Computational Geometry

Lecture 4:
Linear Programming
or
Profit Maximization

Part I:
Introduction to Linear Programming

Philipp Kindermann

Winter Semester 2020

Maximizing Profits

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$$\begin{aligned} M_A: &\quad 4x_1 + 11x_2 \\ M_B: &\quad x_1 + x_2 \\ M_C: &\quad x_2 \end{aligned}$$

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$$\begin{array}{lrcrcl} M_A: & 4x_1 & + & 11x_2 & \leq & 880 \\ M_B: & x_1 & + & x_2 & \leq & 150 \\ M_C: & & & x_2 & \leq & 60 \end{array}$$

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Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

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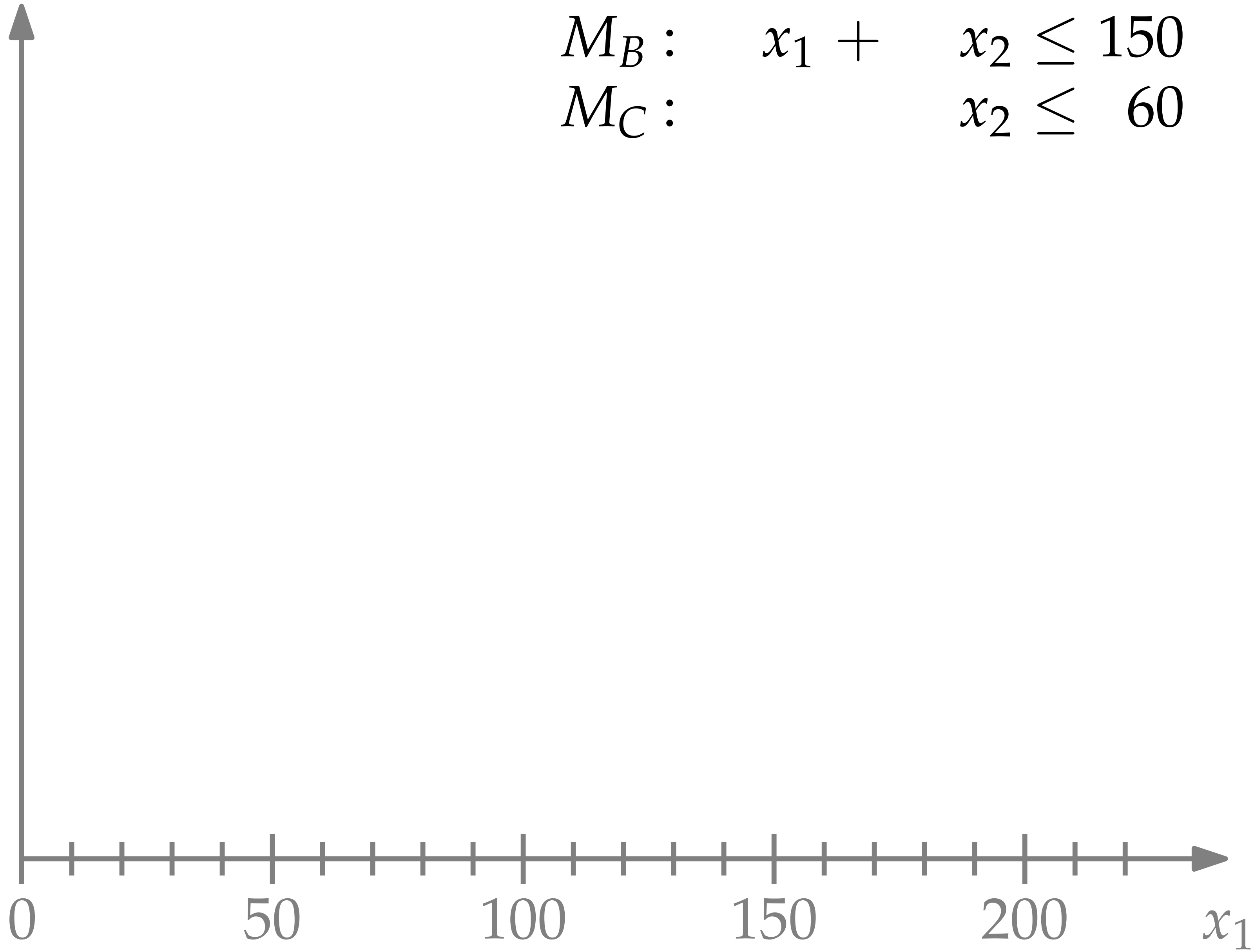

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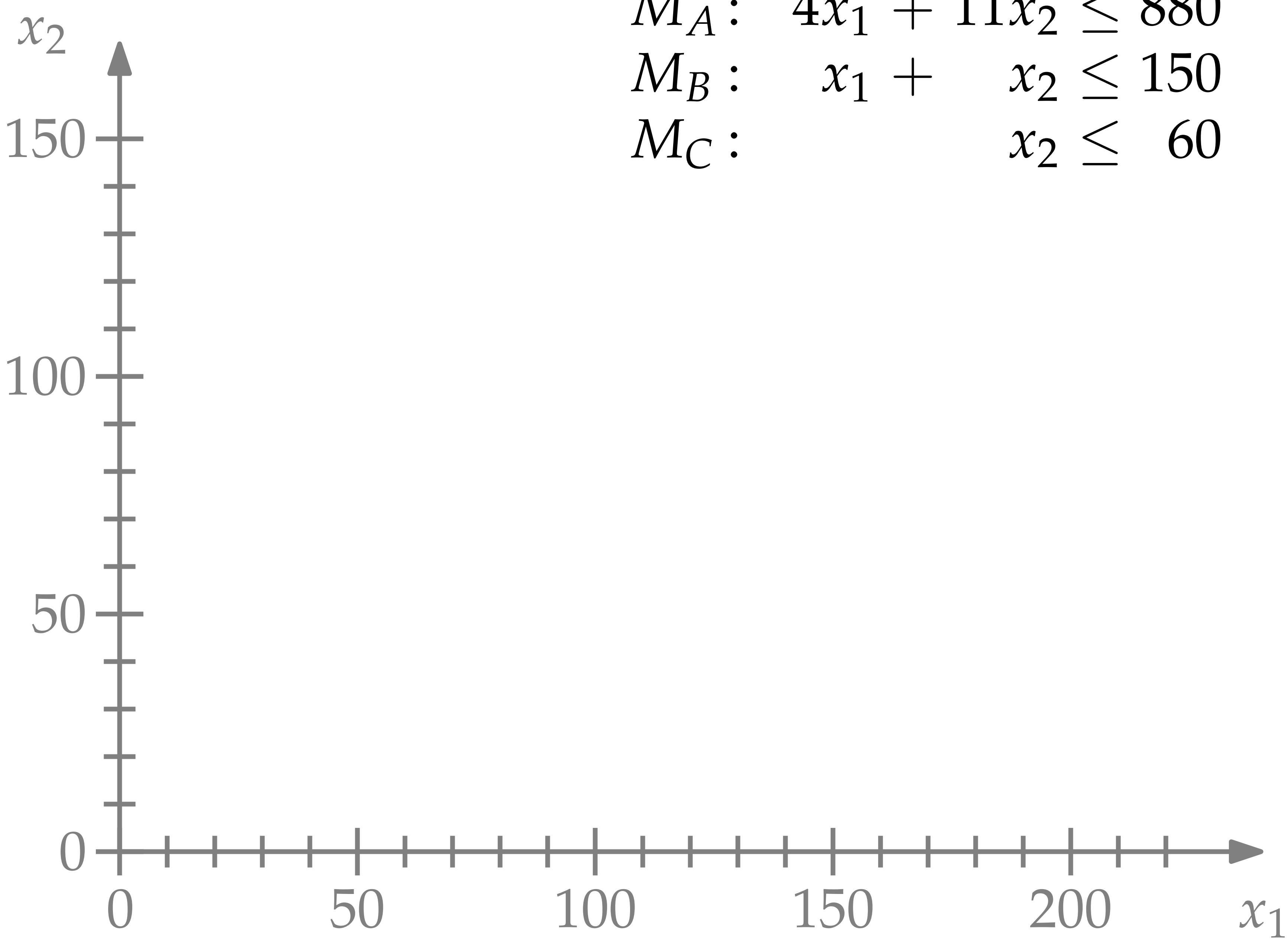

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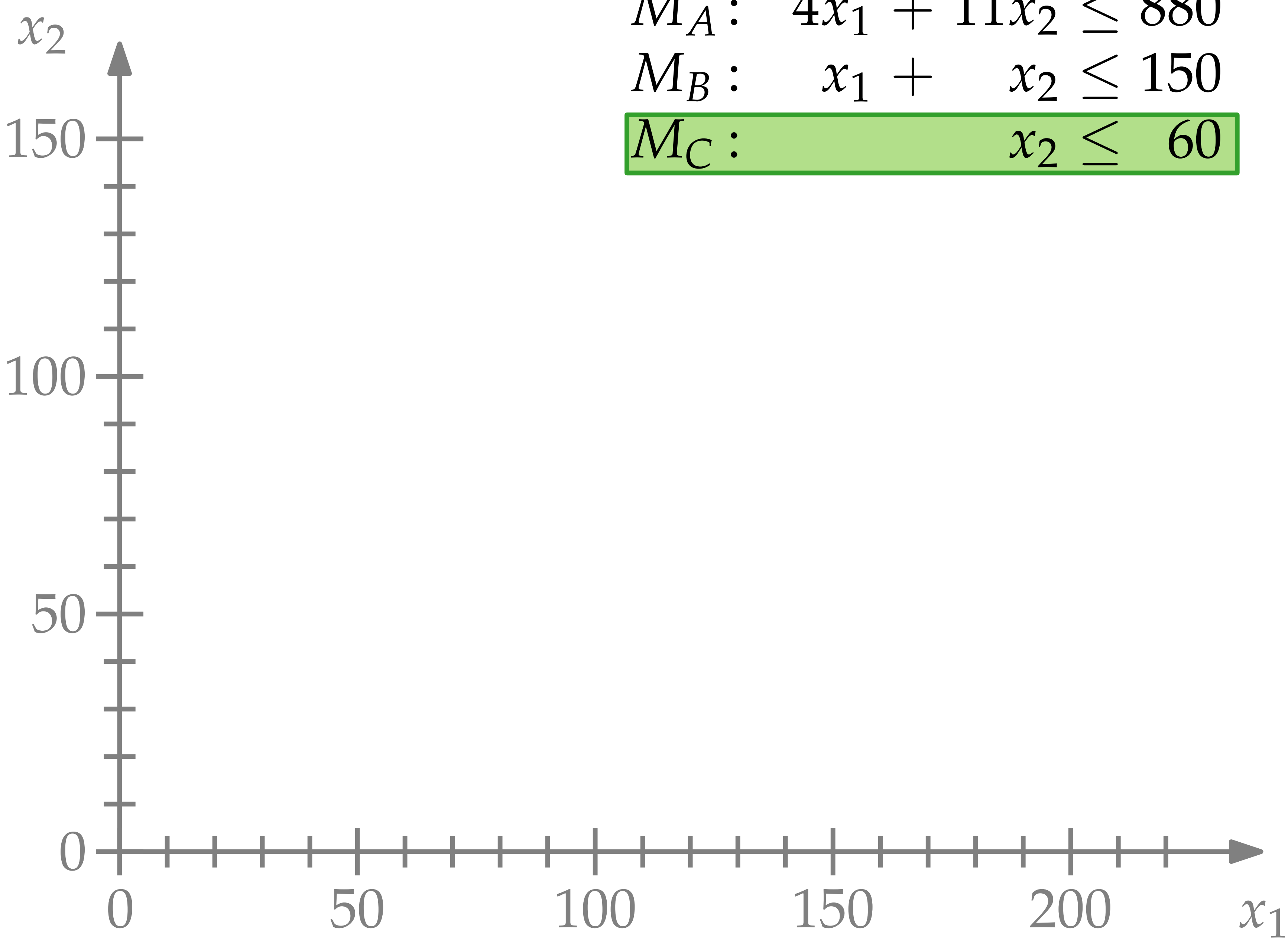

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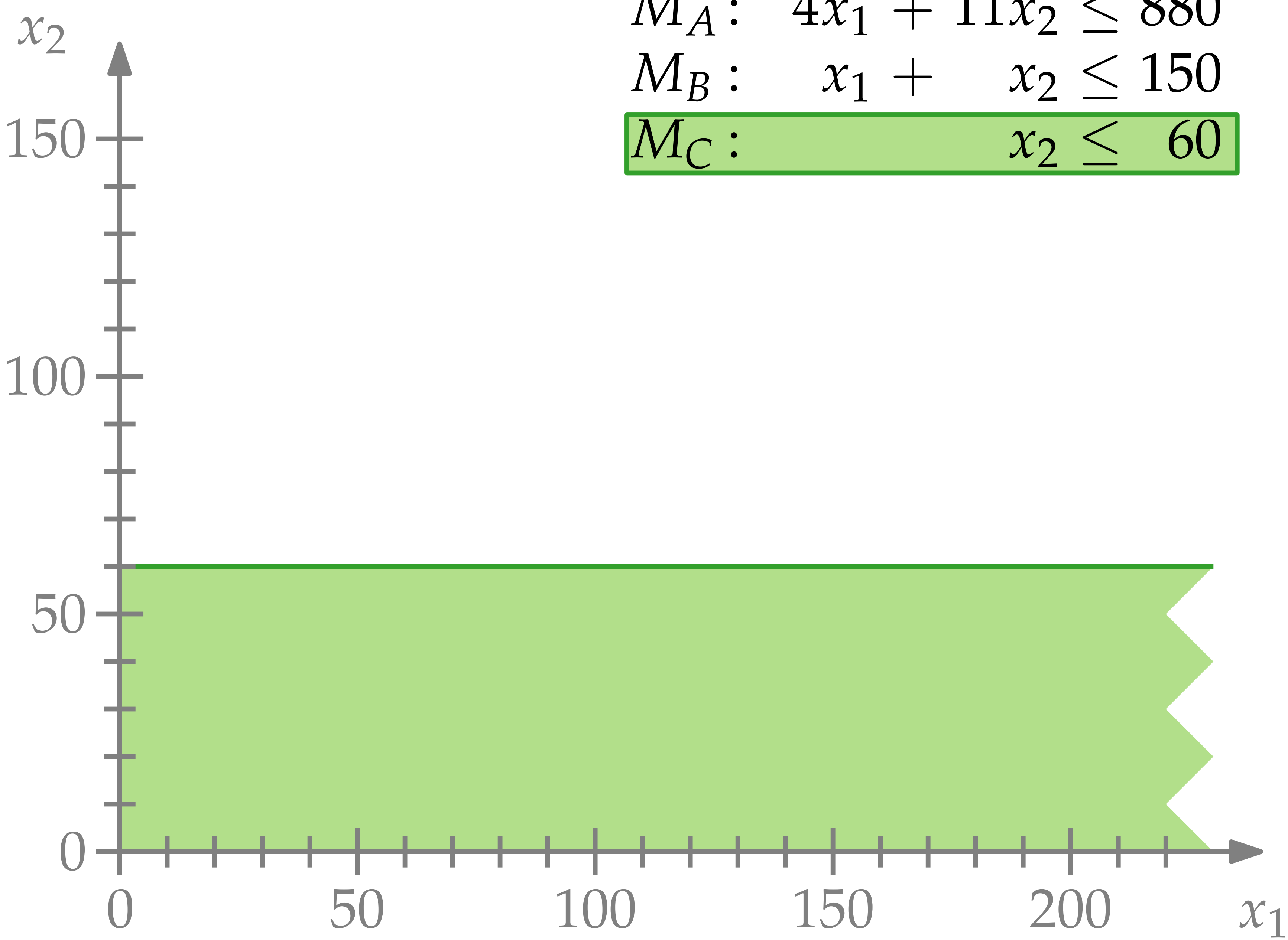

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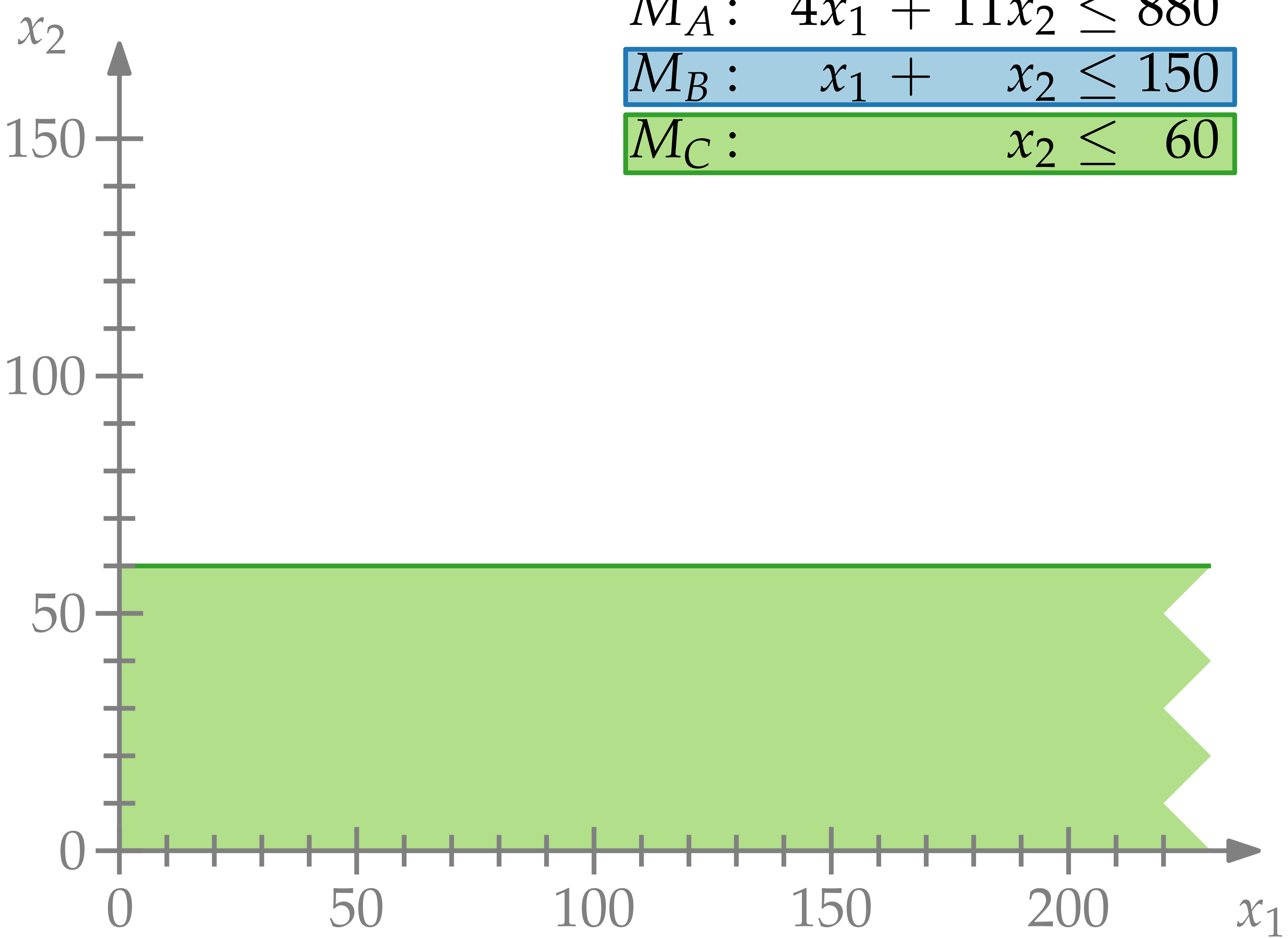

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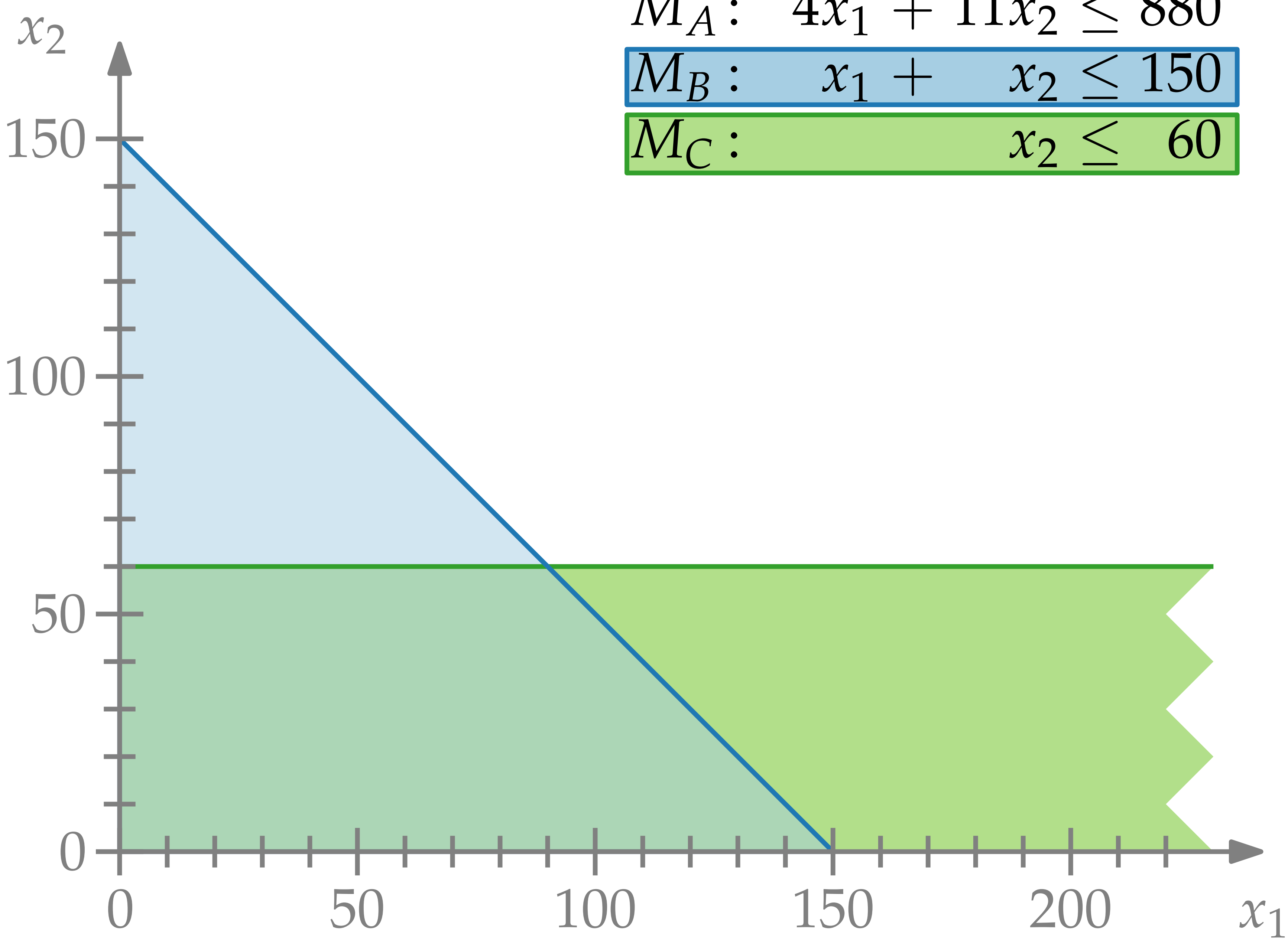

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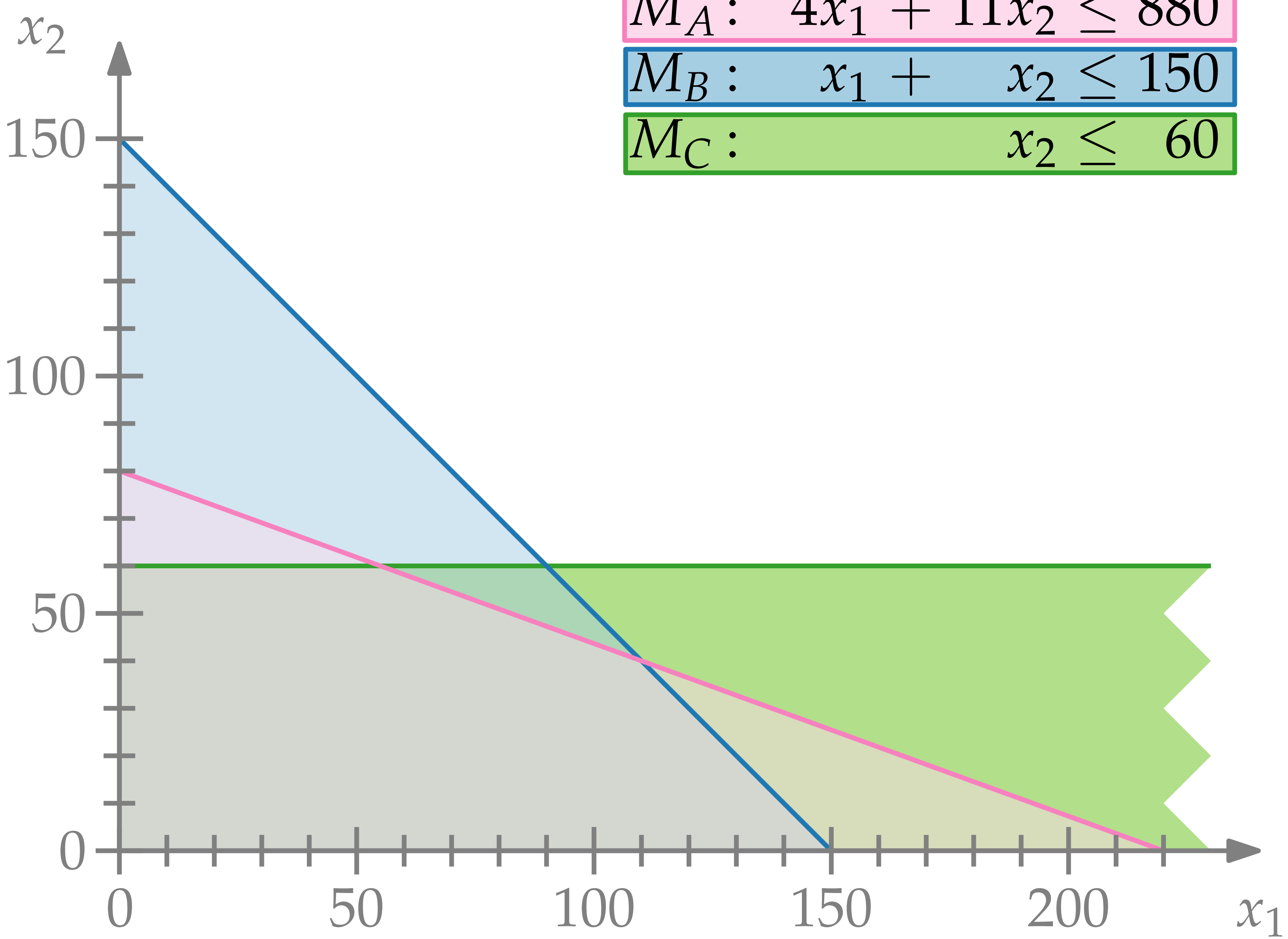

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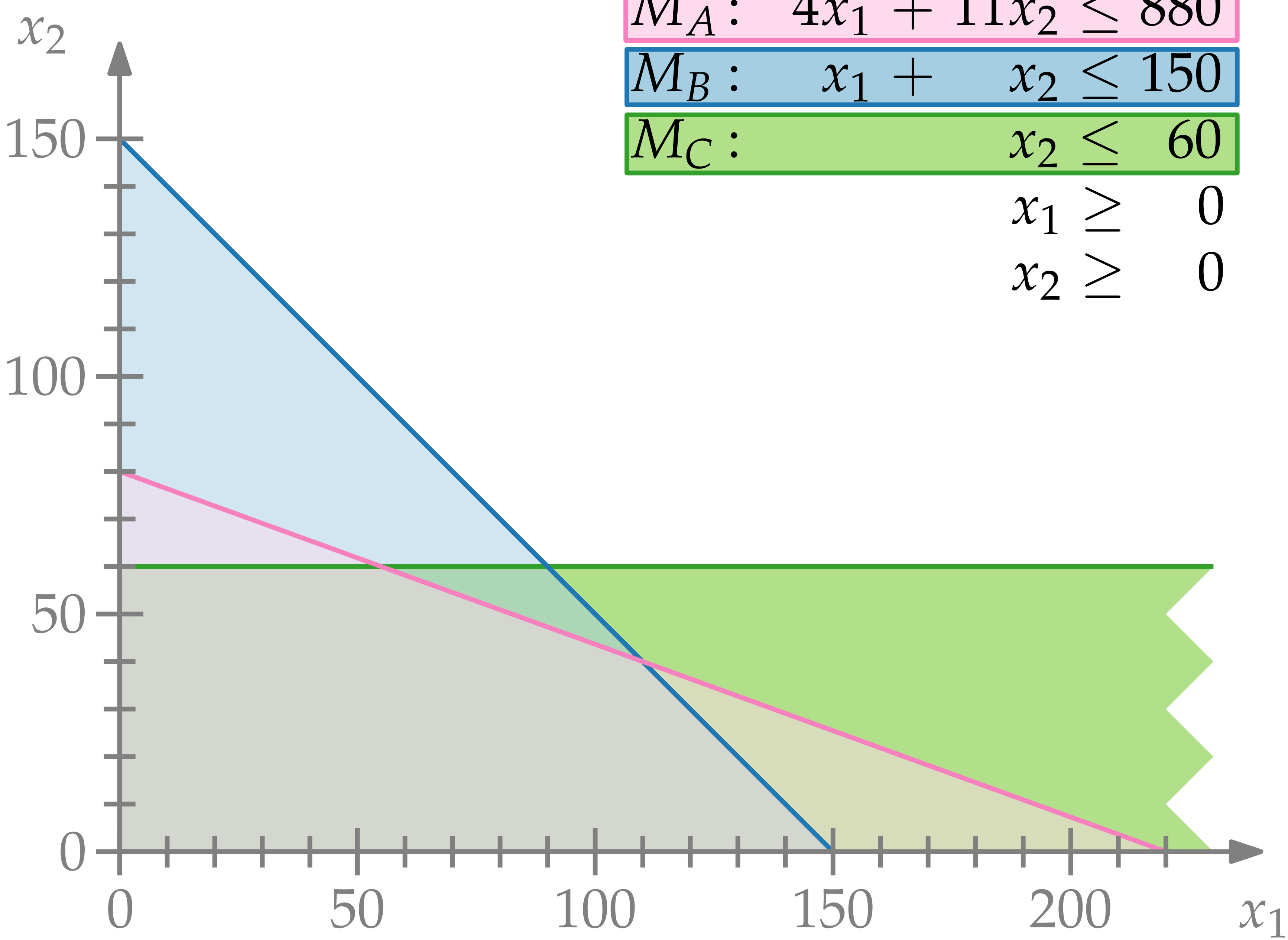

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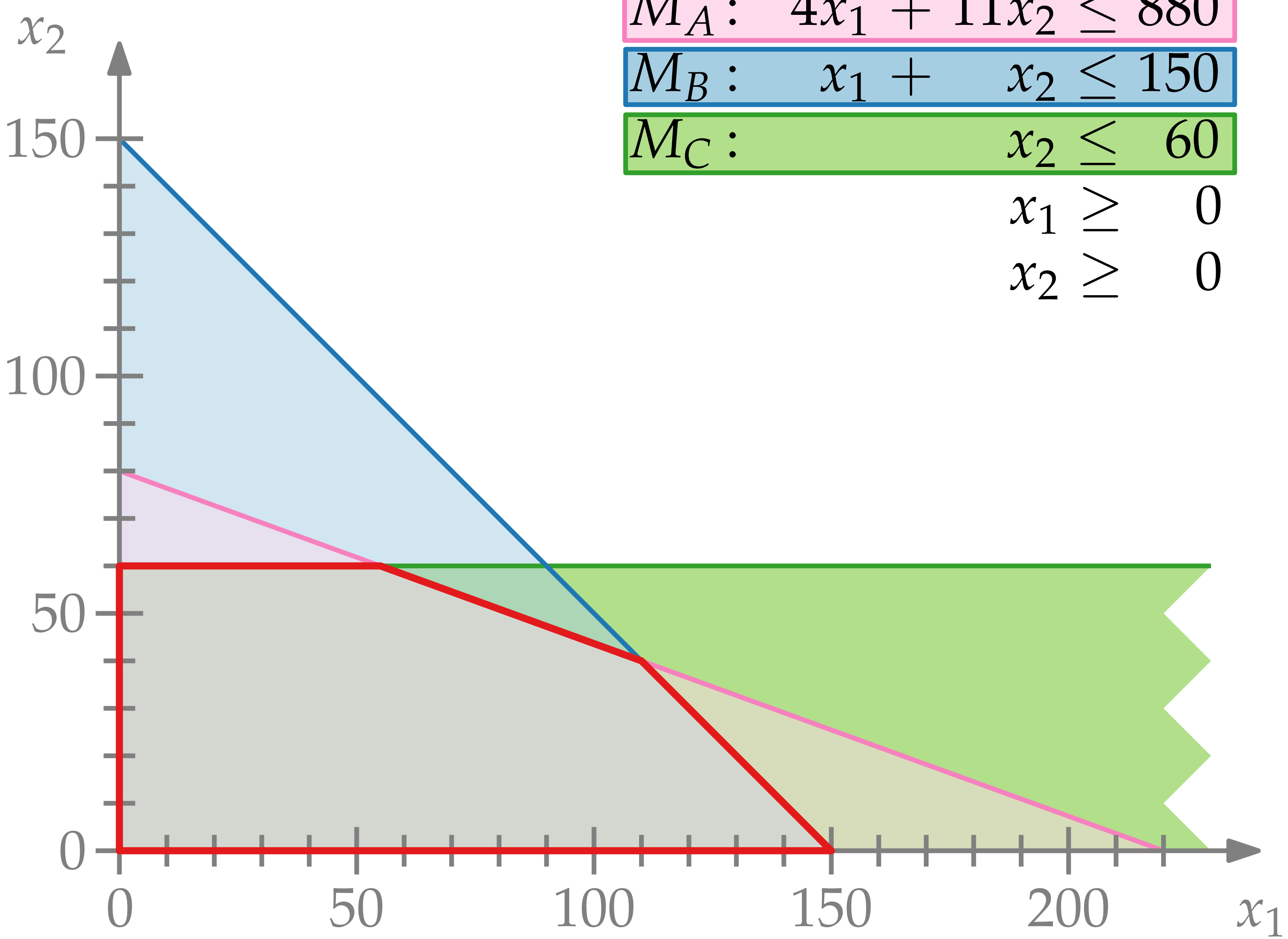

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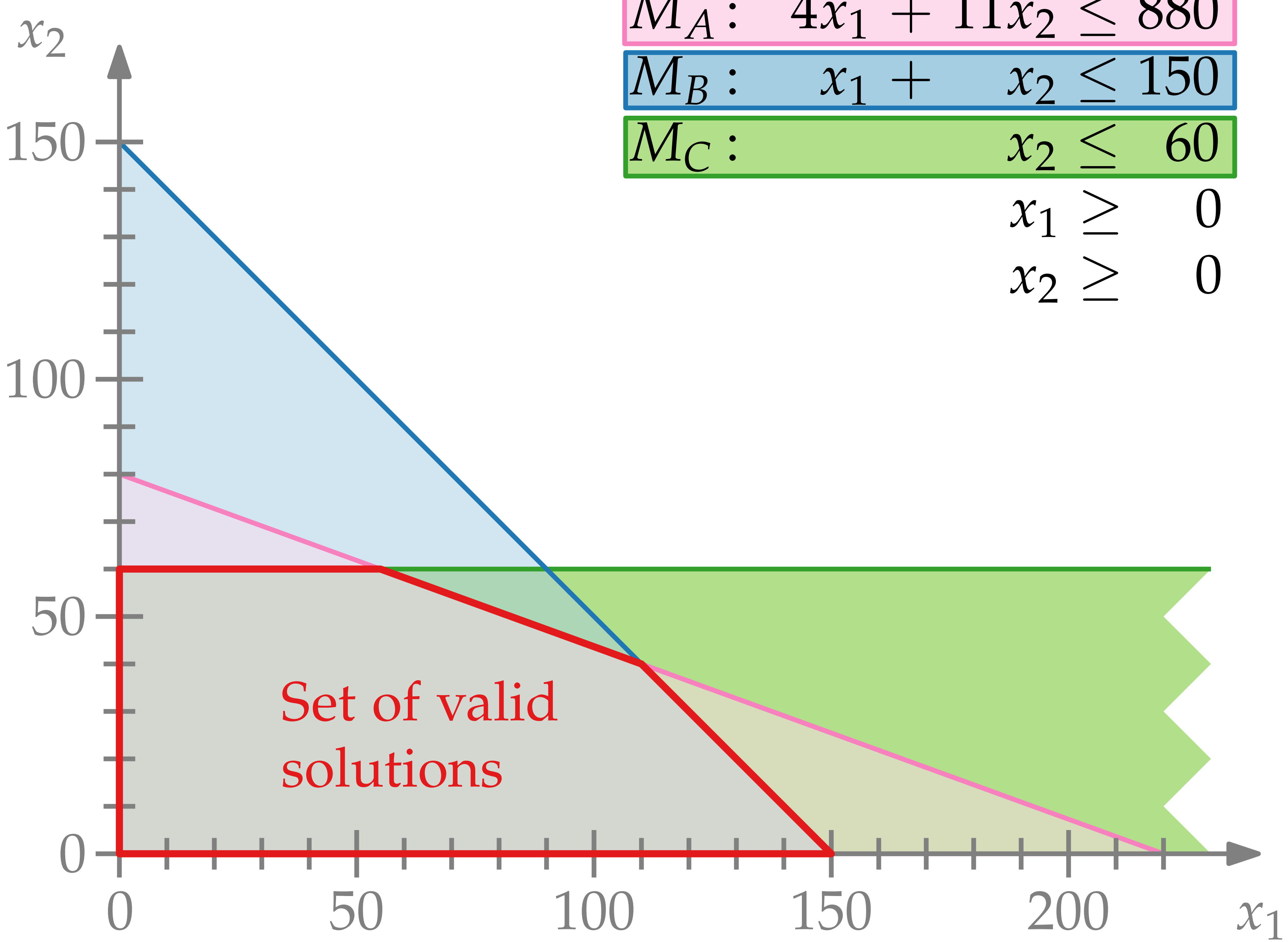

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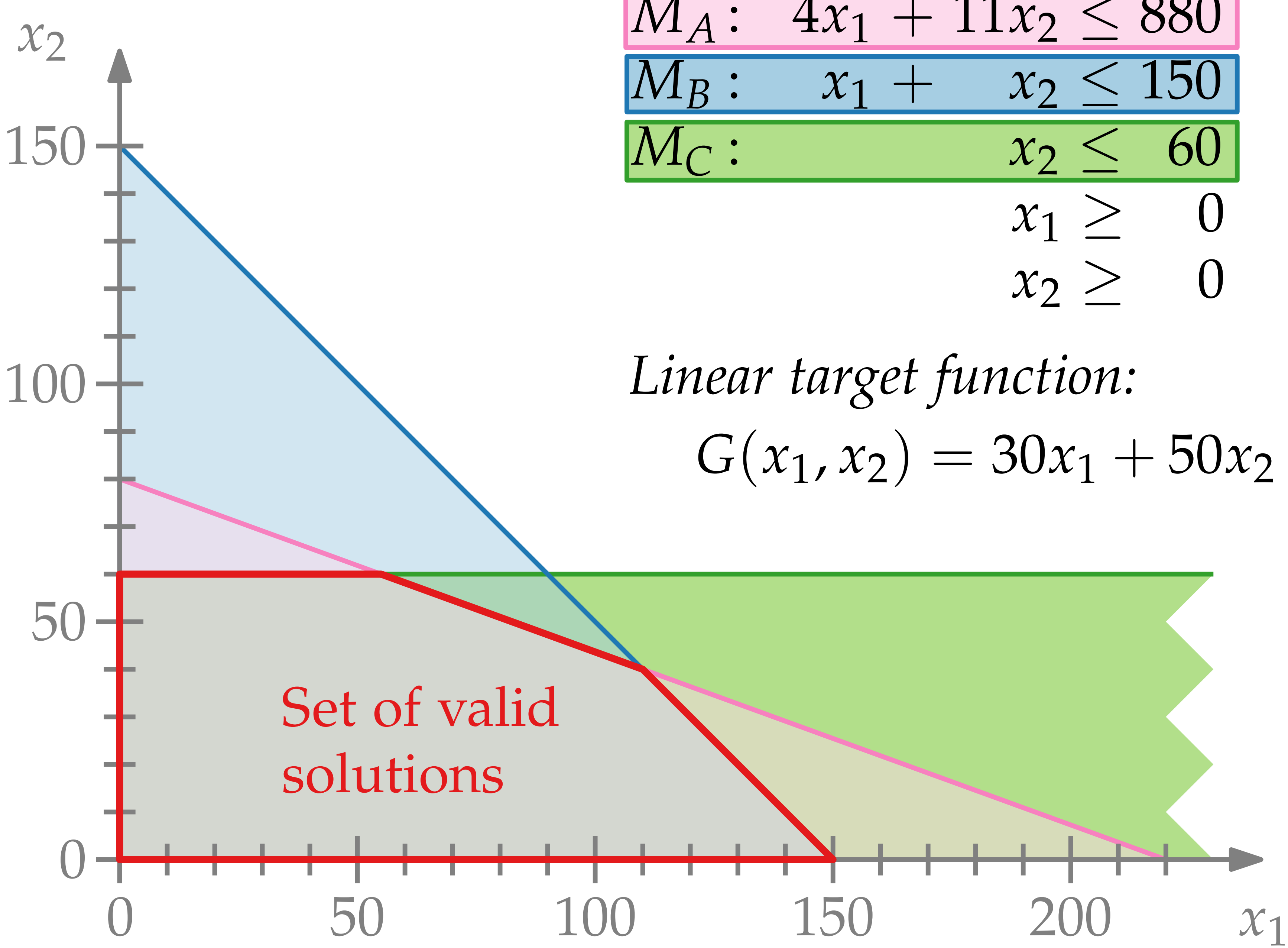

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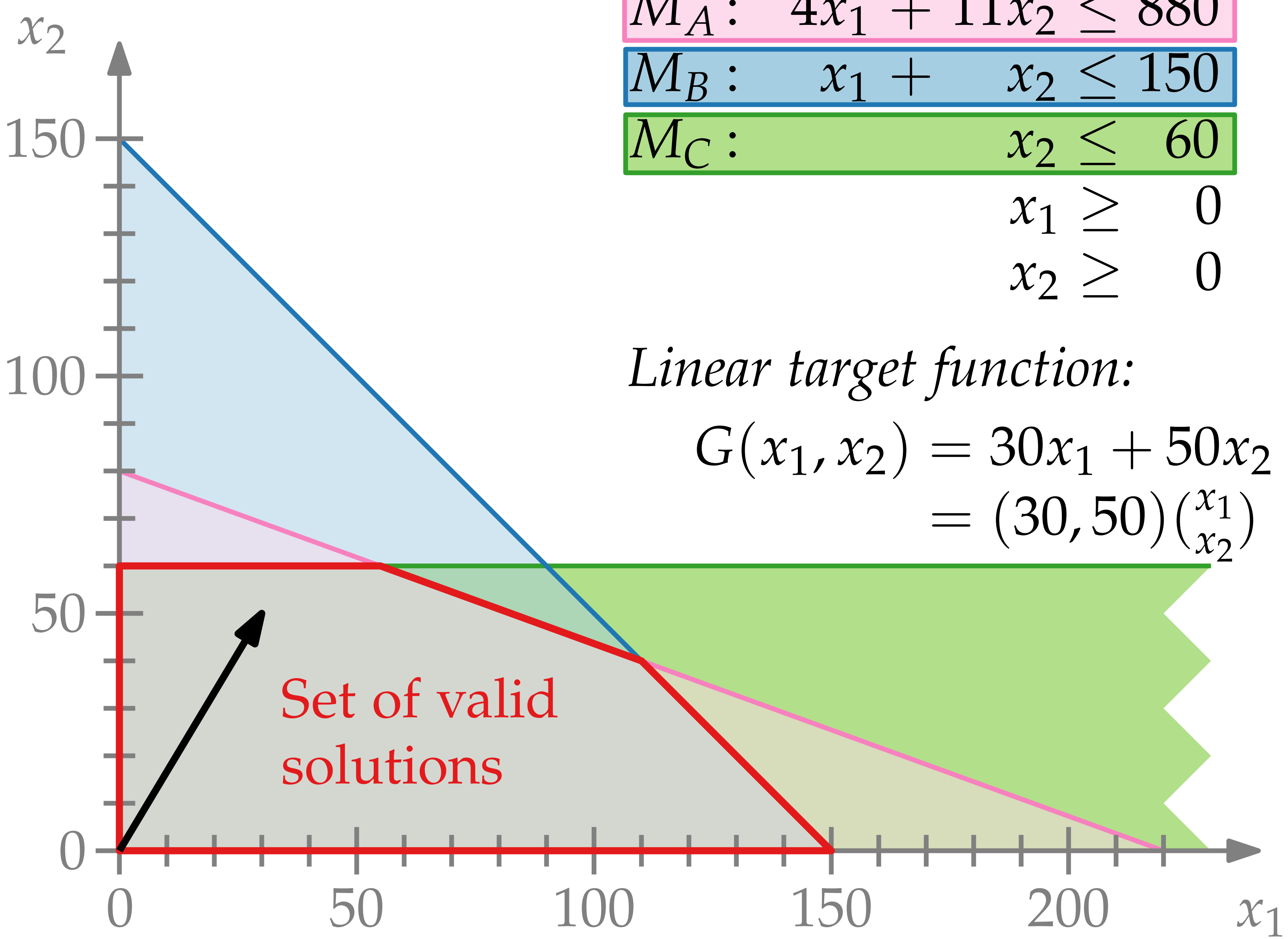

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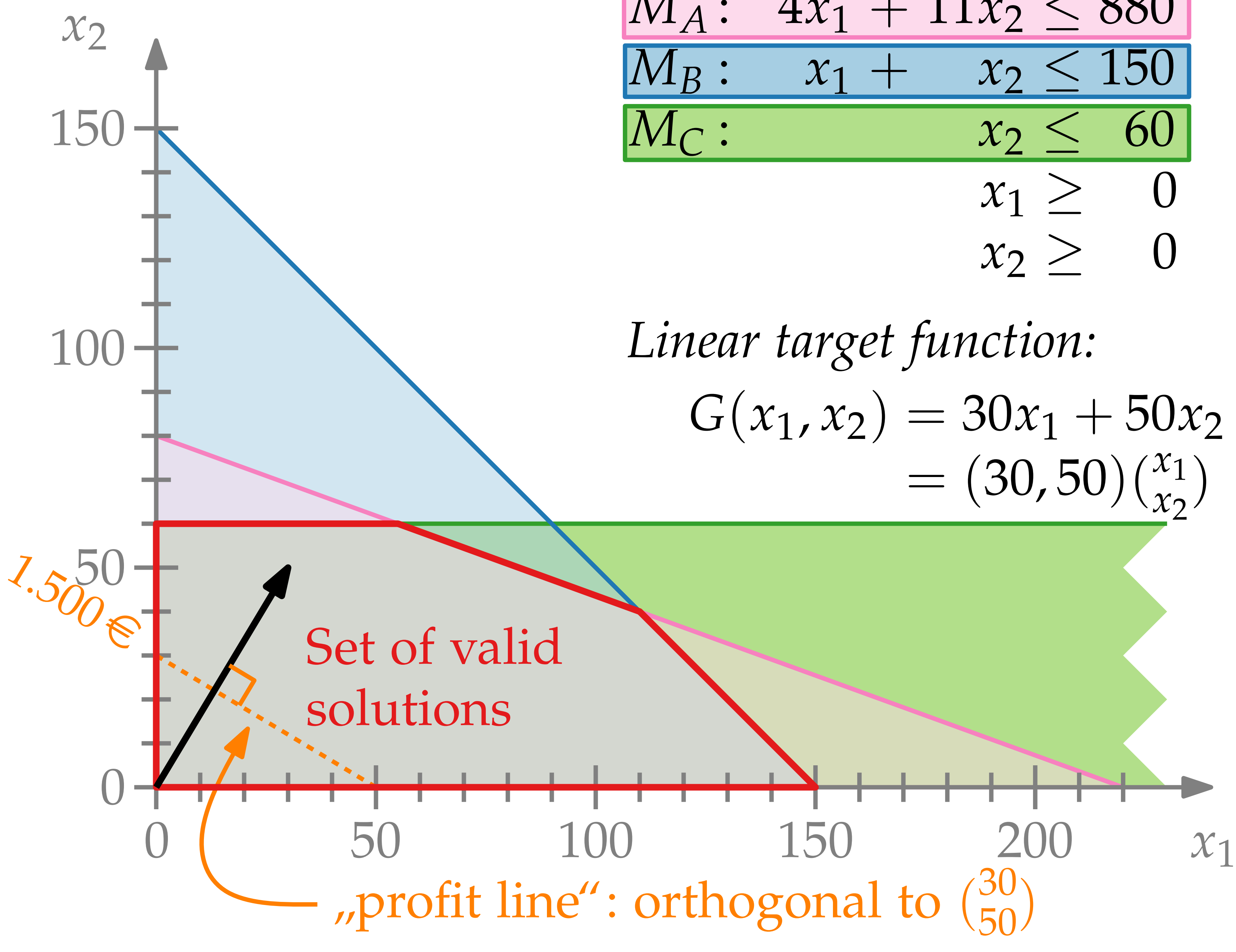

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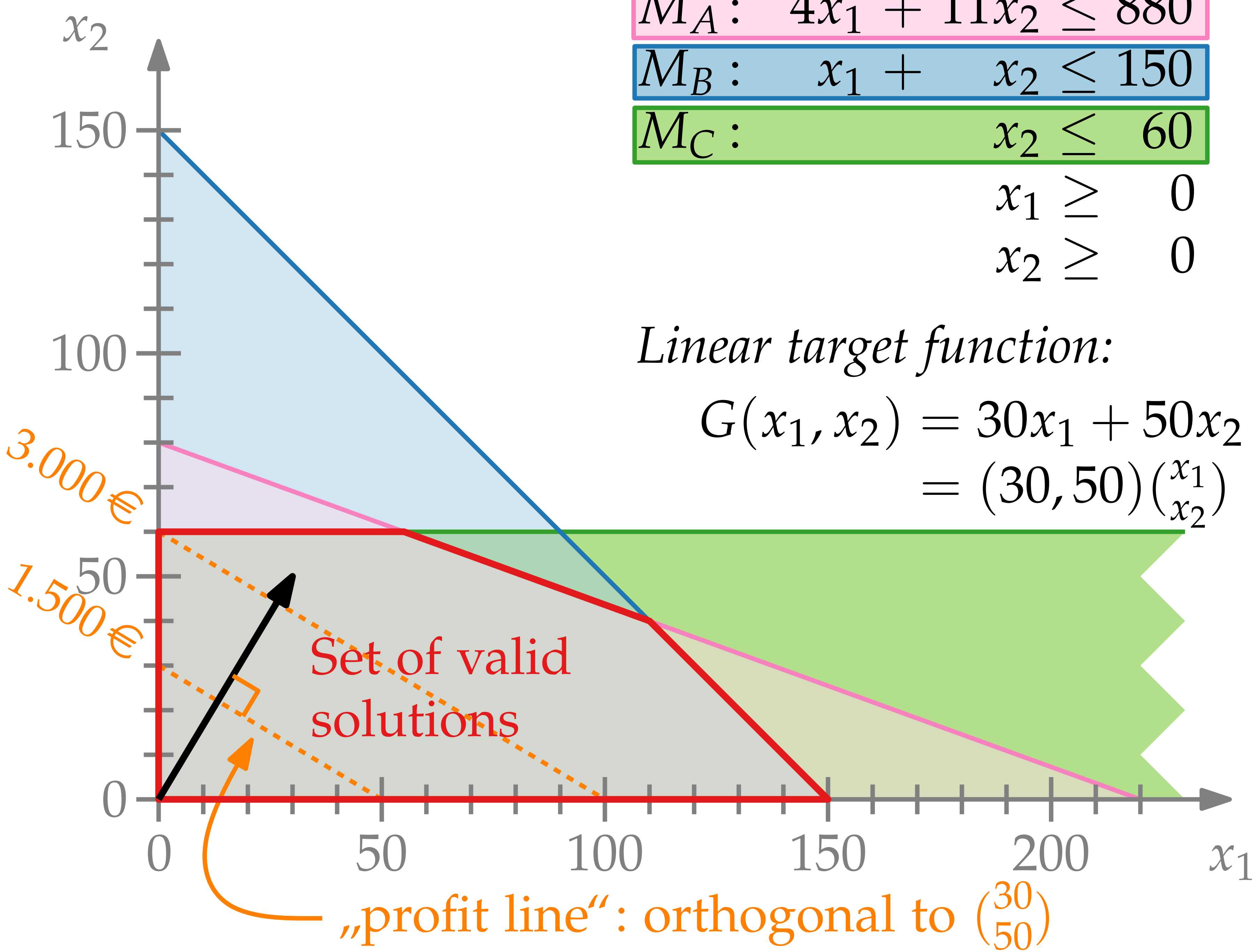

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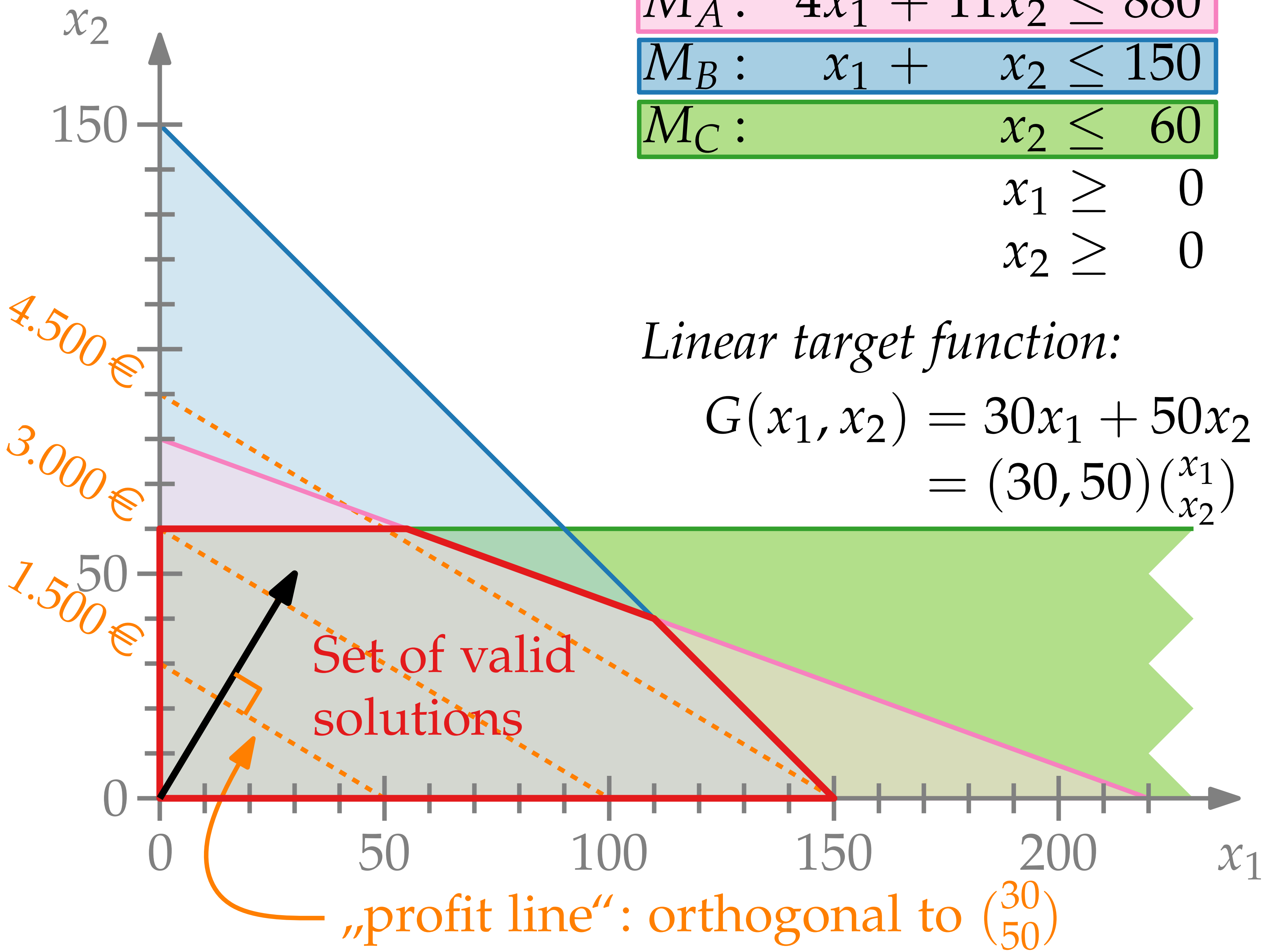

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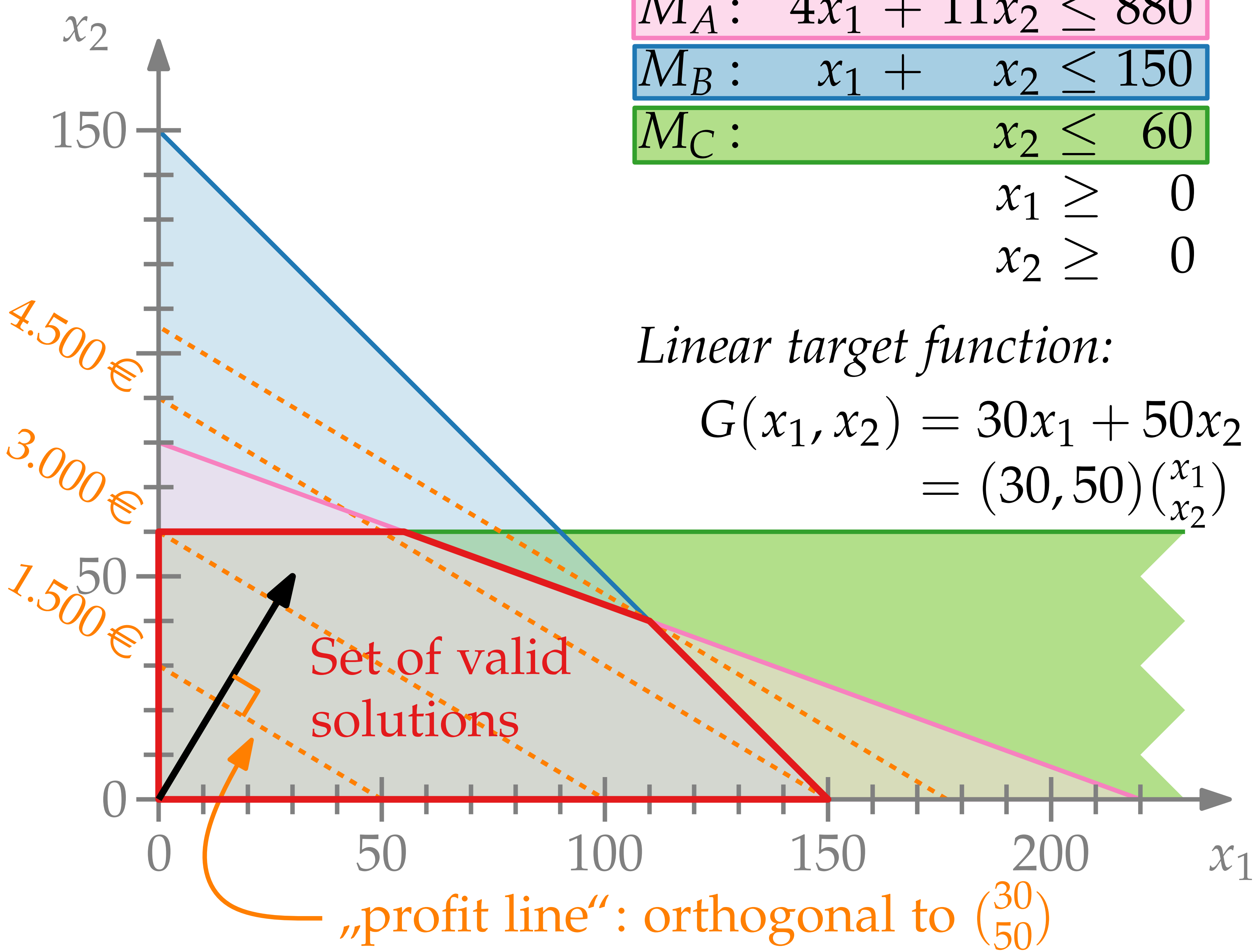

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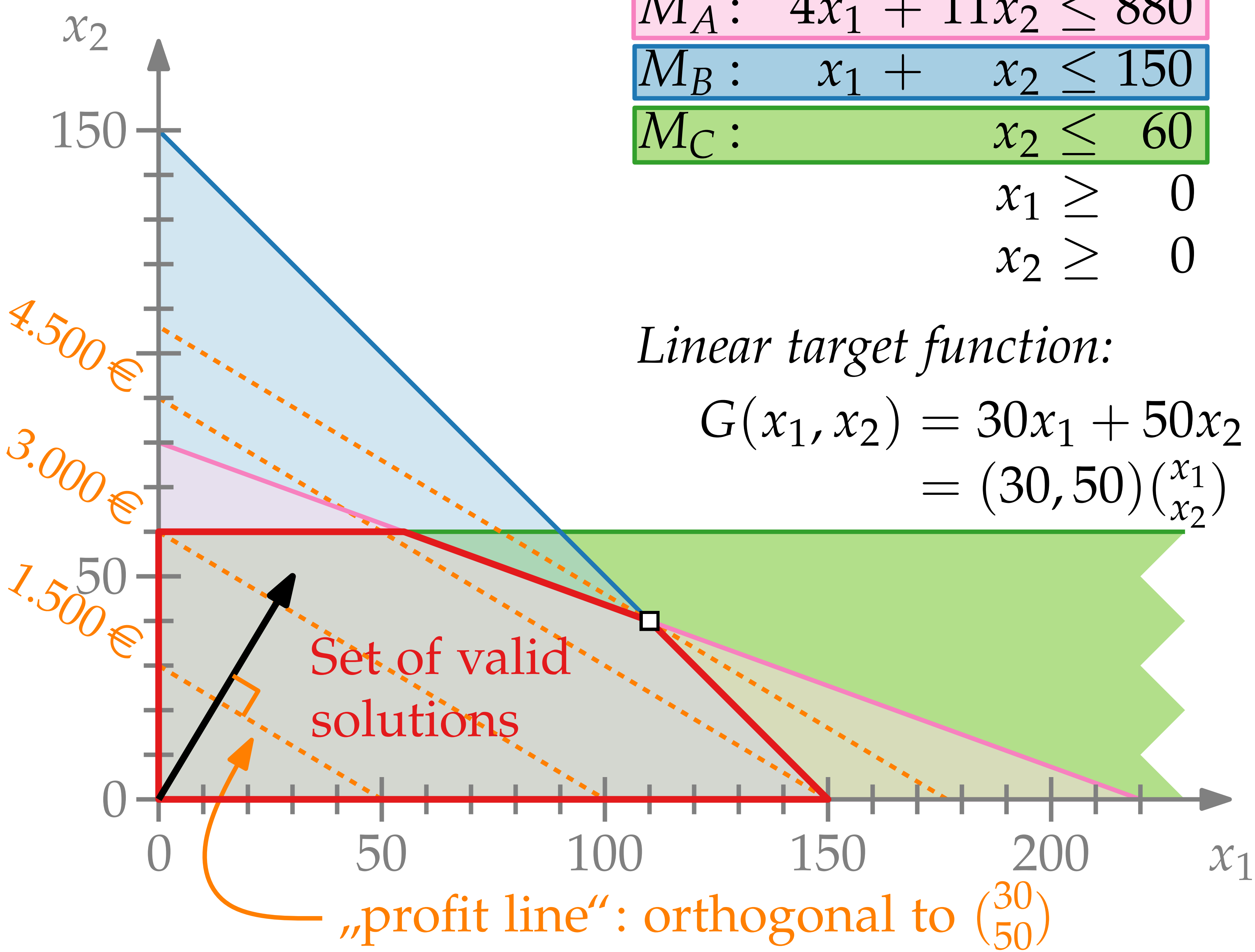

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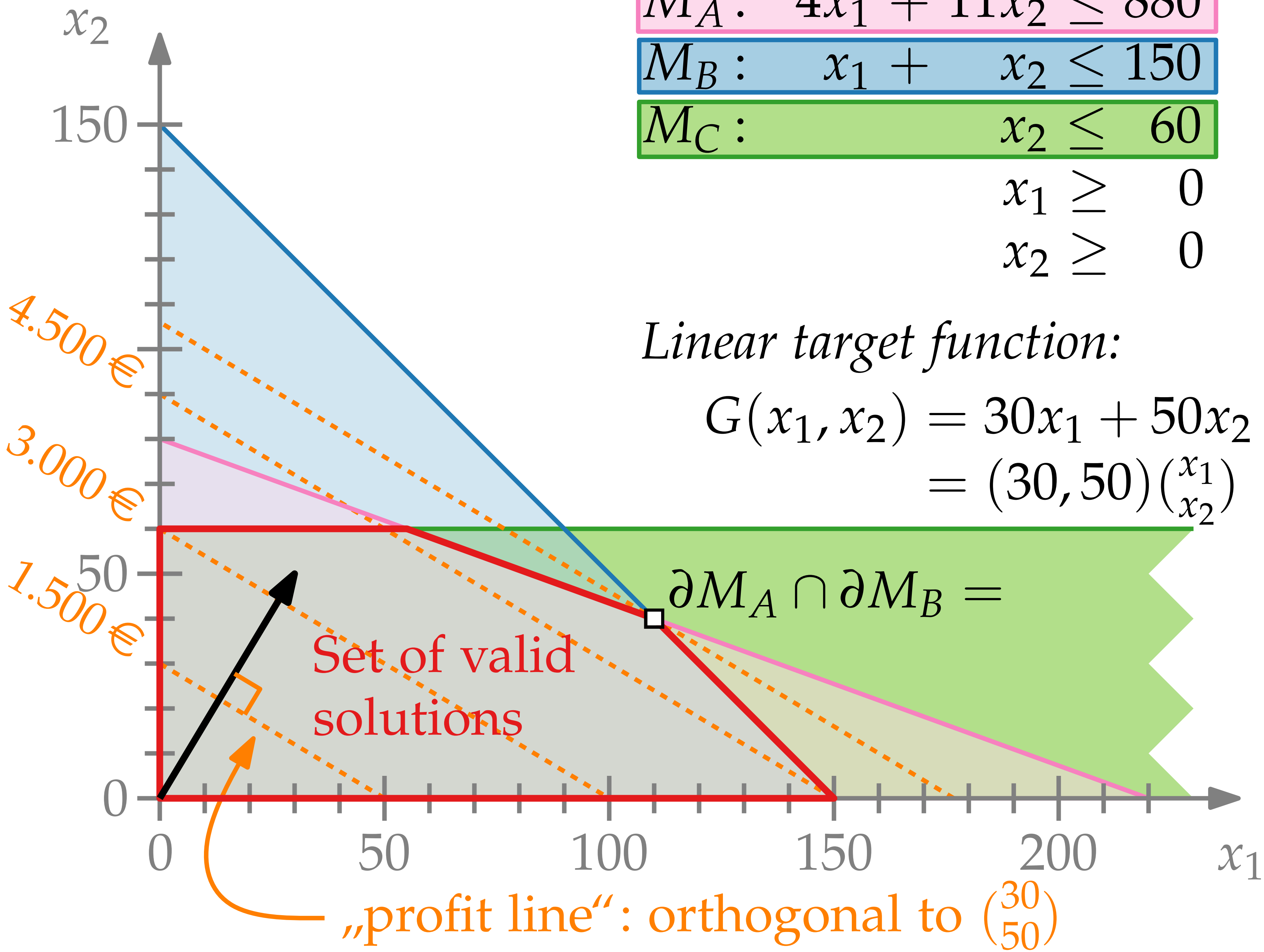

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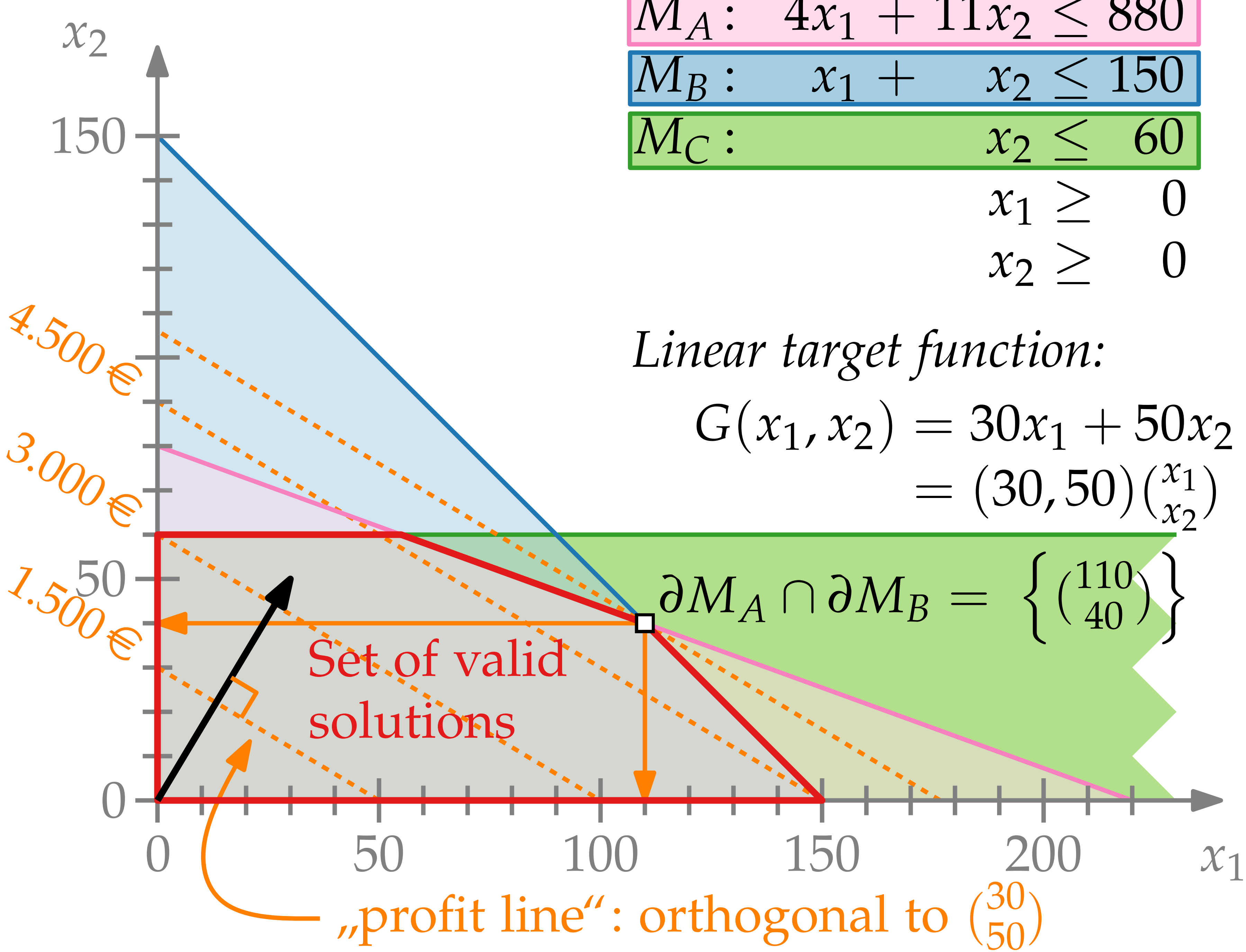
$$\partial M_A \cap \partial M_B = \left\{ \binom{110}{40} \right\}$$



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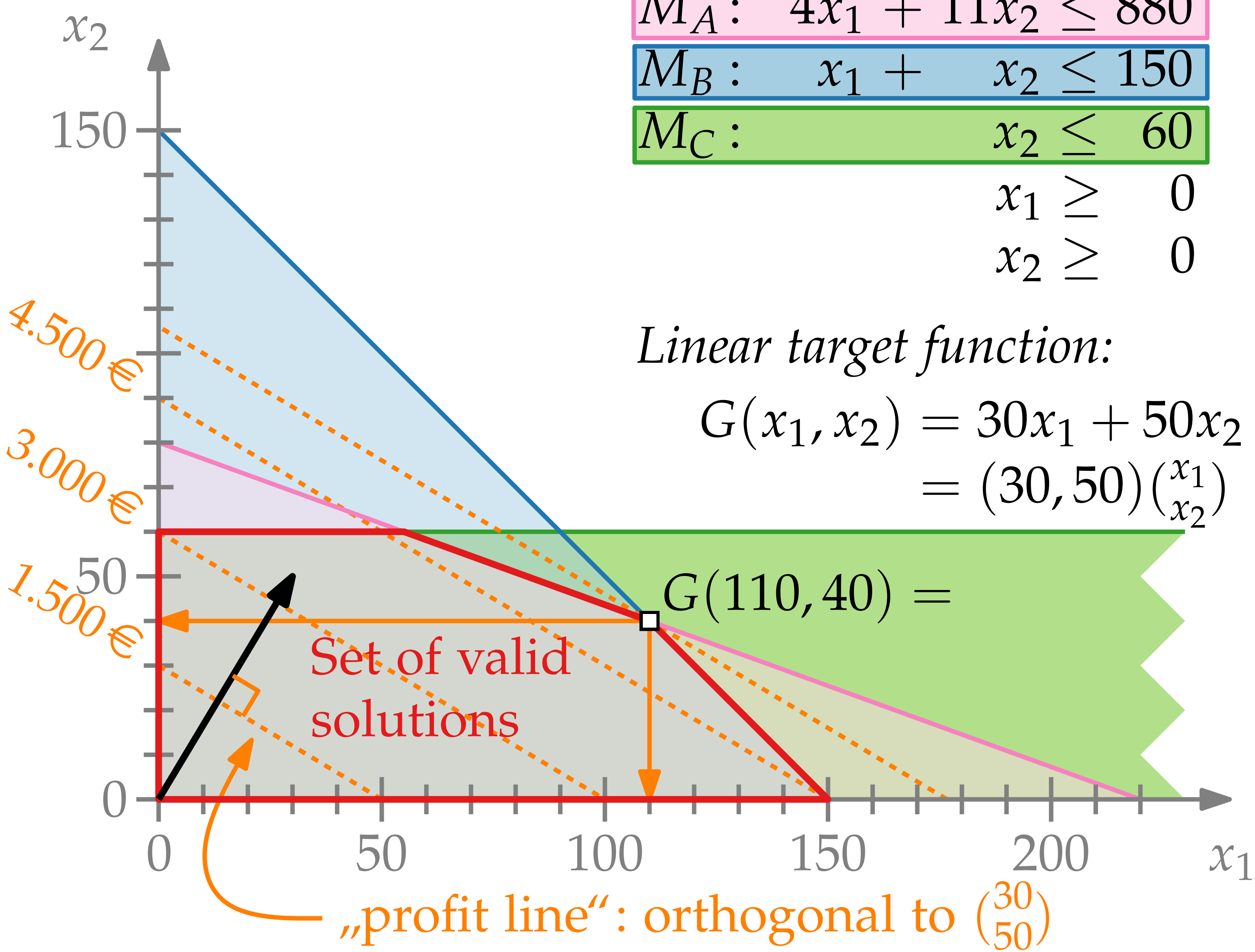

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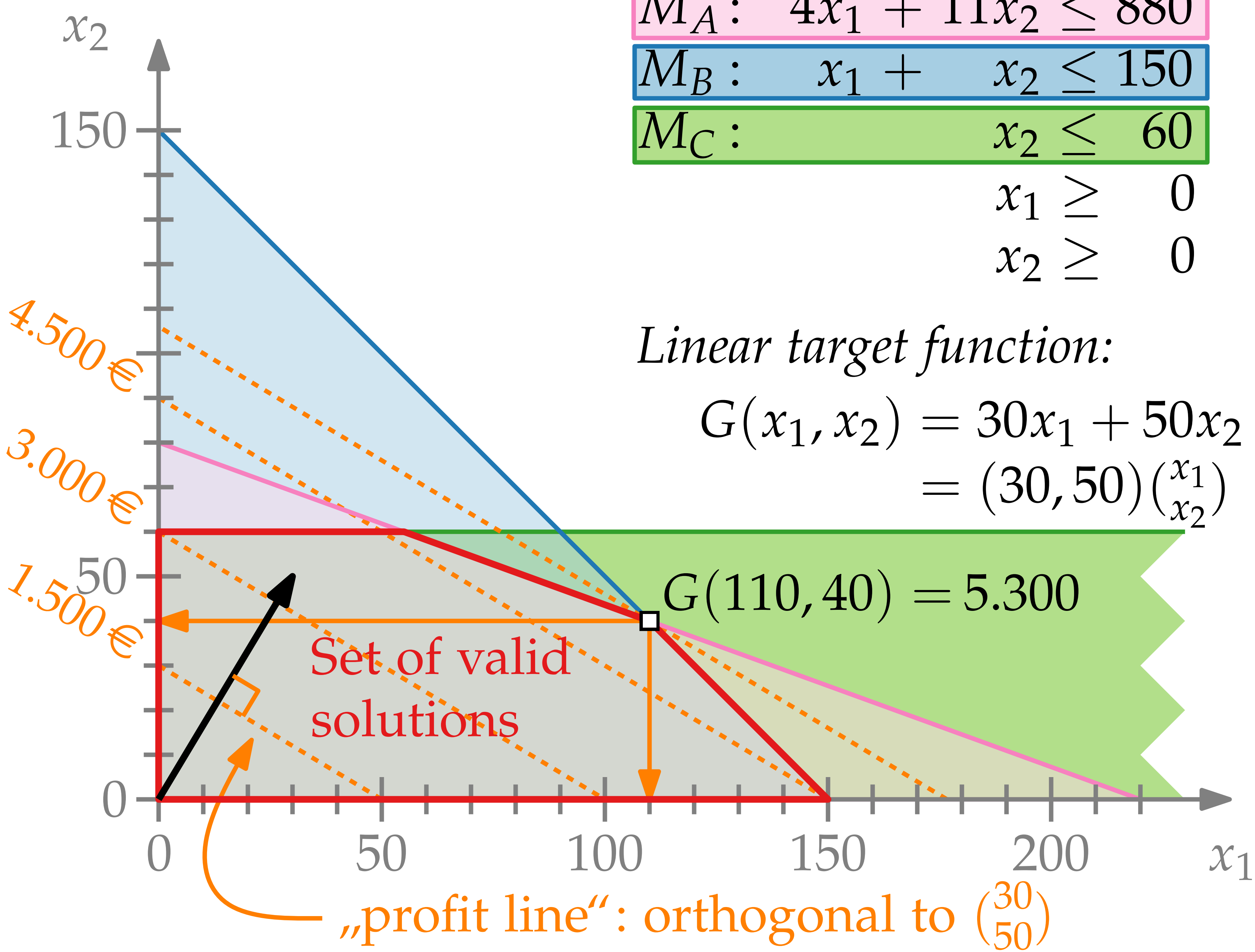

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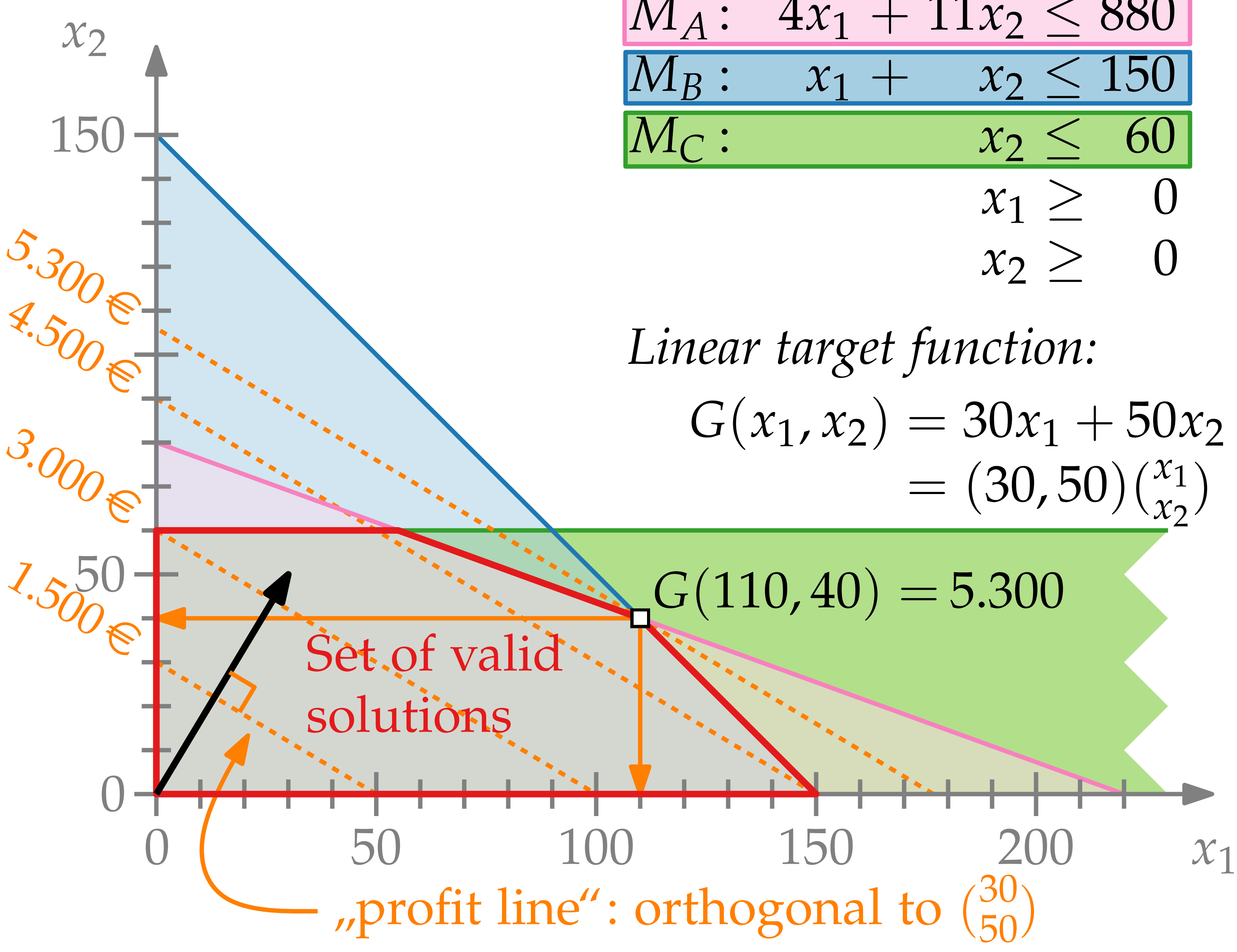

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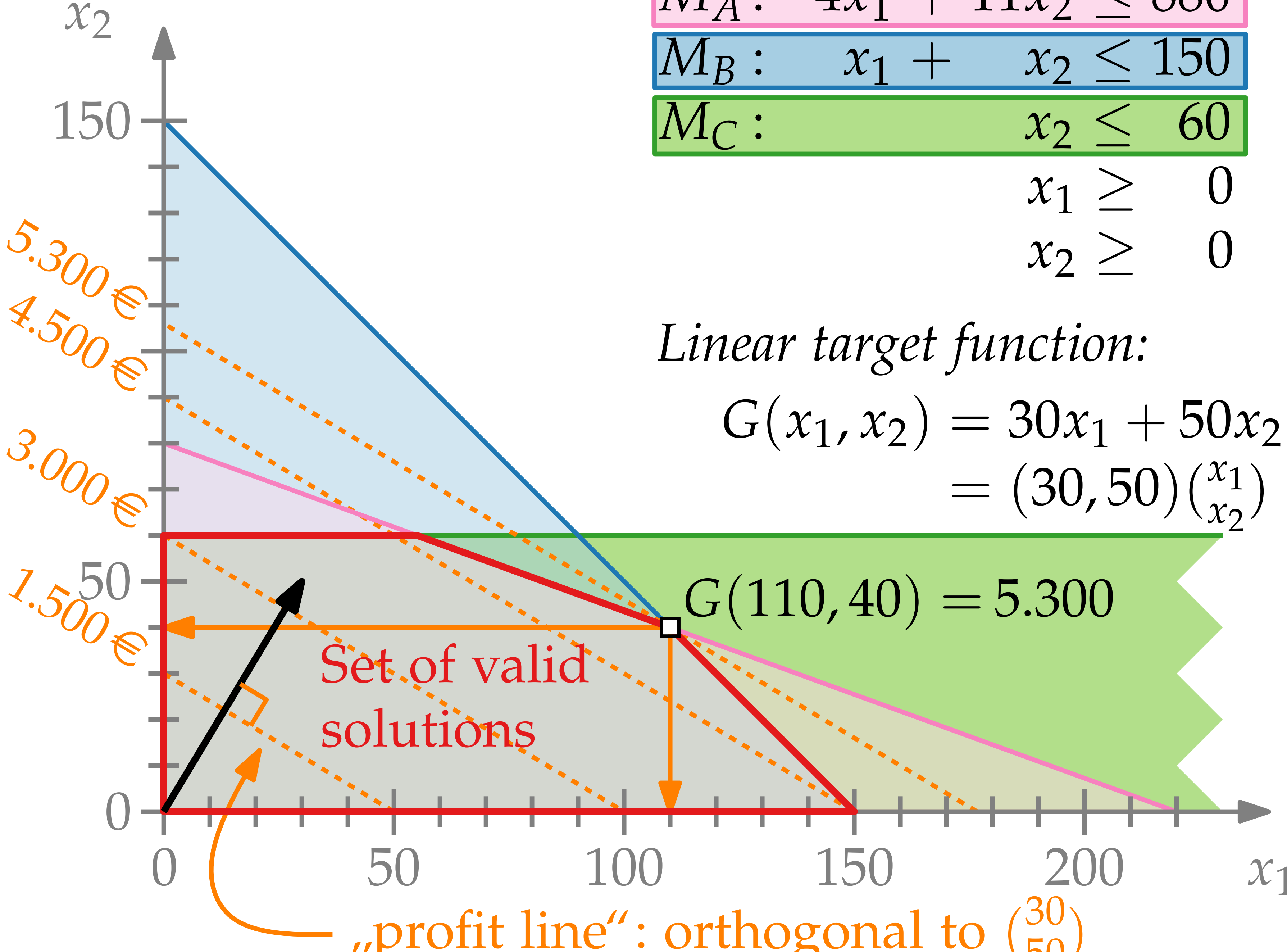

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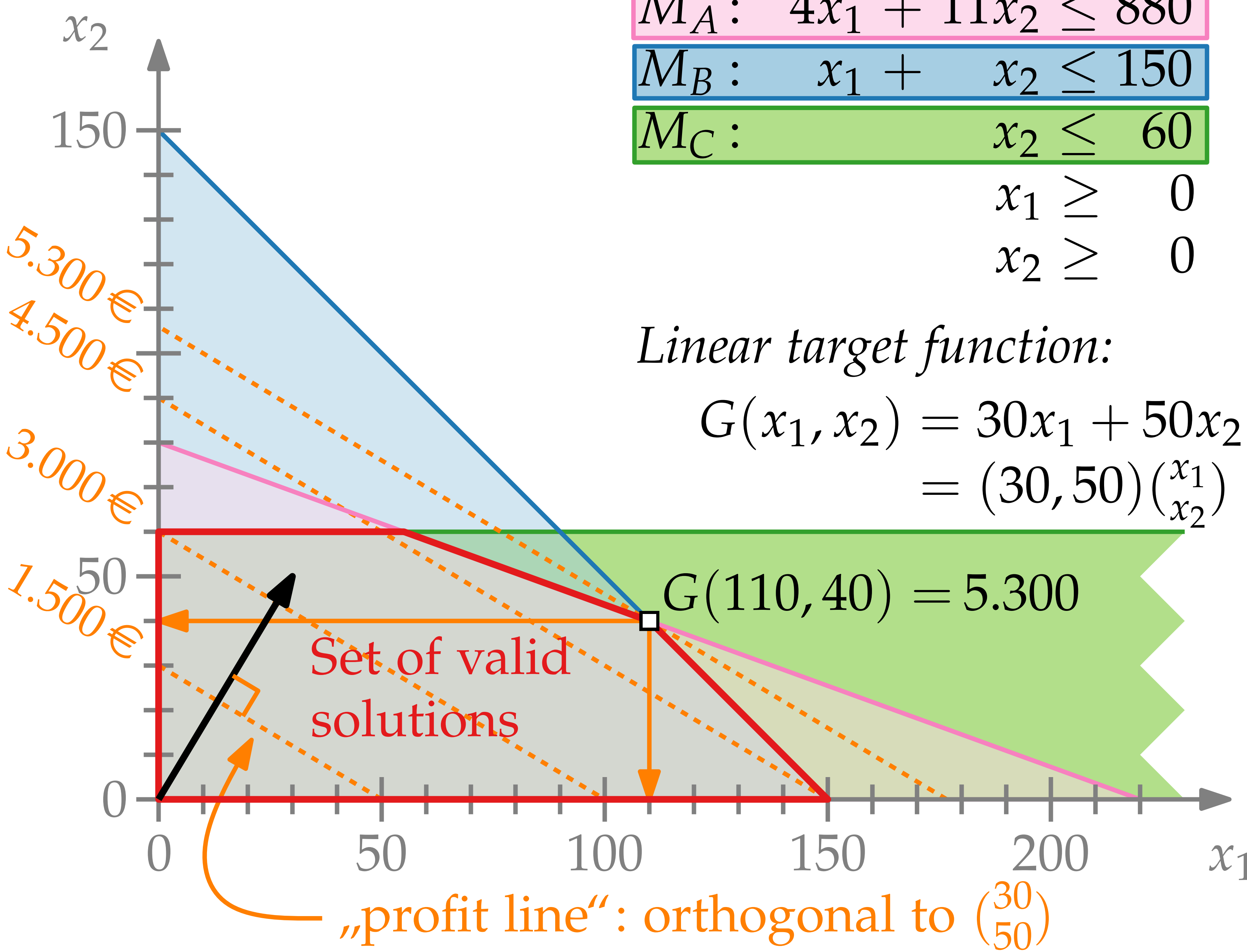

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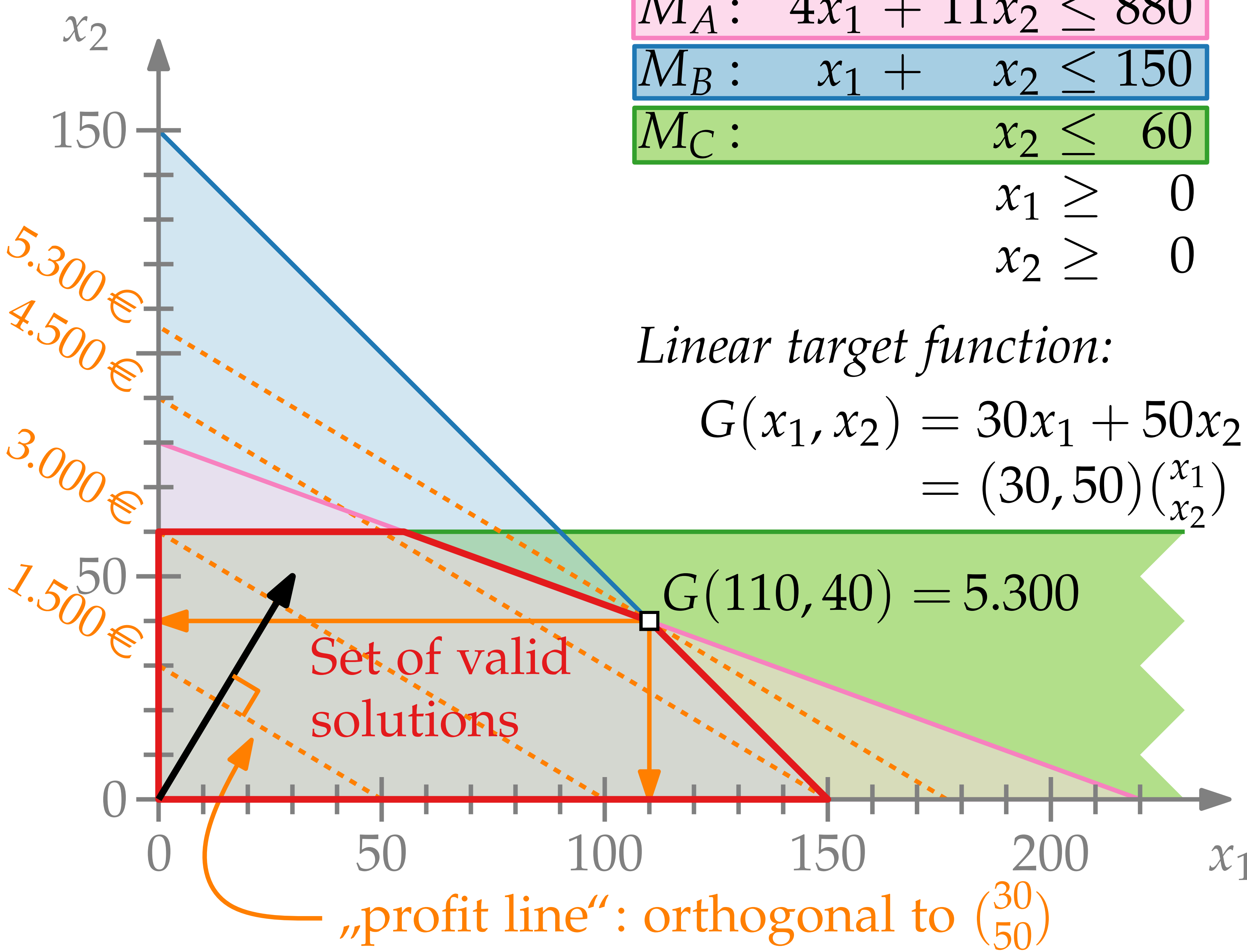

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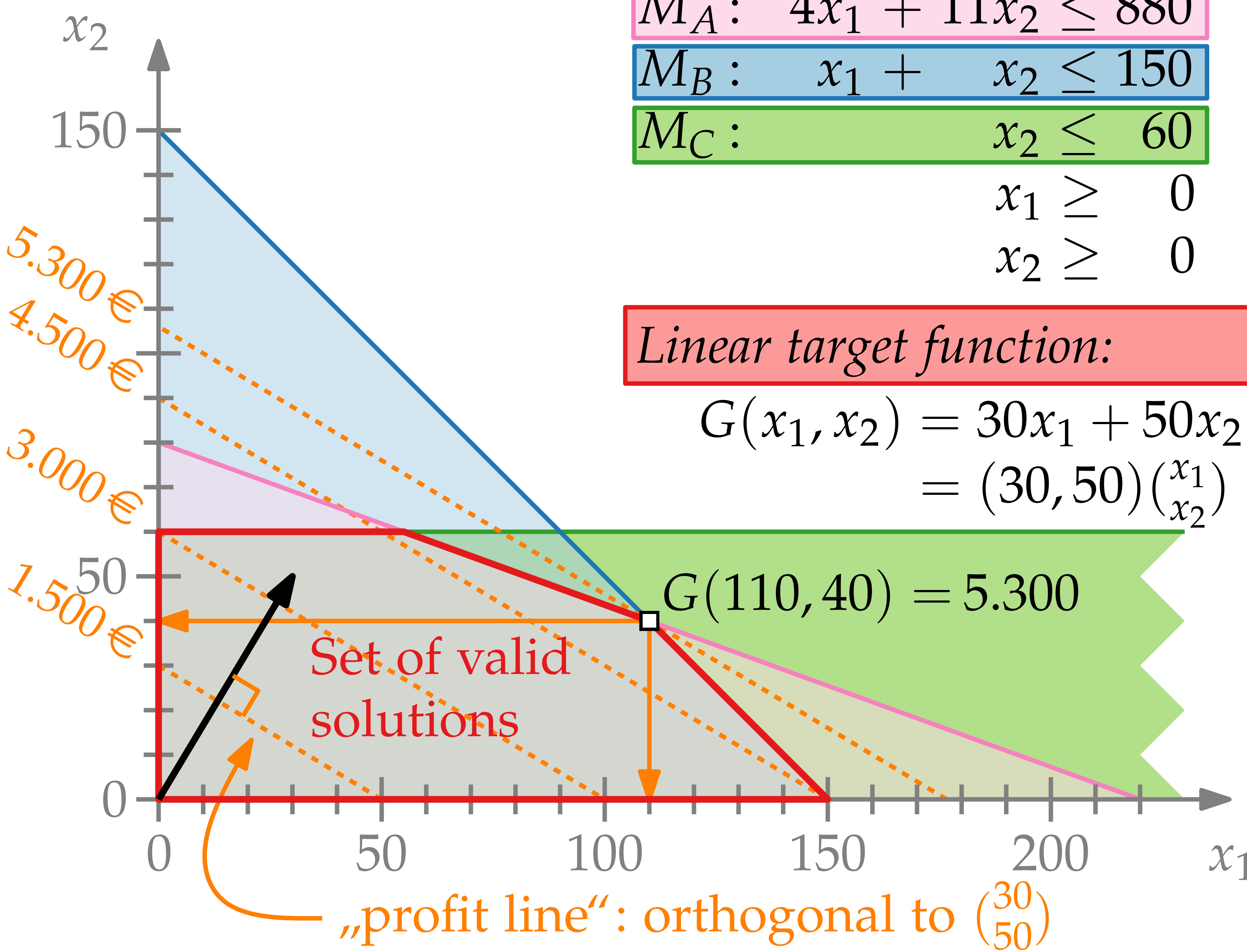

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Linear target function: maximize $c^T x$

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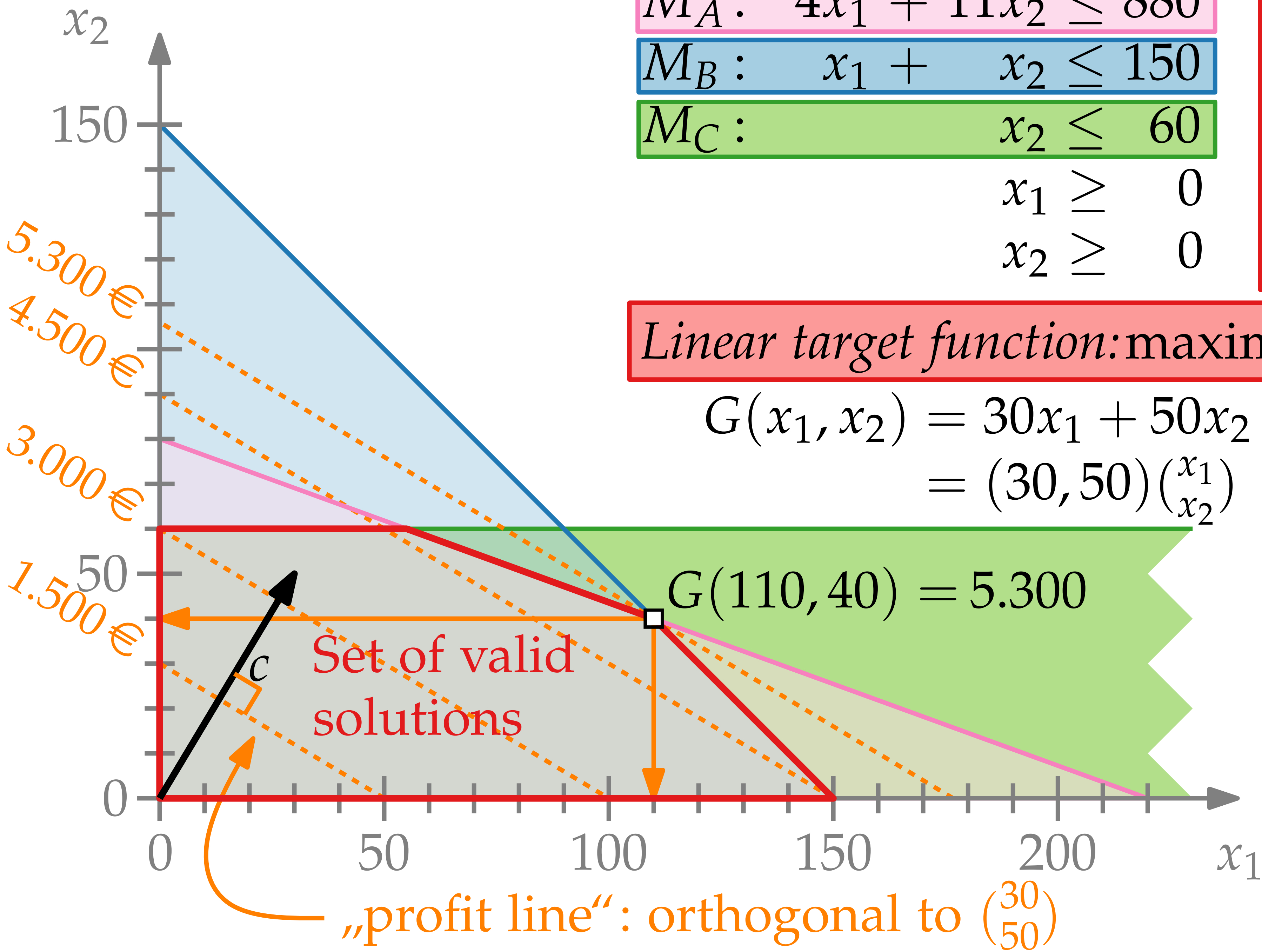

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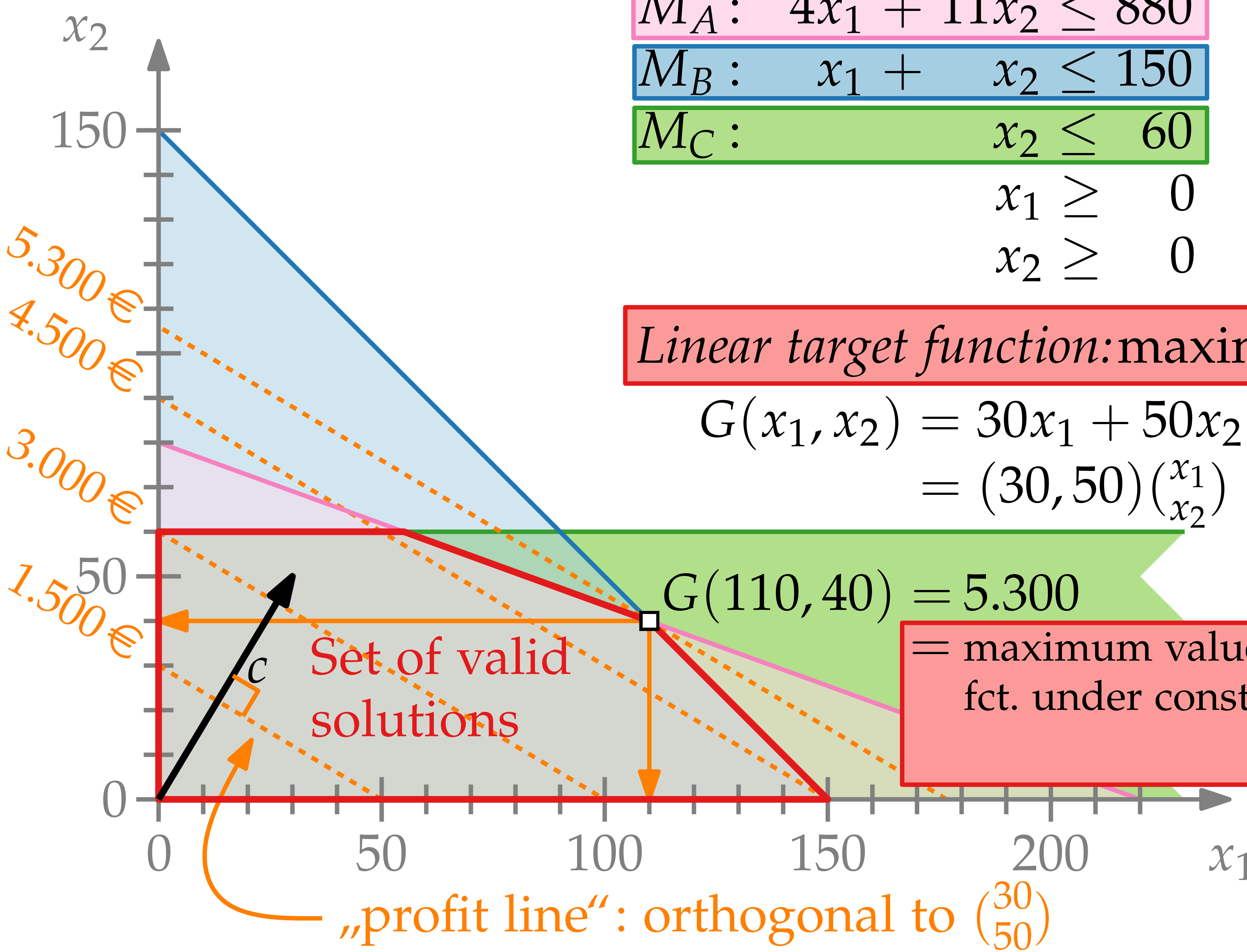

$G(110, 40) = 5.300$

= maximum value of target fct. under constraints.

Solution

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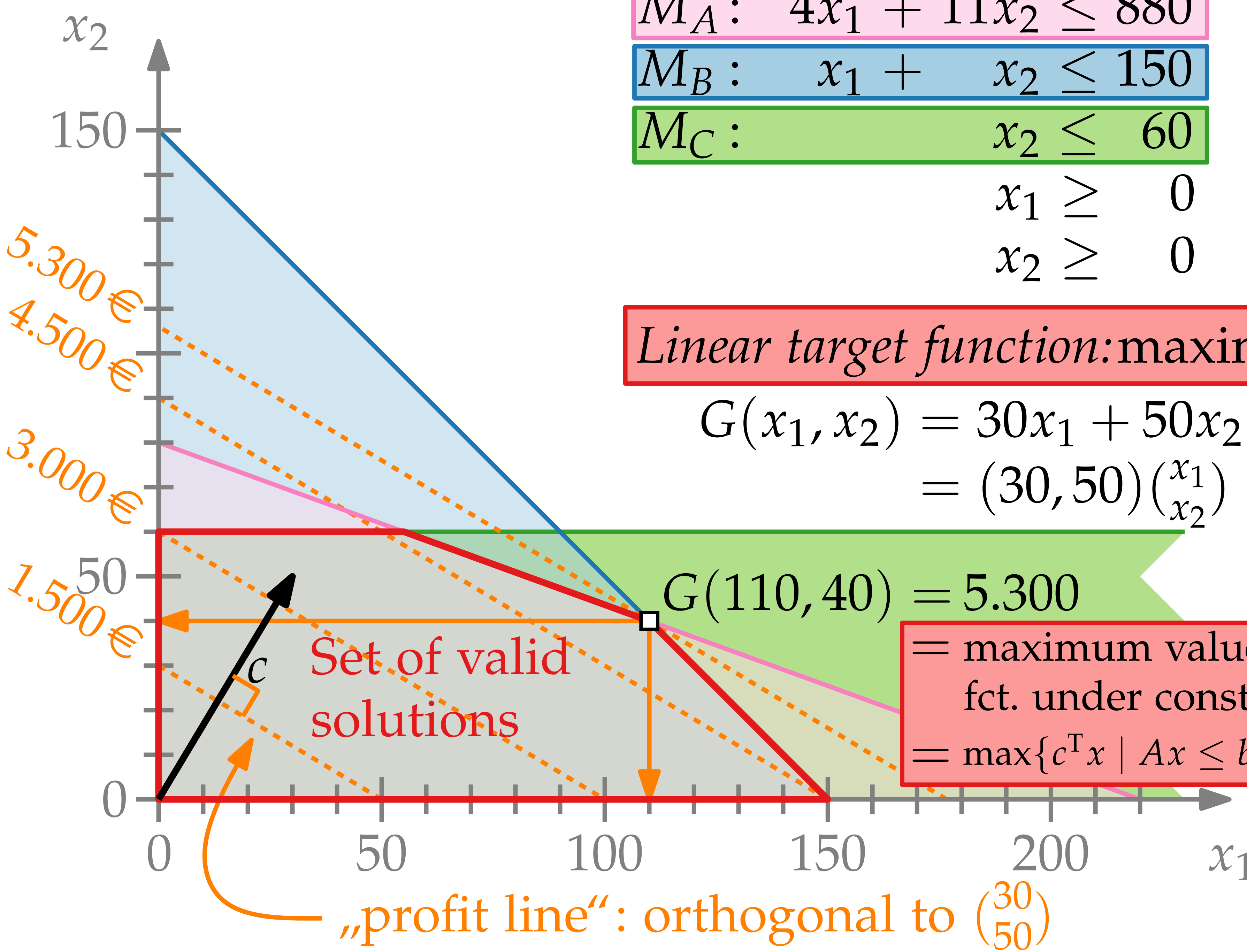

$= $ maximum value of target fct. under constraints.

$= \max\{c^{\mathrm{T}} x \mid Ax \leq b, x \geq 0\}$

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Part II:
A First Approach

Philipp Kindermann

Winter Semester 2020

Definition and Known Algorithms

Given a set H of n halfspaces in $\mathbb{R}^d$ and a direction c, find a point $x \in \bigcap H$ such that cx is maximum (or minimum).

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– Simplex [Dantzig '47]

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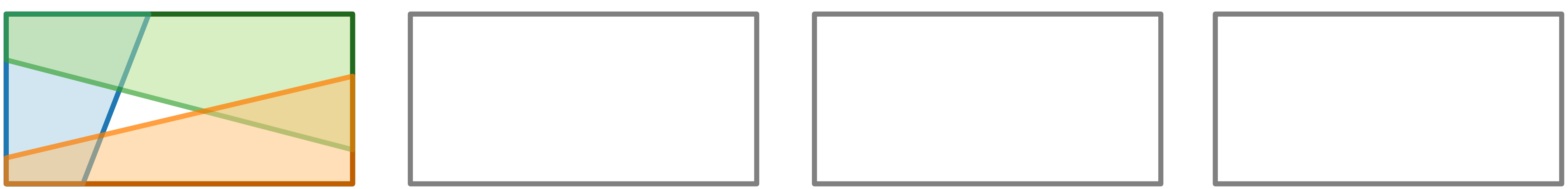

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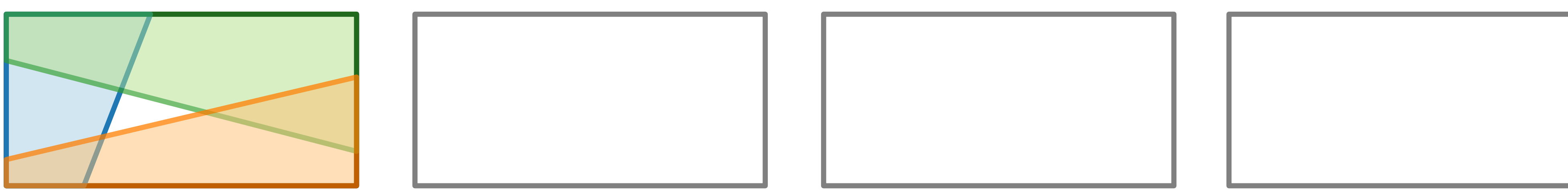

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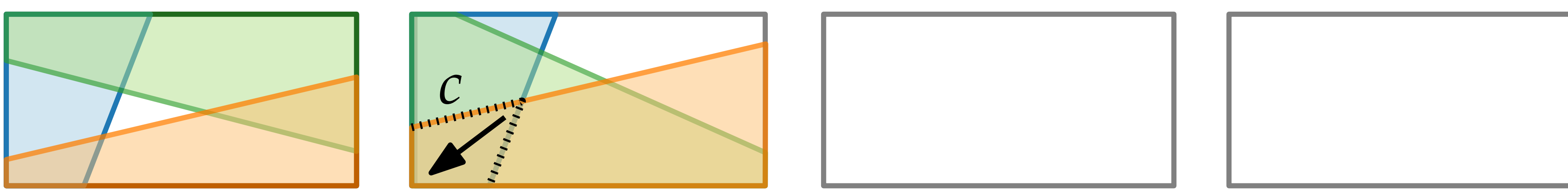

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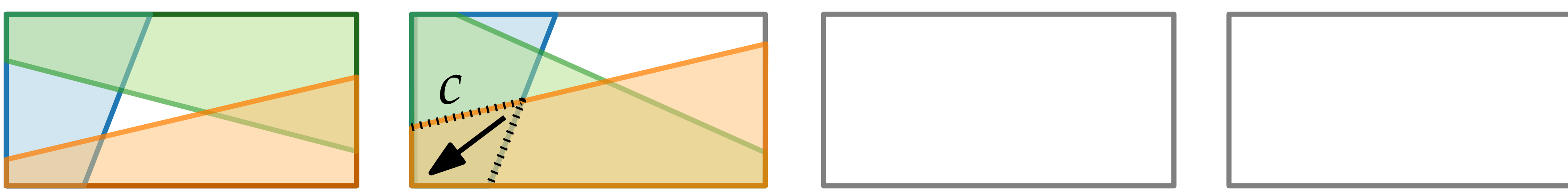

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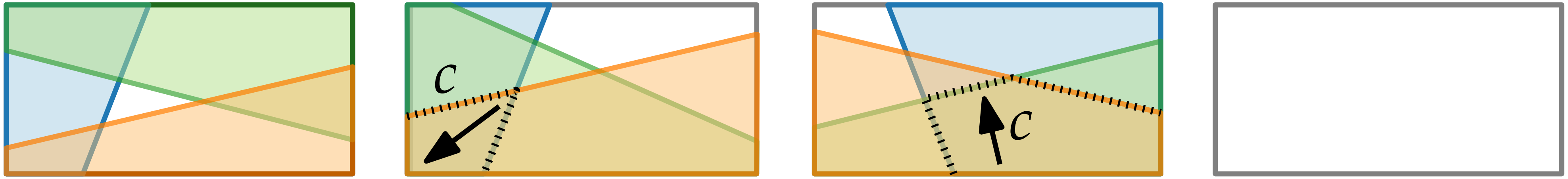

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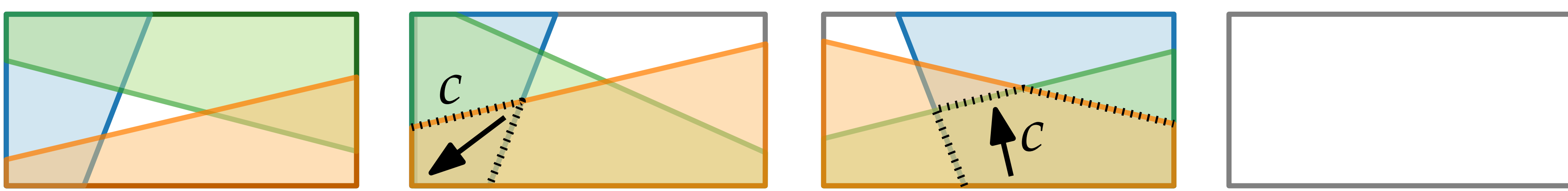

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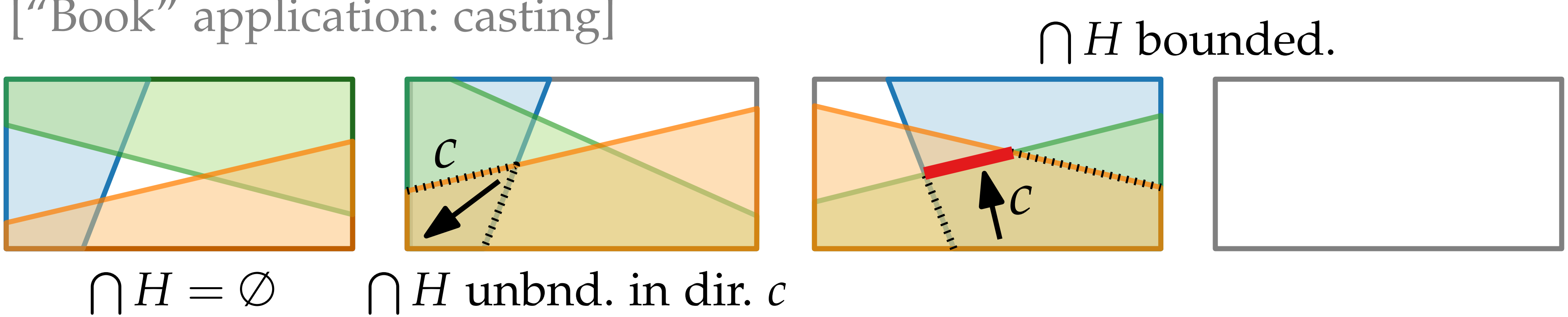

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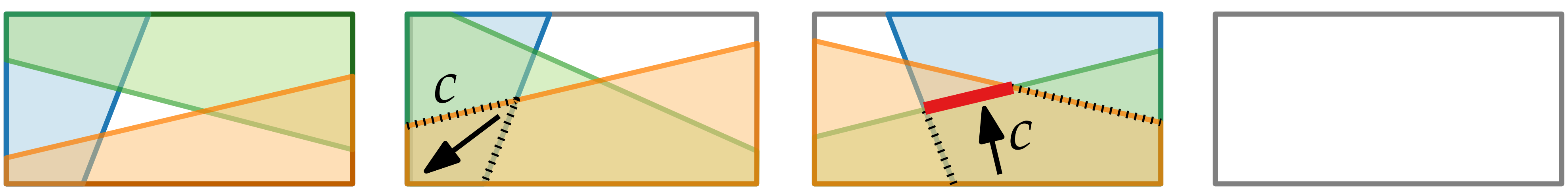

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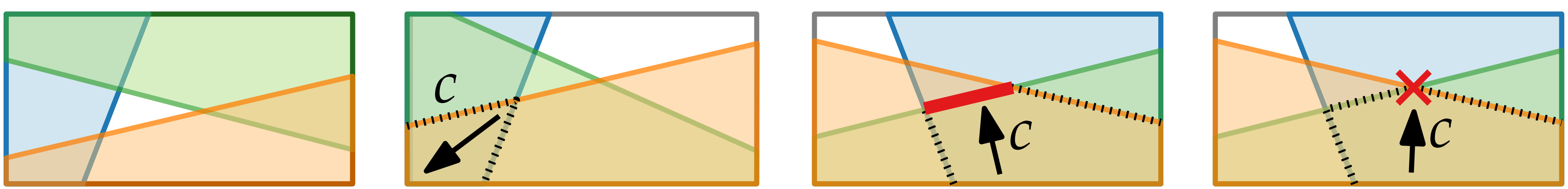

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First Approach

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IntersectHalfplanes(H)

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  if |H| = 1 then
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if $|H| = 1$ **then**
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How??

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Computational Geometry

Lecture 4:
Linear Programming
or
Profit Maximization

Part III:
Intersecting Convex Regions

Philipp Kindermann

Winter Semester 2020

Intersecting Convex Regions

Any ideas?

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Use sweep-line alg. for map overlay (line-segment intersections)!

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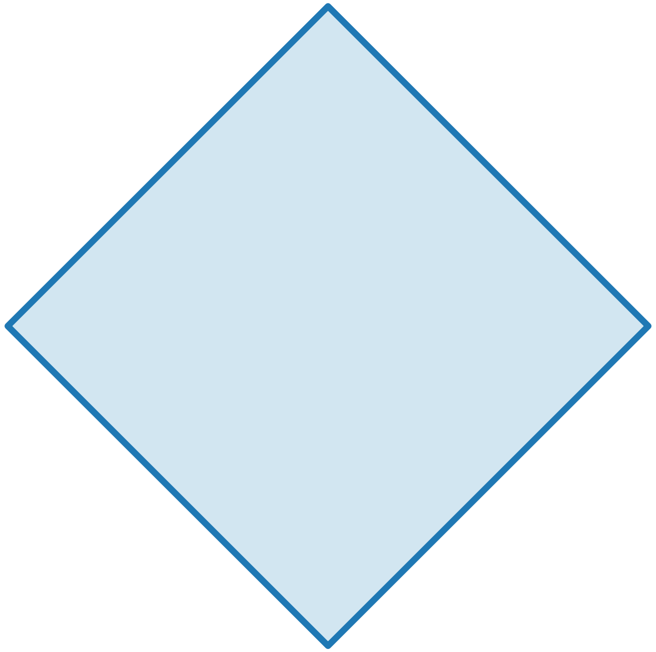

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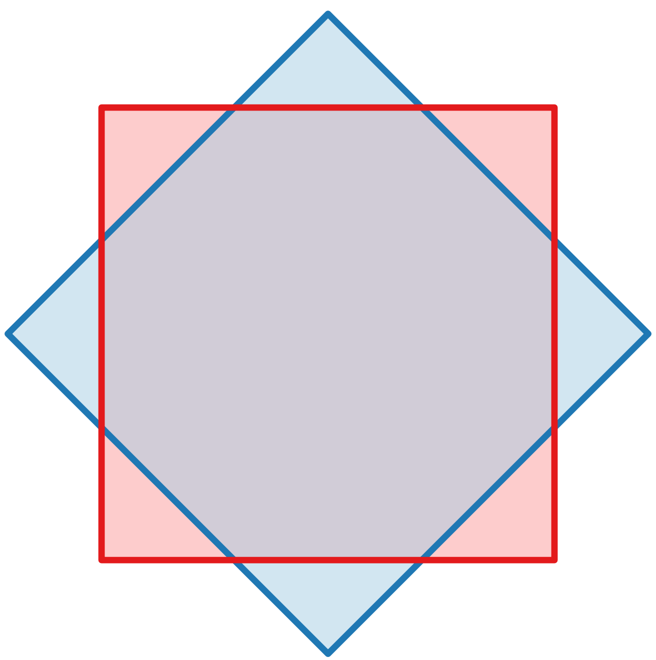

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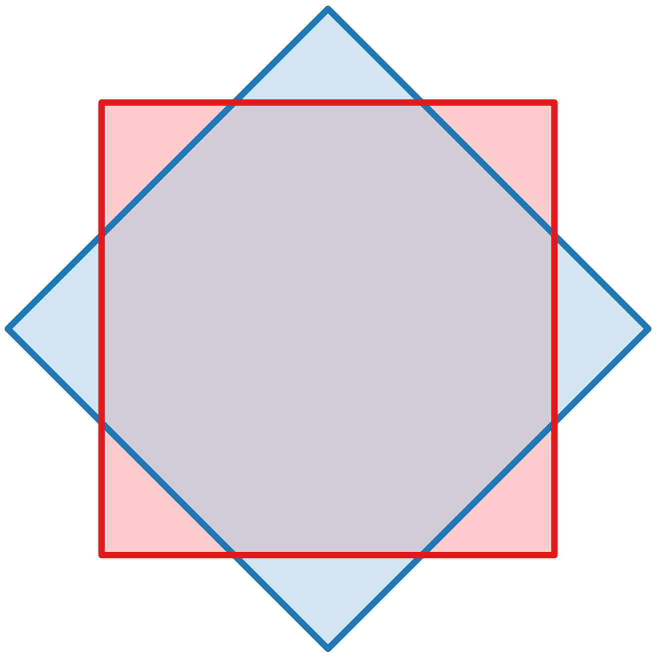

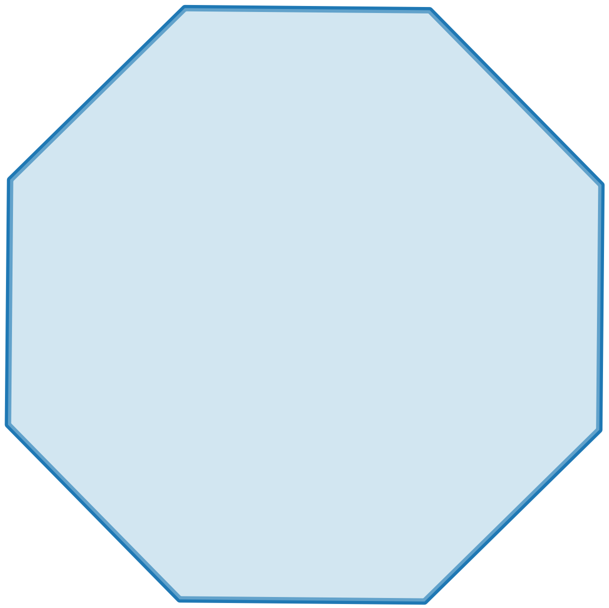

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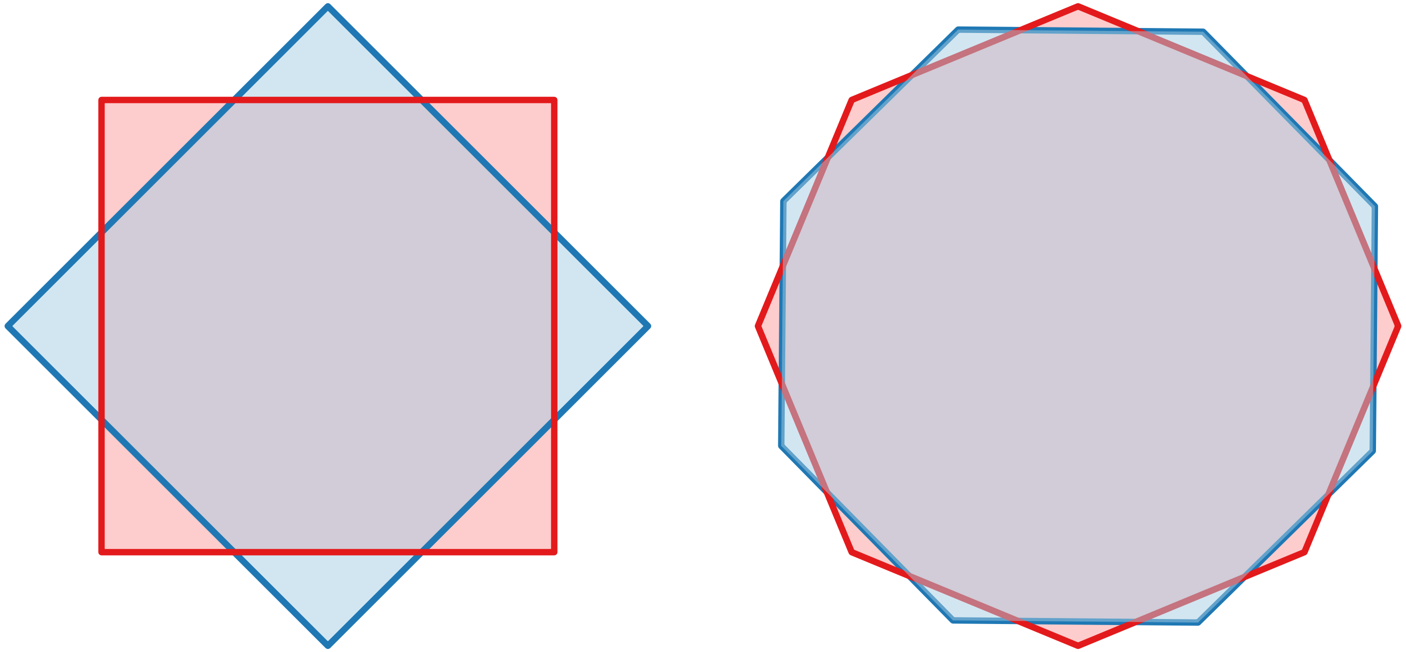

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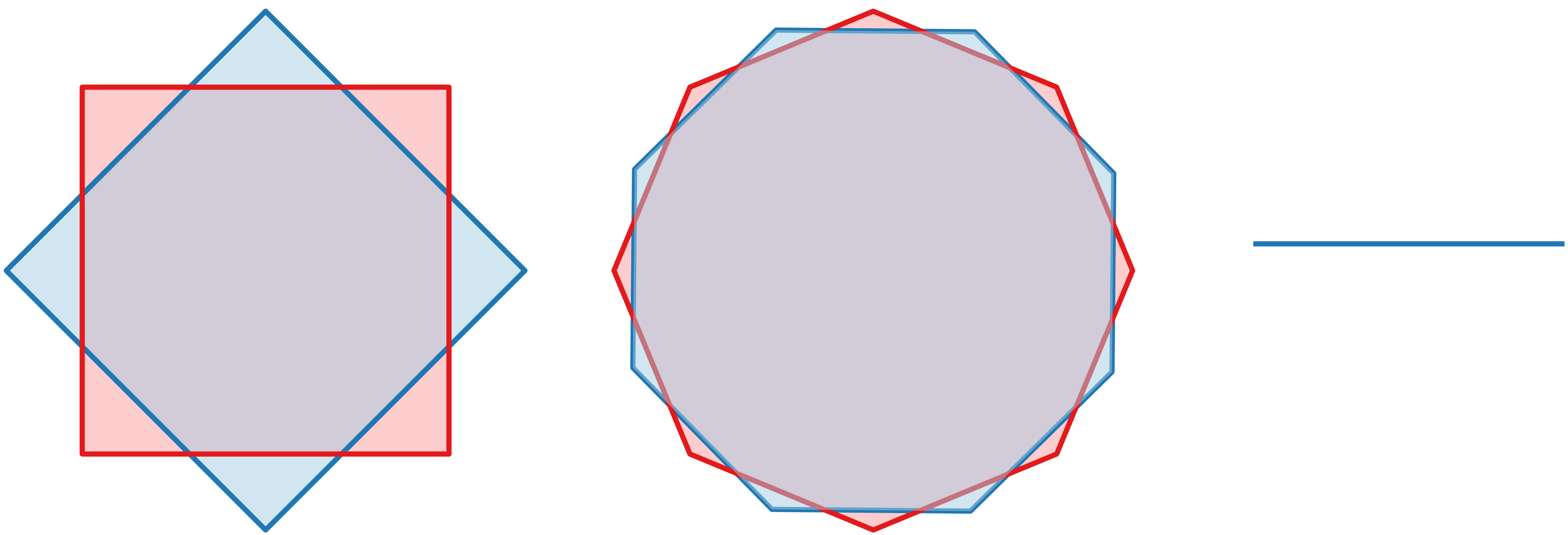

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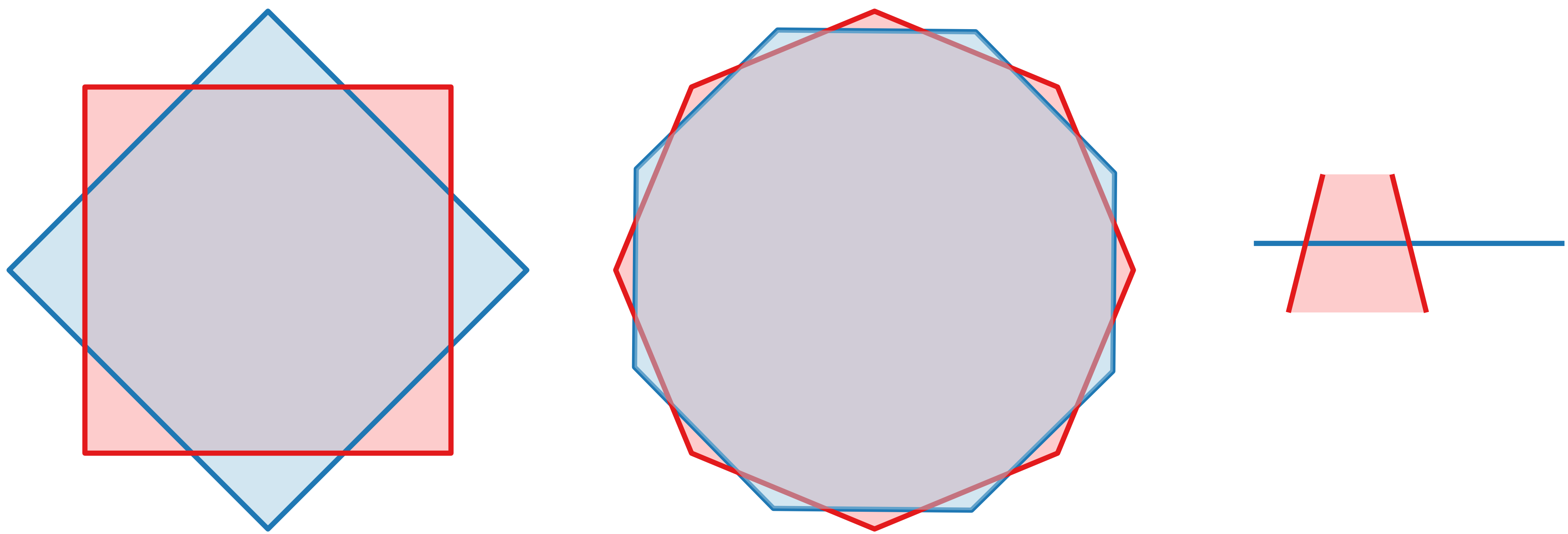

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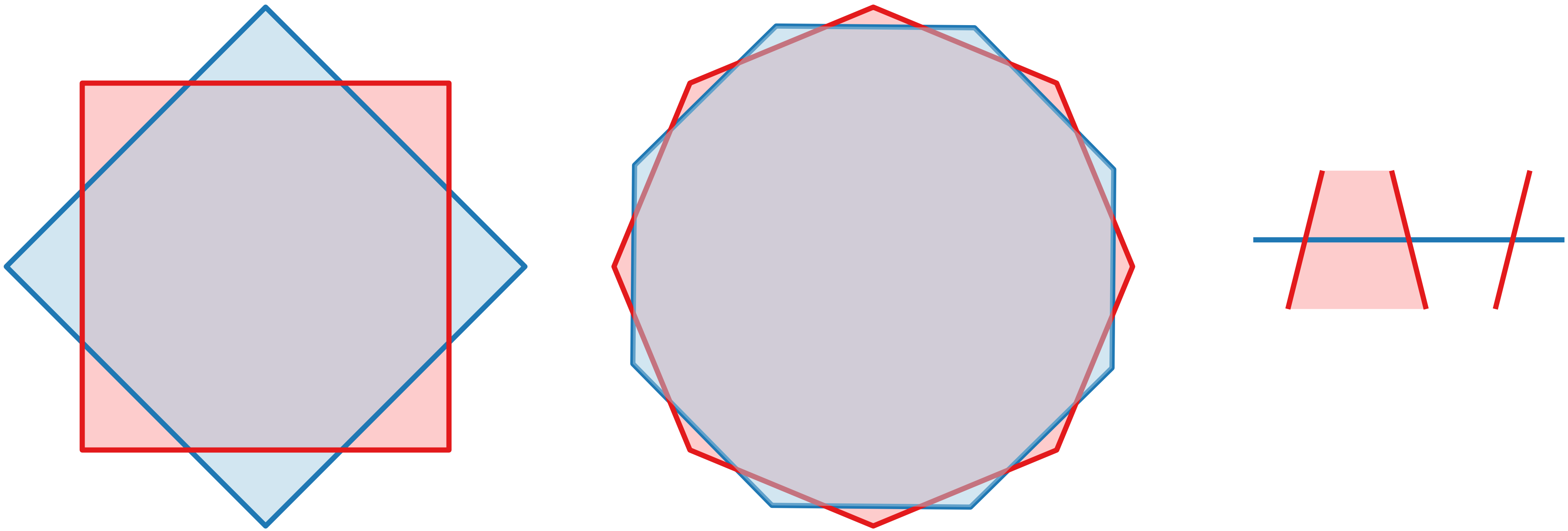

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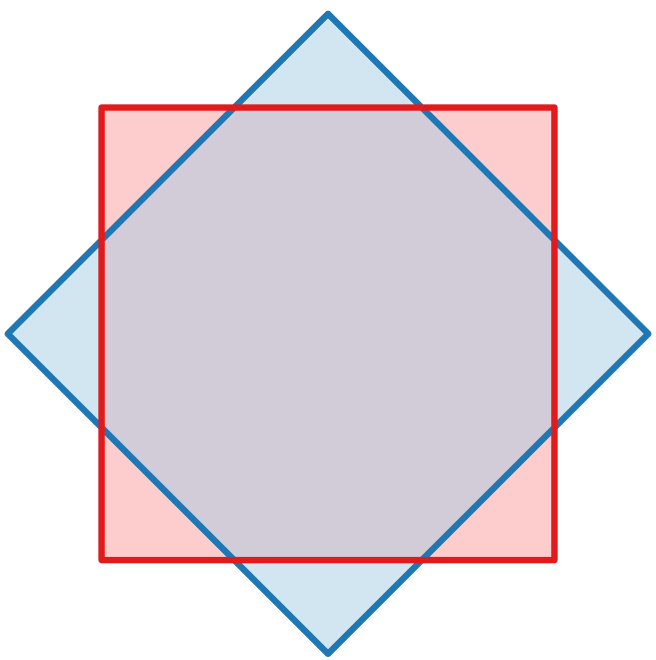

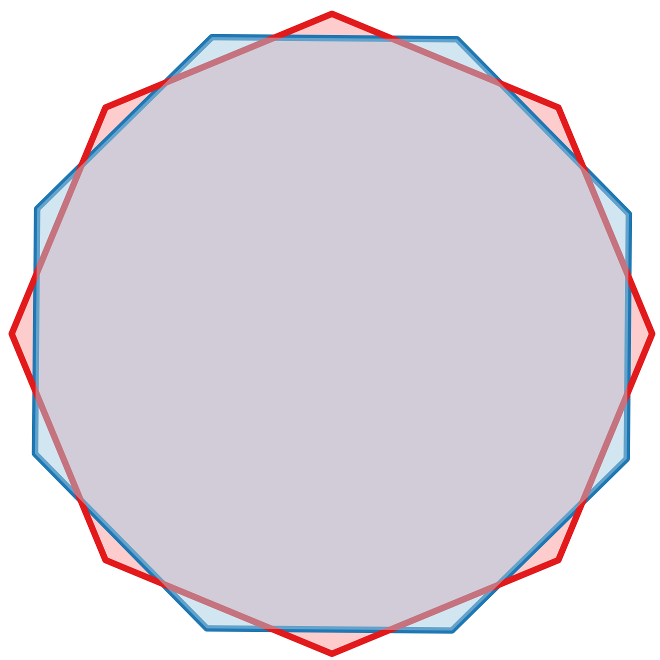

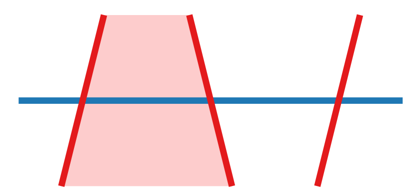

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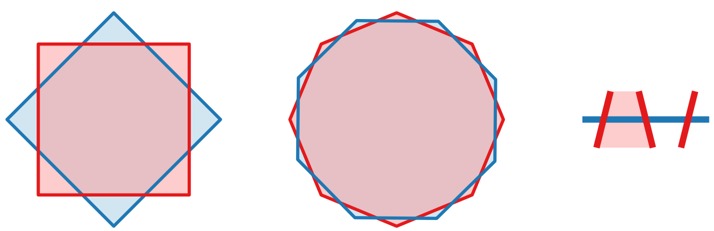

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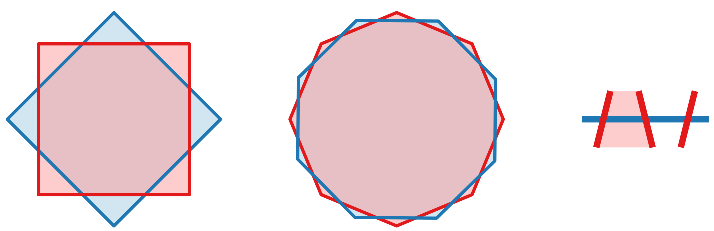

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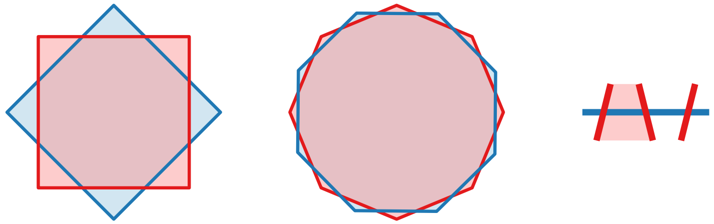

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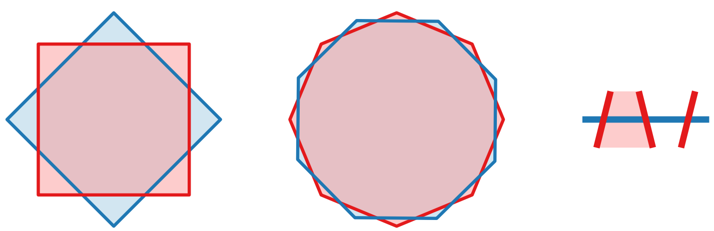

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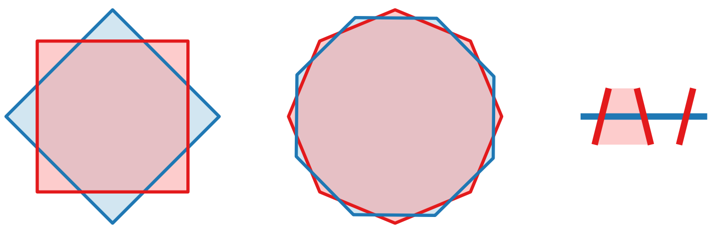

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$$\in O(n \log^2 n)$$

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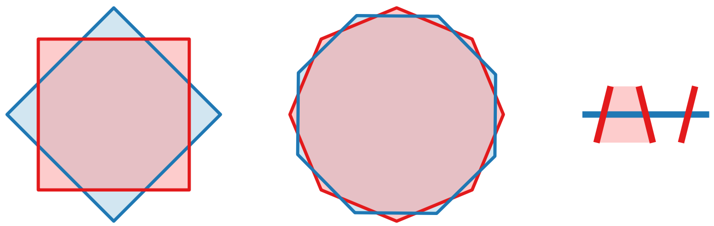

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Better ideas?

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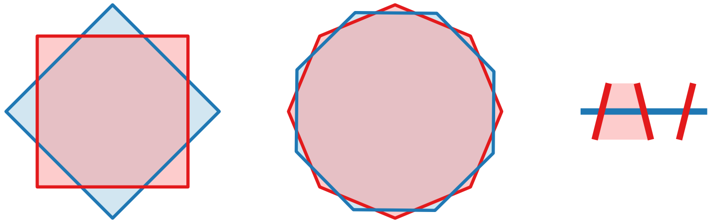

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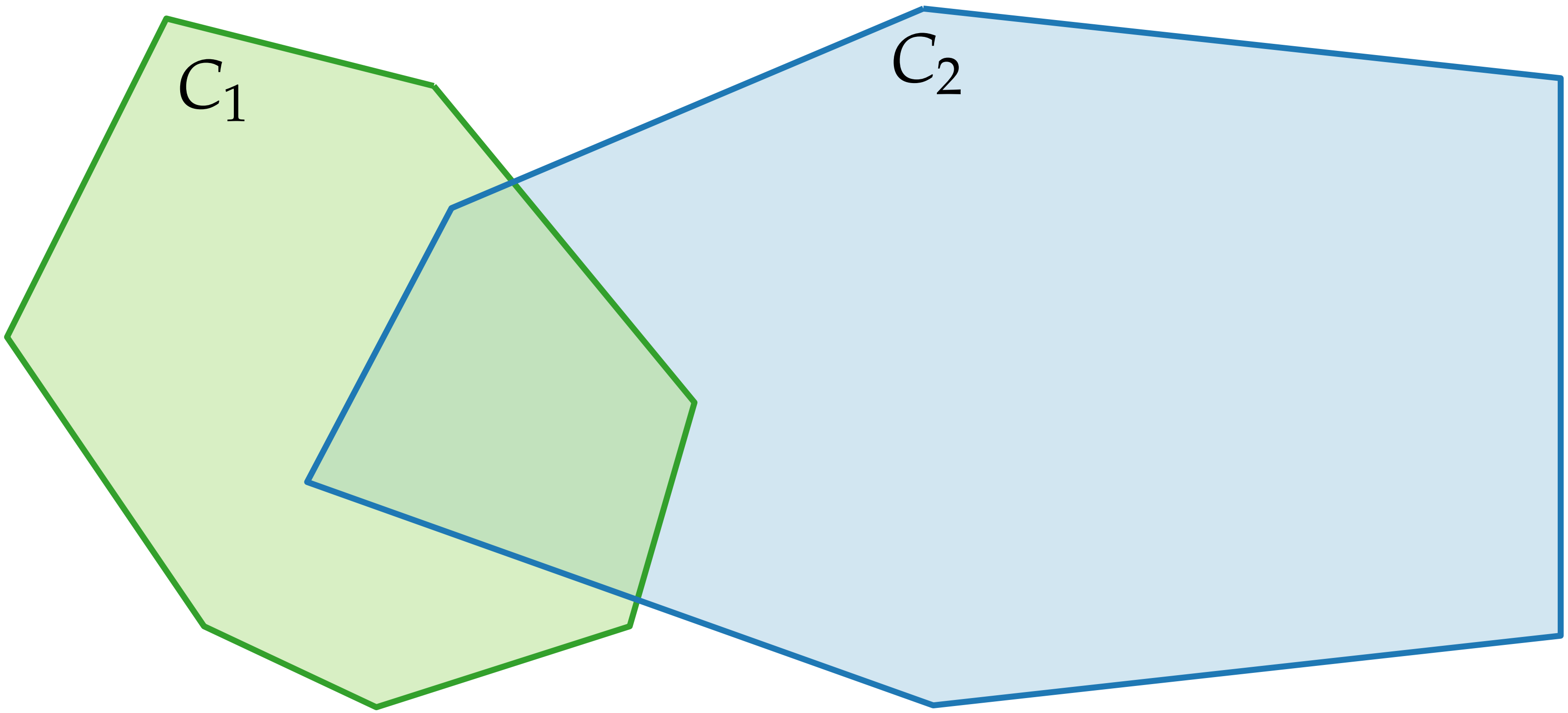
Better ideas?

Better analysis of the sweep-line for *convex* regions/polygons!

Intersecting Convex Regions Faster



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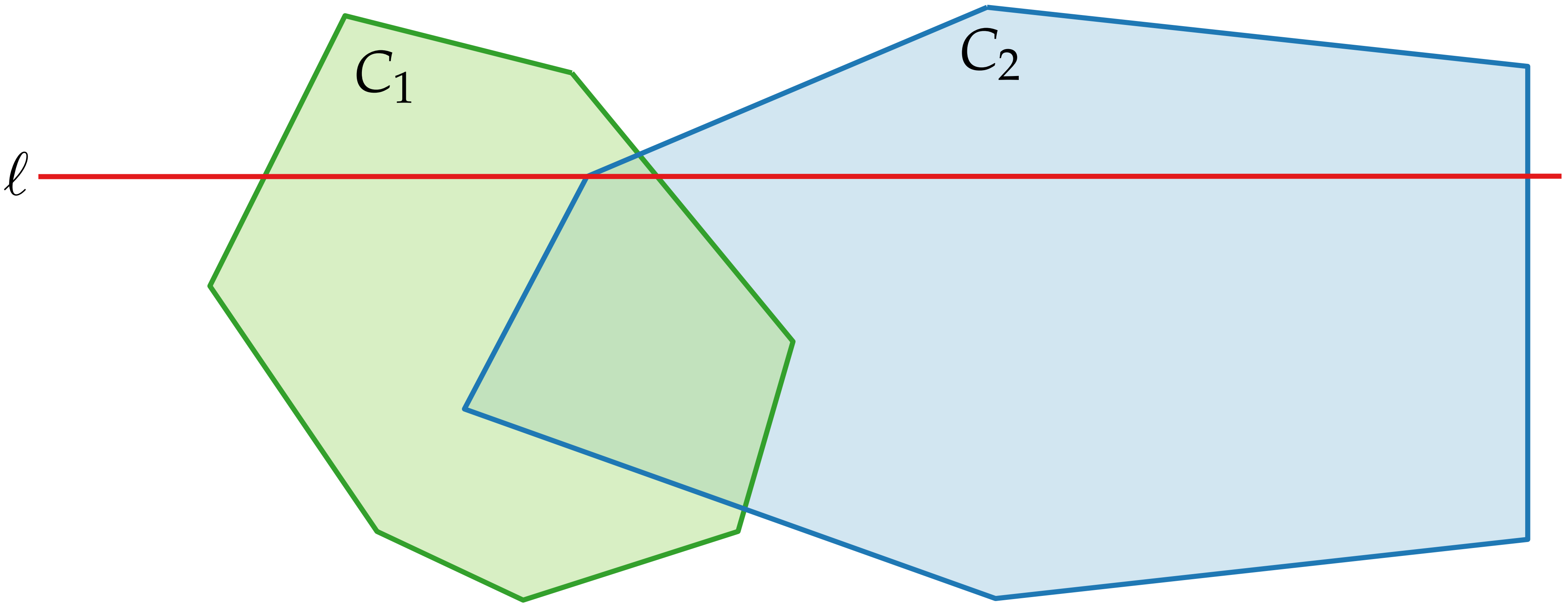

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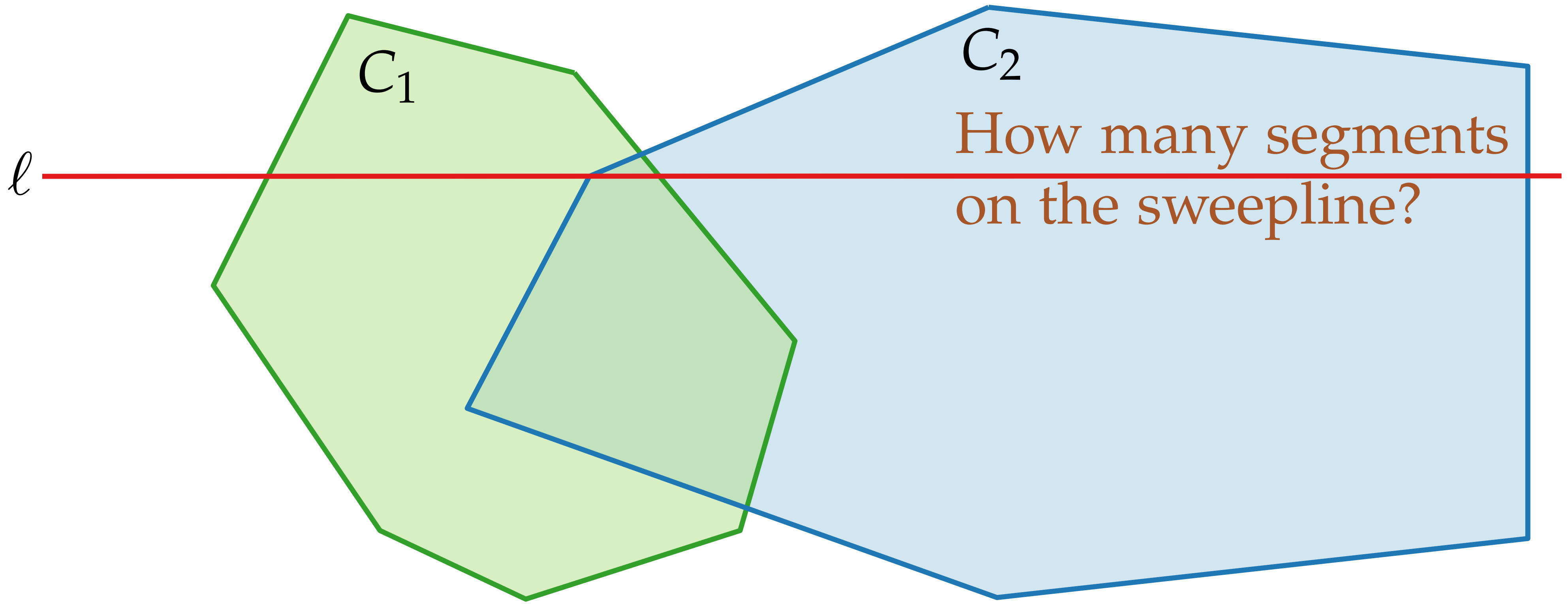

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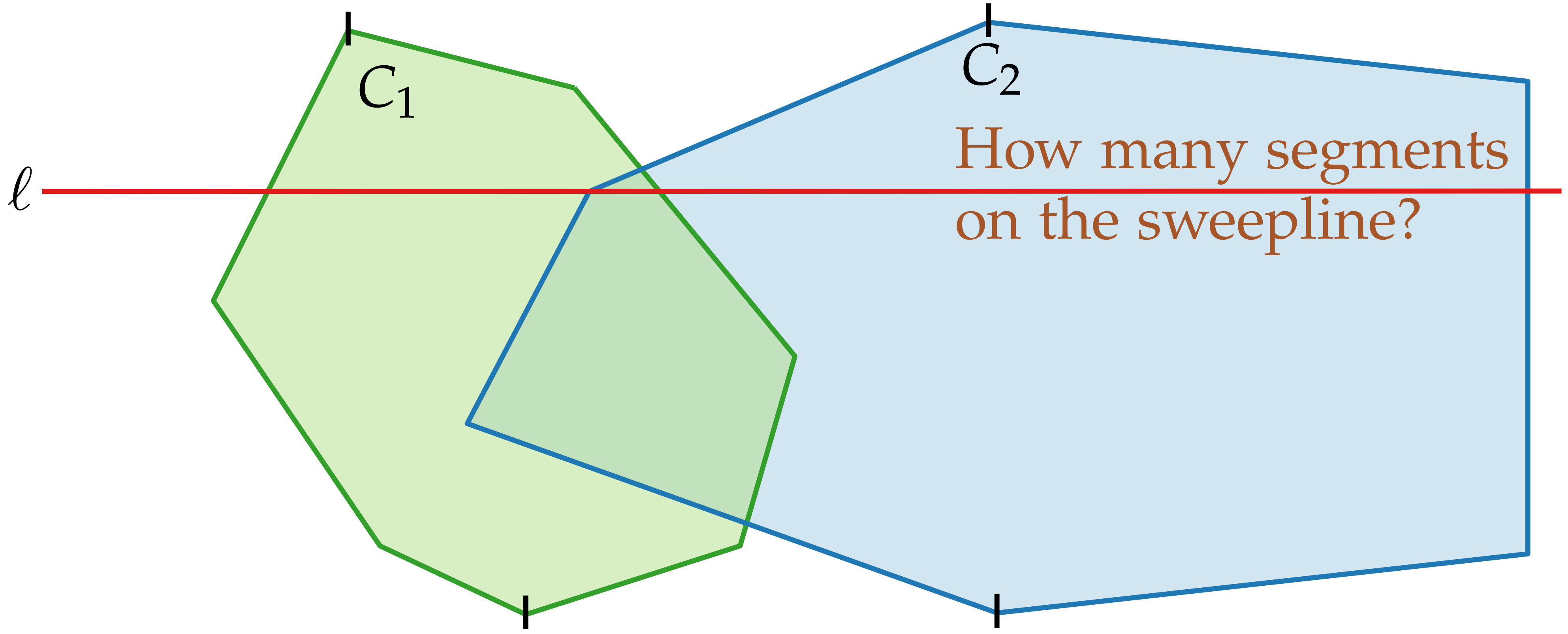

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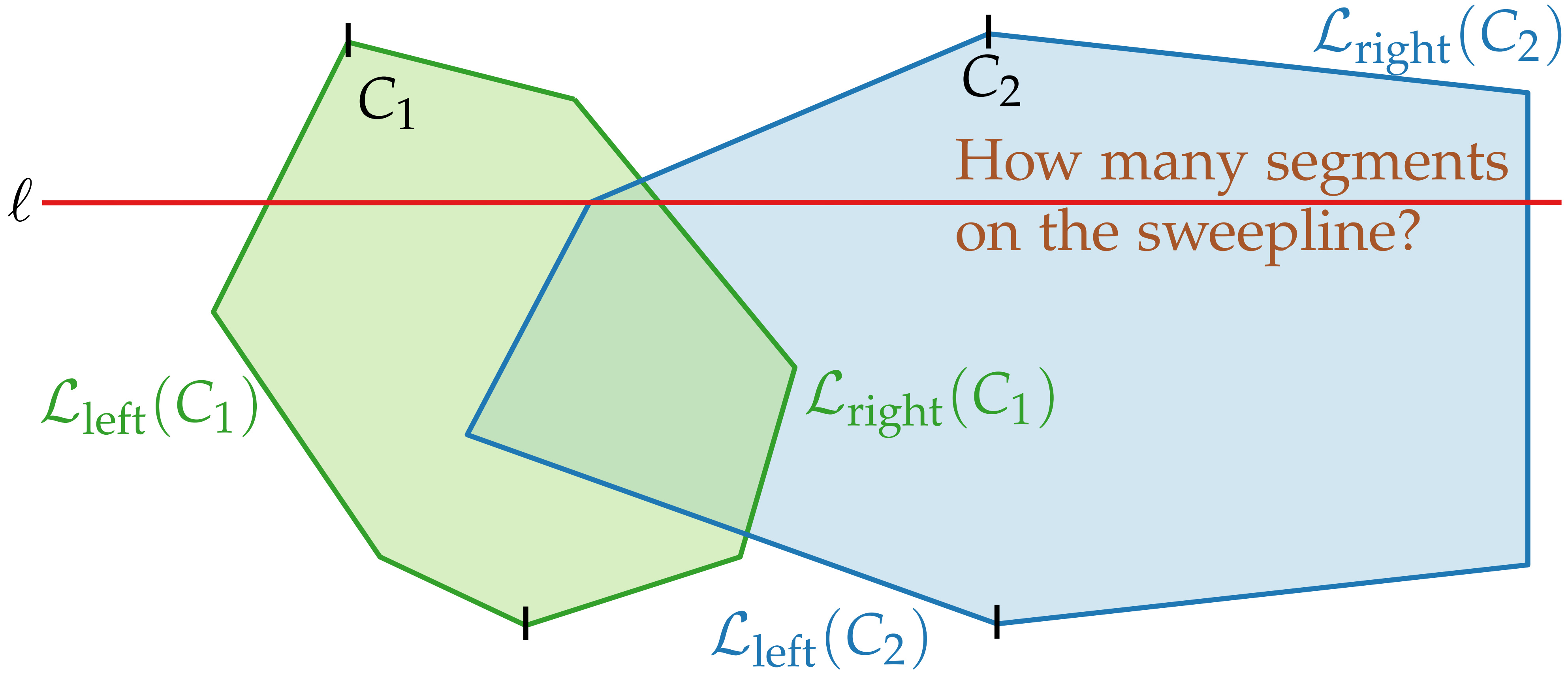

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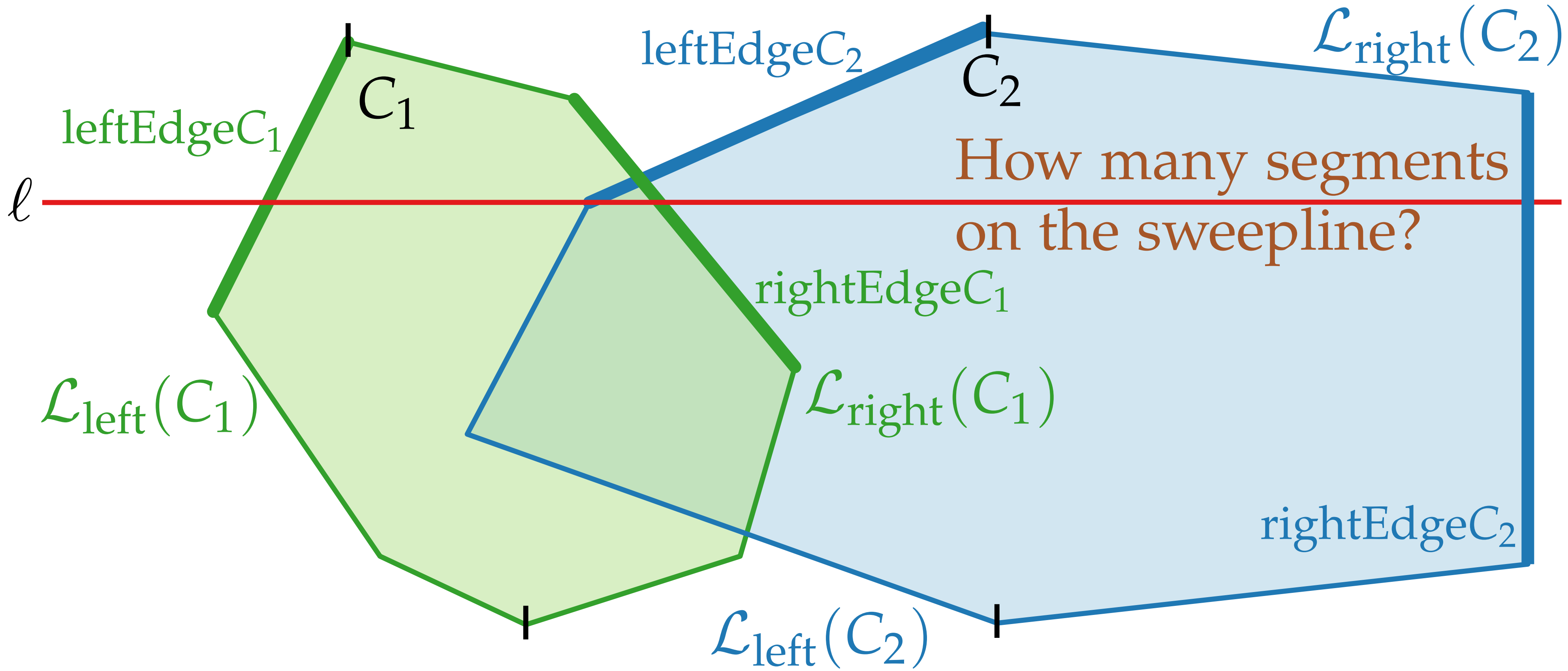

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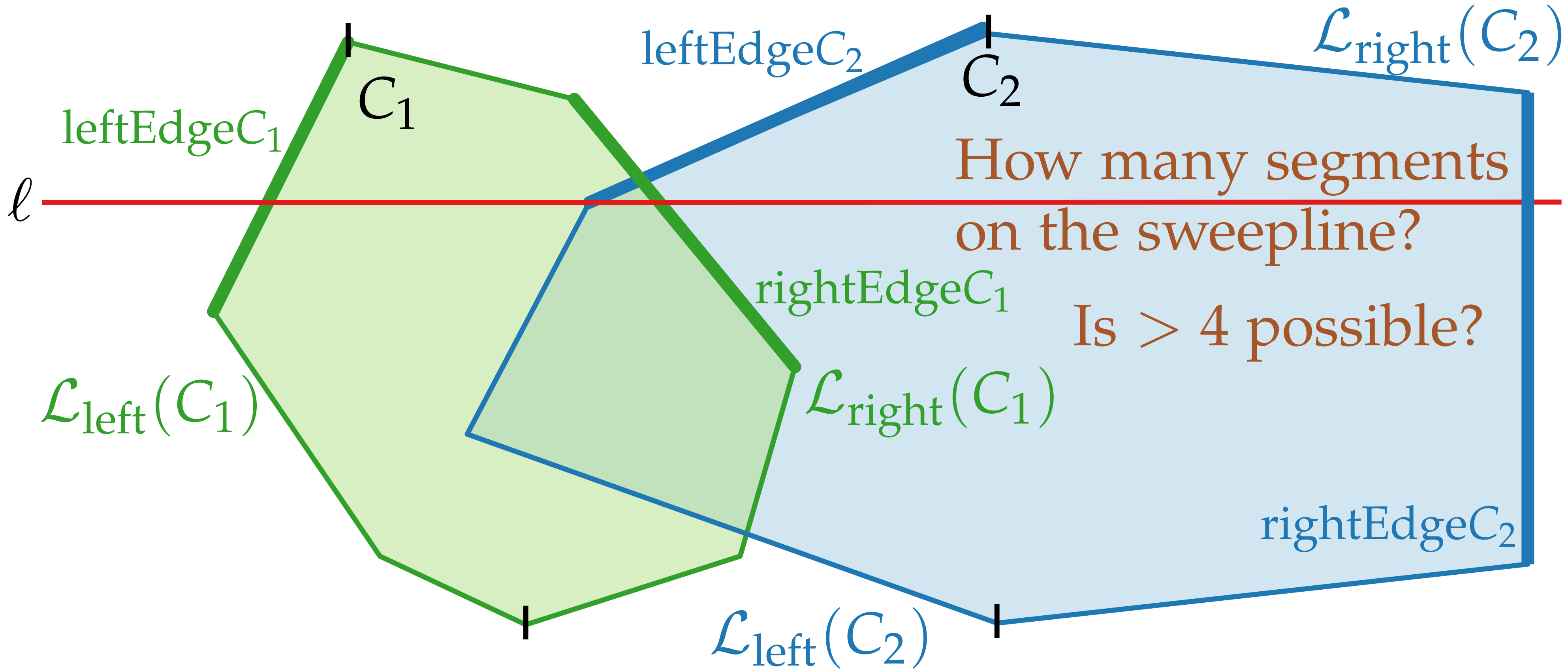

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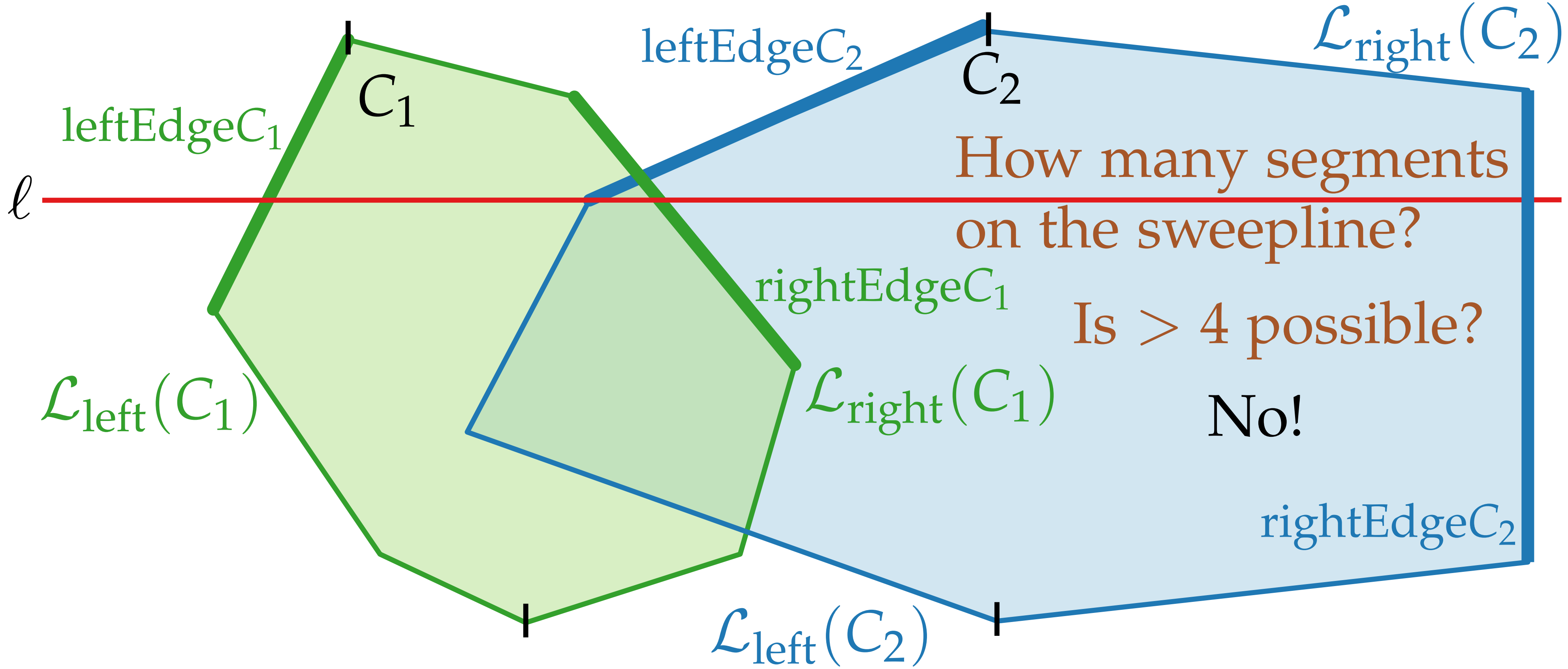

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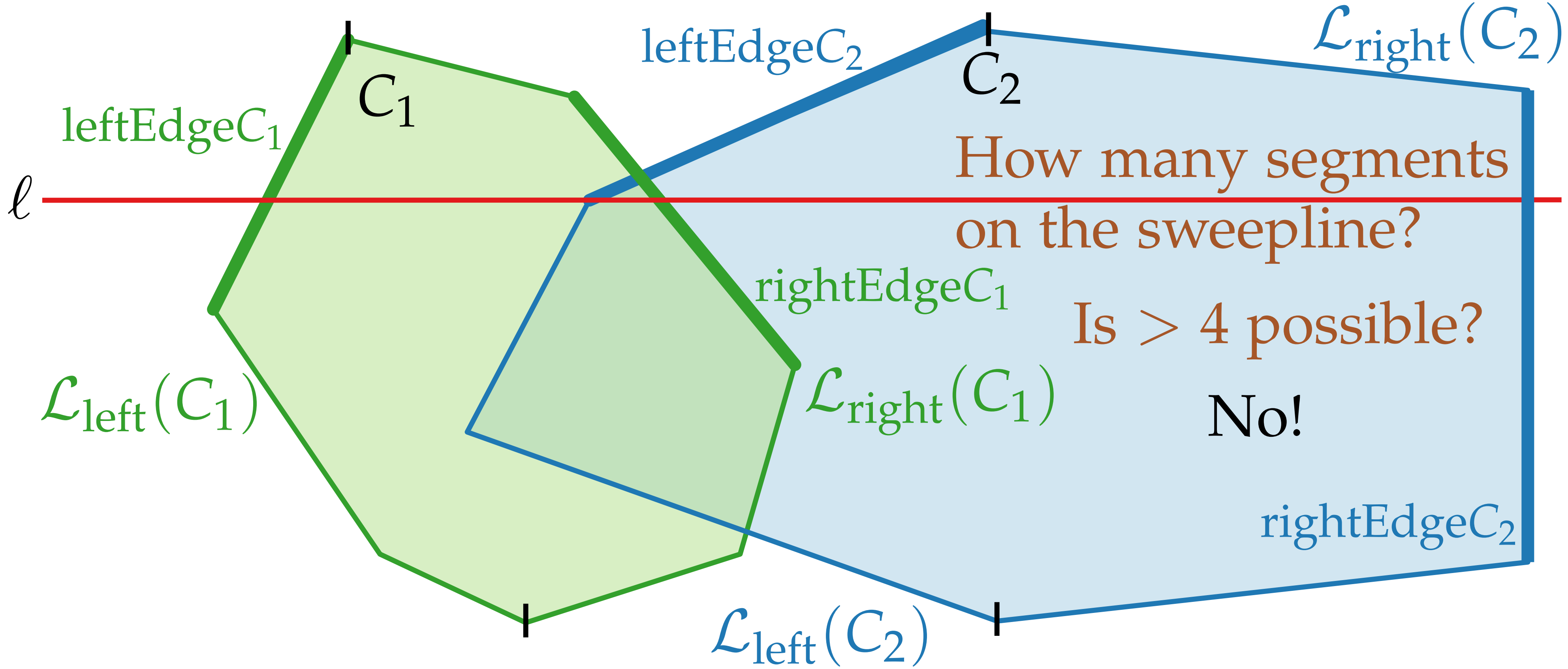
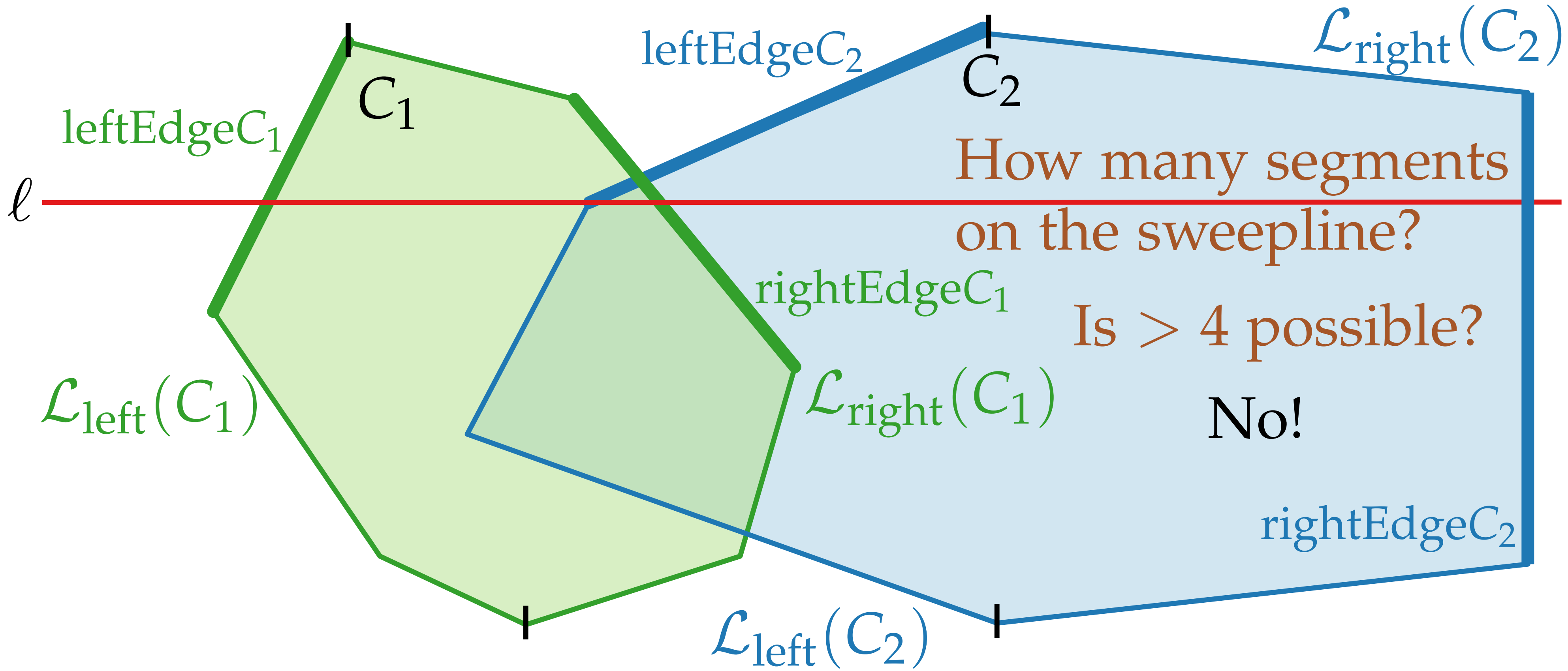


Theorem. The intersection of two convex polygonal regions can be computed in linear time.

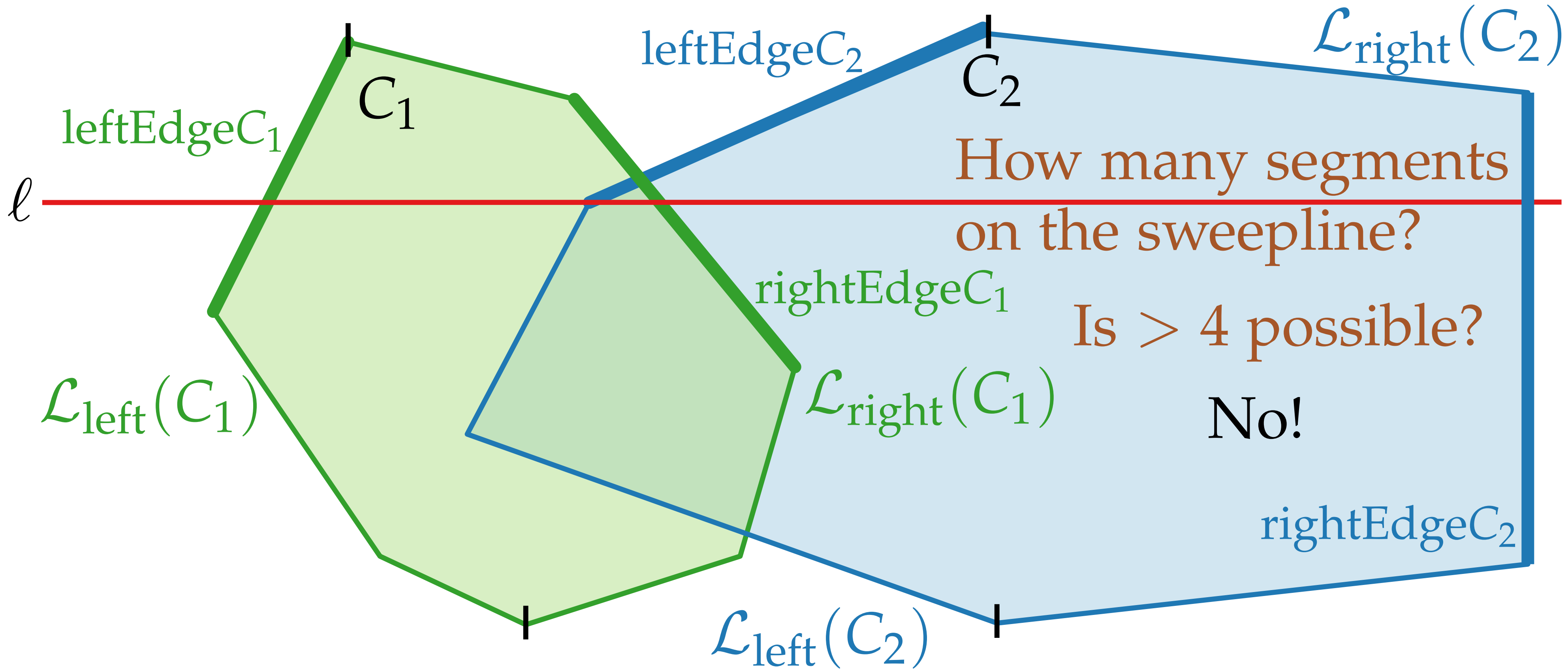
Intersecting Convex Regions Faster



Theorem. The intersection of two convex polygonal regions can be computed in linear time.

Corollary. The intersection of n half planes can be computed in $O(n \log n)$ time.

Intersecting Convex Regions Faster



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Can we do better?

Computational Geometry

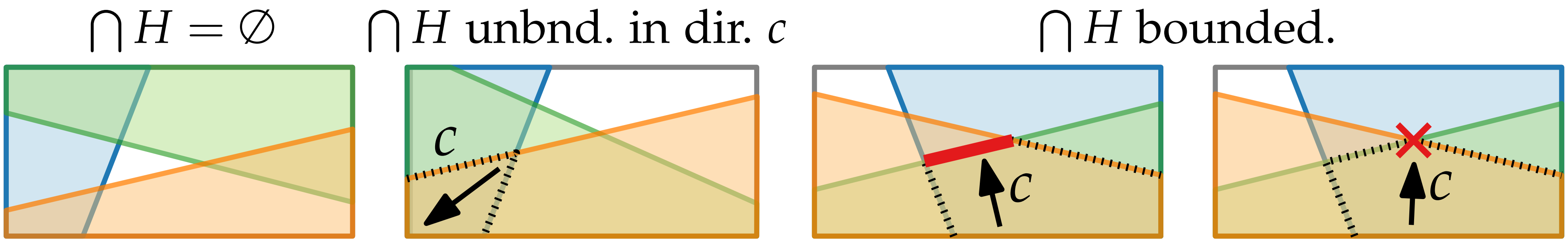
Lecture 4:
Linear Programming
or
Profit Maximization

Part IV:
Incremental Approach

Philipp Kindermann

Winter Semester 2020

A Small Trick: Make Solution Unique



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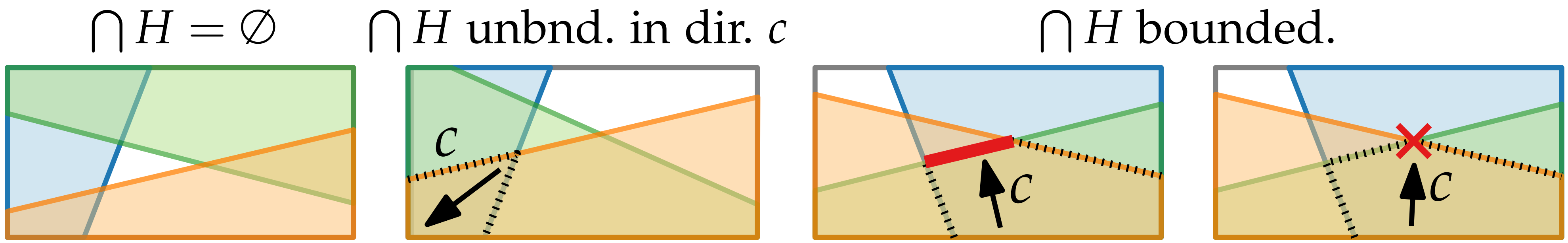

- Add two bounding halfplanes m_1 and m_2

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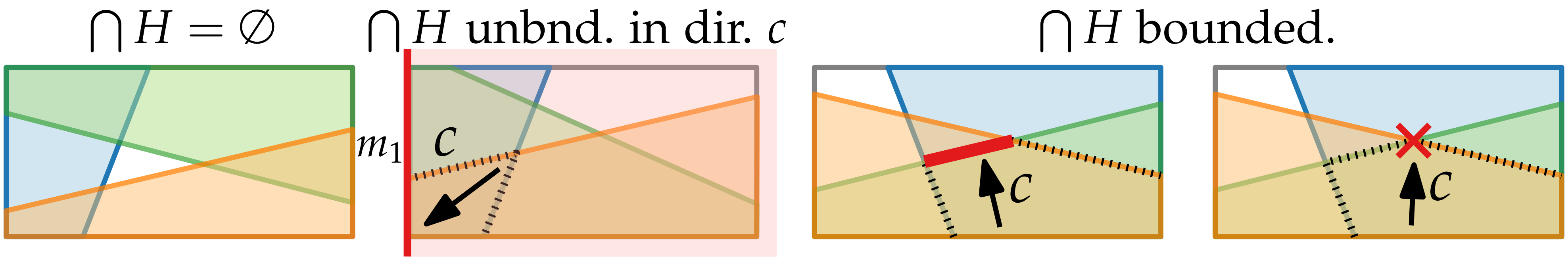

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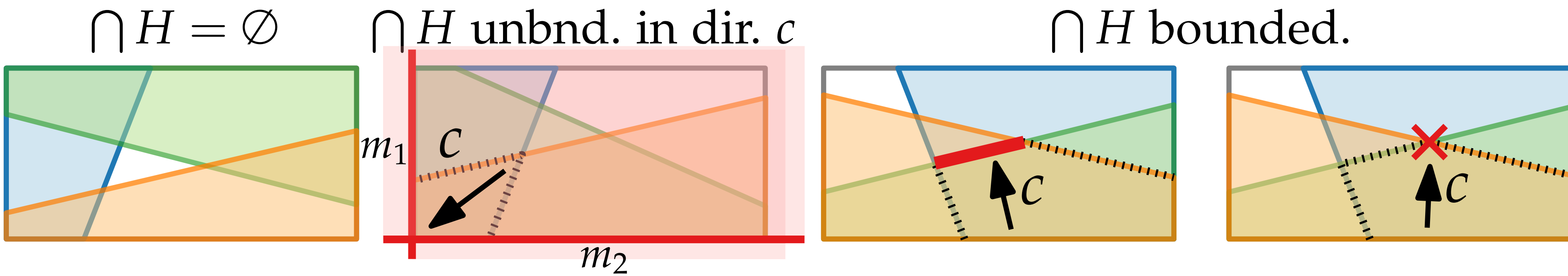

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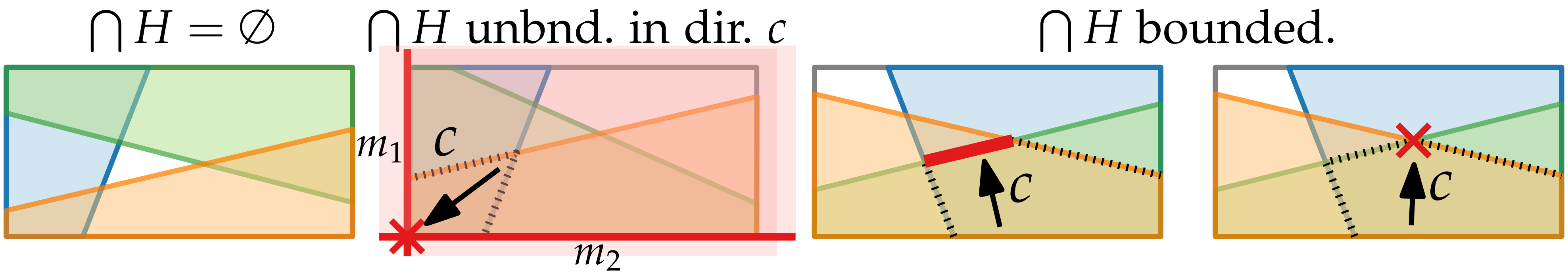

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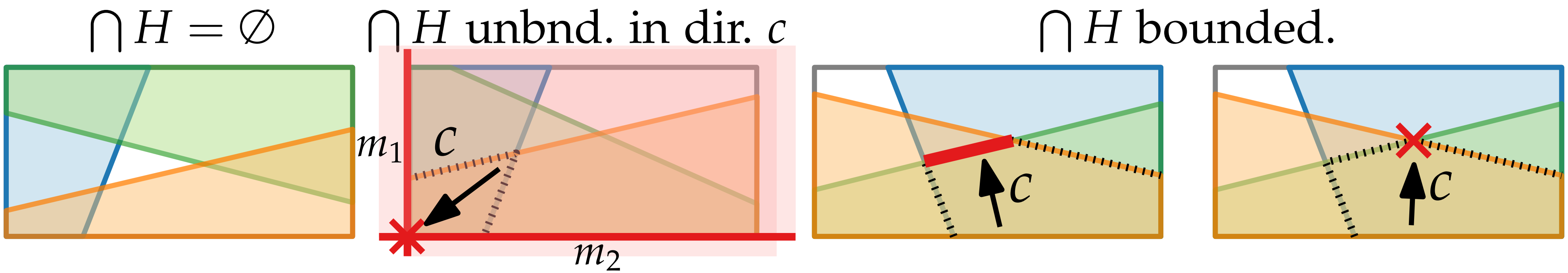

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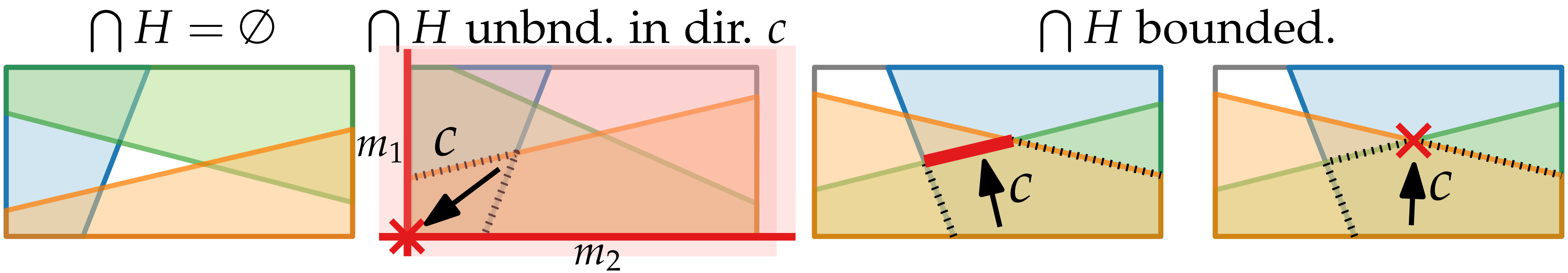

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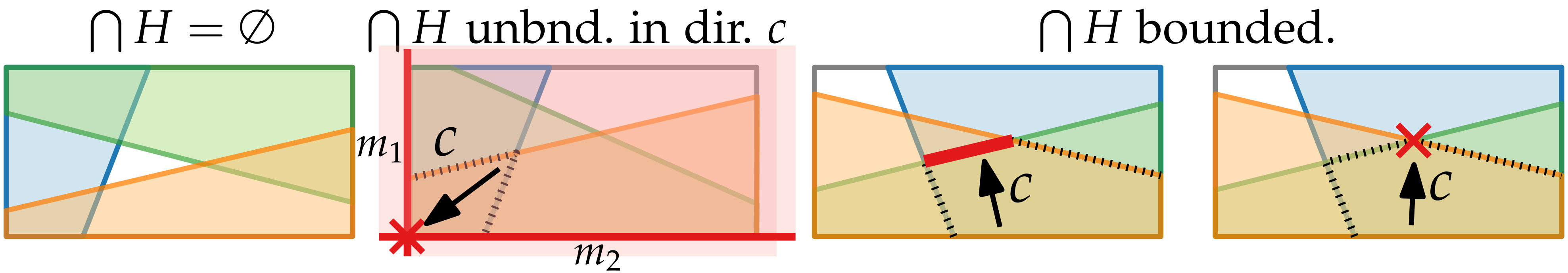

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- Take the lexicographically largest solution.

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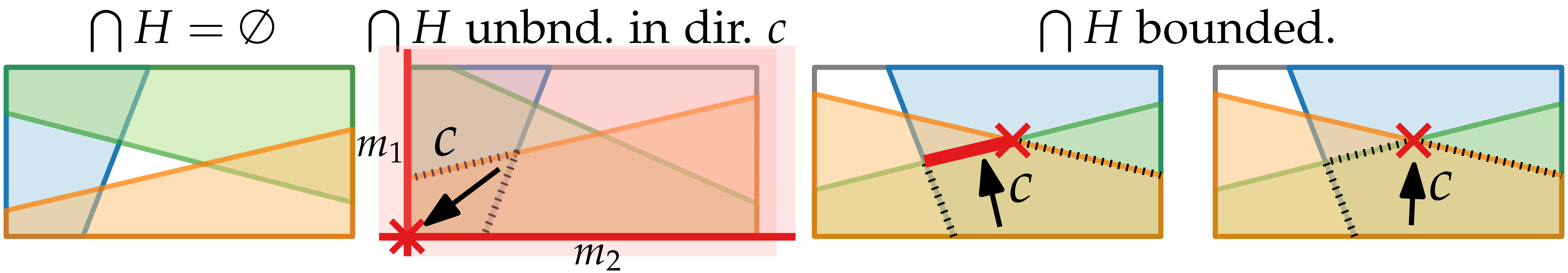

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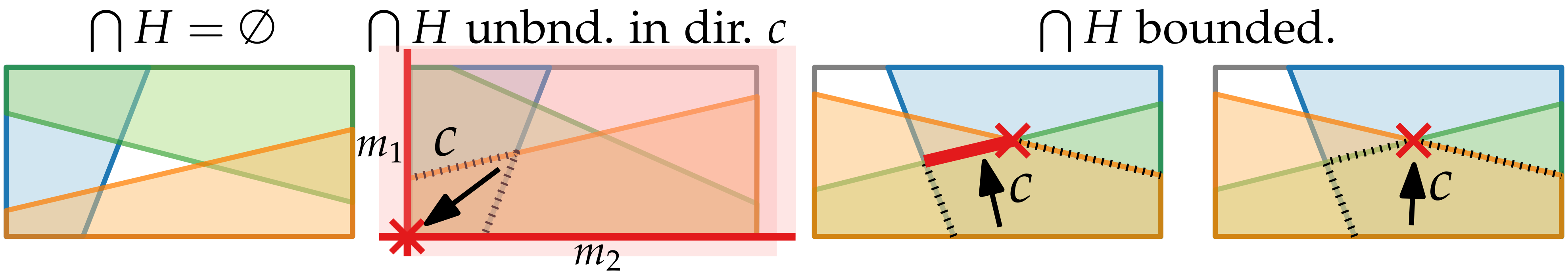

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$\Rightarrow$ Set of solutions is either empty or a uniquely defined pt.

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Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

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2DBoundedLP(H, c, m_1, m_2)

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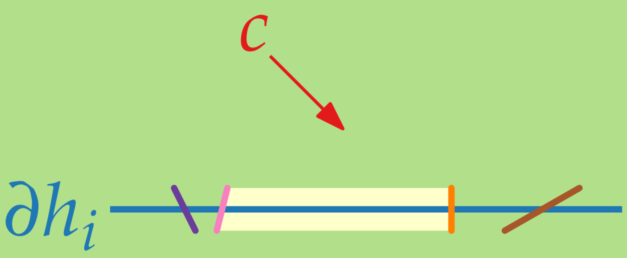

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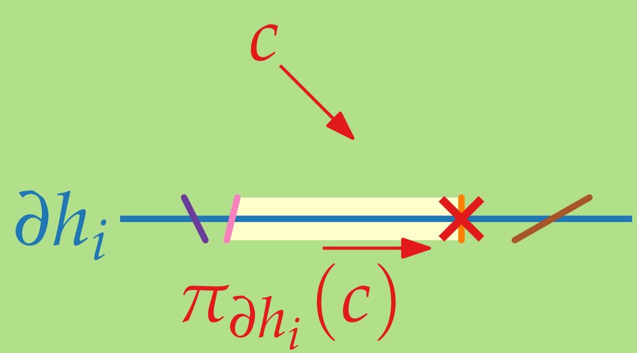

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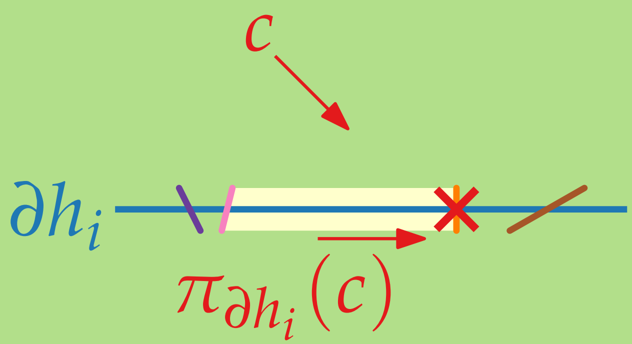

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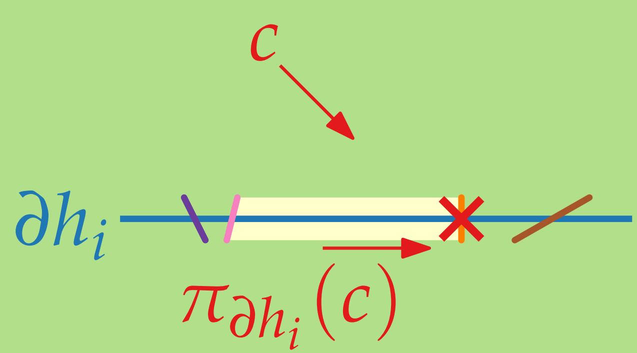

w-c running time:

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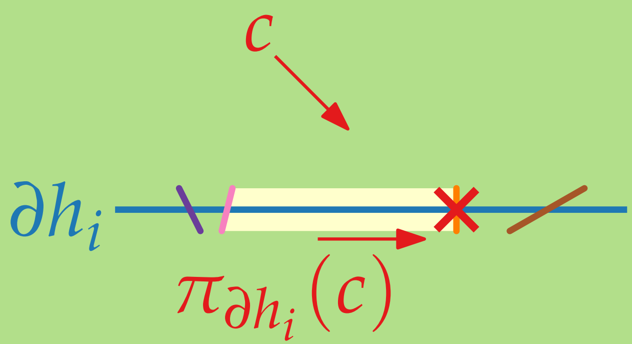

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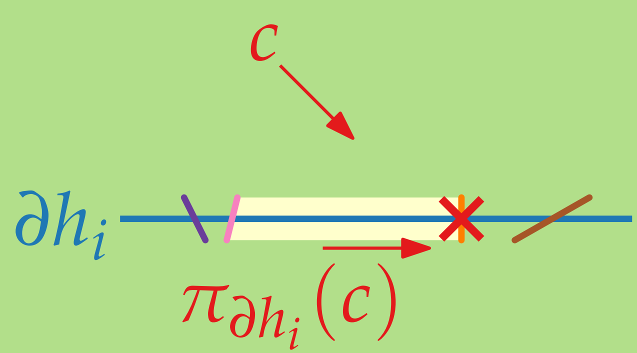

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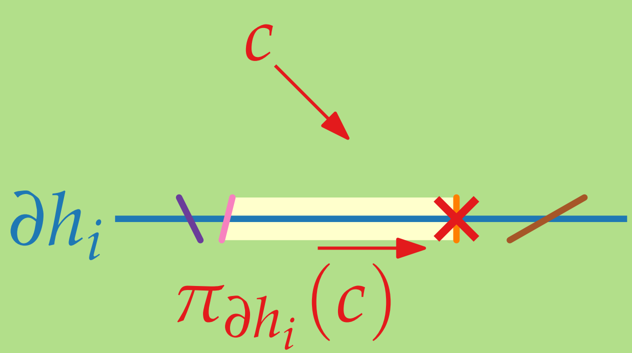

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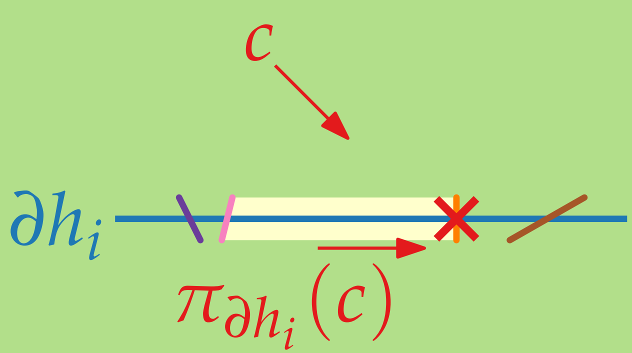

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$T(n) = \sum_{i=1}^{n} O(i) =$

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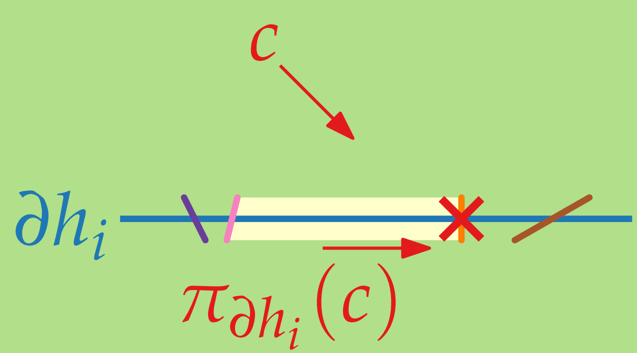


w-c running time:
$T(n) = \sum_{i=1}^{n} O(i) =$
$= O(n^2)$:-(

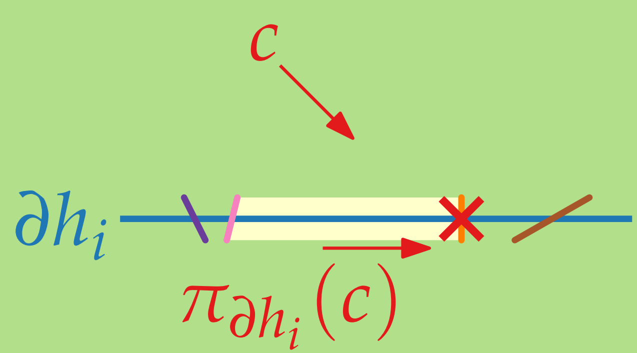
Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

Randomized

$\text{2DBoundedLP}(H, c, m_1, m_2)$

$H_0 = \{m_1, m_2\}$
$v_0 \leftarrow$ corner of $m_1 \cap m_2$
for $i \leftarrow 1$ **to** n **do**
  if $v_{i-1} \in h_i$ **then**
    $v_i \leftarrow v_{i-1}$ — $O(1)$
  else
    $v_i \leftarrow \text{1DBoundedLP}(\pi_{\partial h_i}(H_{i-1}), \pi_{\partial h_i}(c))$ — $O(i)$
    if $v_i =$ nil **then**
      return nil
  $H_i = H_{i-1} \cup \{h_i\}$ — $O(1)$
return v_n



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Incremental Approach

Idea: Don't compute $\bigcap H$, but just *one* (optimal) point!

Randomized

2DBoundedLP(H, c, m_1, m_2)
  compute random permutation of H
  $H_0 = \{m_1, m_2\}$
  $v_0 \leftarrow$ corner of $m_1 \cap m_2$
  for $i \leftarrow 1$ **to** n **do**
    if $v_{i-1} \in h_i$ **then**
      $v_i \leftarrow v_{i-1}$ $O(1)$
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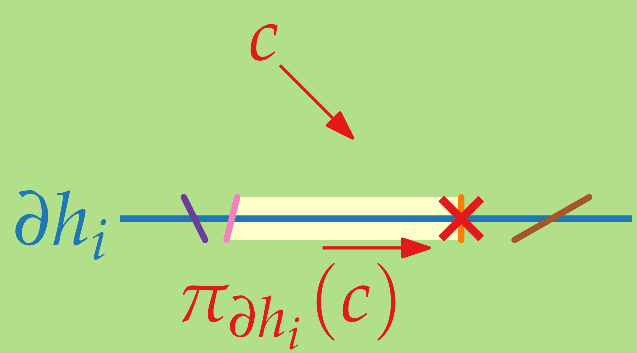

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Computational Geometry

Lecture 4:
Linear Programming
or
Profit Maximization

Part V:
The Randomized-Incremental Approach

Philipp Kindermann Winter Semester 2020

Result

Theorem. The 2D bounded LP problem can be solved in $O(n)$ expected time.

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Proof technique: *Backward analysis!*

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