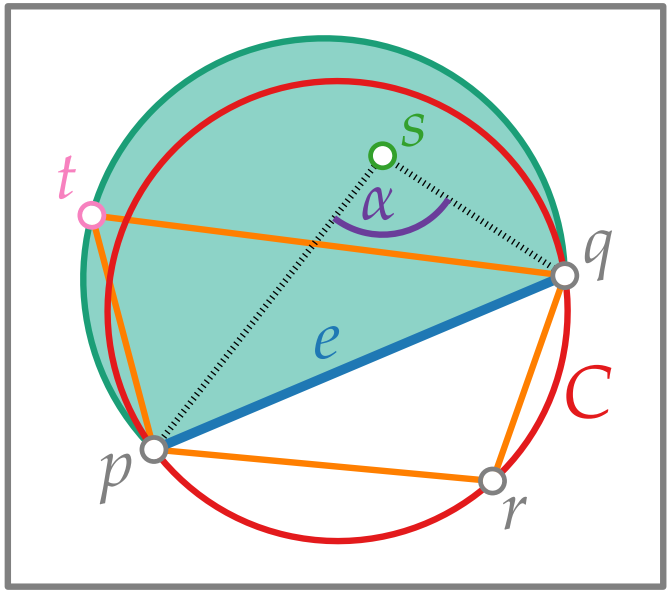

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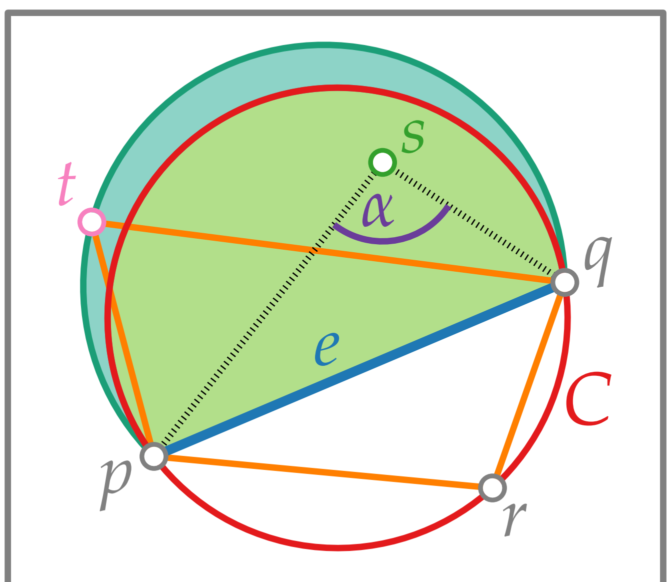

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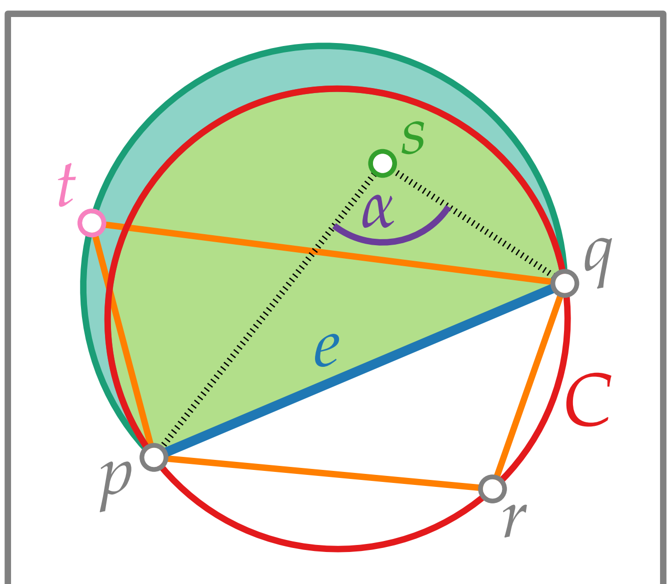

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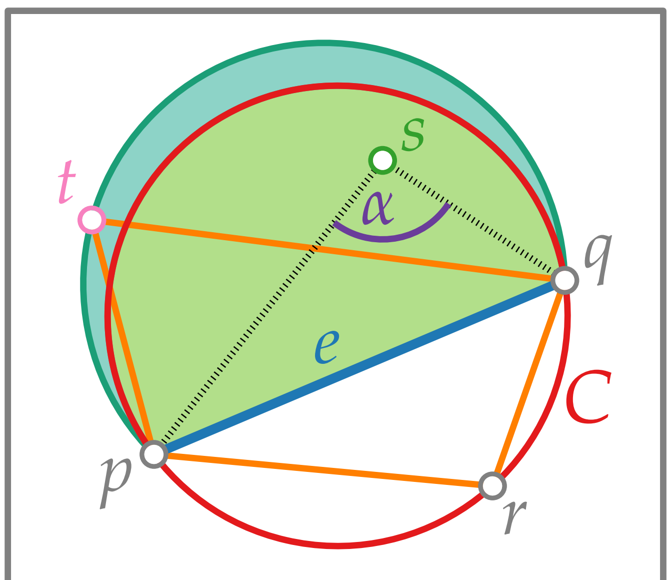

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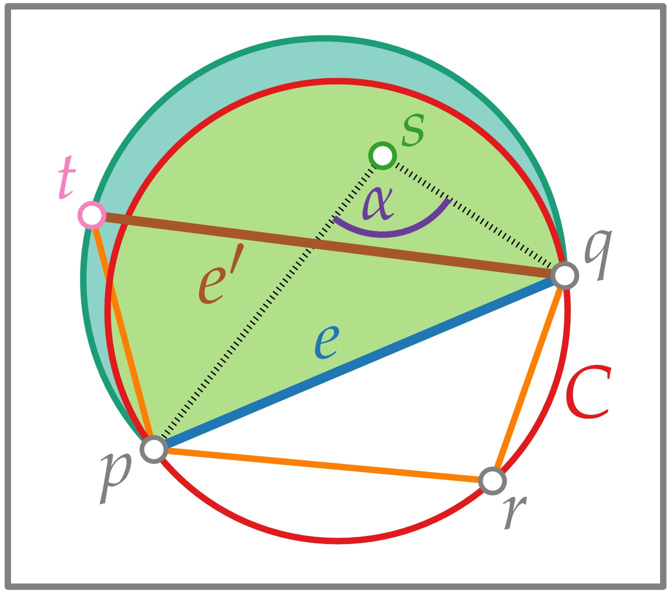

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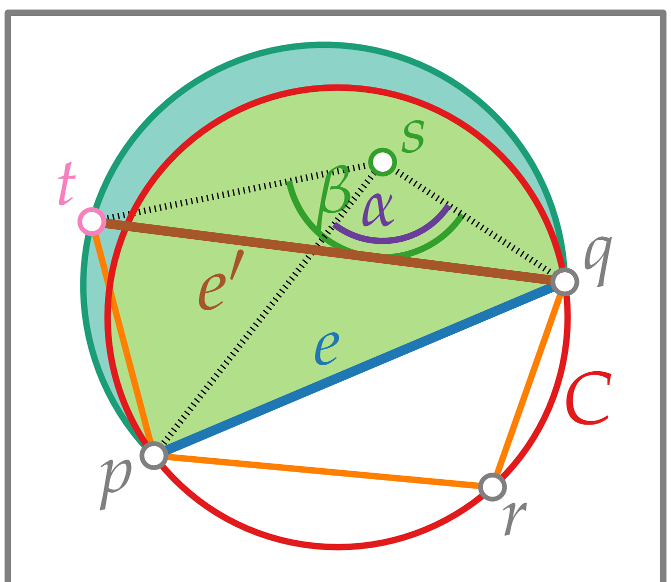

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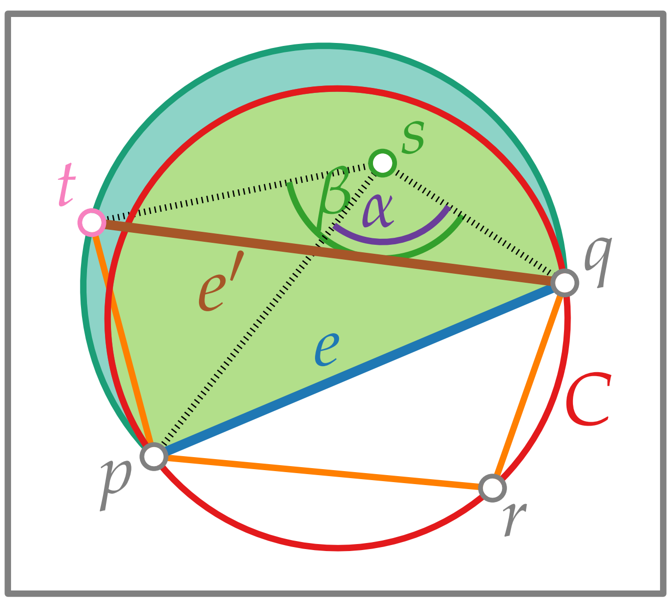

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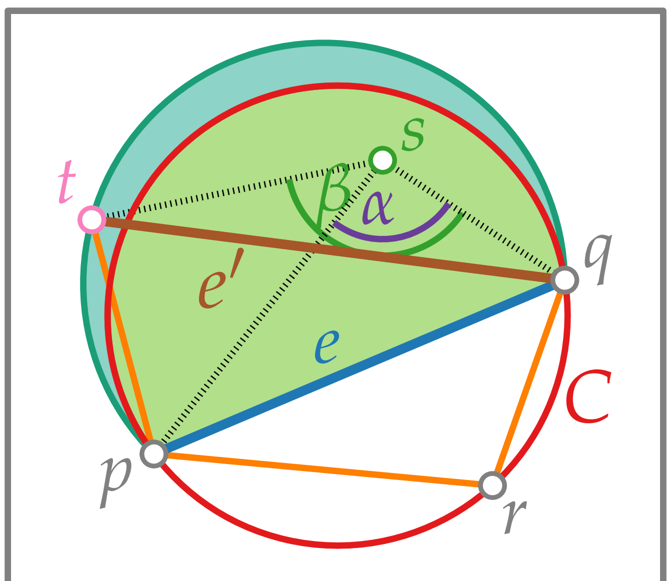

↯ Contradiction to choice of the pair $(\Delta pqr, s)$. $\square$

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no 4 pts on an empty circle!

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$\Rightarrow$ legal triangulation unique

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$\Downarrow$ angle-optimal $\Rightarrow$ legal

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$\Rightarrow$ legal triangulation unique
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Theorem. $P \subset \mathbb{R}^2$ finite, $\mathcal{T}$ triangulation of P.
Then $\mathcal{T}$ legal $\Leftrightarrow$ $\mathcal{T}$ Delaunay.

no 4 pts on an empty circle!

Observation. Suppose P is in general position.

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All Delaunay triang. have same min. angle.

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[How?]