Computational Geometry

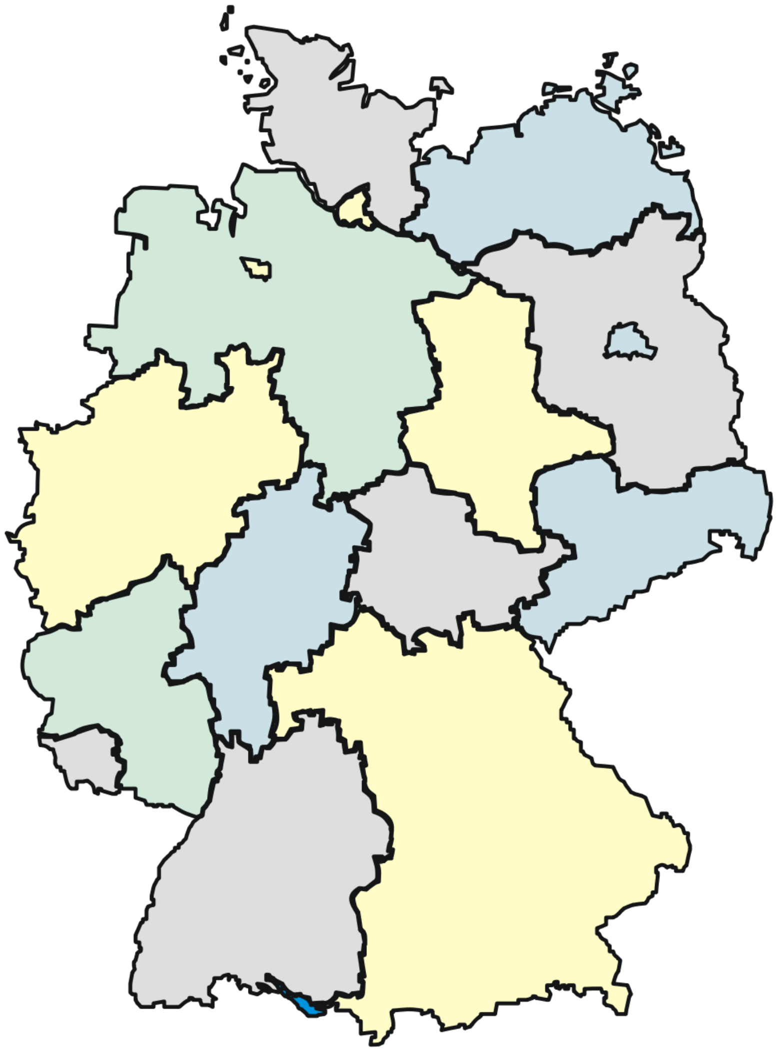
Lecture 6: Point Localization or Where am I?

Part I: Definition & First Approach

Philipp Kindermann

Winter Semester 2020

What's the Problem?



[Stefan-Xp, CC BY-SA 3.0, via wikipedia]

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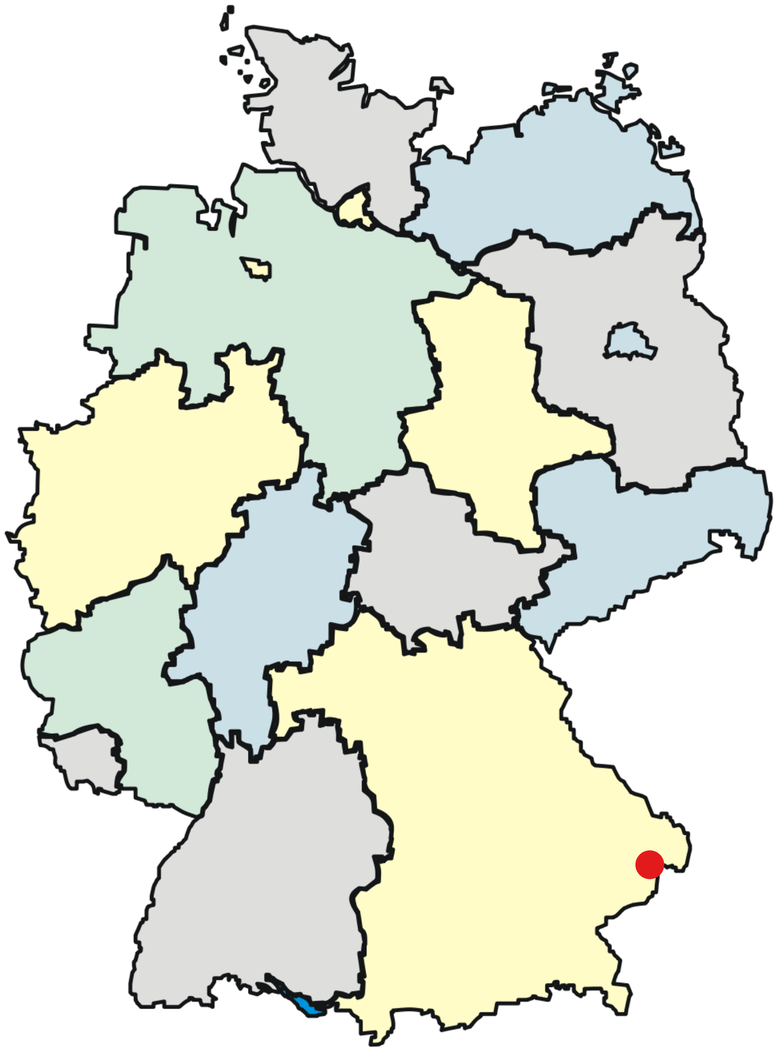

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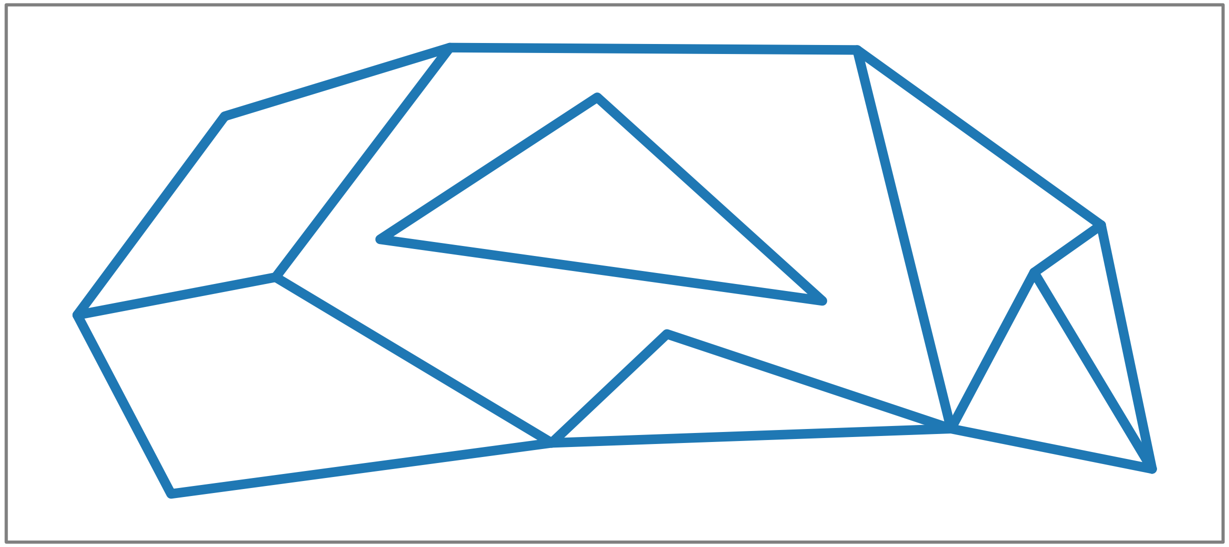

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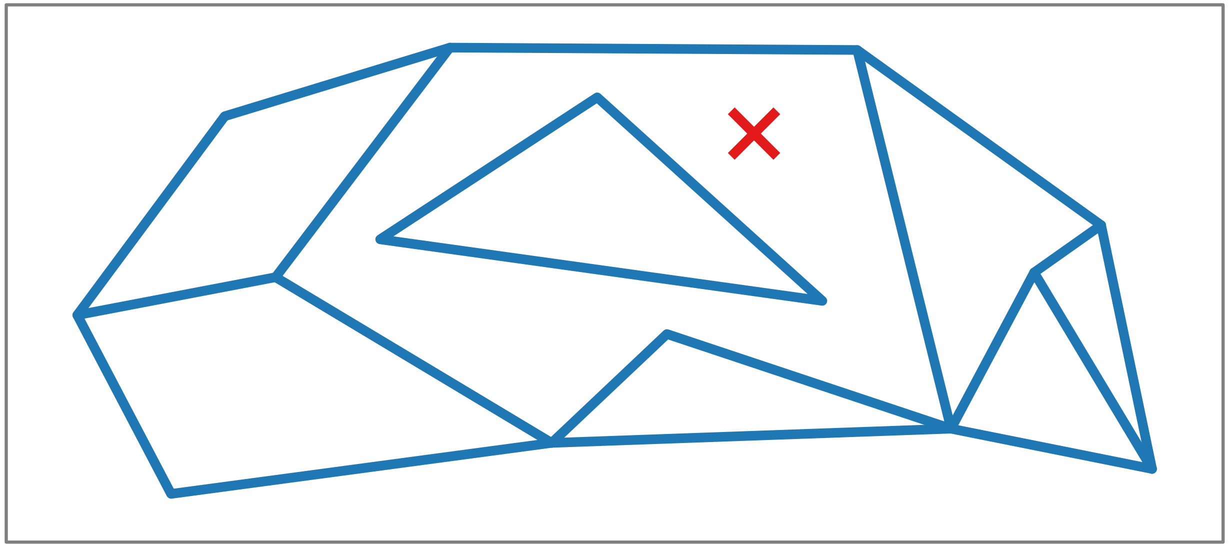

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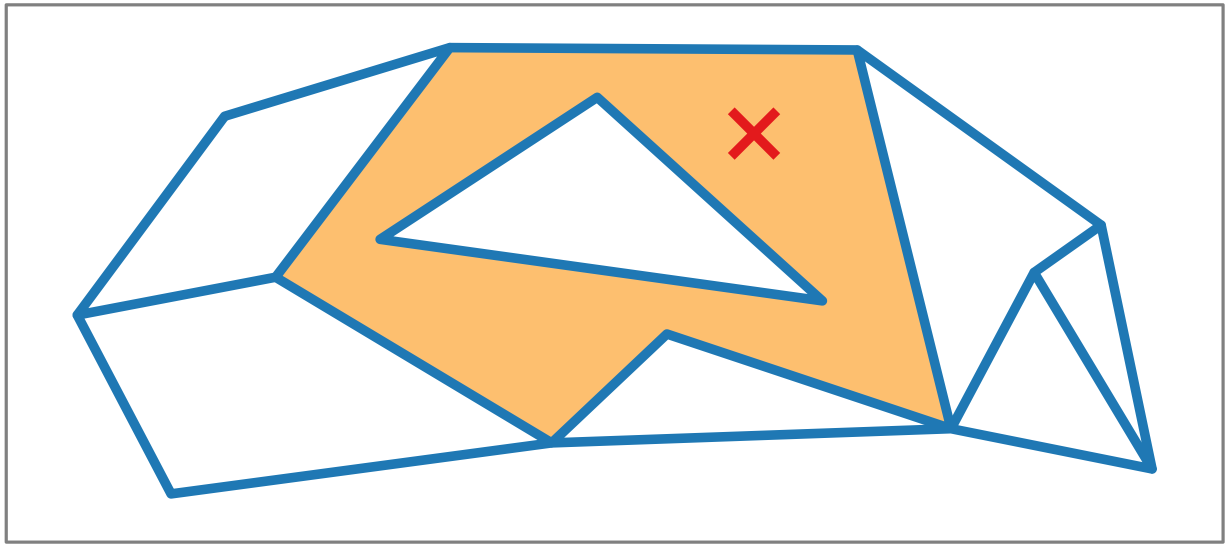

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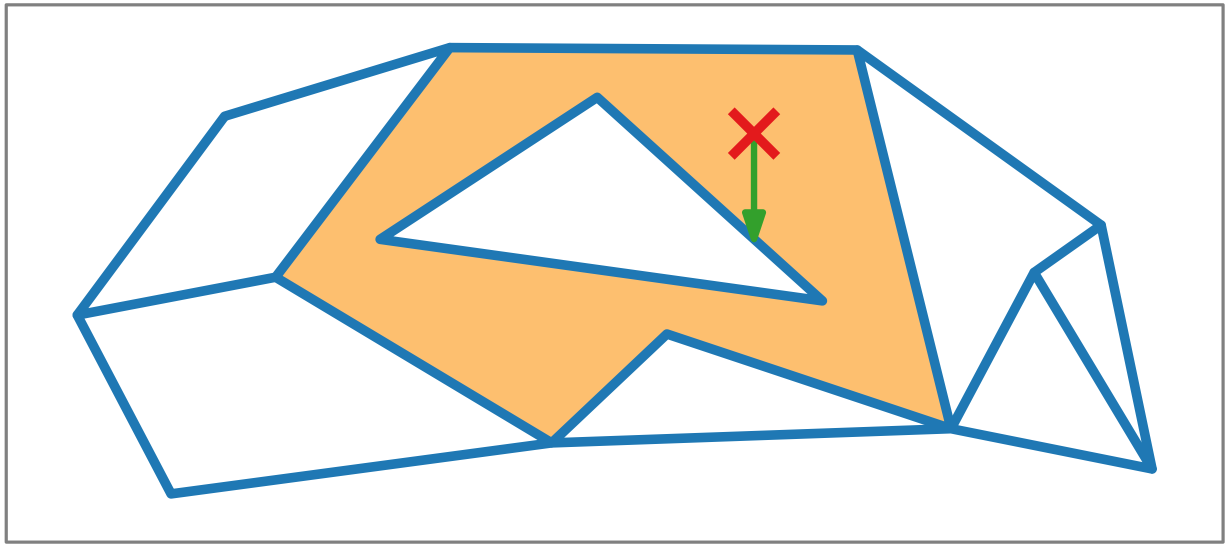

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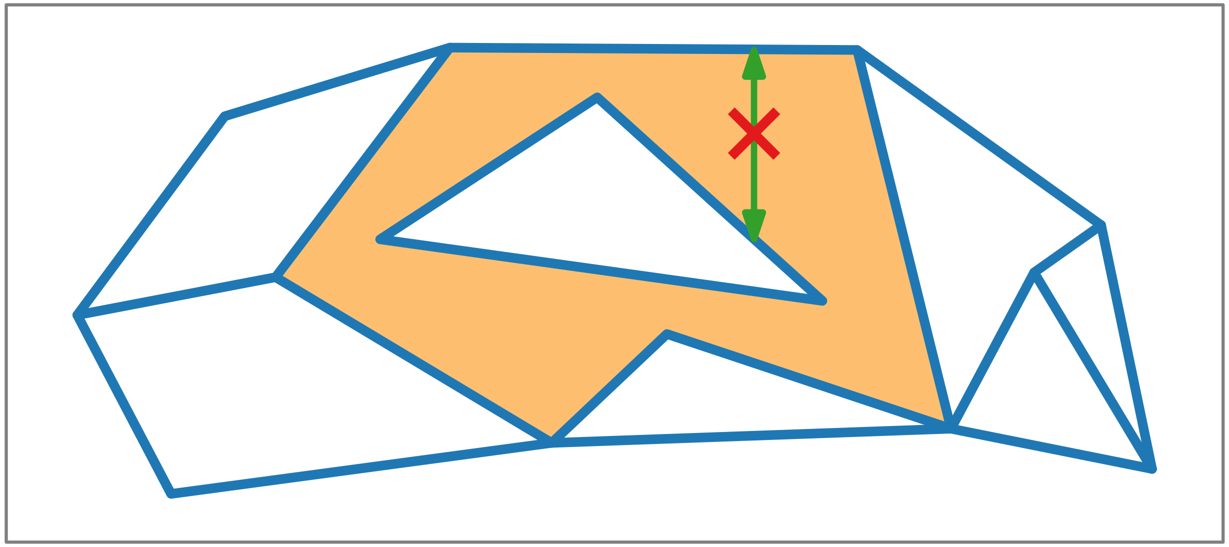

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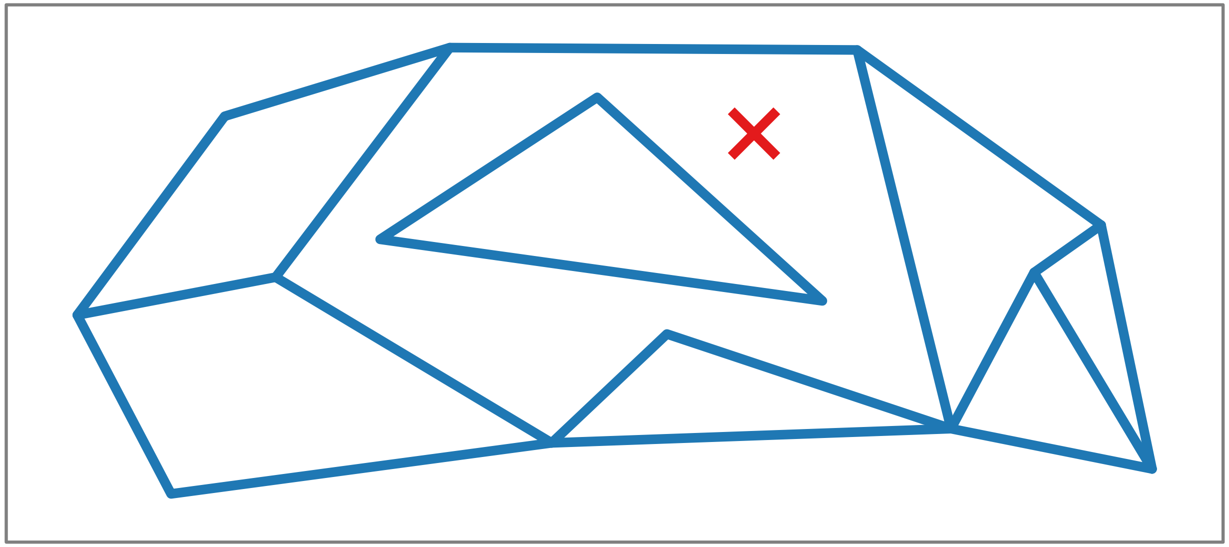

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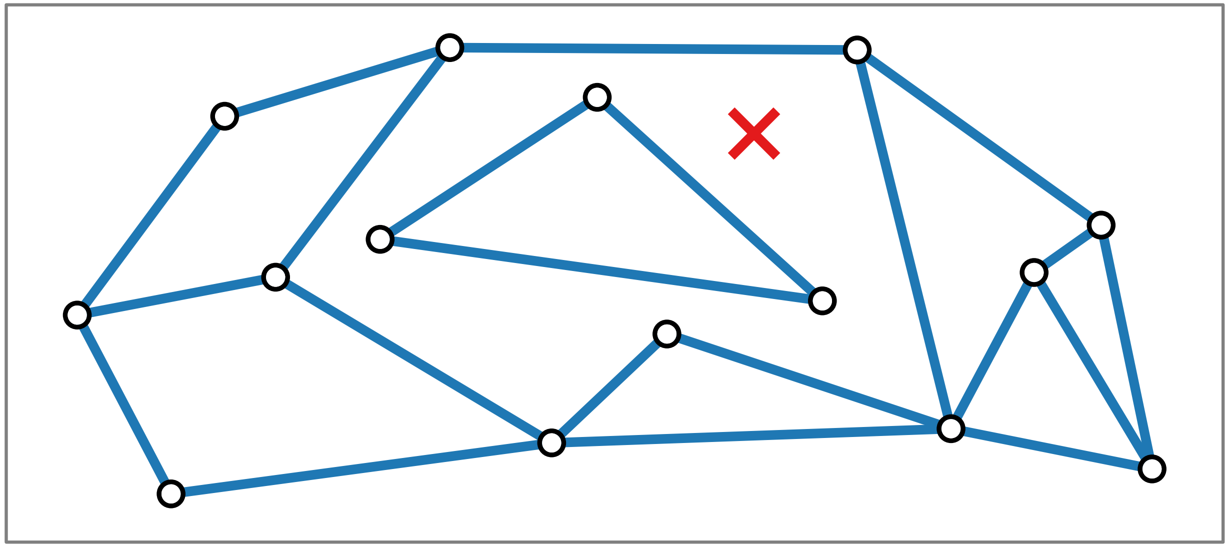

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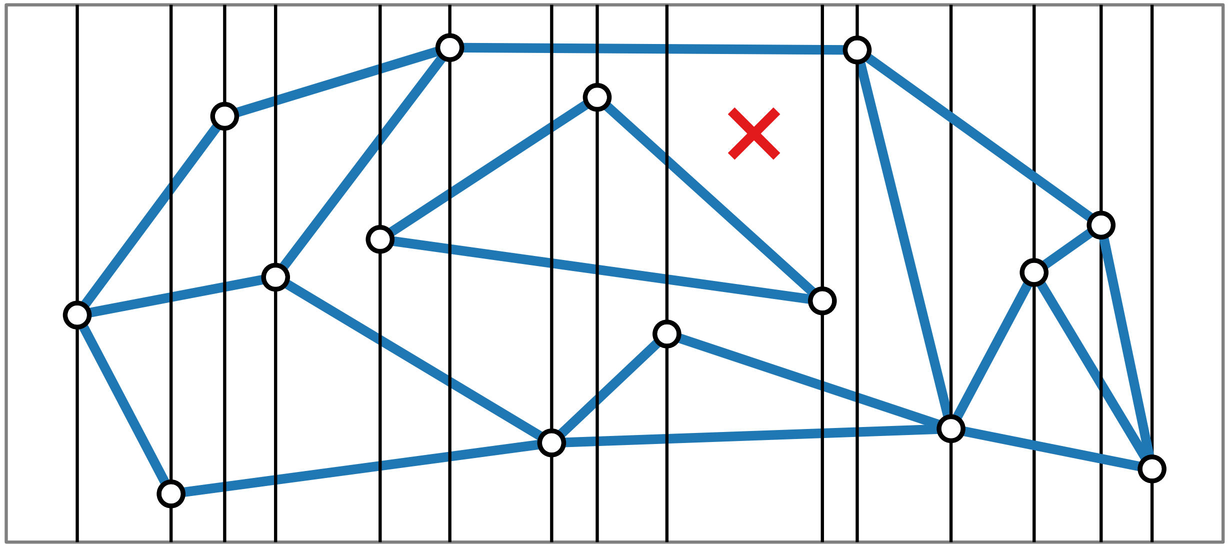

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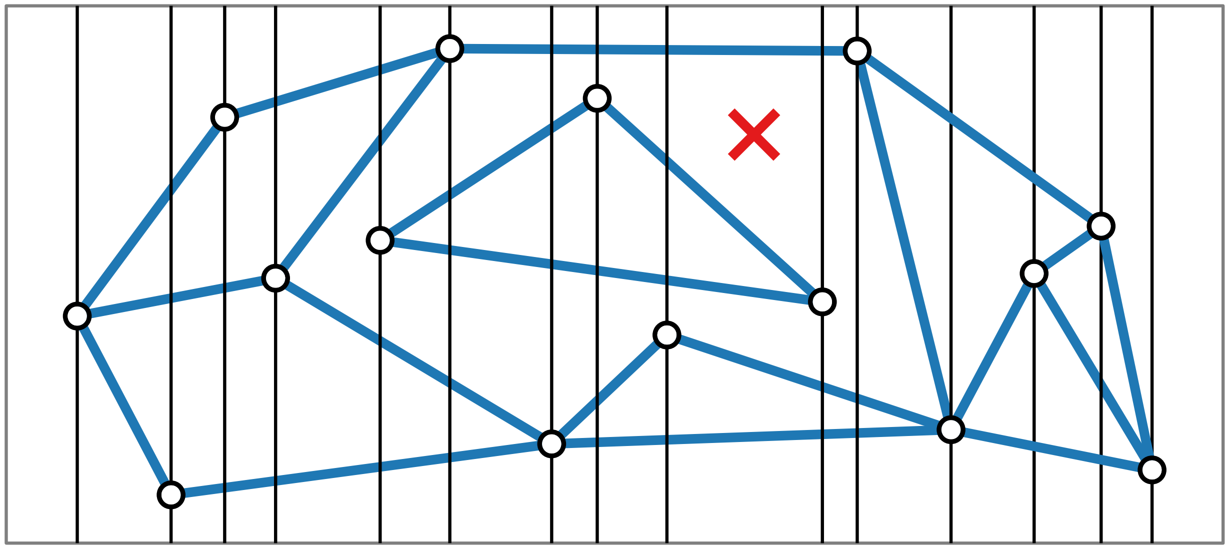

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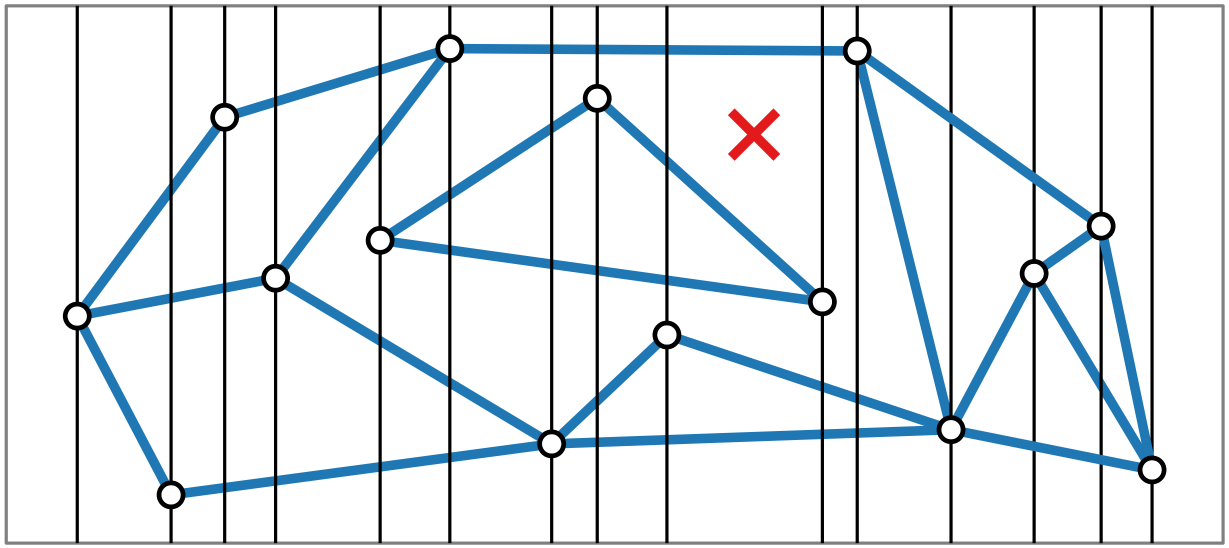

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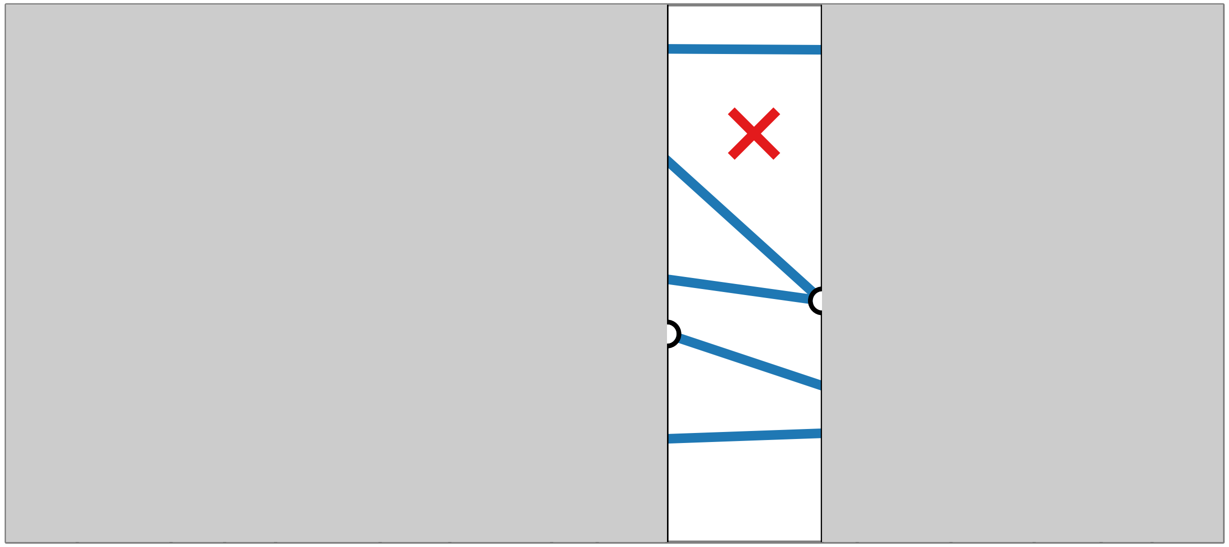

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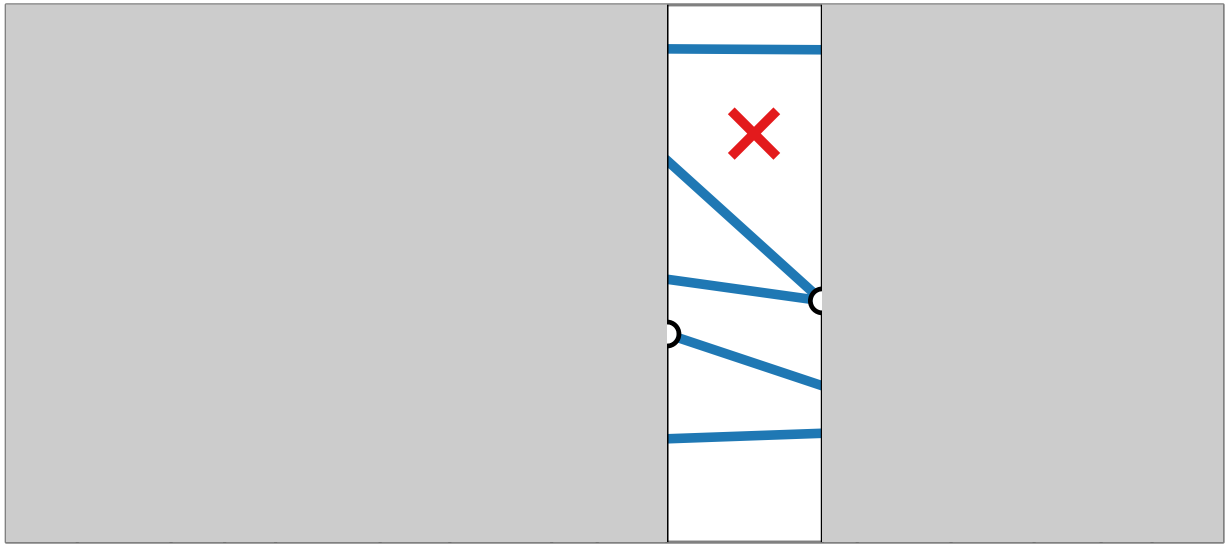

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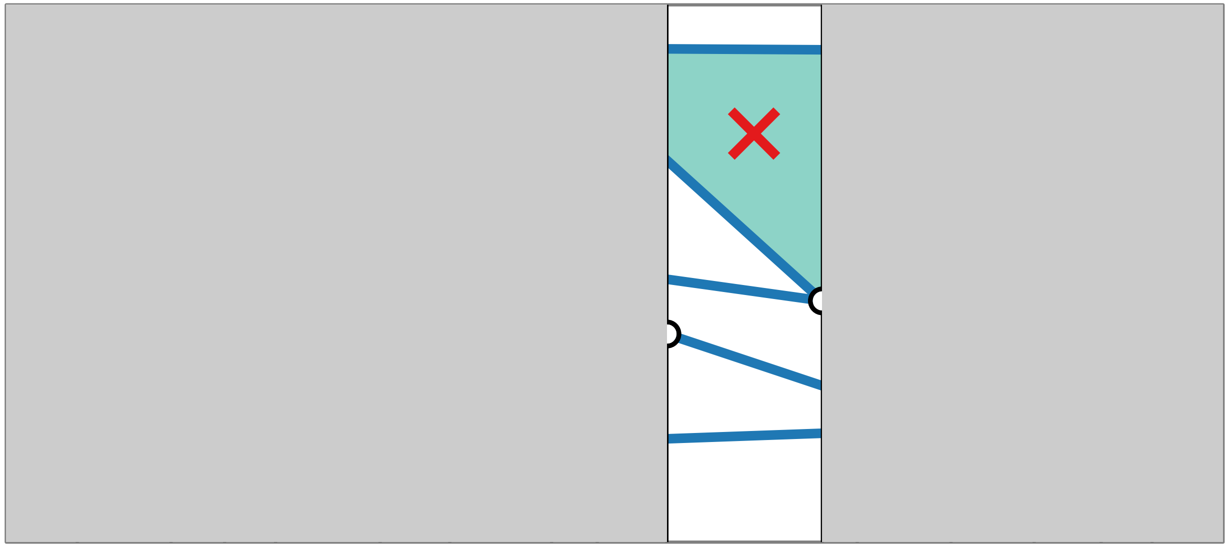

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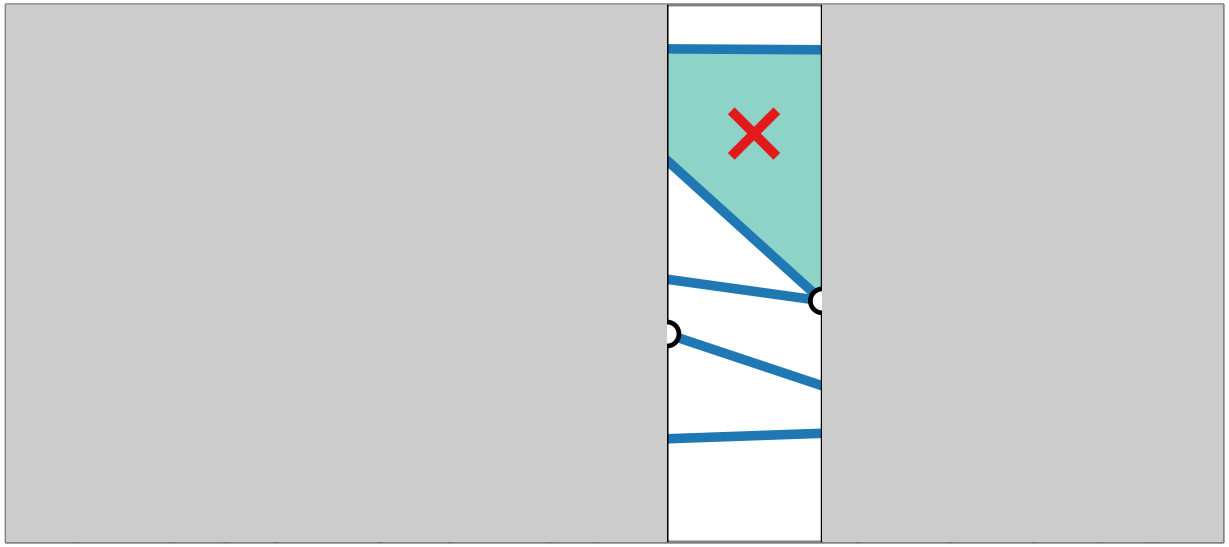

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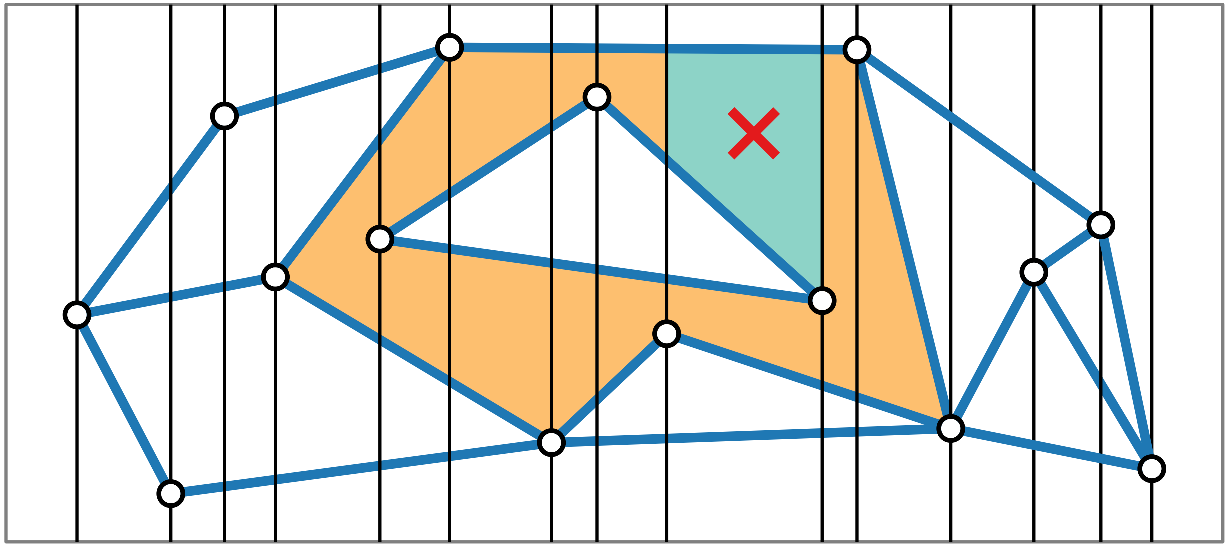

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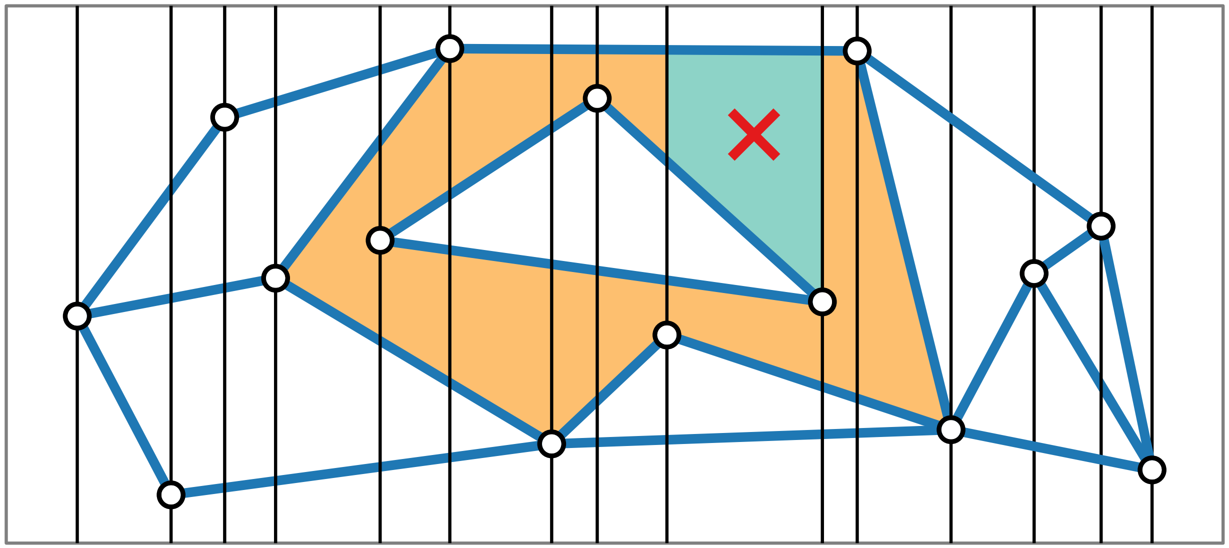

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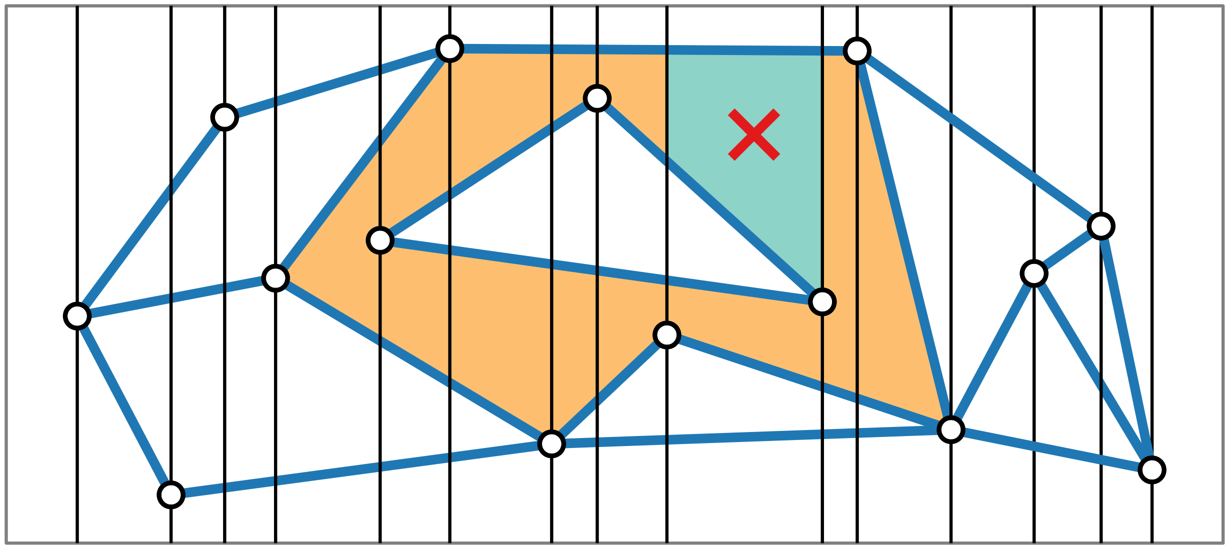

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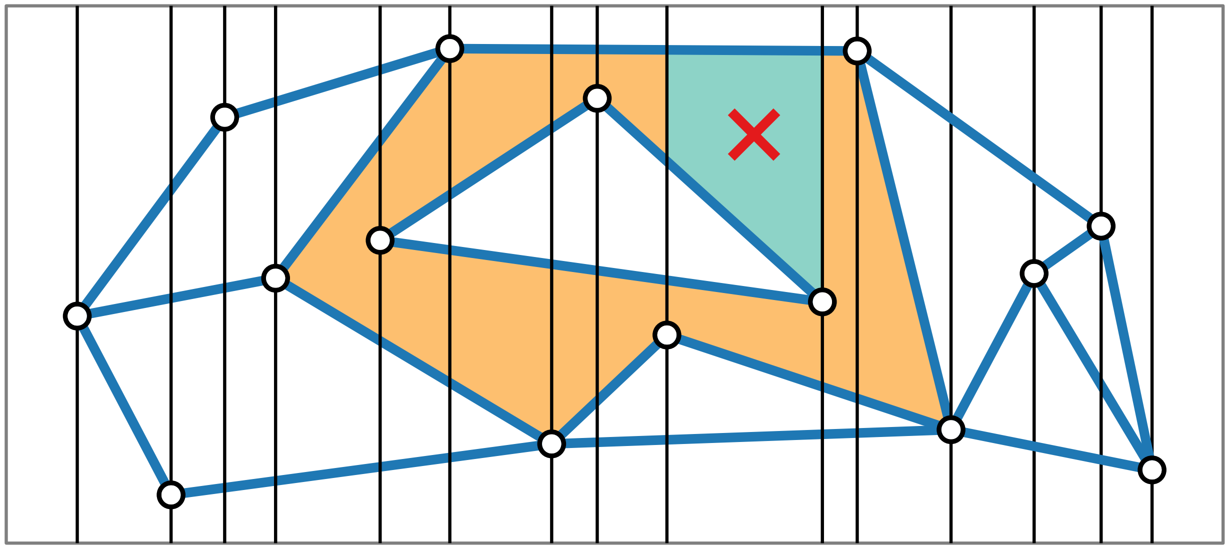

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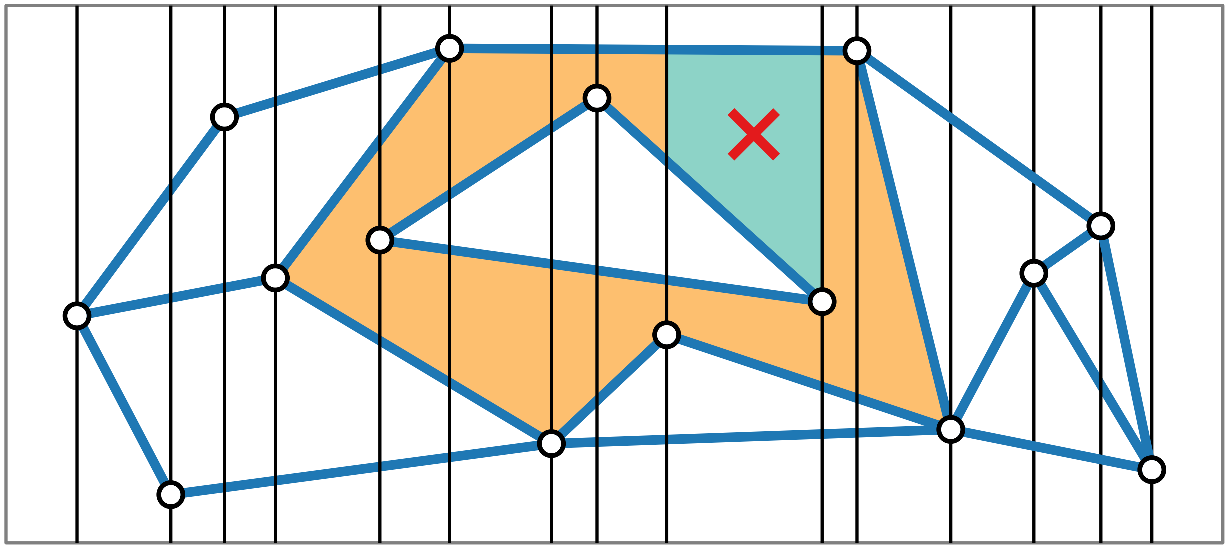

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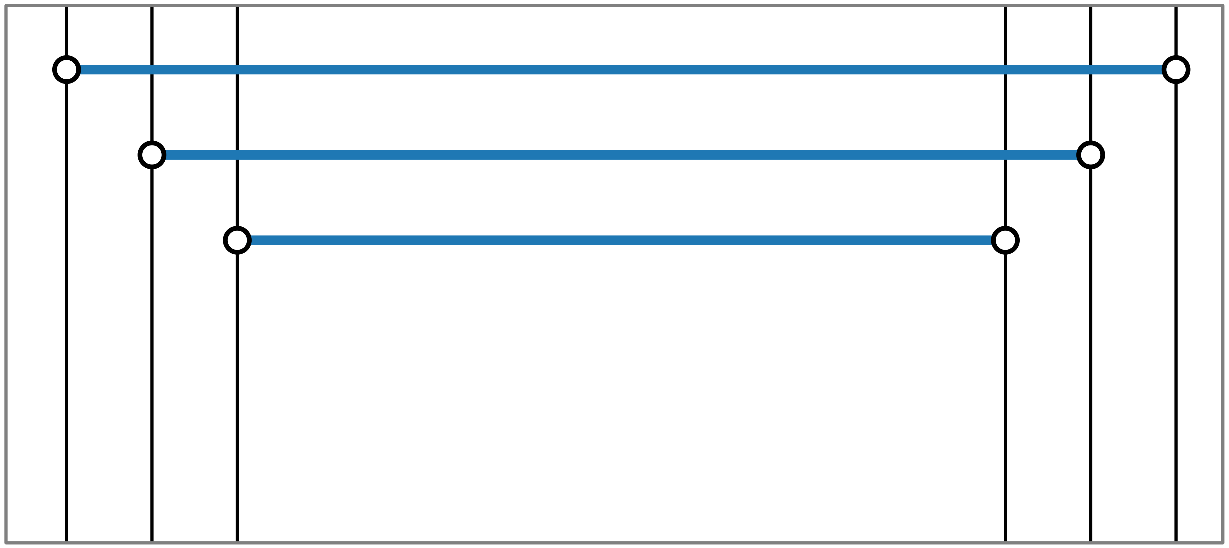

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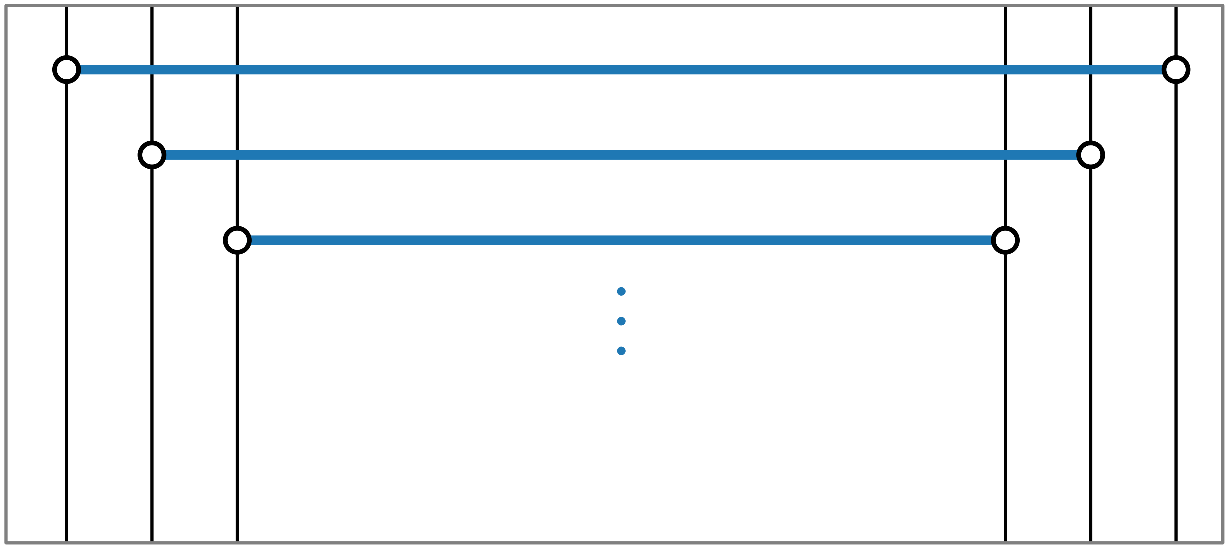

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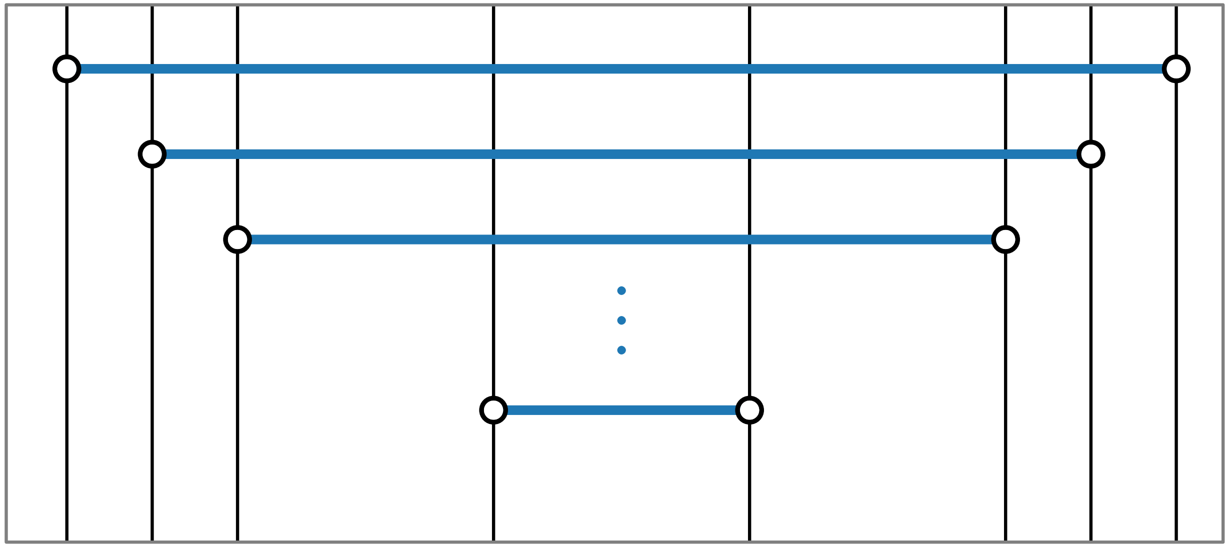

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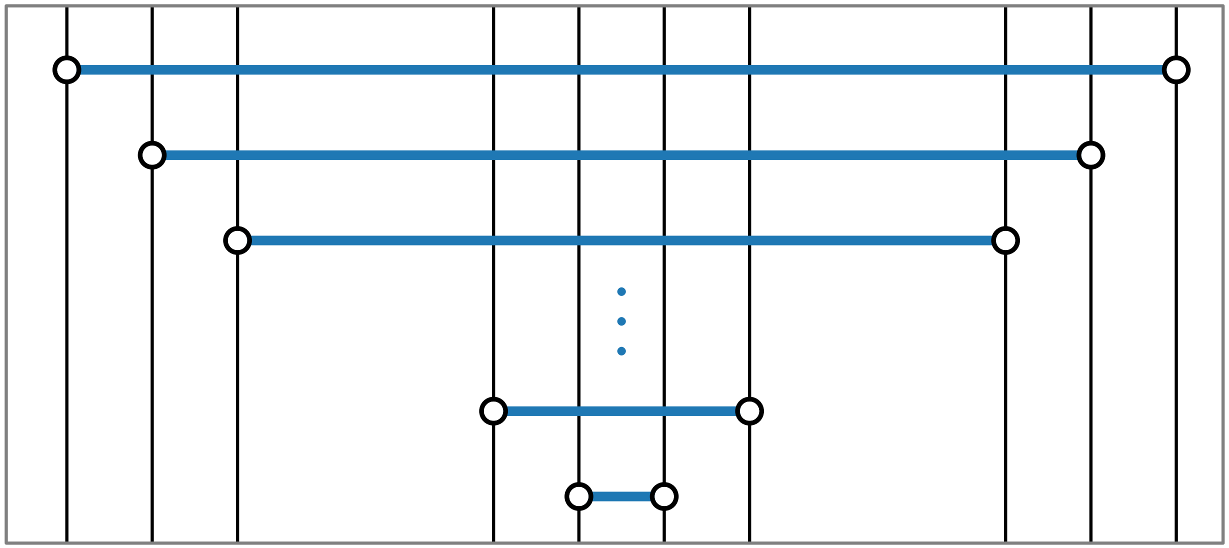

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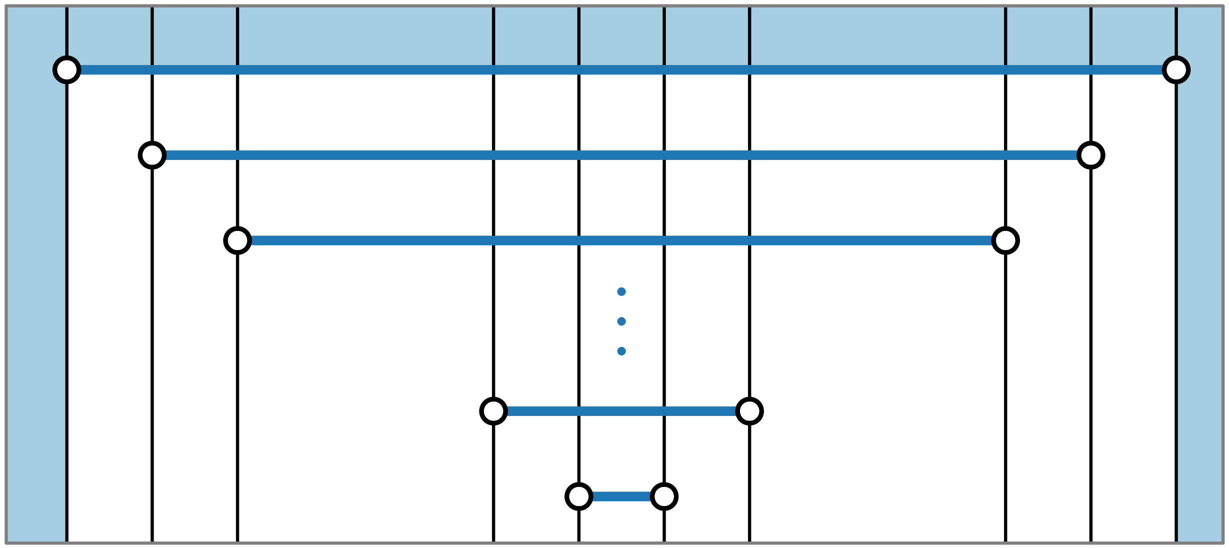

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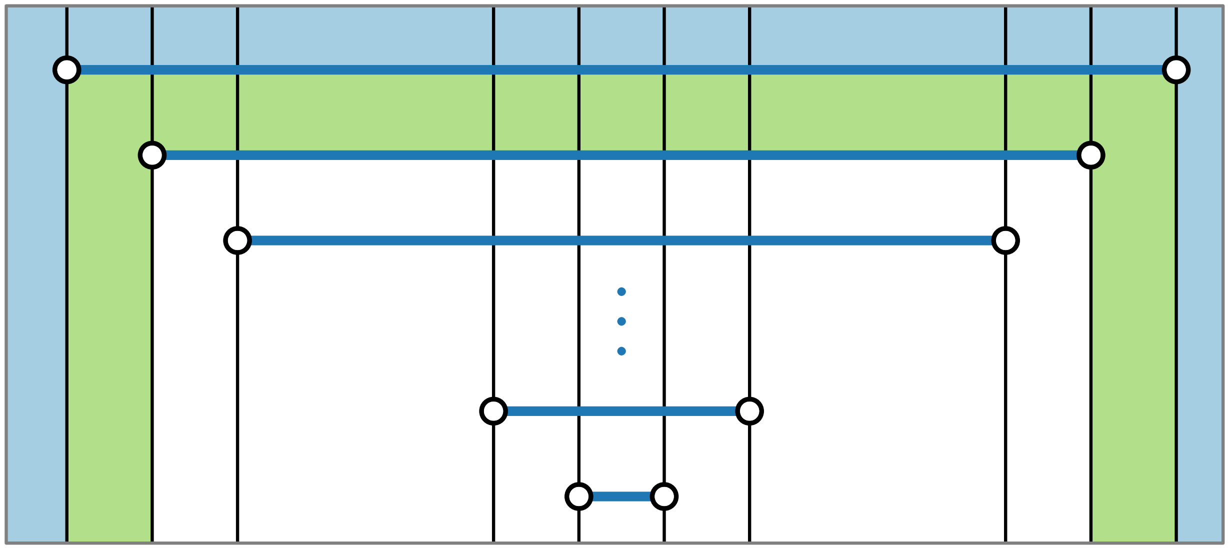

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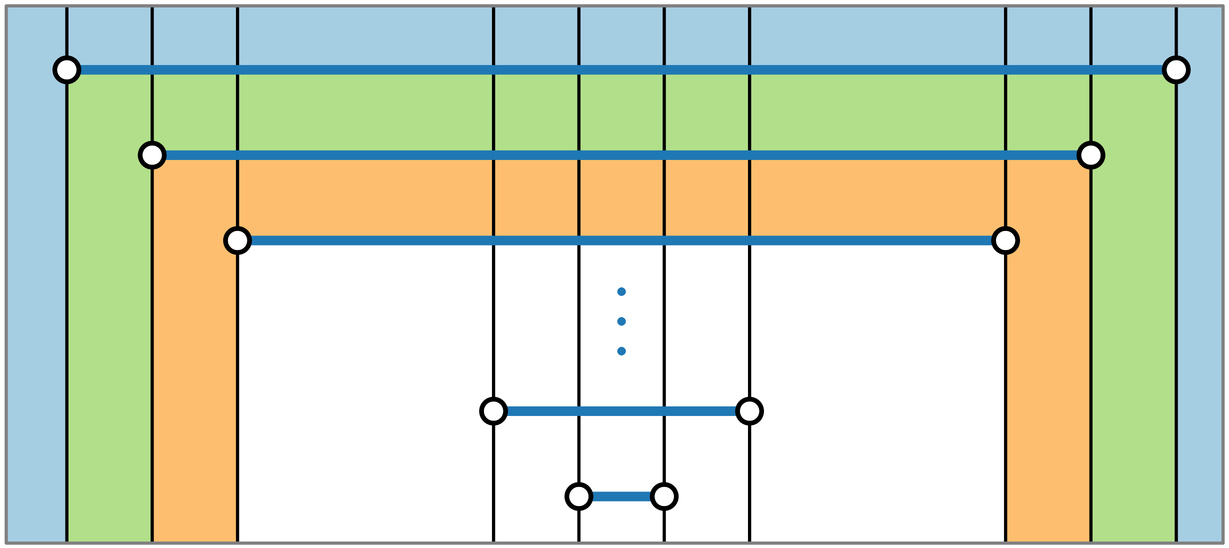

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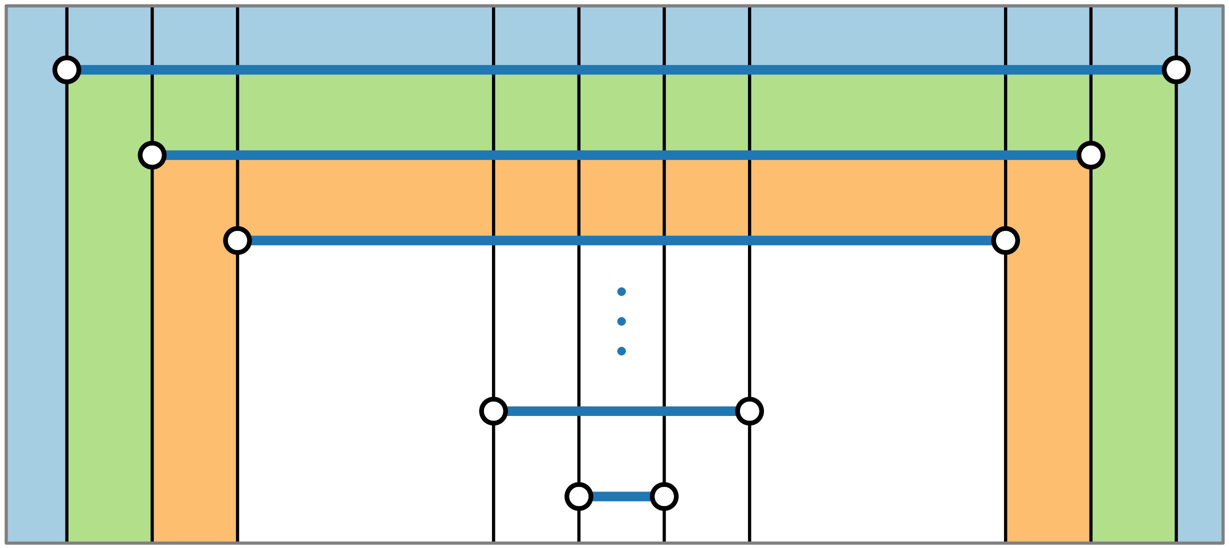

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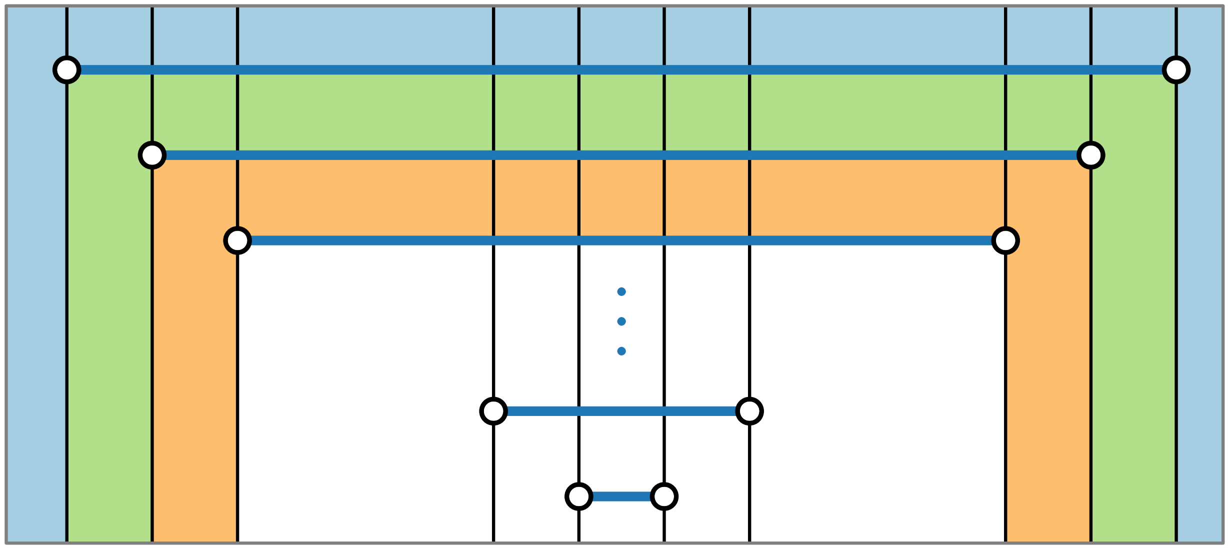

$$n + 1 + n - 1 + n - 3 + \ldots = n^2/4$$

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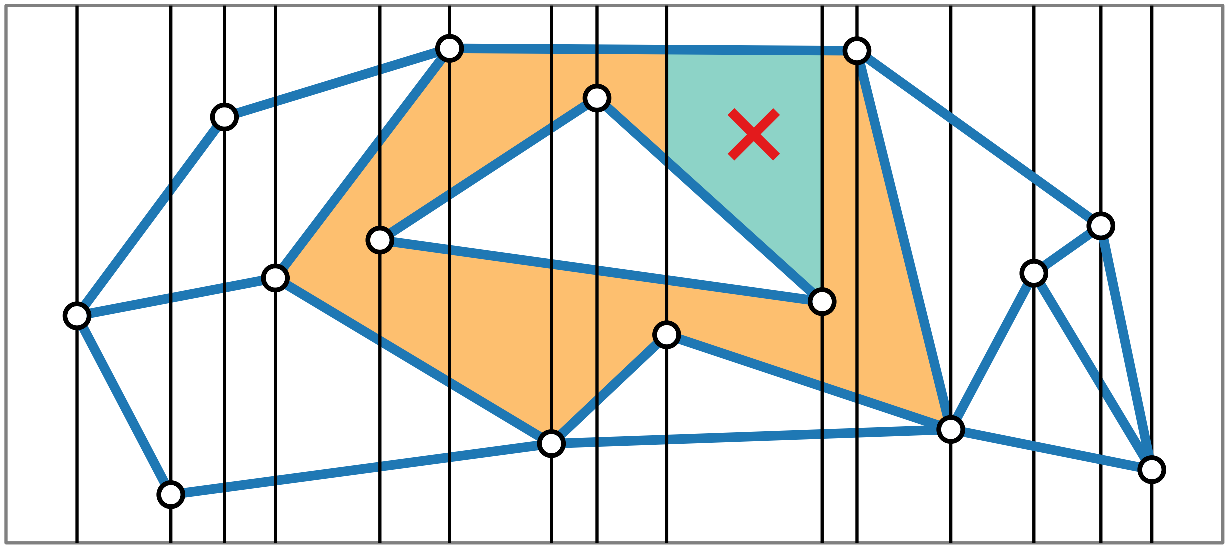

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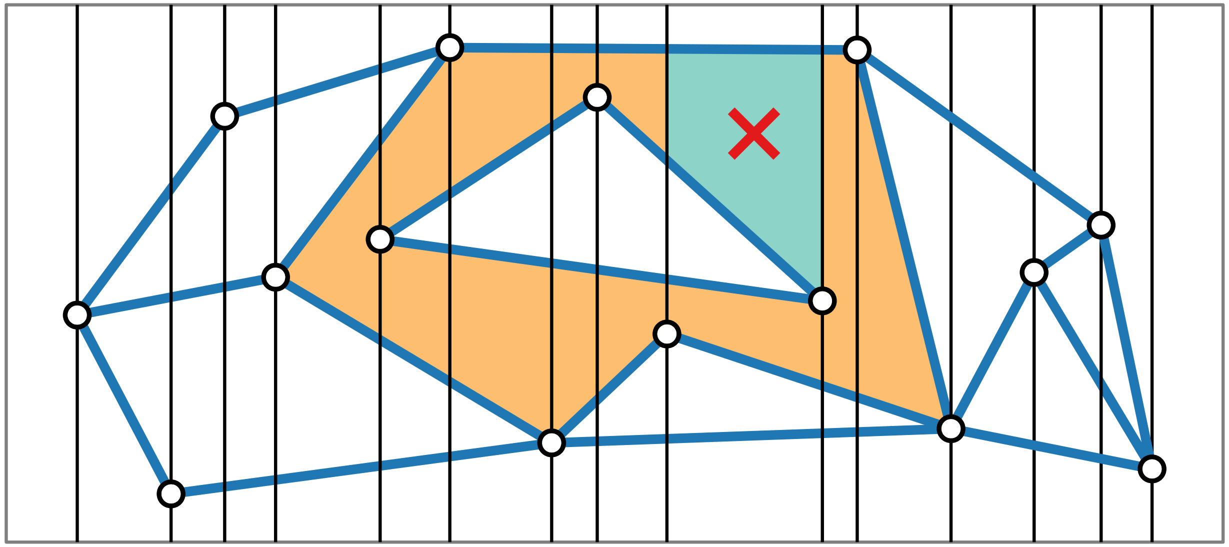

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Computational Geometry

Lecture 6:
Point Localization
or
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Part II:
Decreasing the Complexity

Philipp Kindermann

Winter Semester 2020

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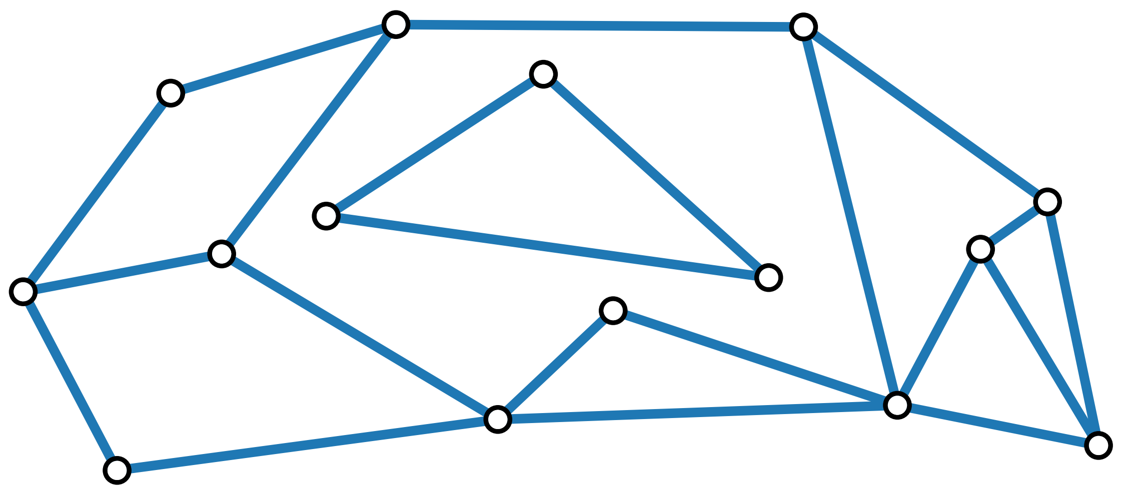

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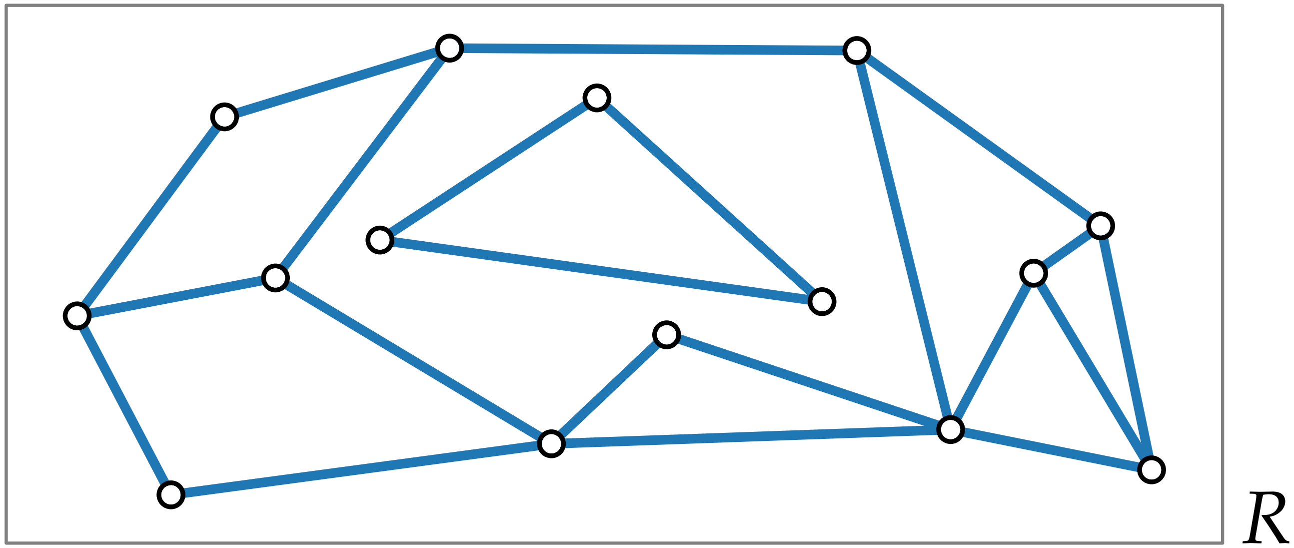

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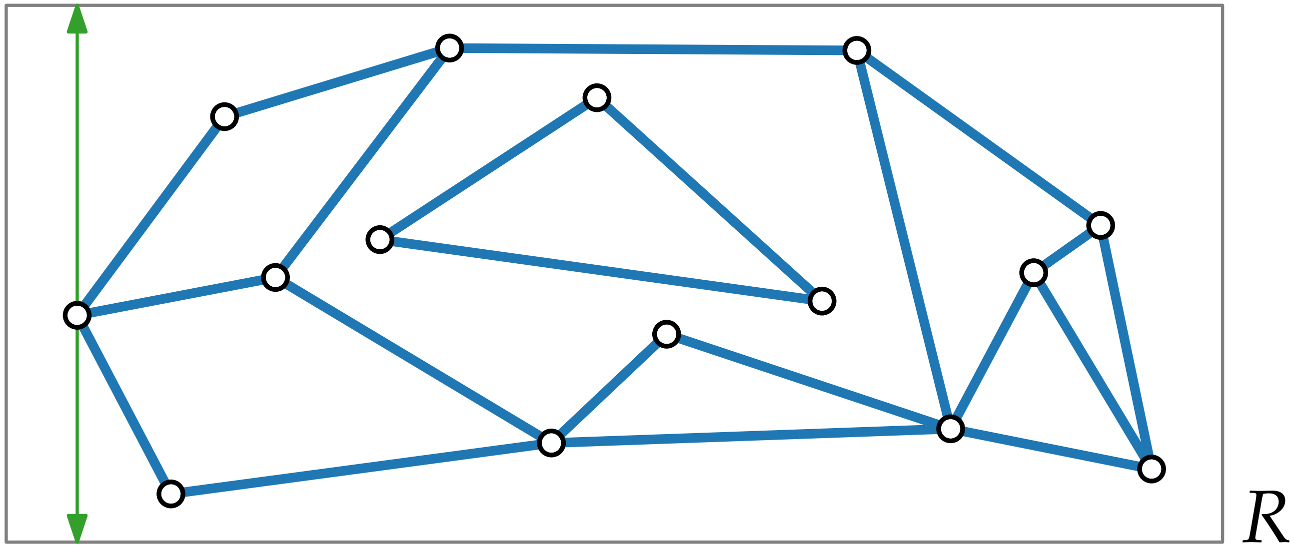

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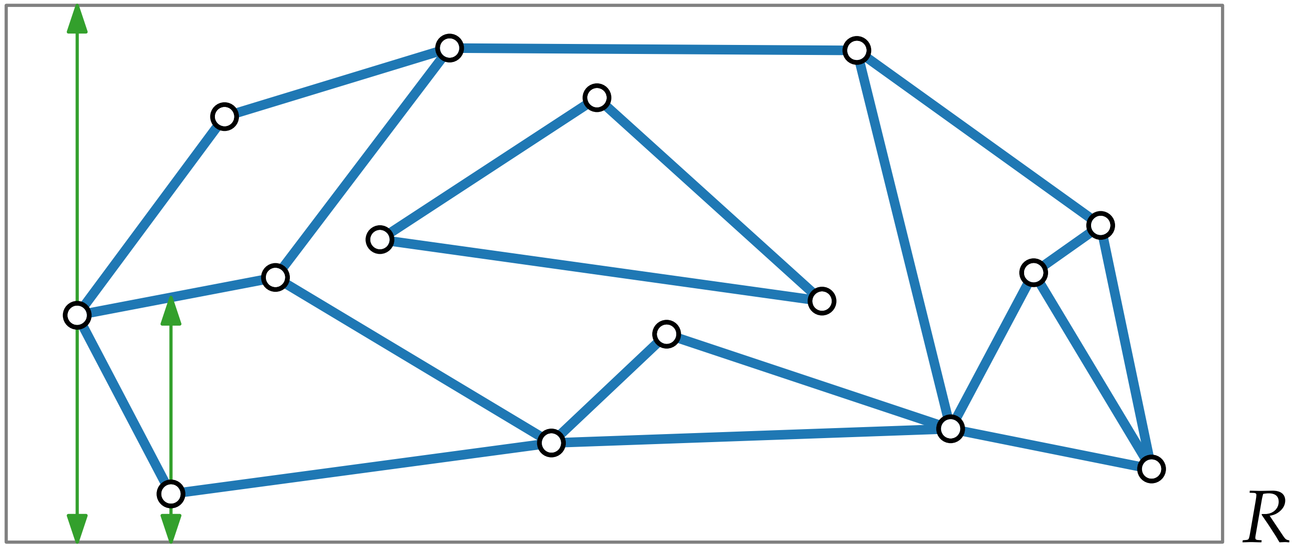

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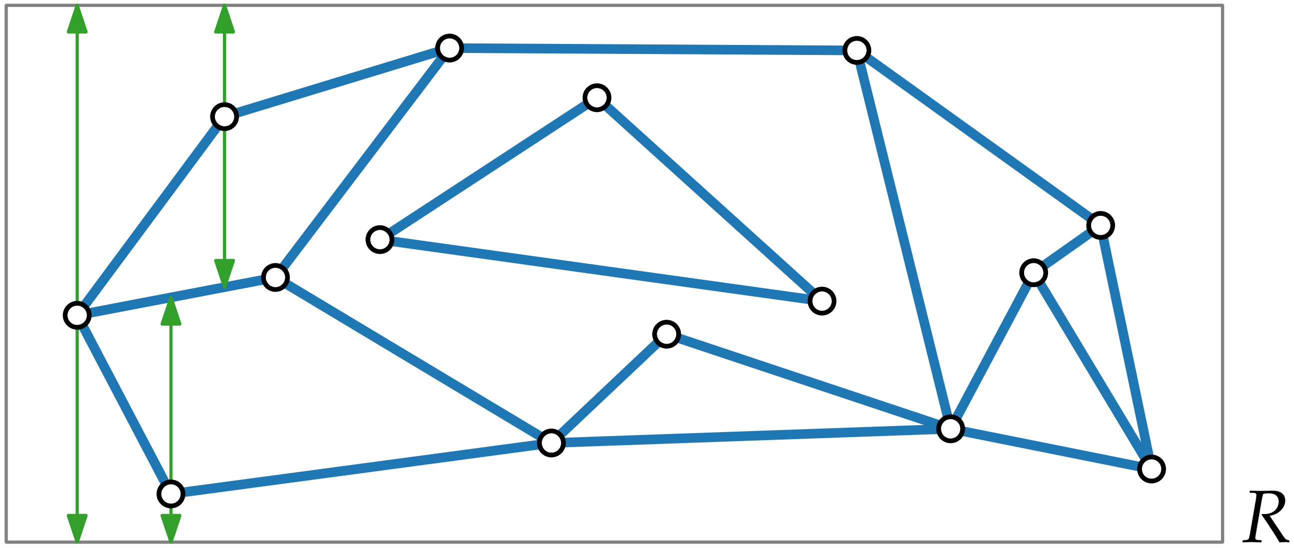

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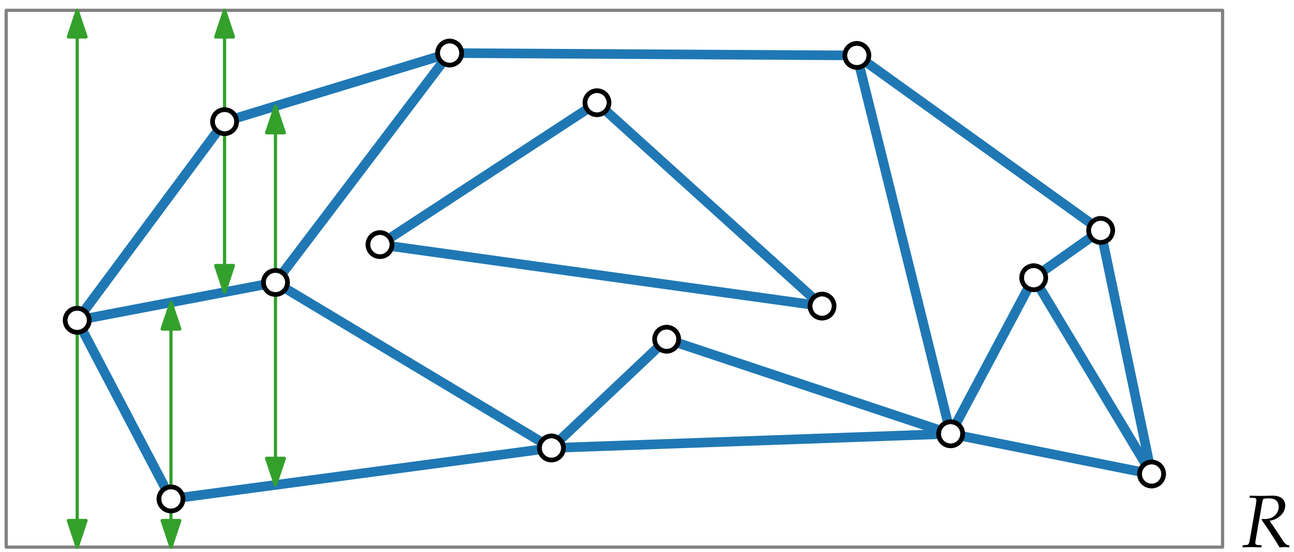

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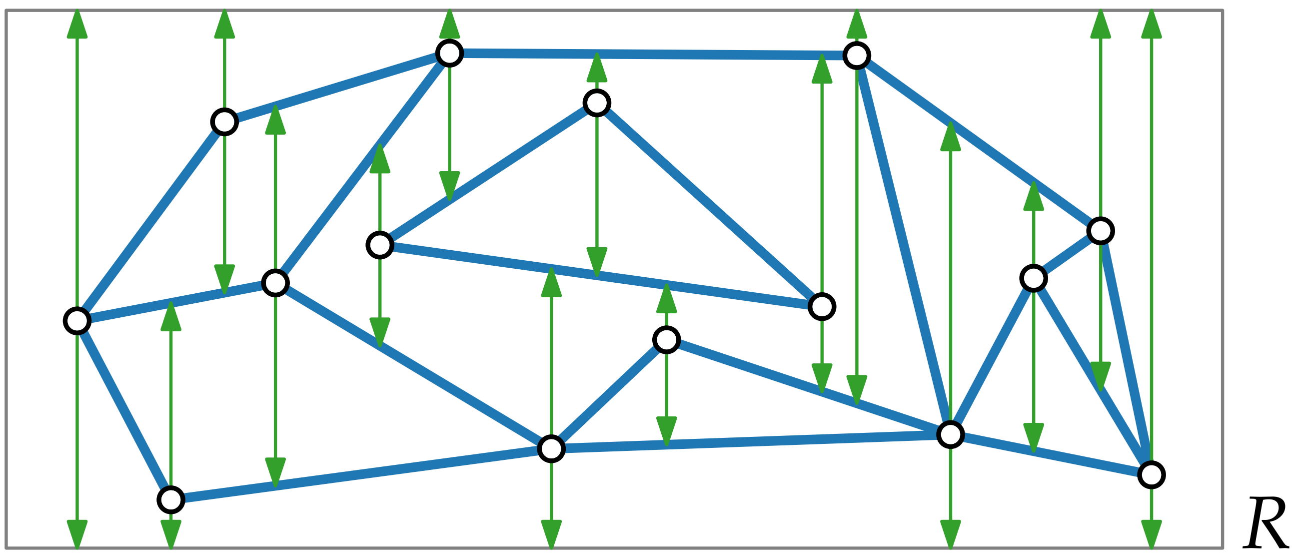

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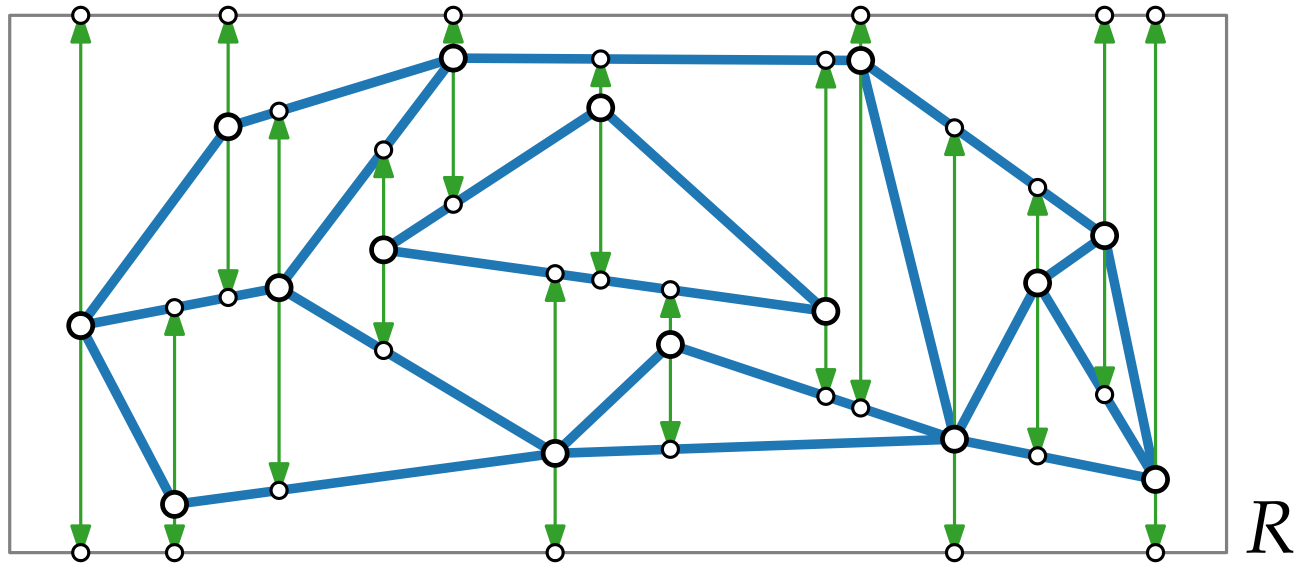

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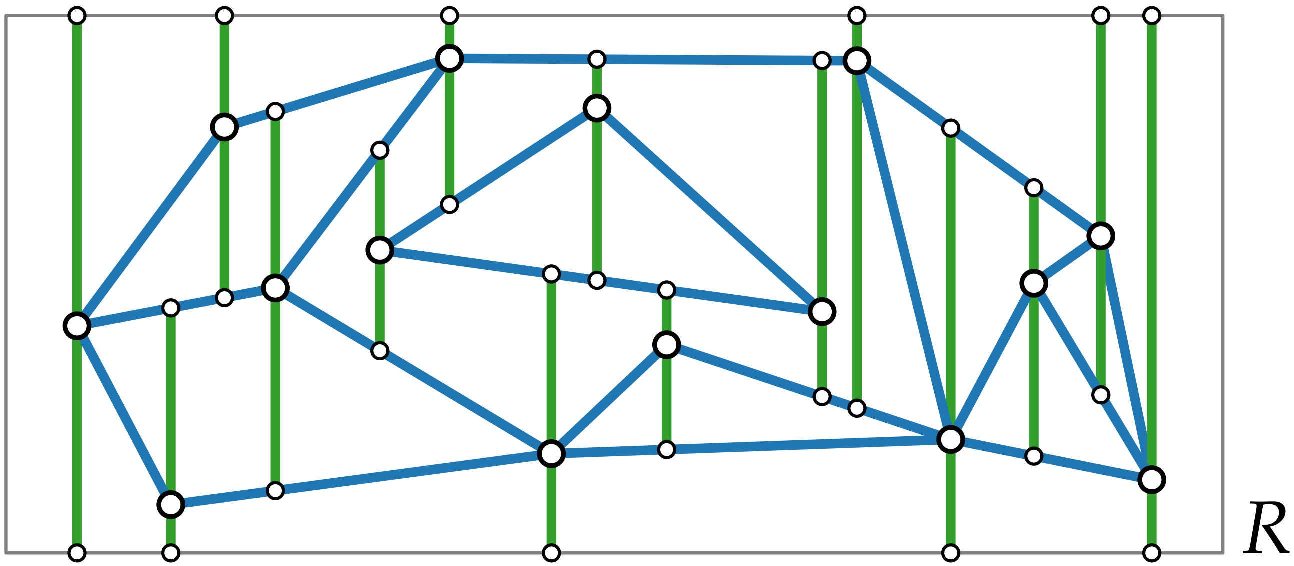

Decreasing the Complexity

Observation: The slab partition of $\mathcal{S}$ is a *refinement* $\mathcal{S}'$ of $\mathcal{S}$ that consists of (possibly degenerate) trapezoids.

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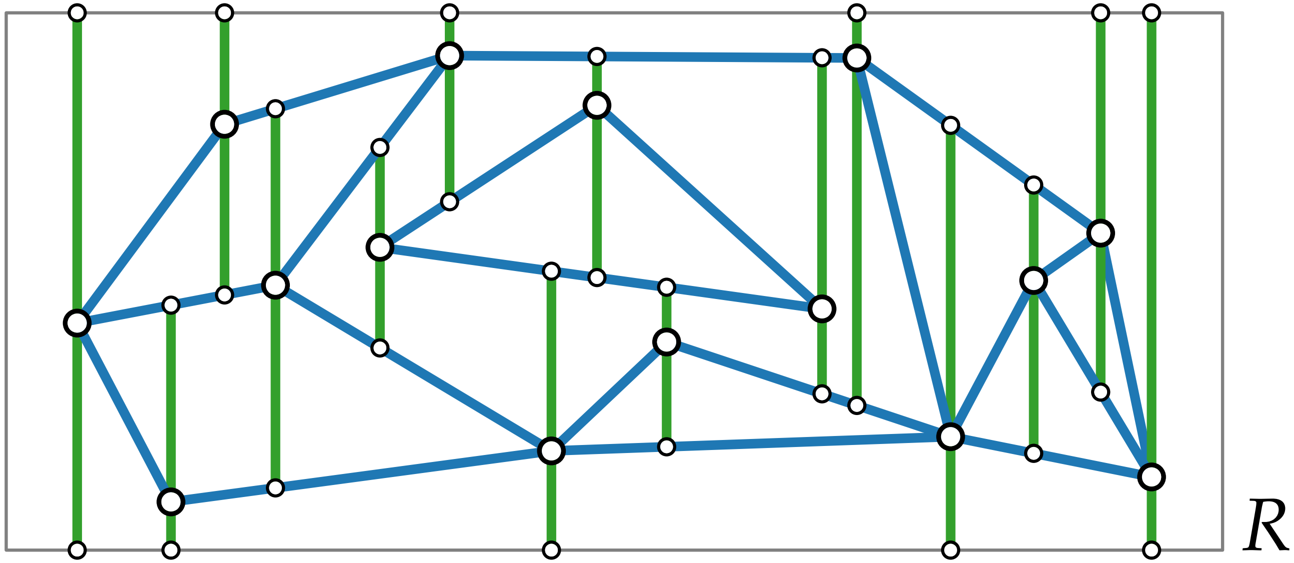

Assumption: $\mathcal{S}$ is in *general position*, that is, no two vertices have the same x-coordinates.

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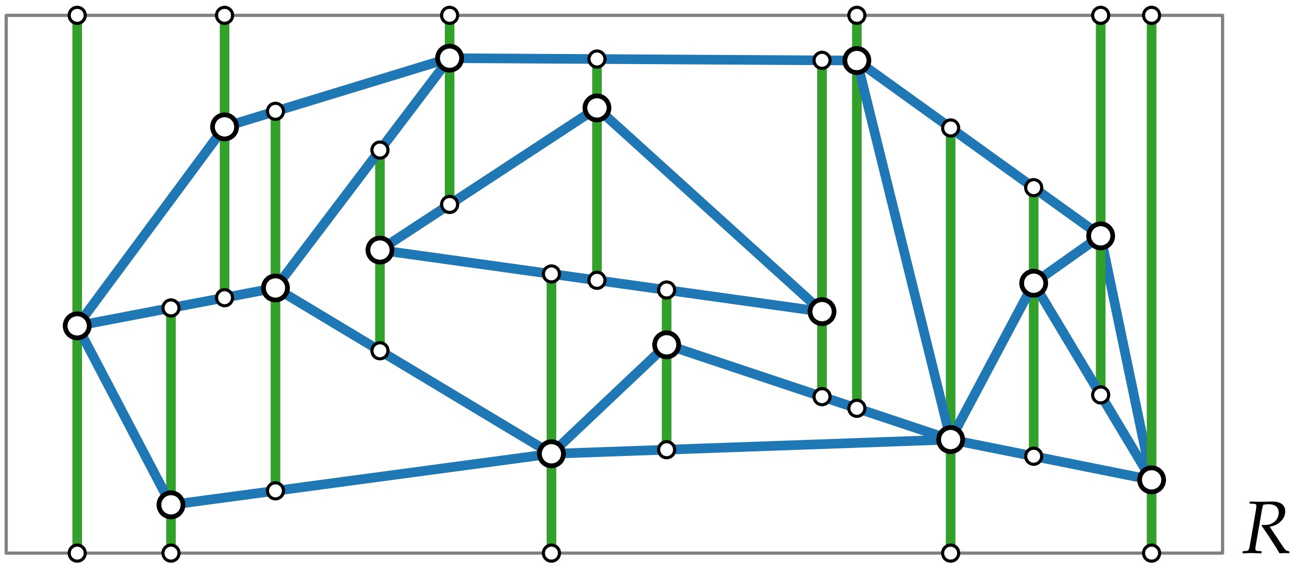

See Lecture 5

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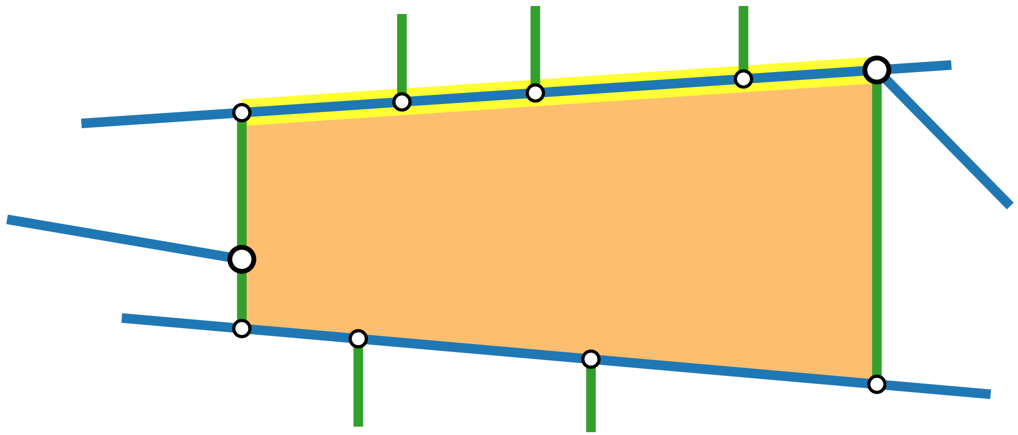

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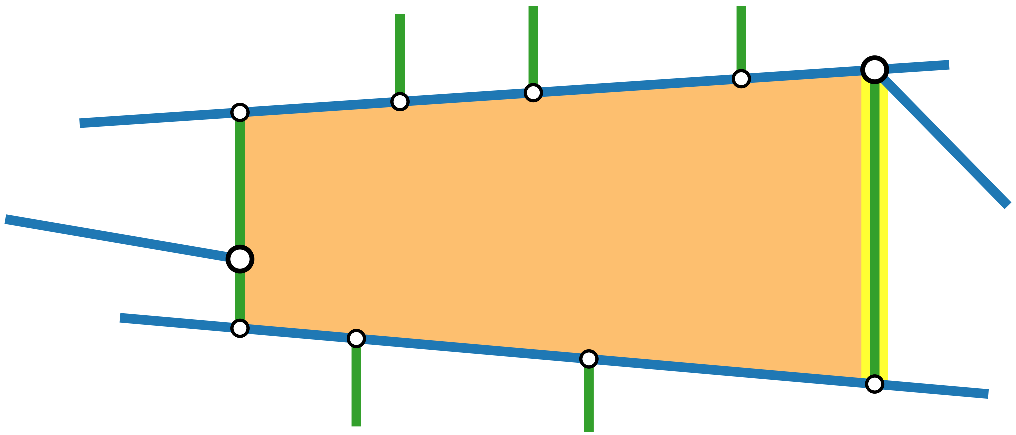

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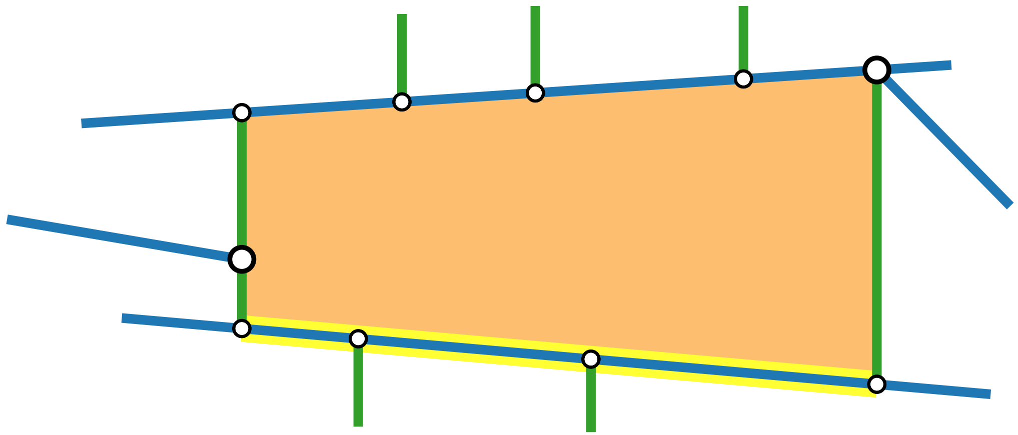

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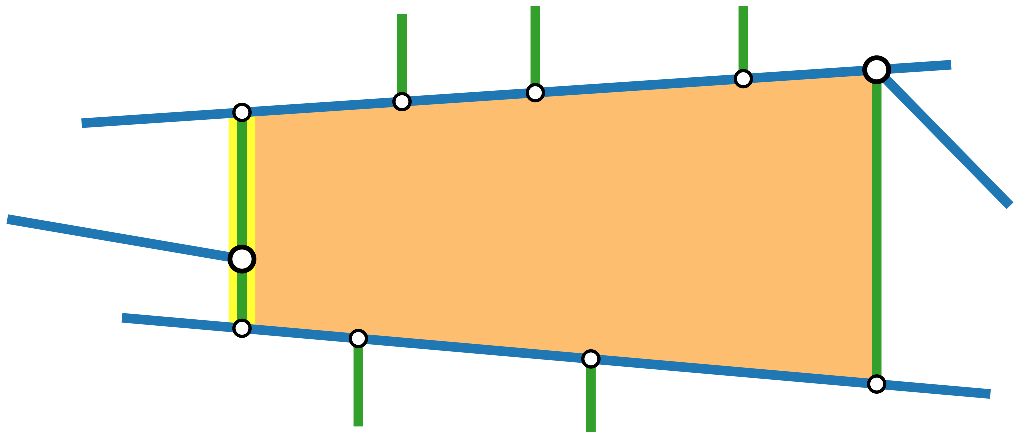

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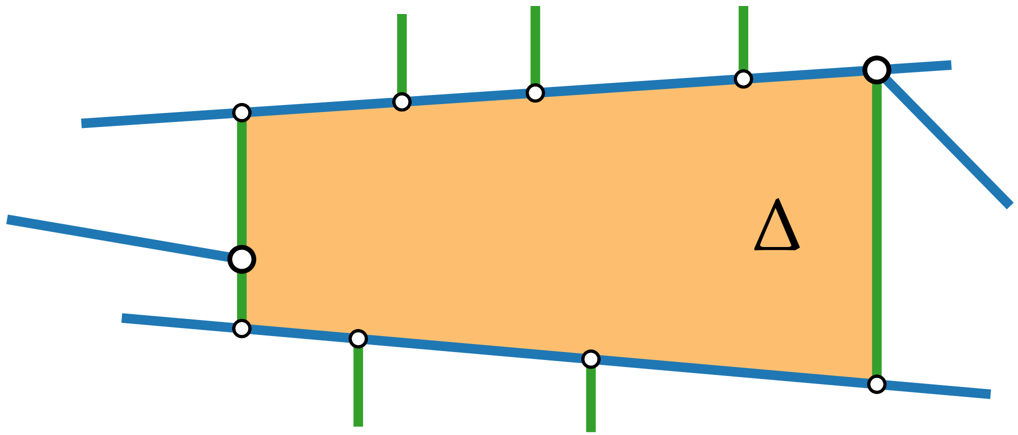

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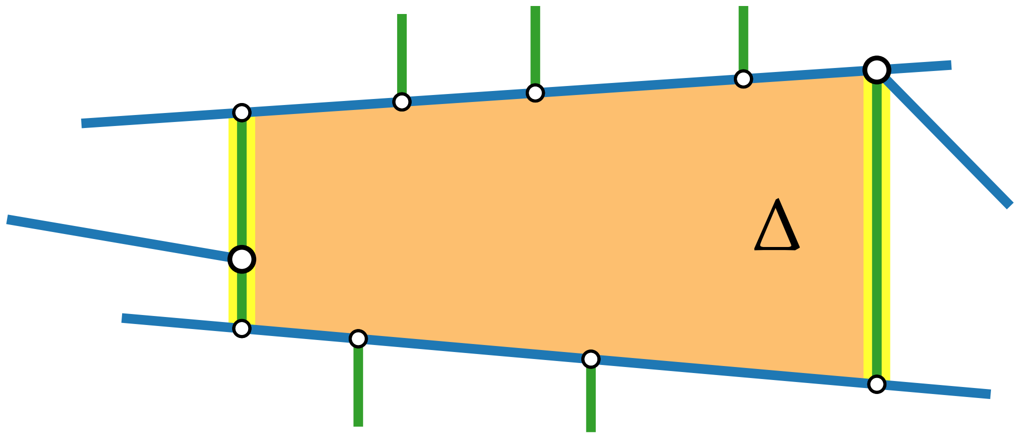

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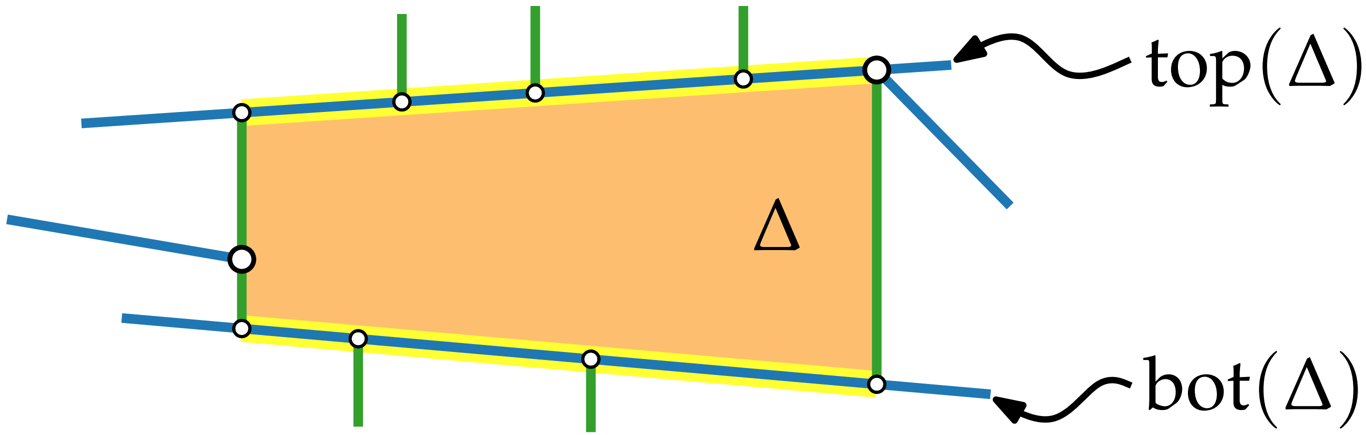

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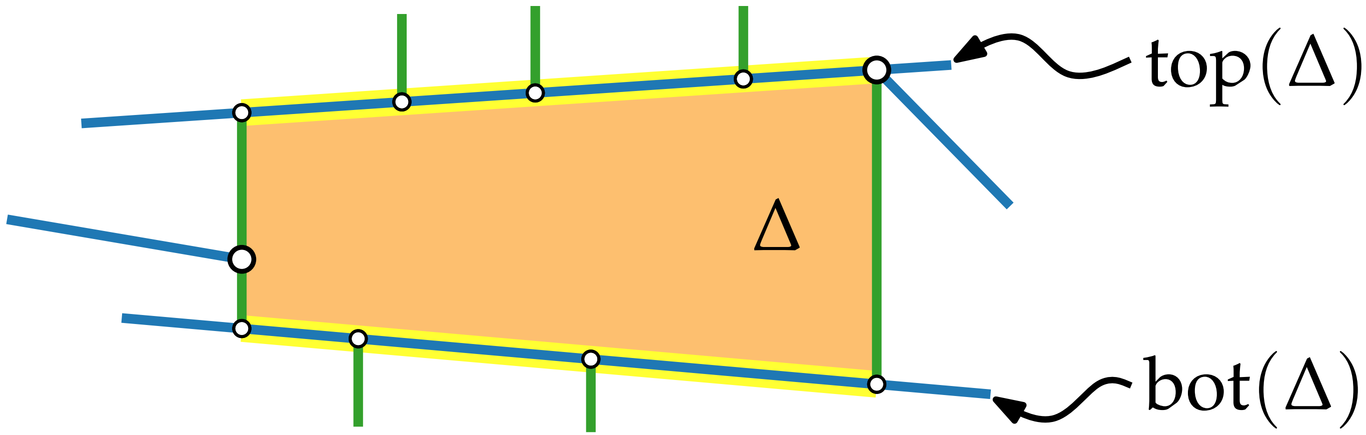

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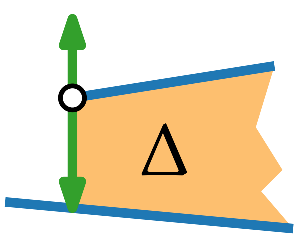

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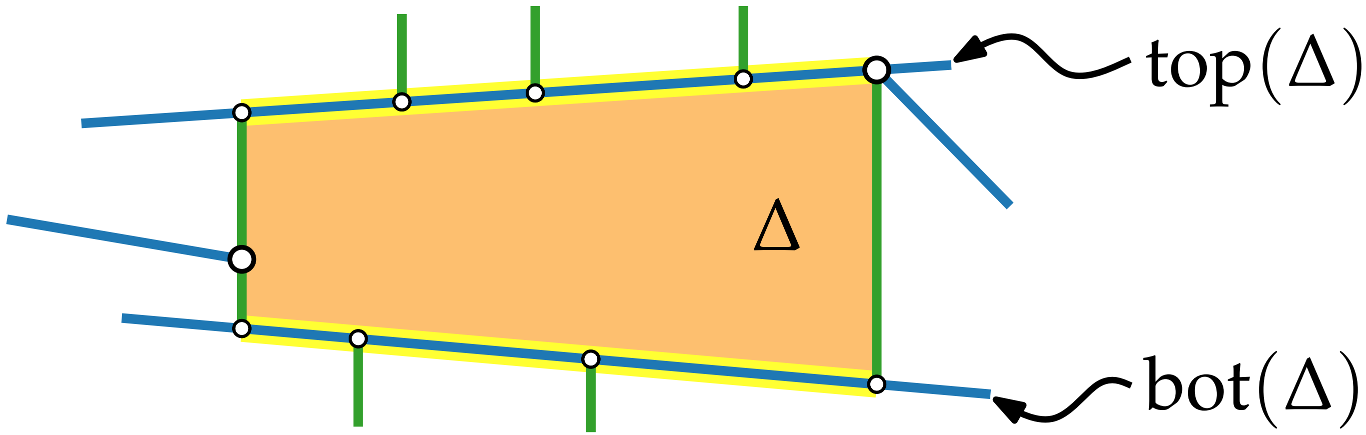

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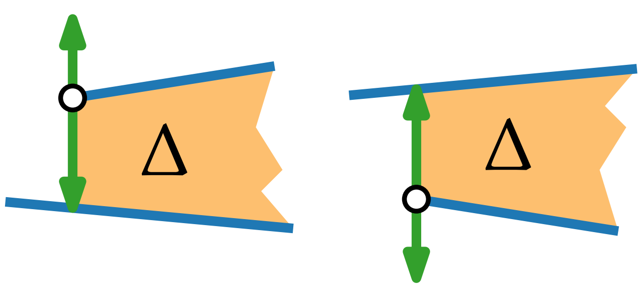

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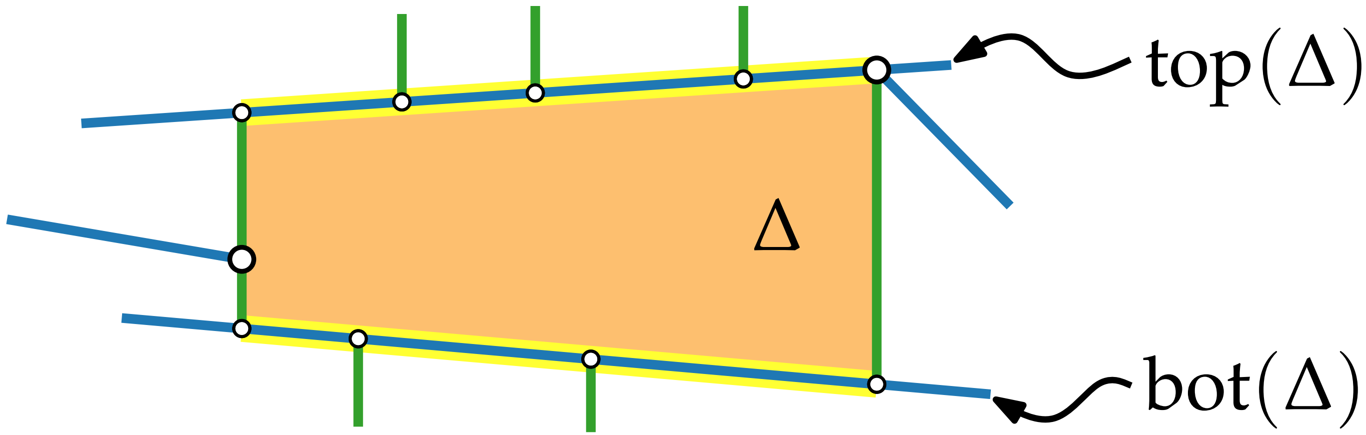

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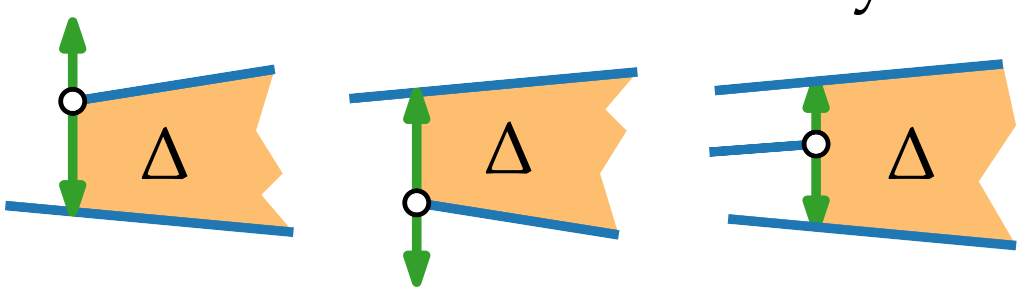

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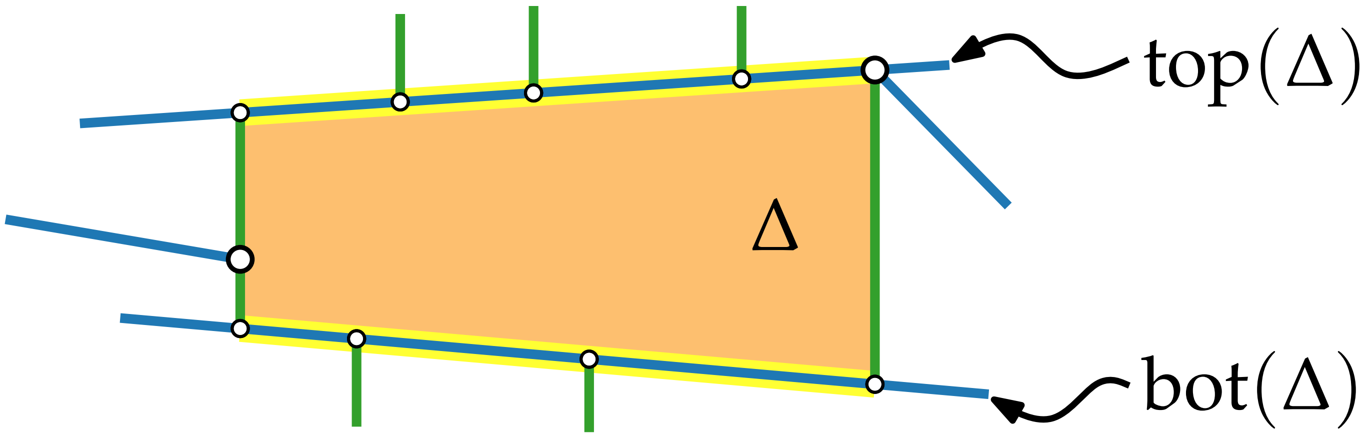

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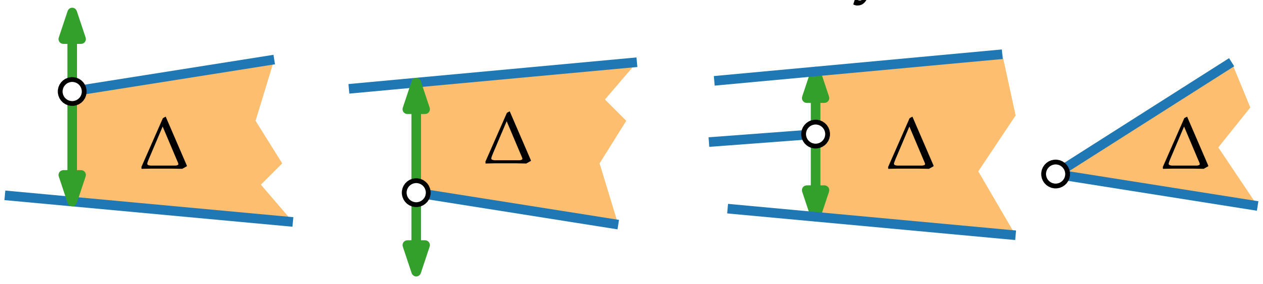

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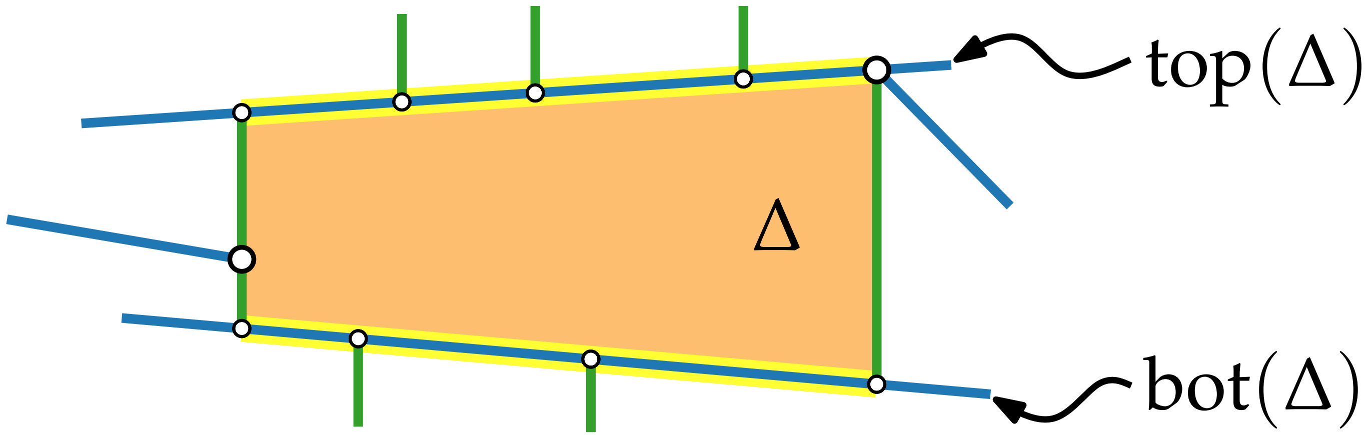

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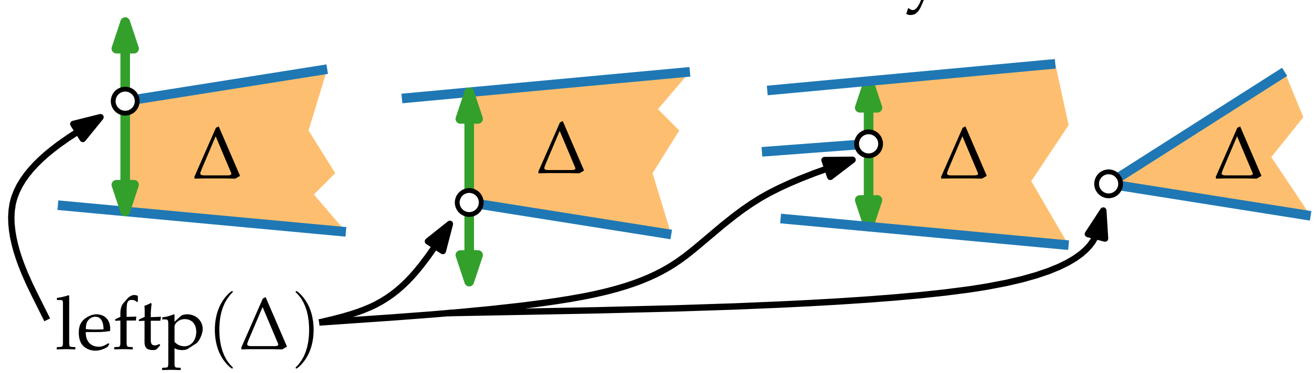

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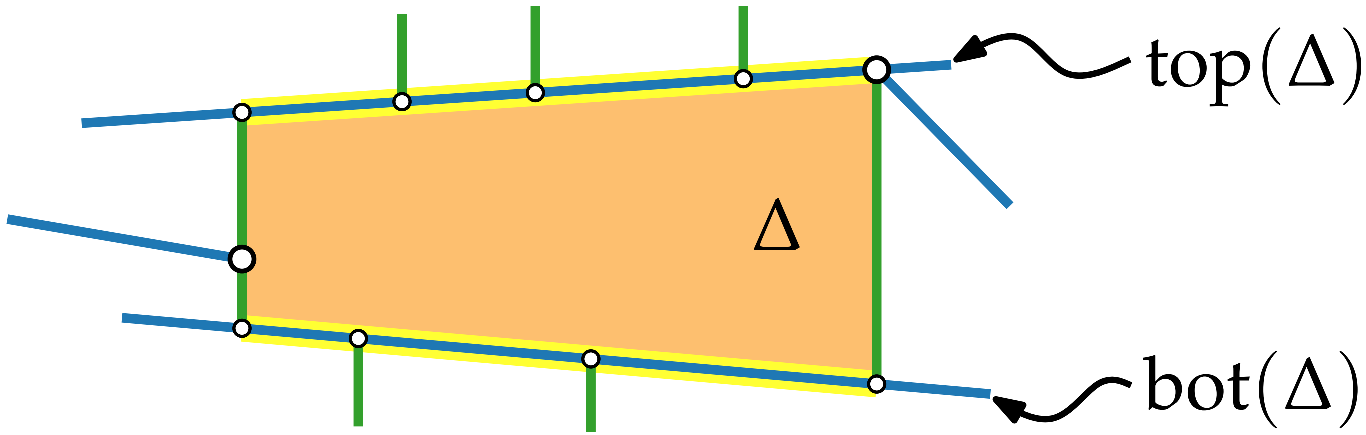

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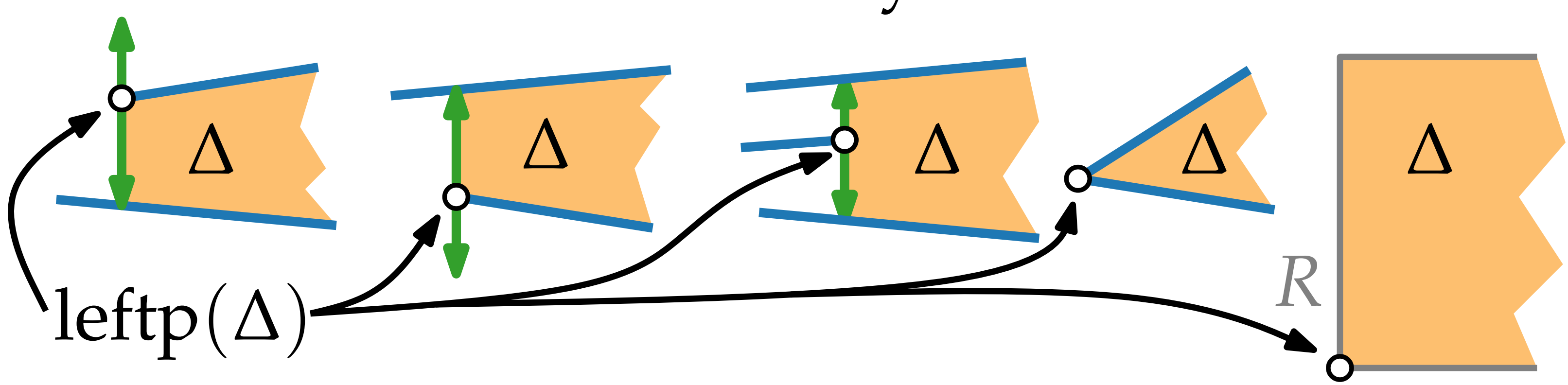

Complexity of $\mathcal{T}(\mathcal{S})$

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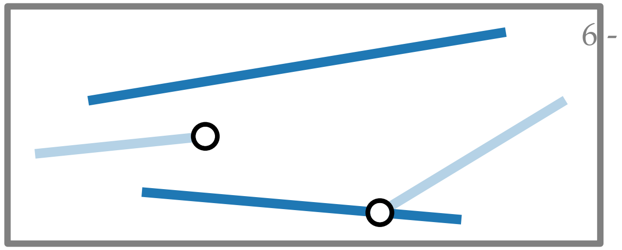

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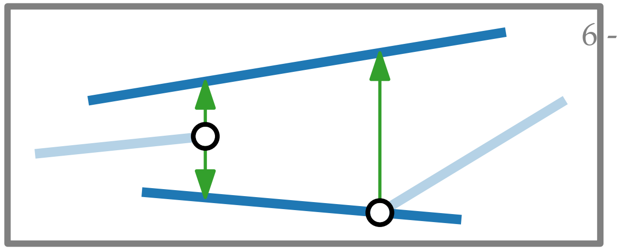

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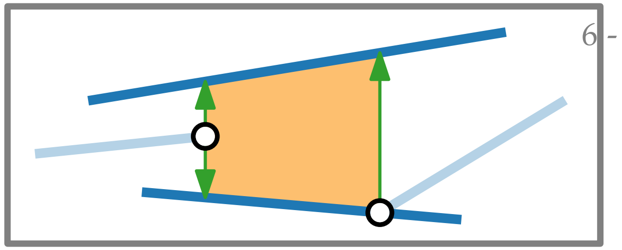

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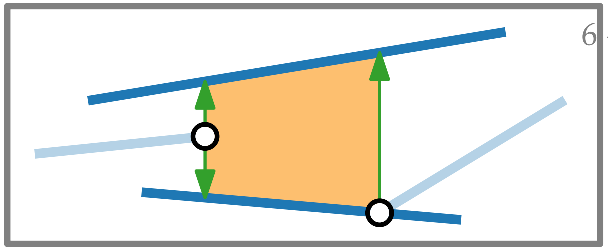

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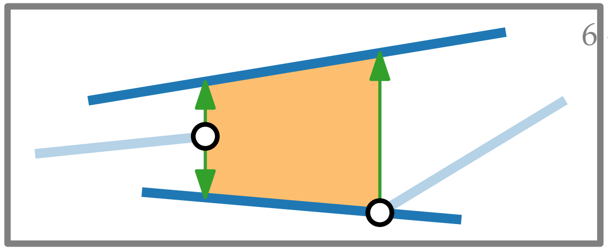

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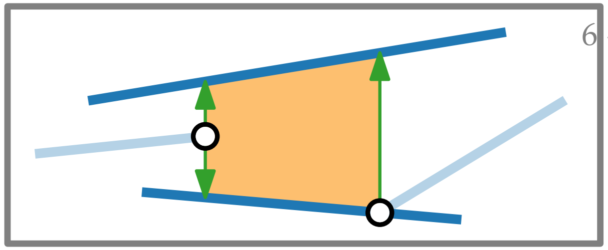

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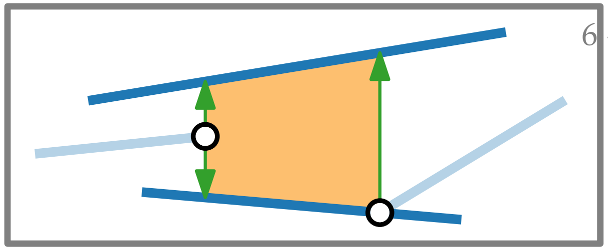

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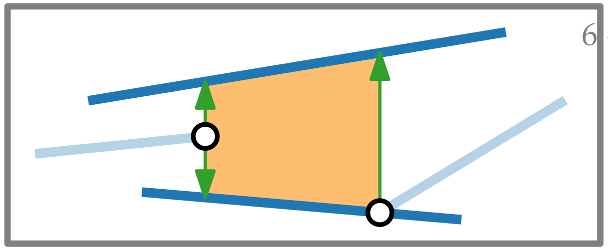

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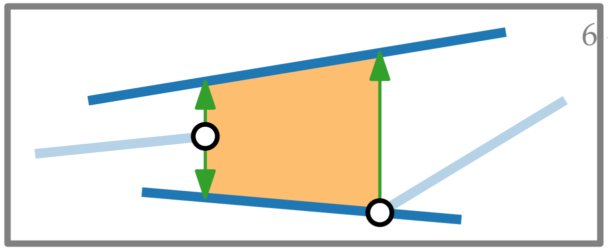

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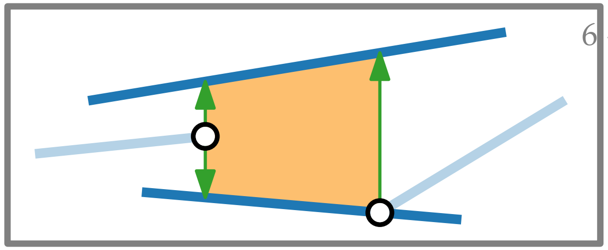

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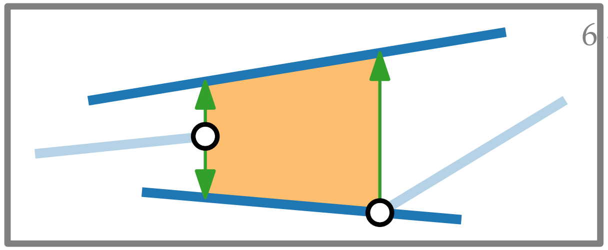

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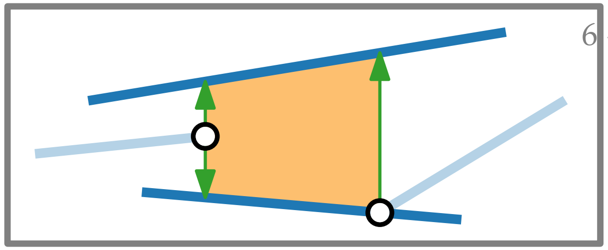

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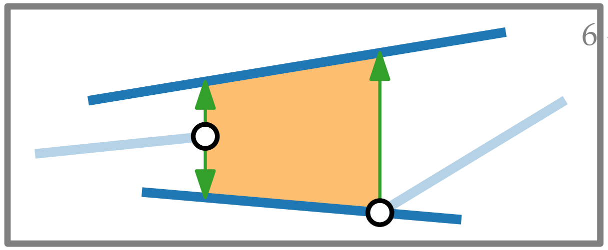

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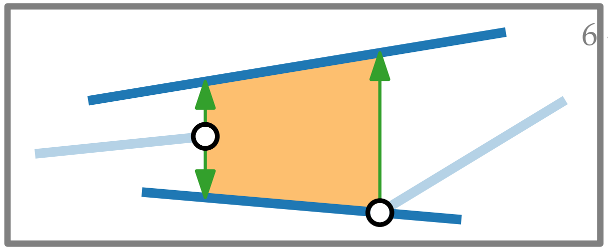

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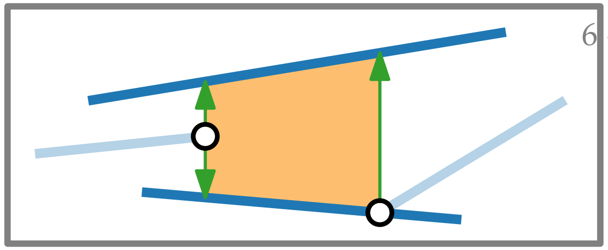

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Approach:

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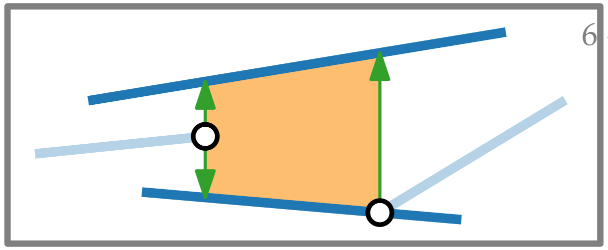

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Approach: Construct trapezoidal map $\mathcal{T}(\mathcal{S})$ and point-location data structure $\mathcal{D}(\mathcal{S})$ for $\mathcal{T}(\mathcal{S})$ *incrementally*!

Computational Geometry

Lecture 6:
Point Localization
or
Where am I?

Part III:
The 1D Problem

Philipp Kindermann

Winter Semester 2020

The 1D Problem

Given a set S of n real numbers...

The 1D Problem

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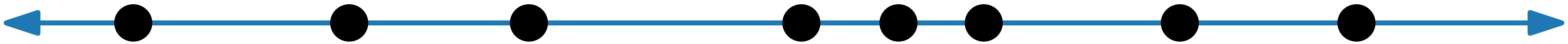

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Given a set S of n real numbers... $i \in \{1, \ldots, n\}$

$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, I_{i-1} := set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

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– pick an arbitrary point s_i from $S \setminus S_{i-1}$

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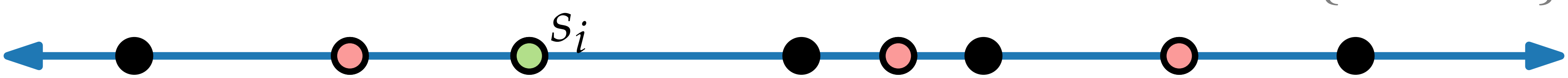

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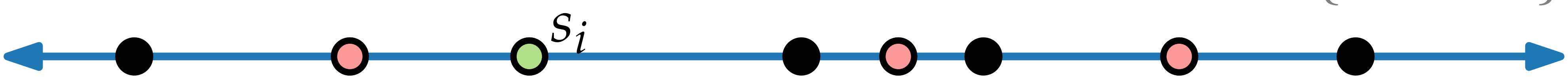

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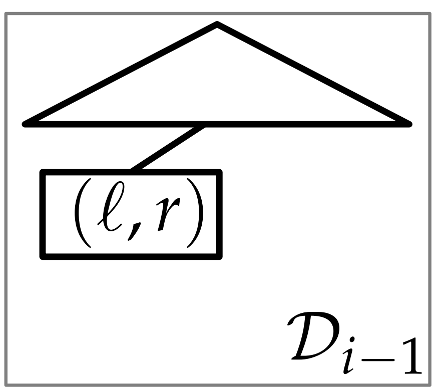

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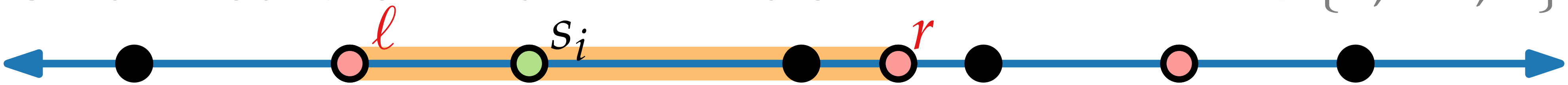

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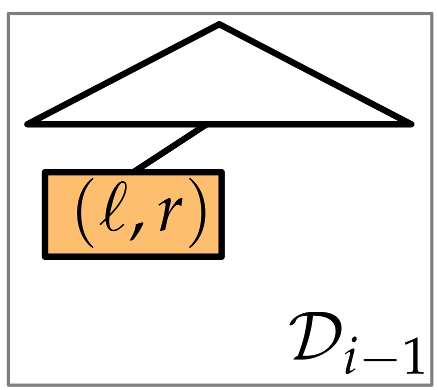

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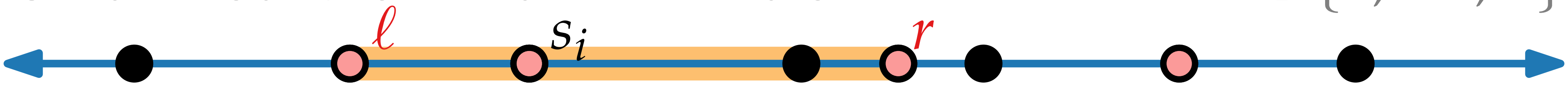

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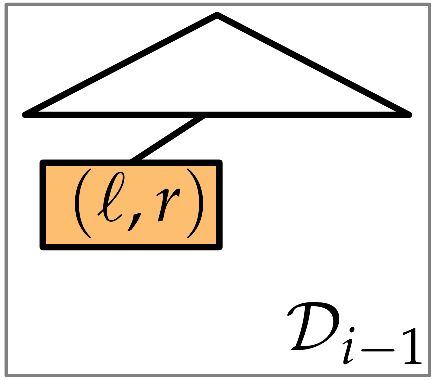

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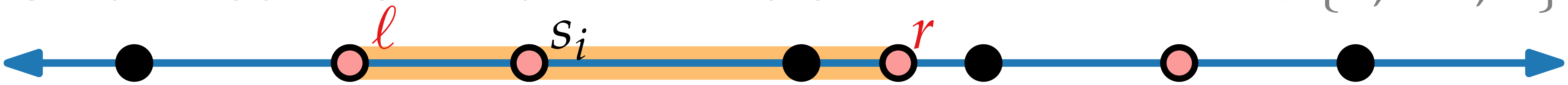

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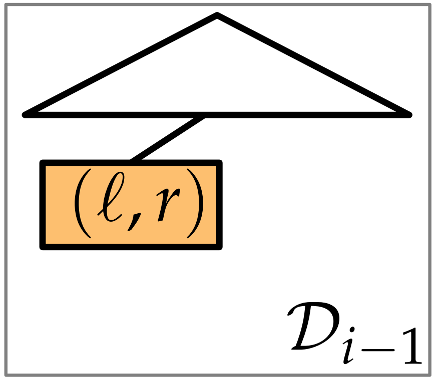

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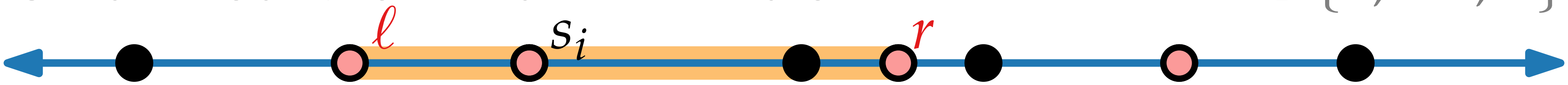

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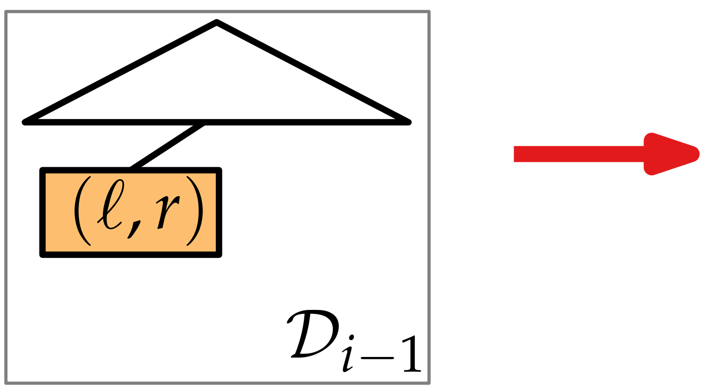

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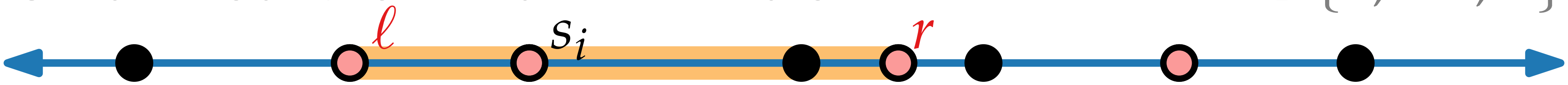

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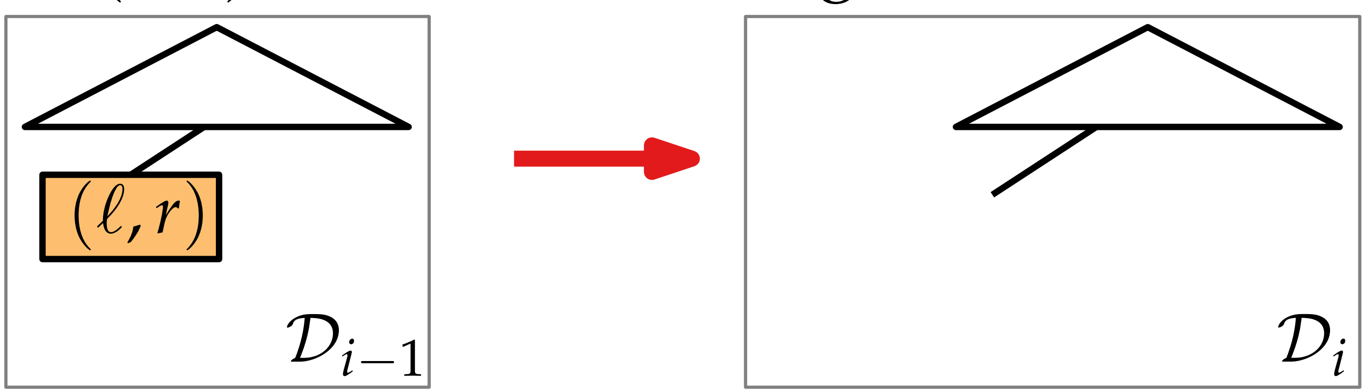

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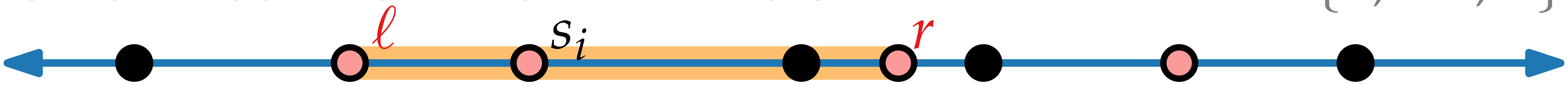

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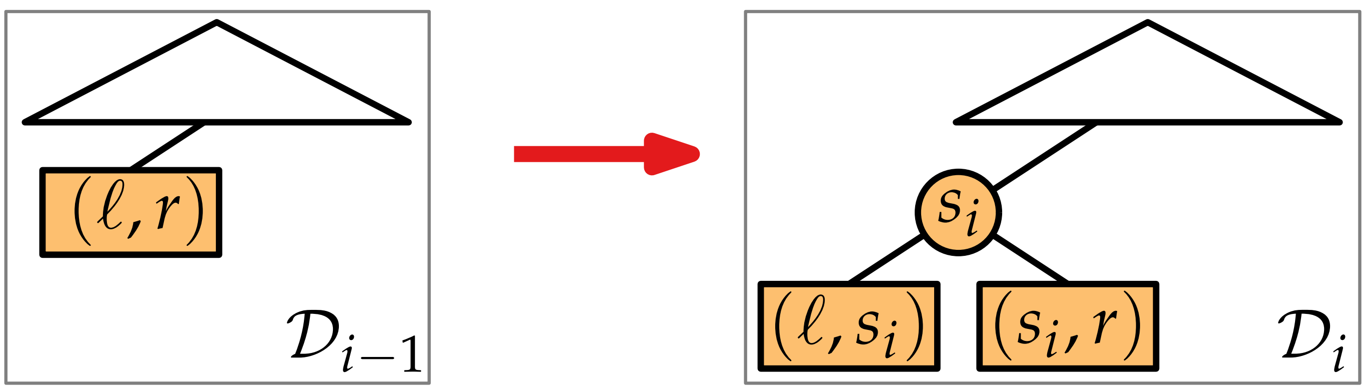

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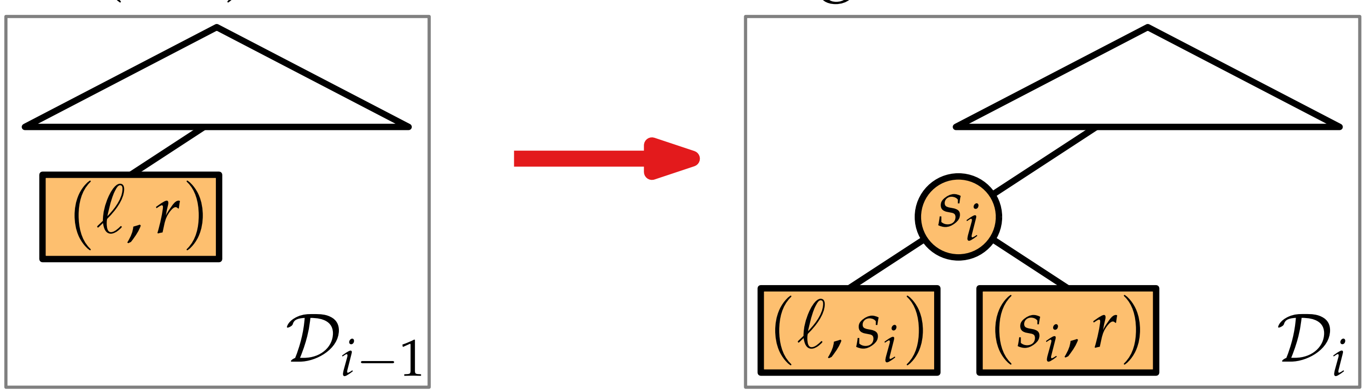

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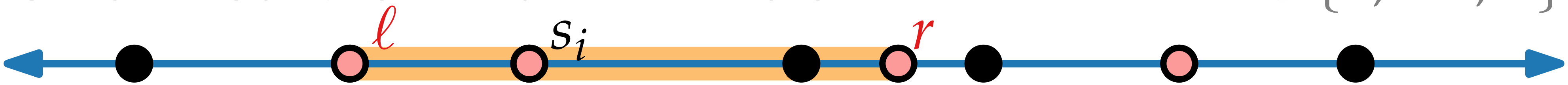

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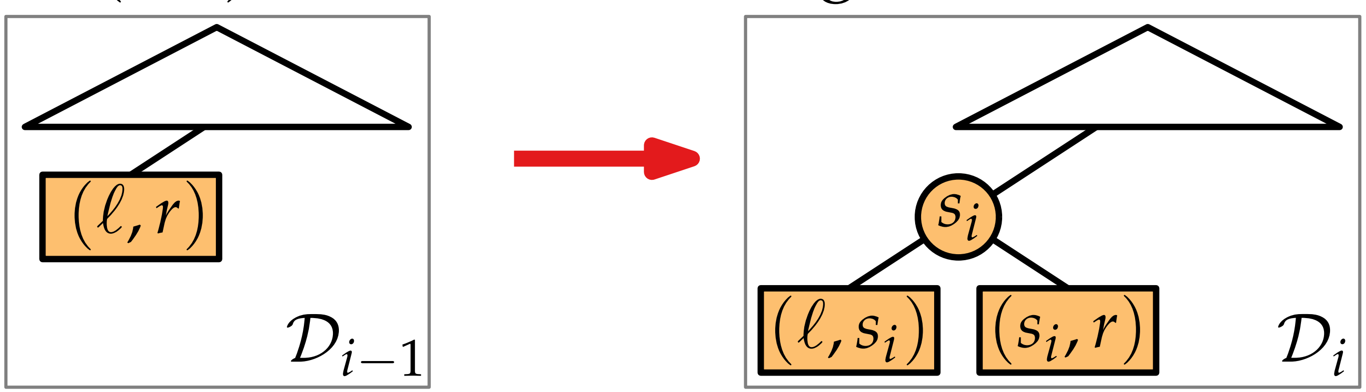

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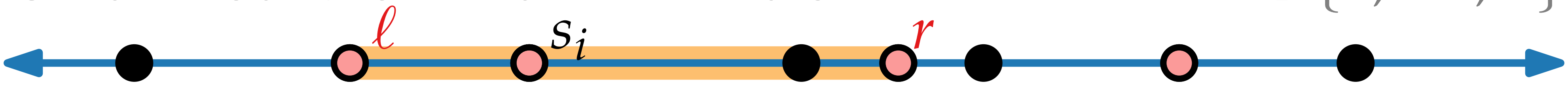

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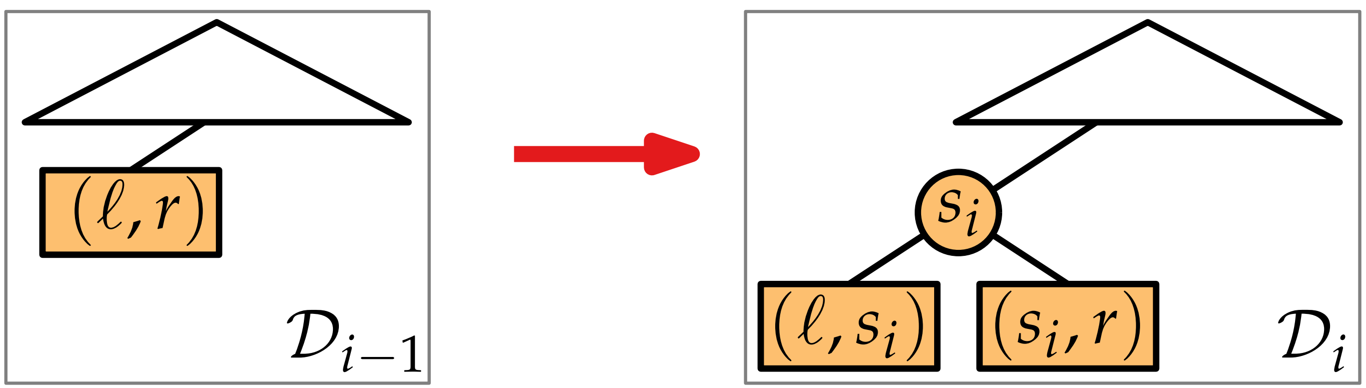

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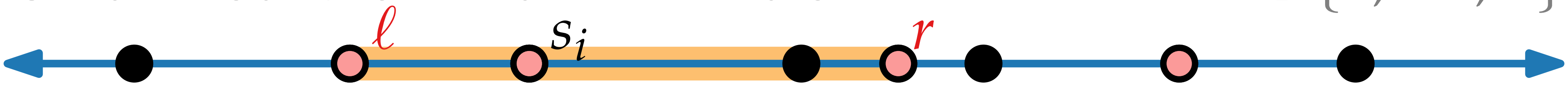

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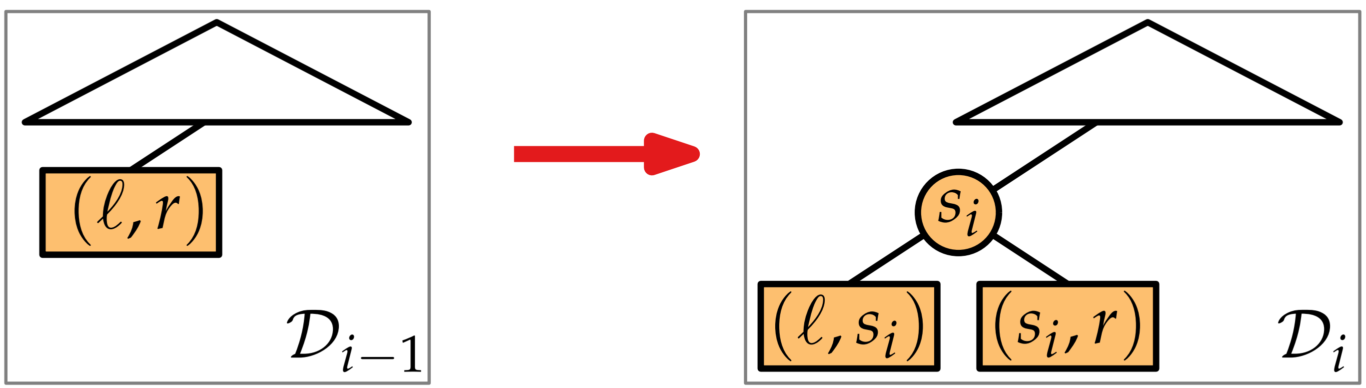

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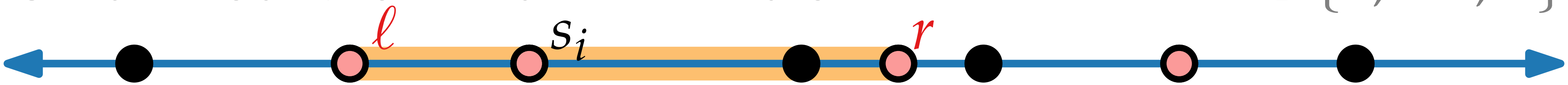

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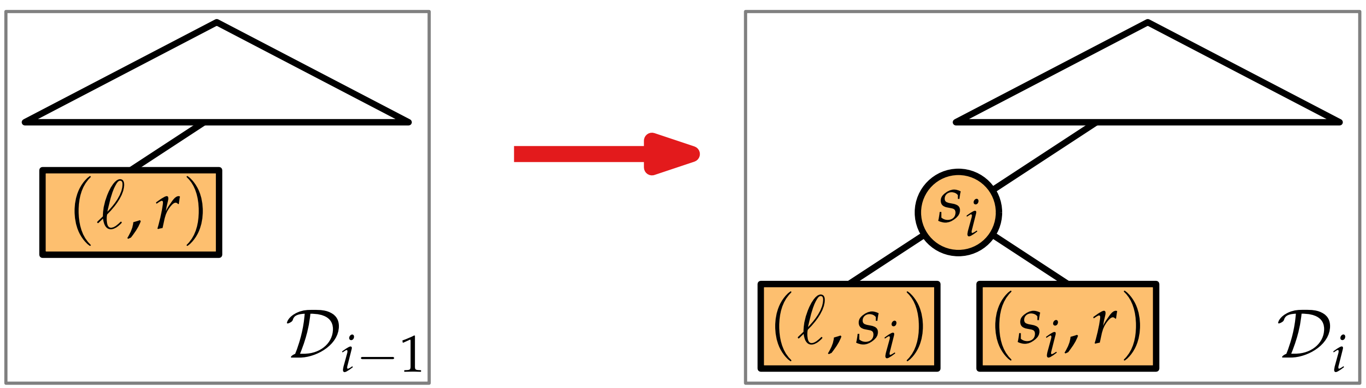

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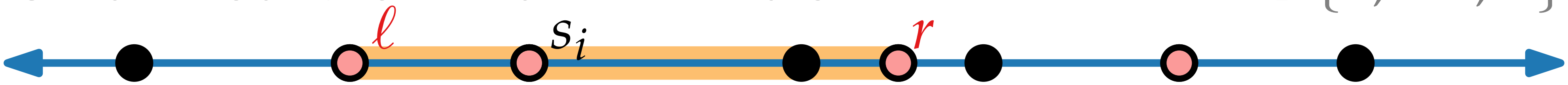

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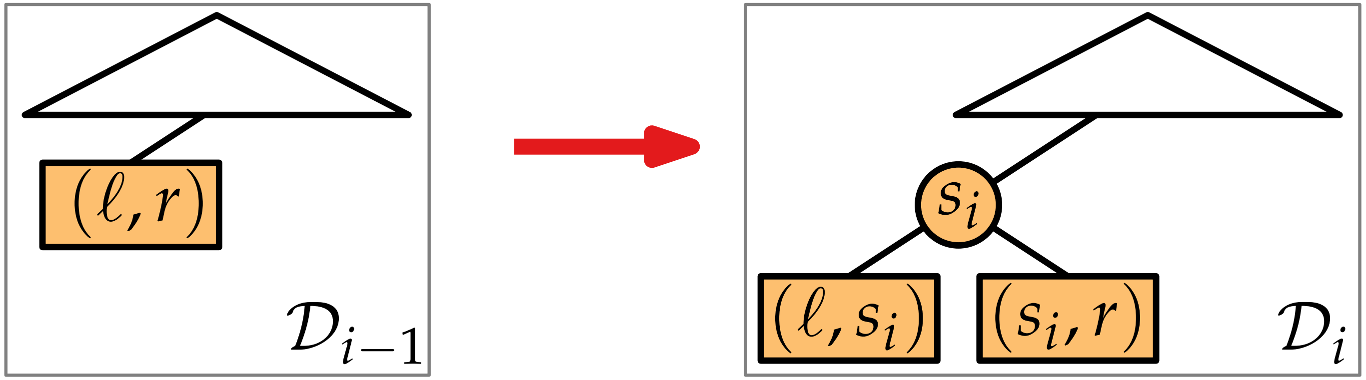

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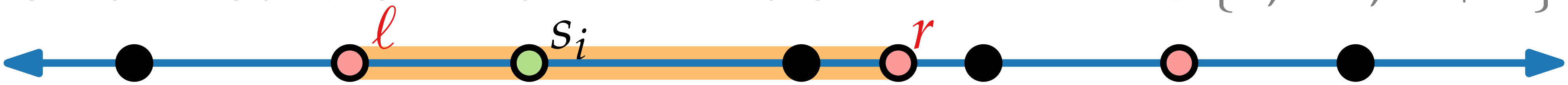

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Theorem. The randomized-incremental alg. preproc. a set S of n reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

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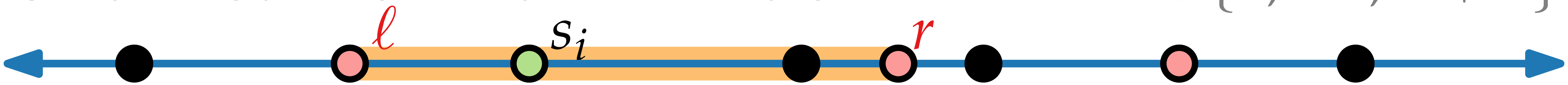

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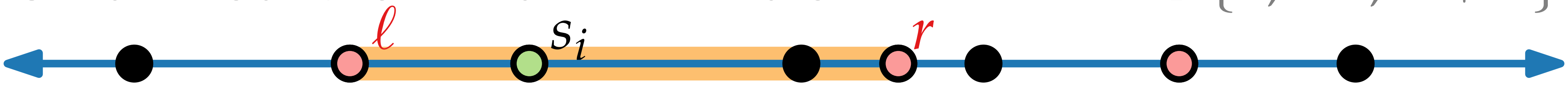

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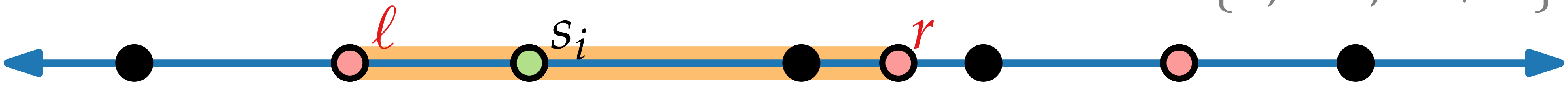

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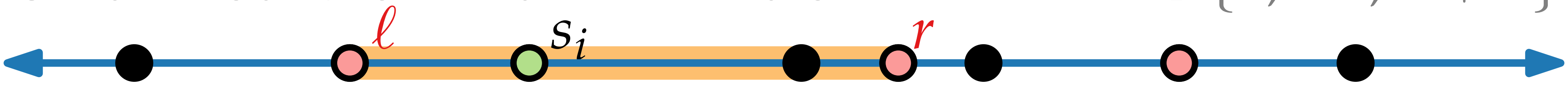

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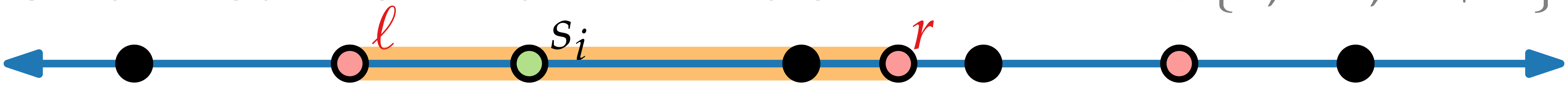

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Backwards analysis:
(see Lecture 04)

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Expected Query Time of $\mathcal{D}_n$

$E[X_i] = P[X_i = 1] =$

$= \text{probability that } I_i(q) \neq I_{i-1}(q)\text{, i.e., } s_i \in I_{i-1}(q).$

Backwards analysis:
(see Lecture 04)

Consider S_i fixed.

Define random variable $X_i = \begin{cases} 1 & \text{if } I_i(q) \neq I_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

$E[\text{query time in } \mathcal{D}_n] = E[\text{length search path in } \mathcal{D}_n] =$

$= E[\sum_{i=1}^{n} X_i] = \sum_{i=1}^{n} E[X_i] = \textbf{?}$

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Consider S_i fixed.
If we *remove* a randomly chosen pt from S_i, what's the probability that the interval containing q changes?

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Backwards analysis: (see Lecture 04)

Consider S_i fixed.
If we *remove* a randomly chosen pt from S_i, what's the probability that the interval containing q changes?
– we have i choices, identically distributed

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Expected Query Time of $\mathcal{D}_n$

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Backwards analysis: (see Lecture 04)

Consider S_i fixed.
If we *remove* a randomly chosen pt from S_i, what's the probability that the interval containing q changes?
– we have i choices, identically distributed
– at most two of these change the interval

Define random variable $X_i = \begin{cases} 1 & \text{if } I_i(q) \neq I_{i-1}(q), \\ 0 & \text{else.} \end{cases}$

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Expected Query Time of $\mathcal{D}_n$

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Expected Query Time of $\mathcal{D}_n$

$$E[X_i] = P[X_i = 1] = 2/i$$
$$= \text{probability that } I_i(q) \neq I_{i-1}(q) \text{, i.e., } s_i \in I_{i-1}(q).$$

Backwards analysis:
(see Lecture 04)

Consider S_i fixed.
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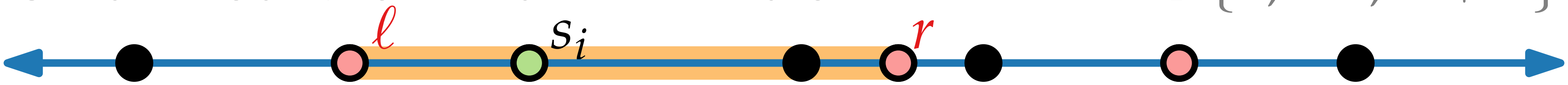
$O(\log n)$

The 1D Result

Given a set S of n real numbers... $i \in \{1, \ldots, n+1\}$



$S_{i-1} := \{s_1, \ldots, s_{i-1}\}$, I_{i-1} := set of conn. comp. of $\mathbb{R} \setminus S_{i-1}$

Theorem. The randomized-incremental alg. preproc. a set S of n reals in $O(n \log n)$ expected time such that a query takes $O(\log n)$ expected time.

Computational Geometry

Lecture 6:
Point Localization
or
Where am I?

Part IV:
The 2D Problem

Philipp Kindermann

Winter Semester 2020

The 2D Problem

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

The 2D Problem

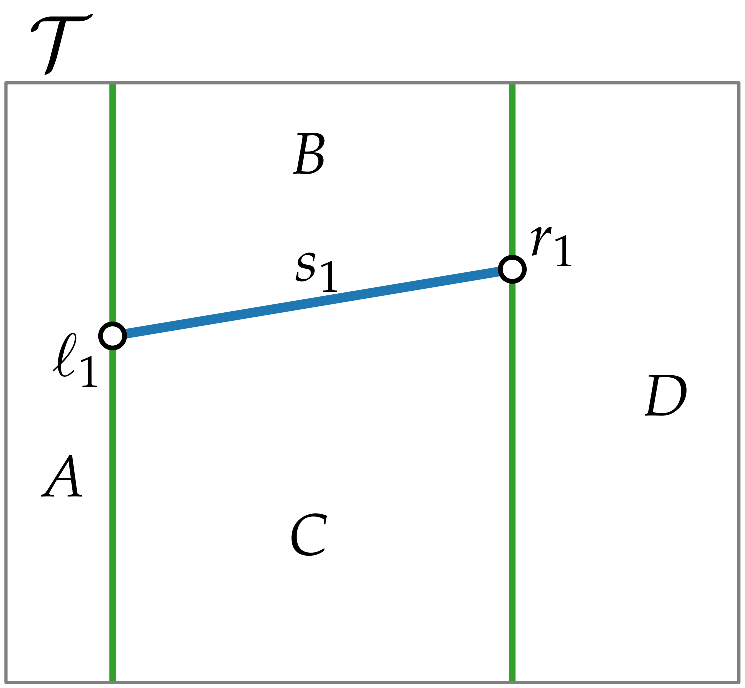
trapezoidal map → $\mathcal{T}$

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

The 2D Problem

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

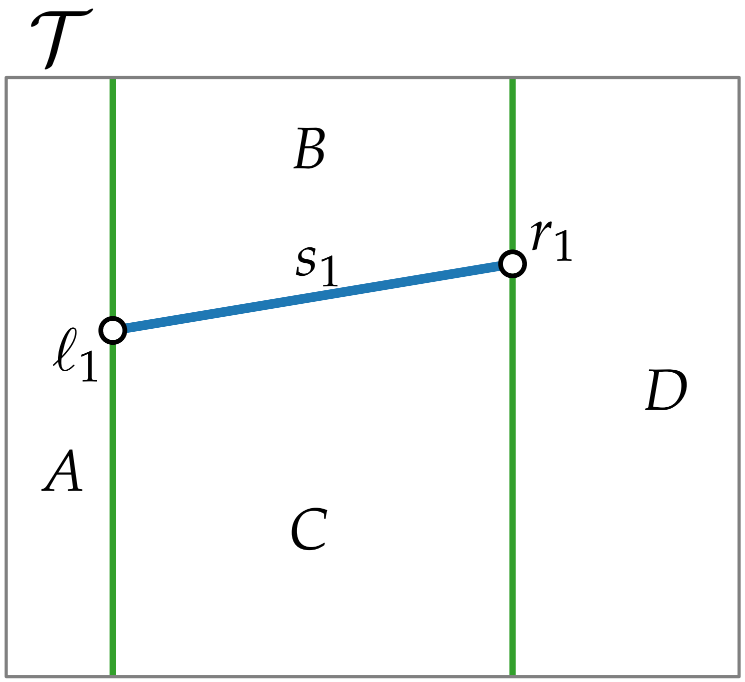


The 2D Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$



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point-location data structure (DAG)

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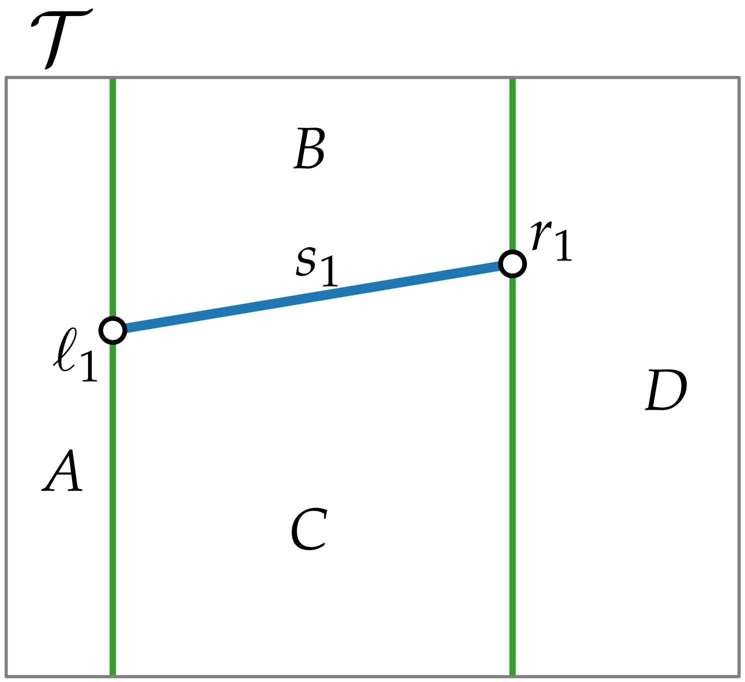
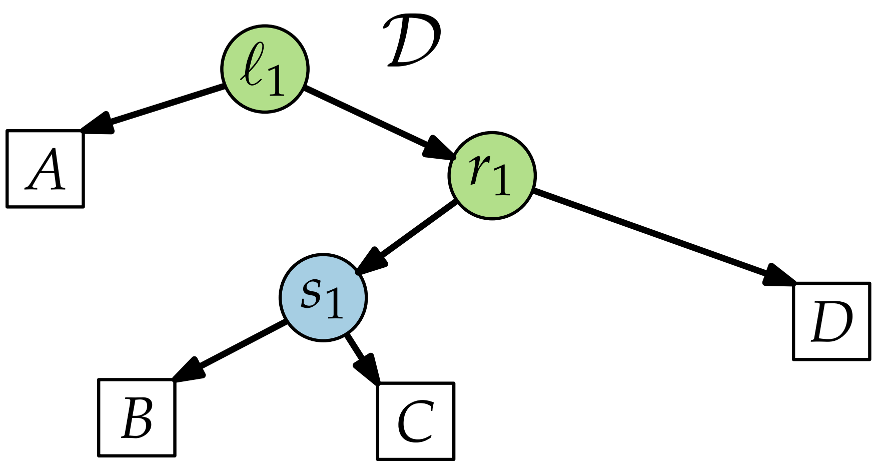
Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

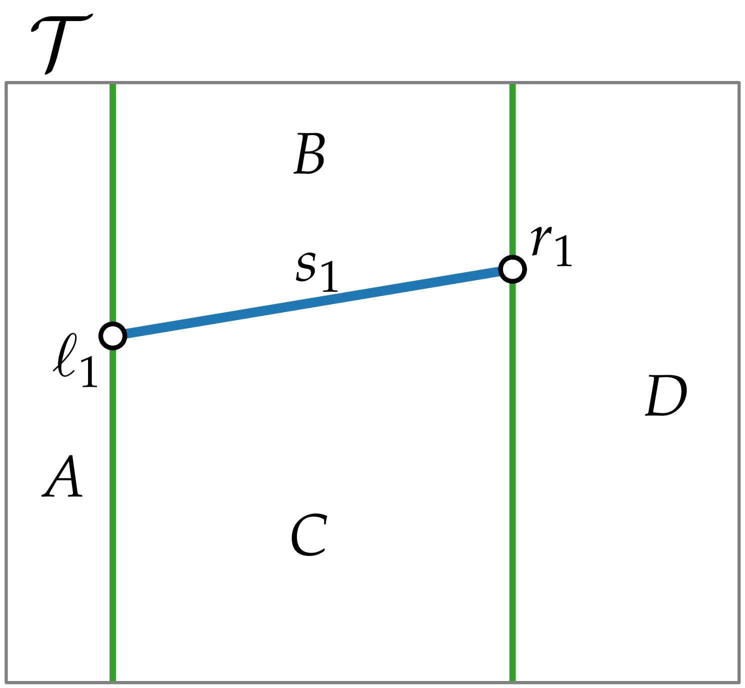


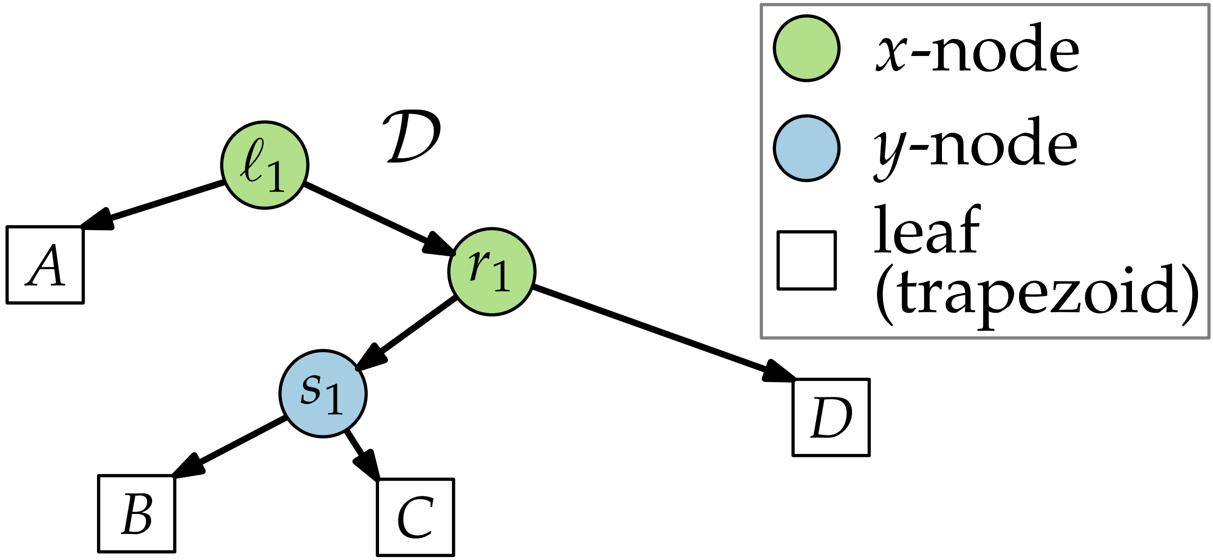


The 2D Problem

point-location data structure (DAG) → $\mathcal{D}$
trapezoidal map → $\mathcal{T}$

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

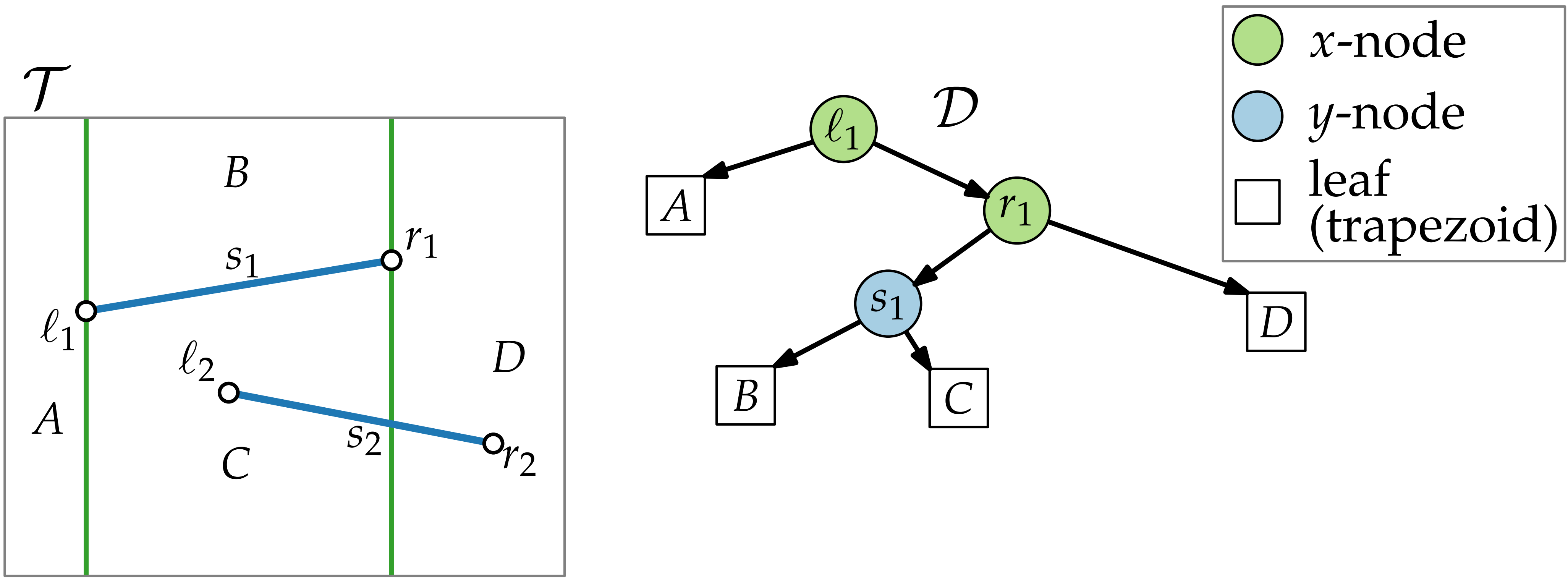




The 2D Problem

point-location data structure (DAG)
trapezoidal map

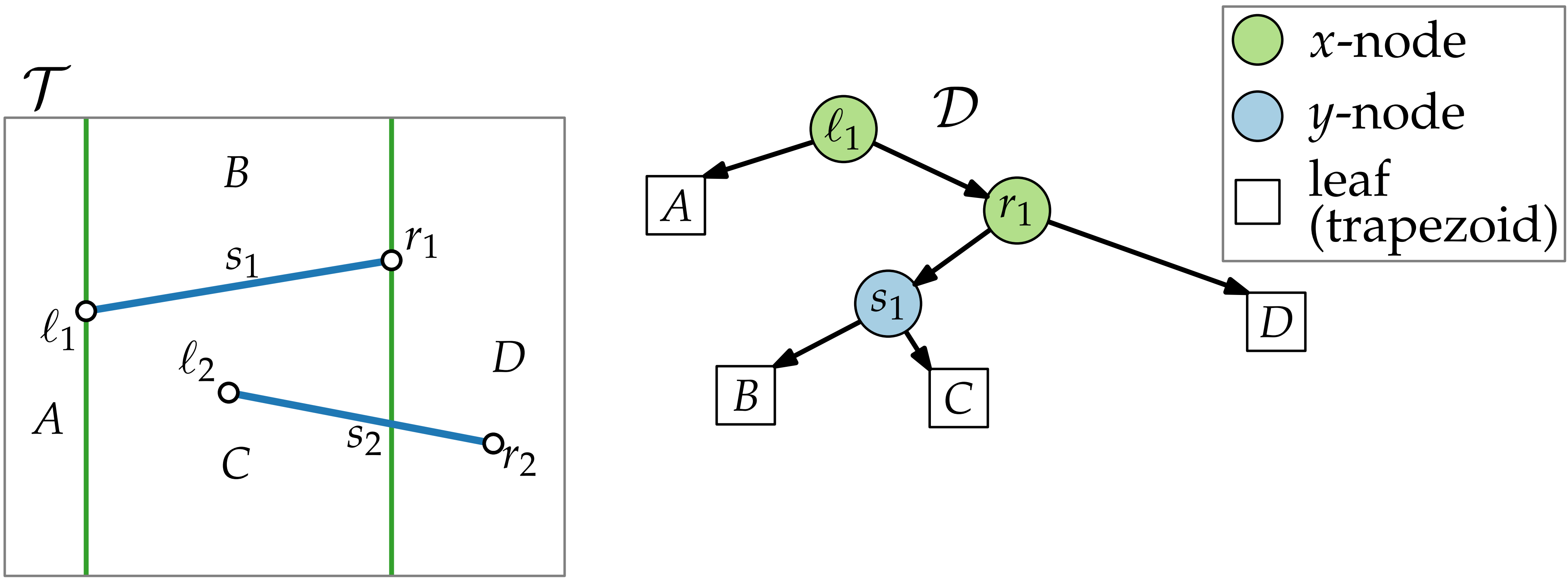
Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$



The 2D Problem

point-location data structure (DAG)
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$
– use $\mathcal{D}$ to locate left endpoint of next segment s



The 2D Problem

point-location data structure (DAG)

trapezoidal map

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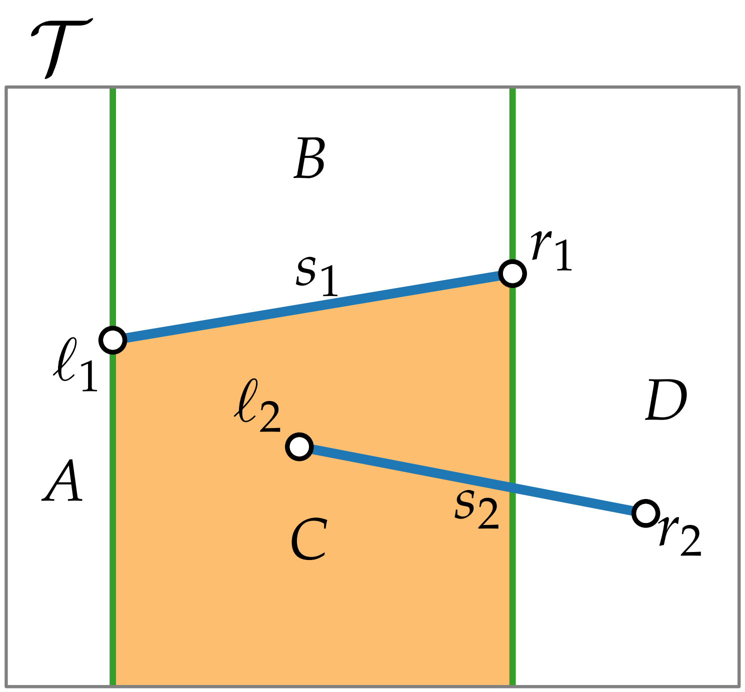
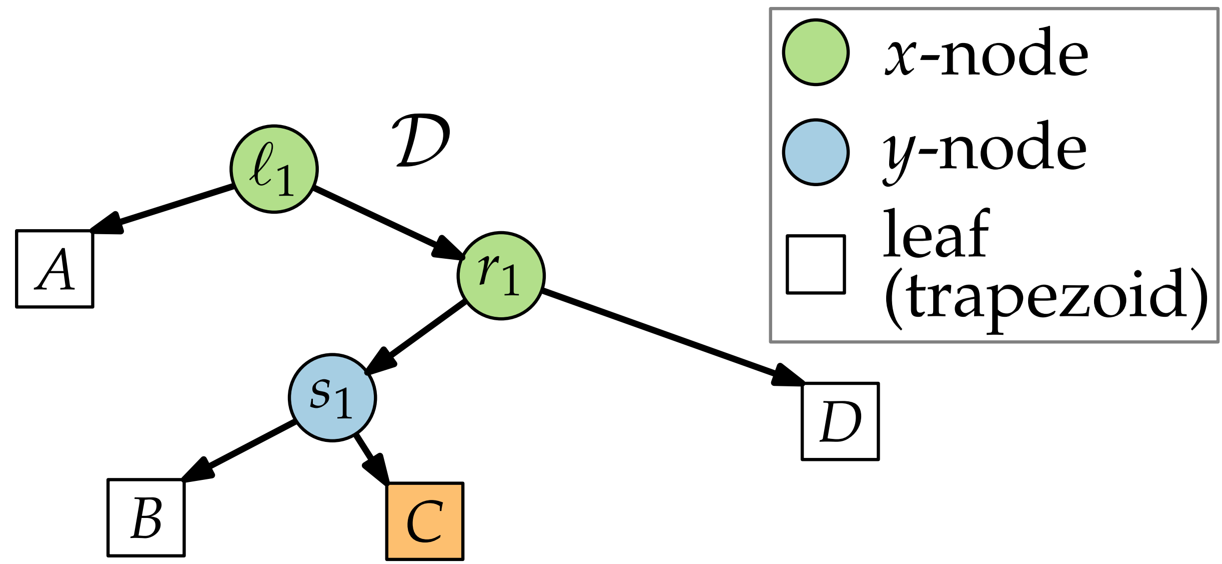




The 2D Problem

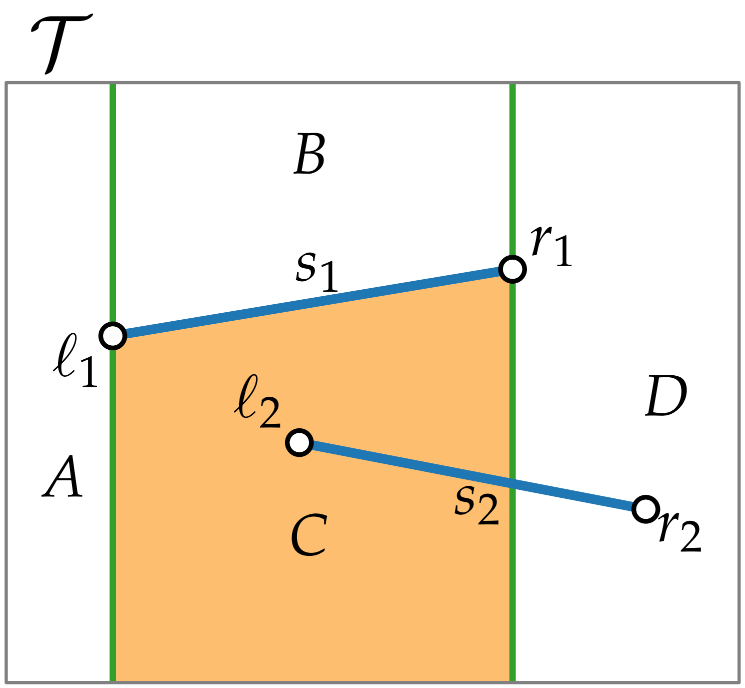
point-location data structure (DAG)

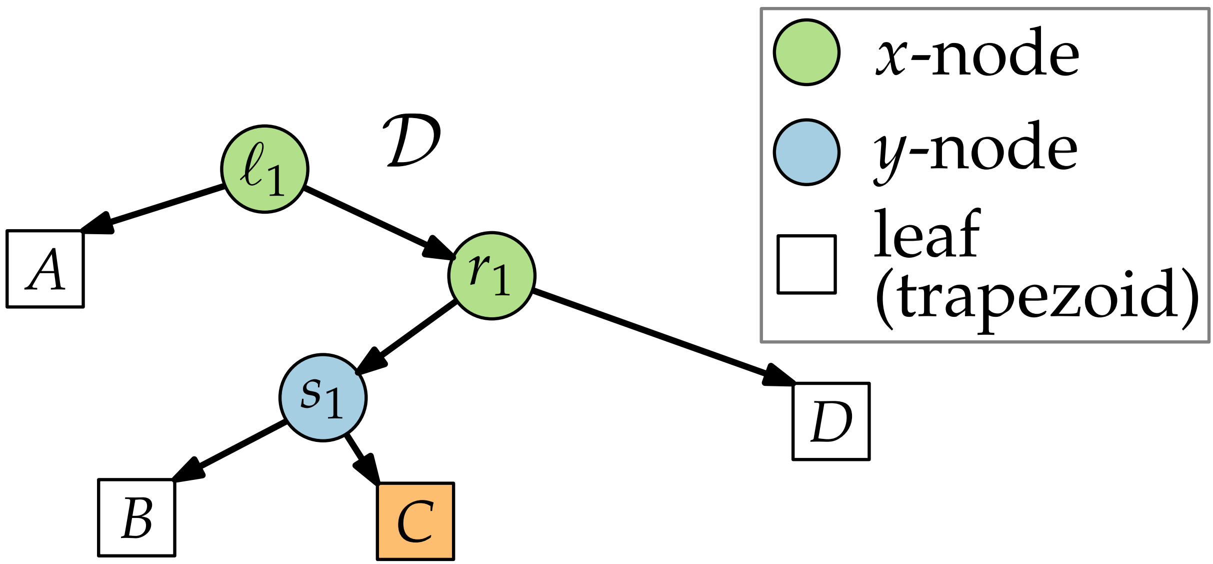
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

– use $\mathcal{D}$ to locate left endpoint of next segment s

– "walk" along s through $\mathcal{T}$





The 2D Problem

point-location data structure (DAG)

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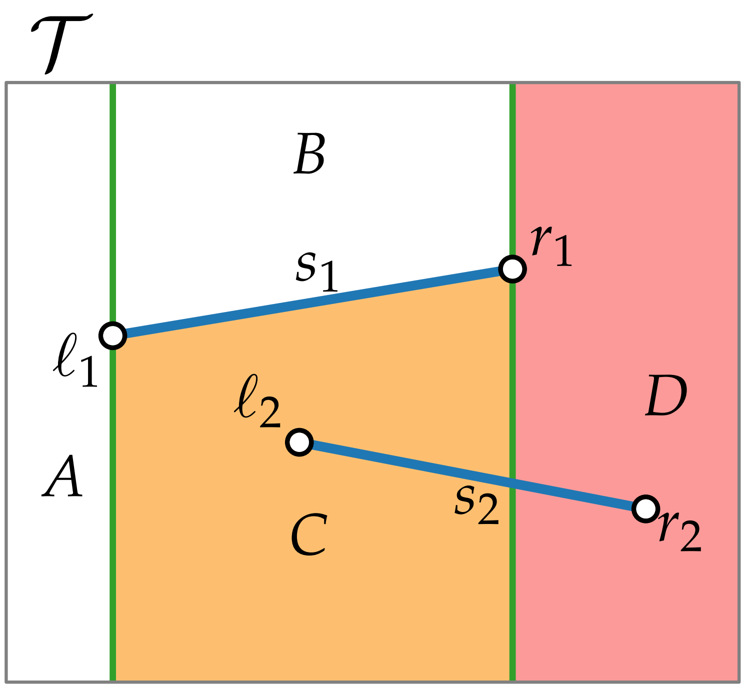
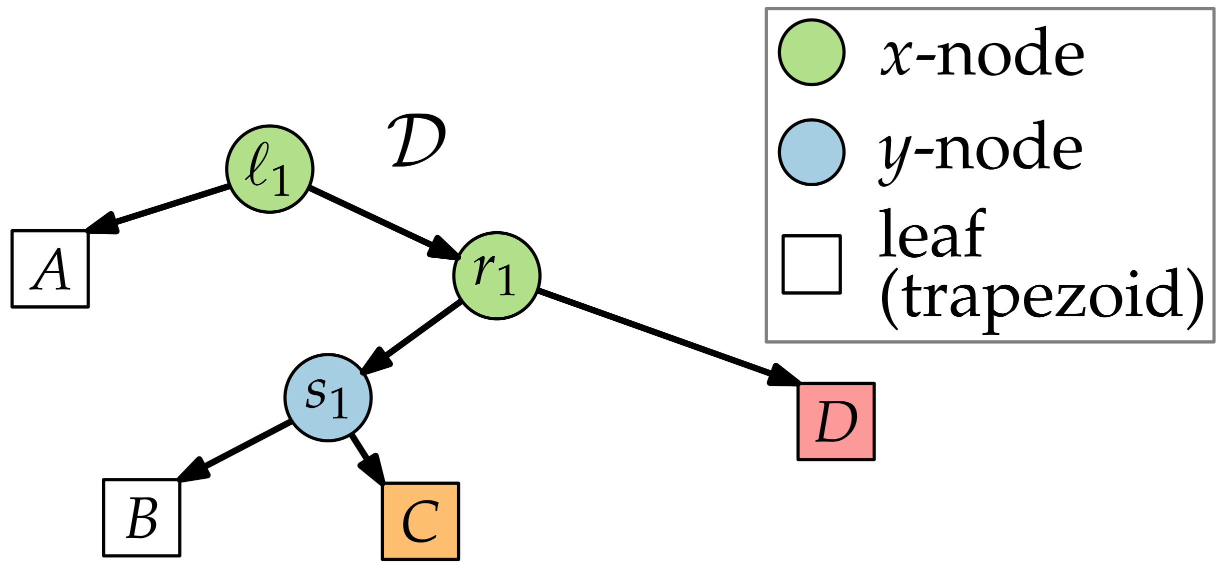




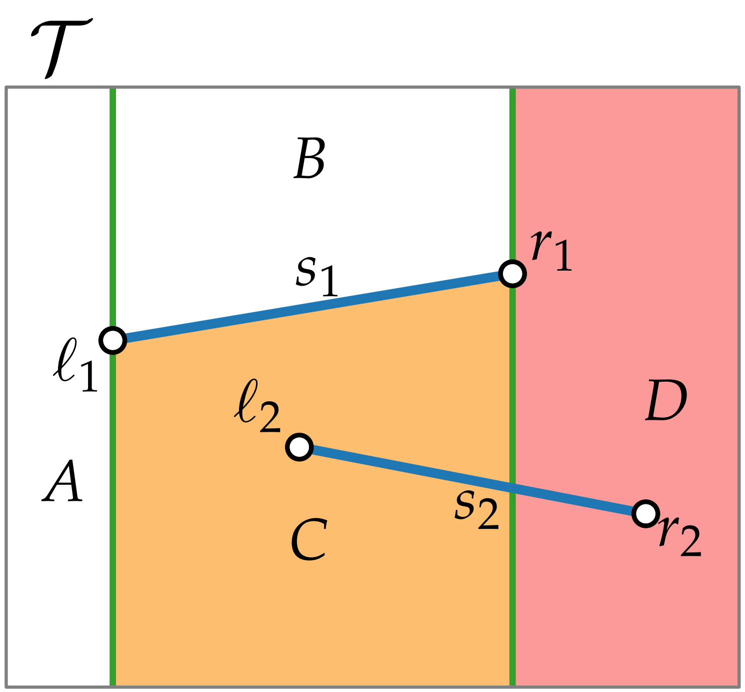
The 2D Problem

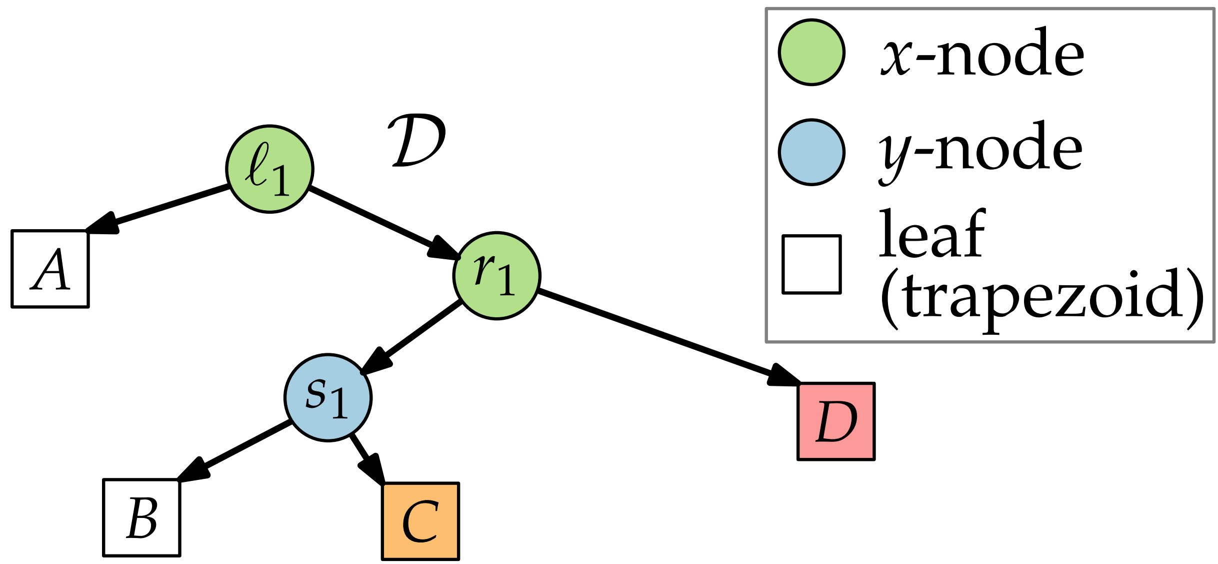
point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
- "walk" along s through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting s





The 2D Problem

point-location data structure (DAG)

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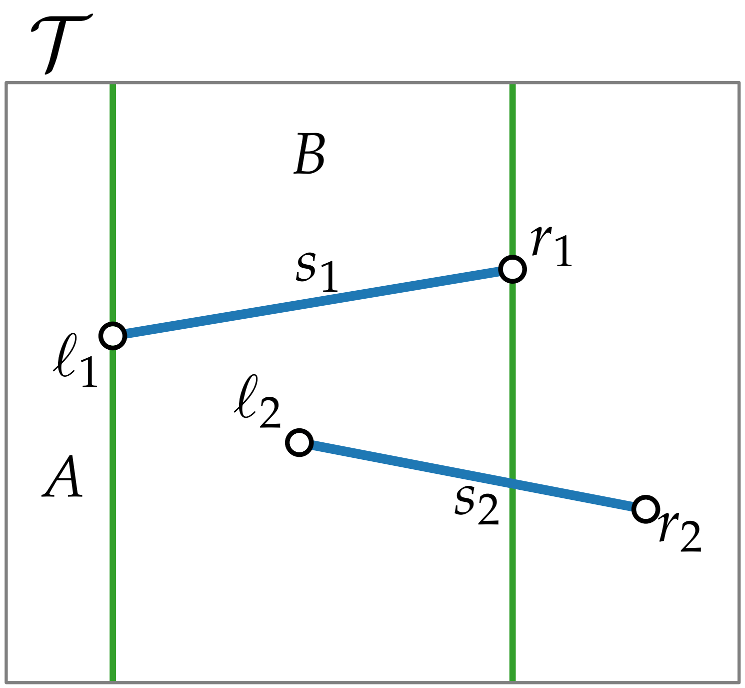

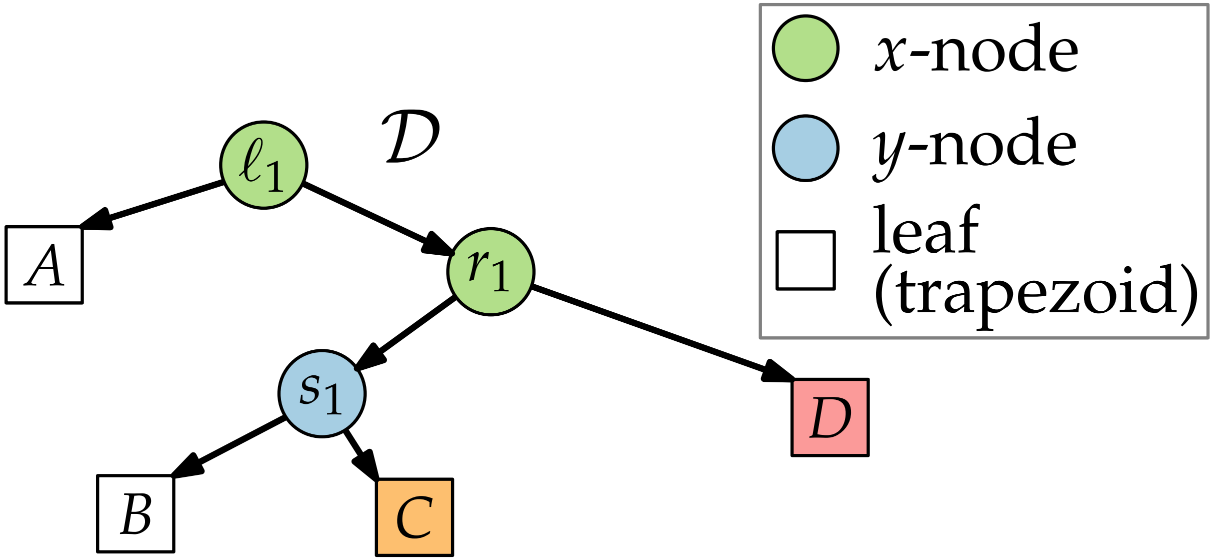

The 2D Problem

point-location data structure (DAG)
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
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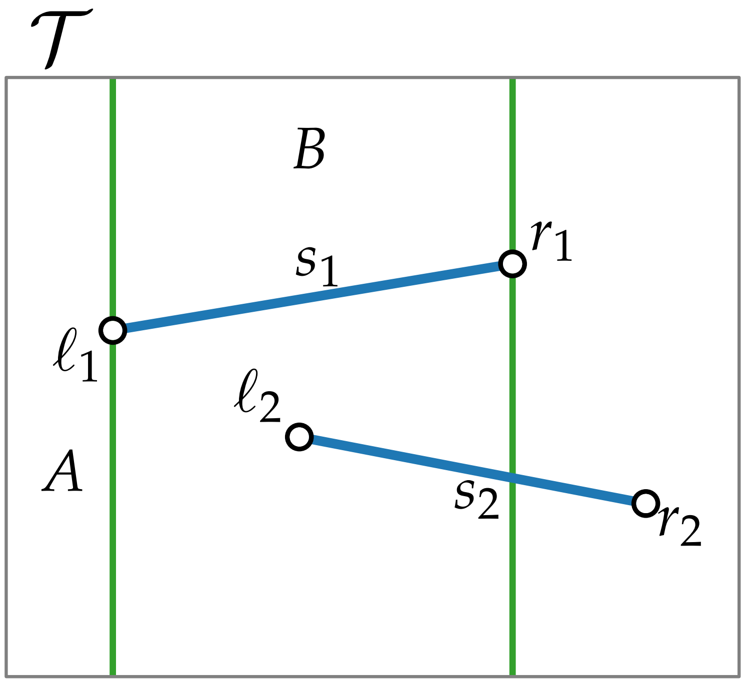

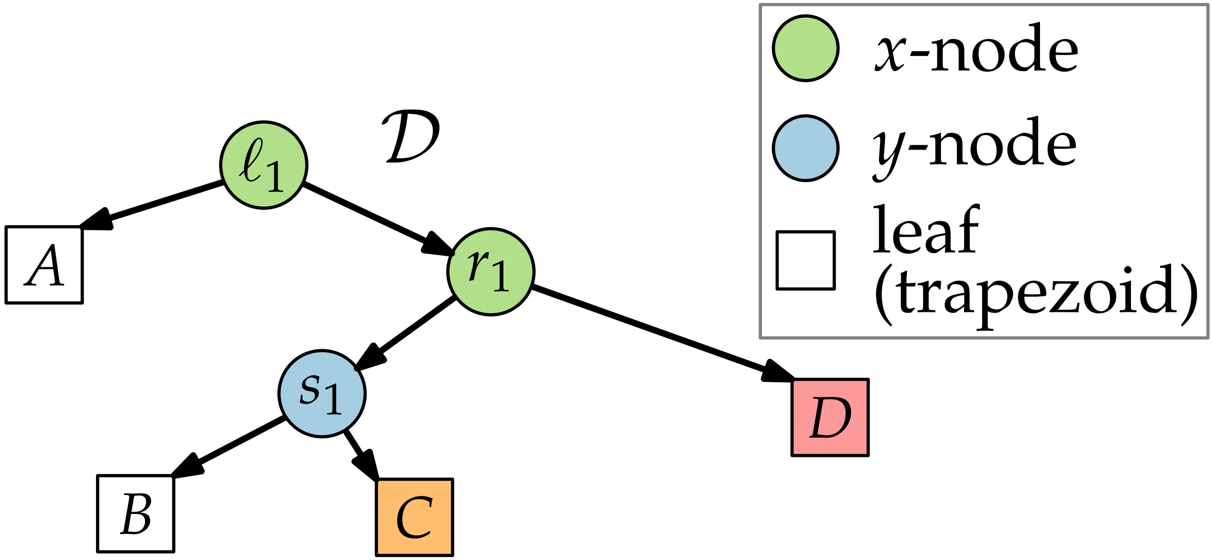

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point-location data structure (DAG)

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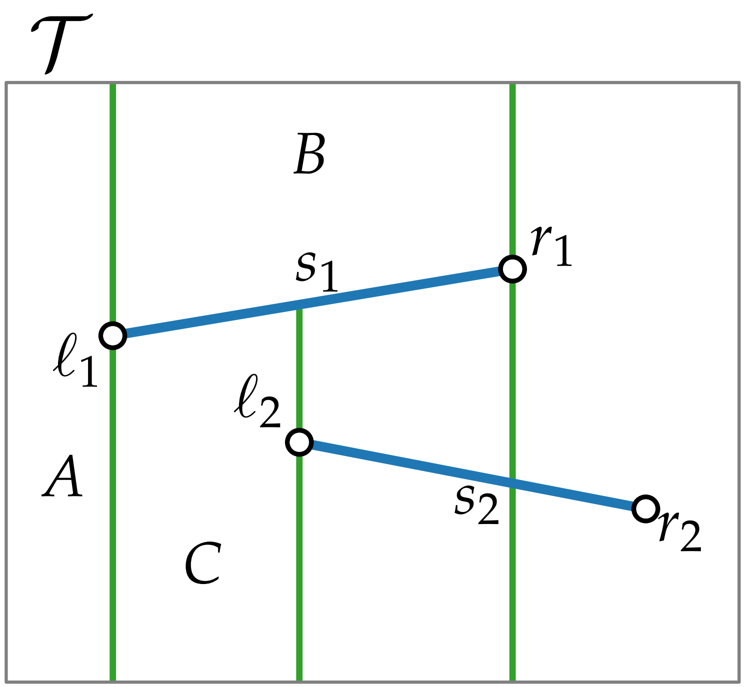

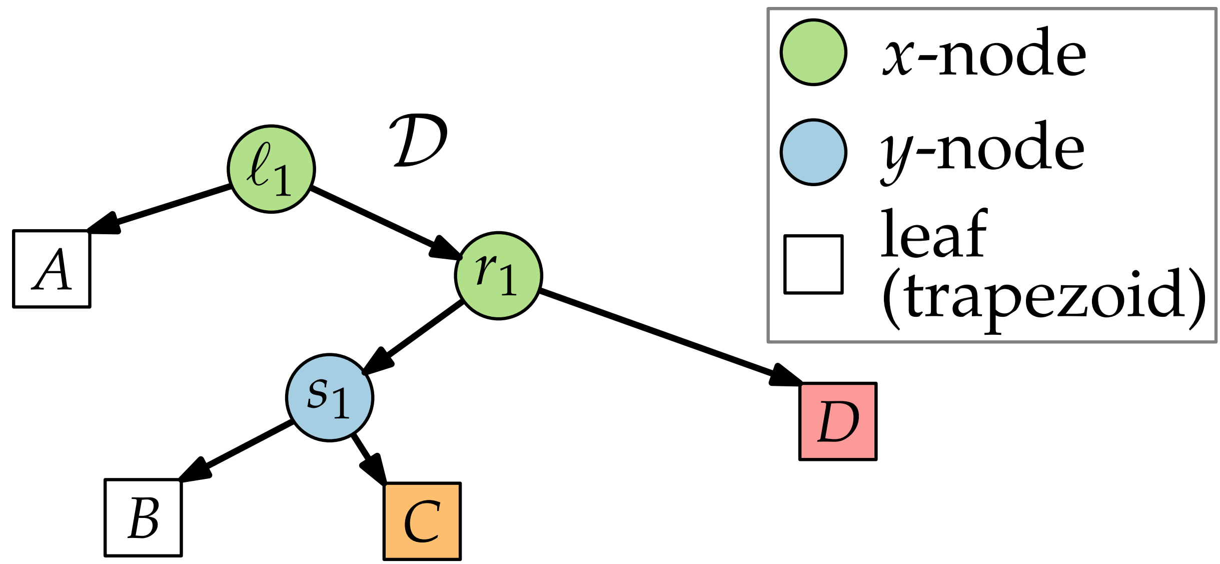

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point-location data structure (DAG)
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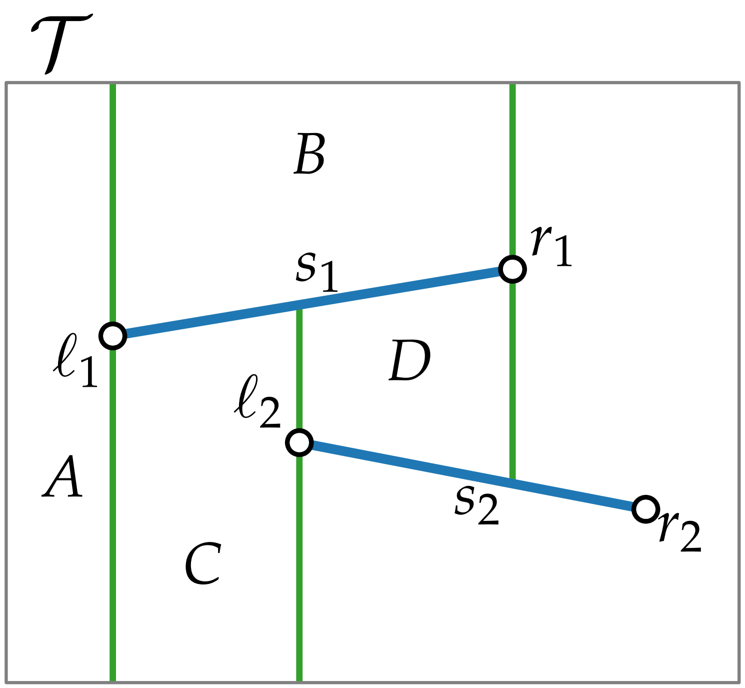

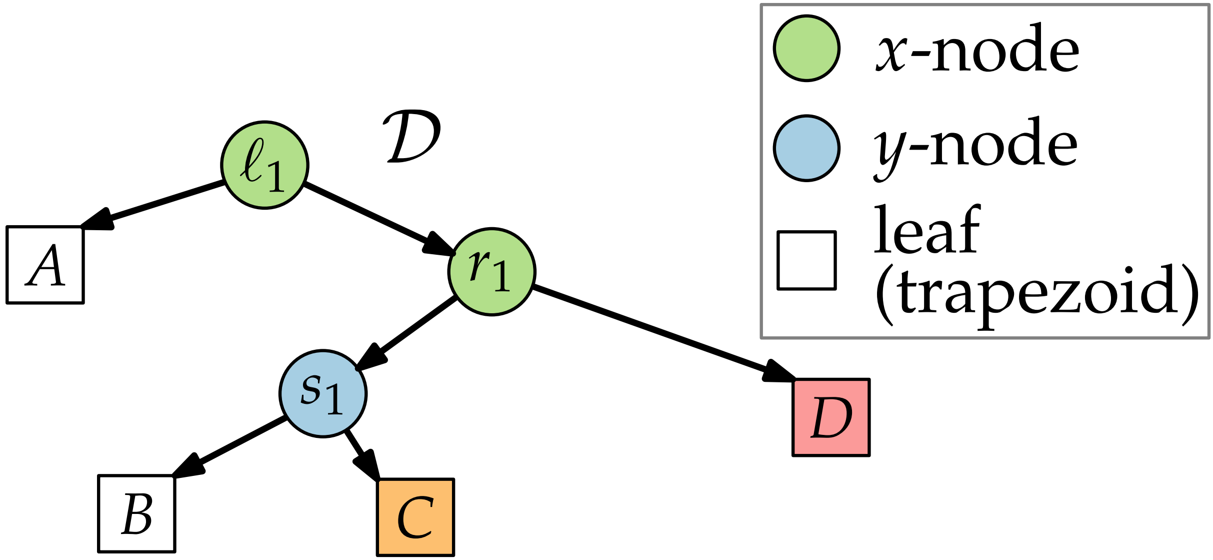

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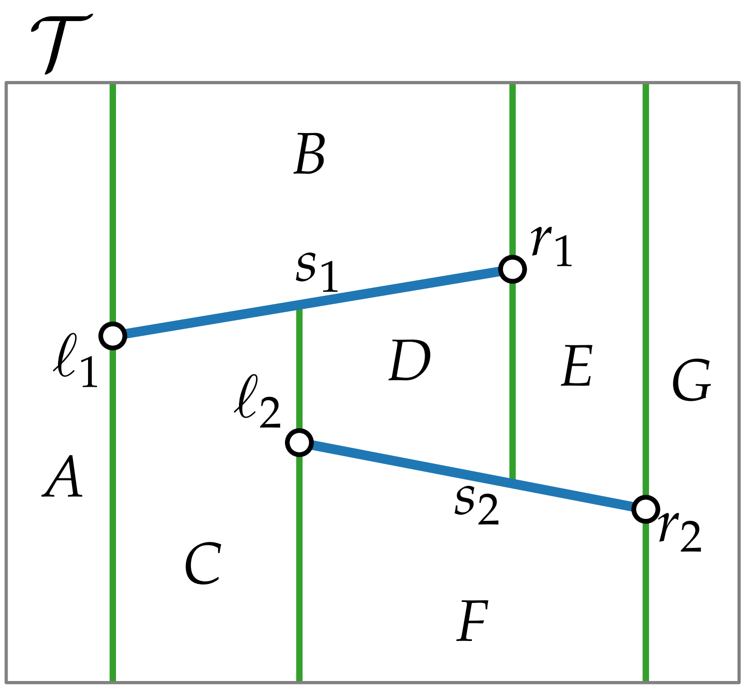

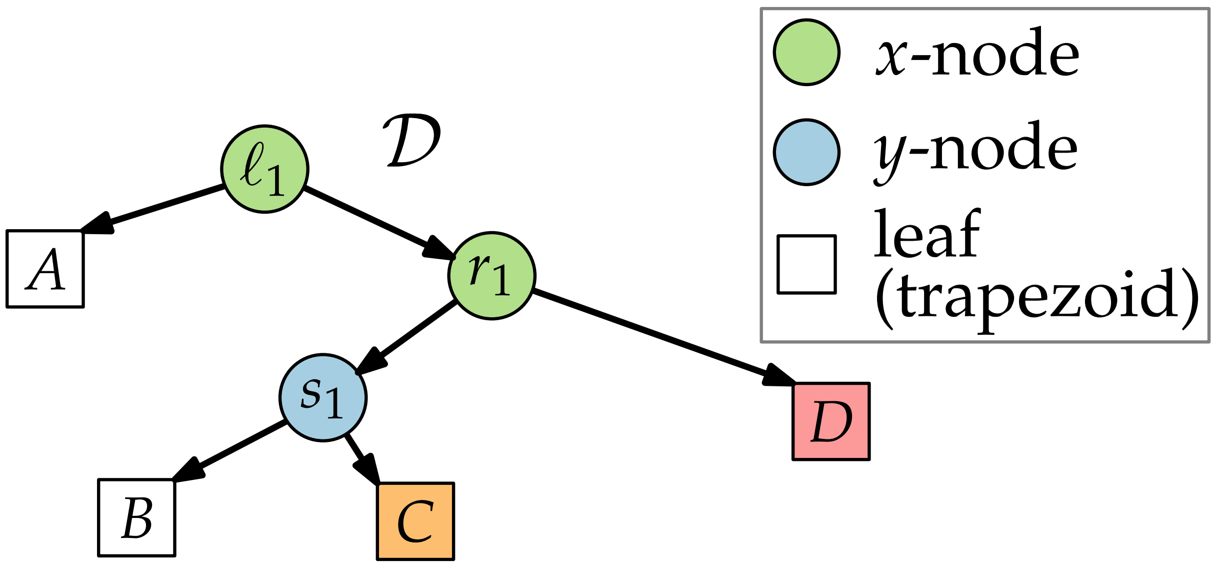

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point-location data structure (DAG)

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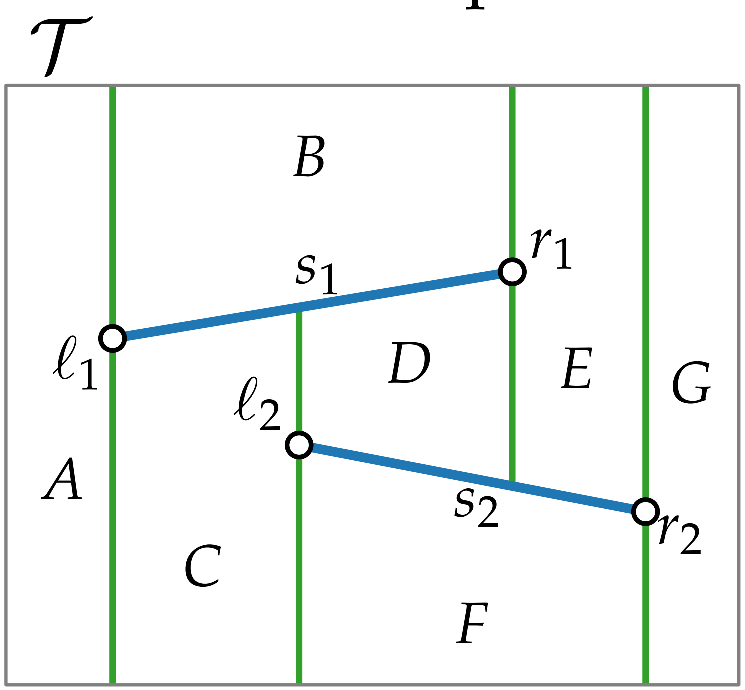

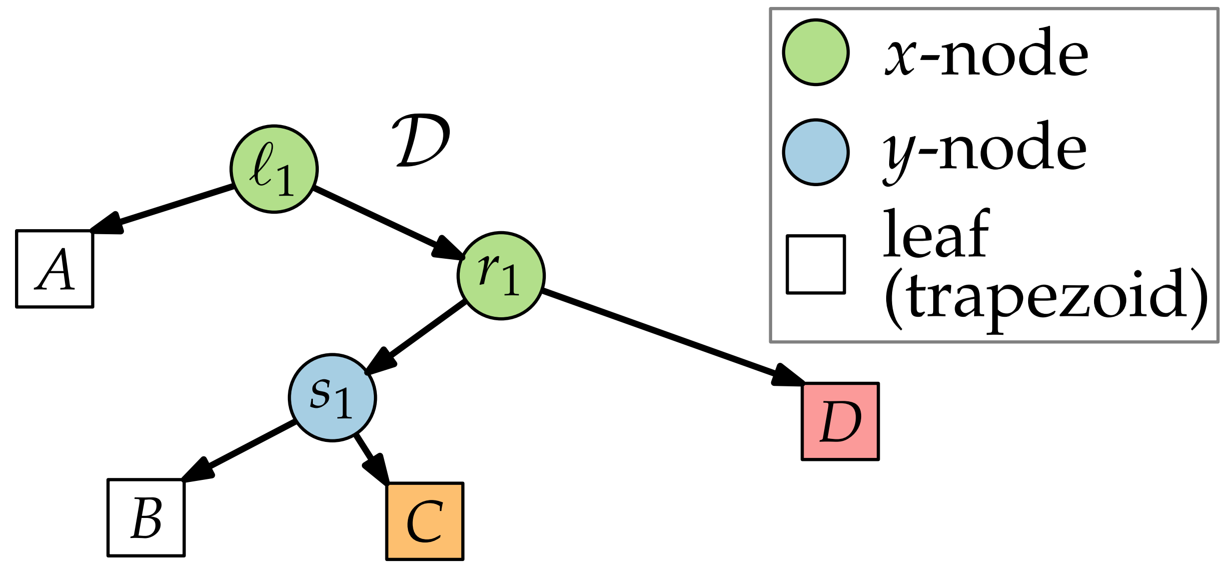

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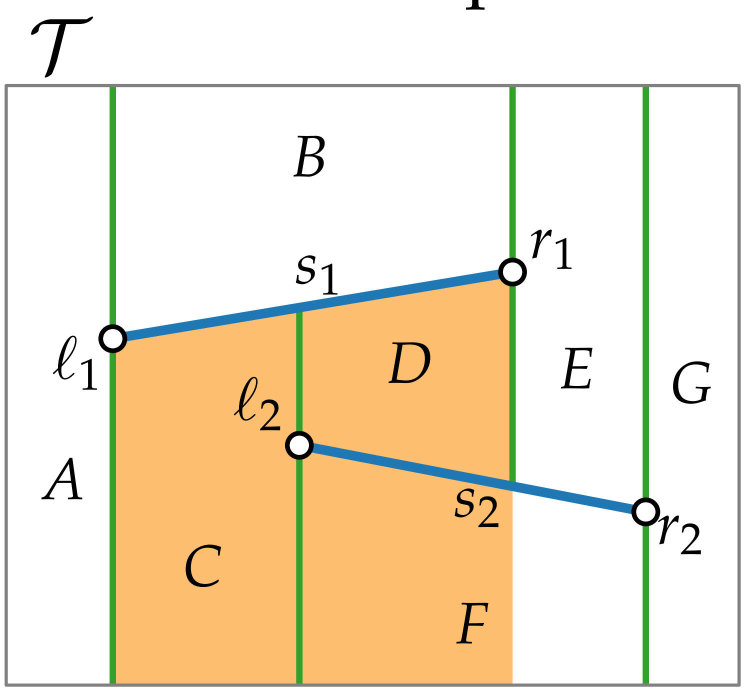

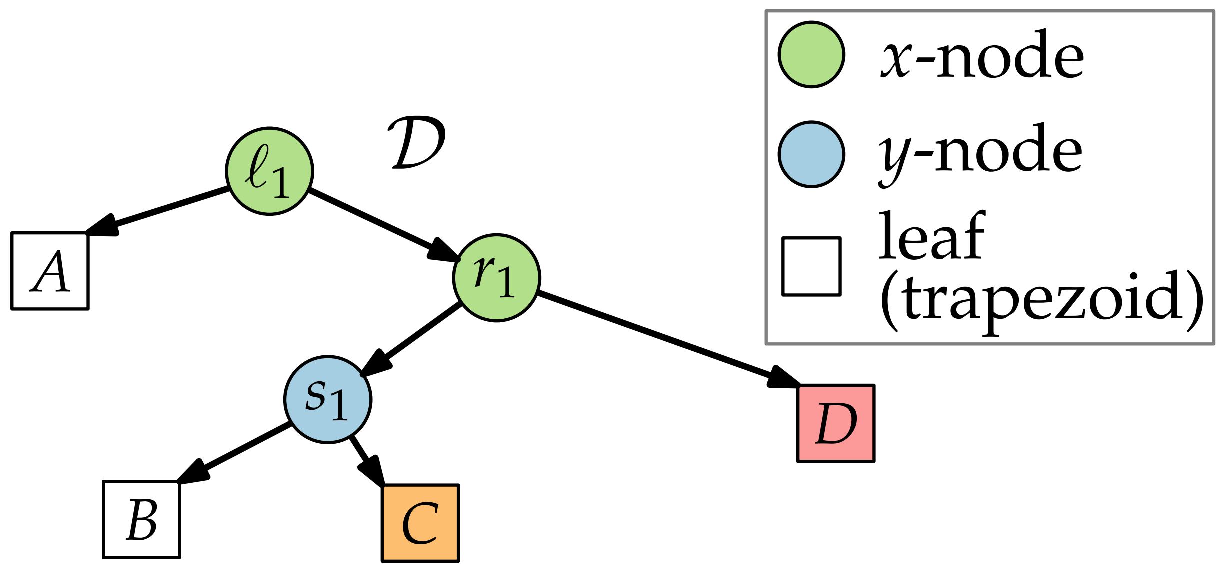

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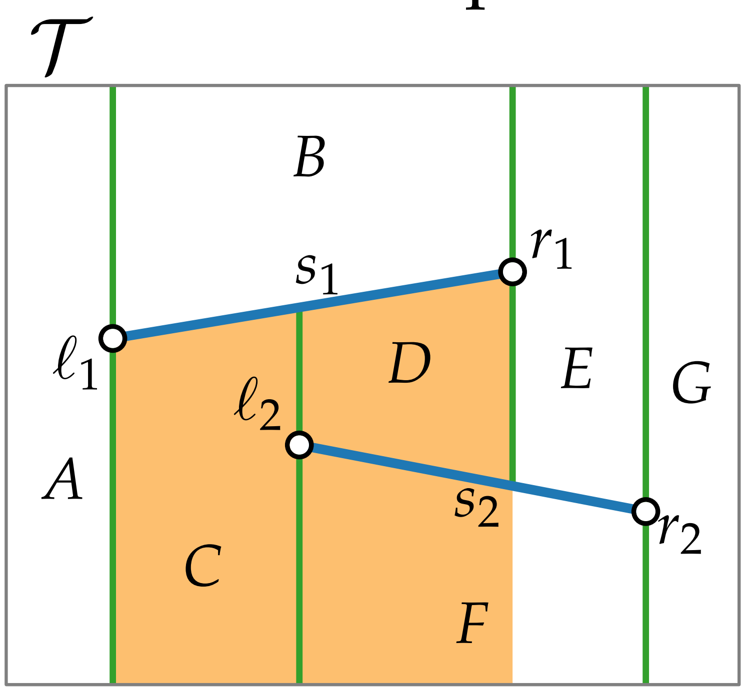

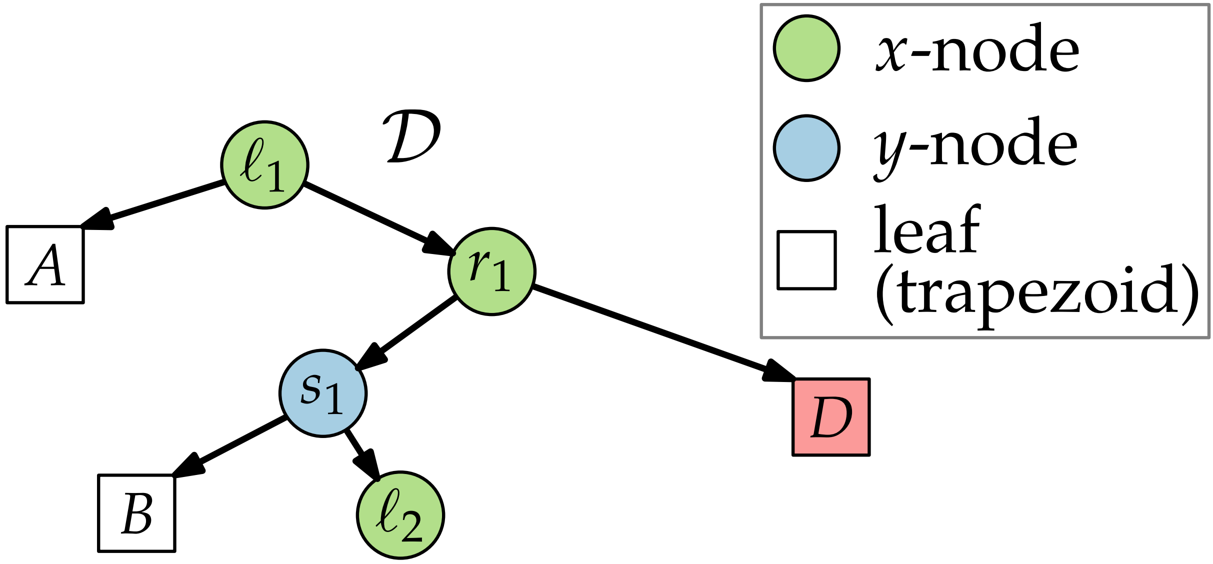

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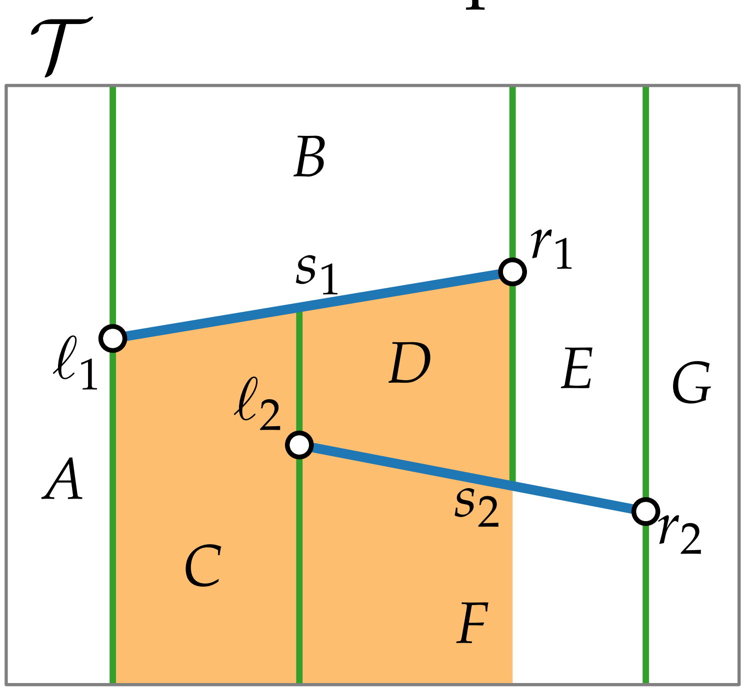

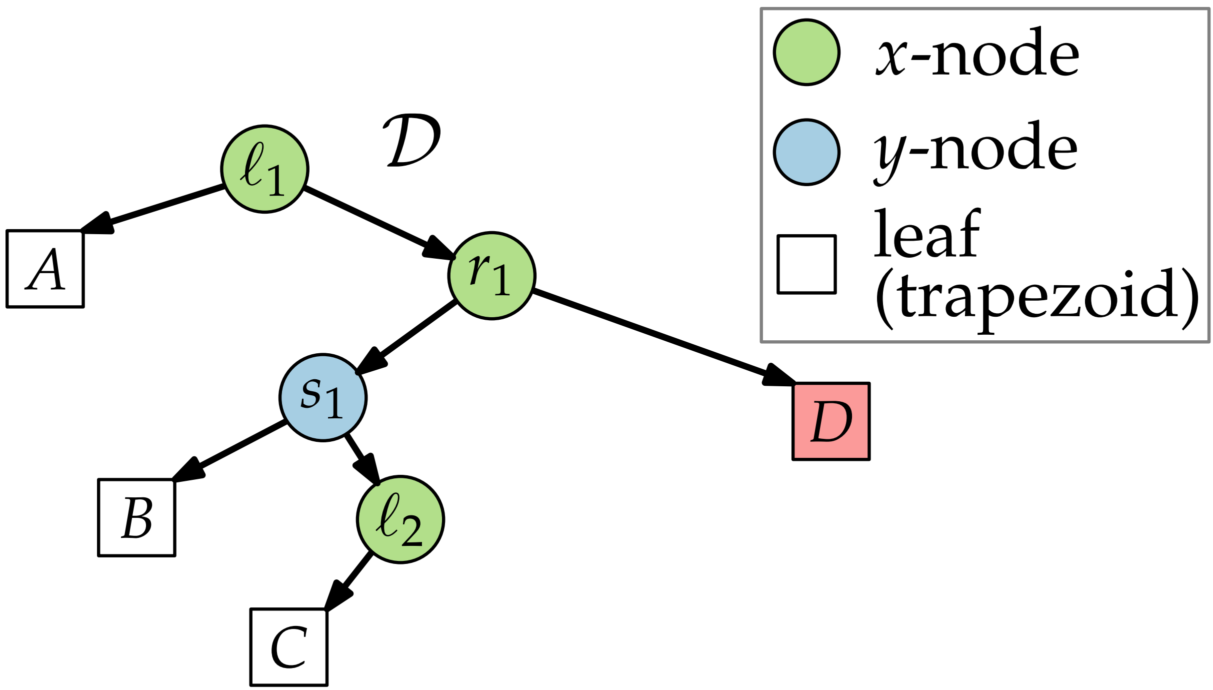

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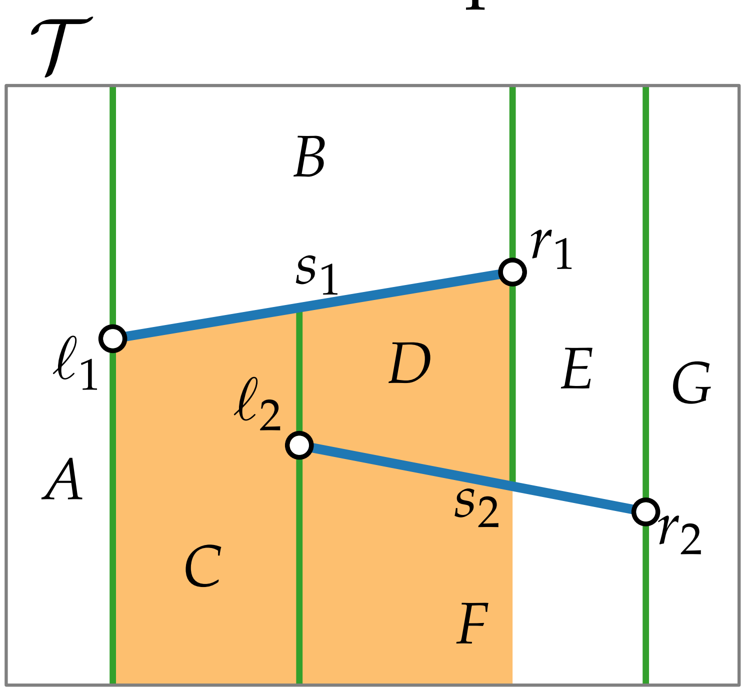
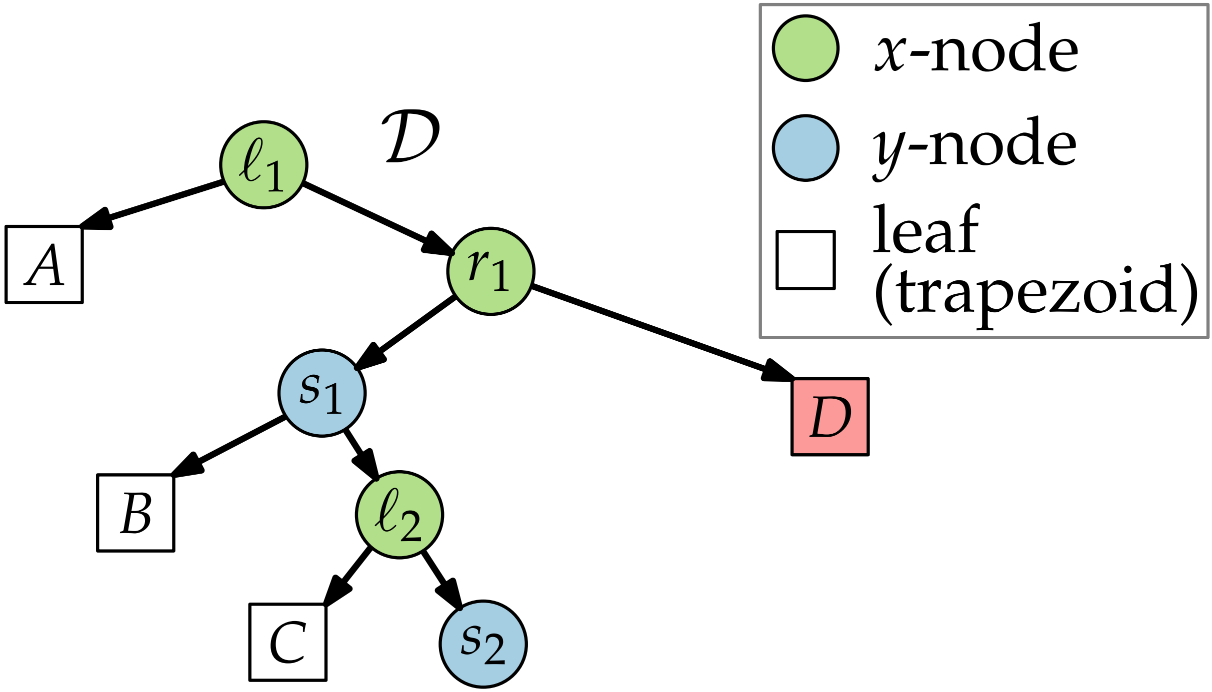


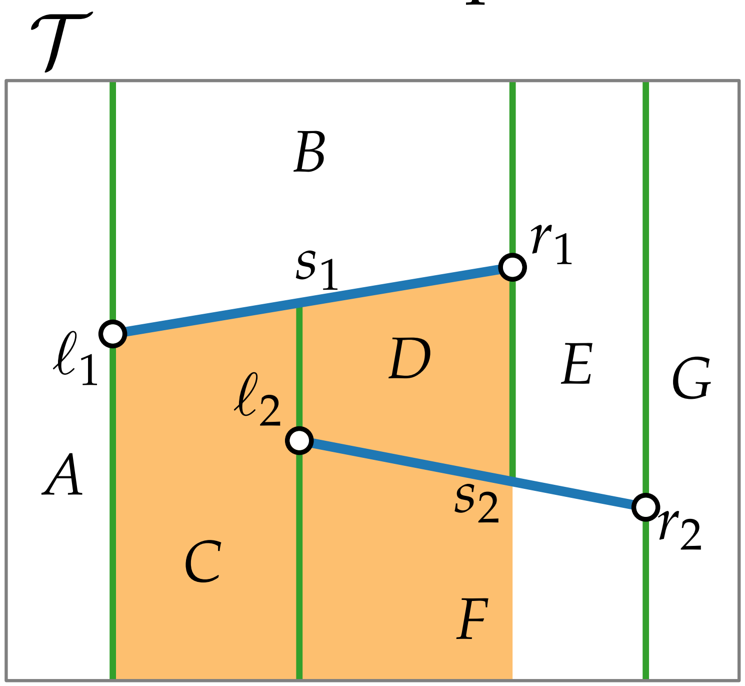


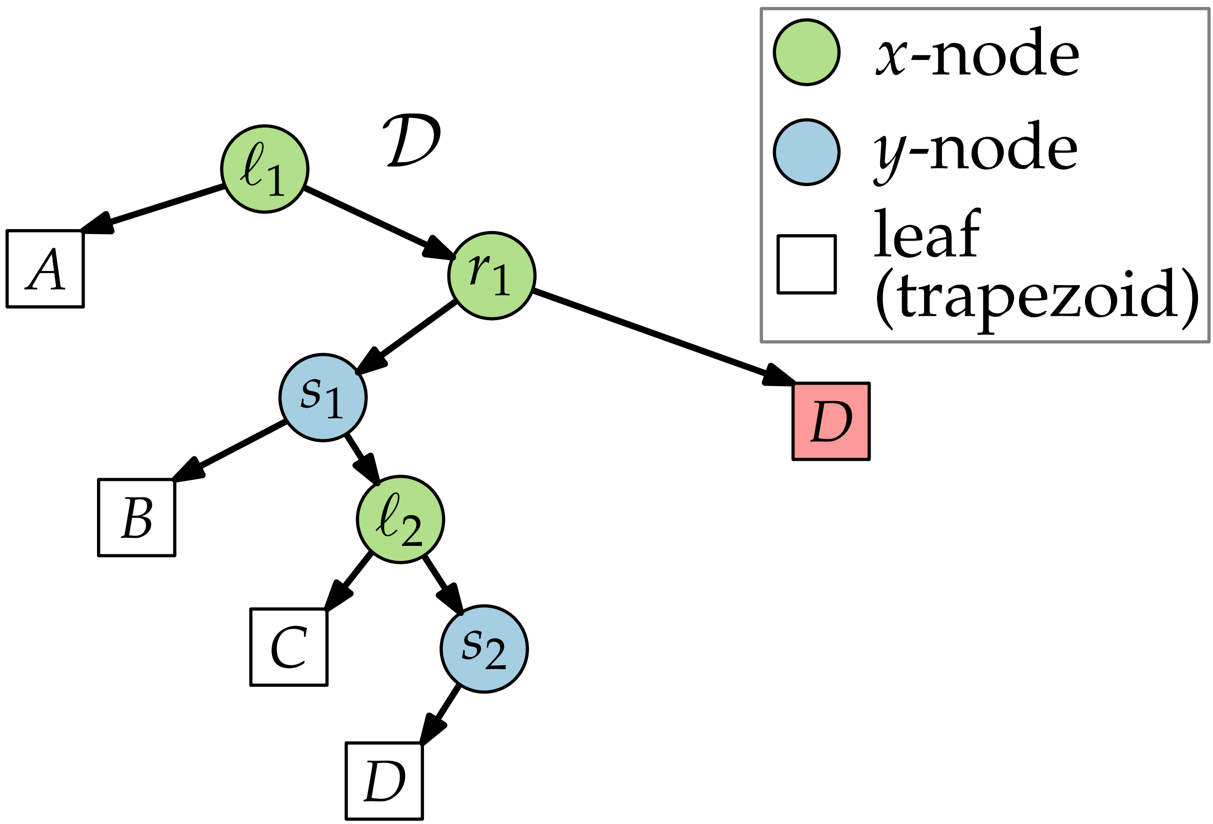
The 2D Problem

point-location data structure (DAG)
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
- "walk" along s through $\mathcal{T}$
- destroy all trapezoids of $\mathcal{T}$ intersecting s
- construct new trapezoids of $\mathcal{T}$ (adjacent to s)
- update $\mathcal{D}$





The 2D Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
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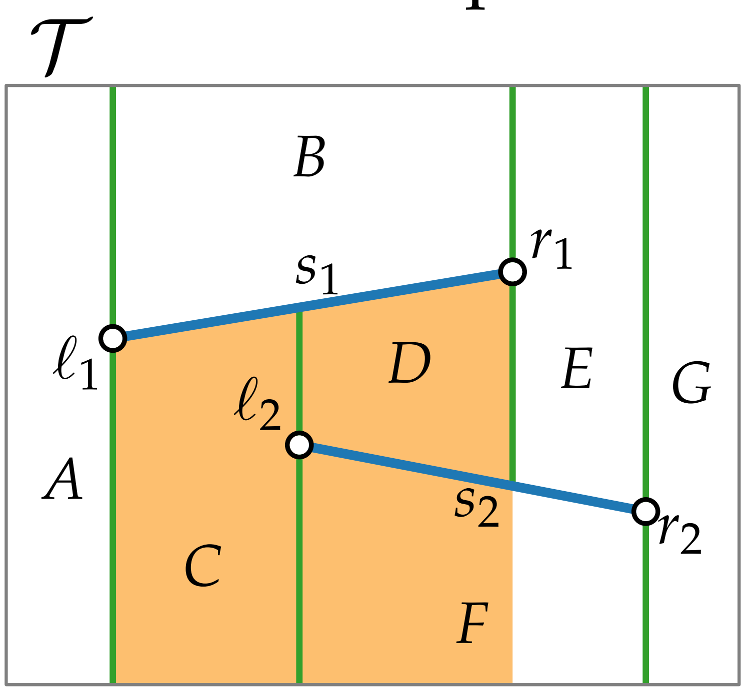

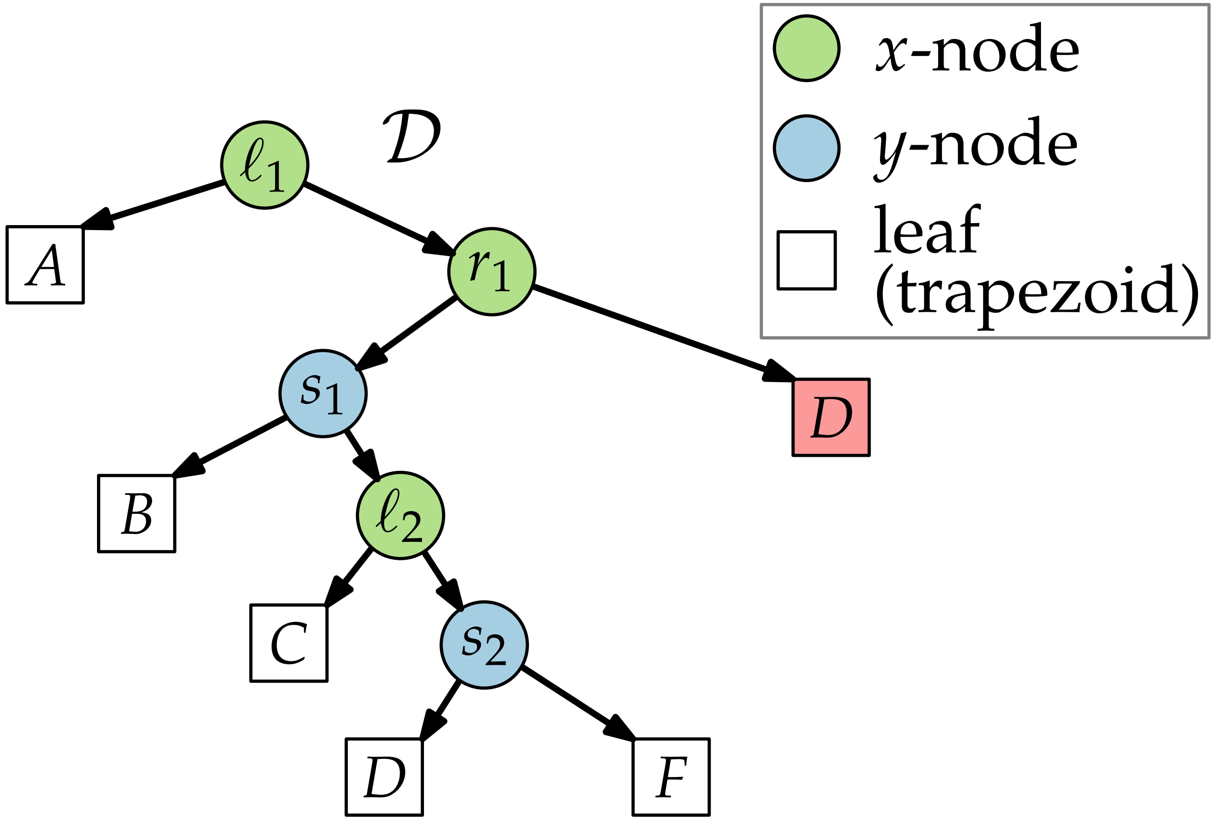

The 2D Problem

point-location data structure (DAG)
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

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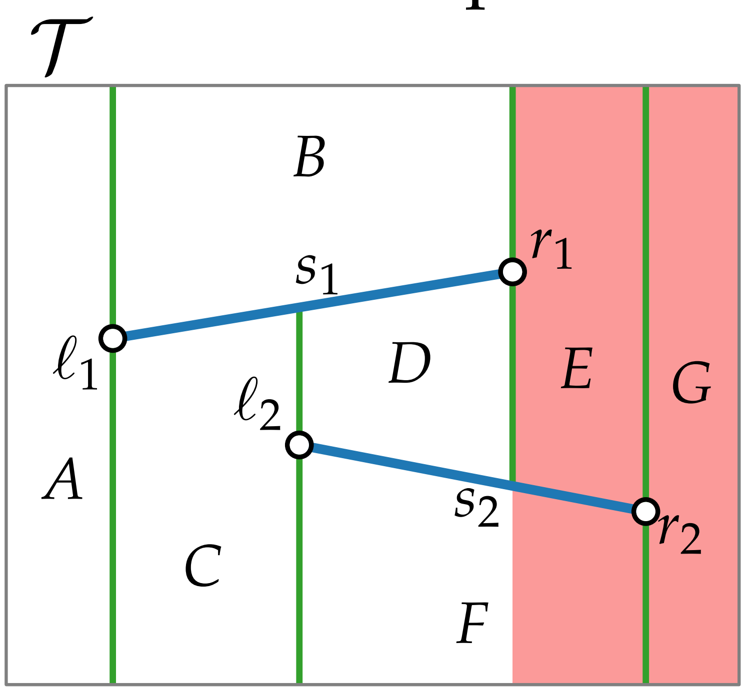
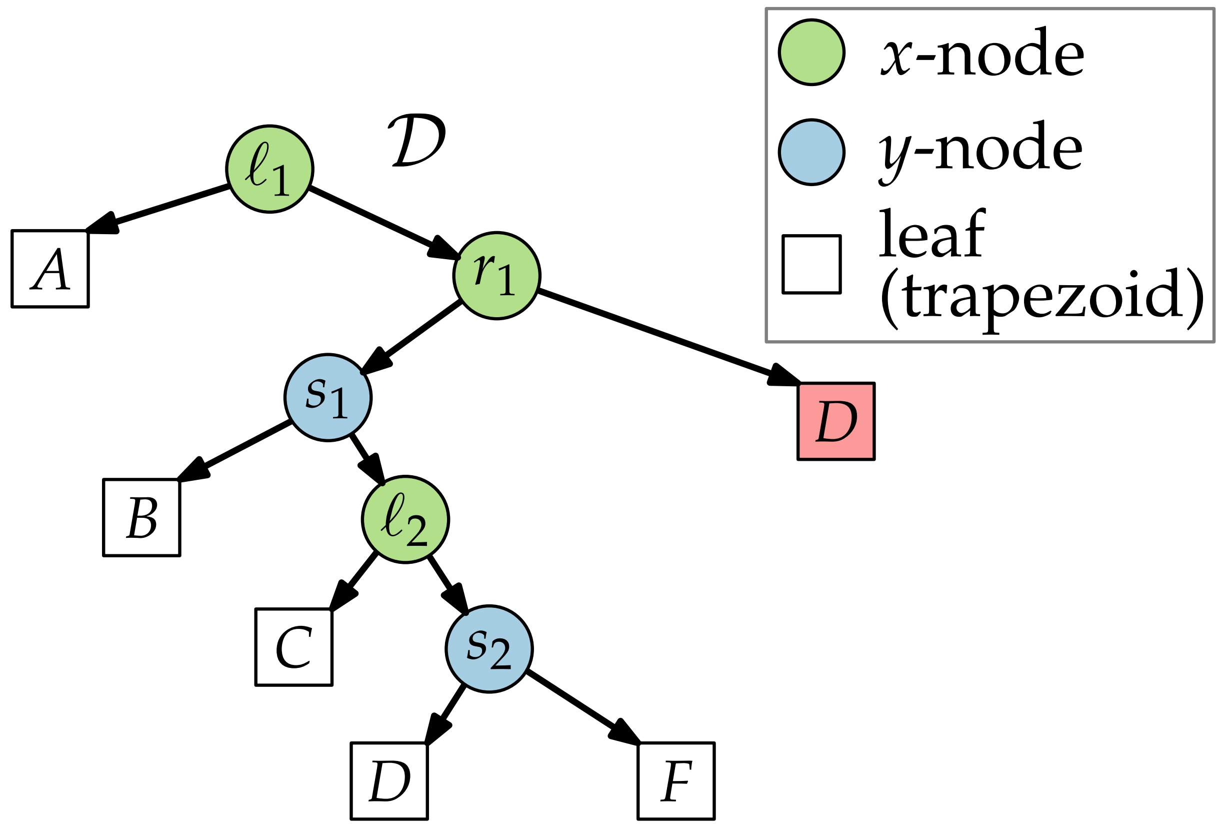
- update $\mathcal{D}$





The 2D Problem

point-location data structure (DAG)
trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

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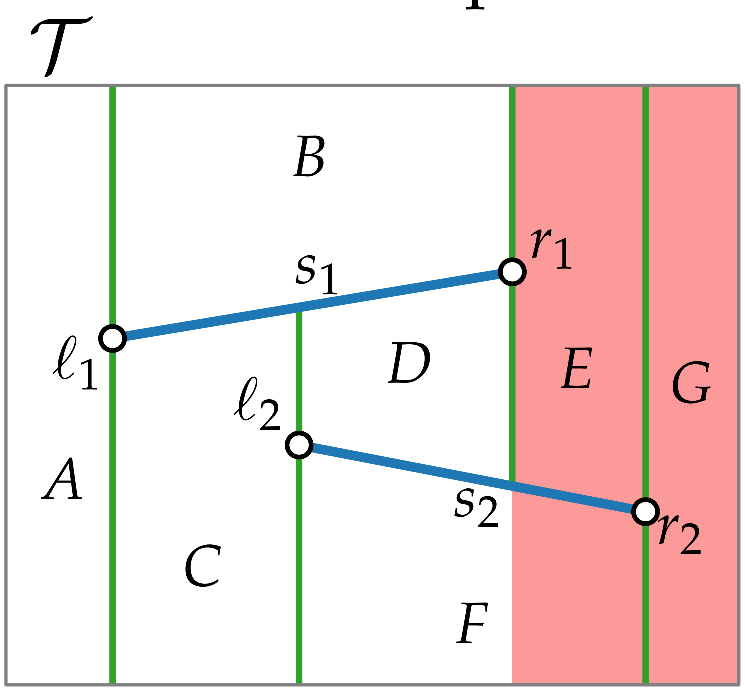

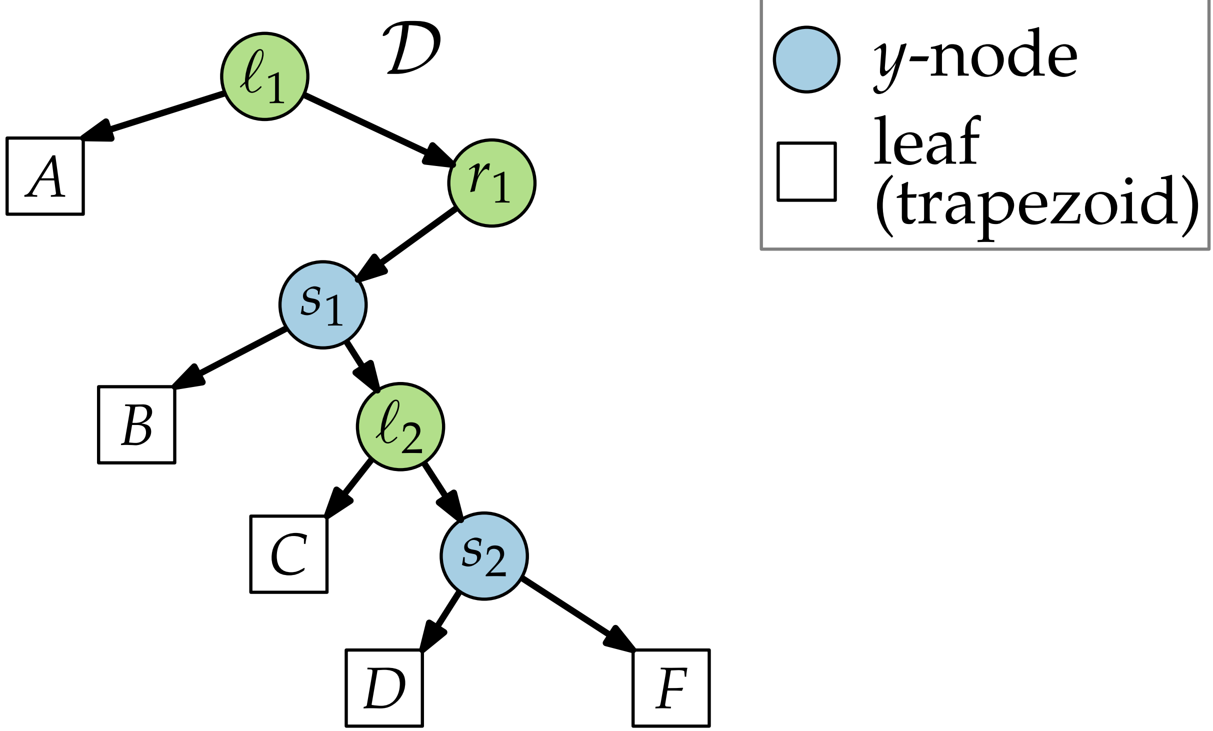

The 2D Problem

point-location data structure (DAG)

trapezoidal map

Approach: randomized-incremental construction of $\mathcal{T}$ and $\mathcal{D}$

- use $\mathcal{D}$ to locate left endpoint of next segment s
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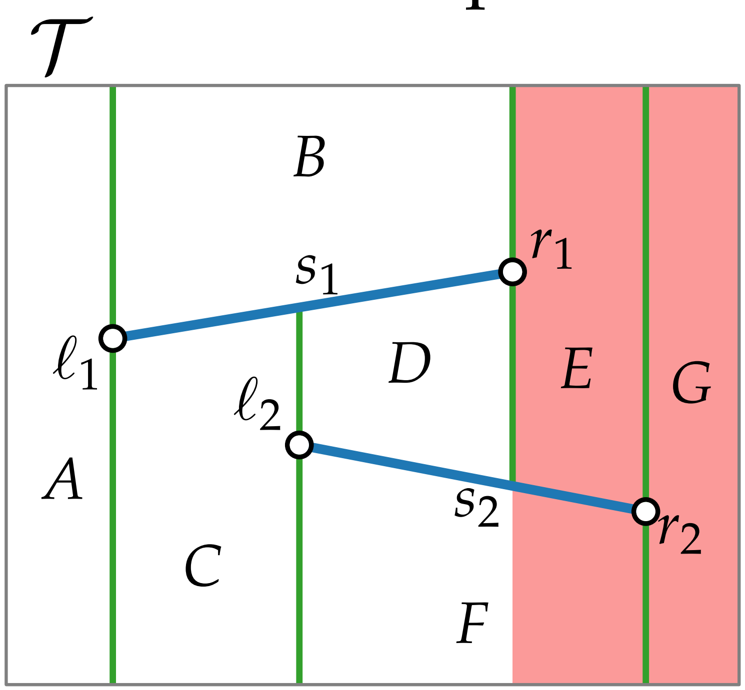

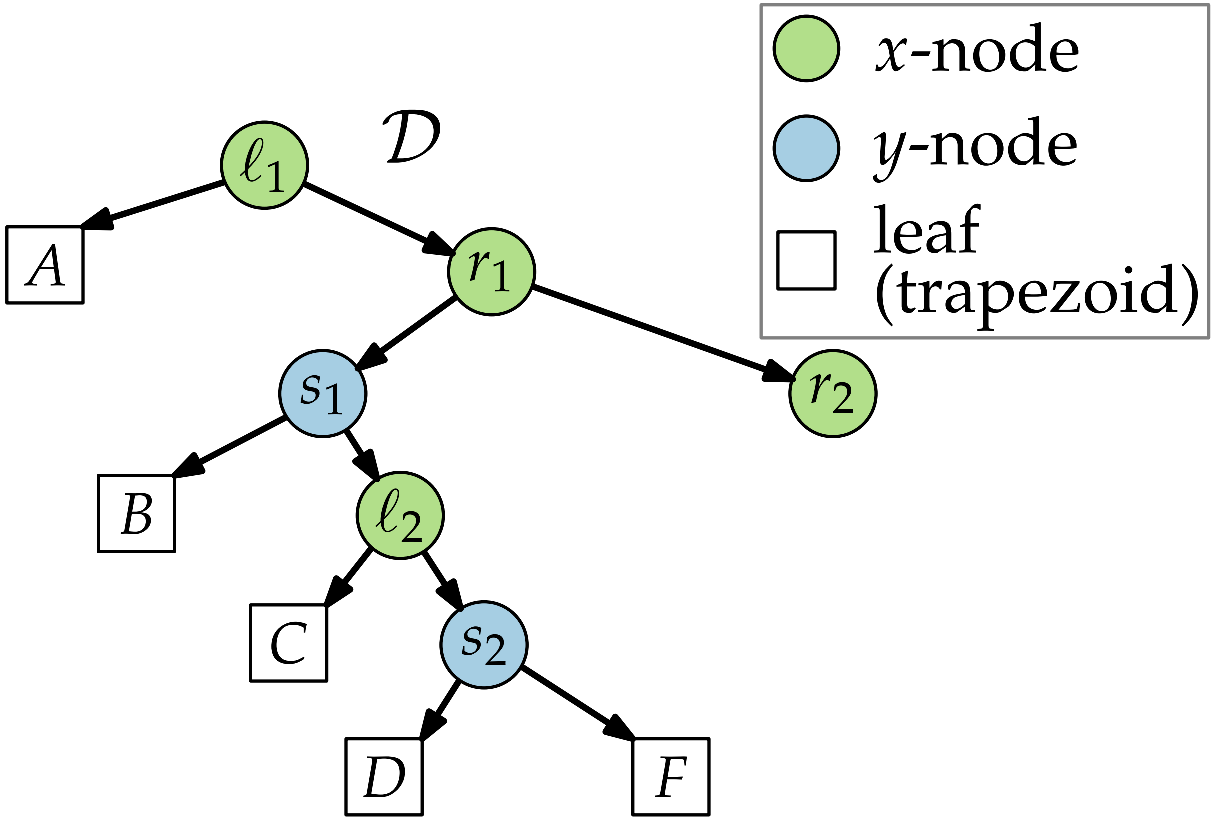

The 2D Problem

point-location data structure (DAG)
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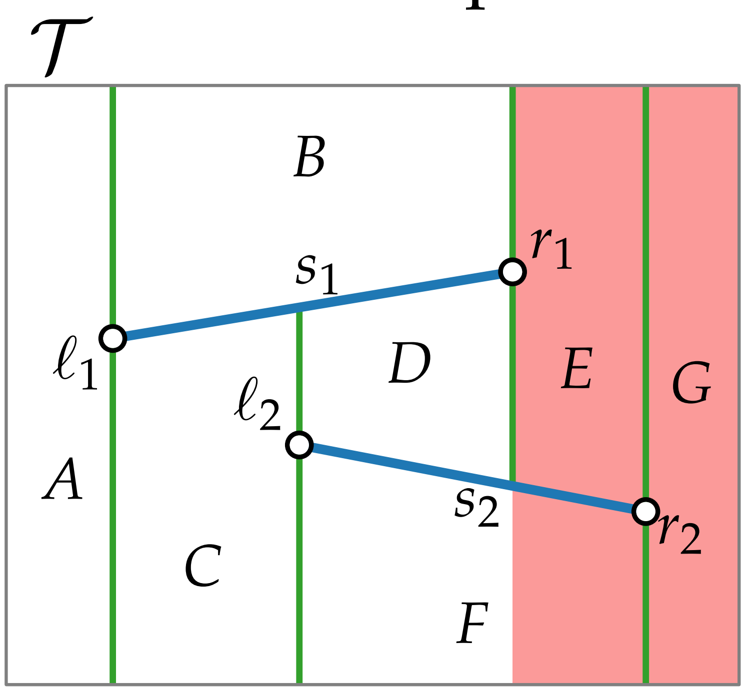

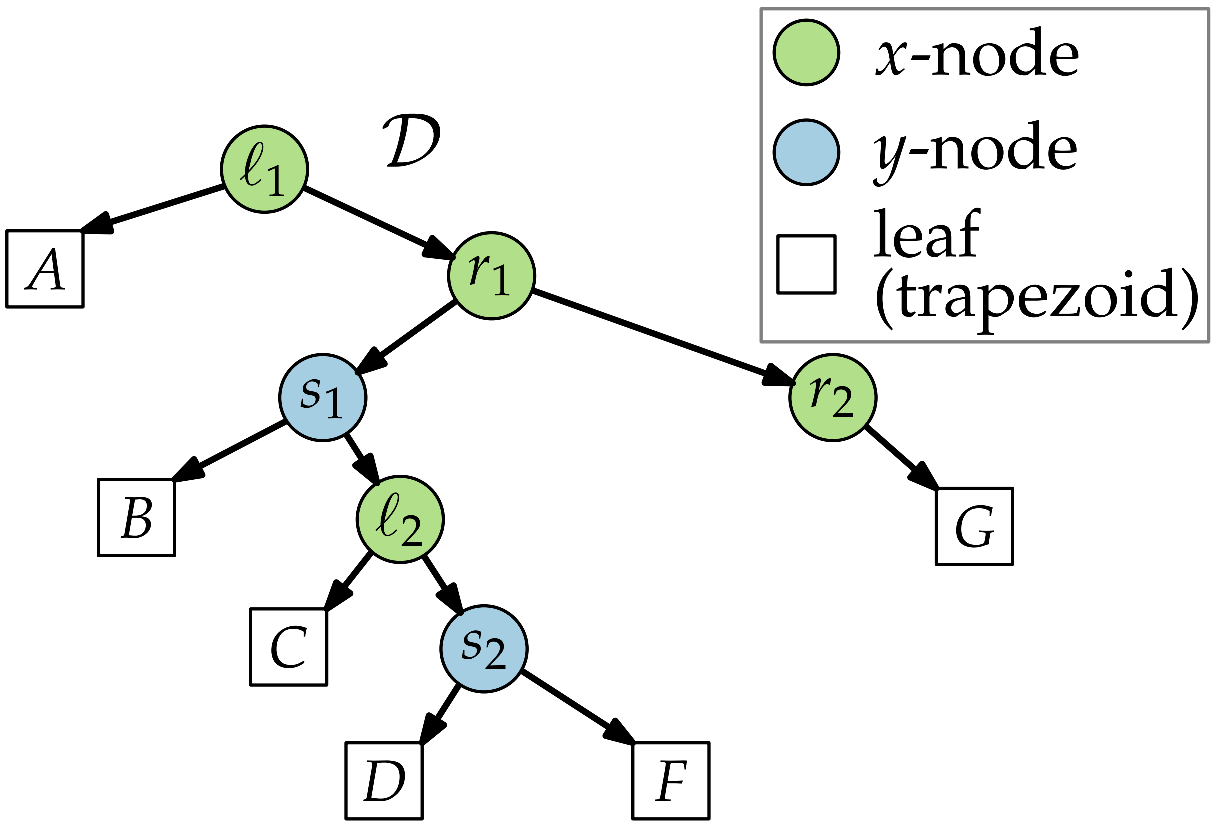

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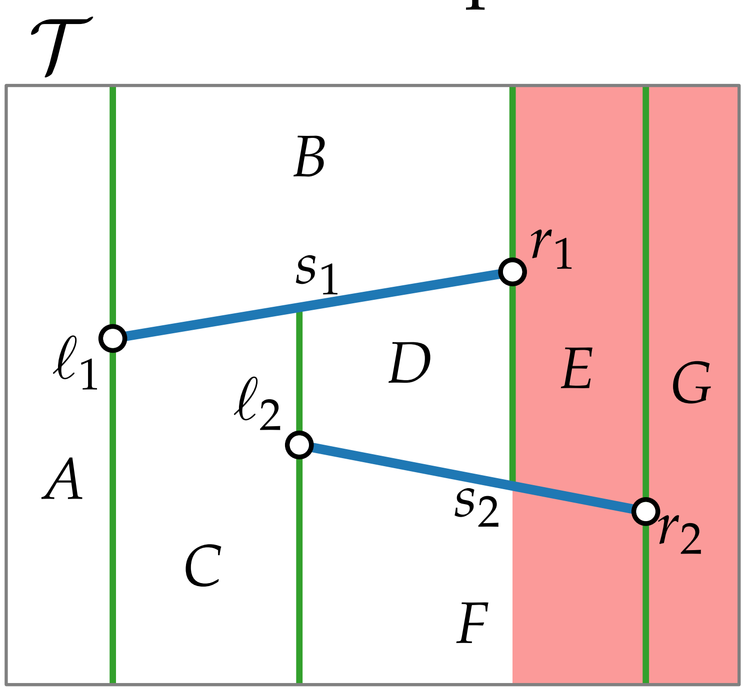

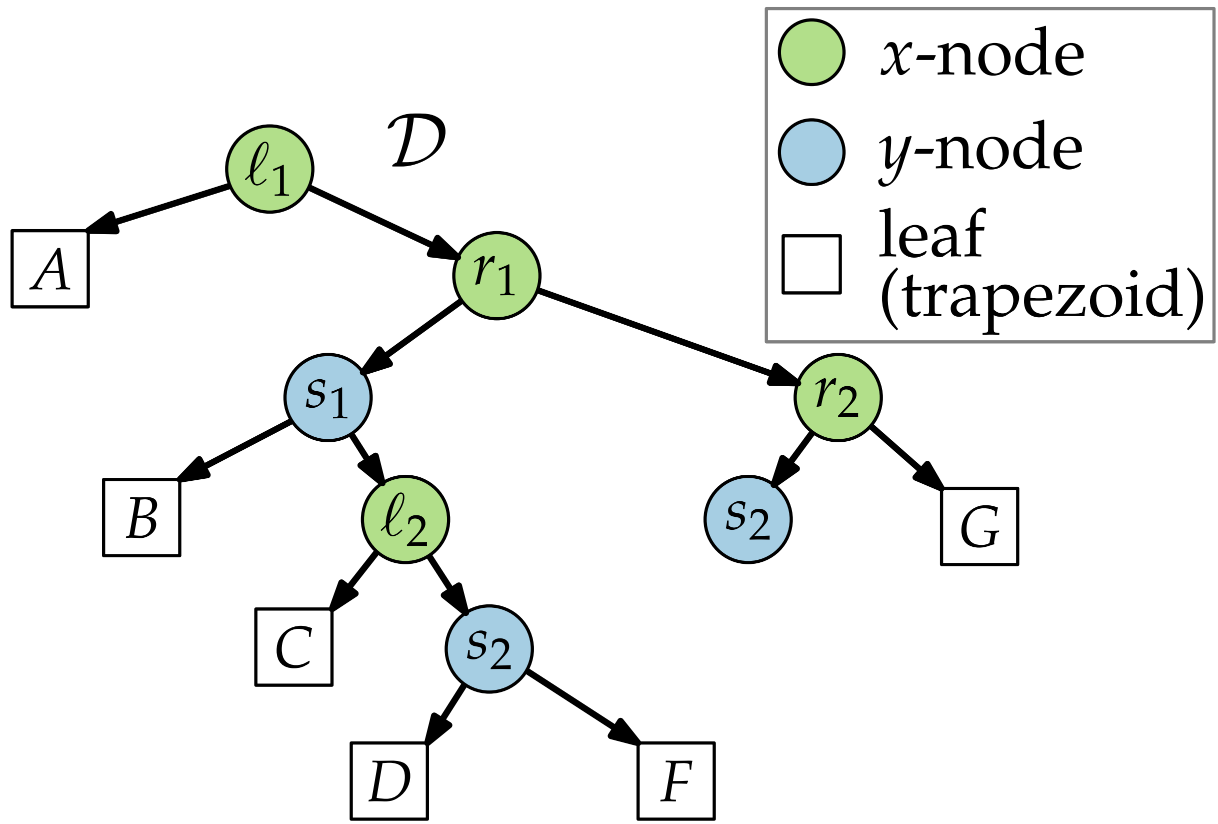

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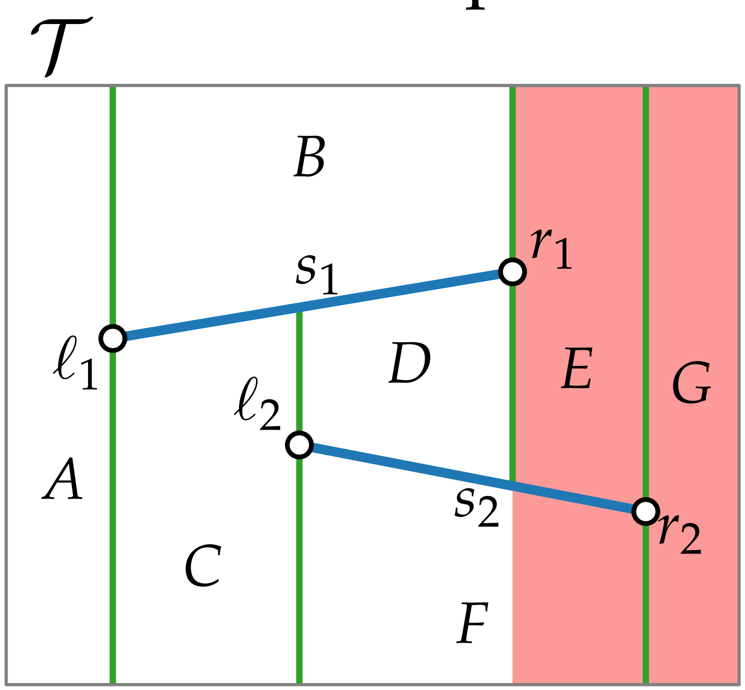

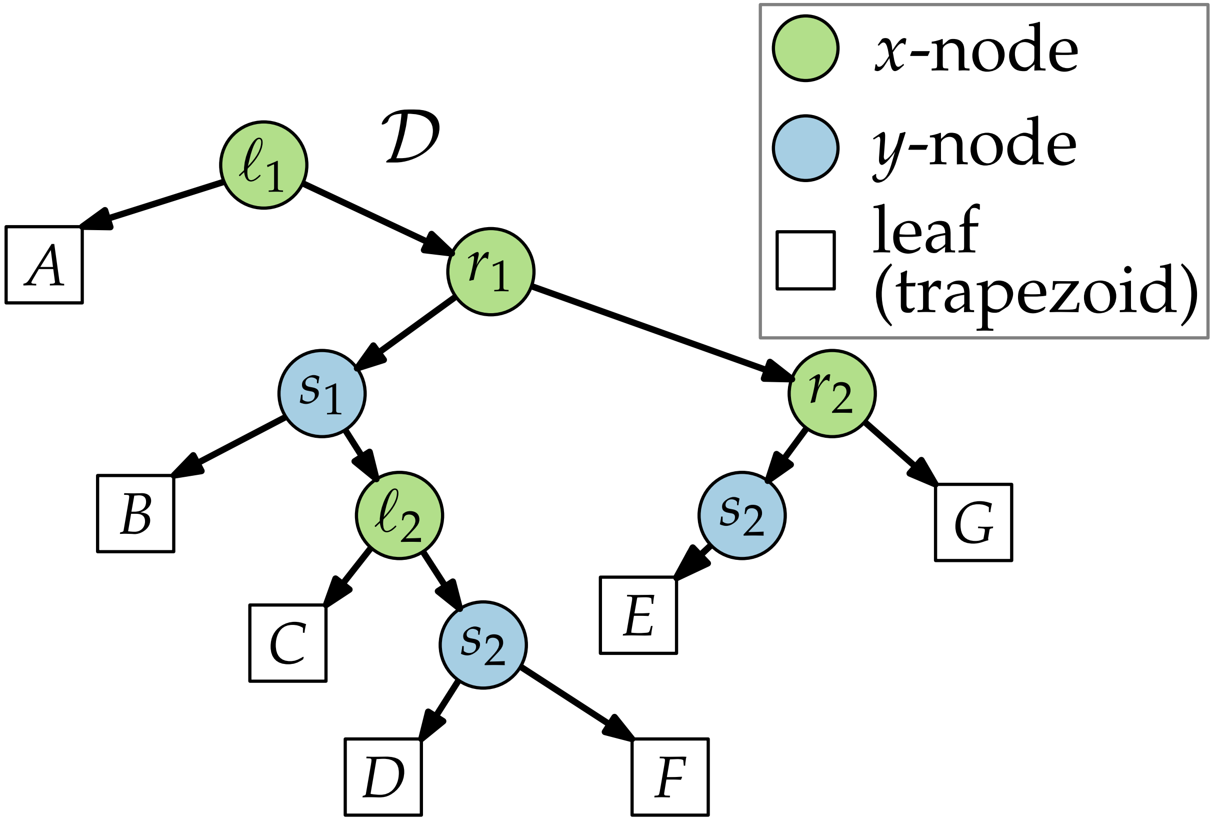

The 2D Problem

point-location data structure (DAG)

trapezoidal map

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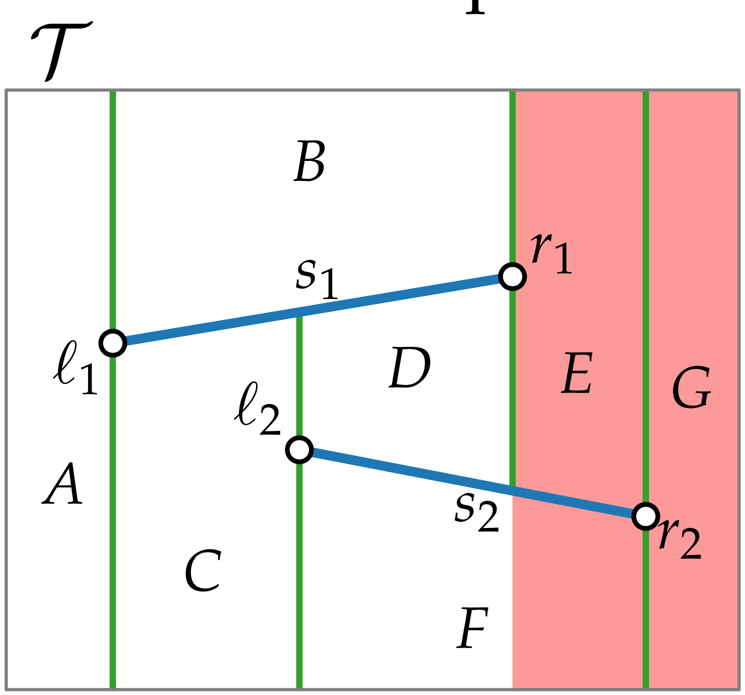

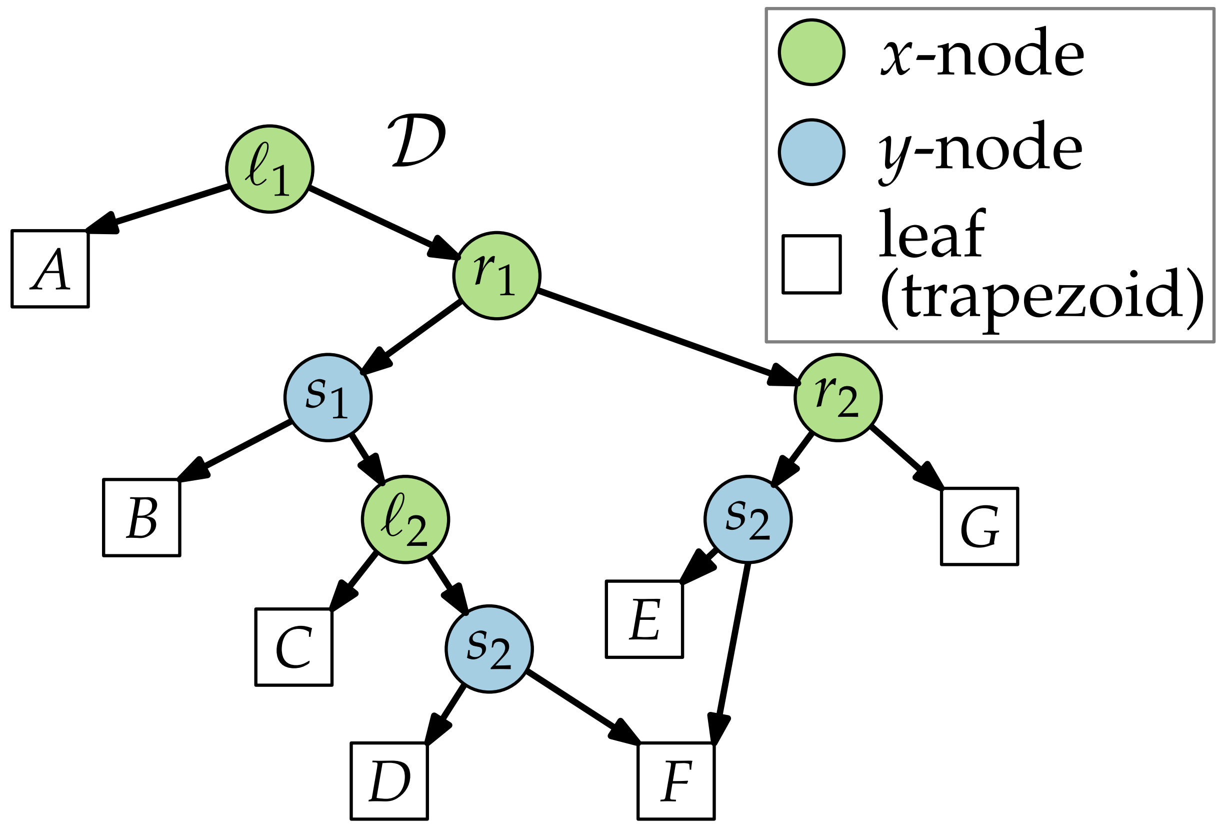

The 2D Problem

point-location data structure (DAG)

trapezoidal map

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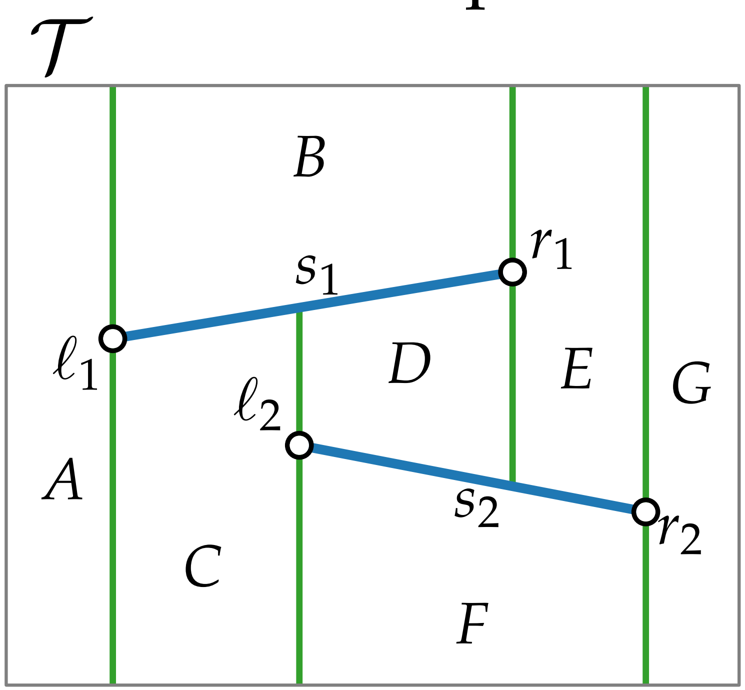
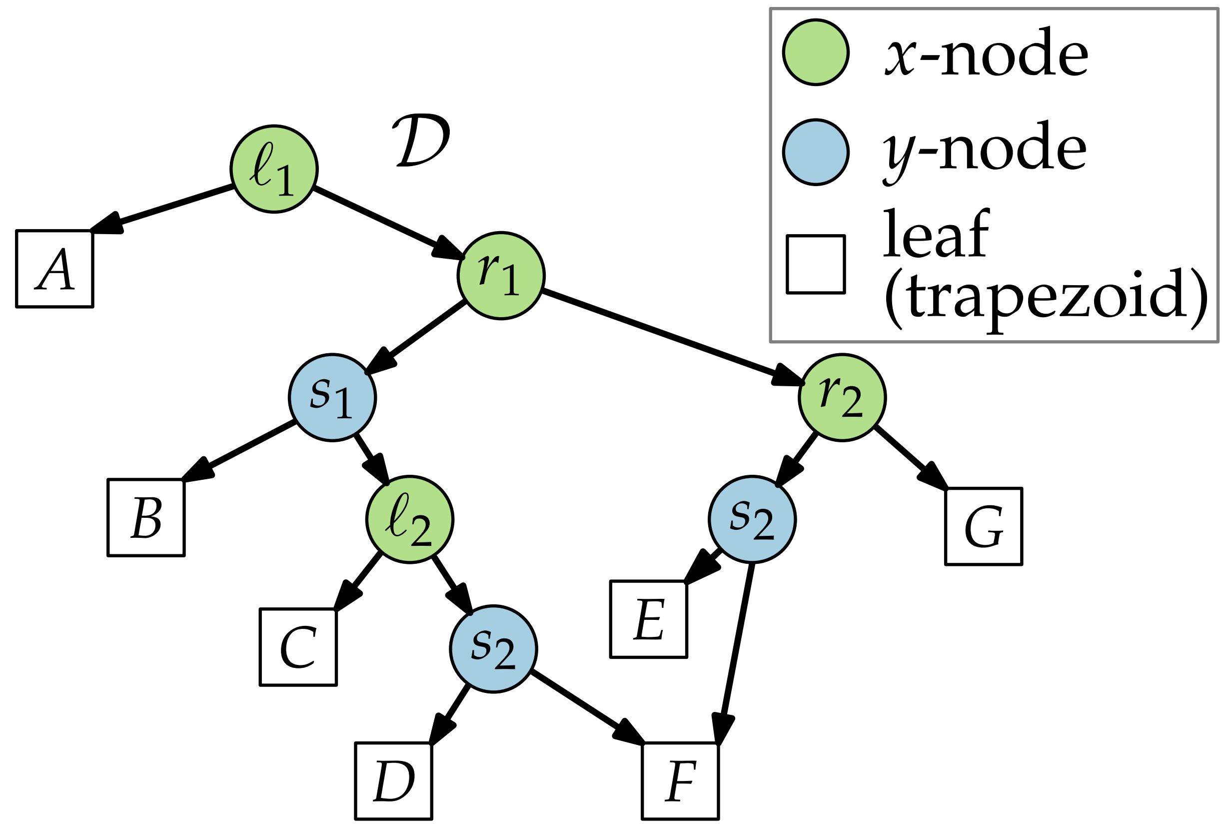
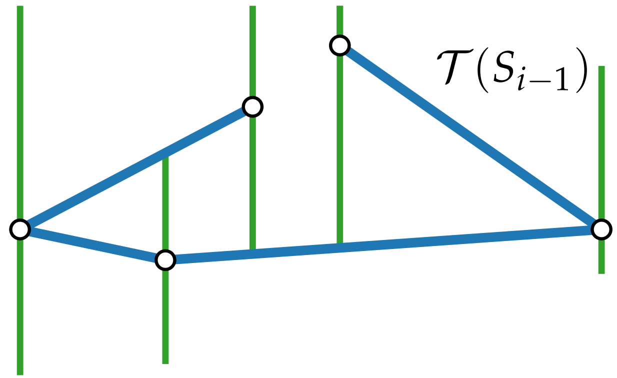
– update $\mathcal{D}$



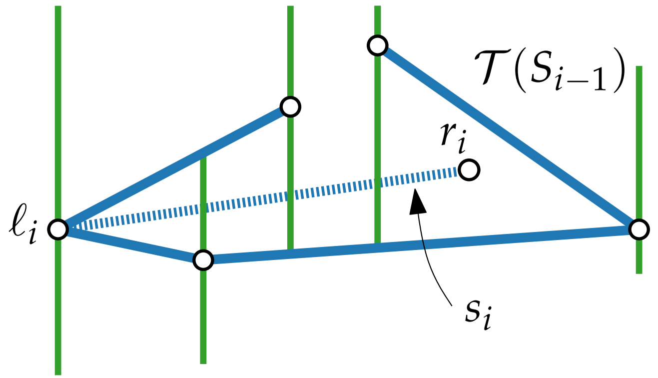


Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



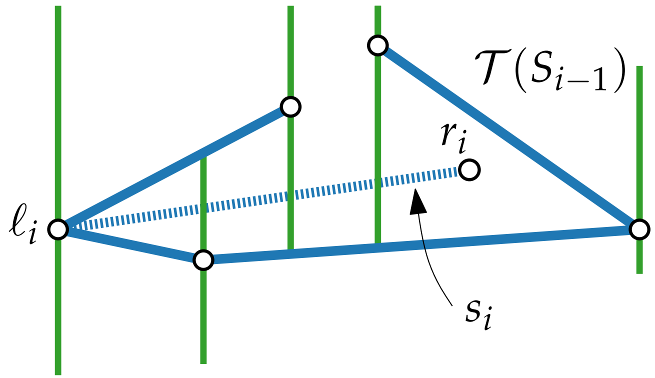
TrapezoidalMap(set S of n non-crossing seg.)
  $R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



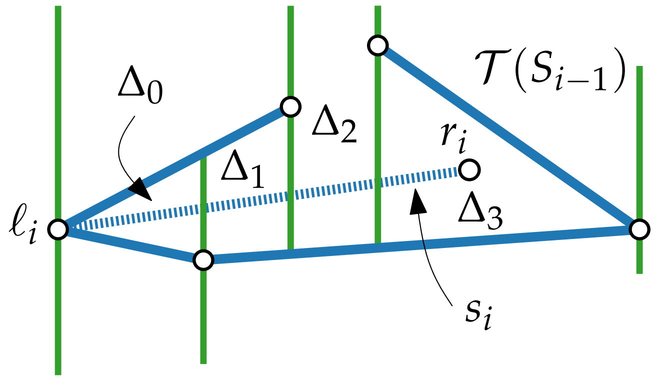
TrapezoidalMap(set S of n non-crossing seg.)
  R = BBox(S); $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n)$ = RandomPermutation(S)
  for $i = 1$ **to** n **do**

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



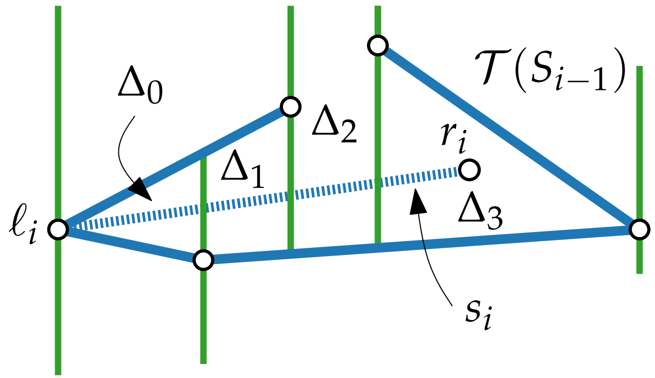
TrapezoidalMap(set S of n non-crossing seg.)
R = BBox(S); $\mathcal{T}$.init(); $\mathcal{D}$.init()
$(s_1, s_2, \ldots, s_n)$ = RandomPermutation(S)
for $i = 1$ **to** n **do**
$(\Delta_0, \ldots, \Delta_k)$ = FollowSegment($\mathcal{T}$, $\mathcal{D}$, s_i)

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



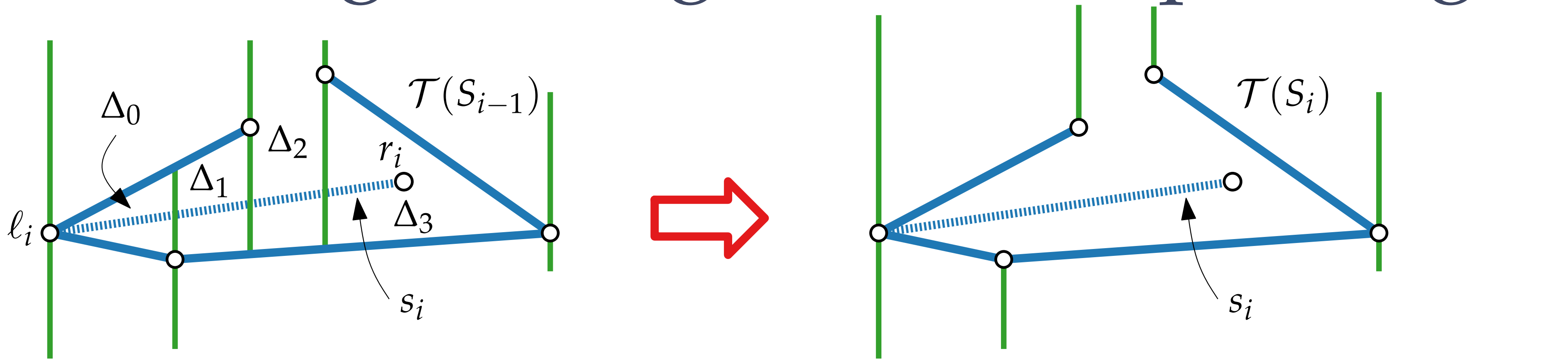
TrapezoidalMap(set S of n non-crossing seg.)
  $R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



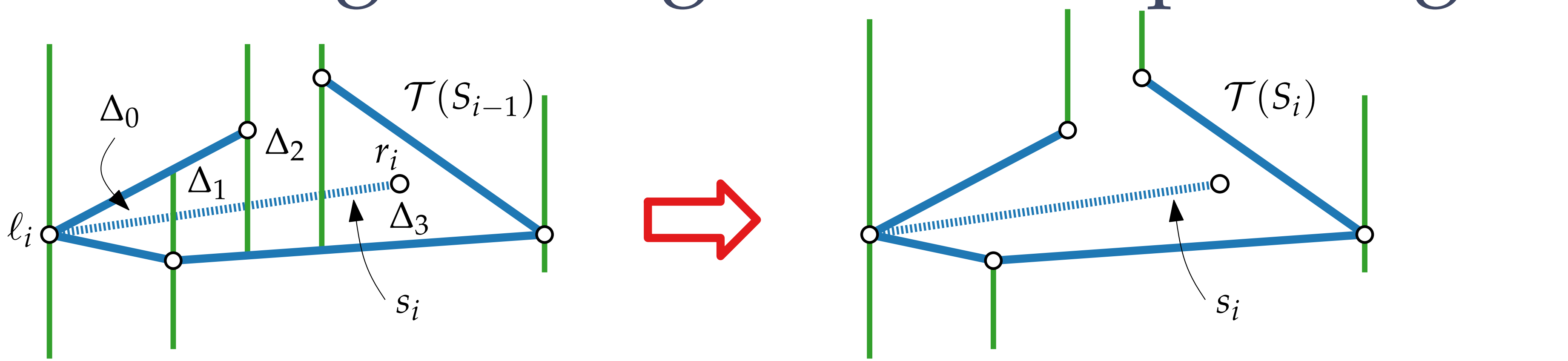
TrapezoidalMap(set S of n non-crossing seg.)
  $R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
    $\mathcal{T}.\text{remove}(\Delta_0, \ldots, \Delta_k)$

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



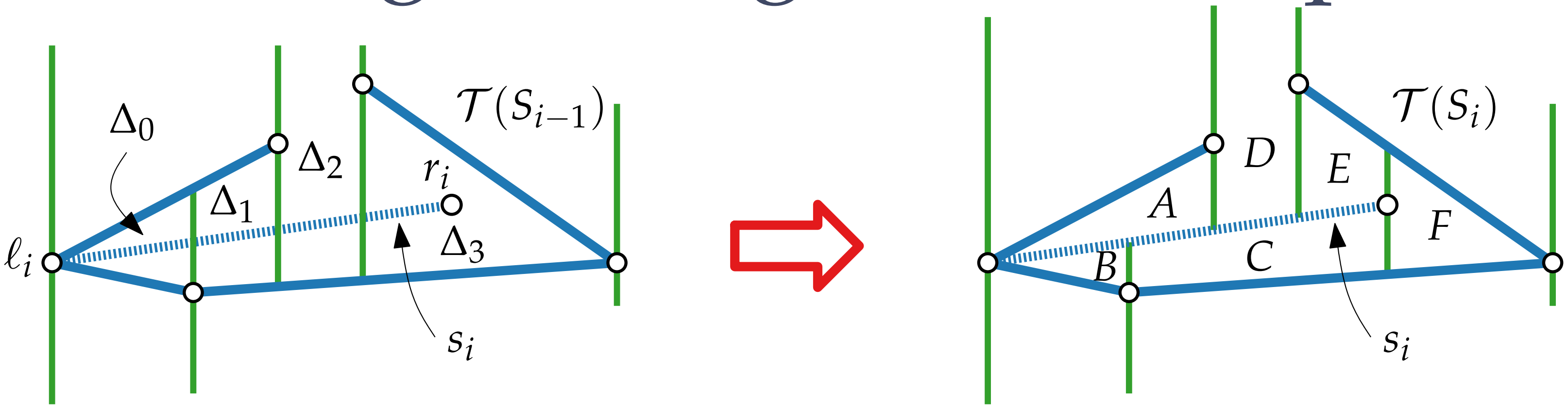
TrapezoidalMap(set S of n non-crossing seg.)
  $R = \mathrm{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n) = \mathrm{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \mathrm{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



TrapezoidalMap(set S of n non-crossing seg.)
$R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
for $i = 1$ **to** n **do**
  $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
  $\mathcal{T}$.remove($\Delta_0, \ldots, \Delta_k$)
  $\mathcal{T}$.add(new trapezoids incident to s_i)

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



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  $R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
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  for $i = 1$ **to** n **do**
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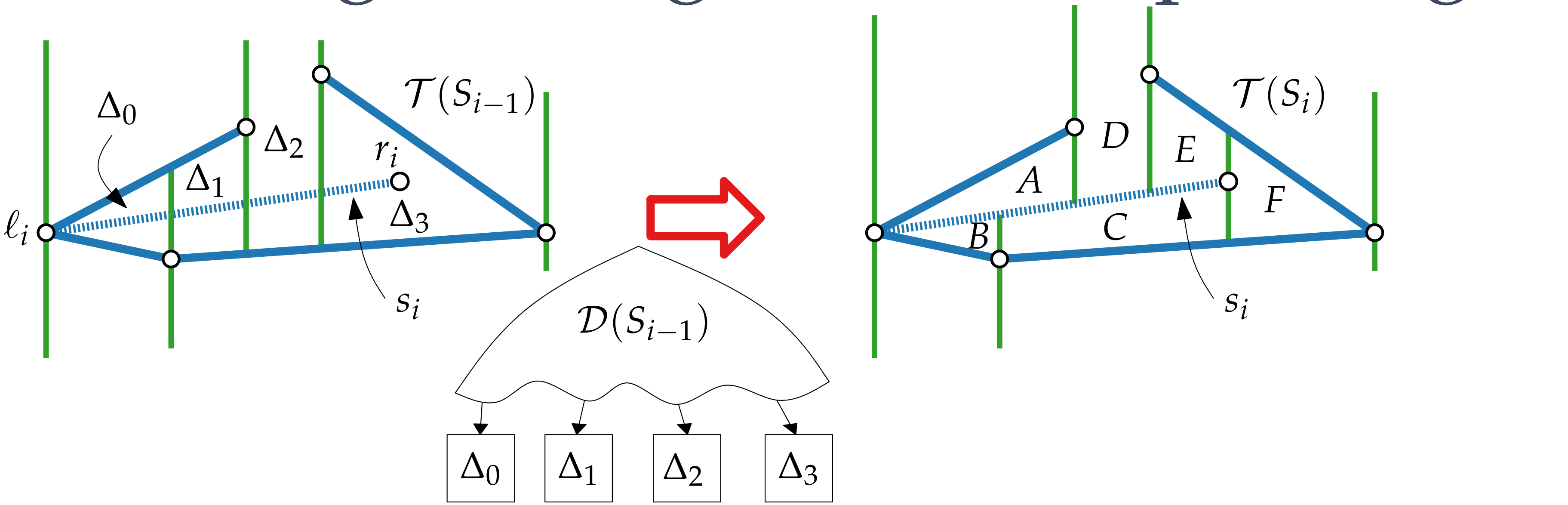
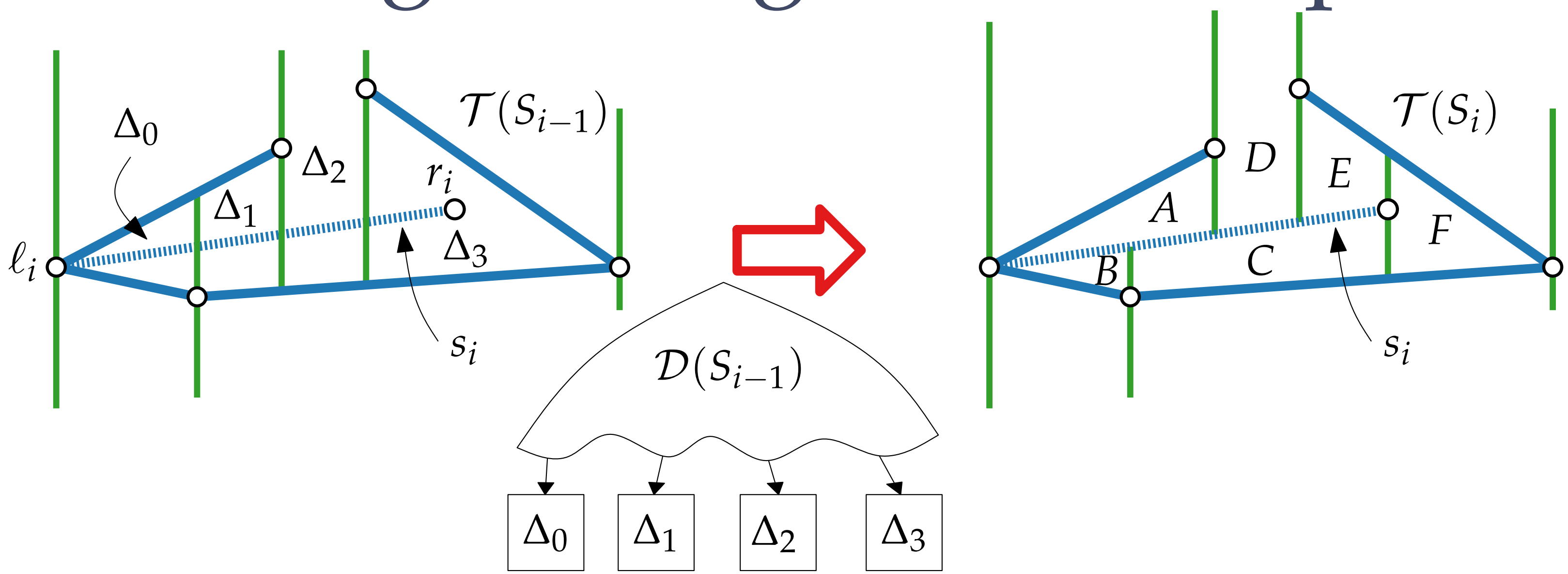
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{T}$.add(new trapezoids incident to s_i)

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$




```
TrapezoidalMap(set S of n non-crossing seg.)
   R = BBox(S); T.init(); D.init()
   (s_1, s_2, ..., s_n) = RandomPermutation(S)
   for i = 1 to n do
      (Δ_0, ..., Δ_k) = FollowSegment(T, D, s_i)
      T.remove(Δ_0, ..., Δ_k)
      T.add(new trapezoids incident to s_i)
```

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$



TrapezoidalMap(set S of n non-crossing seg.)

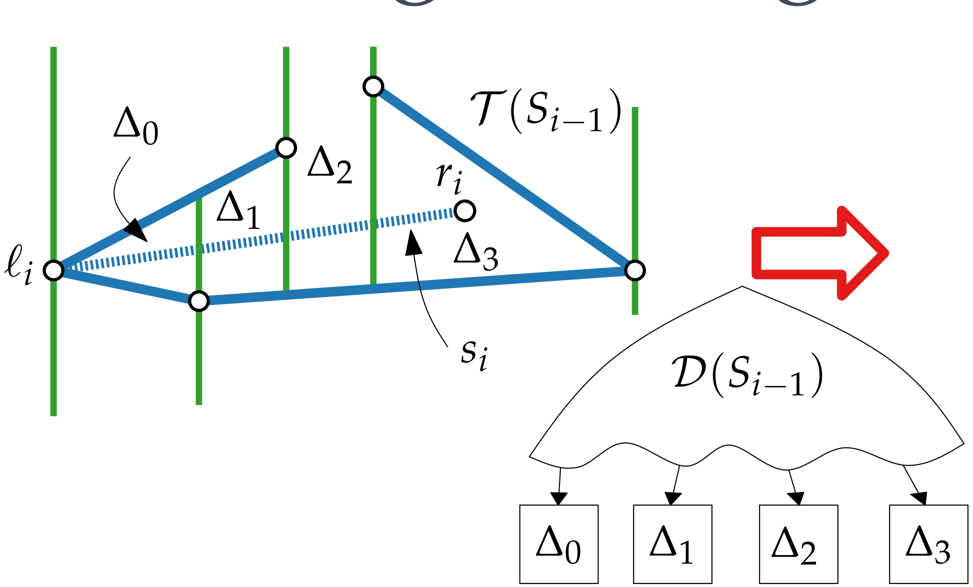
$R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()

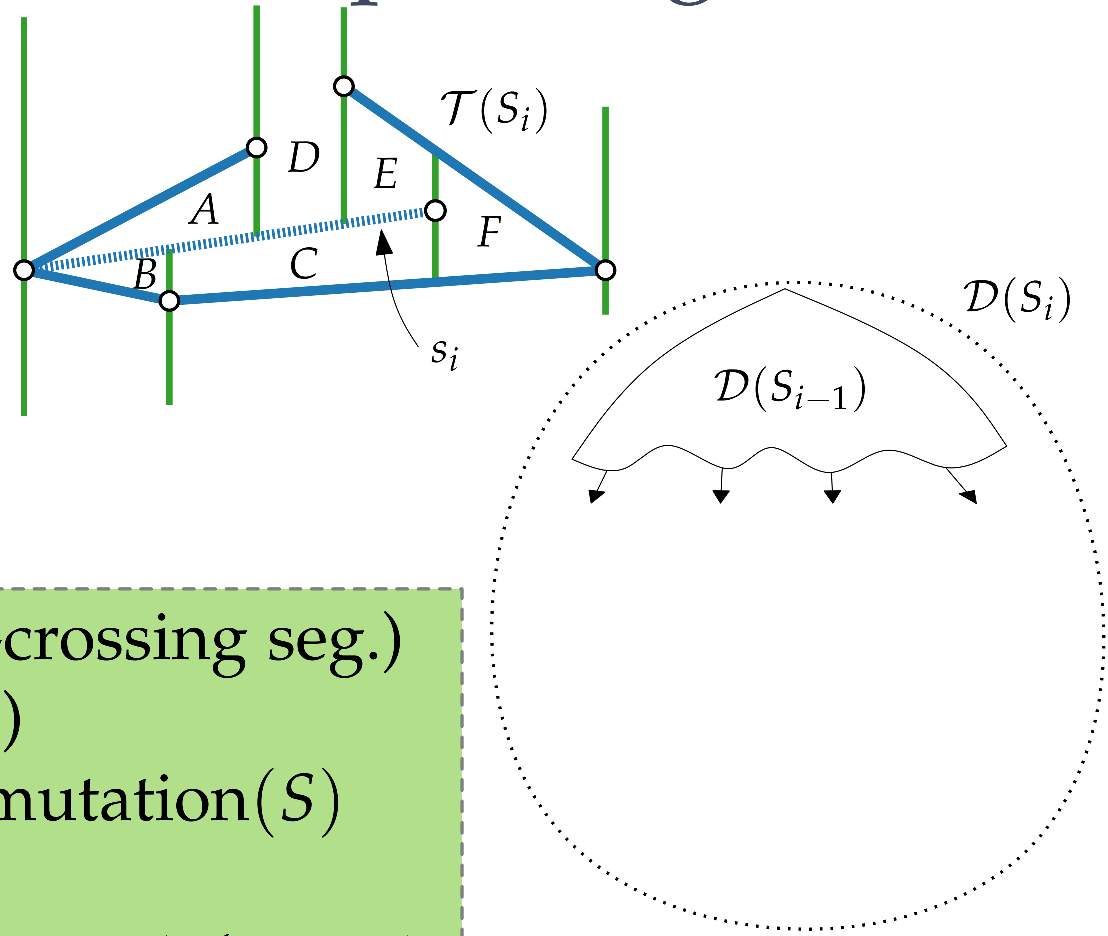
$(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

for $i = 1$ **to** n **do**

- $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
- $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
- $\mathcal{T}$.add(new trapezoids incident to s_i)
- $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$





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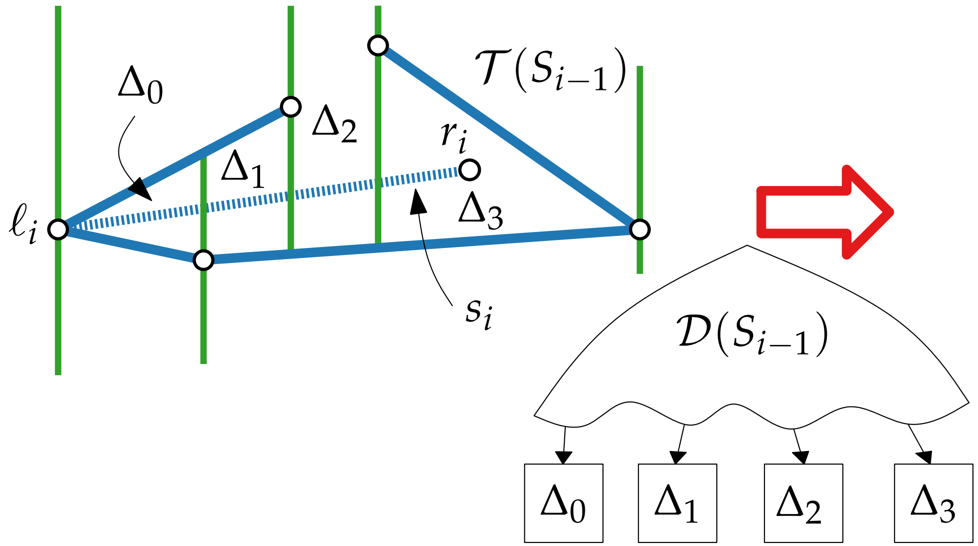
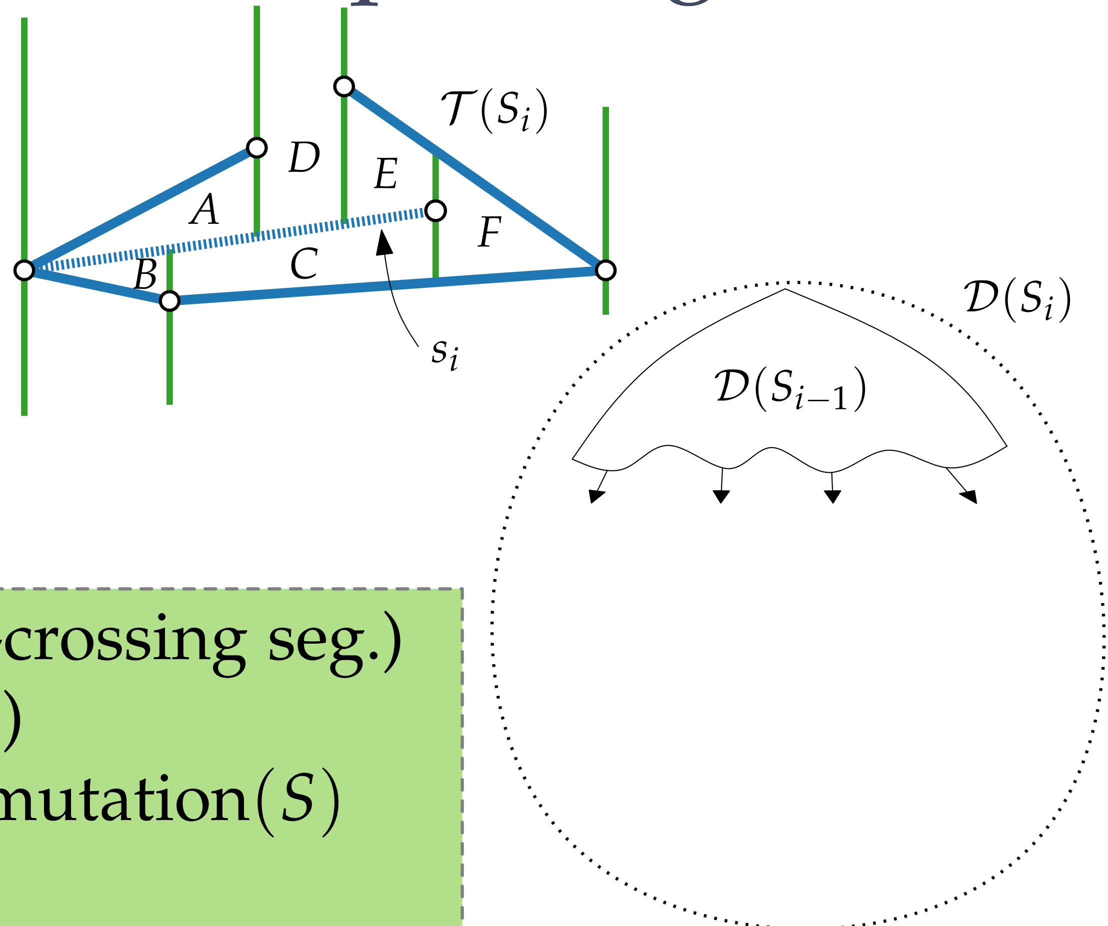
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{T}$.add(new trapezoids incident to s_i)
    $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$

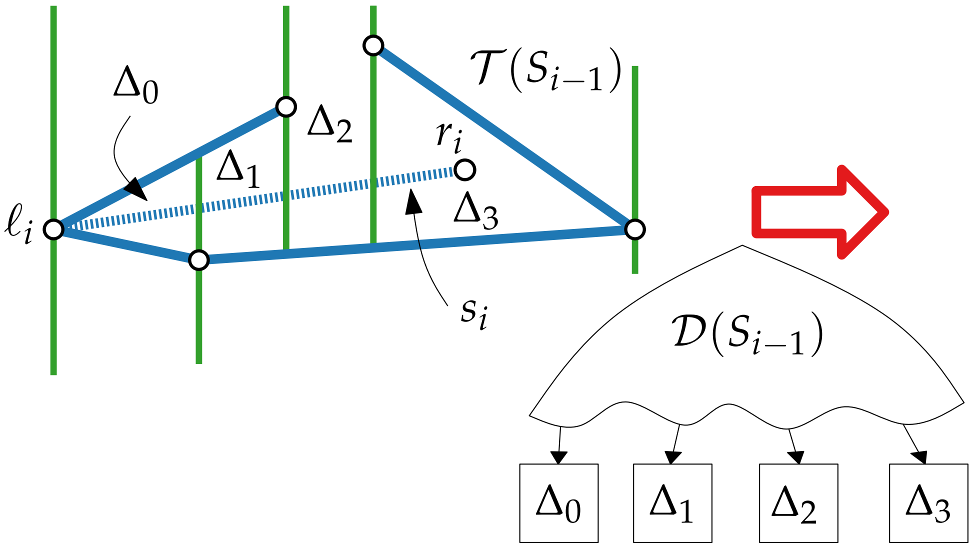
Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$

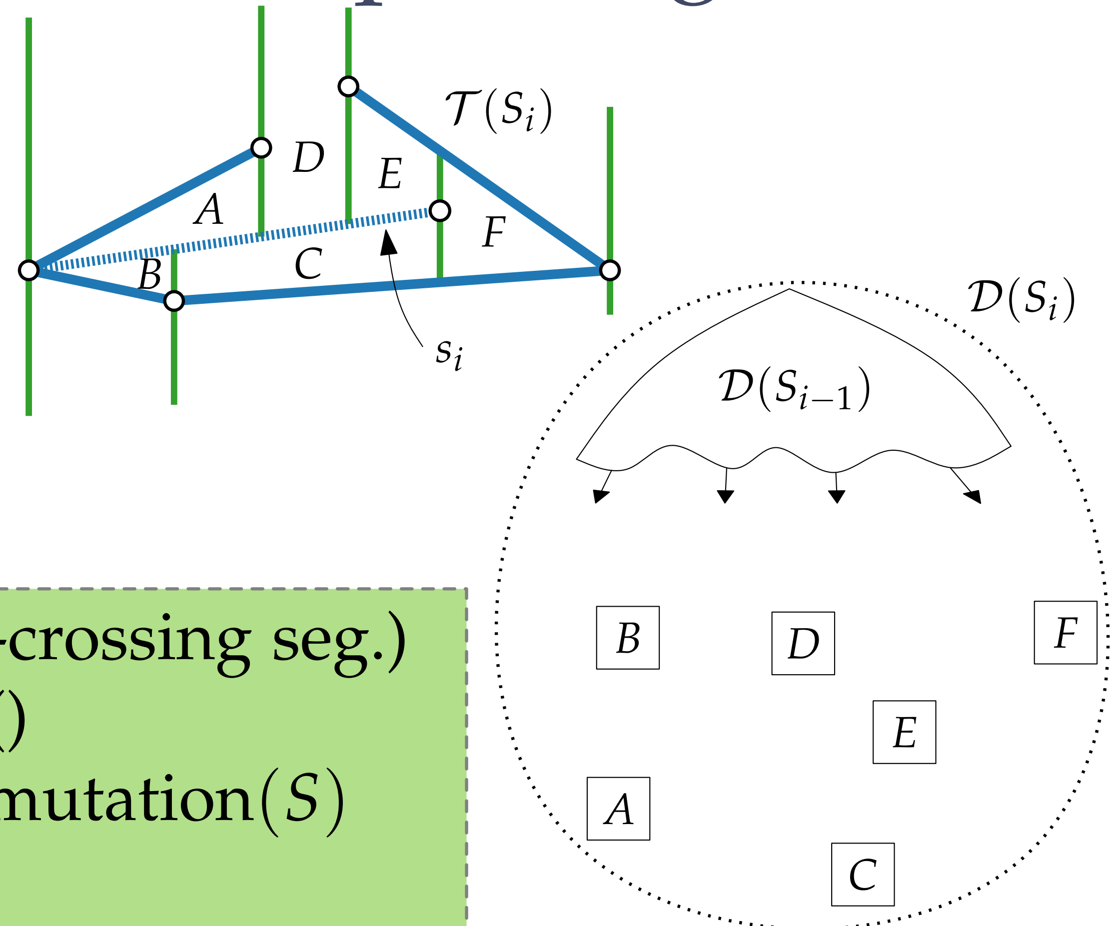




TrapezoidalMap(set S of n non-crossing seg.)
  $R = \text{BBox}(S)$; $\mathcal{T}$.init(); $\mathcal{D}$.init()
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{T}$.add(new trapezoids incident to s_i)
    $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{D}$.add_leaves(new trapez. incident to s_i)

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$





TrapezoidalMap(set S of n non-crossing seg.)

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  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$

  for $i = 1$ **to** n **do**

    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$

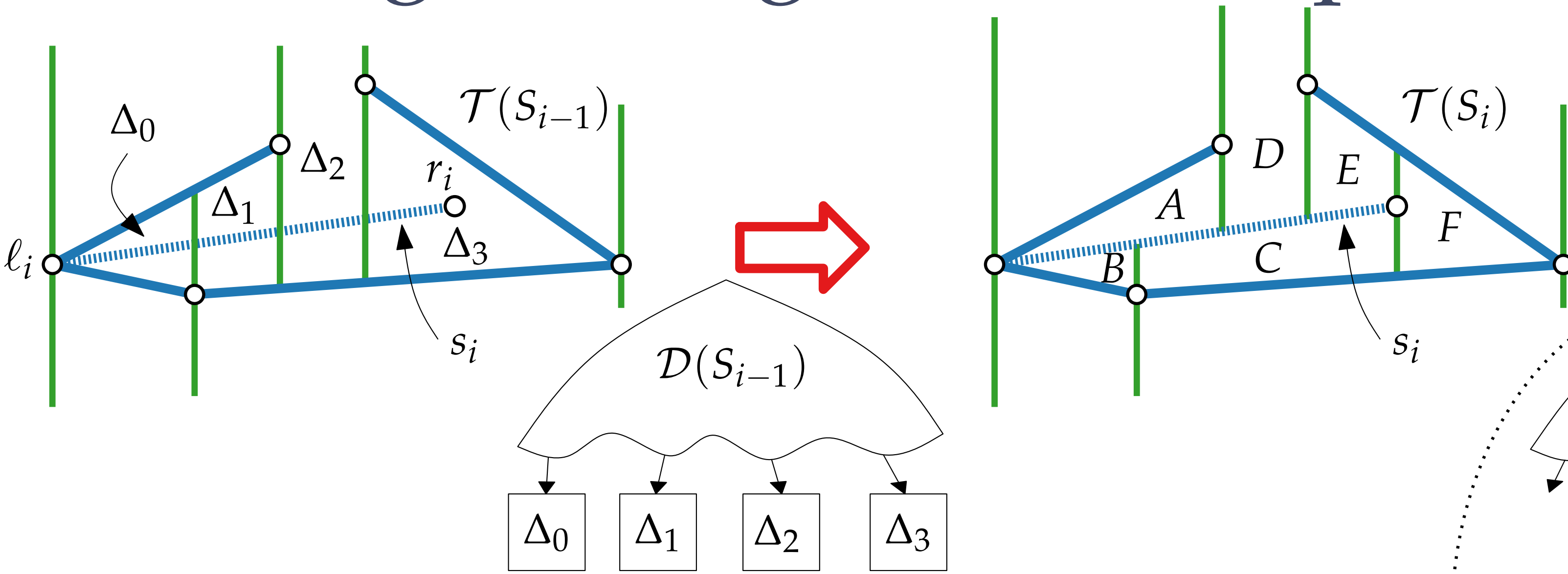
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$

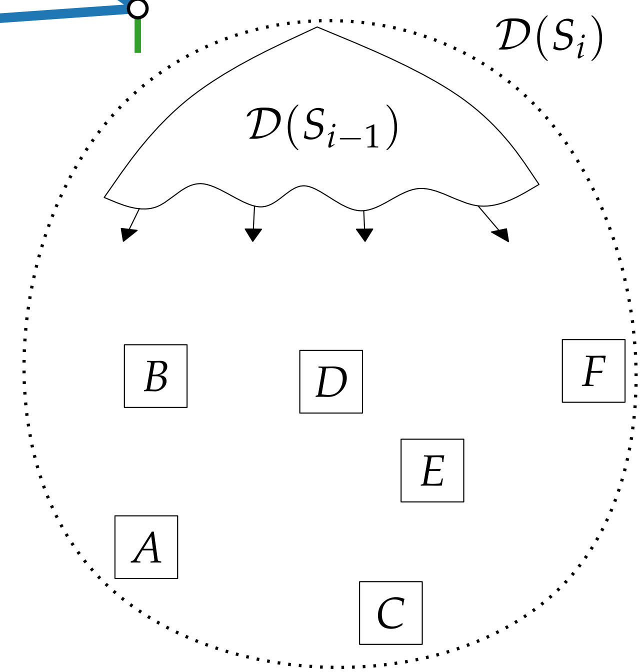
    $\mathcal{T}$.add(new trapezoids incident to s_i)

    $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$

    $\mathcal{D}$.add_leaves(new trapez. incident to s_i)

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$





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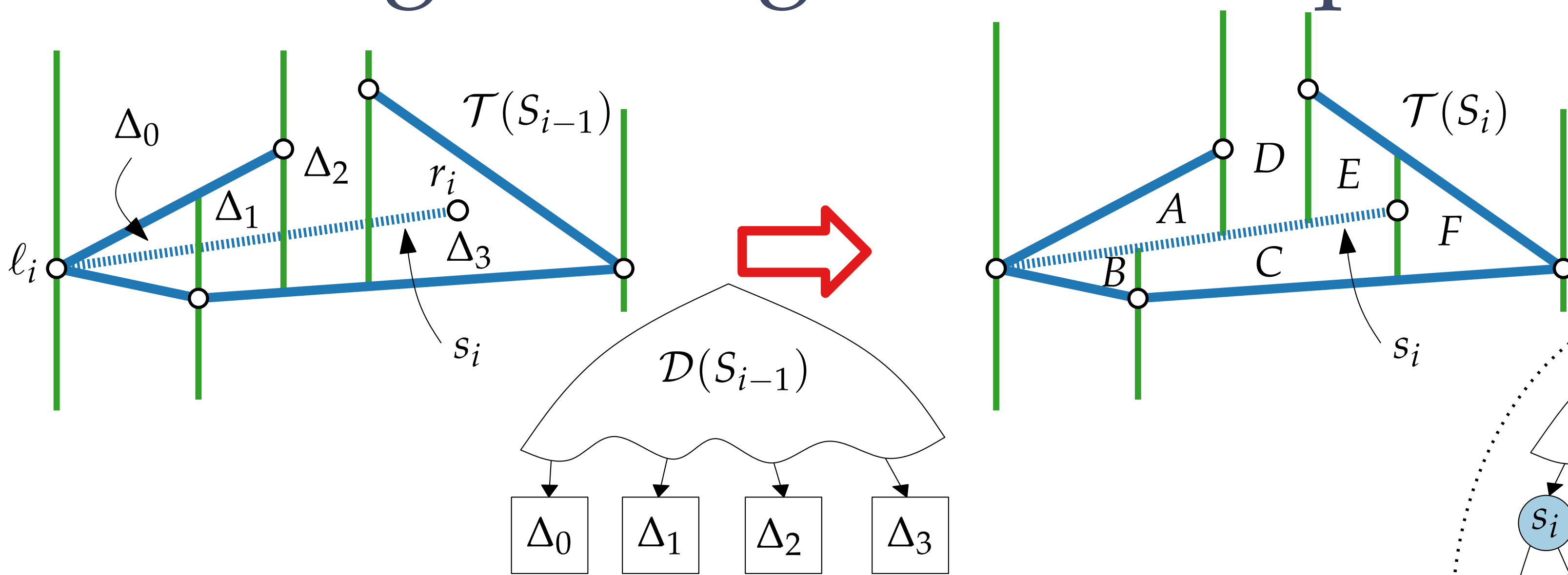
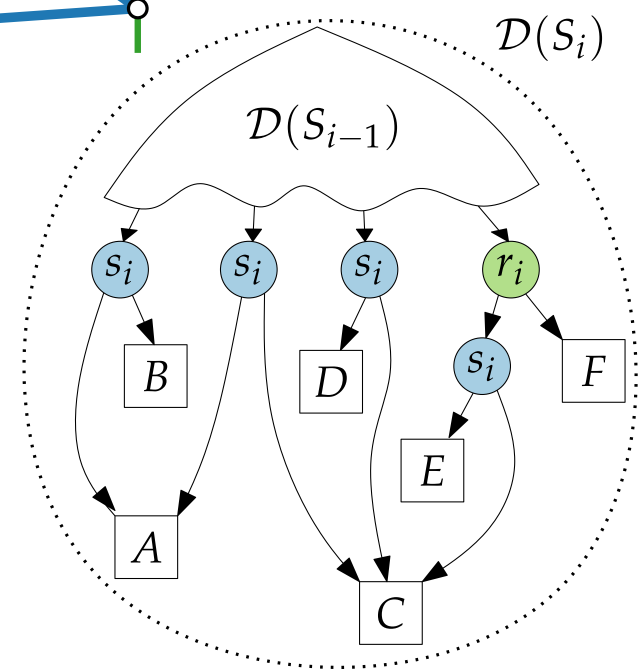
  $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
  for $i = 1$ **to** n **do**
    $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
    $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{T}$.add(new trapezoids incident to s_i)
    $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$
    $\mathcal{D}$.add_leaves(new trapez. incident to s_i)
    $\mathcal{D}$.add_new_inner_nodes()

Walking Through $\mathcal{T}$ and Updating $\mathcal{D}$





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    $(s_1, s_2, \ldots, s_n) = \text{RandomPermutation}(S)$
    for $i = 1$ **to** n **do**
        $(\Delta_0, \ldots, \Delta_k) = \text{FollowSegment}(\mathcal{T}, \mathcal{D}, s_i)$
        $\mathcal{T}$.remove$(\Delta_0, \ldots, \Delta_k)$
        $\mathcal{T}$.add(new trapezoids incident to s_i)
        $\mathcal{D}$.remove_leaves$(\Delta_0, \ldots, \Delta_k)$
        $\mathcal{D}$.add_leaves(new trapez. incident to s_i)
        $\mathcal{D}$.add_new_inner_nodes()

The 2D Result

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  $\Rightarrow$ construction time

Computational Geometry

Lecture 6:
Point Localization
or
Where am I?

Part V:
Query Time and Size

Philipp Kindermann

Winter Semester 2020

Query Time

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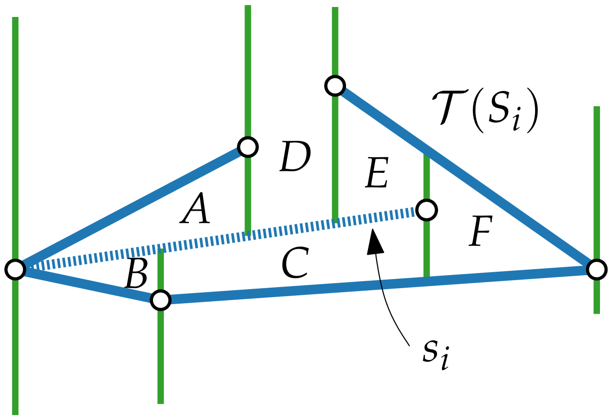

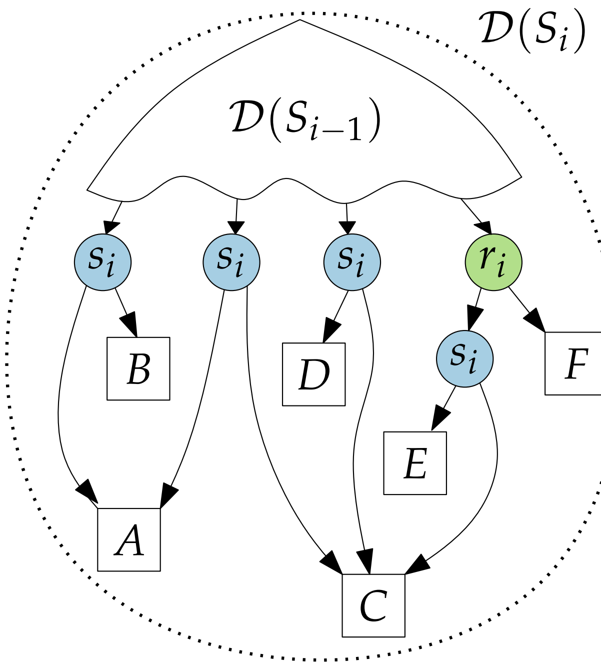

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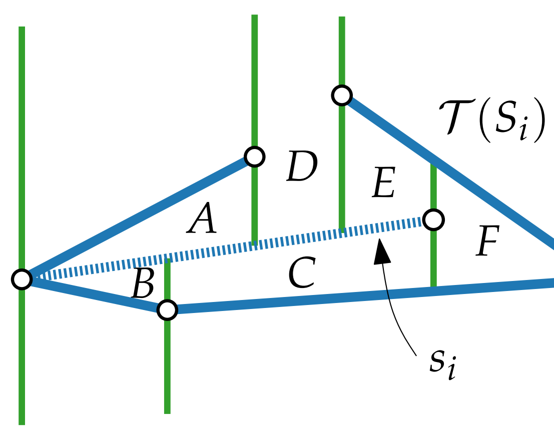

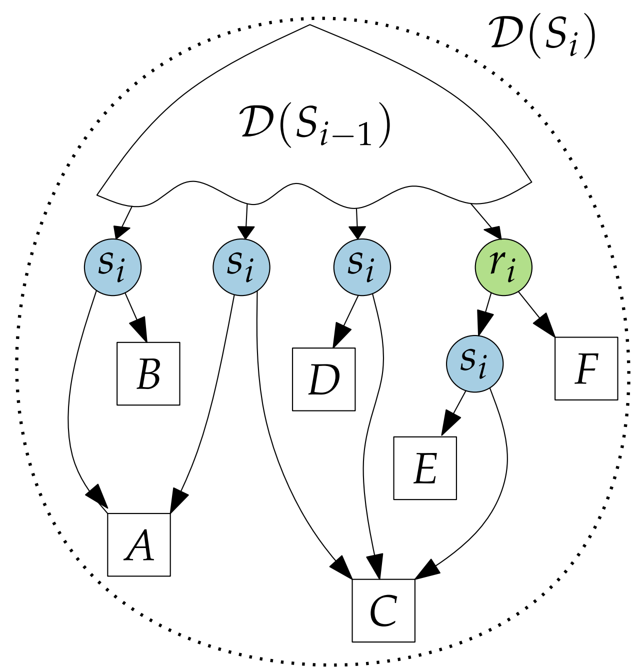

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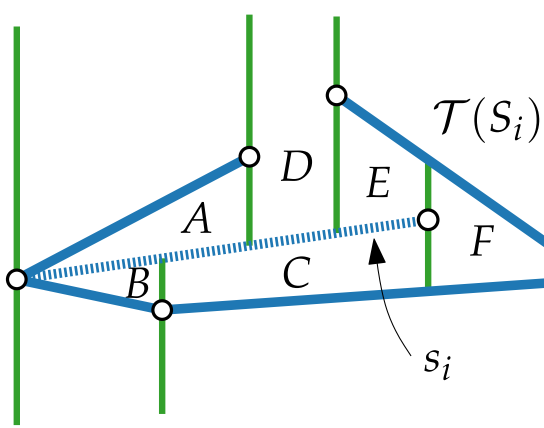

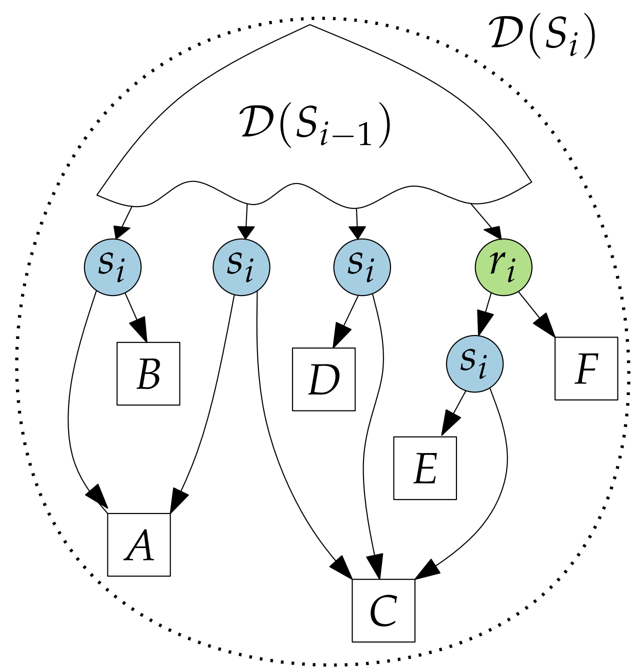

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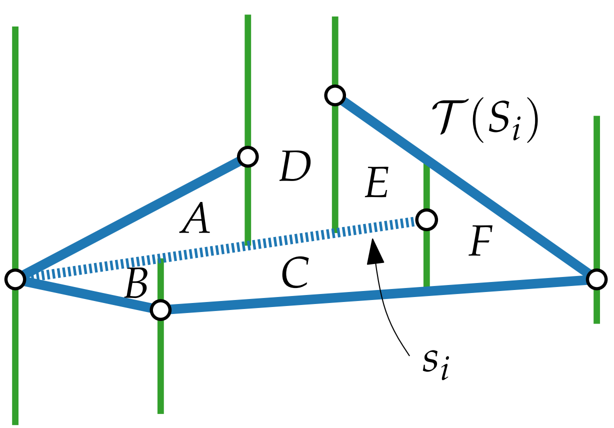

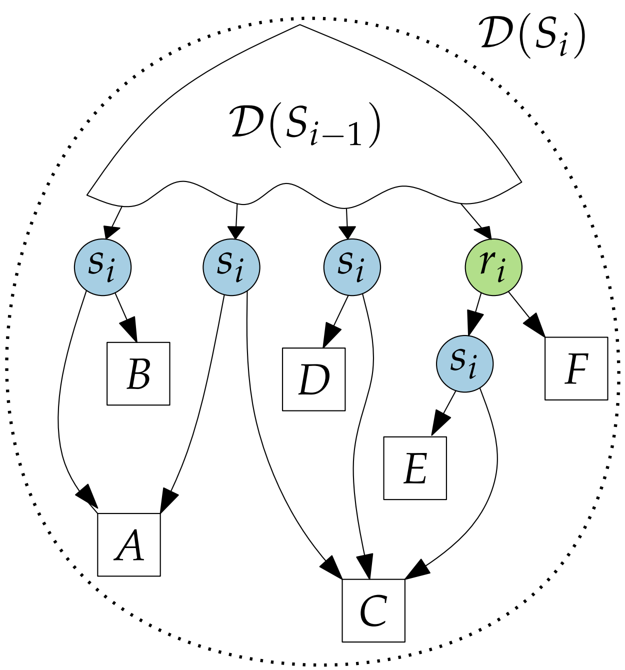

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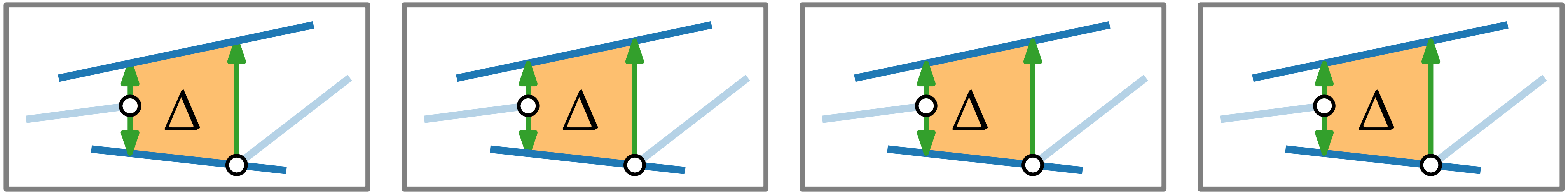

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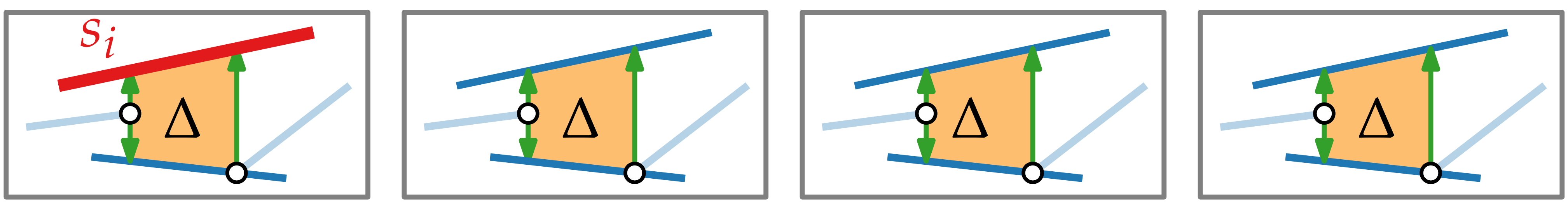

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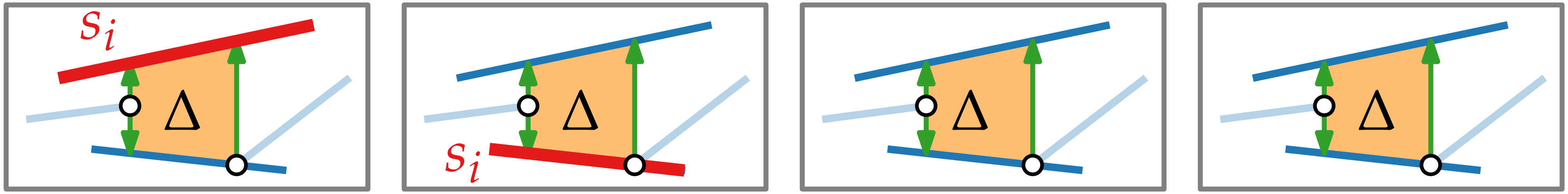

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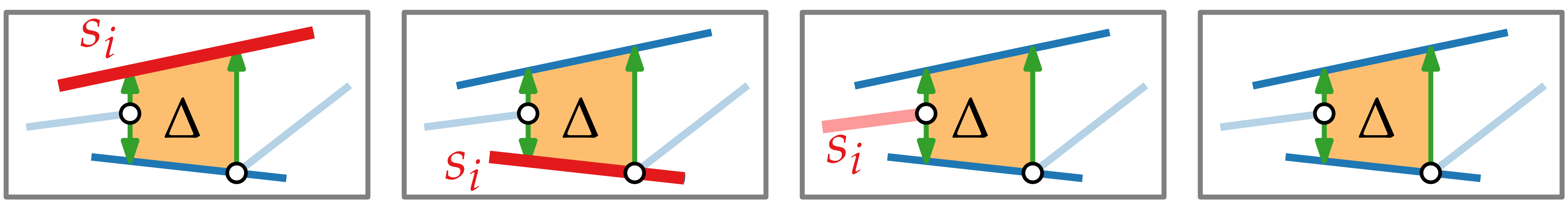

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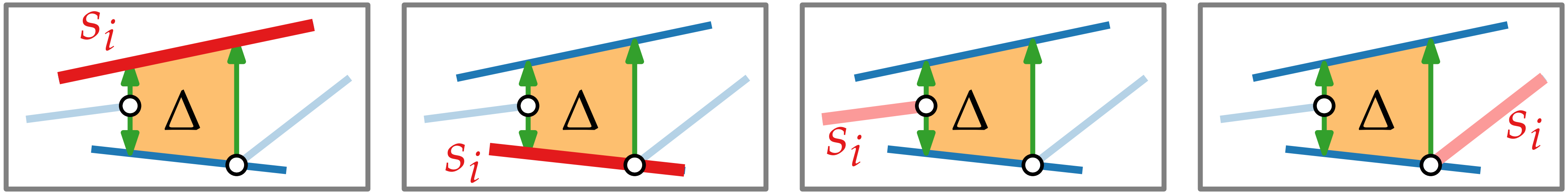

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Trick: $\mathcal{T}(S_i)$ (and thus Δ) is uniquely determined by S_i.

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$\Rightarrow \Delta$ does *not* depend on insertion order.

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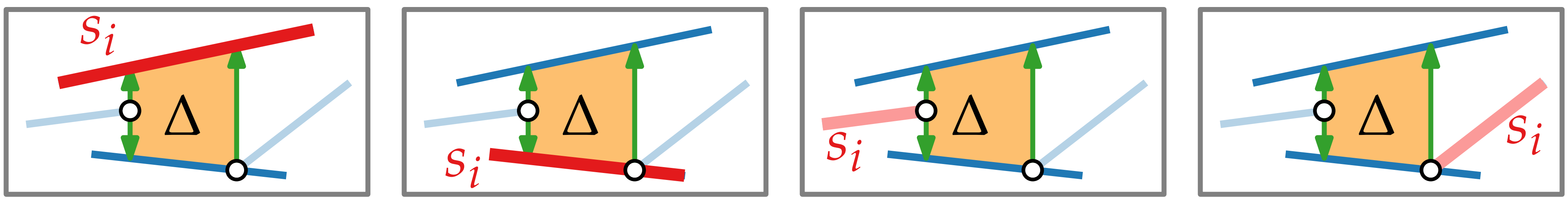
Four cases:



Query Time (cont'd)

p_i = prob. that the search path Π_q of q in $\mathcal{D}$ contains a node that was created in iteration i,
i.e., prob. that Δ changes when inserting s_i.

Aim: bound p_i.

Tool: *Backwards analysis!*

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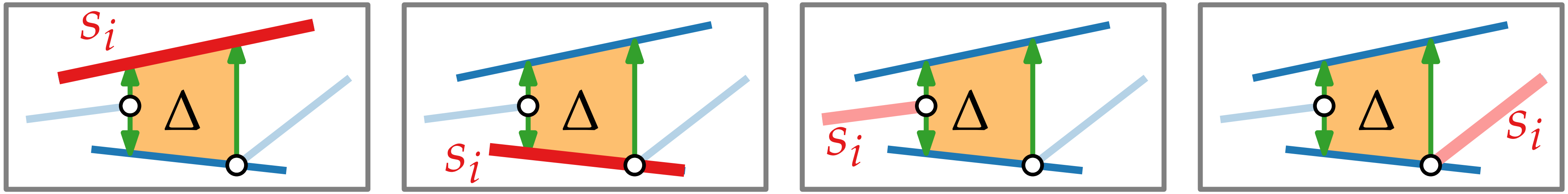

$\mathbf{P}(\text{top}(\Delta) = s_i) = ?$

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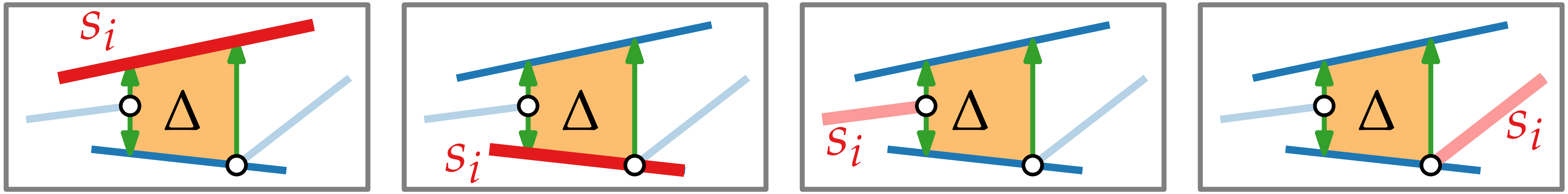

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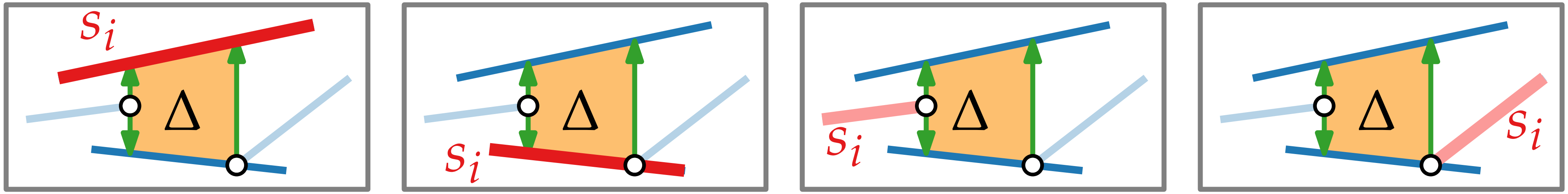

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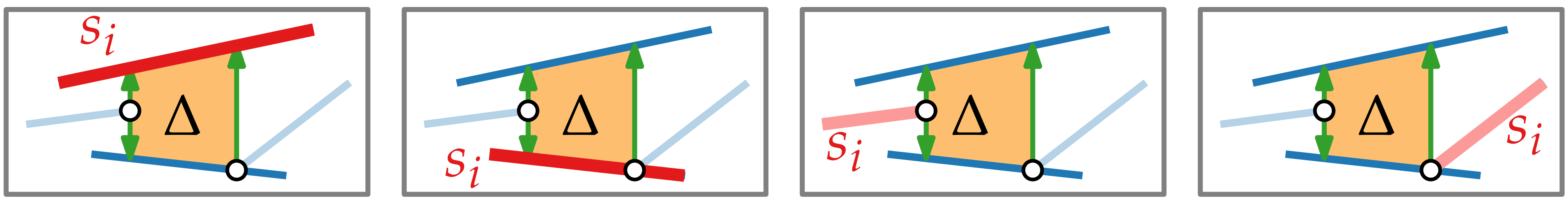

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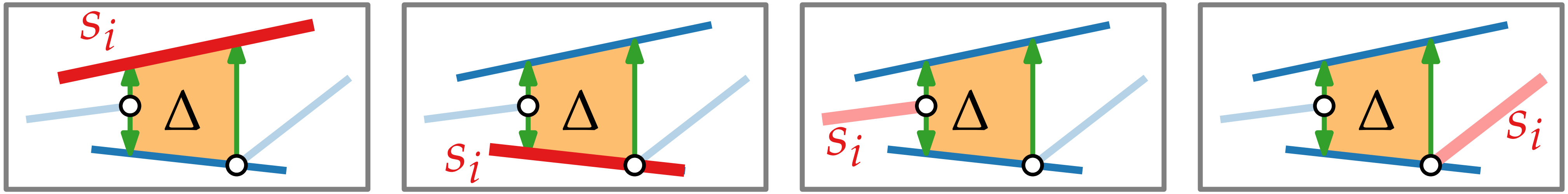

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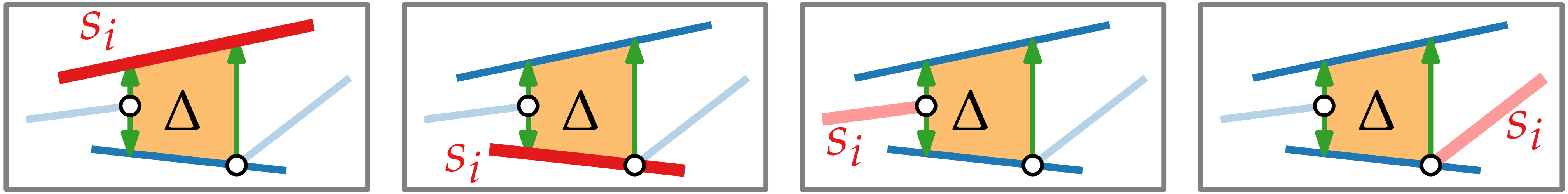

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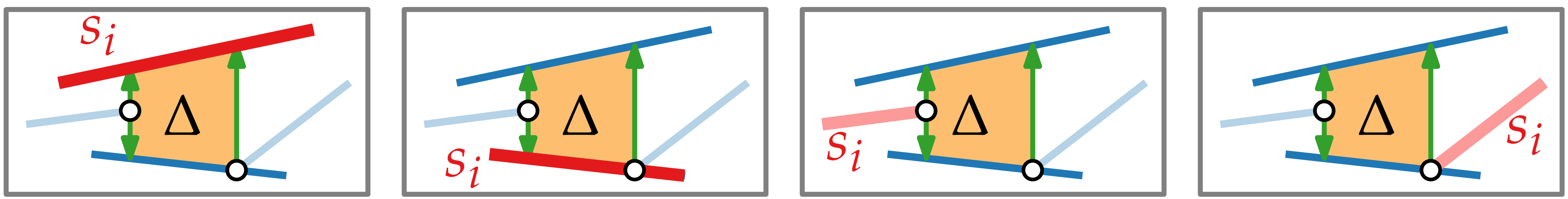

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$\square$

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$\Rightarrow$ Size: $O(n) + \sum_{i=1}^{n} \mathbf{E}[X_i] \leq O(n) + 13n \in O(n)$ $\square$

The 2D Result

Theorem. TrapezoidalMap(S) computes $\mathcal{T}(S)$ for a set of n line segments in general position and a search structure $\mathcal{D}$ for $\mathcal{T}(S)$ in $O(n \log n)$ expected time. The expected size of $\mathcal{D}$ is $O(n)$ and the expected query time is $O(\log n)$.