Computational Geometry

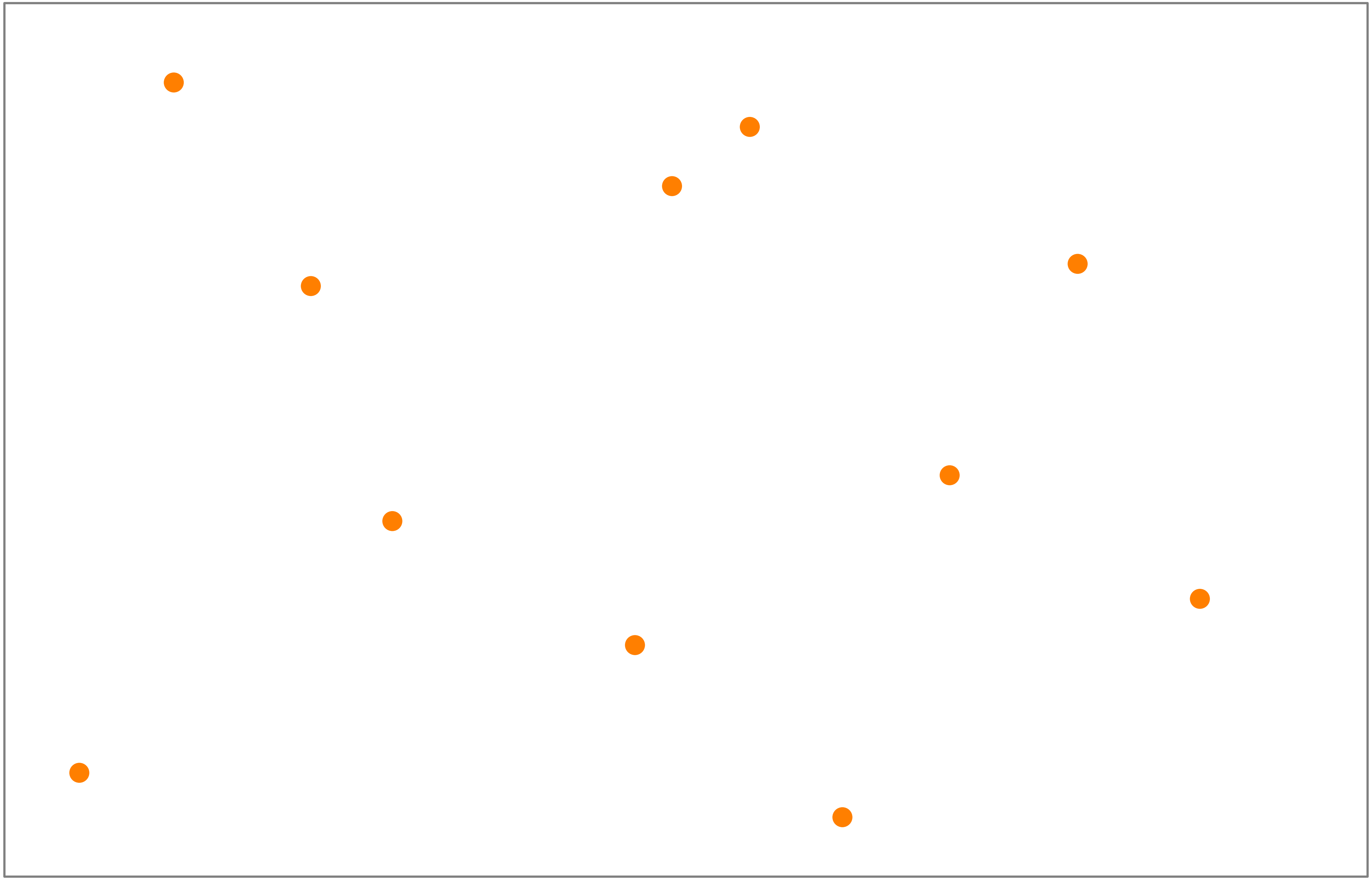
Lecture 7:
Voronoi Diagrams
or
The Post-Office Problem

Part I:
The Post-Office Problem

Philipp Kindermann

Winter Semester 2020

The Post-Office Problem



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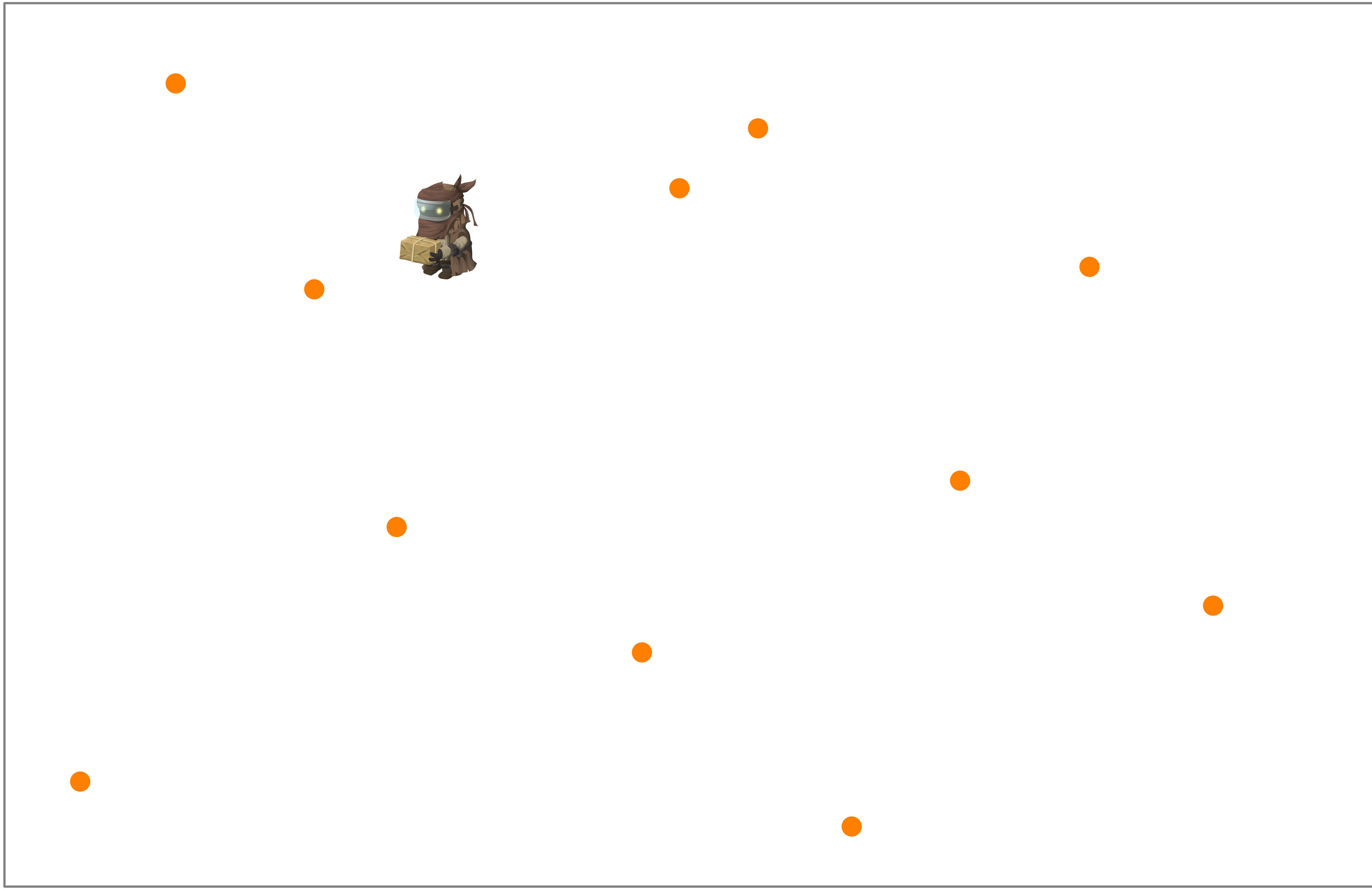

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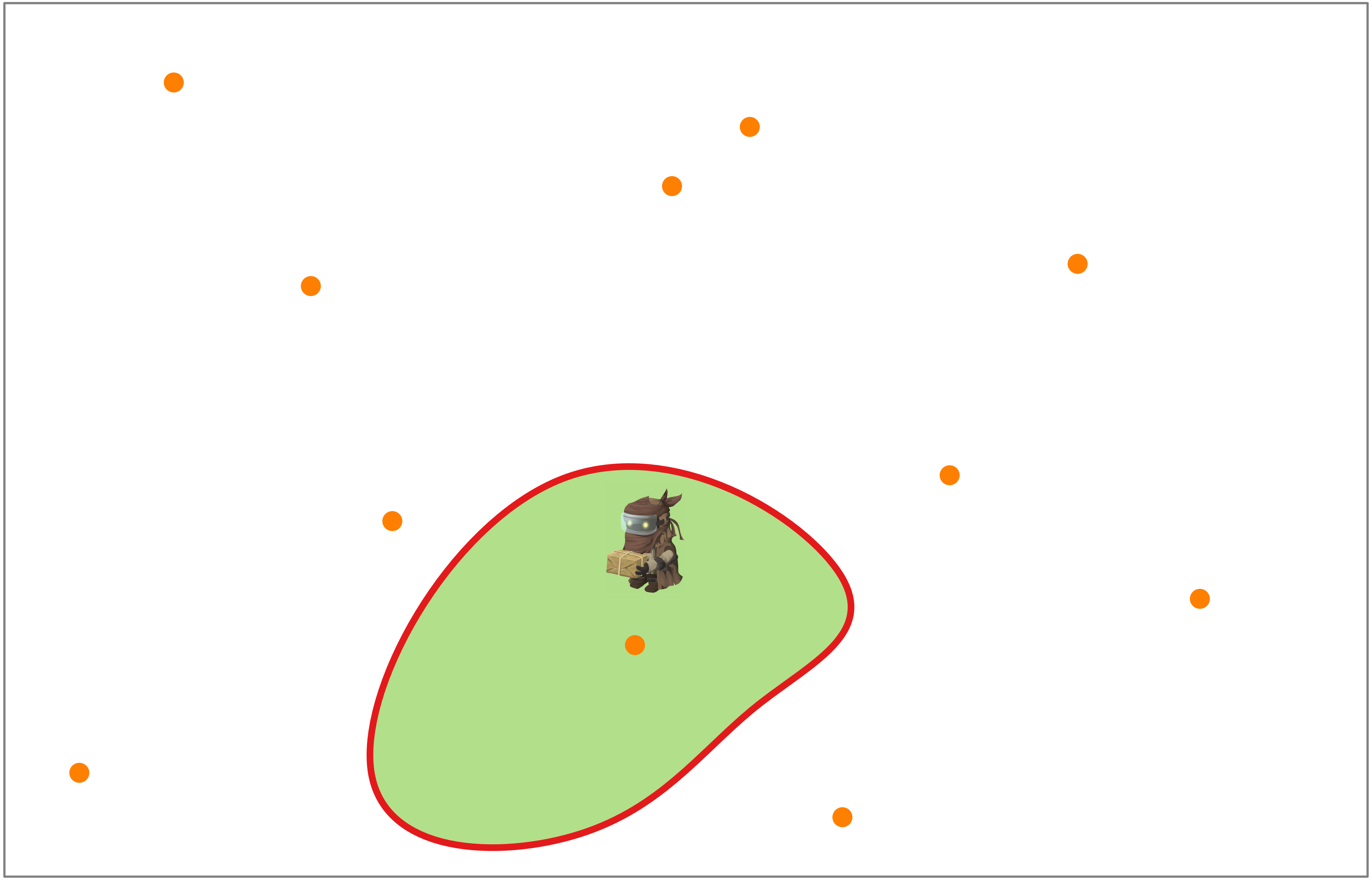

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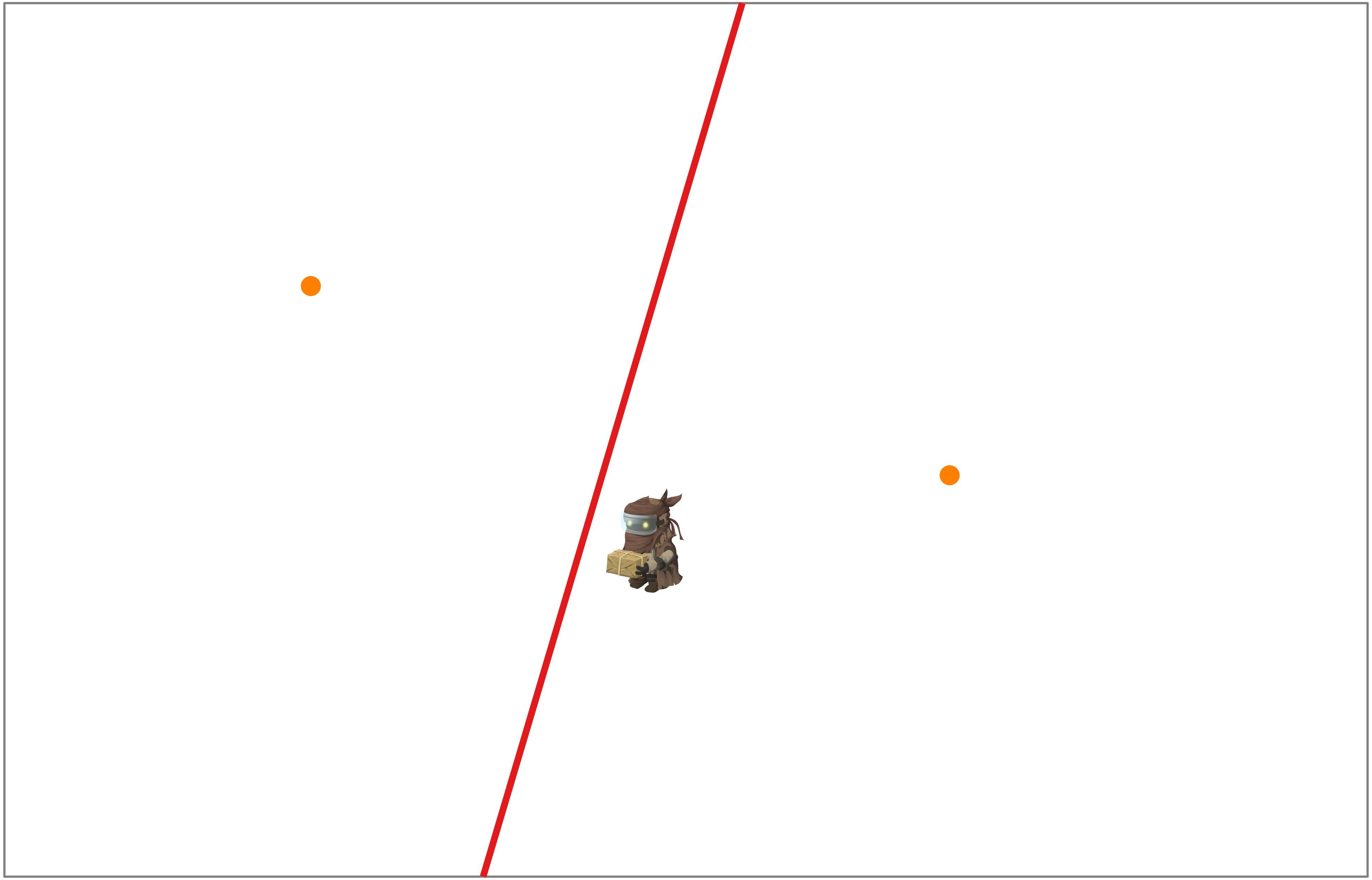

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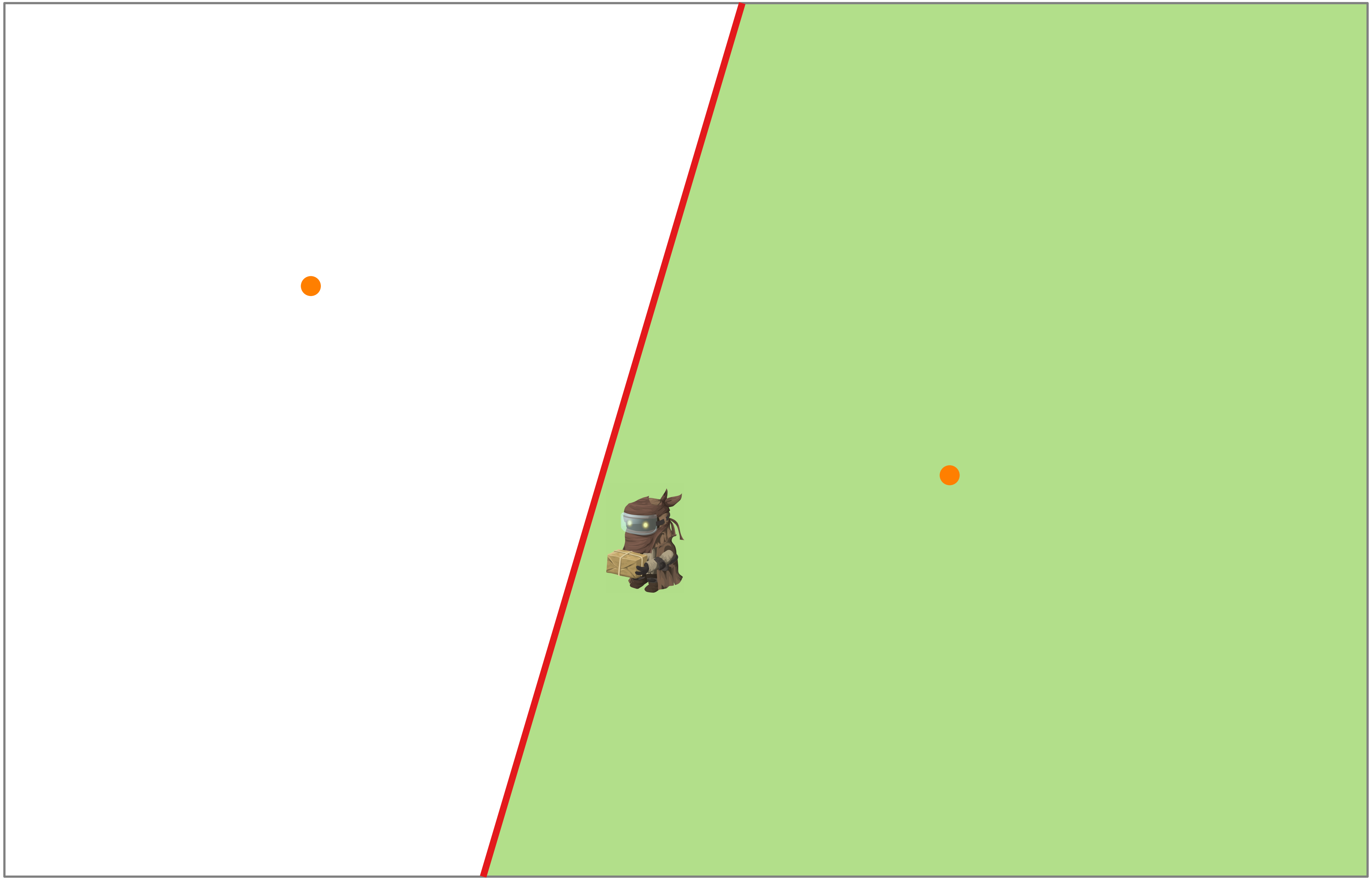

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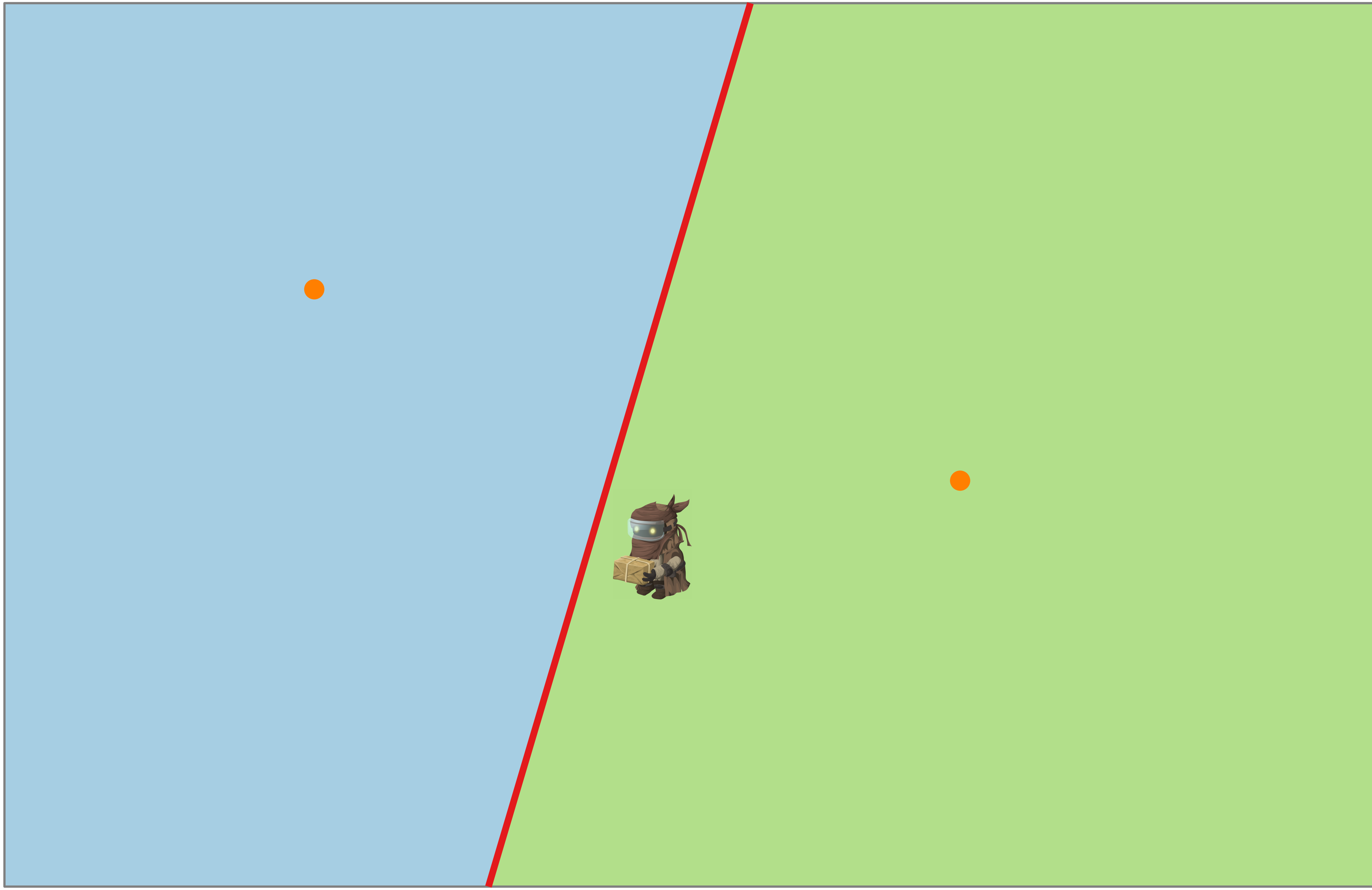

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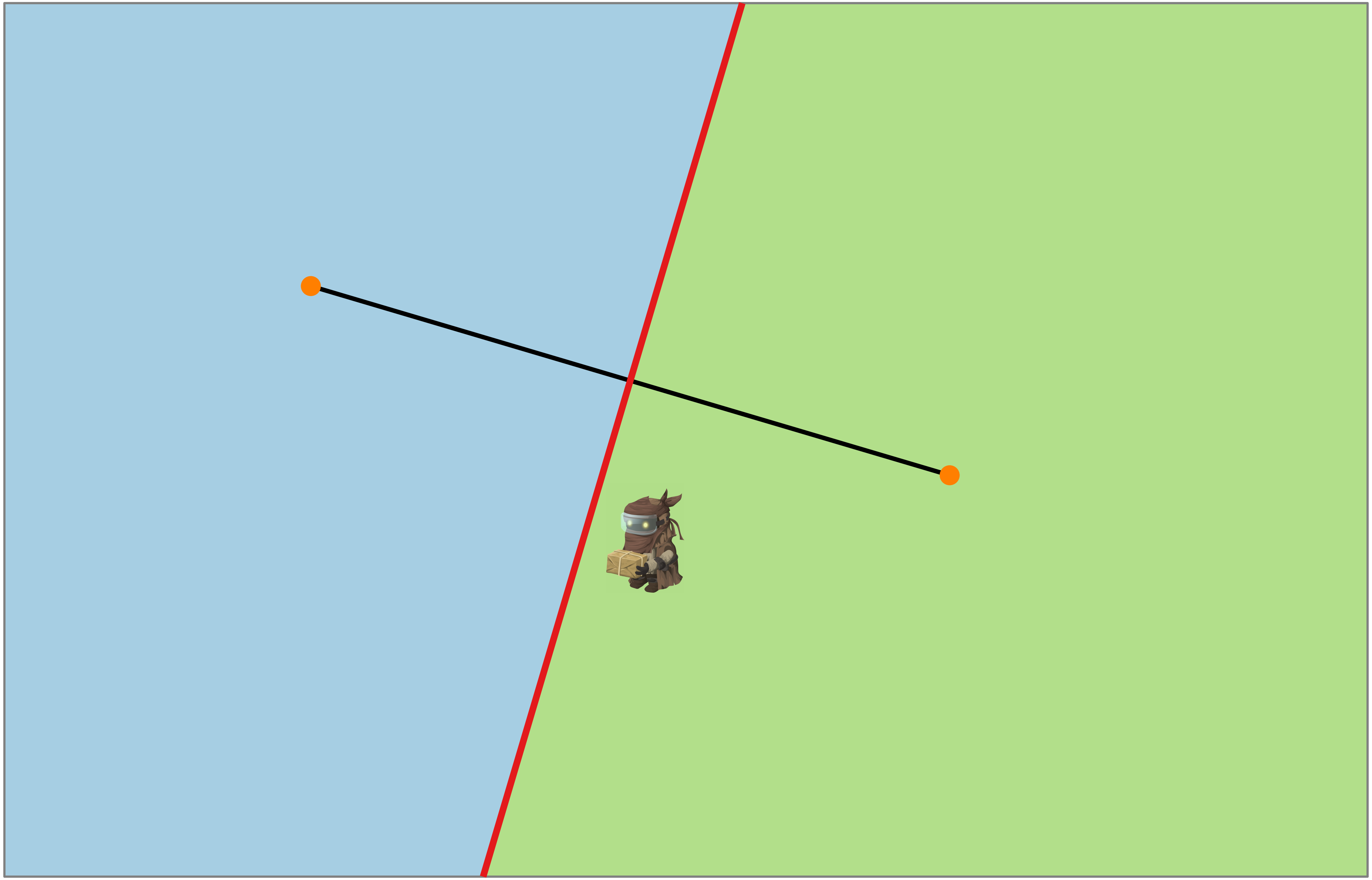

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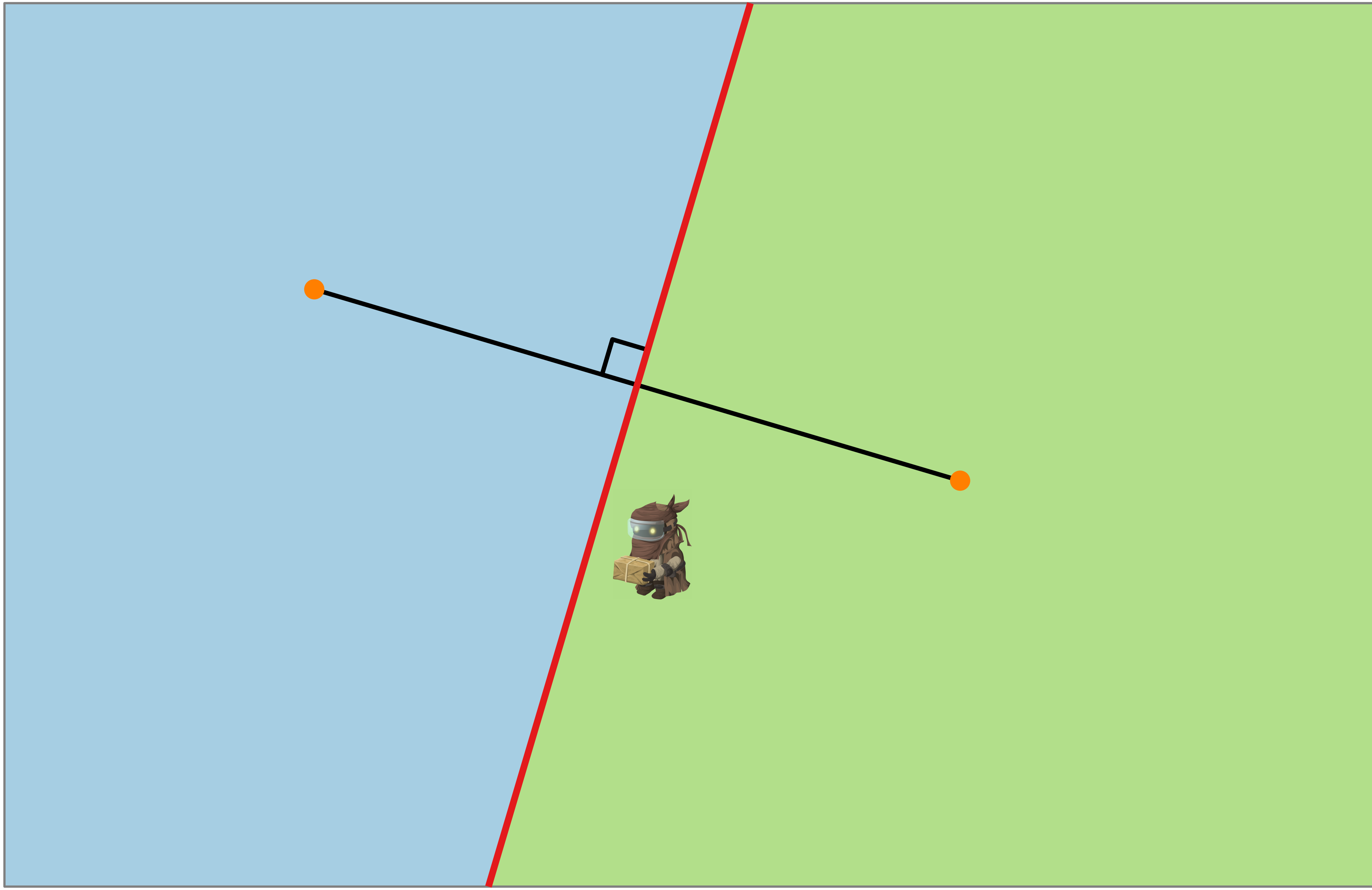

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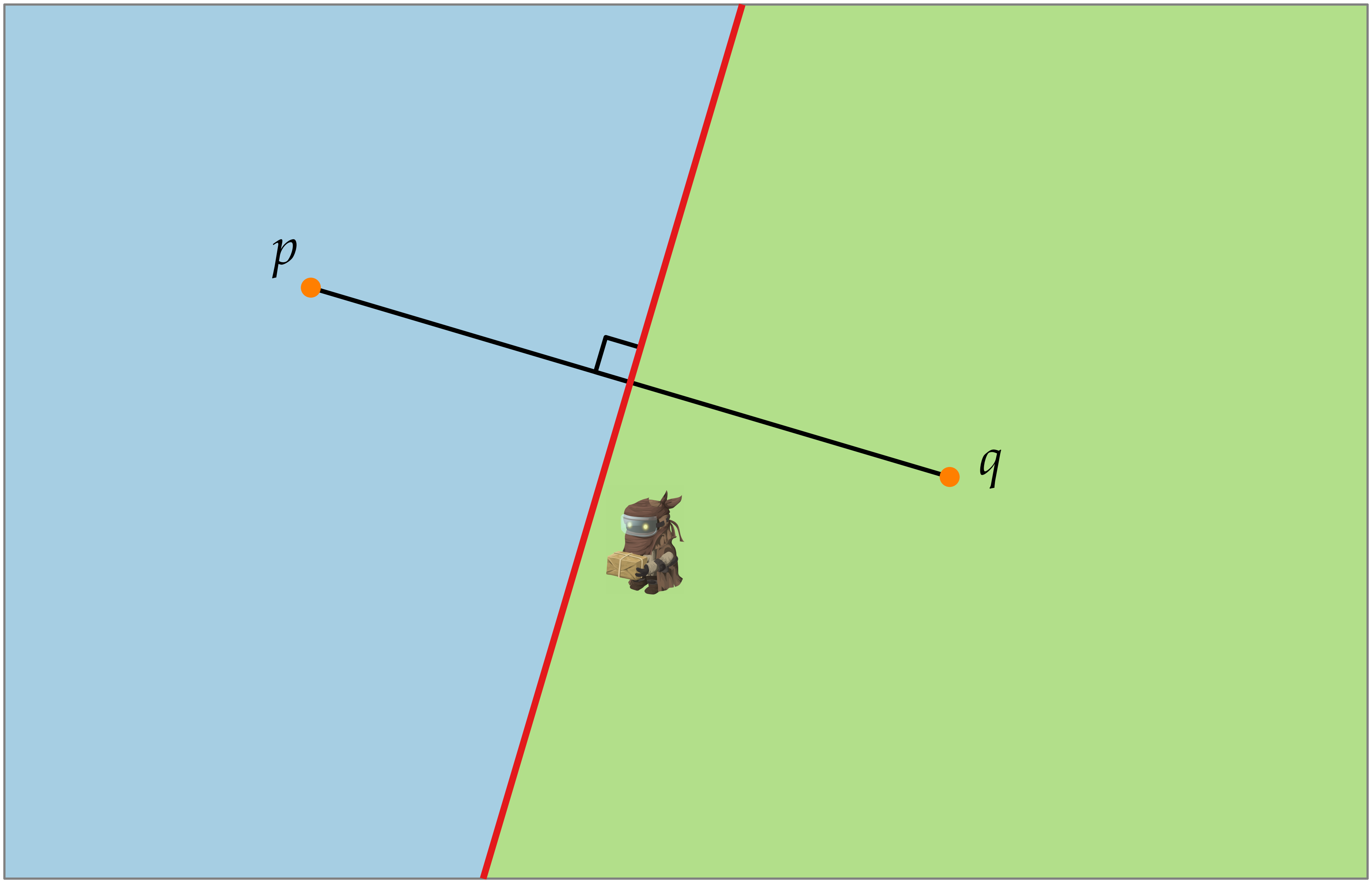

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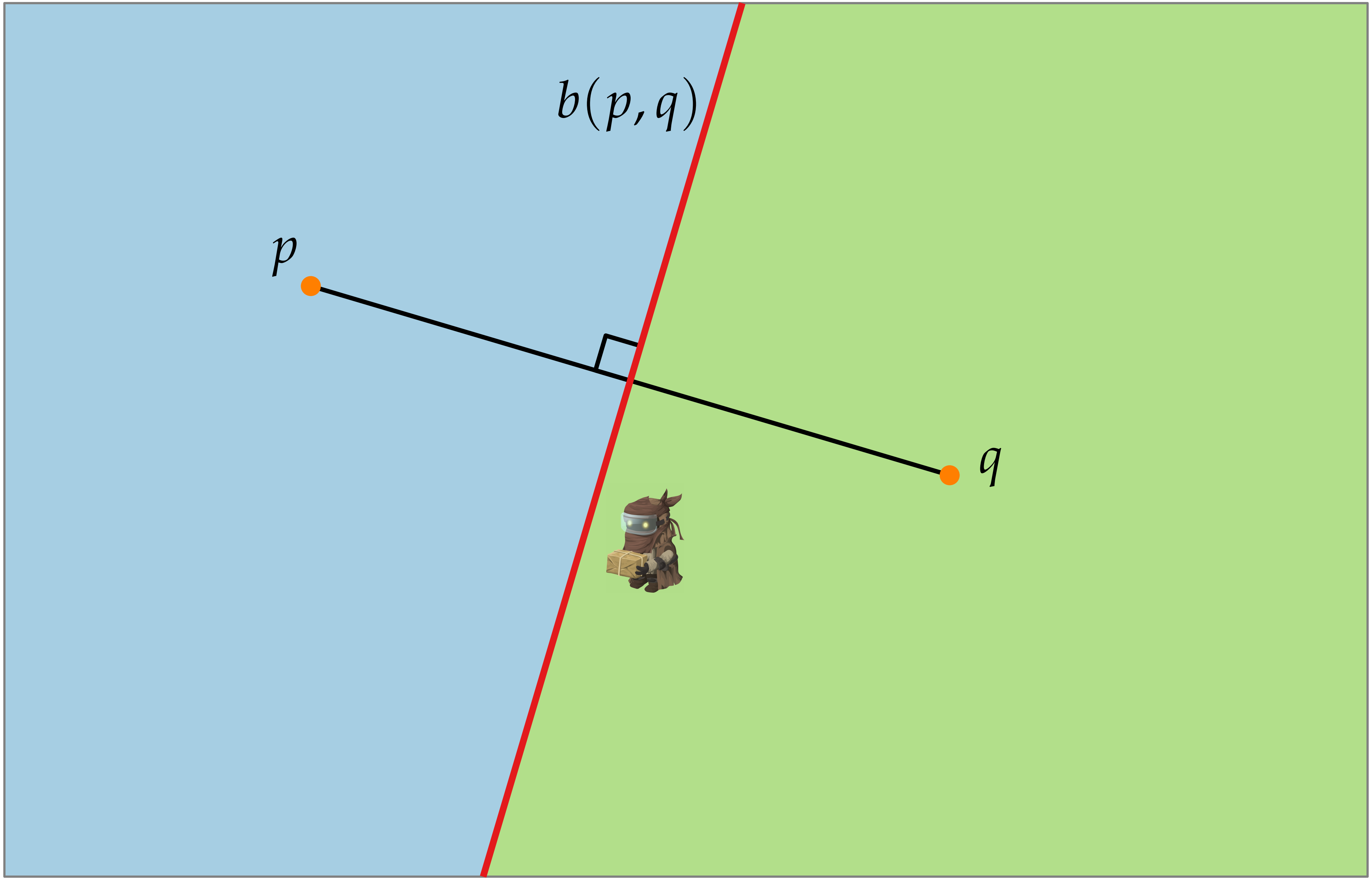

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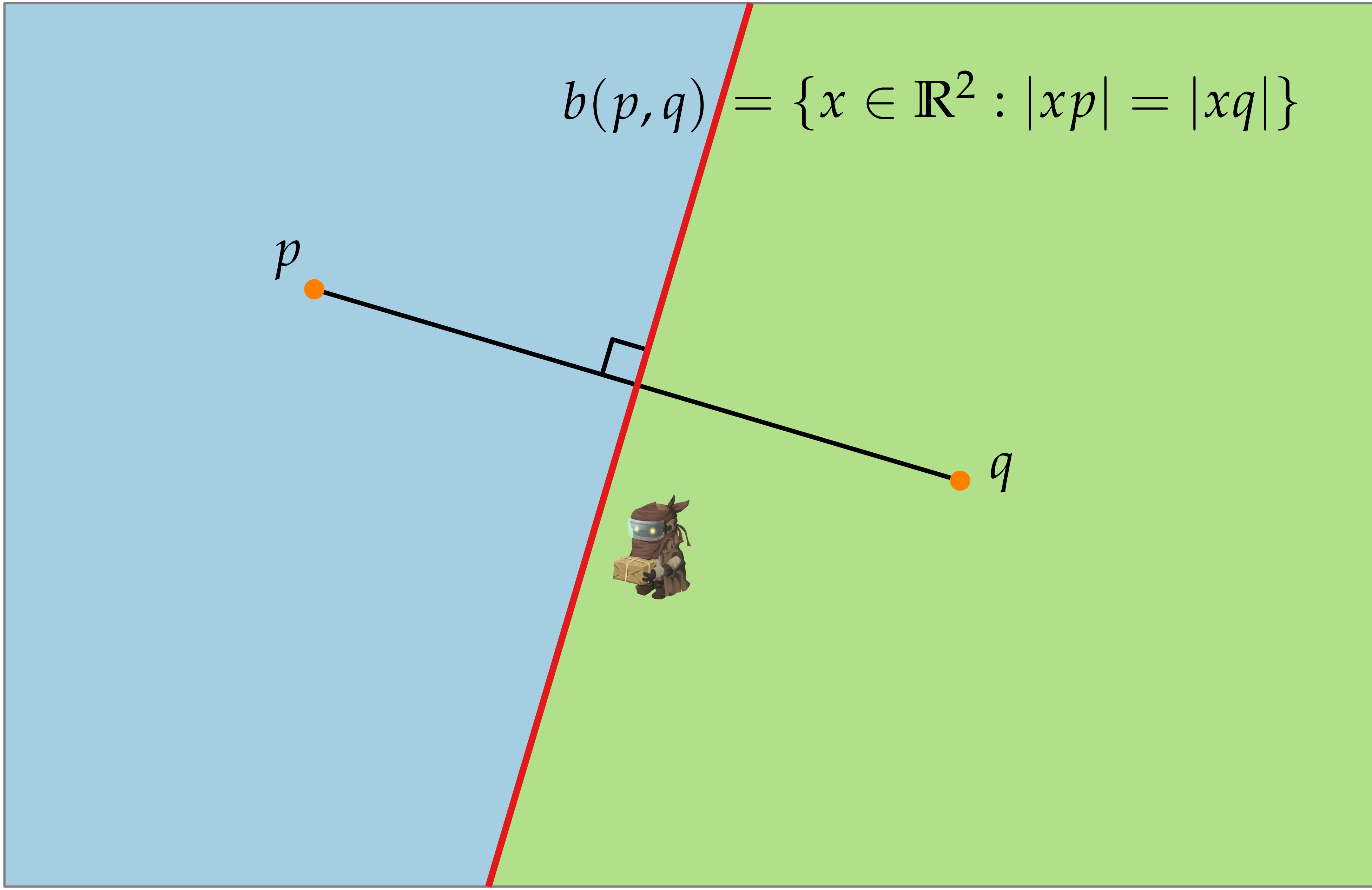

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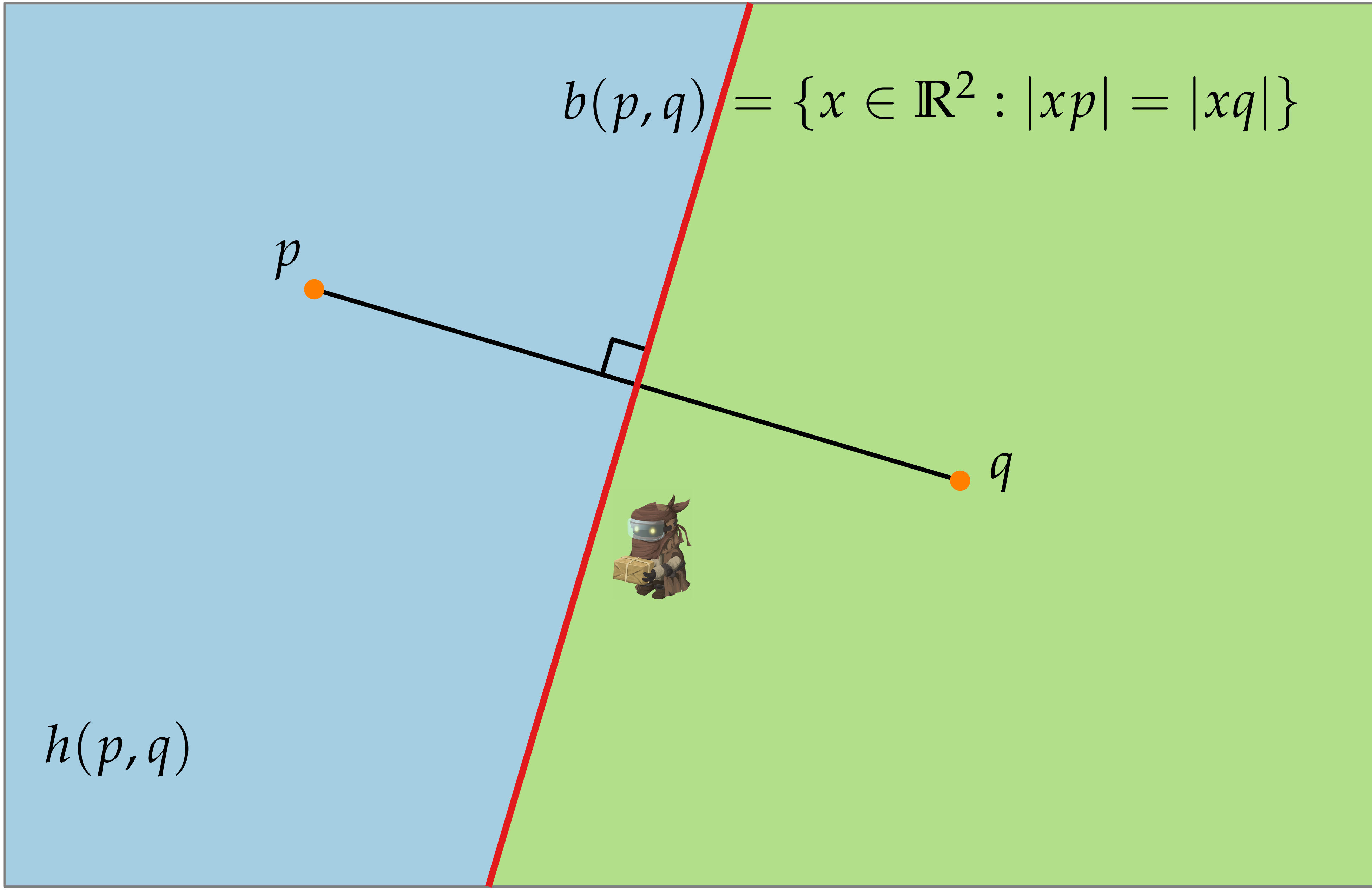

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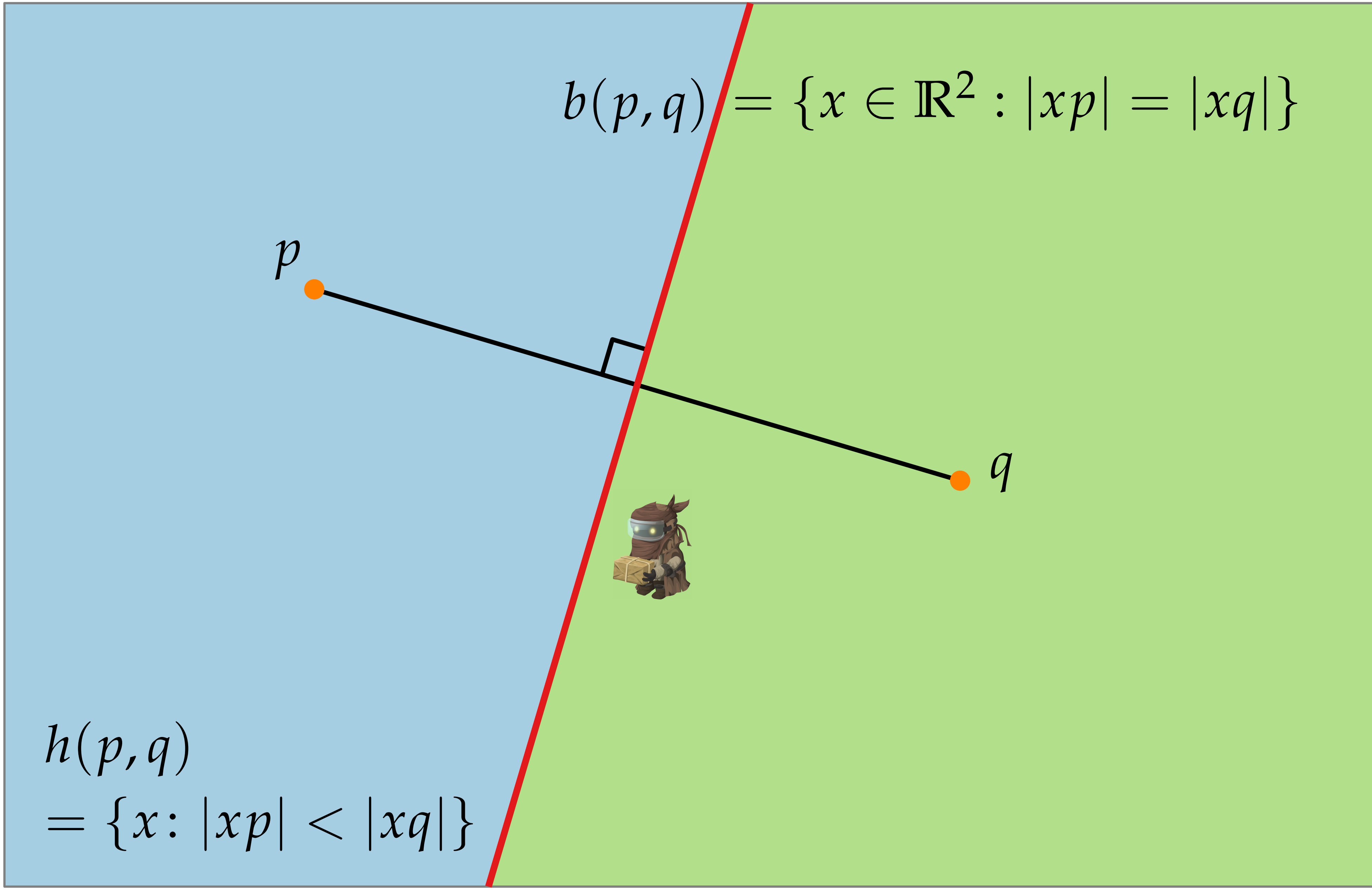

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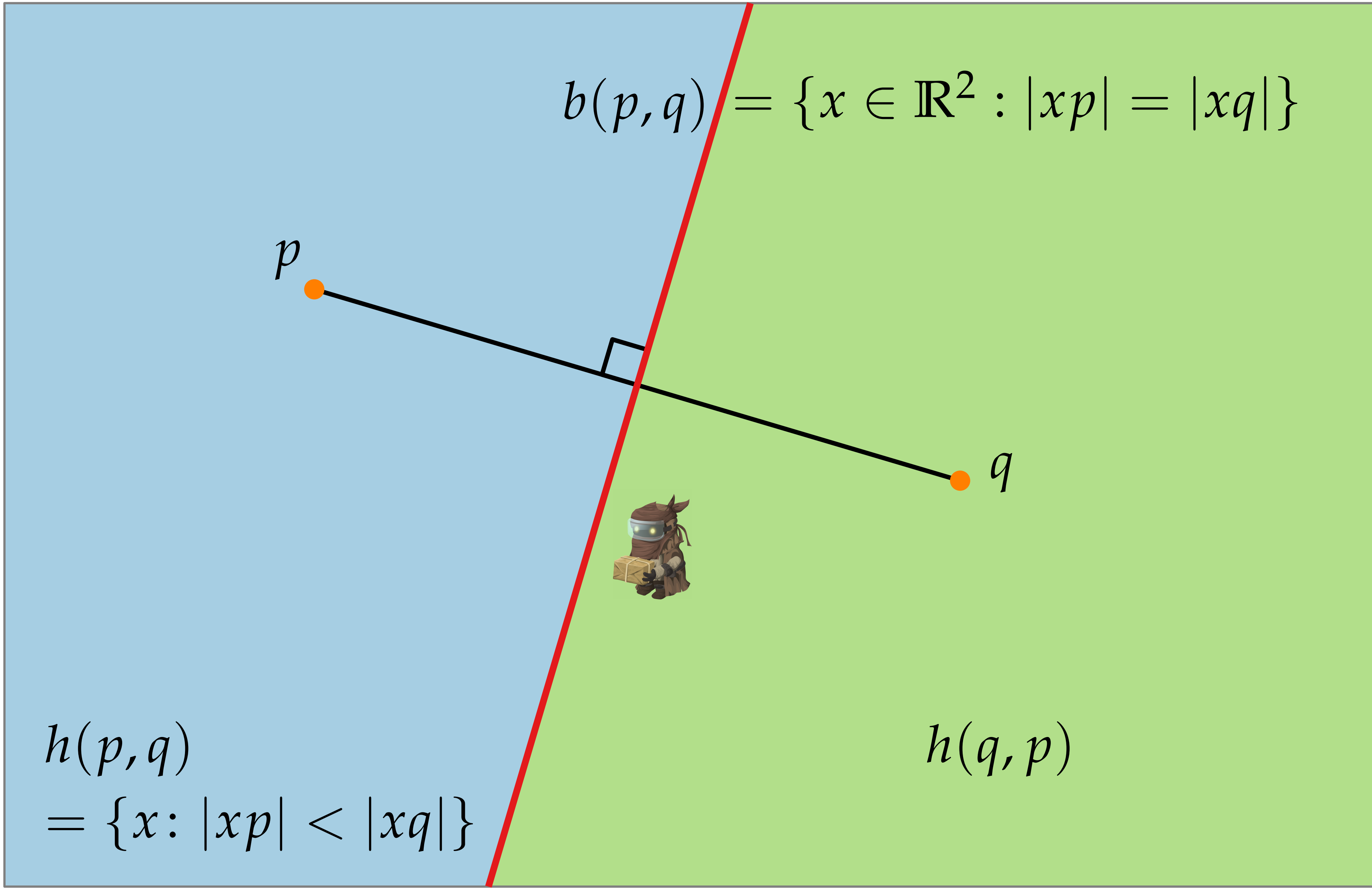

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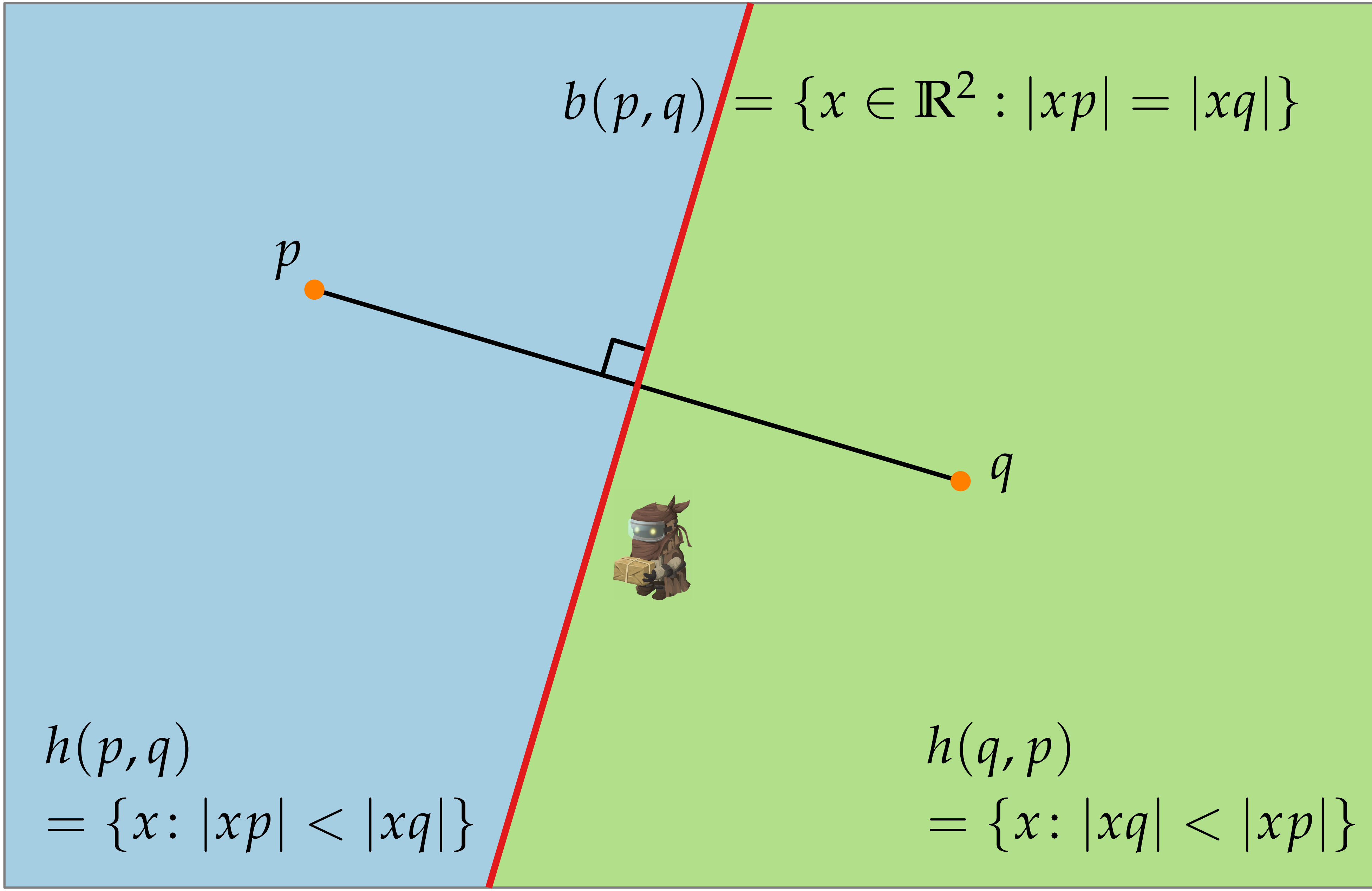

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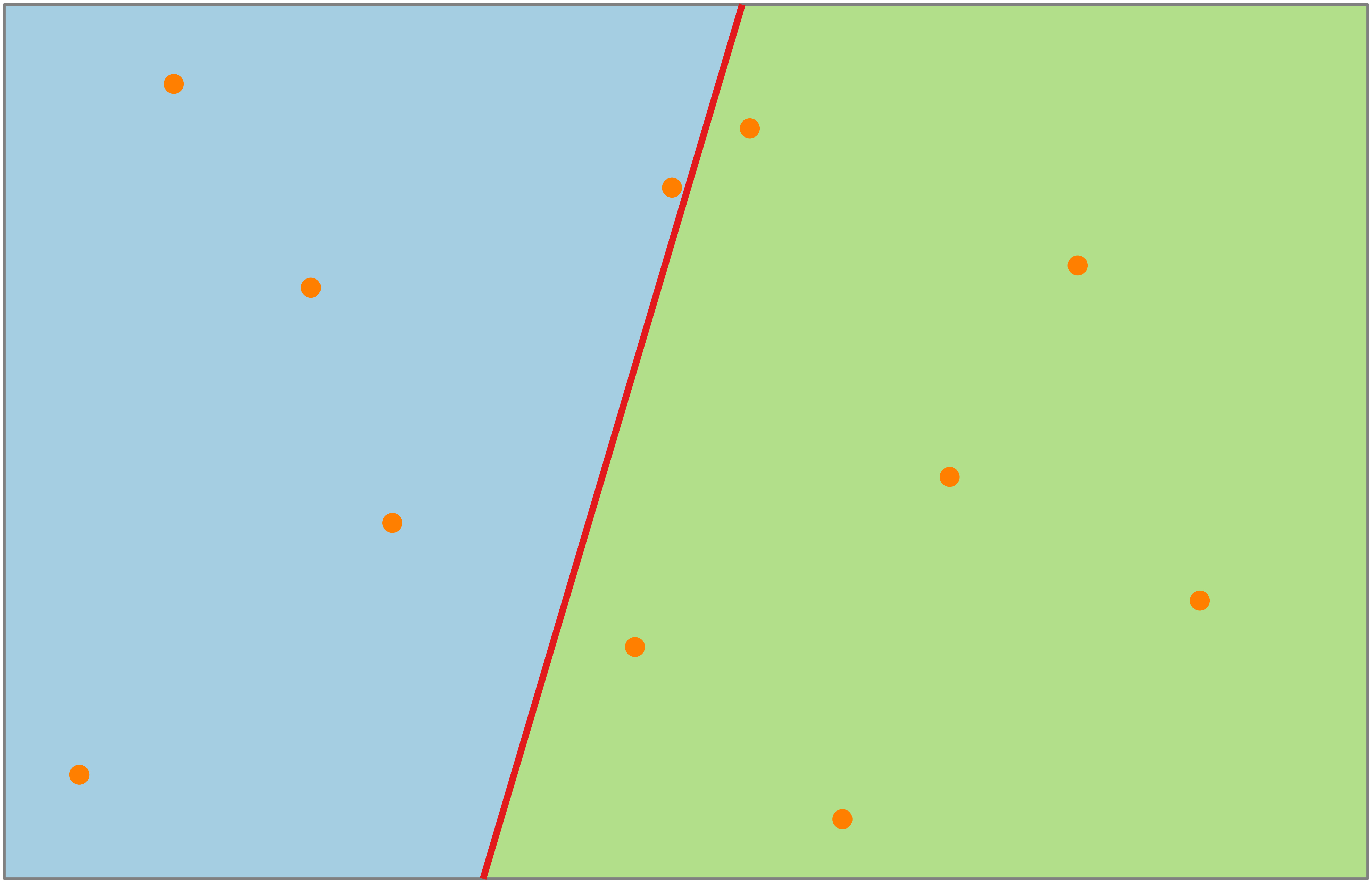

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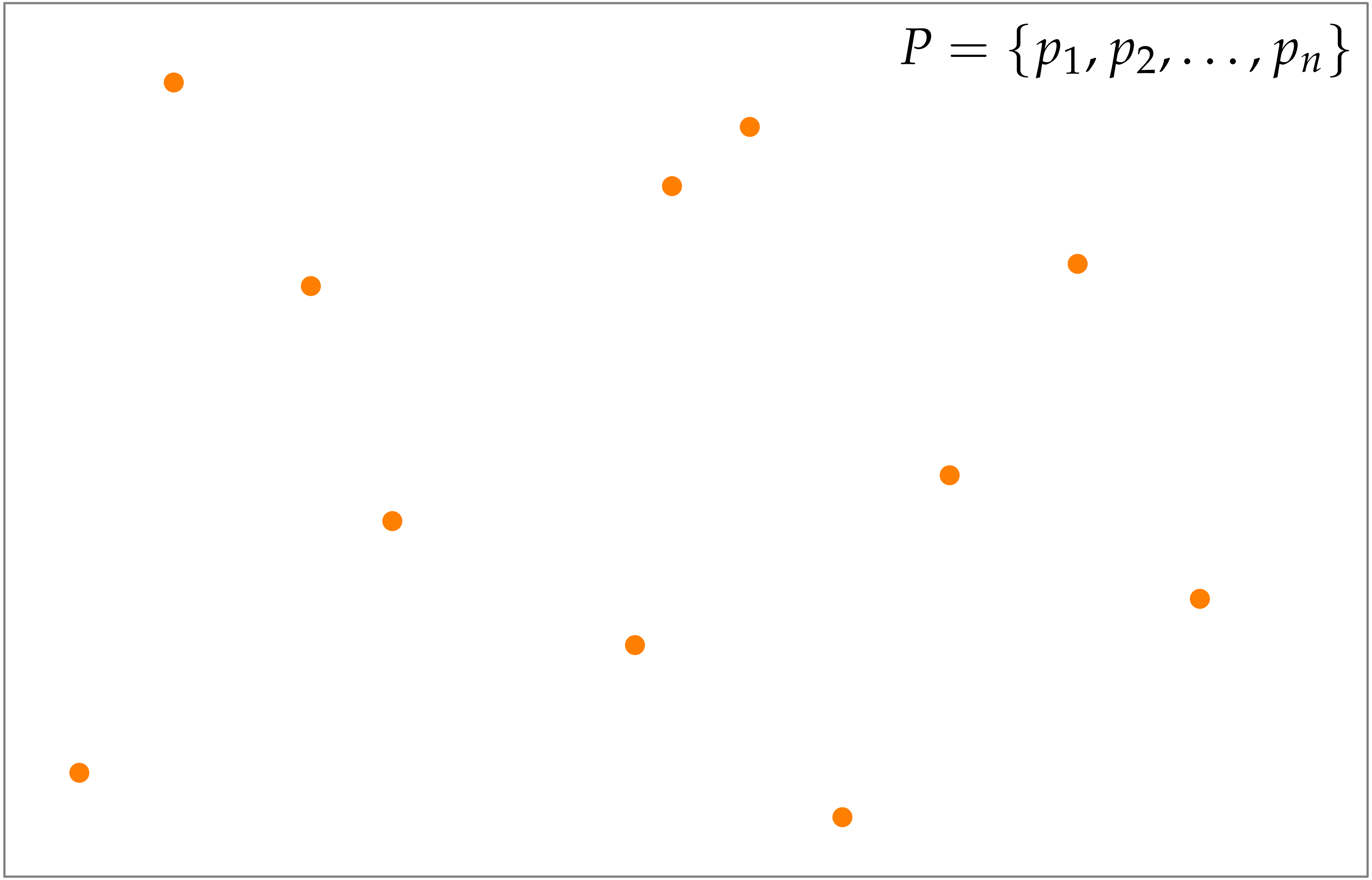

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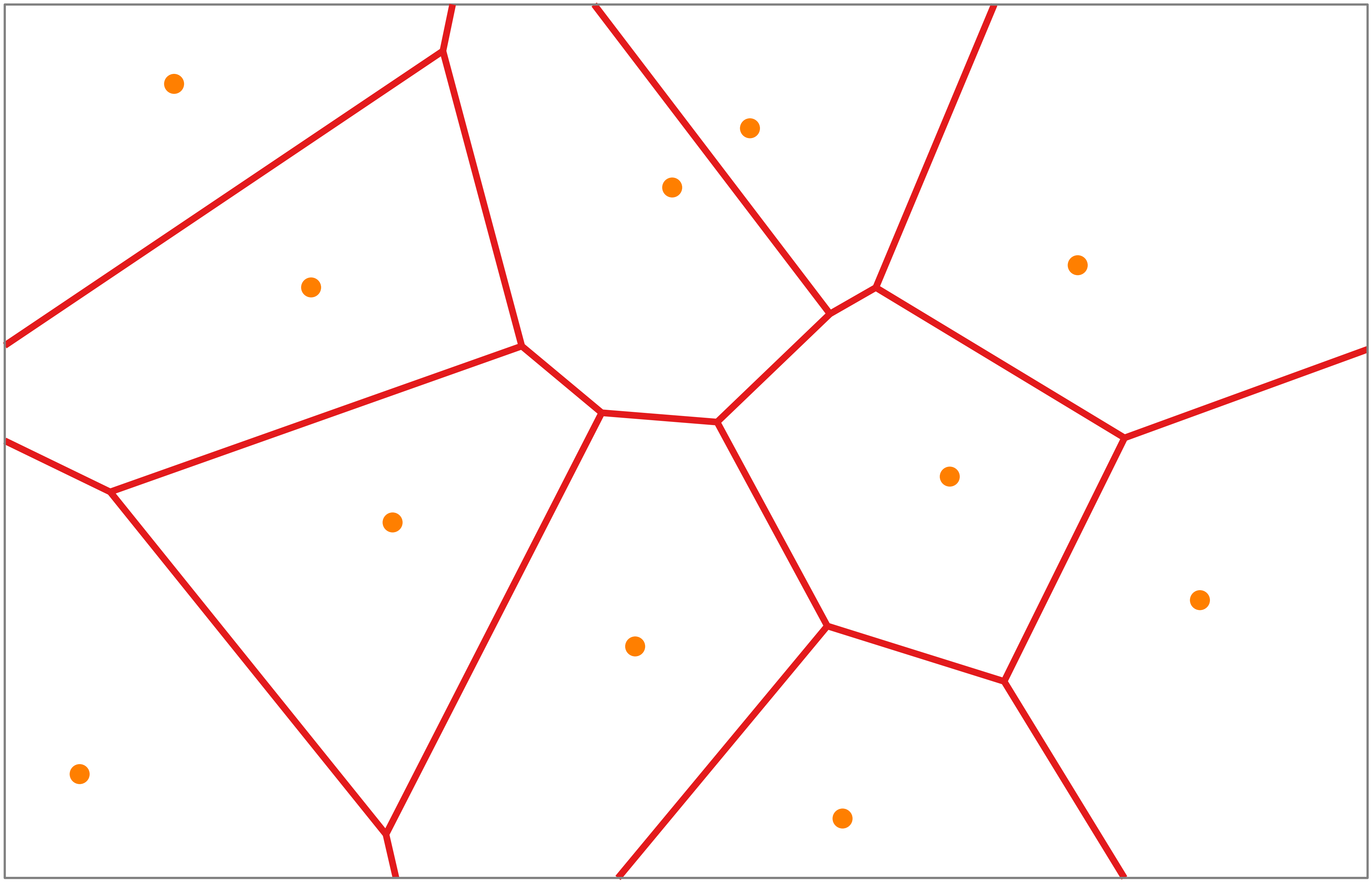

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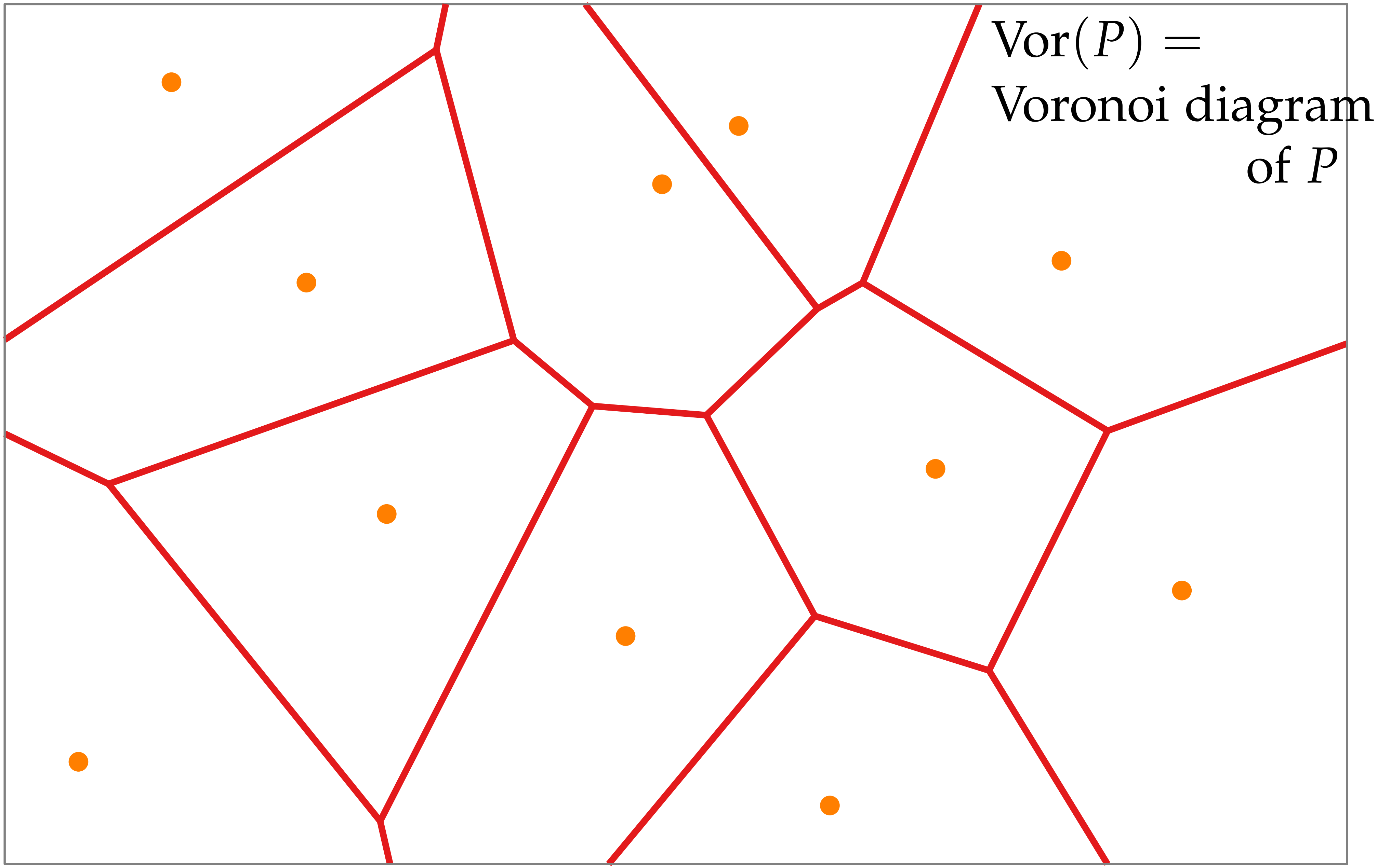

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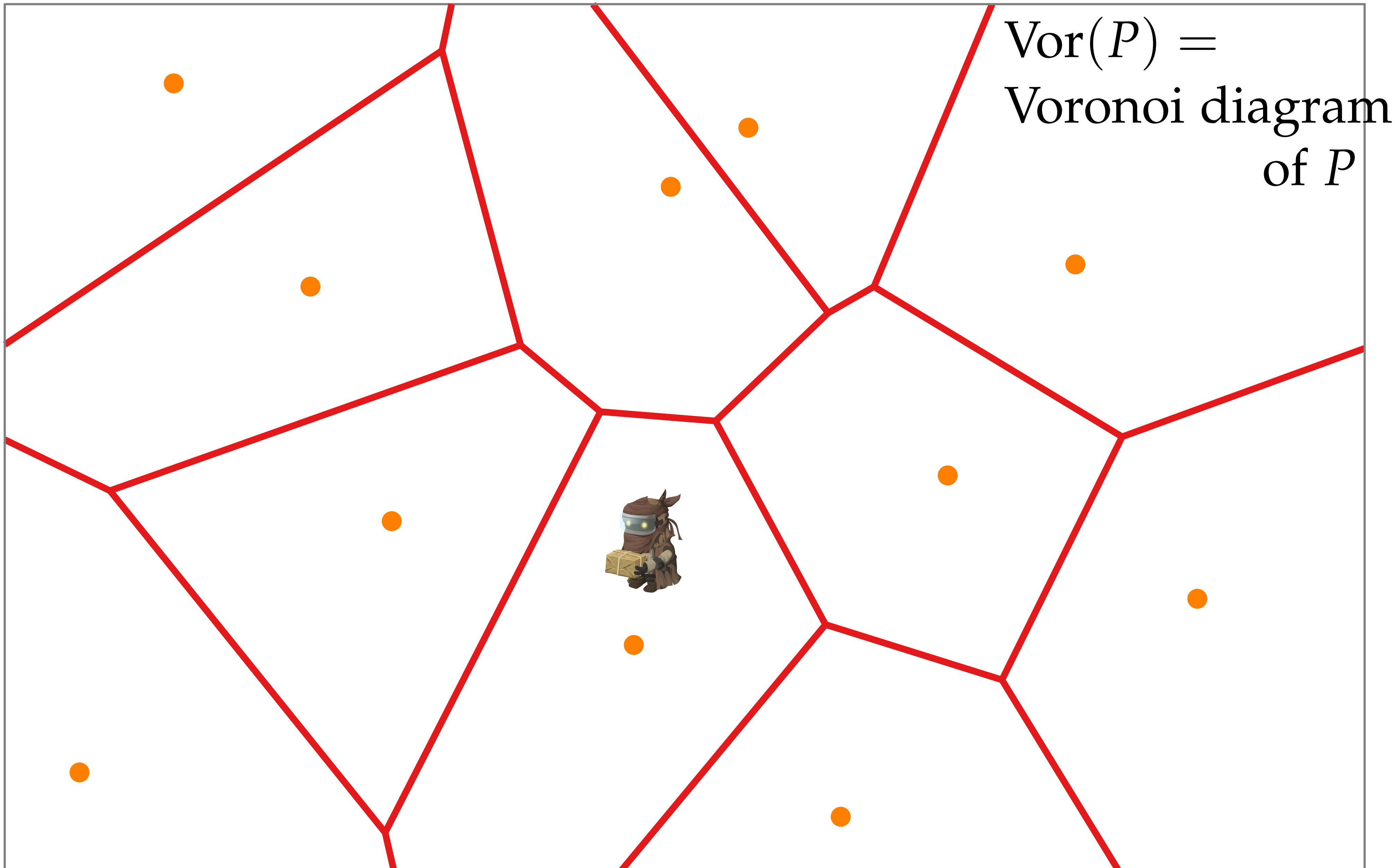

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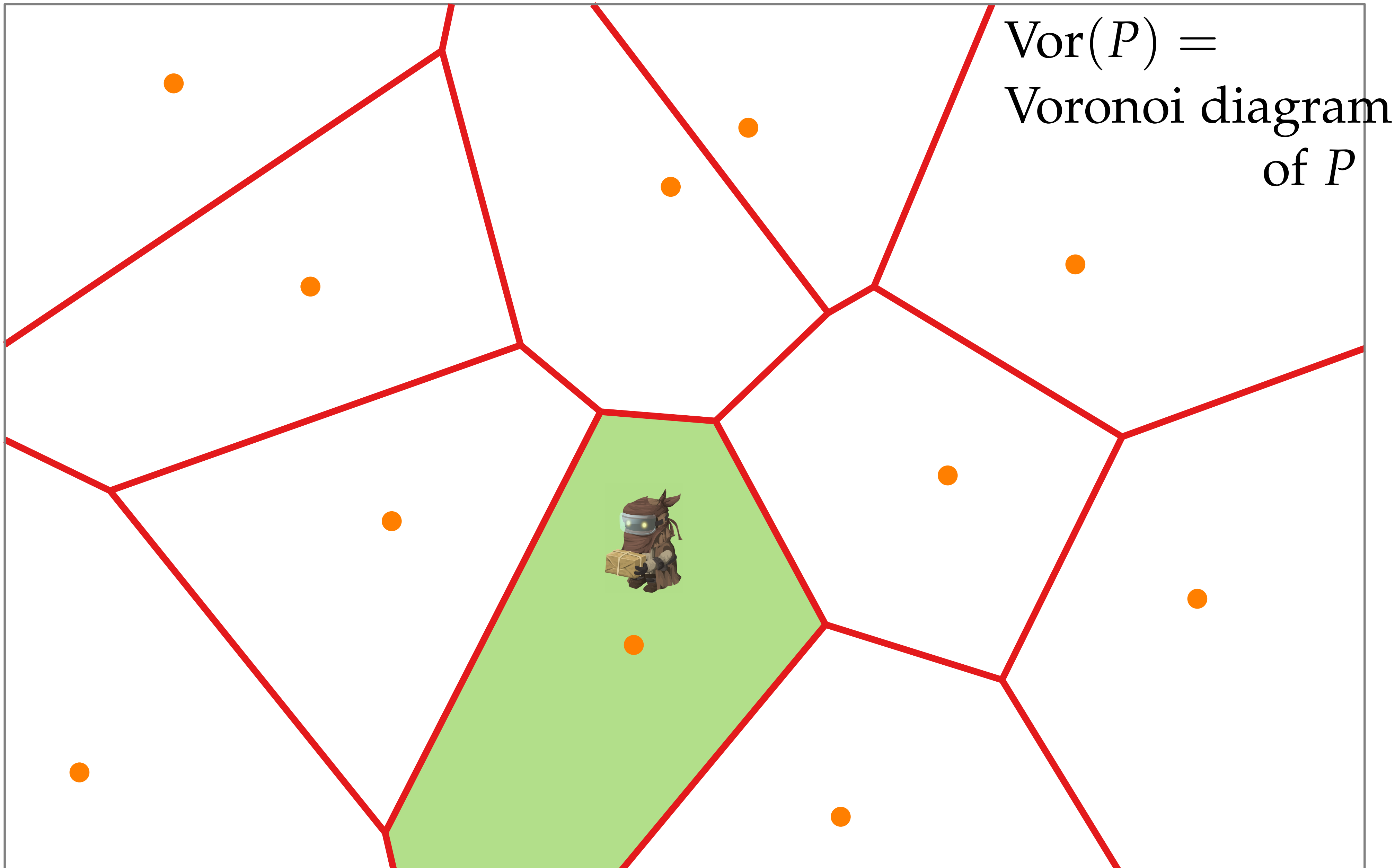

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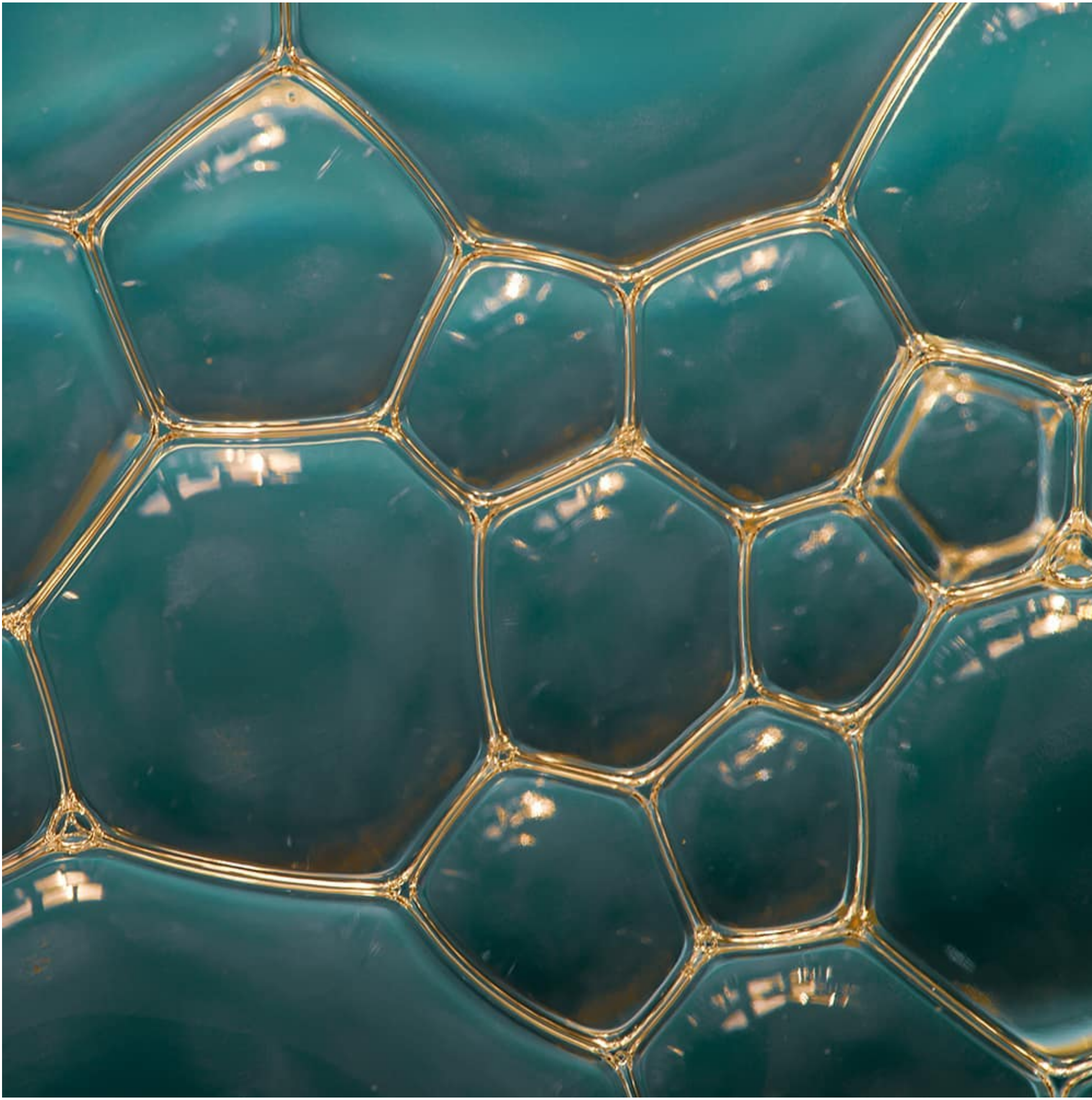
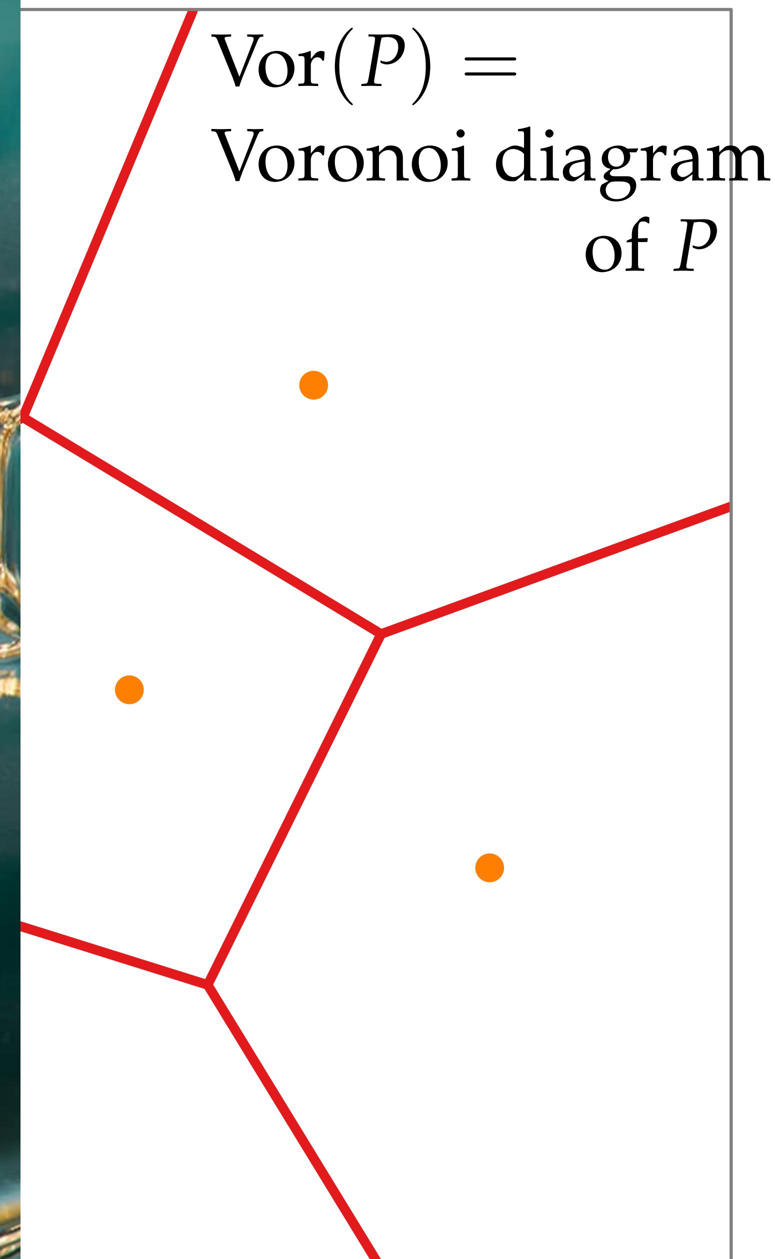


@kakteen007 on picuki.com



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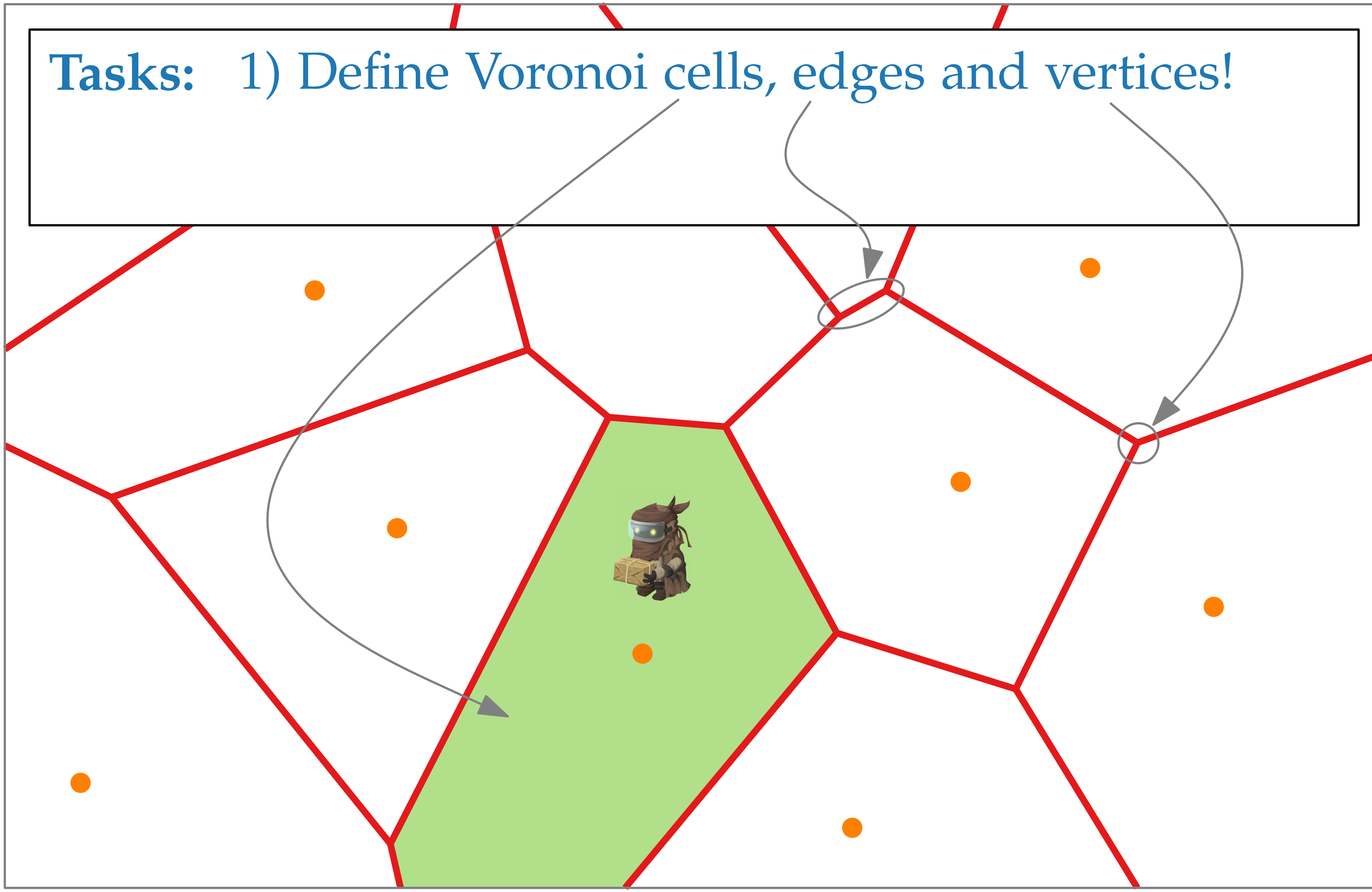
The Post-Office Problem

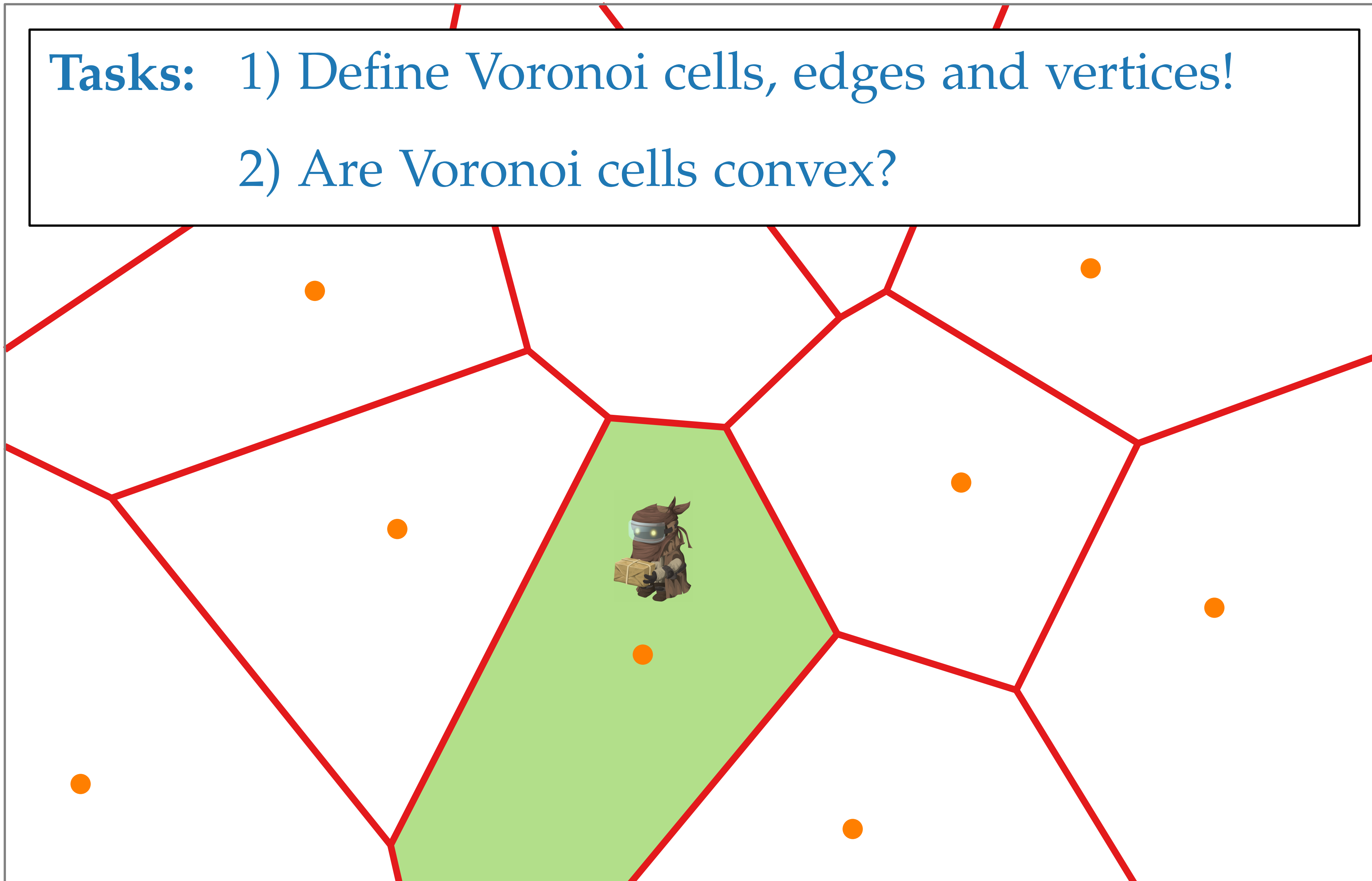
Tasks: 1) Define Voronoi cells, edges and vertices!



The Post-Office Problem

Tasks: 1) Define Voronoi cells, edges and vertices!

2) Are Voronoi cells convex?



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Part II:
The Voronoi Diagram

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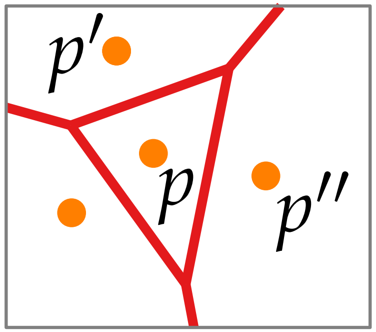

$\mathrm{Vor}(P)$

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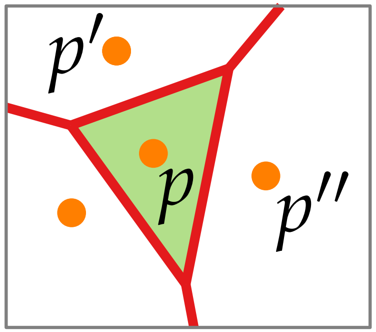

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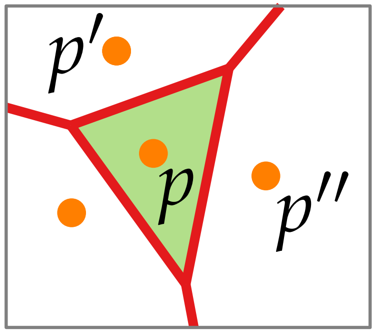

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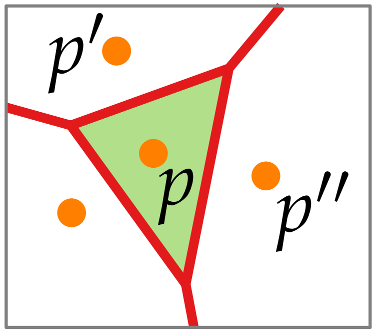

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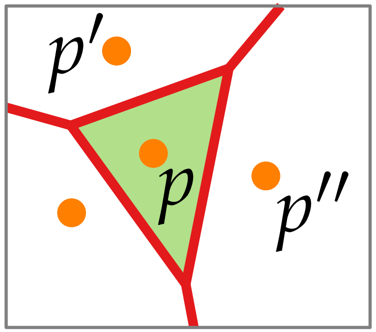

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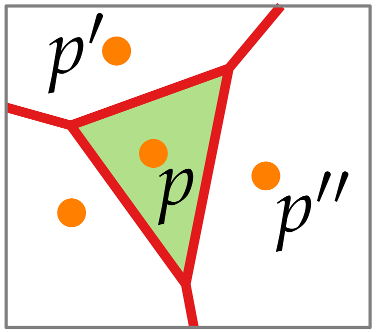

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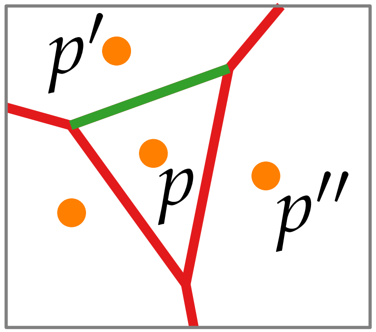

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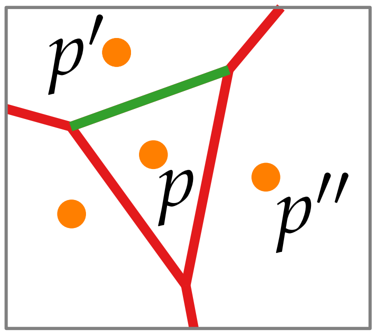

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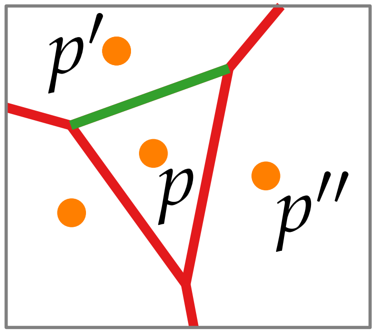

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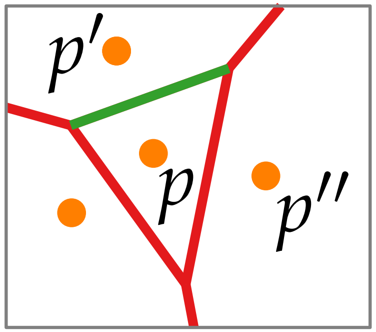

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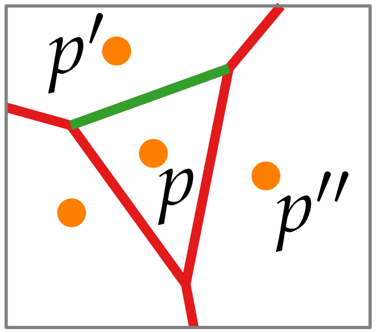

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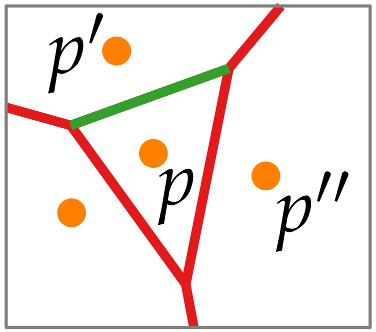

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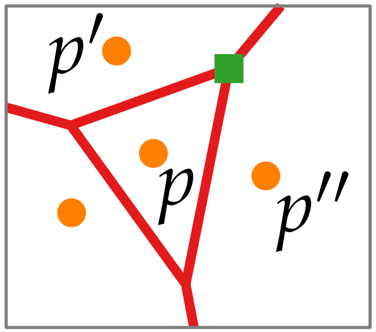
[*Voronoi diagram*] 

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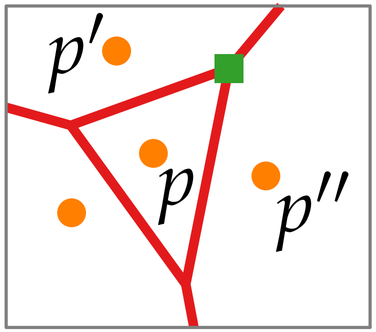

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[Voronoi vertex]

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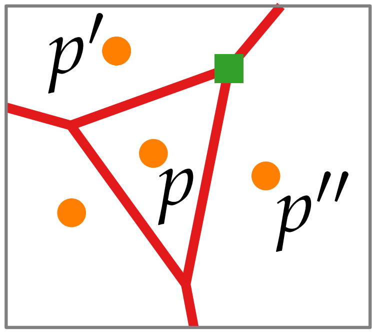

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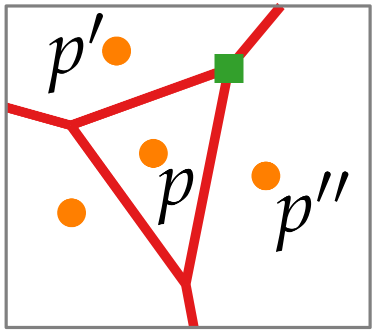

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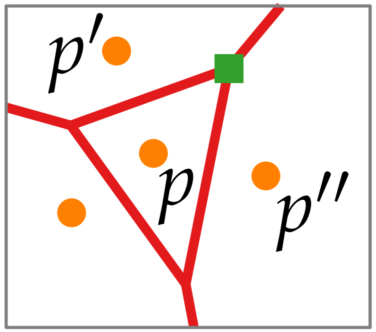

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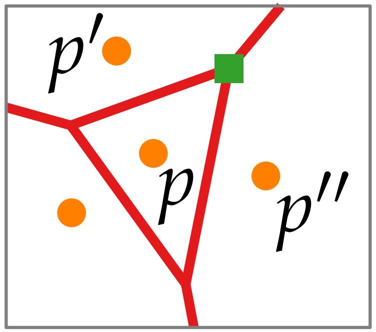

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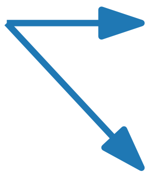

[*Voronoi cell*]

$$\mathcal{V}(\{p\}) = \mathcal{V}(p) = \{x \in \mathbb{R}^2 : |xp| < |xq| \text{ for all } q \in P \setminus \{p\}\}$$
$$= \bigcap_{q \neq p} h(p, q)$$

[*Voronoi edge*]

$$\mathcal{V}(\{p, p'\}) = \{x : |xp| = |xp'| \text{ and } |xp| < |xq| \;\; \forall q \neq p, p'\}$$
$$= \text{rel-int}(\partial\mathcal{V}(p) \cap \partial\mathcal{V}(p')) \;\; \text{(w/o the endpts)}$$

[*Voronoi vertex*]

$$\mathcal{V}(\{p, p', p''\}) = \partial\mathcal{V}(p) \cap \partial\mathcal{V}(p') \cap \partial\mathcal{V}(p'')$$
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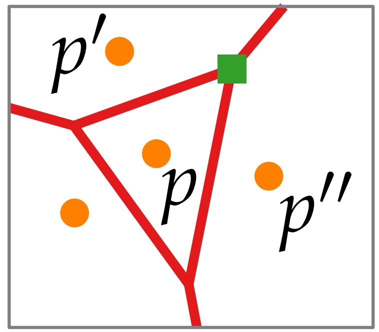
The Voronoi Diagram

Let P be a set of points in the plane and let $p, p', p'' \in P$.

[Voronoi diagram]



$\text{Vor}(P)$ → subdivision of $\mathbb{R}^2$

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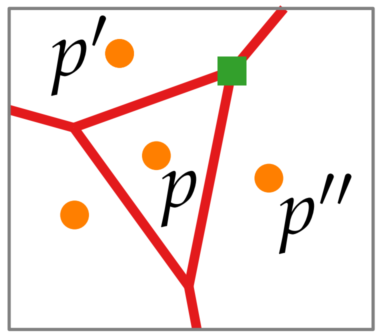

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→ geometric graph

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Computational Geometry

Lecture 7:
Voronoi Diagrams
or
The Post-Office Problem

Part III:
Shape and Complexity

Philipp Kindermann

Winter Semester 2020

Overall Shape of Vor(P)

Theorem. Let $P \subset \mathbb{R}^2$ be a set of n pts (called *sites*). If all sites are collinear, Vor(P) consists of $n-1$ parallel lines. Otherwise, Vor(P) is connected and its edges are line segments or half-lines.

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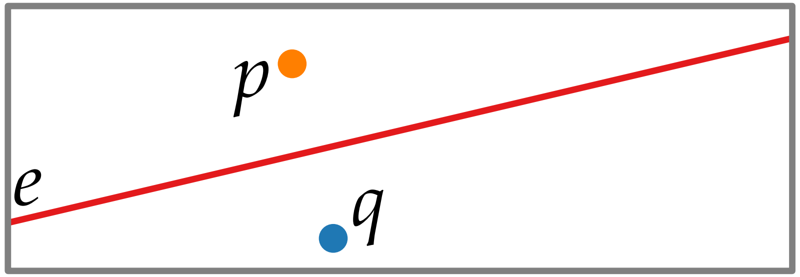

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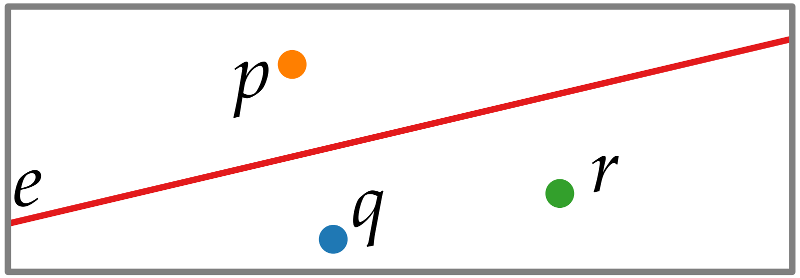

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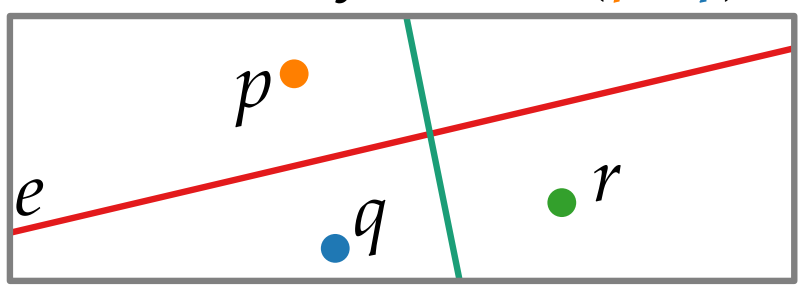

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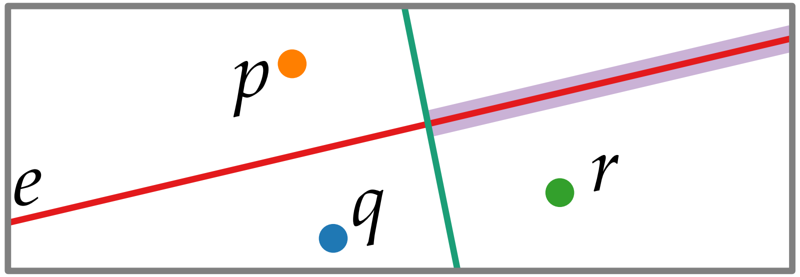

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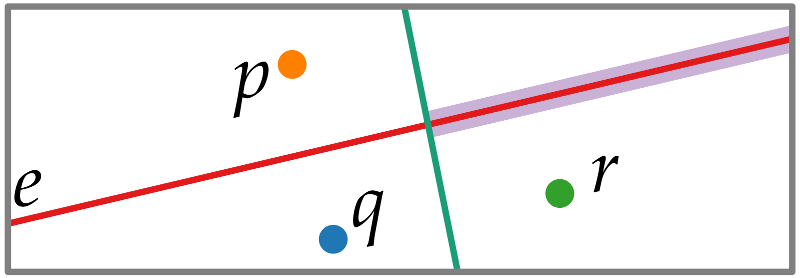

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$\Rightarrow$ e is bounded on at least one side. $\square$

Complexity

Task: Construct a set P of sites such that $\mathrm{Vor}(P)$ has a cell of linear complexity!

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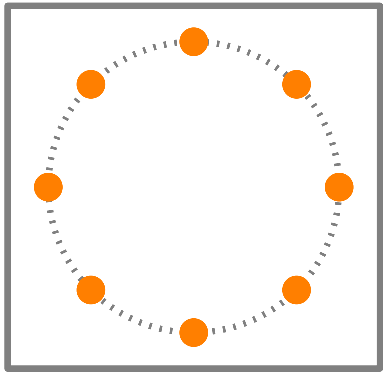

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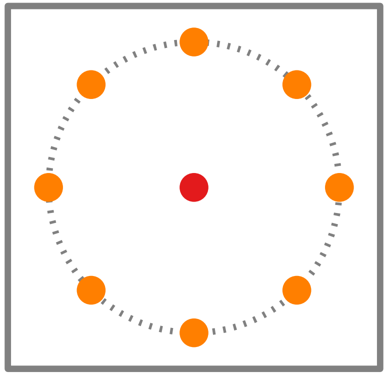

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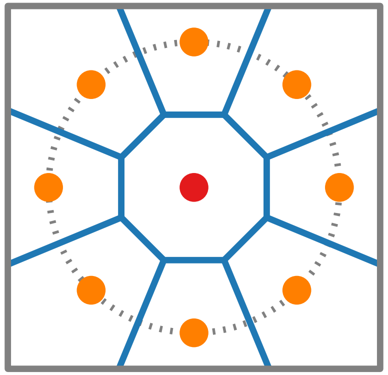

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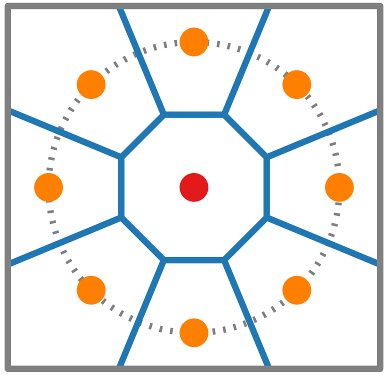

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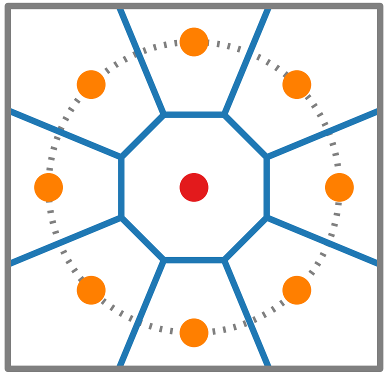

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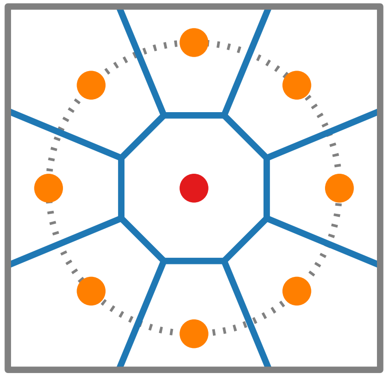

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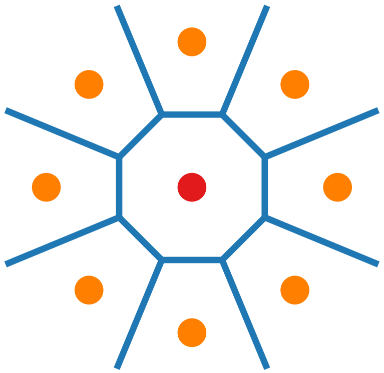
Euler



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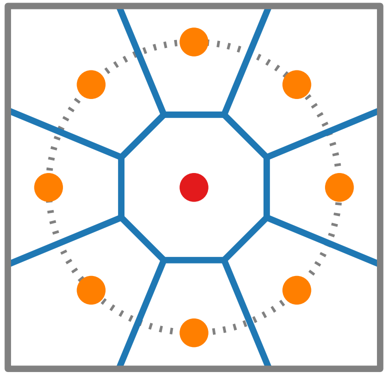

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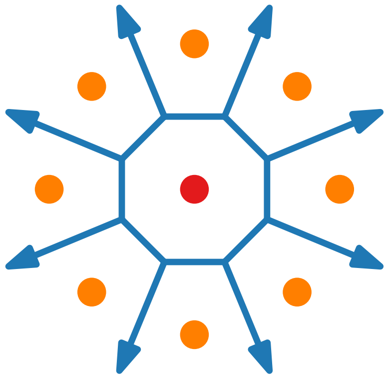

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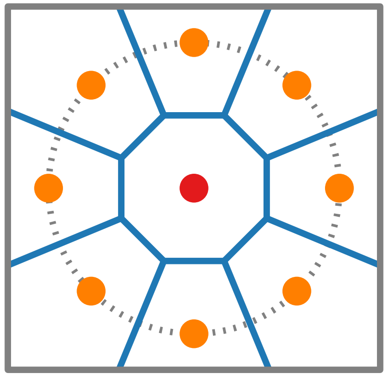

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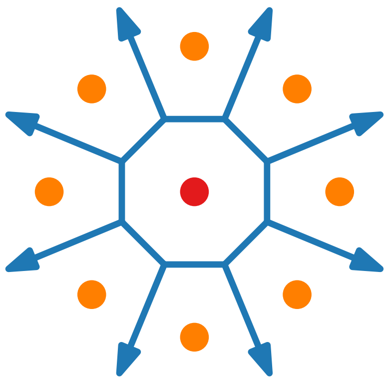

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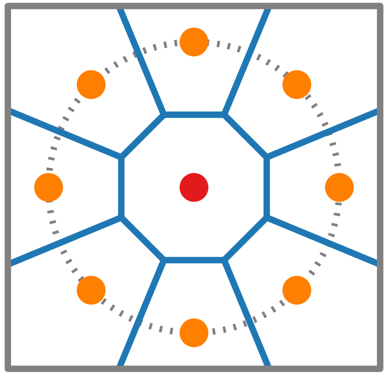

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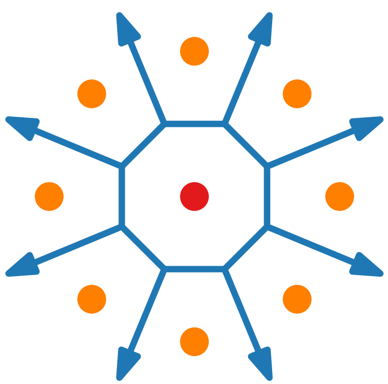

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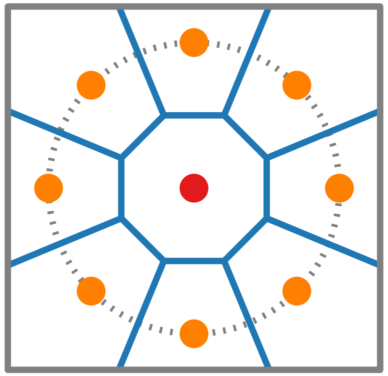

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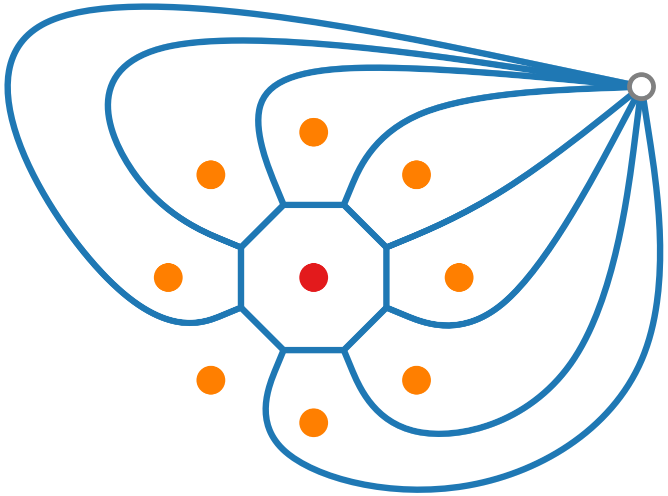

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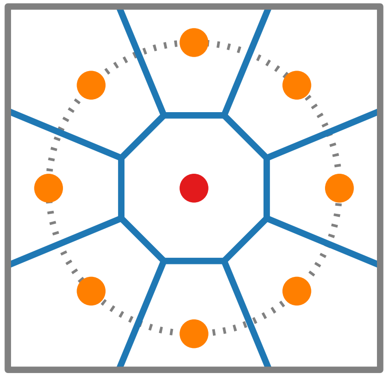

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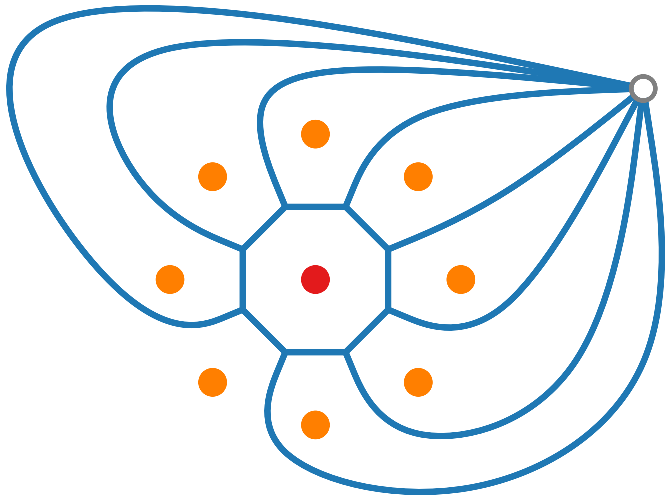

$|F| = n$

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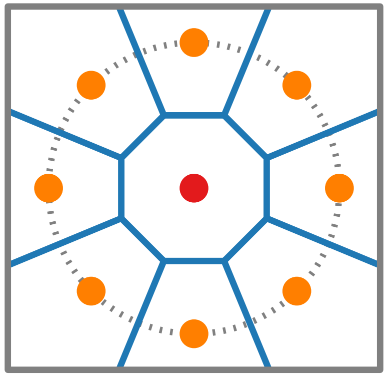

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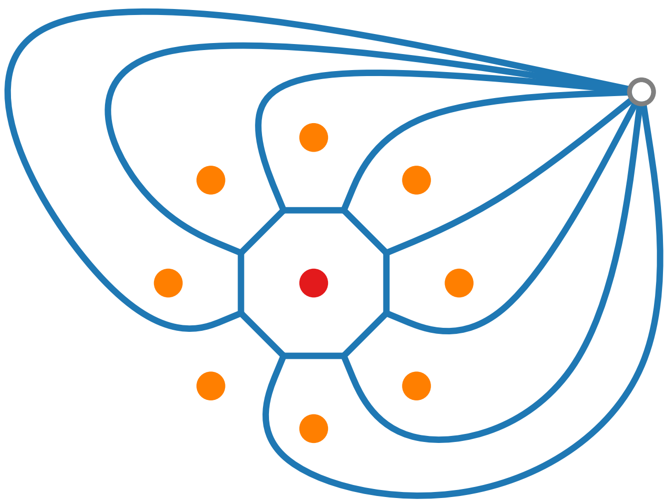

$$|F| = n \Rightarrow (|V|+1) - |E| + n = 2$$

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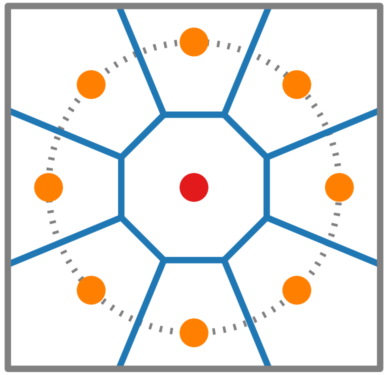

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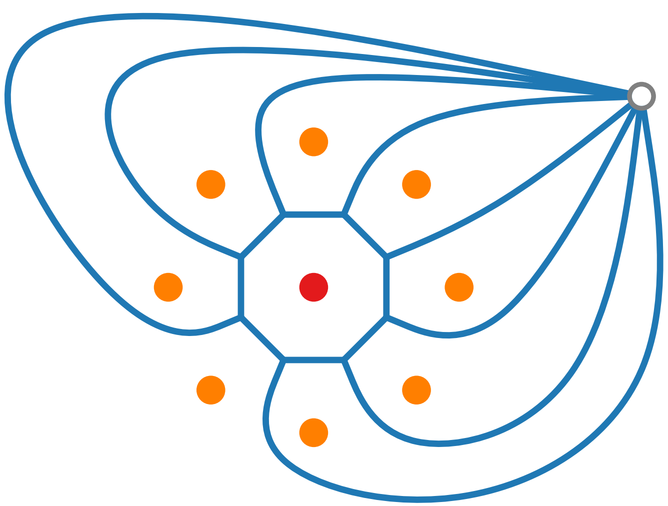

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min. degree 3

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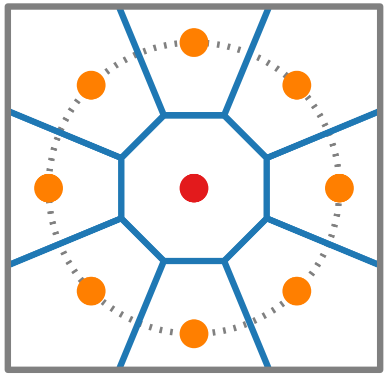

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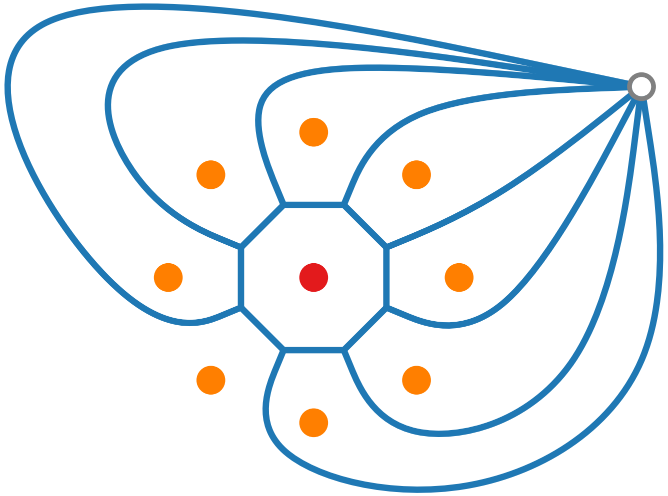

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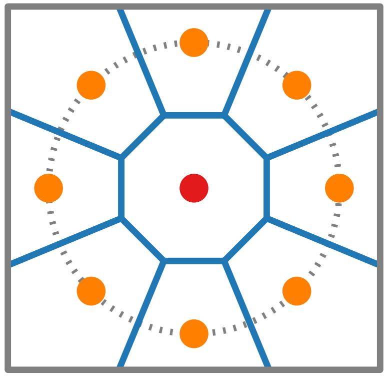

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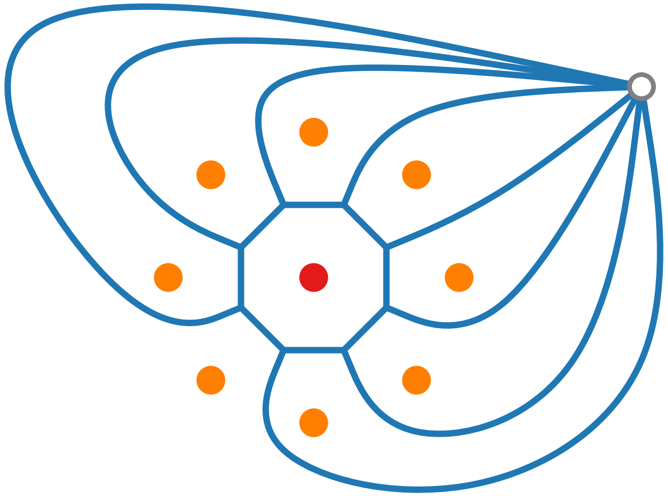

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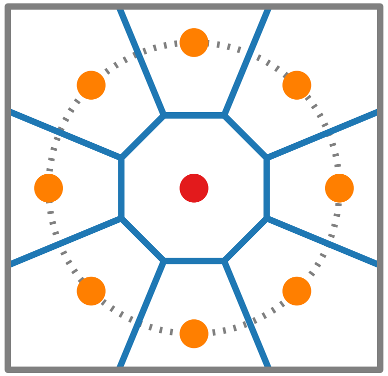

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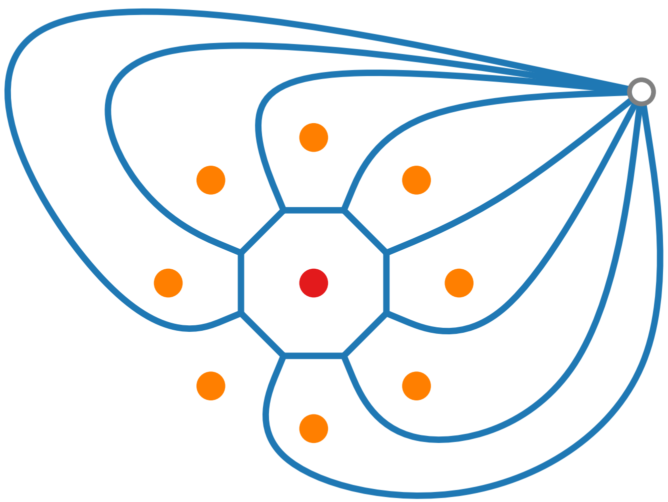

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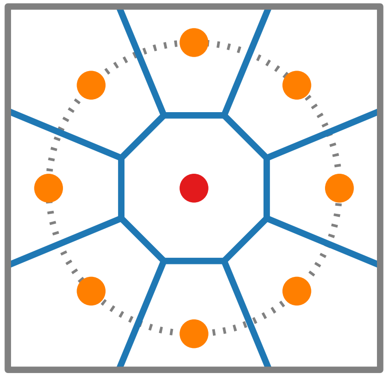

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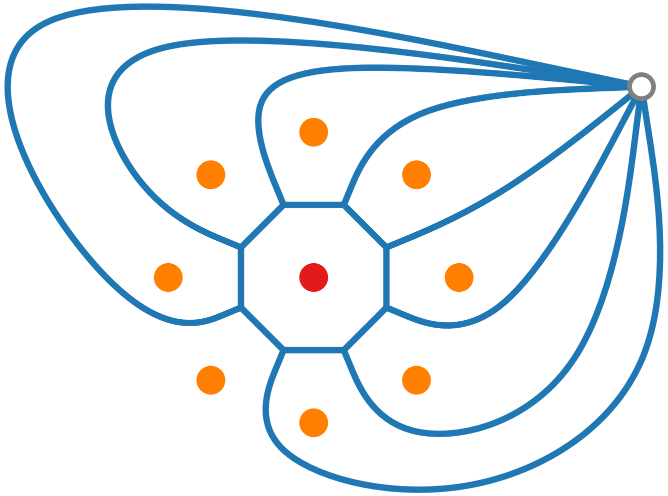

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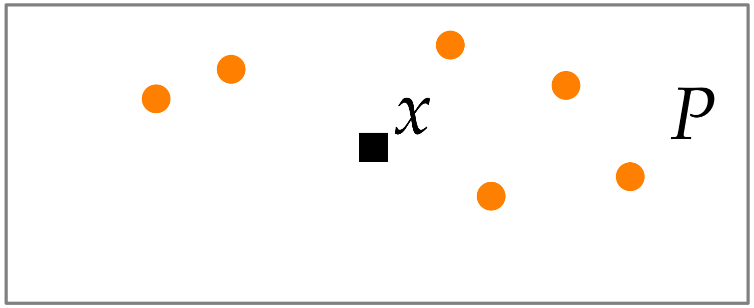
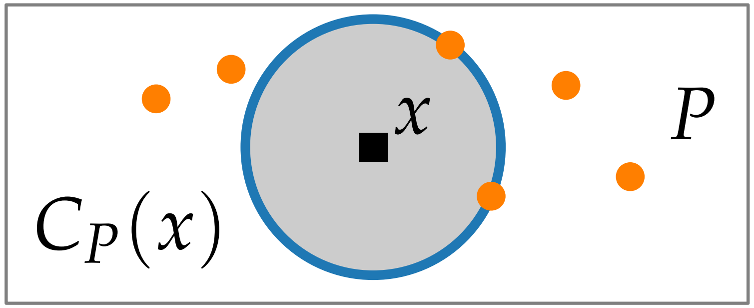
Characterization of Voronoi vtc and edges

$C_P(x) :=$ largest circle centered at x w/o sites in its interior



Characterization of Voronoi vtc and edges

$C_P(x) :=$ largest circle centered at x w/o sites in its interior



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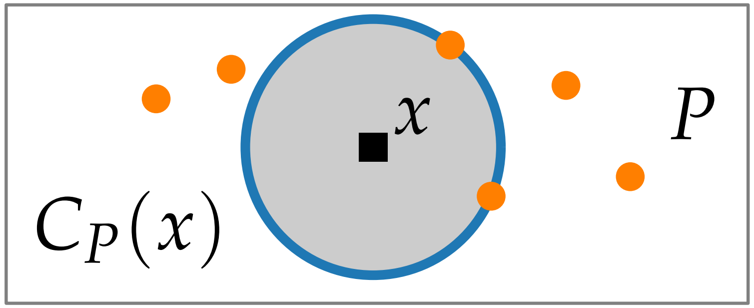

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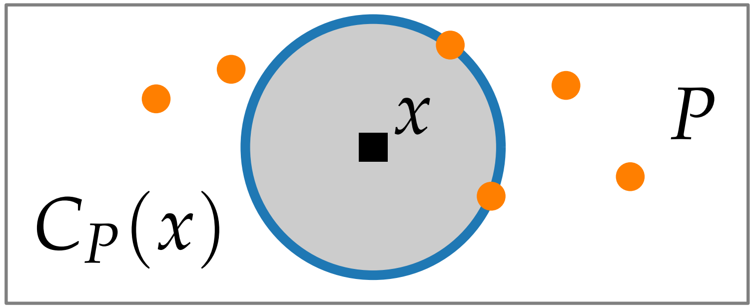

Theorem: (i) x Voronoi vtx $\Leftrightarrow |C_P(x) \cap P| \geq 3$

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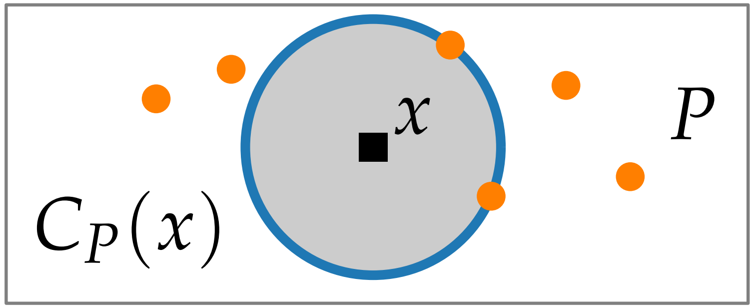

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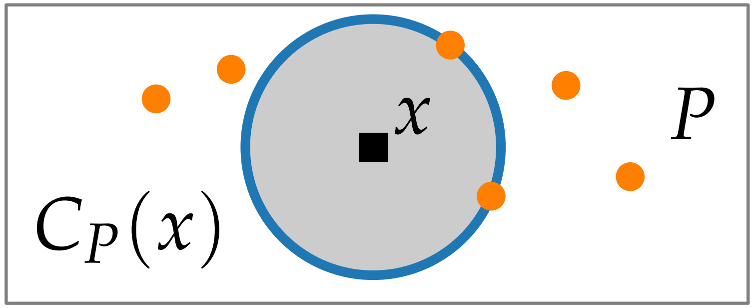

Theorem: (i) x Voronoi vtx $\Leftrightarrow |C_P(x) \cap P| \geq 3$

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$\exists x \in b(p, p') \colon C_P(x) \cap P = \{p, p'\}$

Computational Geometry

Lecture 7:
Voronoi Diagrams
or
The Post-Office Problem

Part IV:
The Beachline

Philipp Kindermann

Winter Semester 2020

Computation

Brute force:

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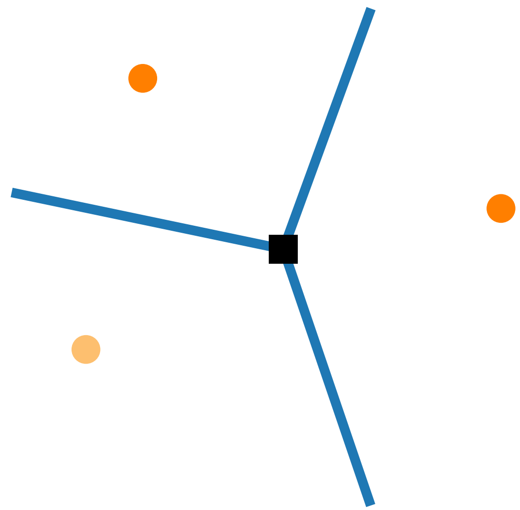

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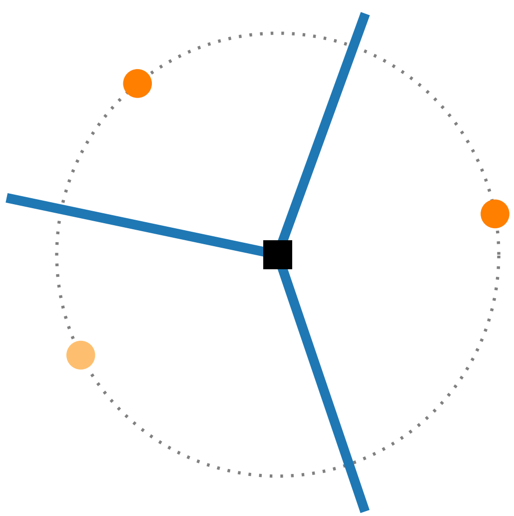

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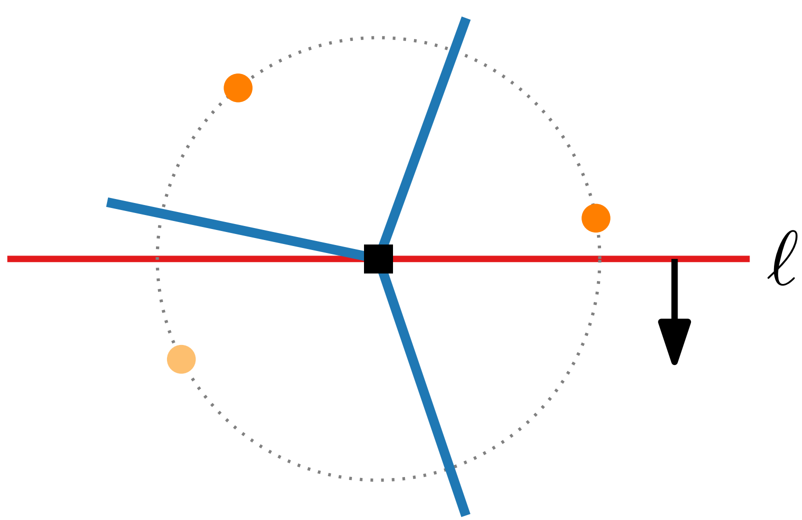

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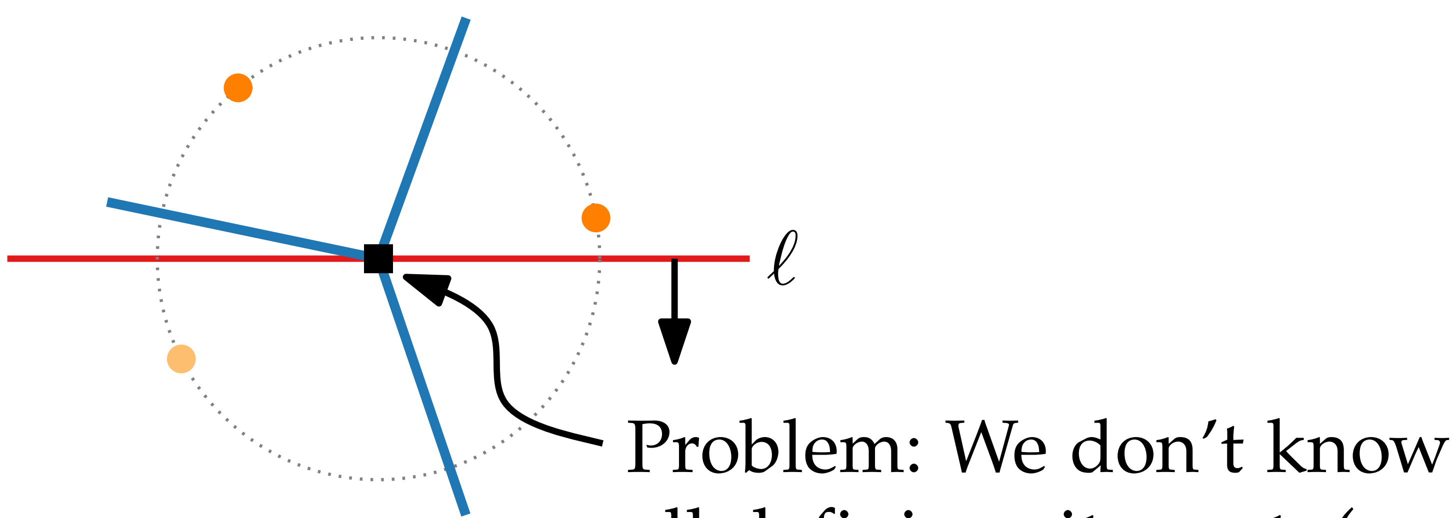

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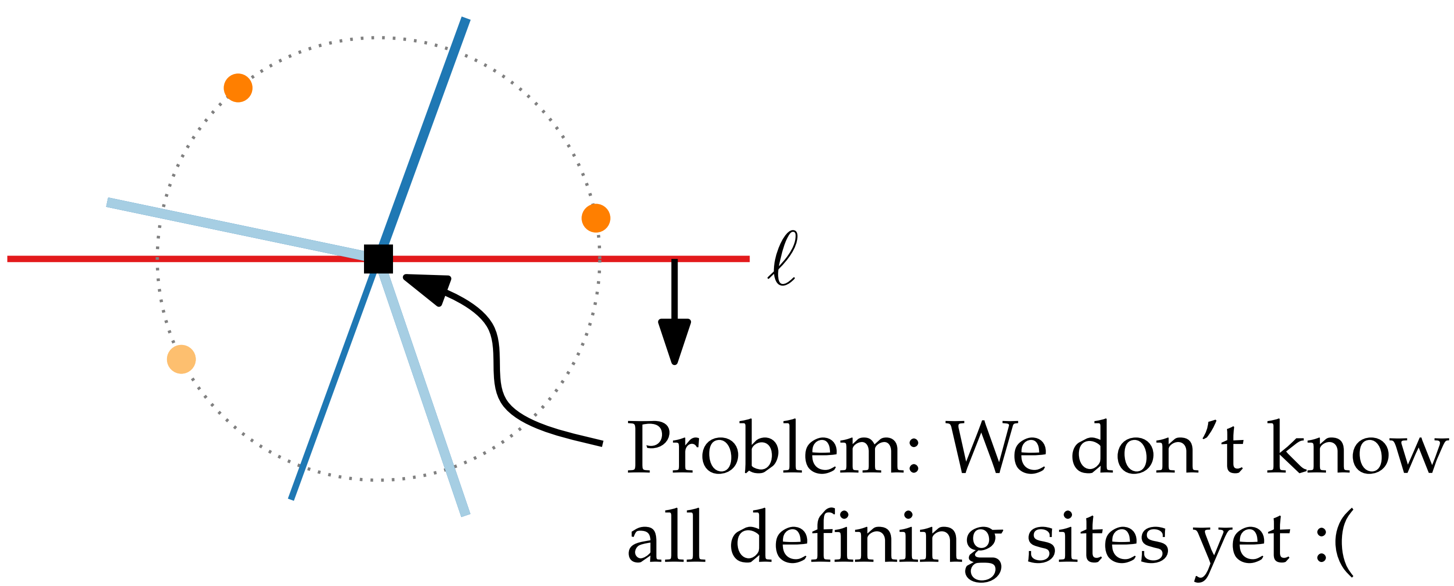

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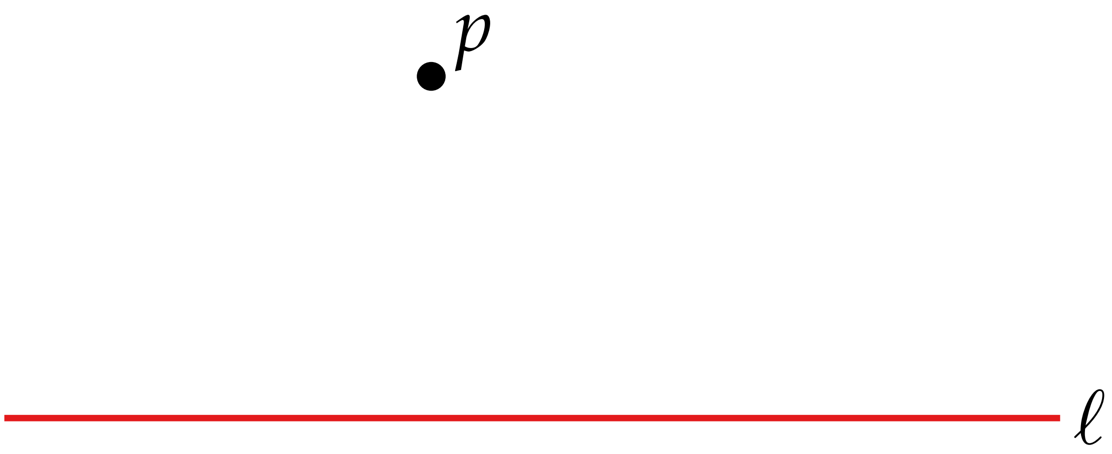

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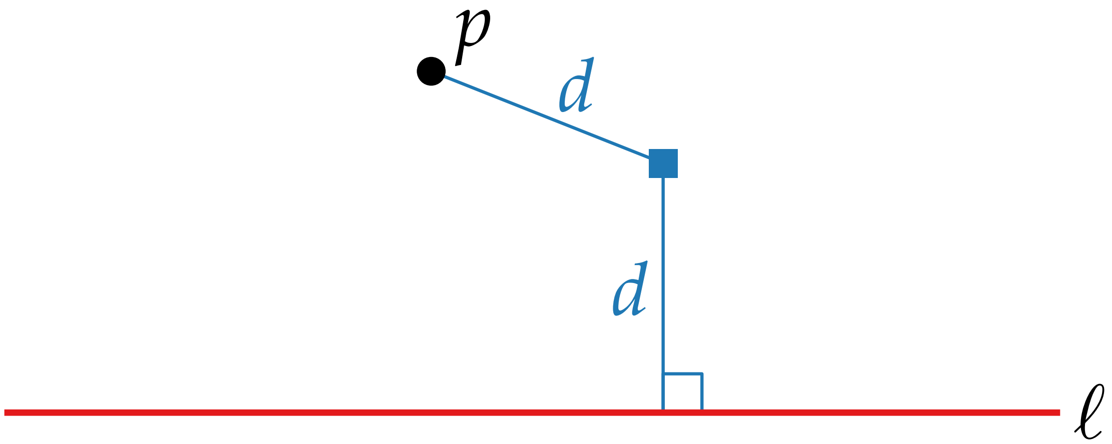

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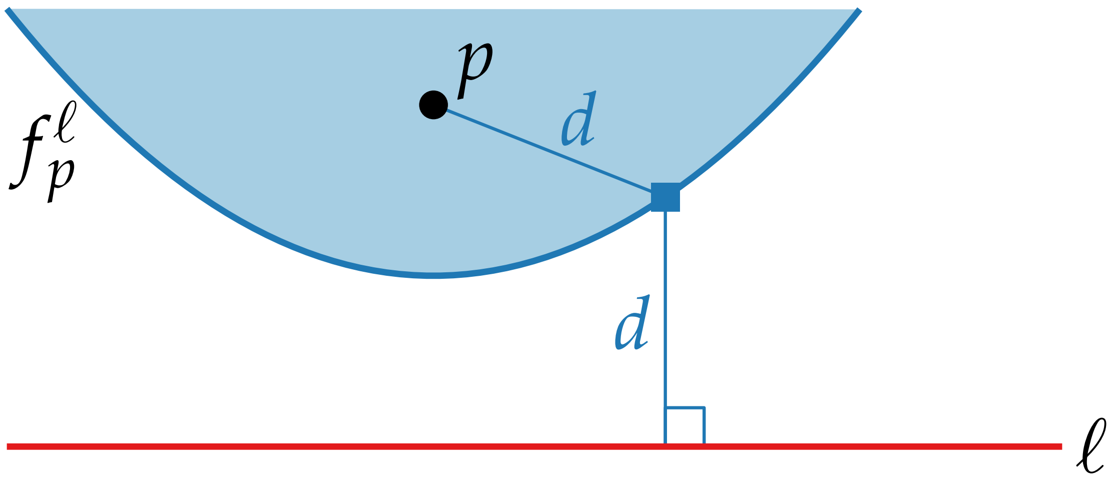

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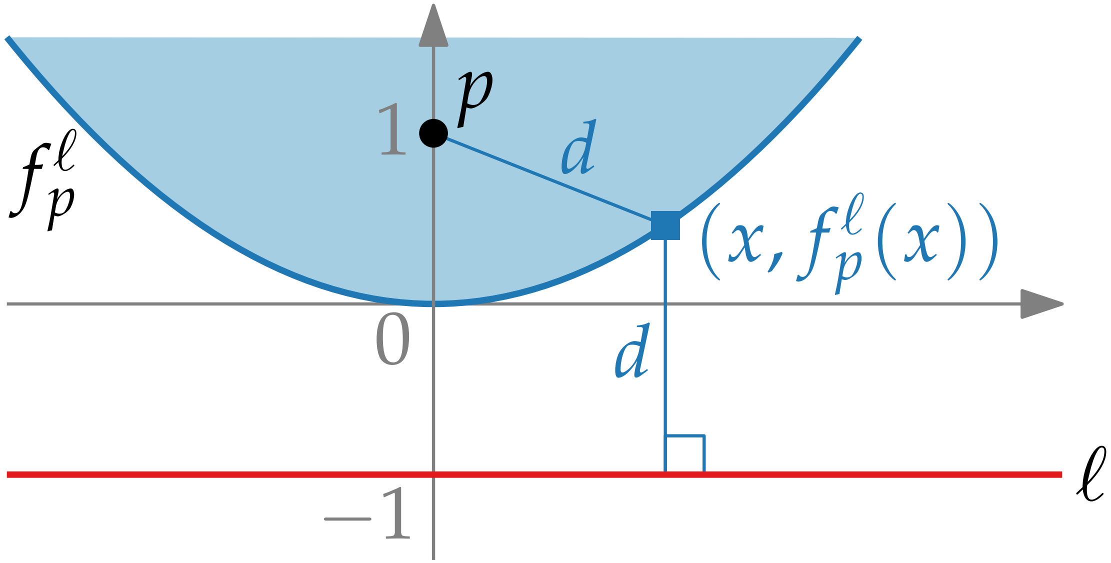

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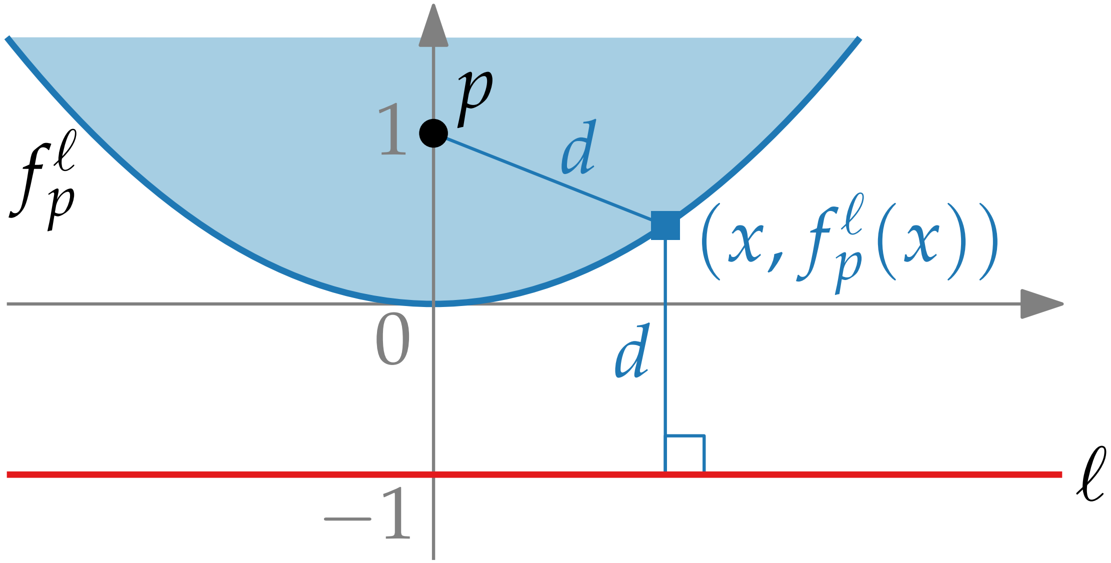

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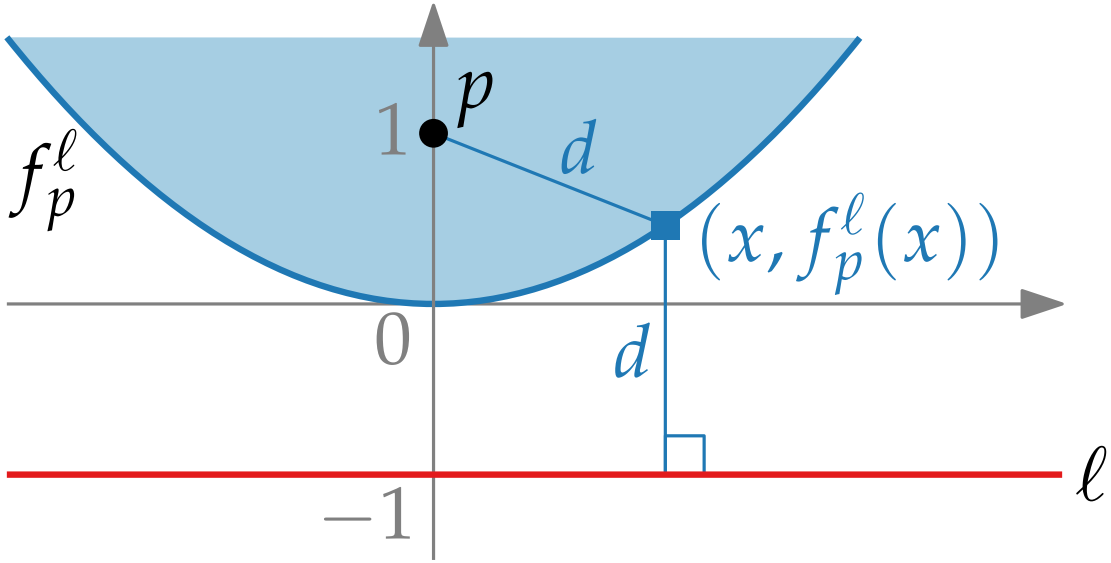

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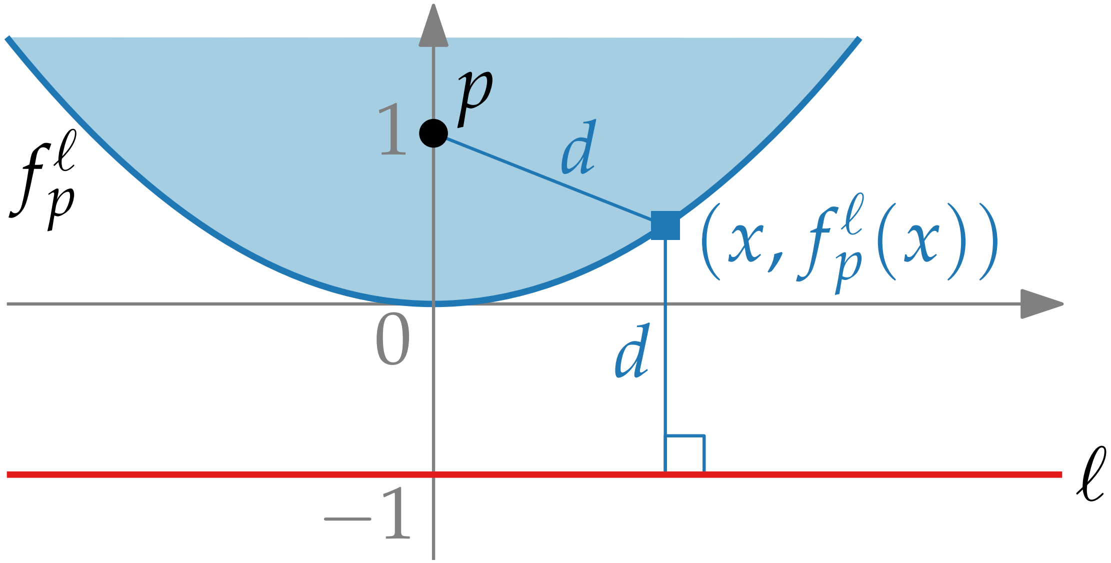

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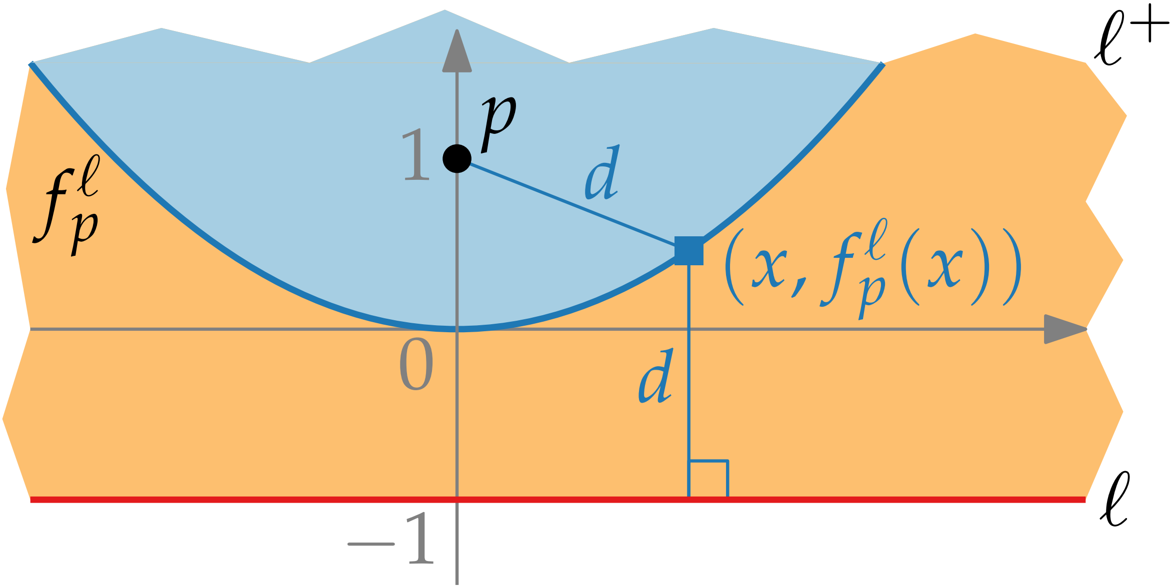

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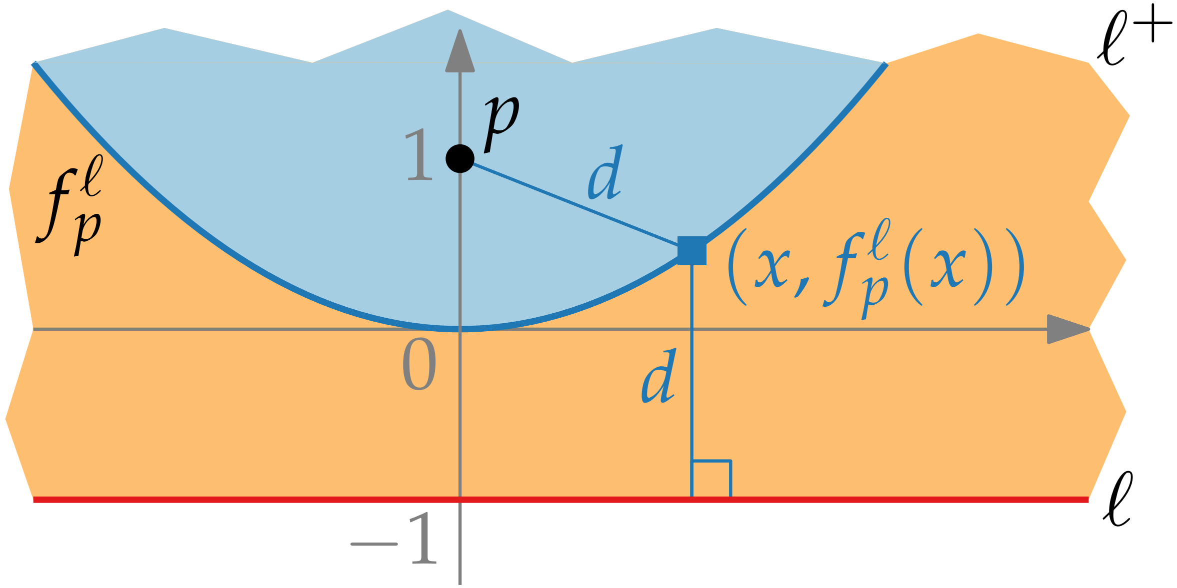

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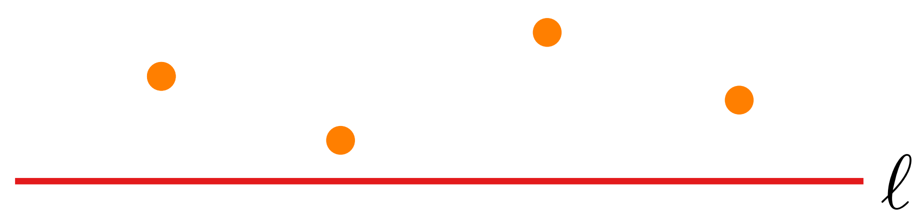

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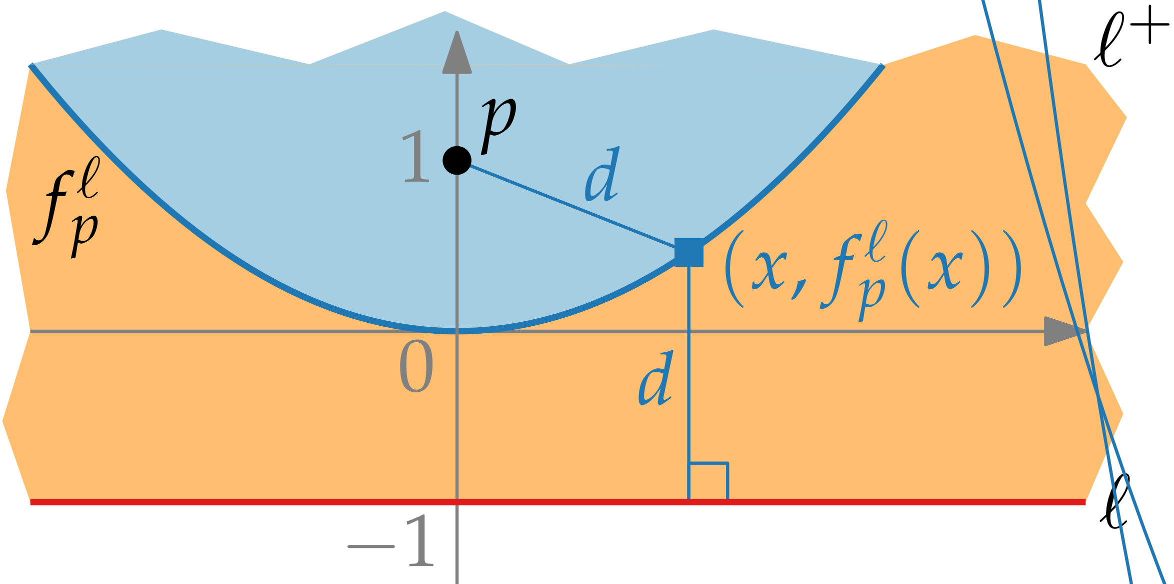

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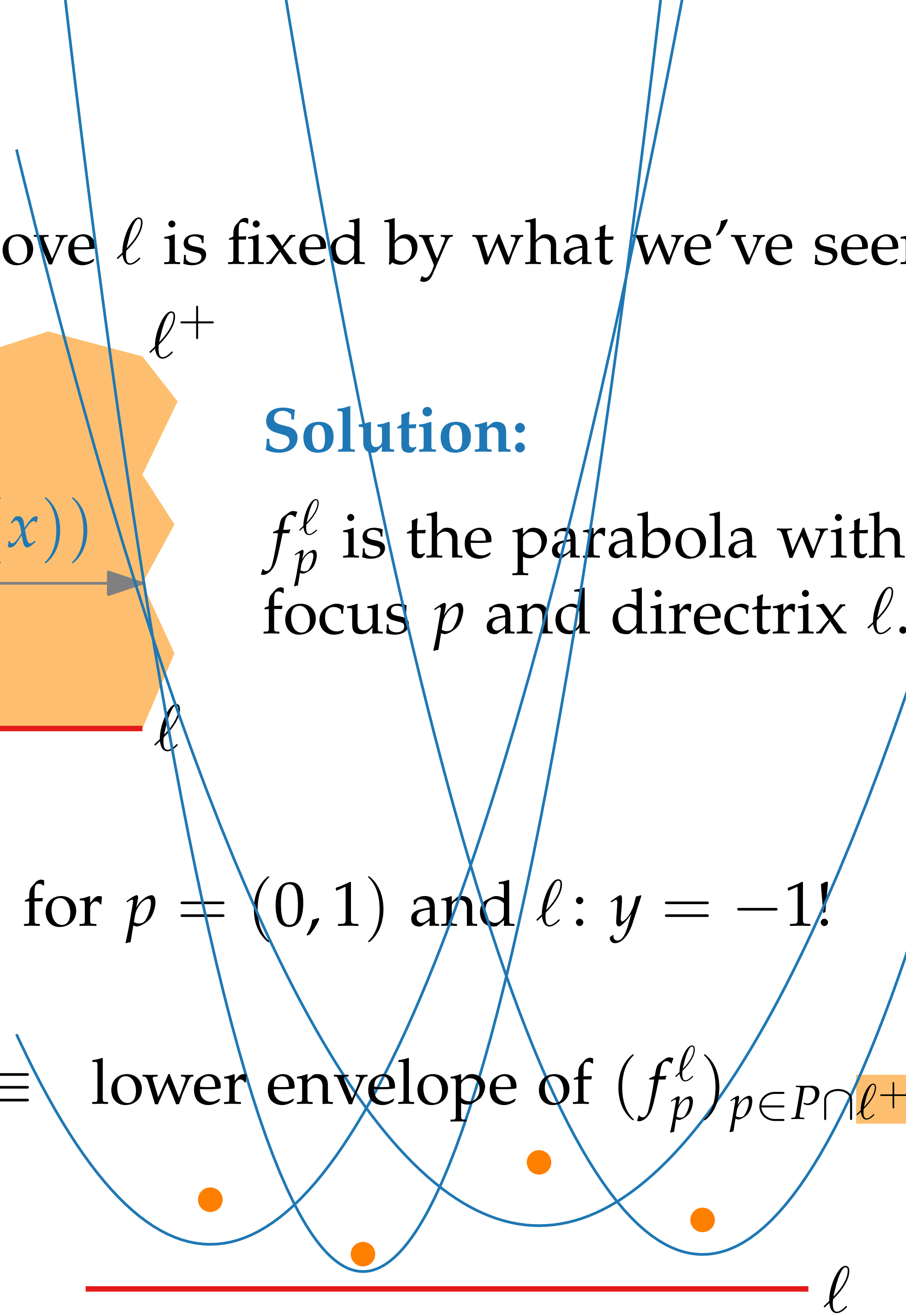

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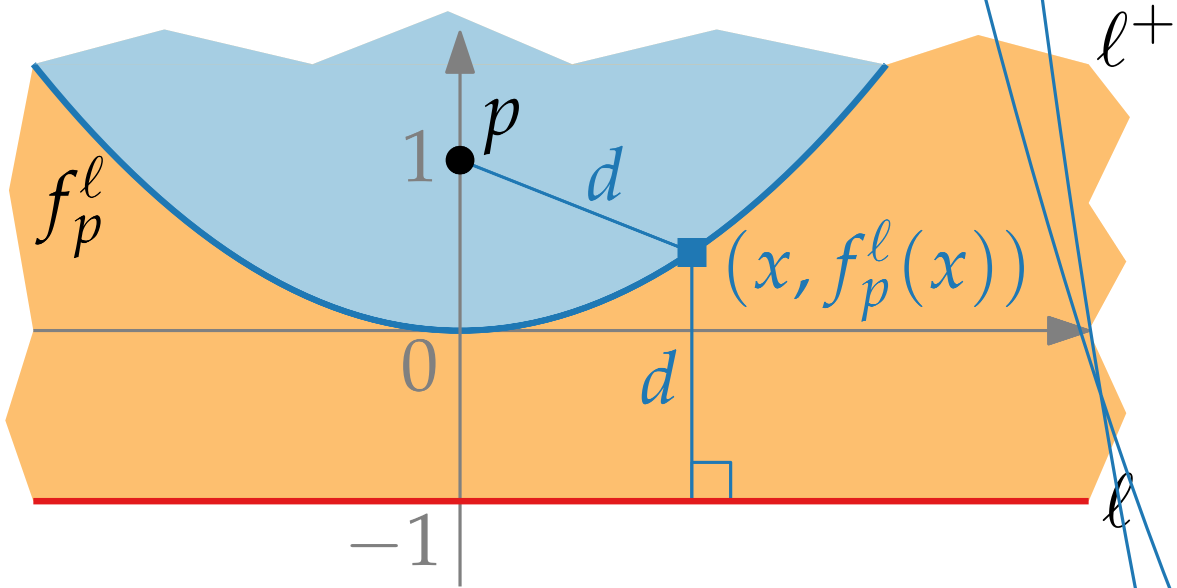

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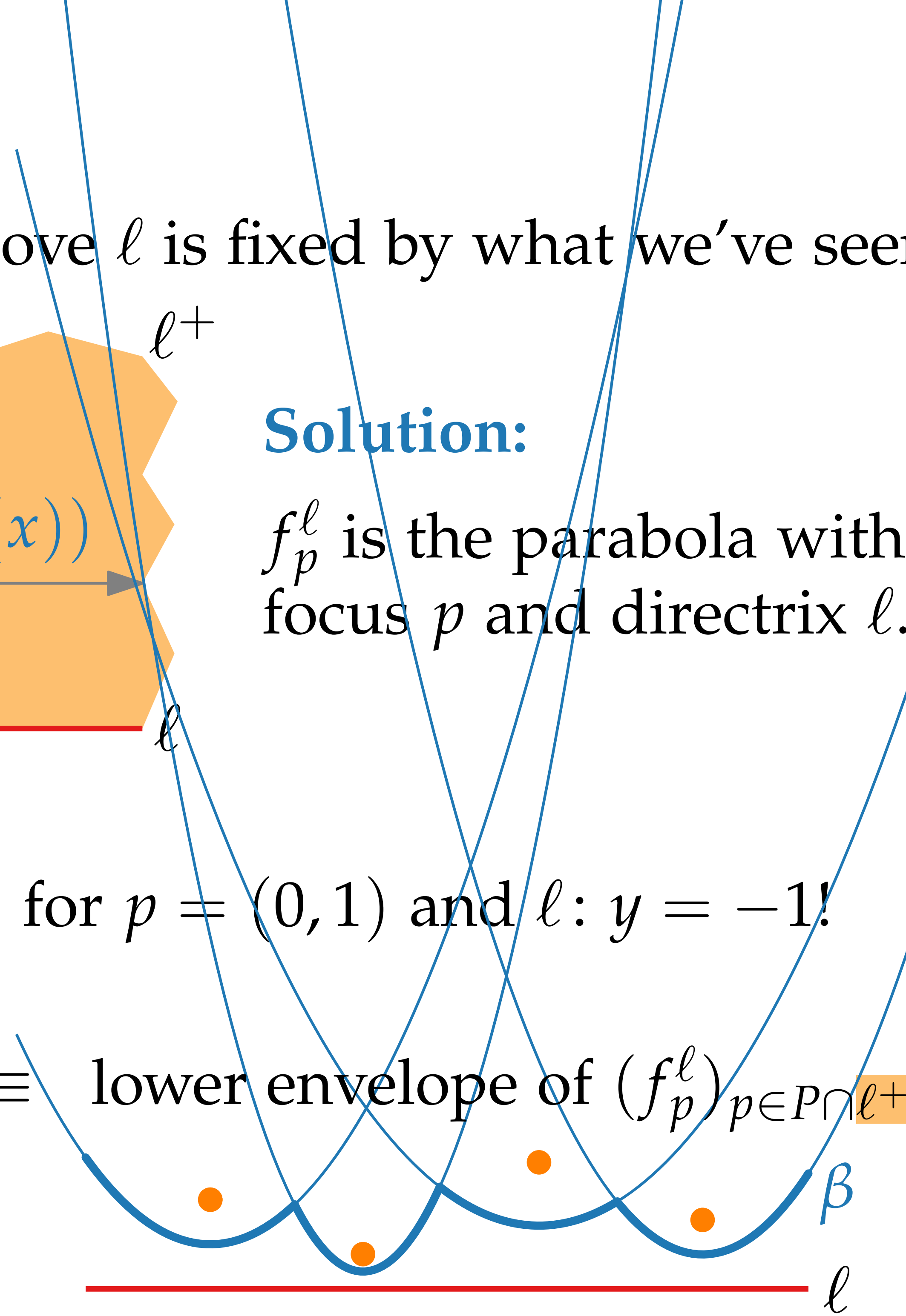

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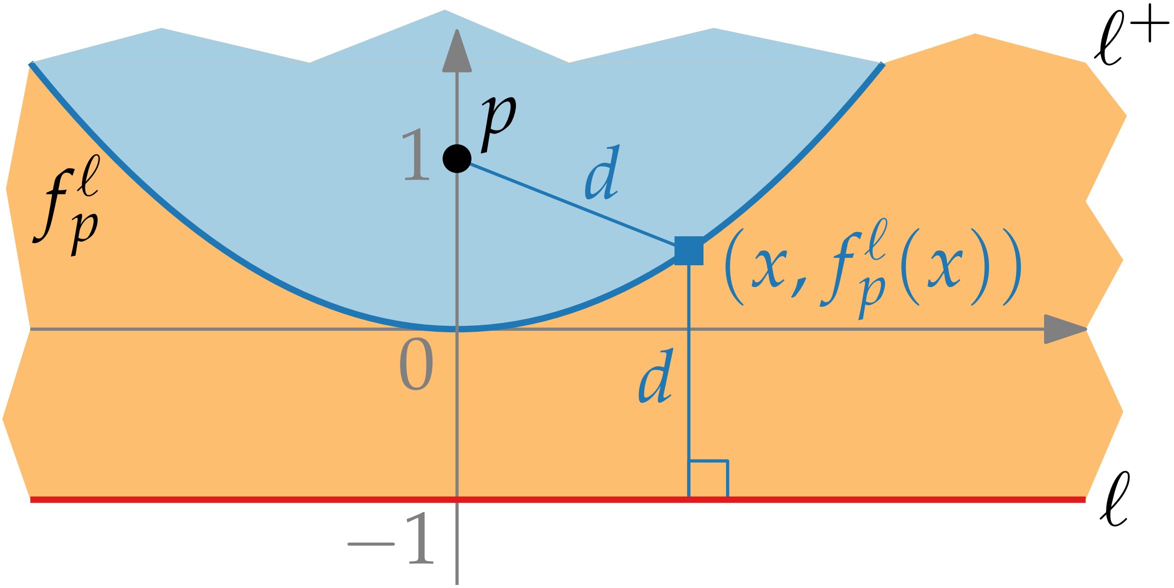

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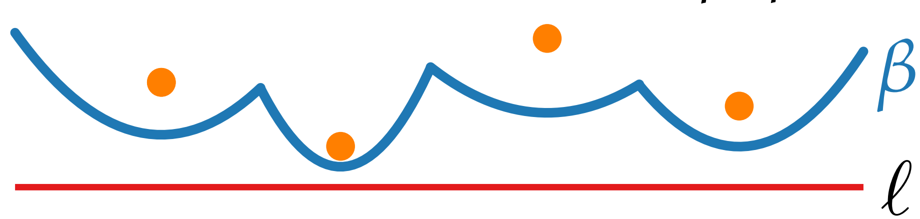

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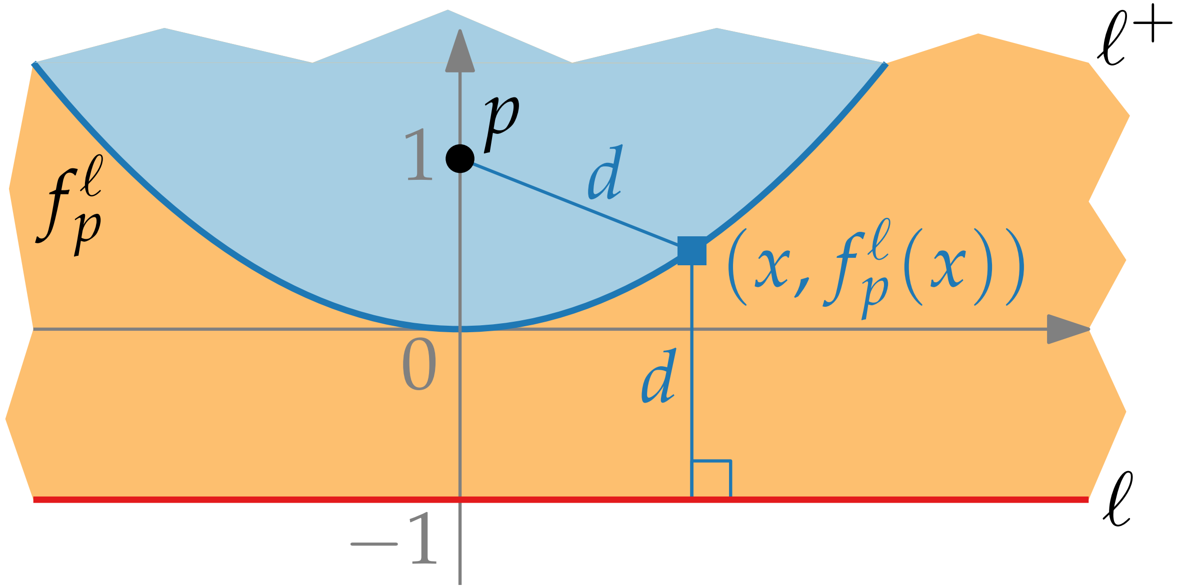

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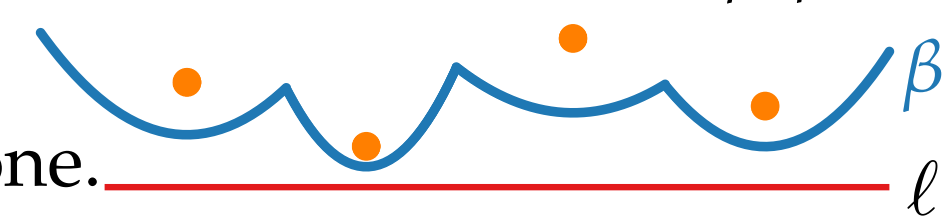
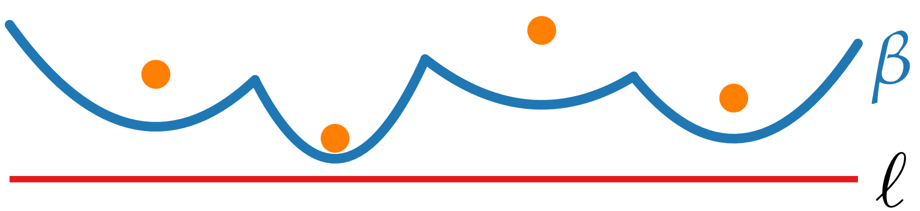
Observation. β is x-monotone.



The Beachline β

Question: What does β have to do with $\mathrm{Vor}(P)$?



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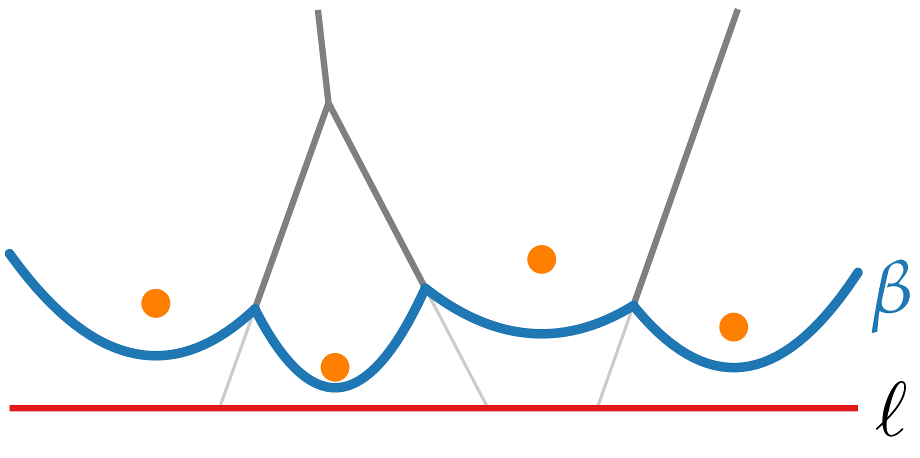

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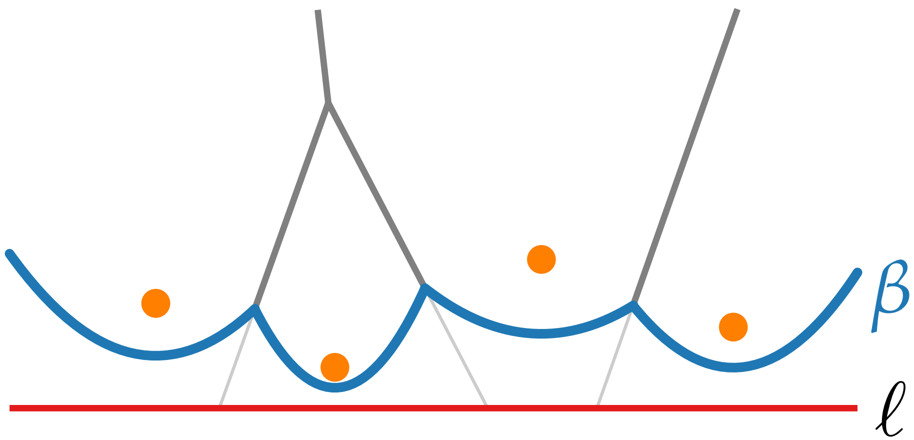

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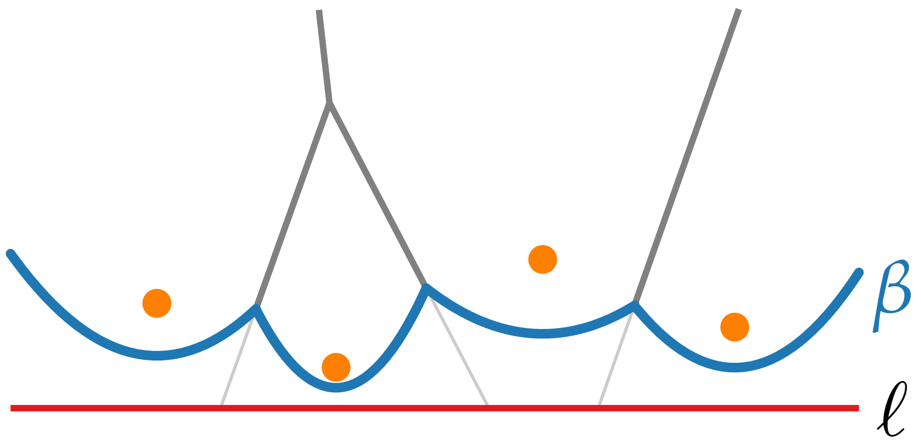

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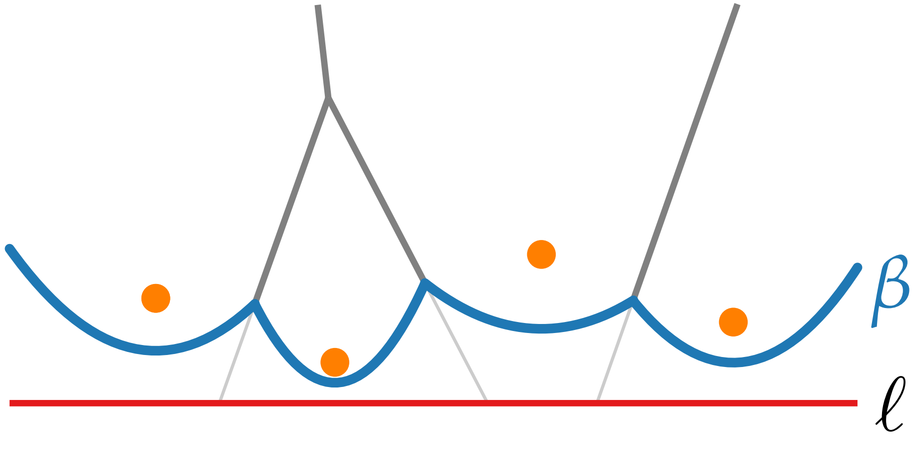

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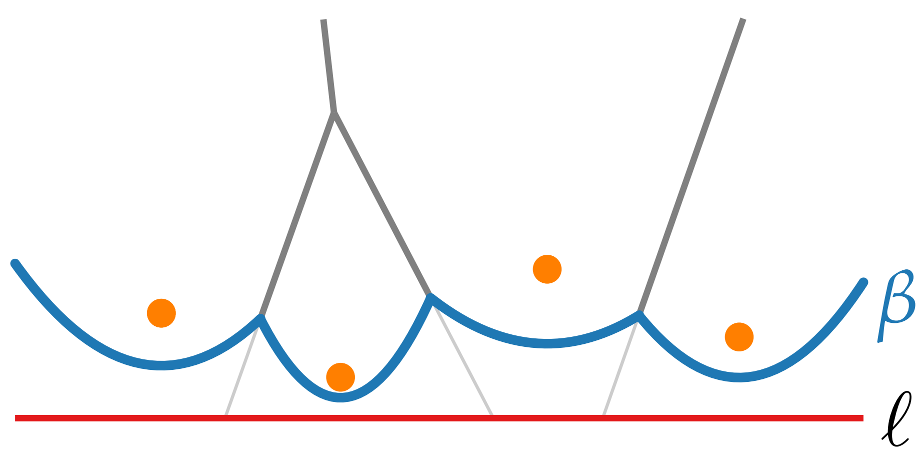

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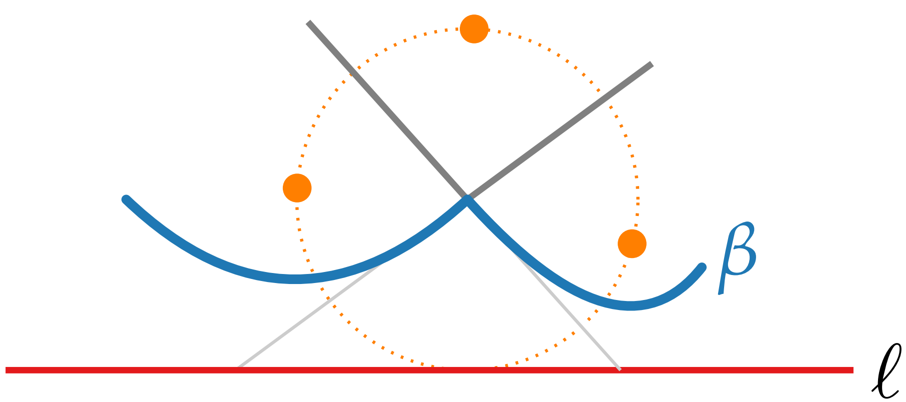

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Lemma. The Voronoi vtc correspond 1:1 to circle events.

Computational Geometry

Lecture 7:
Voronoi Diagrams
or
The Post-Office Problem

Part V:
Fortune's Sweep

Philipp Kindermann

Winter Semester 2020

Fortune's Sweep

VoronoiDiagram($P \subset \mathbb{R}^2$)

$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted by y-coord.

Fortune's Sweep

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while not $\mathcal{Q}$.empty() **do**

  $p \leftarrow \mathcal{Q}$.ExtractMax()

Fortune's Sweep

VoronoiDiagram($P \subset \mathbb{R}^2$)

$\mathcal{Q} \leftarrow$ new PriorityQueue(P) // site events sorted by y-coord.

$\mathcal{T} \leftarrow$ new BalancedBinarySearchTree() // sweep status (β)

$\mathcal{D} \leftarrow$ new DCEL() // to-be Vor(P)

while not $\mathcal{Q}$.empty() **do**

  $p \leftarrow \mathcal{Q}$.ExtractMax()

  if p site event **then**

    HandleSiteEvent(p)

Fortune's Sweep

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VoronoiDiagram(P ⊂ ℝ²)
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         α ← arc on β that will disappear
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  treat remaining int. nodes of T (≡ unbnd. edges of Vor(P))
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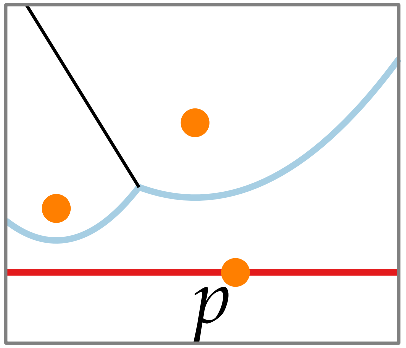
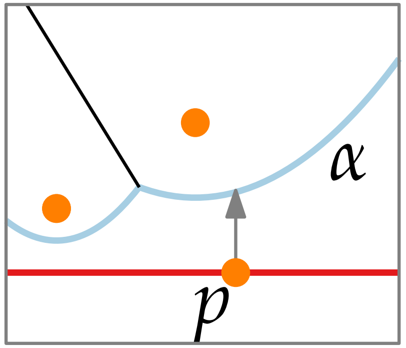
Handling Events



HandleSiteEvent(point p)

HandleCircleEvent(arc α)

Handling Events



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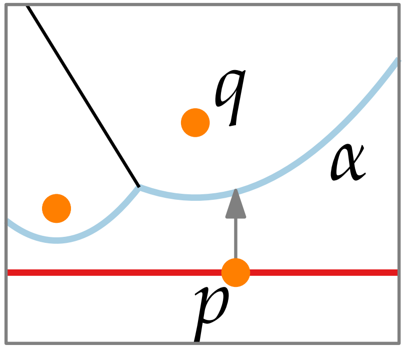

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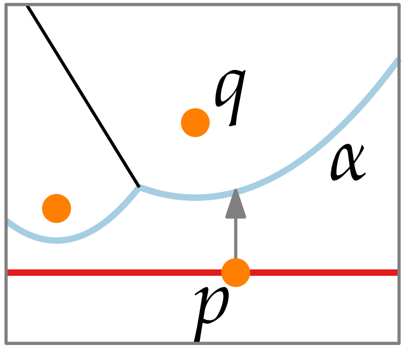

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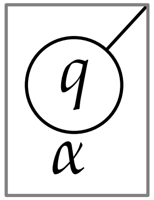

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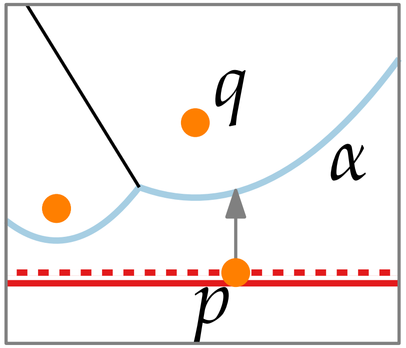

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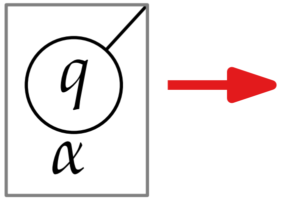

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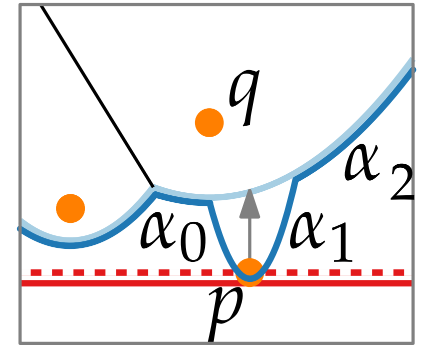

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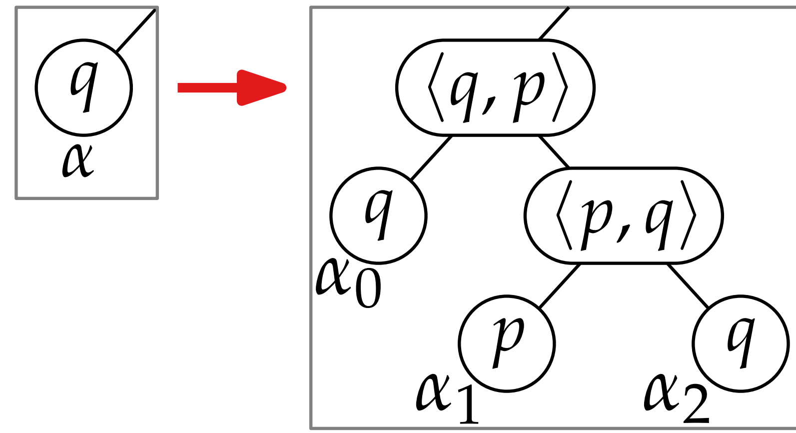

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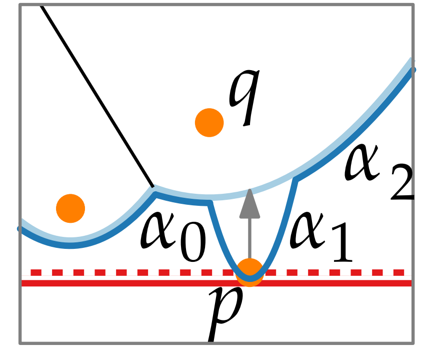

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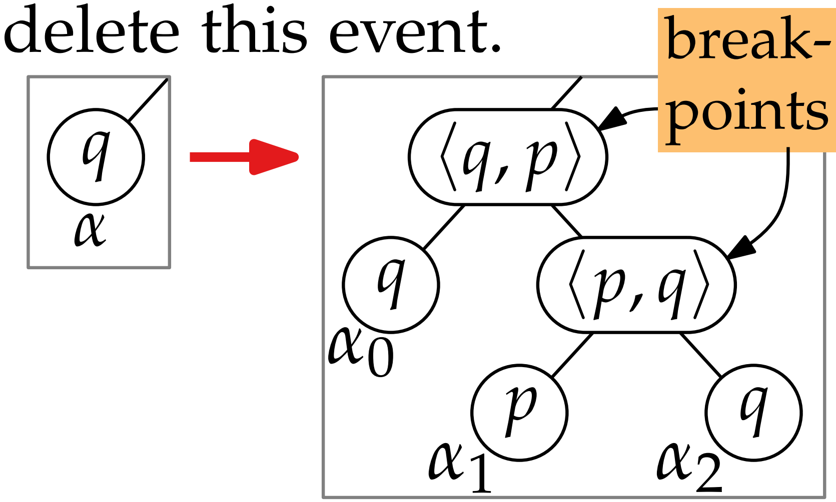

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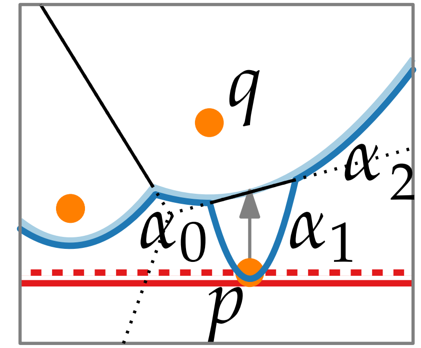

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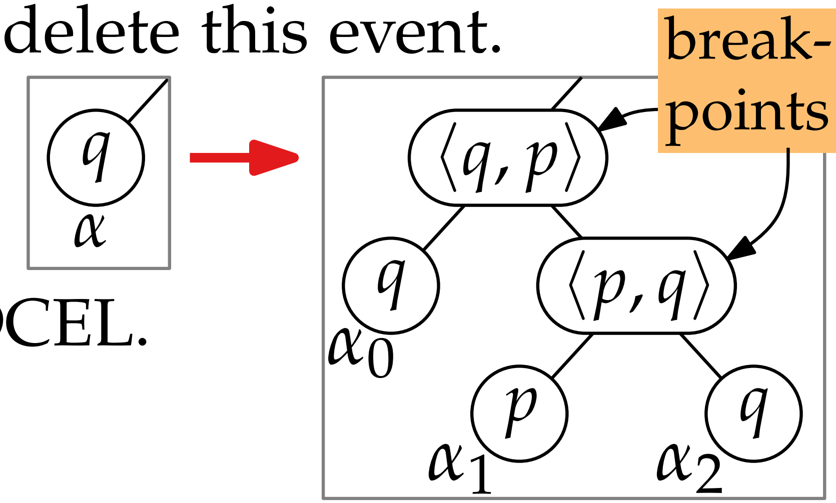

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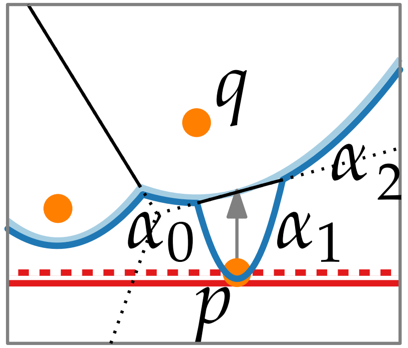

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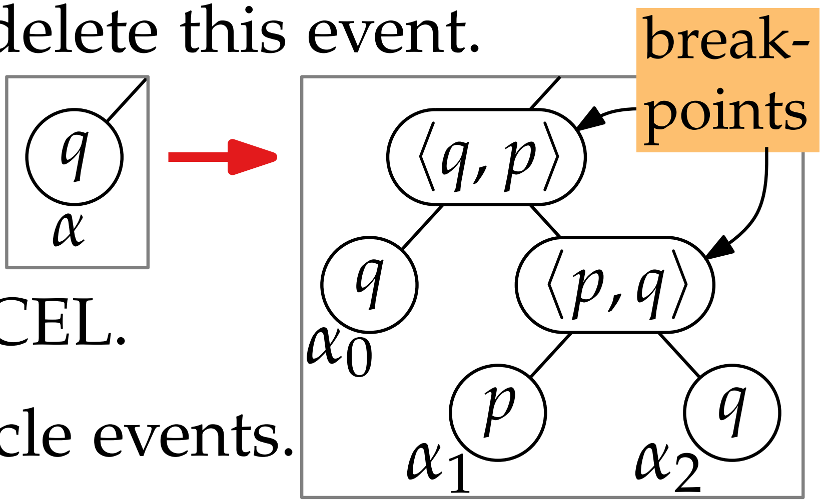

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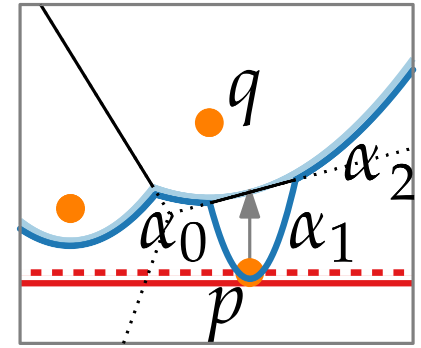

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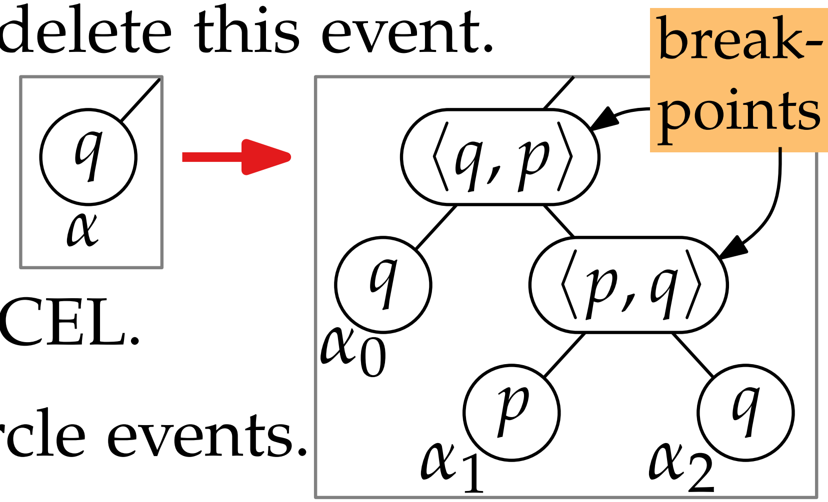

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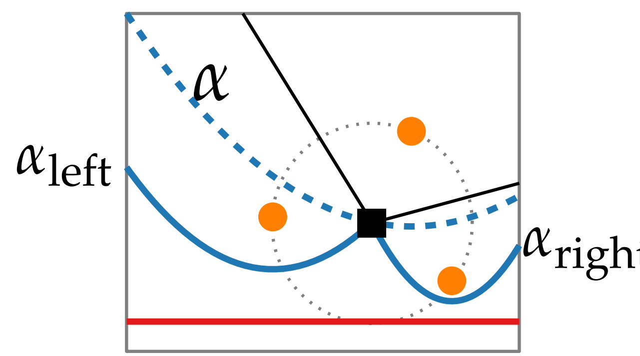

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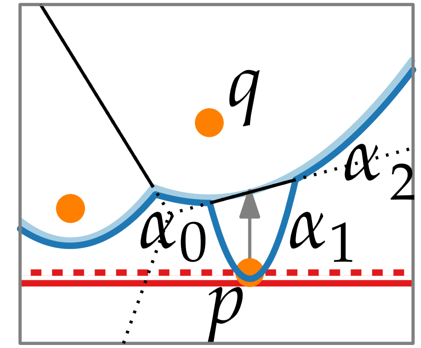

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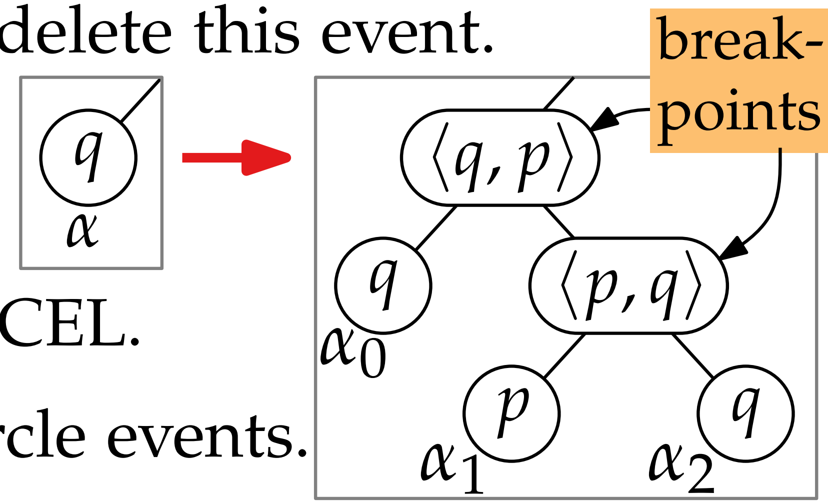

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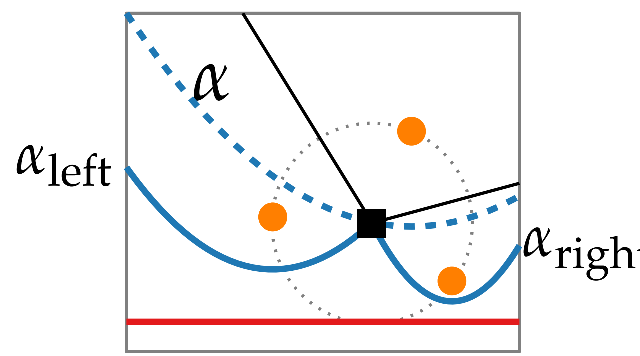

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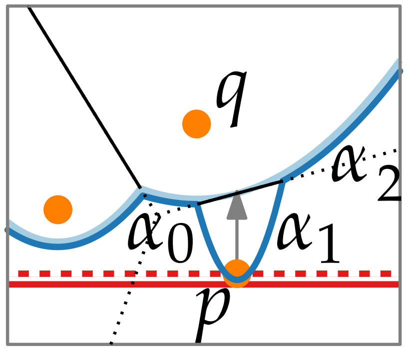

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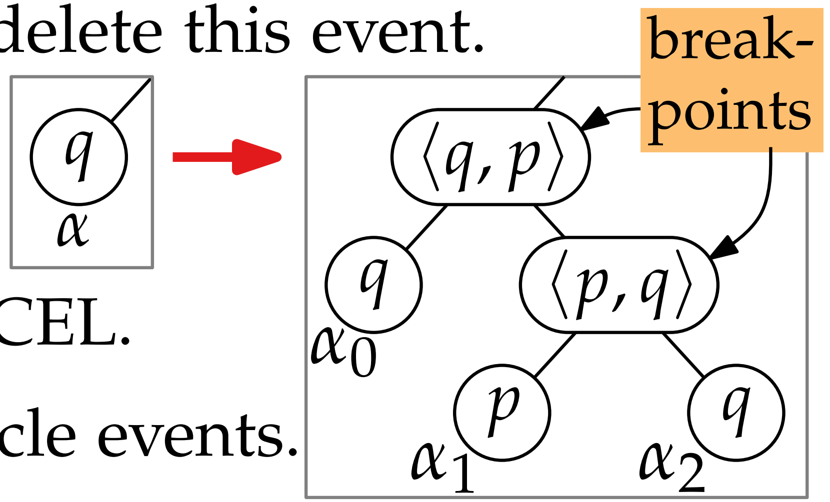

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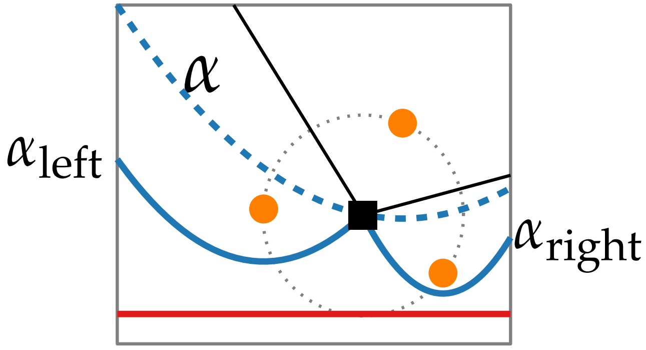

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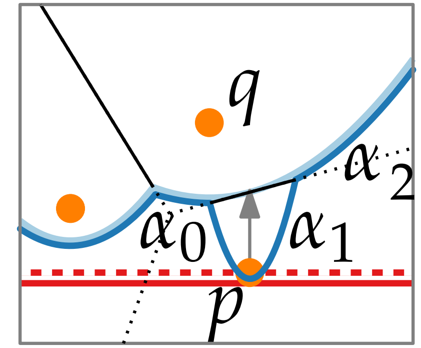

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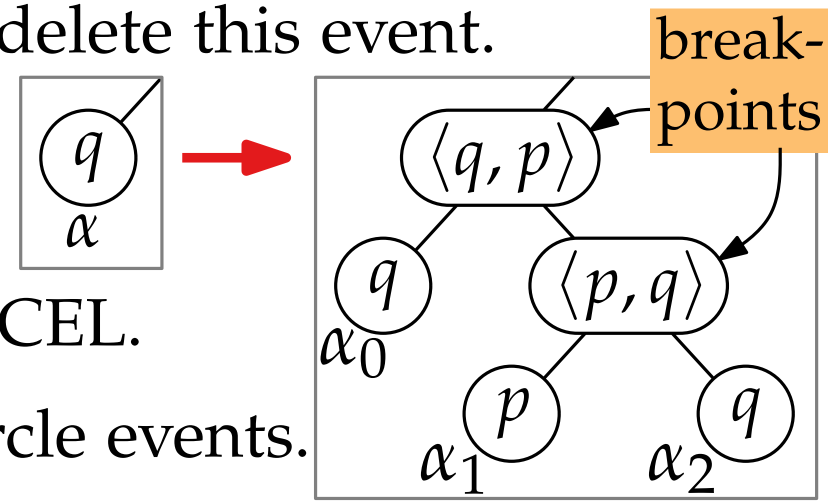

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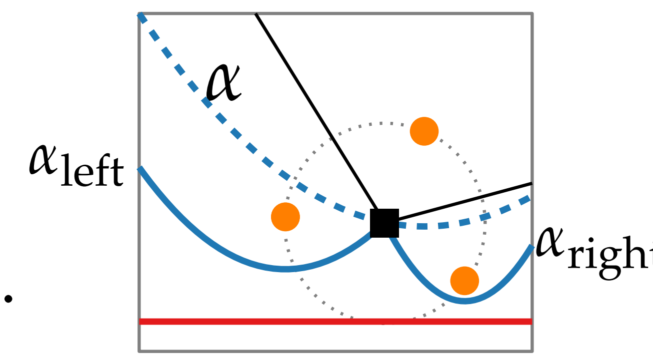

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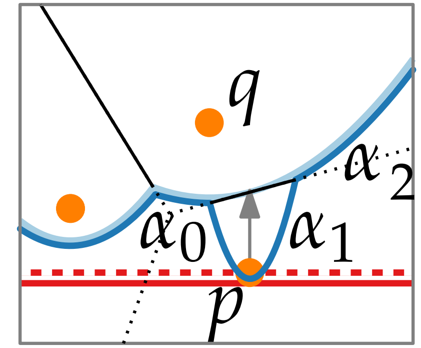

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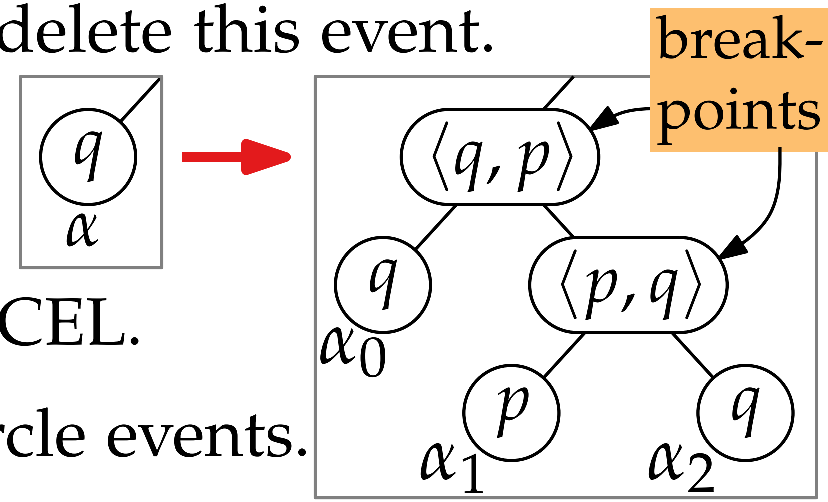

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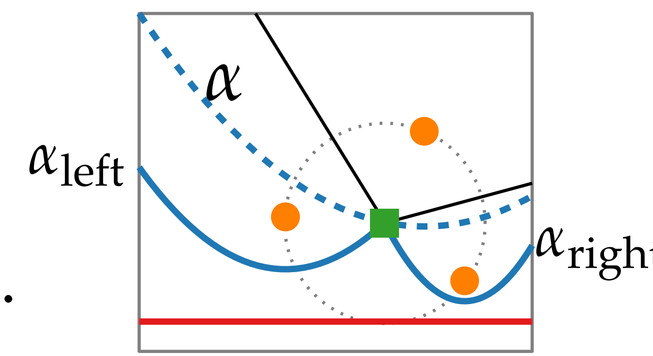

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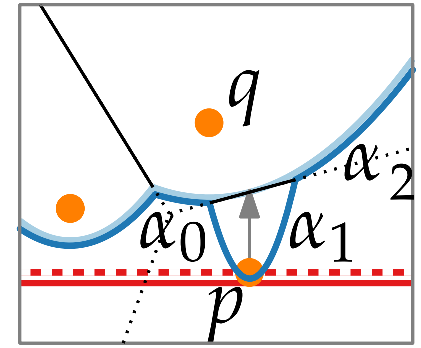

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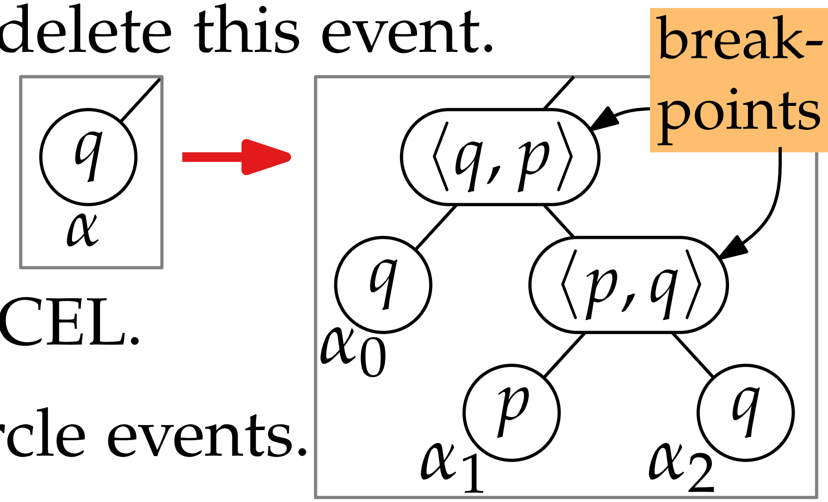

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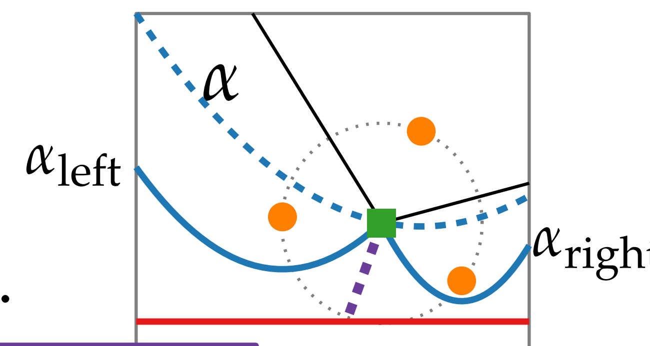

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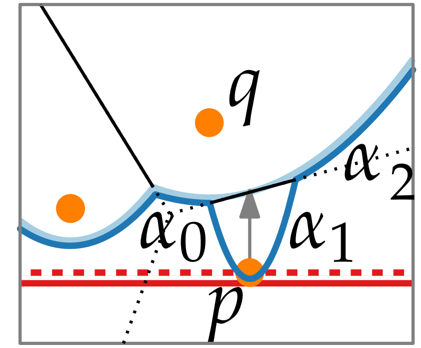

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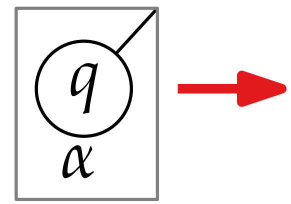

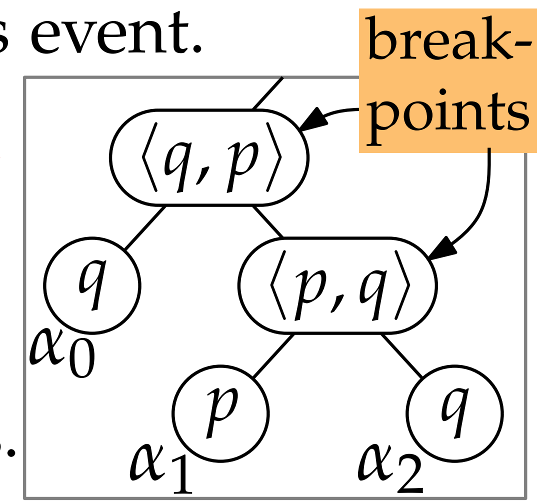

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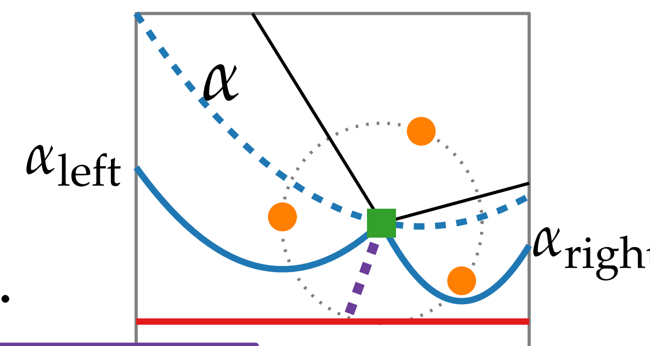

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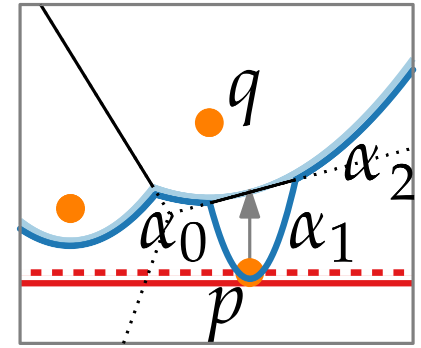

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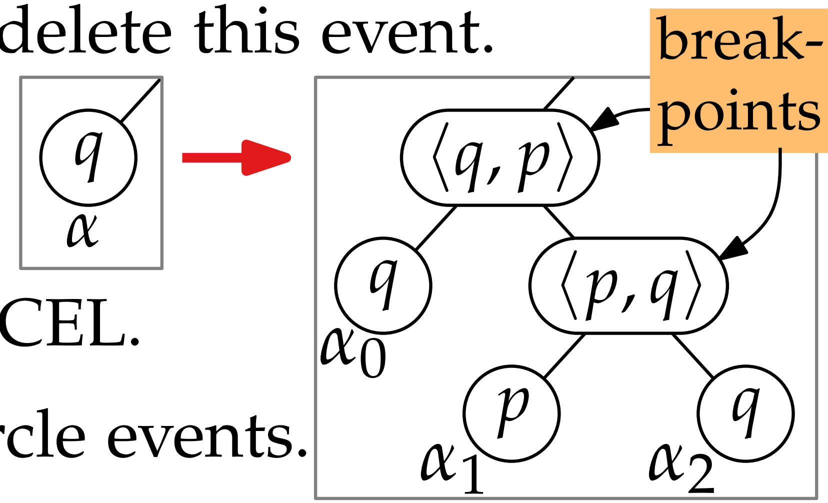

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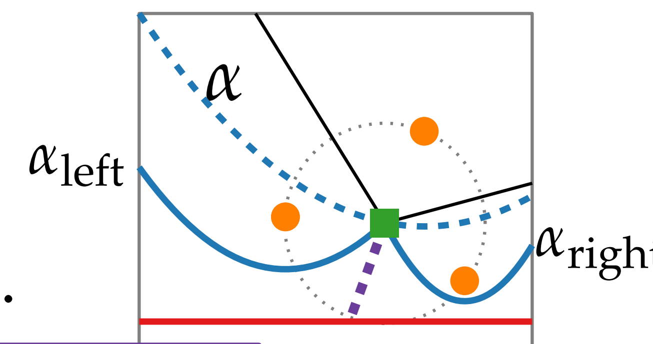

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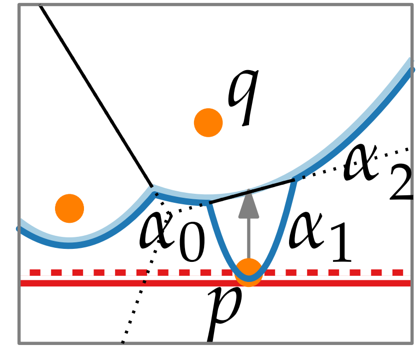
Running time?

Handling Events



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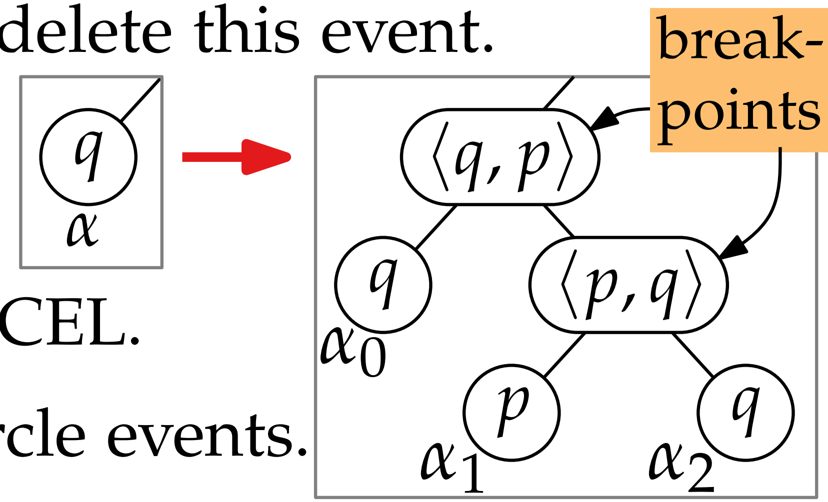

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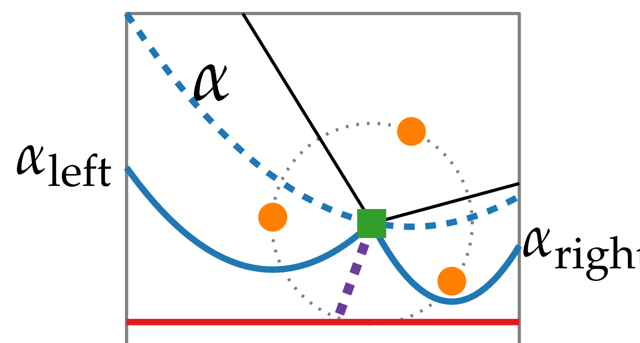

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Running time? $O(\log n)$ per event…

Running Time?

```
VoronoiDiagram(P ⊂ ℝ²)
  Q ← new PriorityQueue(P)   // site events sorted by y-coord.
  T ← new BalancedBinarySearchTree()   // sweep status (β)
  D ← new DCEL()   // to-be Vor(P)
  while not Q.empty() do
      p ← Q.ExtractMax()
      if p site event then
          HandleSiteEvent(p)
      else
          α ← arc on β that will disappear
          HandleCircleEvent(α)
  treat remaining int. nodes of T (≡ unbnd. edges of Vor(P))
  return D
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 while not Q.empty() do
     p ← Q.ExtractMax()
     if p site event then
         HandleSiteEvent(p)    exactly n such events
     else
         α ← arc on β that will disappear
         HandleCircleEvent(α)
 treat remaining int. nodes of T (≡ unbnd. edges of Vor(P))
 return D
```

Running Time?

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VoronoiDiagram(P ⊂ ℝ²)
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      p ← Q.ExtractMax()
      if p site event then
          HandleSiteEvent(p)    exactly n such events
      else
          α ← arc on β that will disappear
          HandleCircleEvent(α)  at most 2n − 5 such events
  treat remaining int. nodes of T (≡ unbnd. edges of Vor(P))
  return D
```

Summary

Theorem. Given a set P of n pts in the plane, Fortune's sweep computes $\mathrm{Vor}(P)$ in $O(n \log n)$ time and $O(n)$ space.

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Steven Fortune. A sweepline algorithm for Voronoi diagrams. *Proc. 2nd Annual ACM Symposium on Computational Geometry.* Yorktown Heights, NY, pp. 313–322. 1986.