Computational Geometry

Lecture 1:
Convex Hull or Mixing Things

Part I:
Organizational & Overview

Philipp Kindermann

Winter Semester 2020

Organizational

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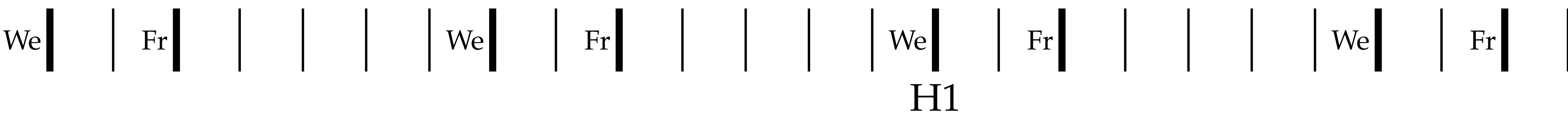

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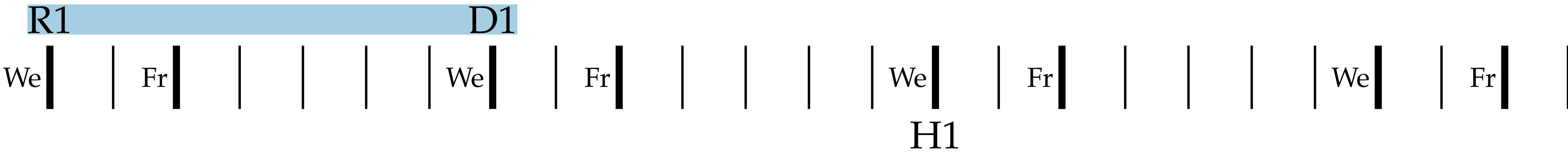

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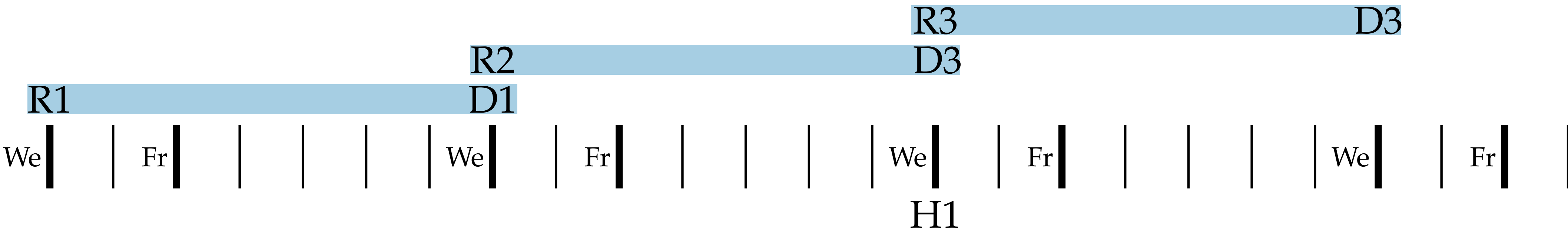

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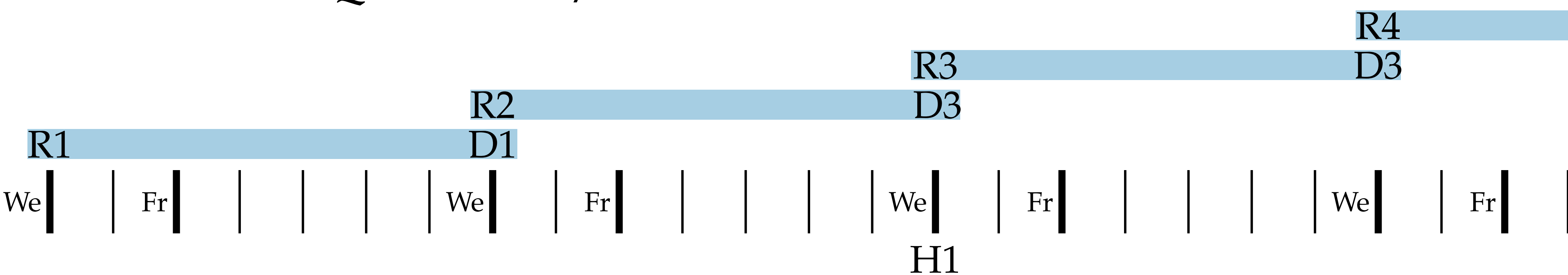

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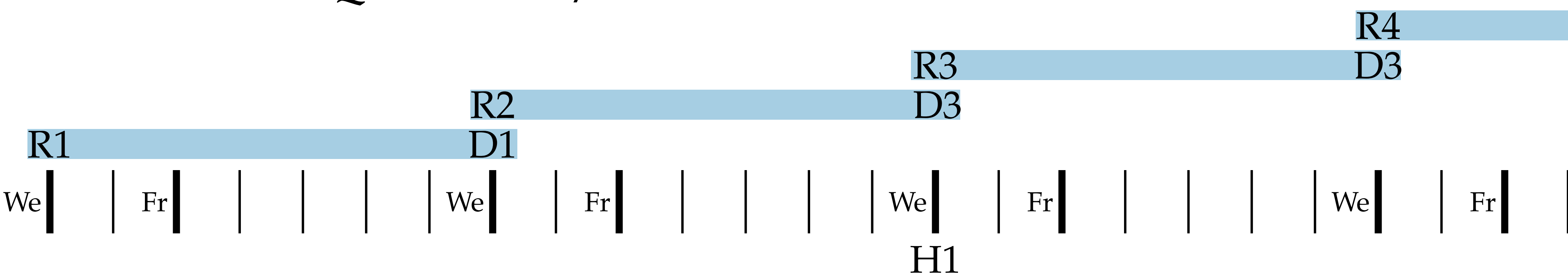

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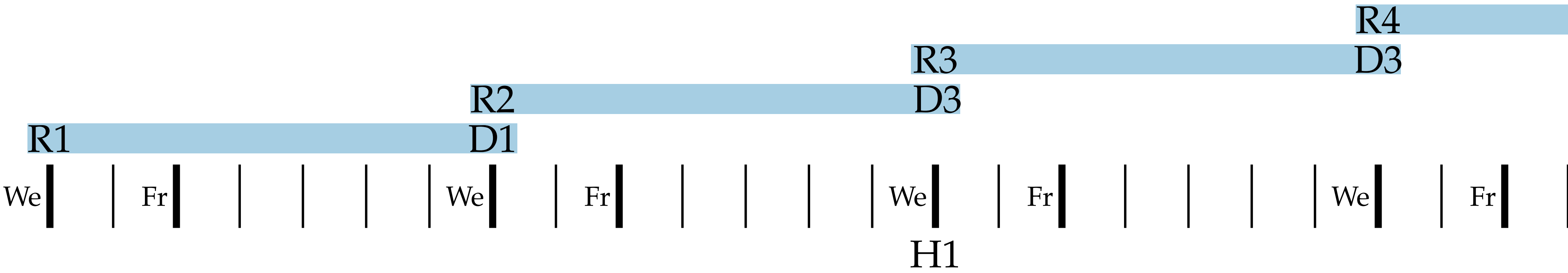

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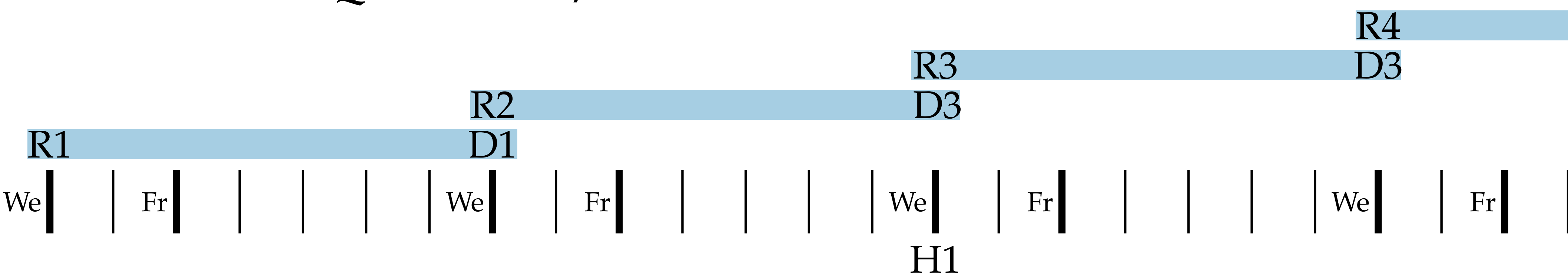

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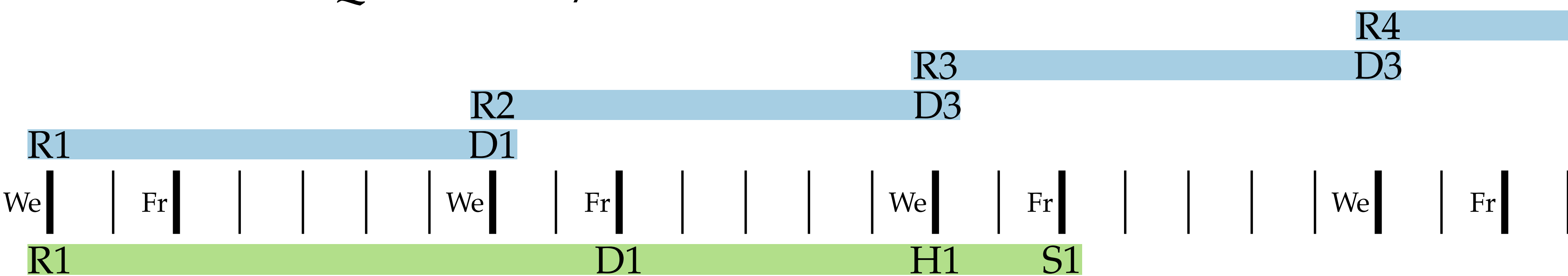

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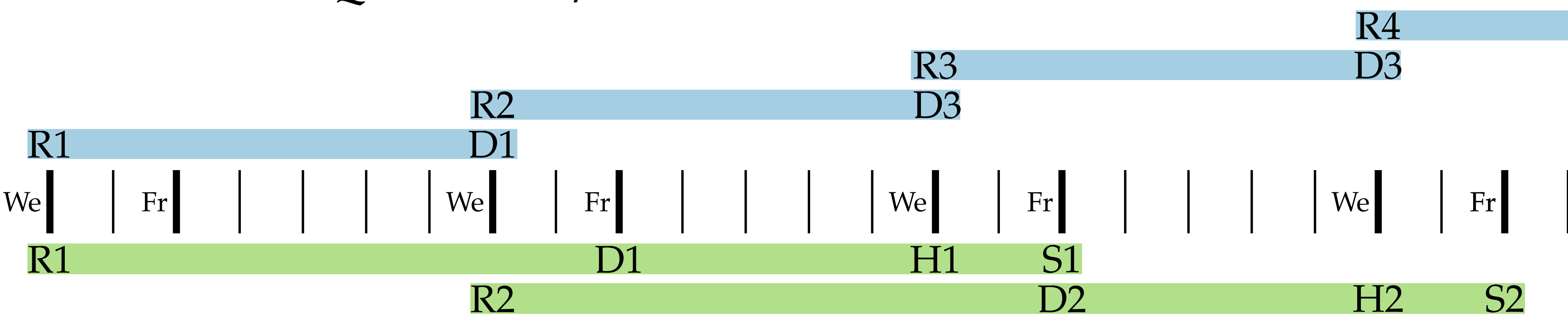

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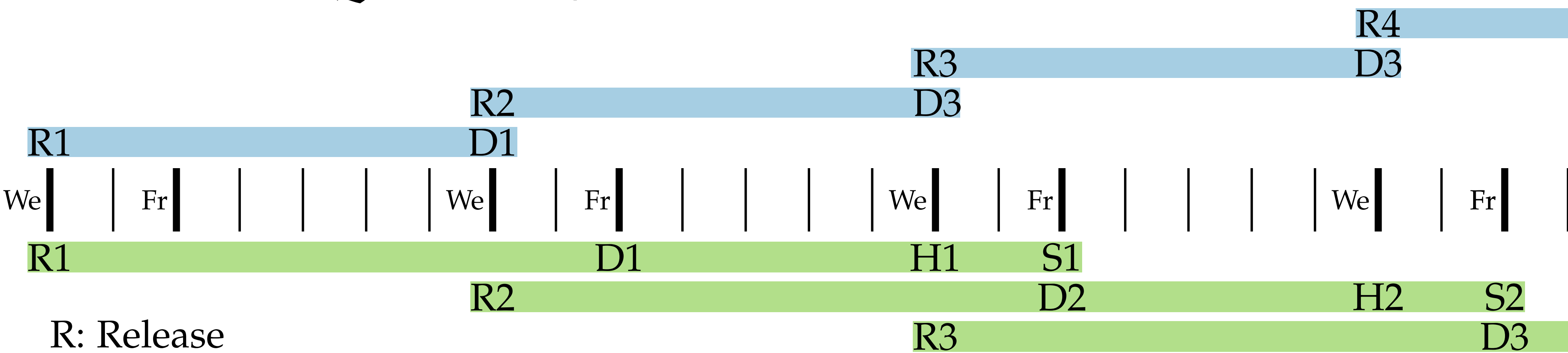

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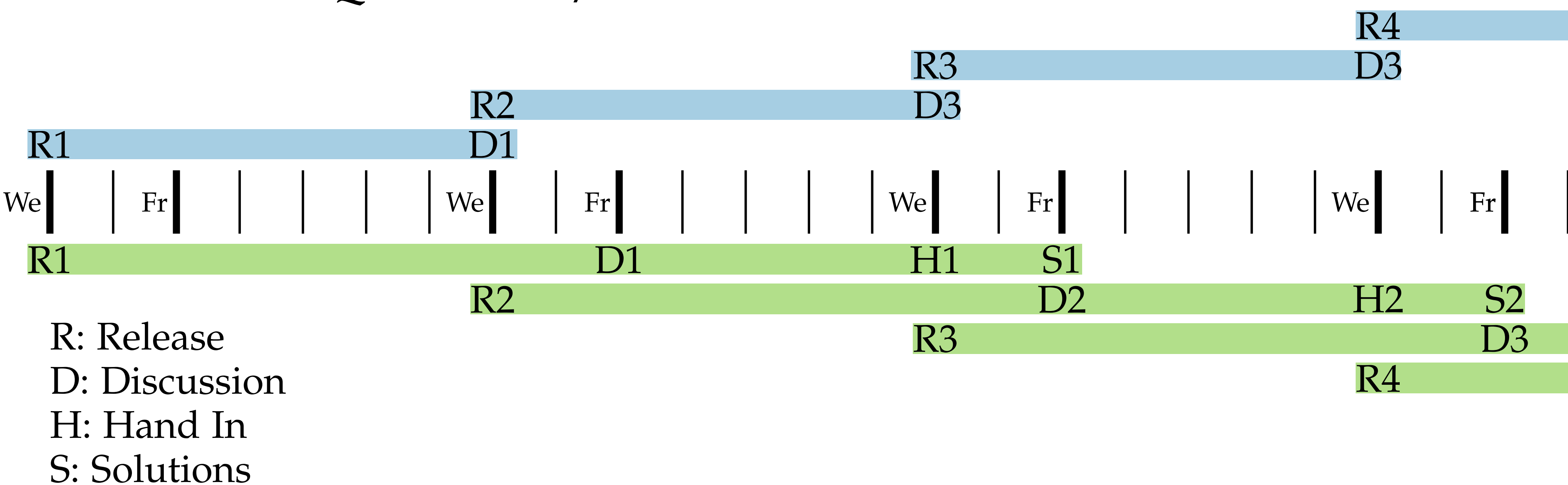

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Our Lectures and Seminars

Algorithms and Data Structures

Algorithmic Graph Theory

WS: Advanced Algorithms | Computational Geometry | Approximation Algorithms

SS: Exact Algorithms | Graph Visualization | Algorithms for Geographic Information Systems

Seminar Graph Visualization | Seminar Algorithms for Programming Contests

Master Project | Master Thesis

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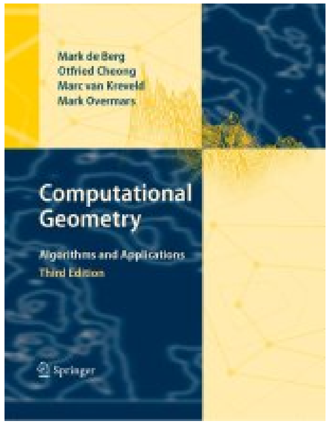
- Some basic *Algorithms & Data Structures*
  (Balanced) binary search tree, priority queue
- Some basic *Algorithmic Graph Theory*
  Breadth-first search, Dijkstra's algorithm

Content (Prelim.)

1. Convex Hull in 2D
2. Segment Intersection
3. Polygon Triangulation
4. Linear Programming
5. Orthogonal Range Queries
6. Point Location
7. Voronoi Diagram
8. Delaunay Triangulation
9. Convex Hull in 3D
10. Motion Planning
11. Simplex Range Searching
12. Visibility Graph & Shortest Path

Literature



M. de Berg, O. Cheong, M. van Kreveld, M. Overmars: Computational Geometry: Algorithms & Applications. Springer, 3rd edition, 2008

Main resource for this course!

Abbreviated as: Comp. Geom A&A

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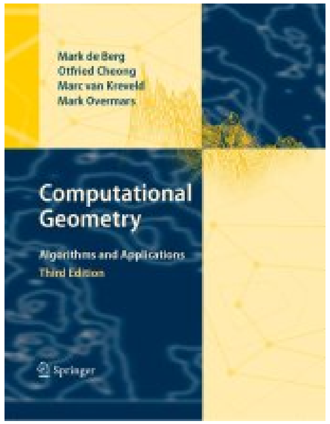

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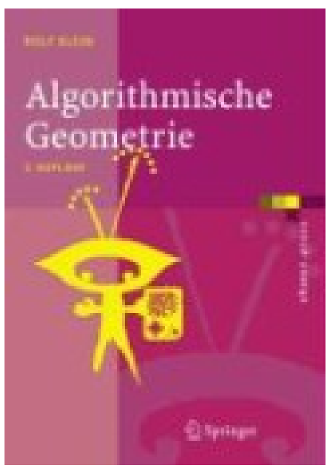

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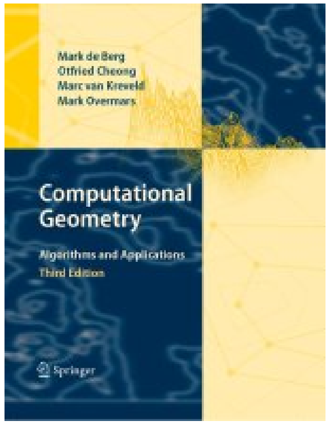

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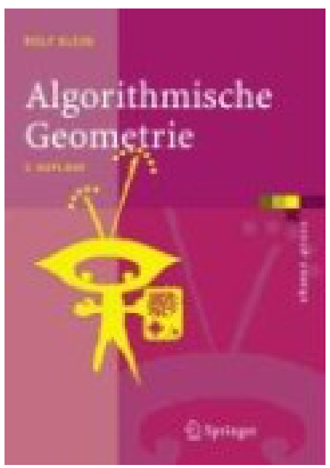

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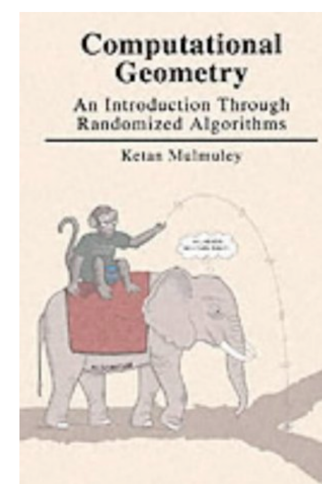

Ketan Mulmuley:
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Computational Geometry

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Part II:
Mixing Things

Philipp Kindermann

Winter Semester 2020

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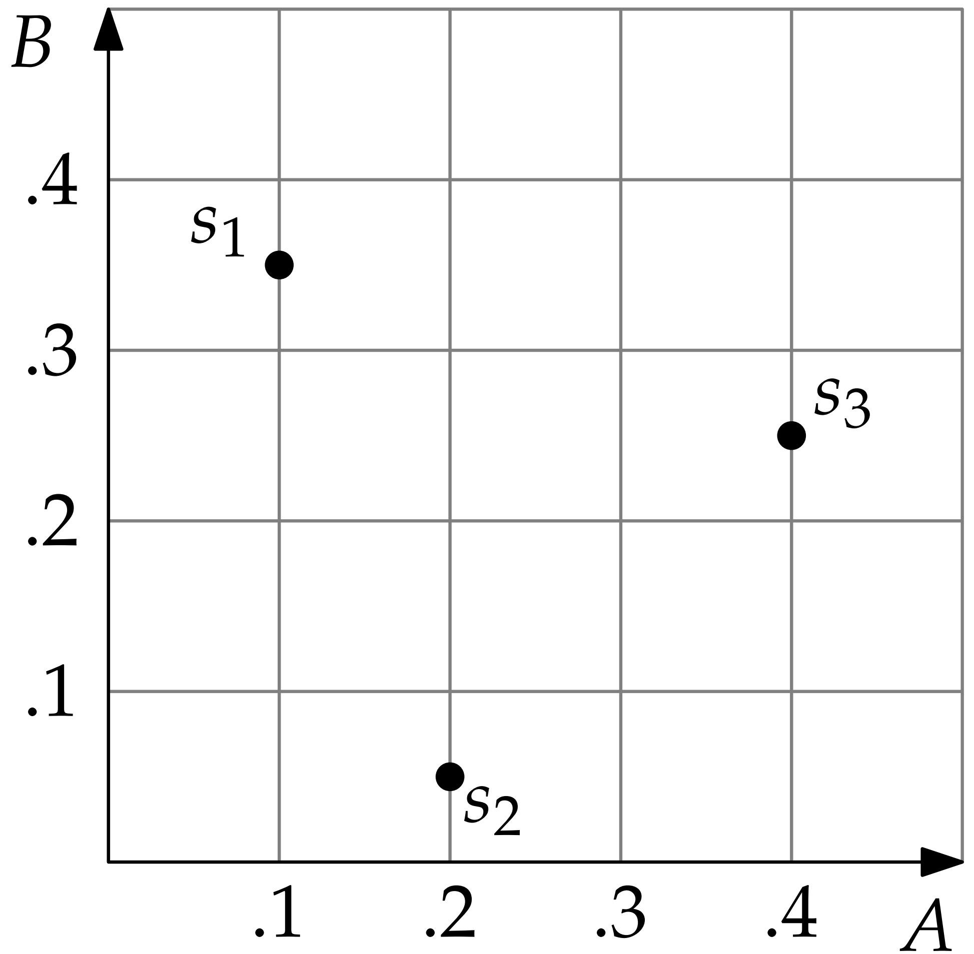

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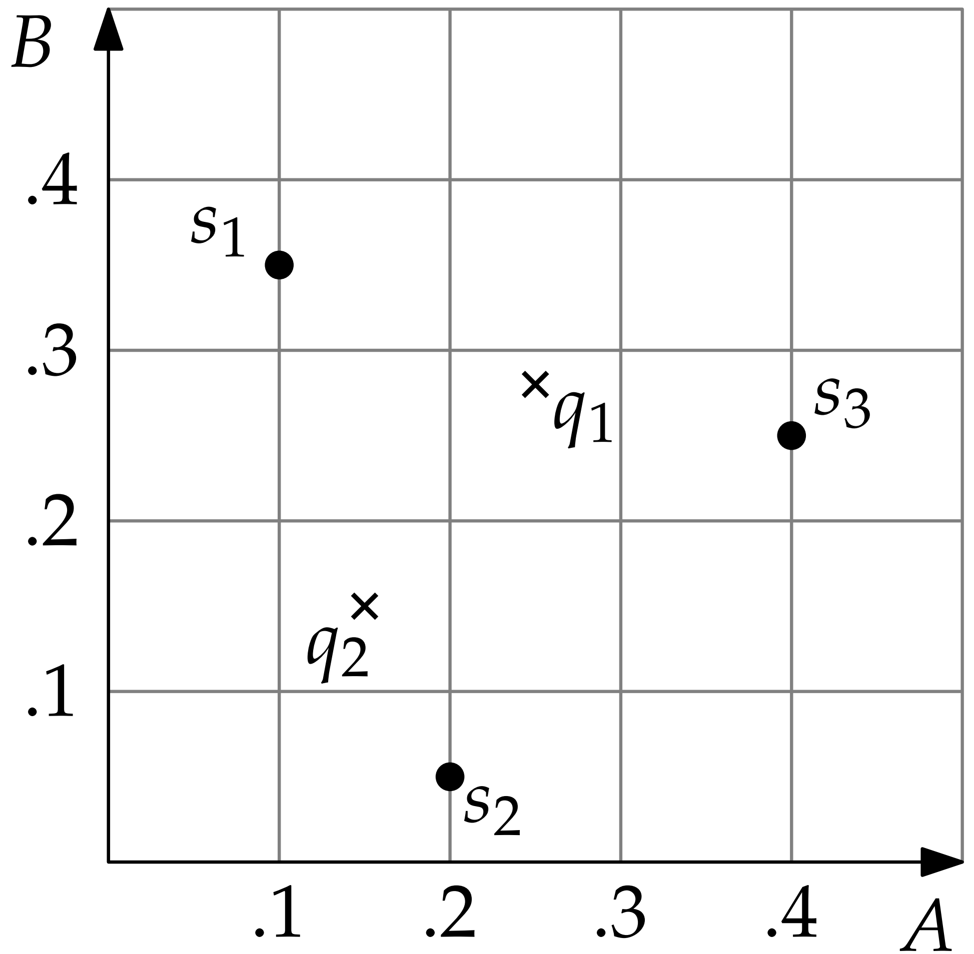

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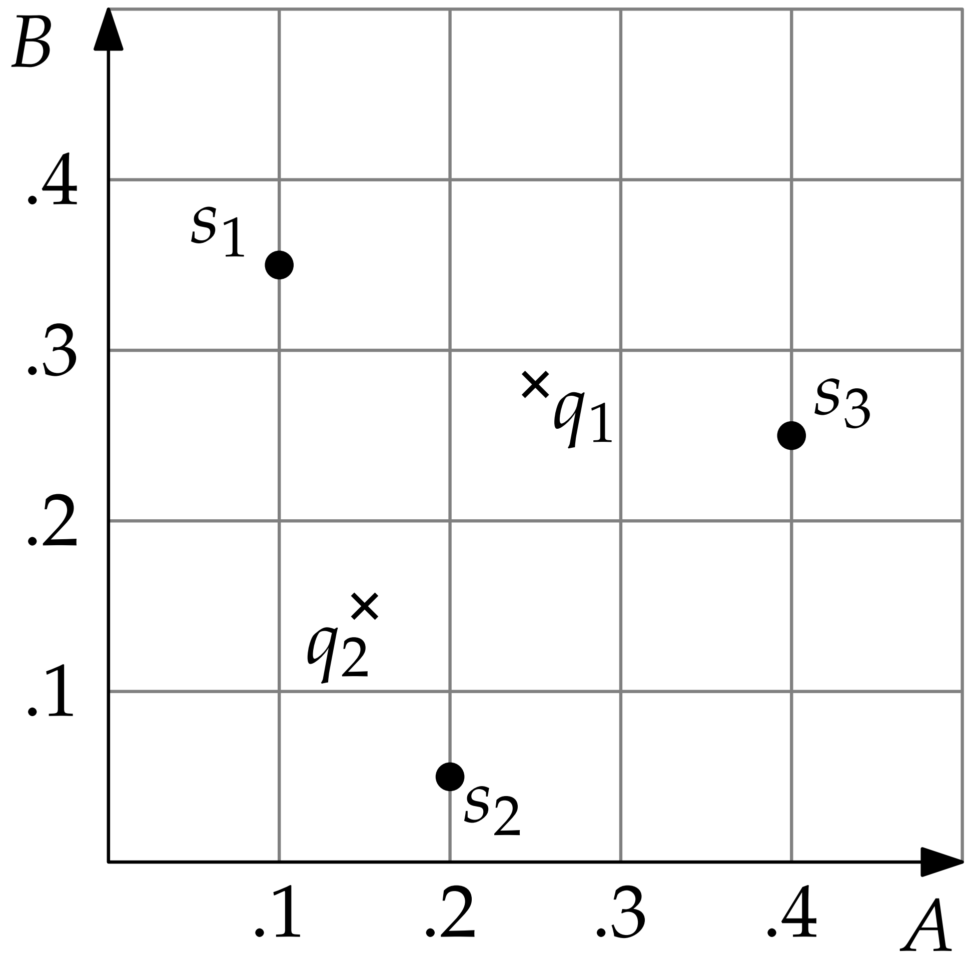

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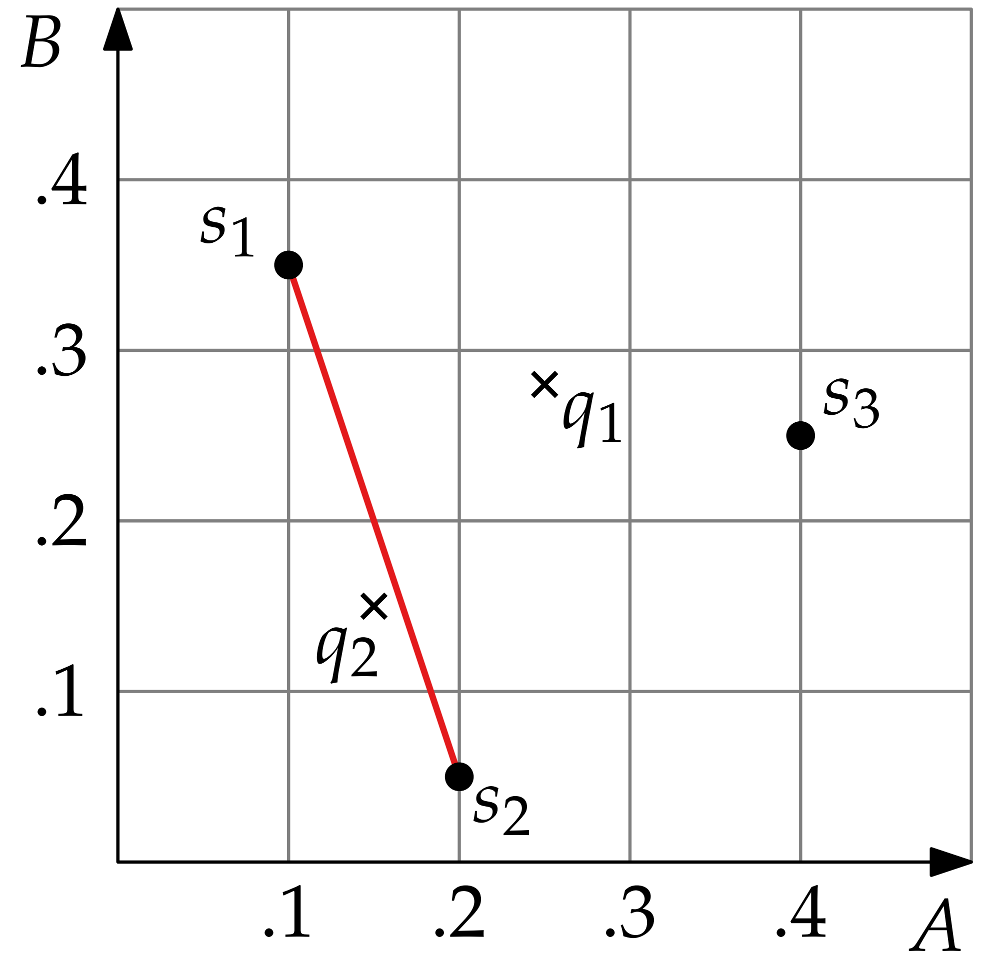

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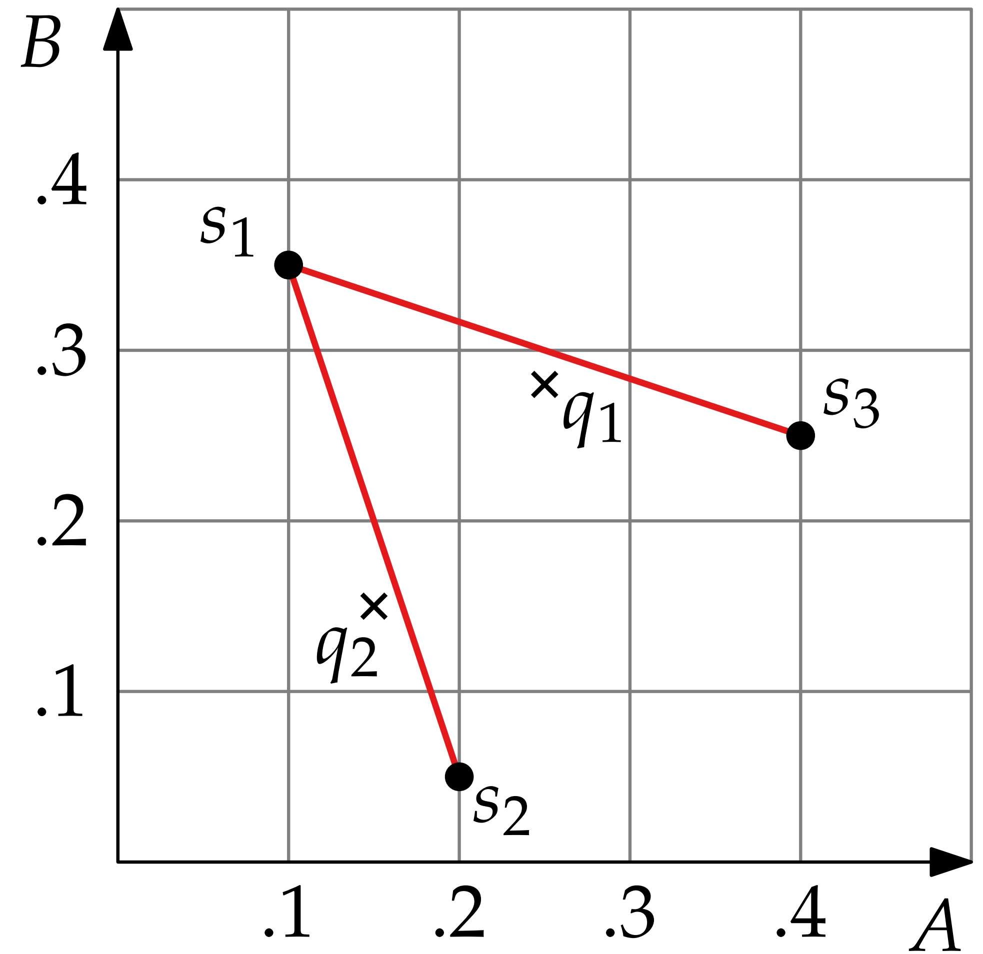

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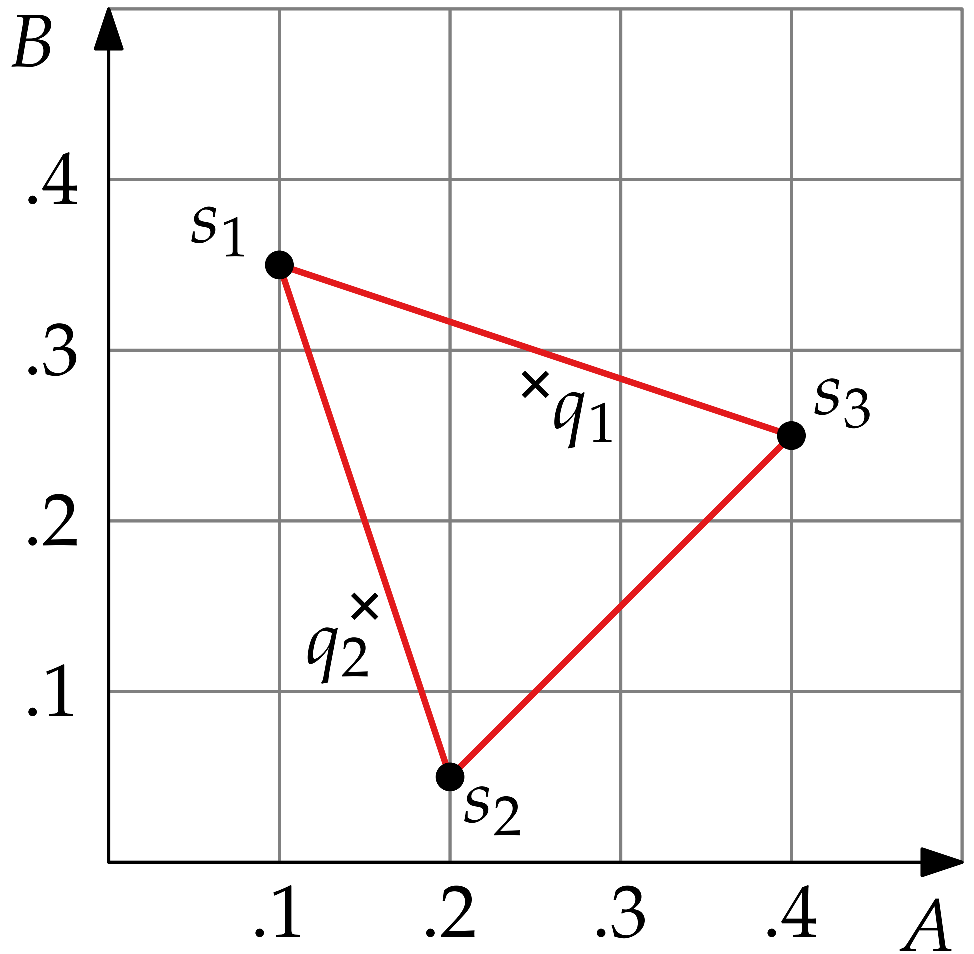

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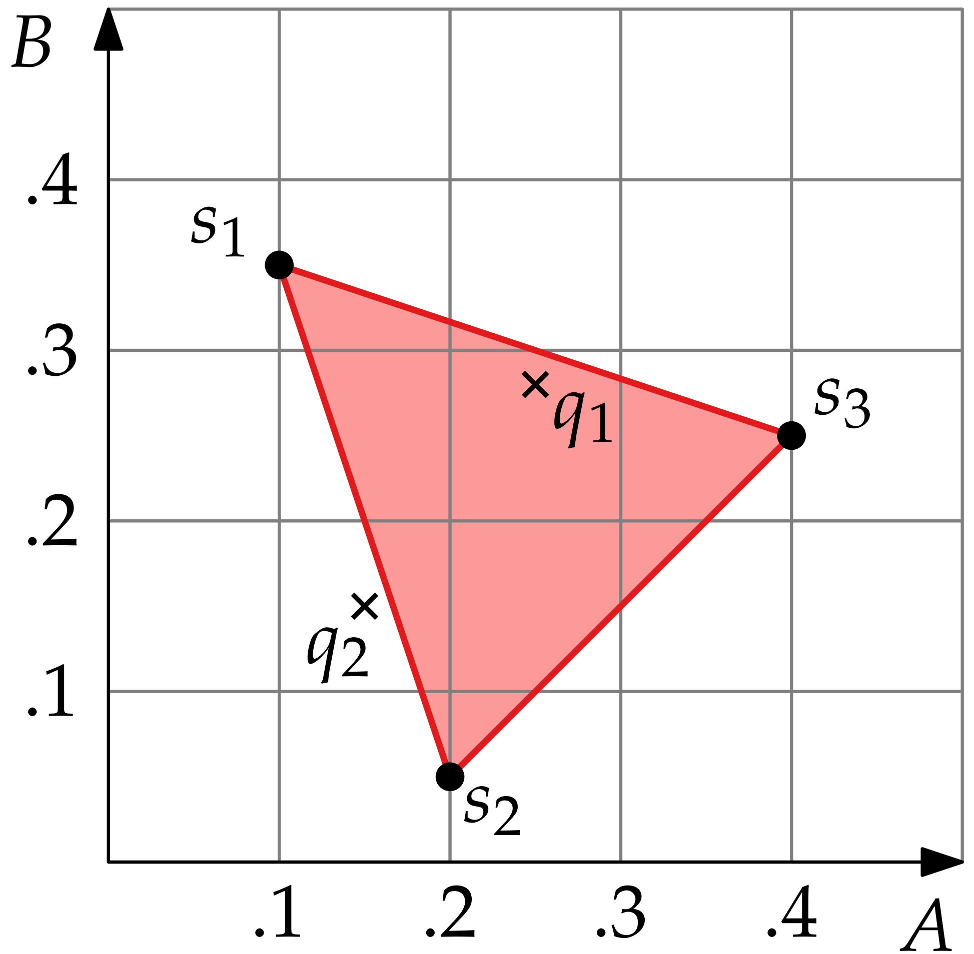

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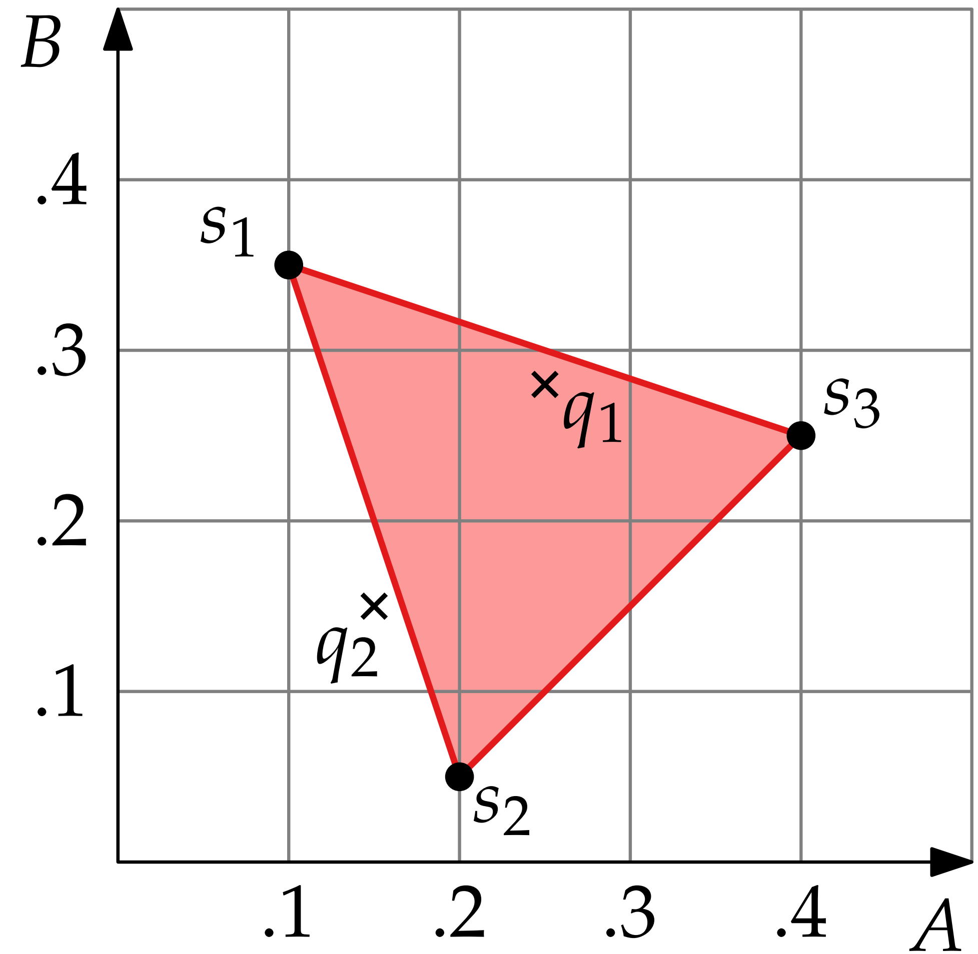

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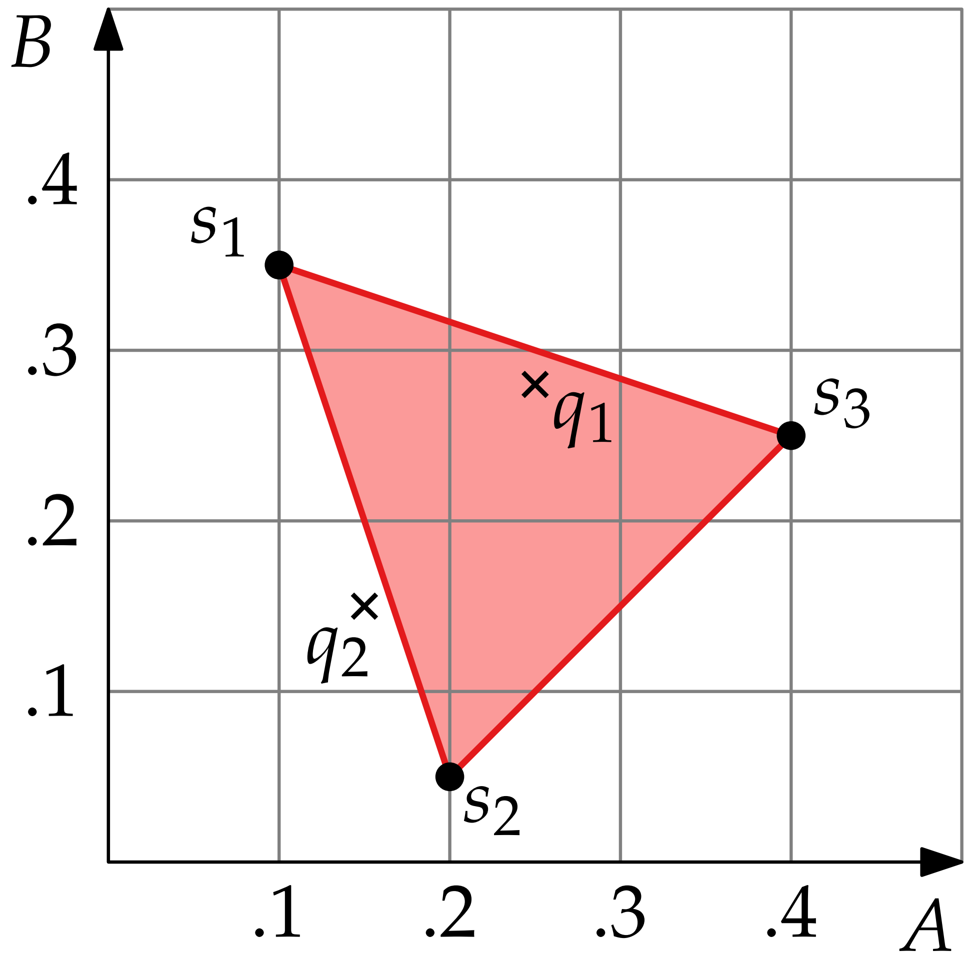

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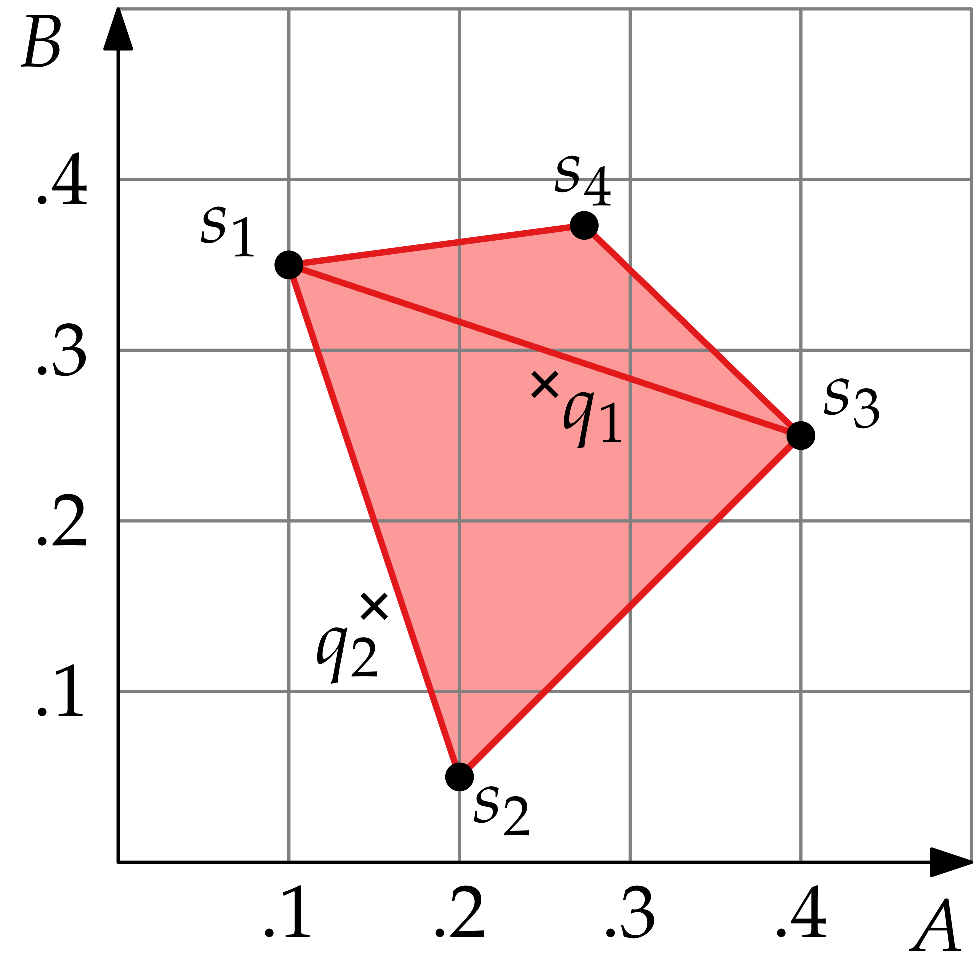

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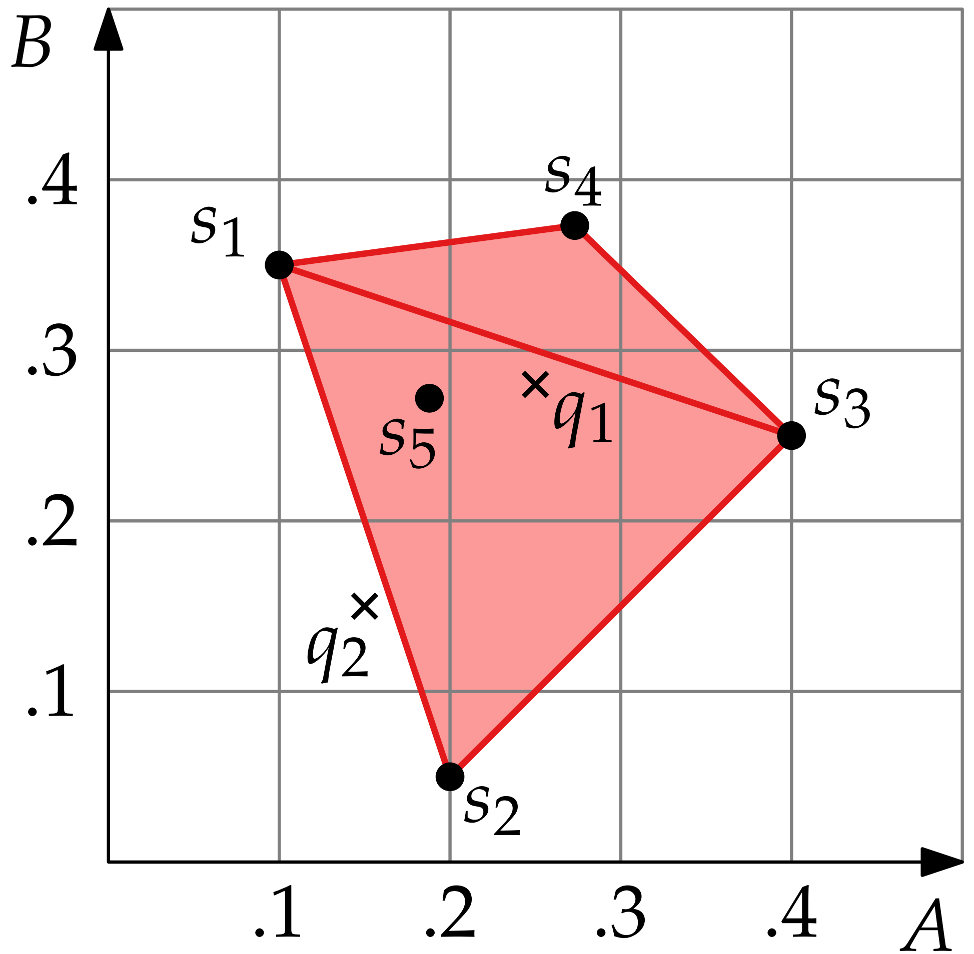

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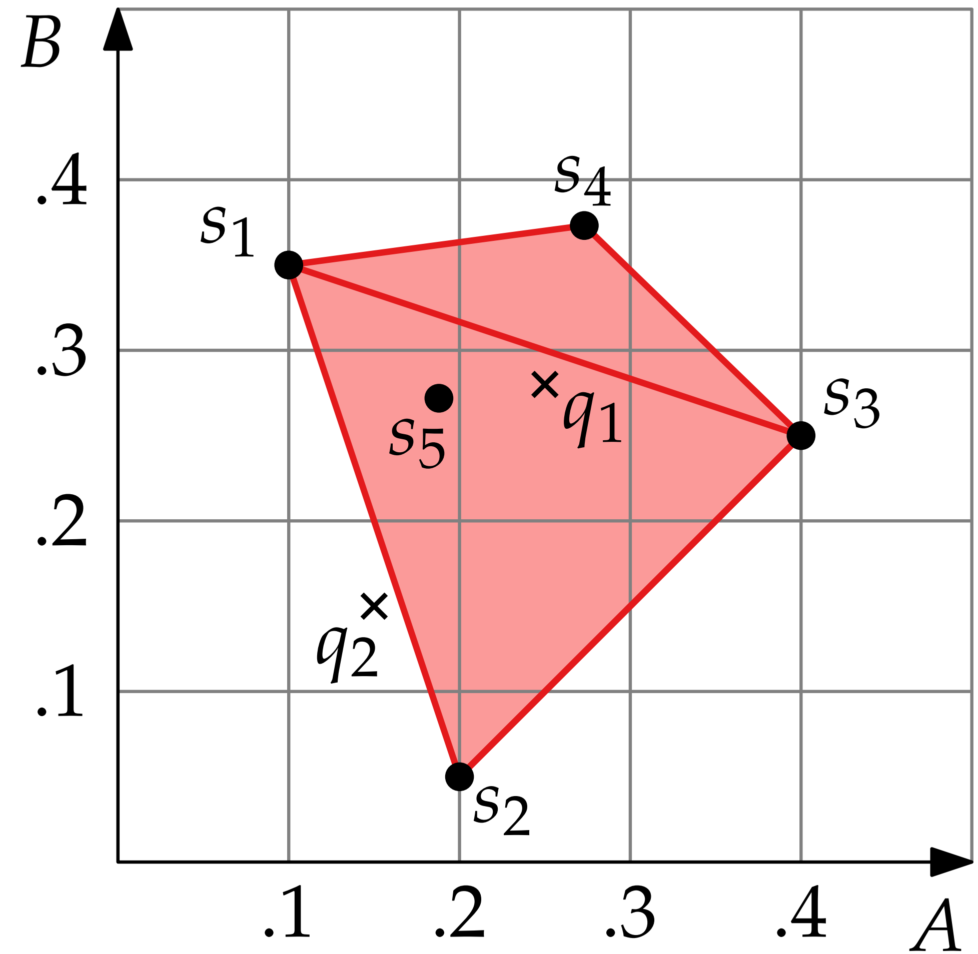
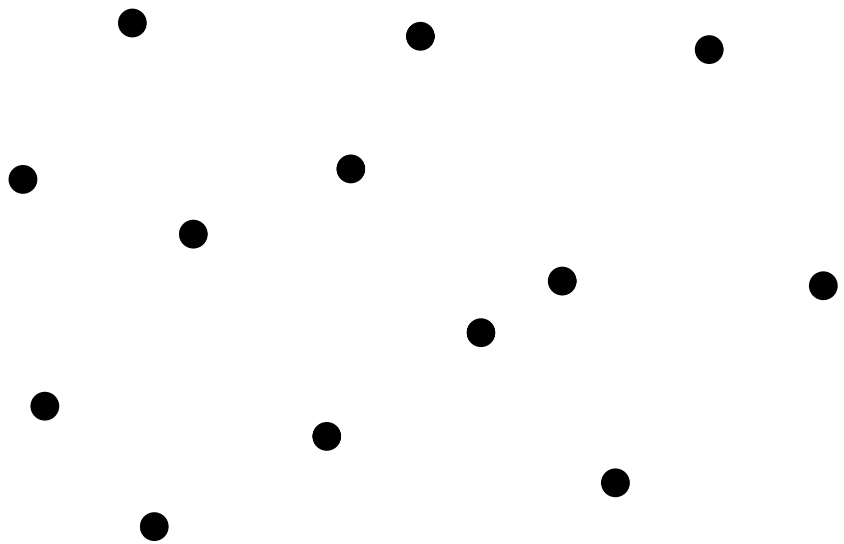


Observation. Given a set $S \subset \mathbb{R}^{\not{2}d}$ of substances, we can mix a substance $q \in \mathbb{R}^{\not{2}d}$ using the substances in $S \quad \Leftrightarrow \quad q \in \mathrm{CH}(S)$.

Formally. . .



Given $S \subset \mathbb{R}^2$, how do we define the *convex hull* $\mathrm{CH}(S)$?

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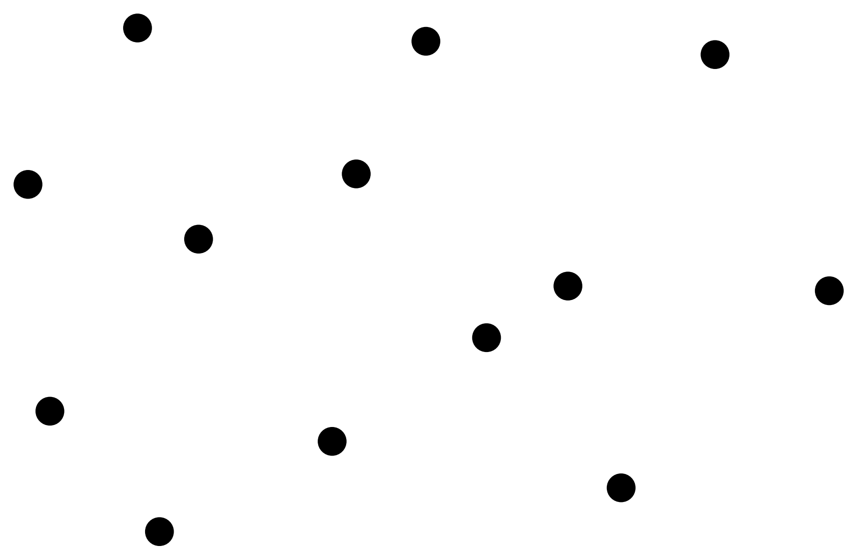

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Physics approach:

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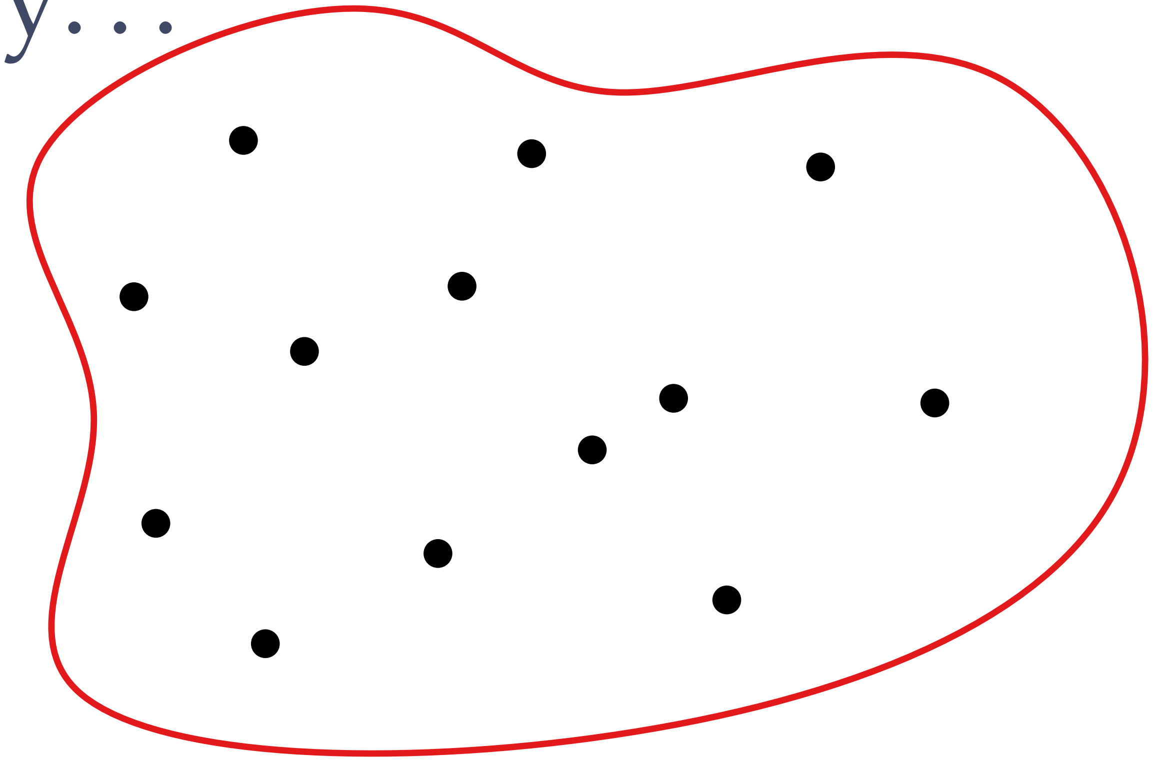

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Physics approach: – take (large enough) elastic rope

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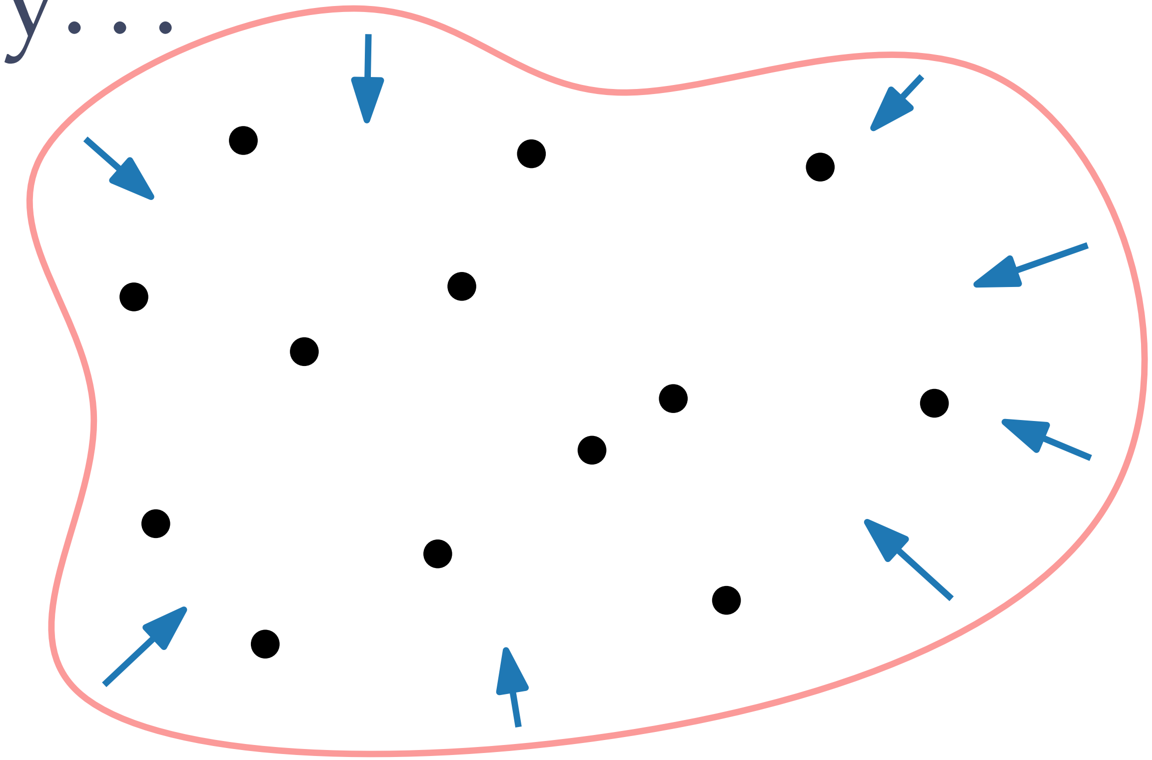

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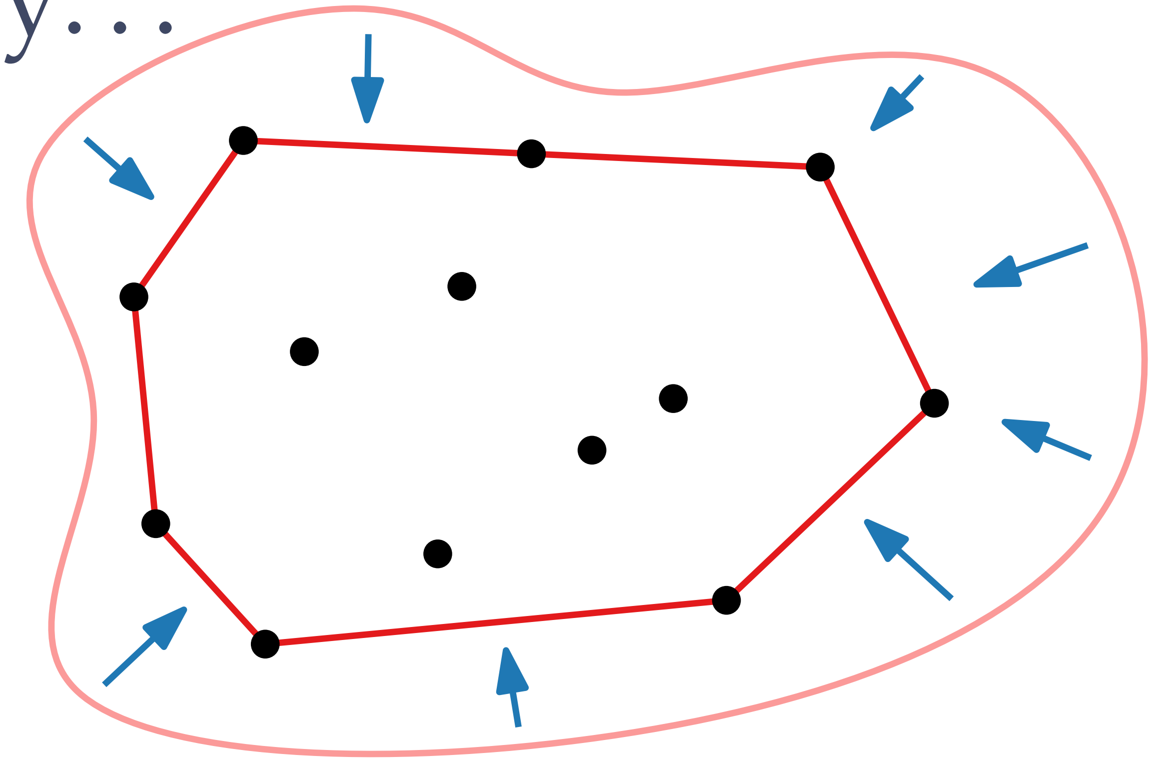

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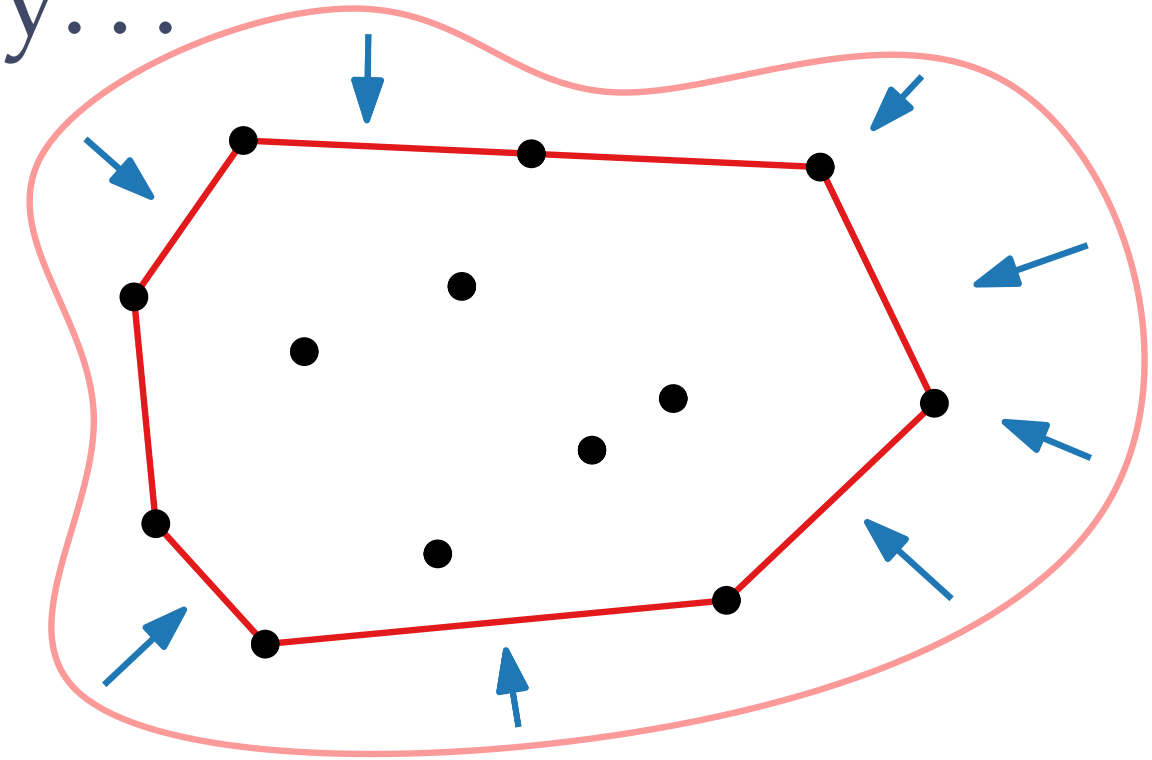

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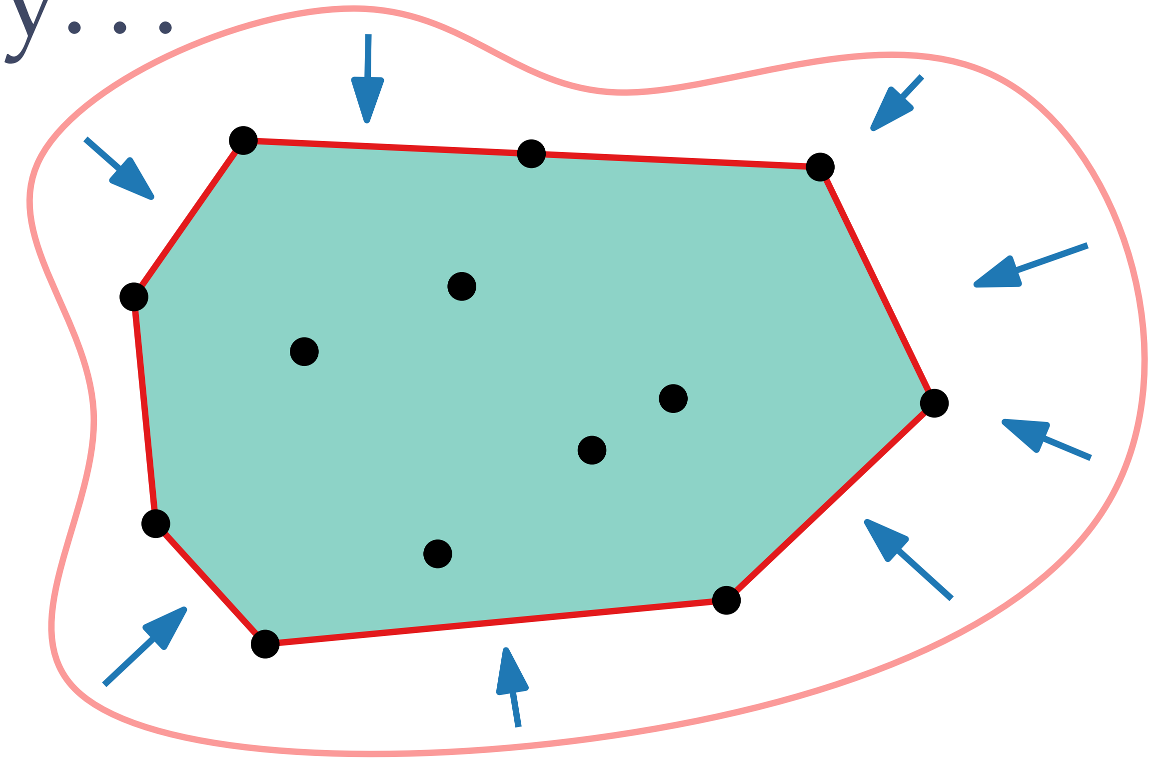

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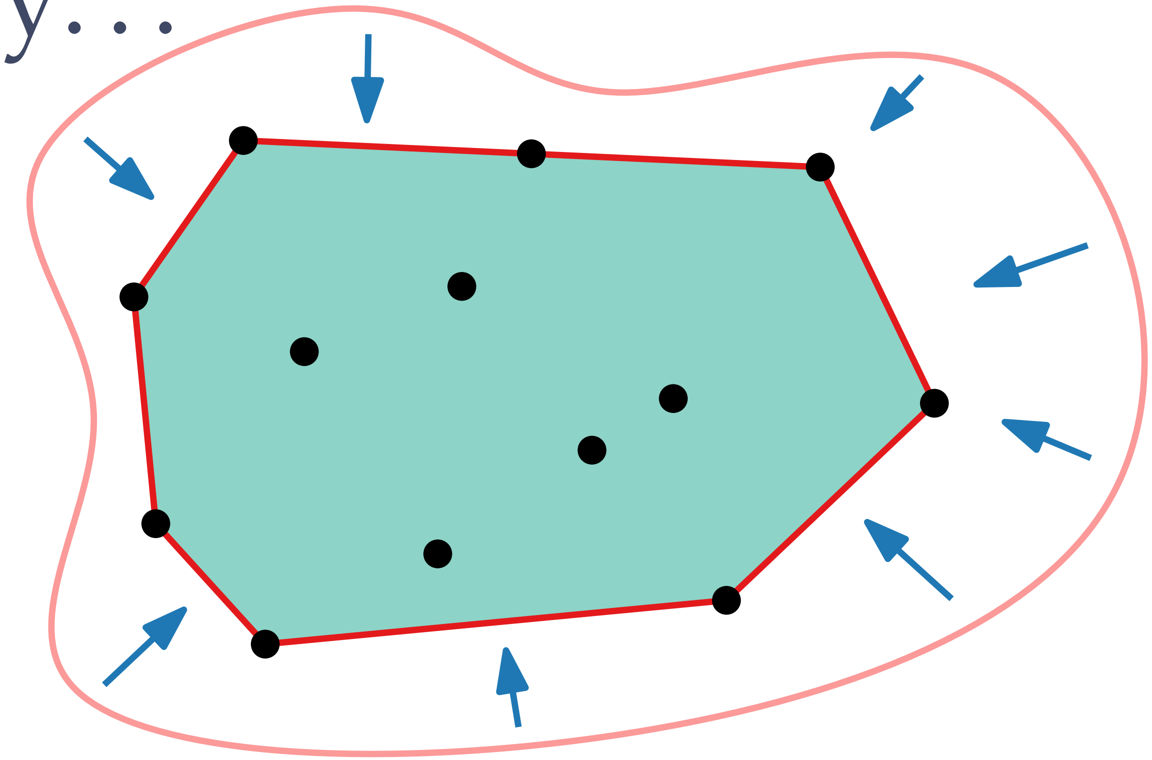

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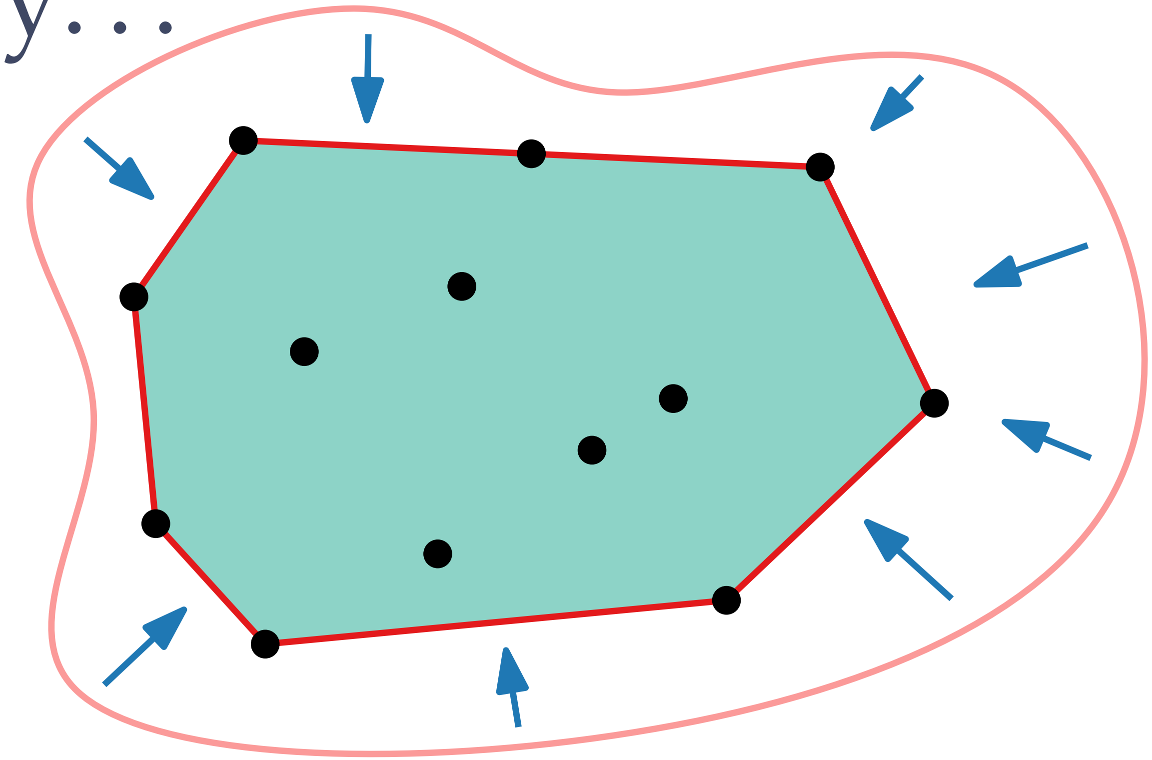
Maths approach:

Formally...



Given $S \subset \mathbb{R}^2$, how do we define the *convex hull* $\mathrm{CH}(S)$?

Physics approach: – take (large enough) elastic rope
– stretch and let go
– take area inside (and on) the rope

Maths approach: – define *convex*

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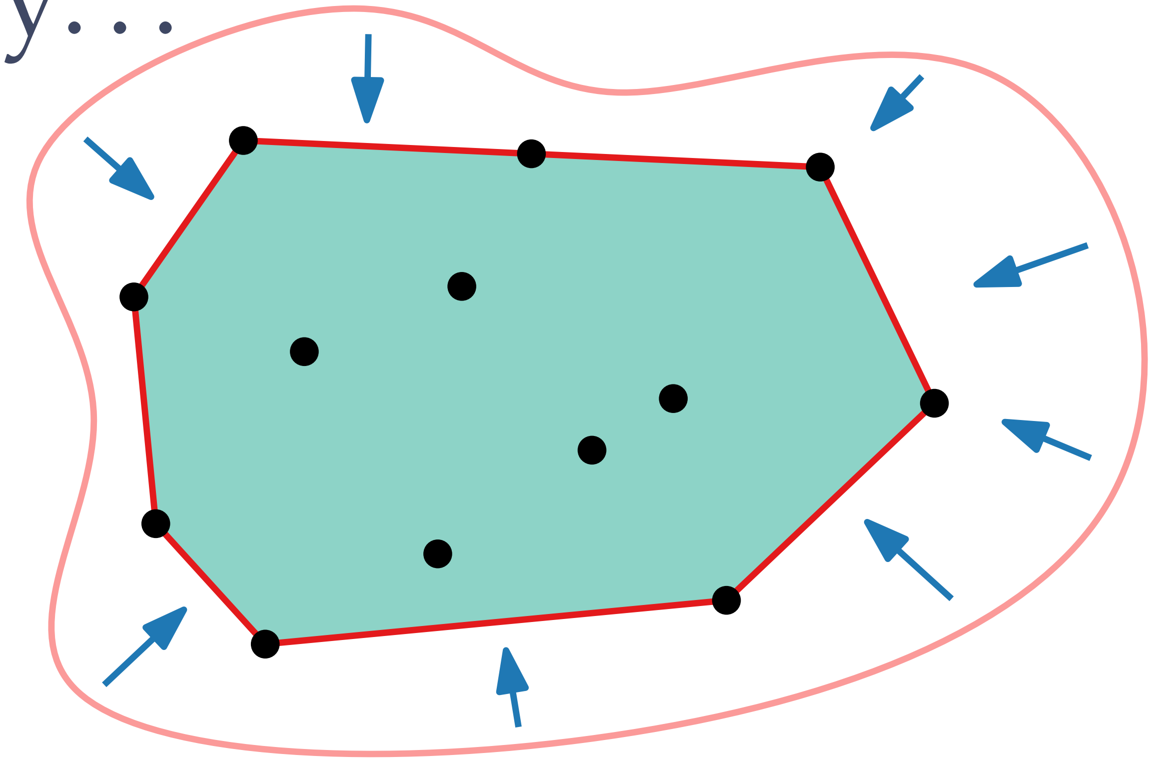

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Physics approach: – take (large enough) elastic rope
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Maths approach: – define *convex*
– define $\mathrm{CH}(S) = \bigcap_{C \supseteq S\,:\,C \text{ convex}} C$

Towards Computation

$$\mathrm{CH}(S) \stackrel{\text{def}}{=} \bigcap_{C \supseteq S\,:\; C \text{ convex}} C$$

Problem with maths approach:

Towards Computation

$$\mathrm{CH}(S) \stackrel{\text{def}}{=} \bigcap_{C \supseteq S:\, C \text{ convex}} C$$

Problem with maths approach: ← *This set is* HUGE!

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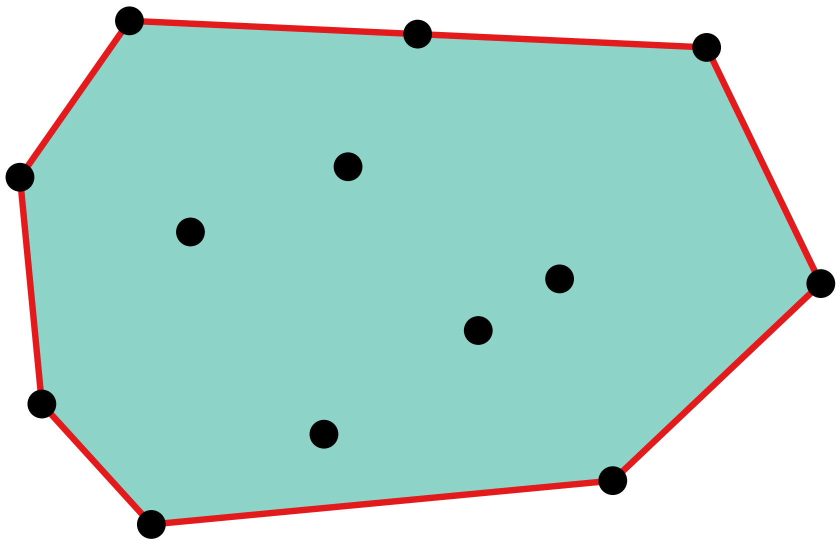

Maybe we can do with a little less?

Towards Computation

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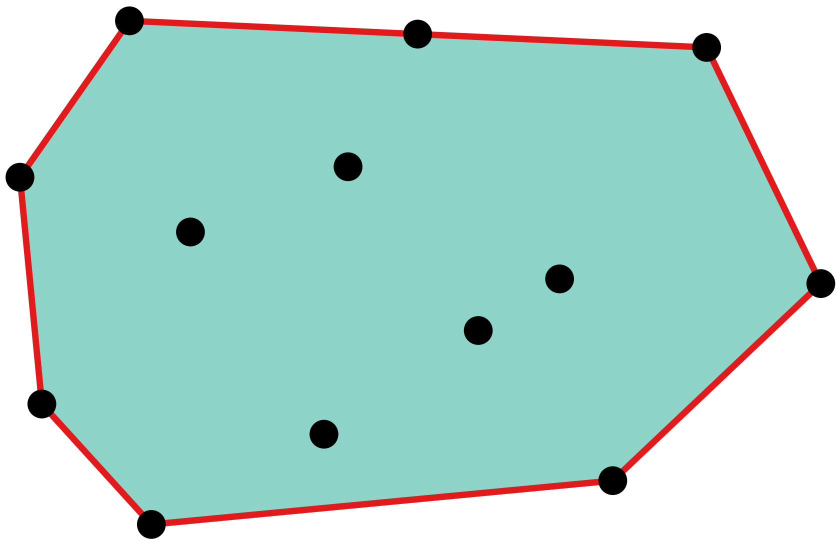

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Claim. $\mathrm{CH}(S) =$

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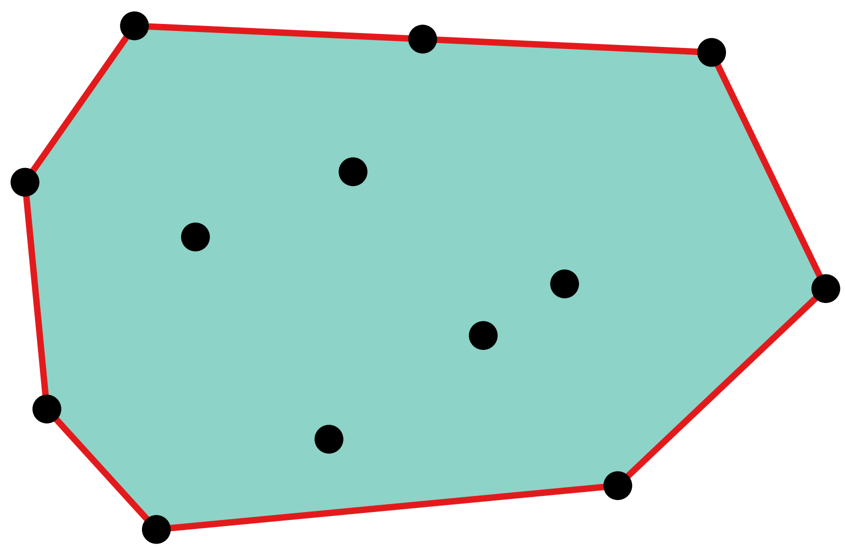

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Claim. $$\mathrm{CH}(S) = \bigcap_{\substack{H \supseteq S: \\ H \text{ closed halfplane}}} H$$

Towards Computation

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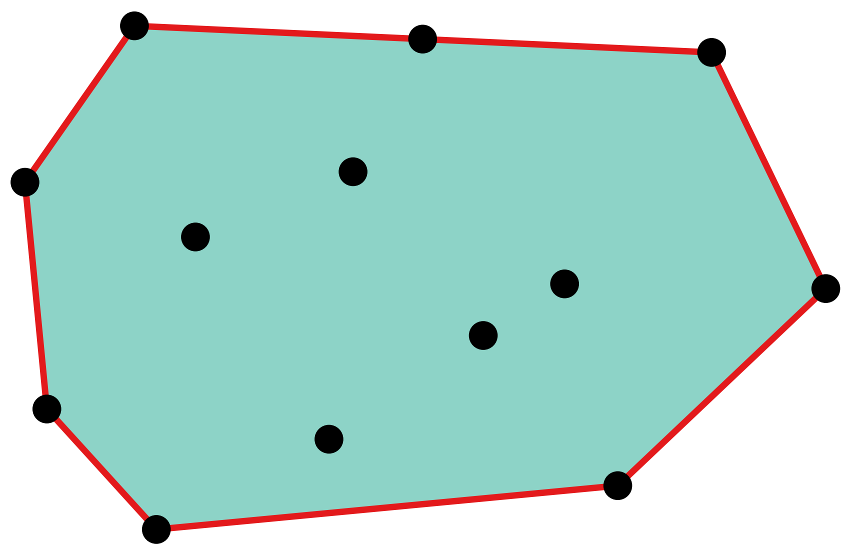

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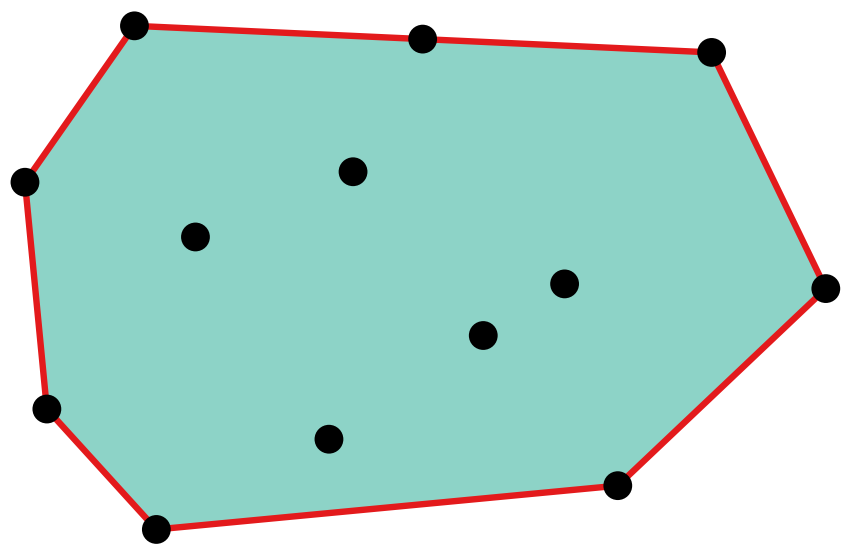

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Claim. $$\mathrm{CH}(S) = \bigcap_{\substack{H \supseteq S: \\ H \text{ closed halfplane}}} H = \bigcap_{\substack{H \supseteq S:\, H \text{ cl. halfplane}, \\ |\partial H \cap S| \geq 2}} H$$

Computational Geometry

Lecture 1:
Convex Hull or Mixing Things

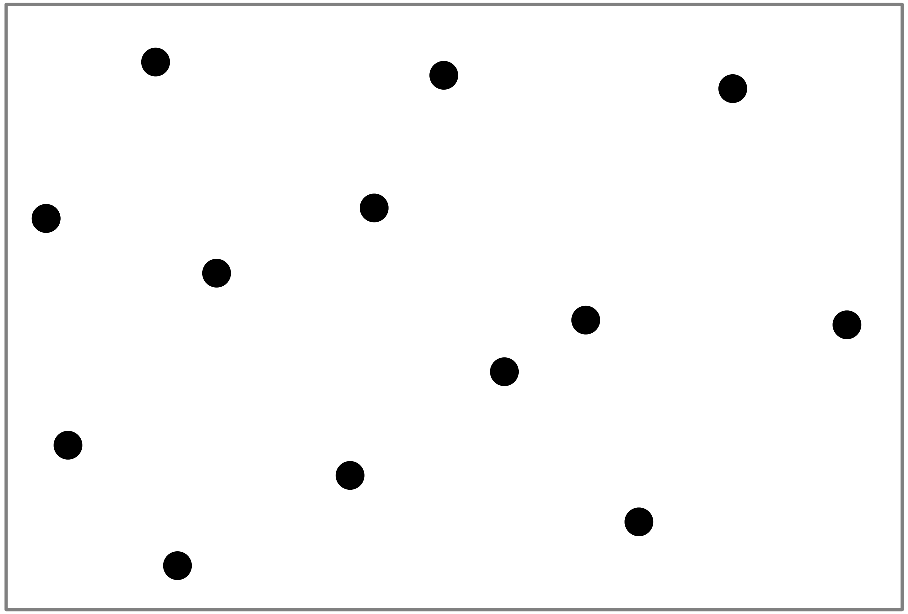
Part III:
Algorithmic Approach

Philipp Kindermann

Winter Semester 2020

Algorithmic Approach

Input: set S of n points in the plane, that is, $S \subset \mathbb{R}^2$



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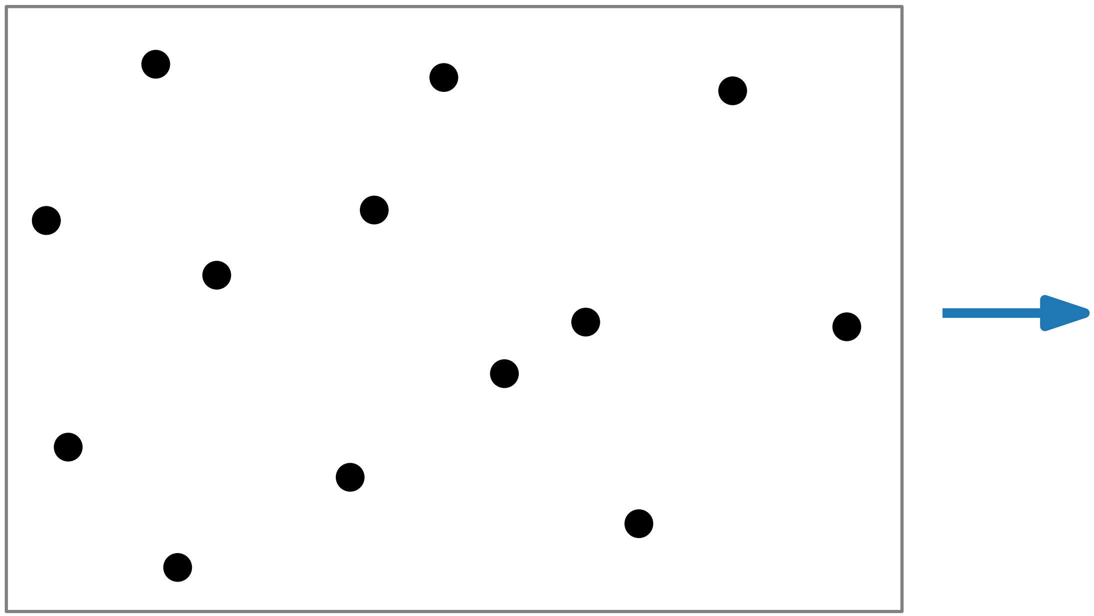

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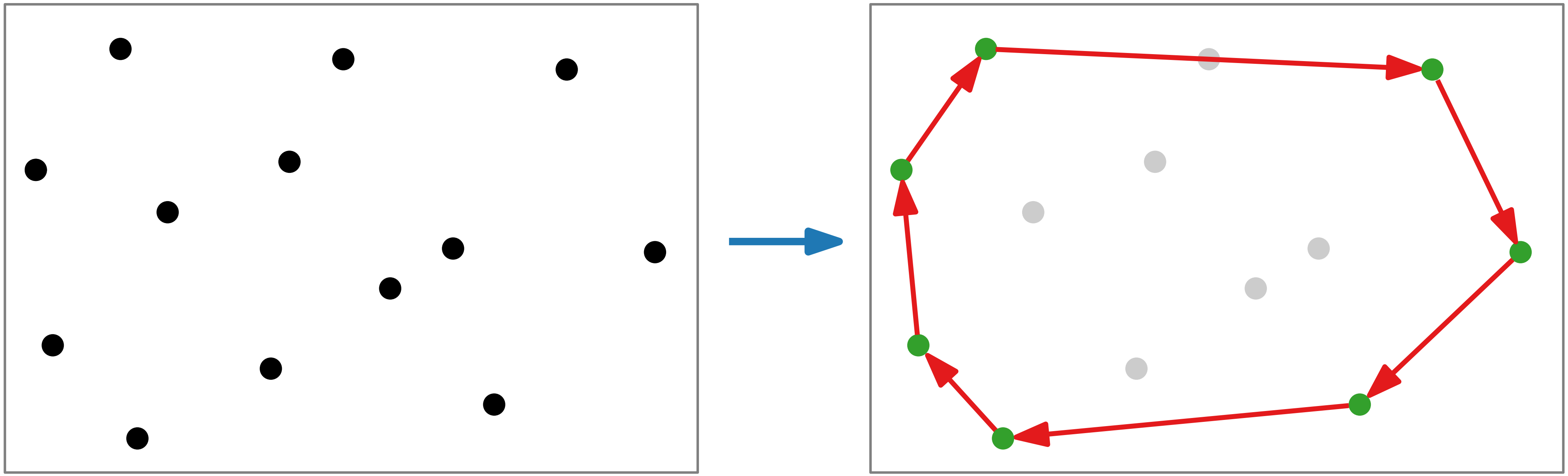

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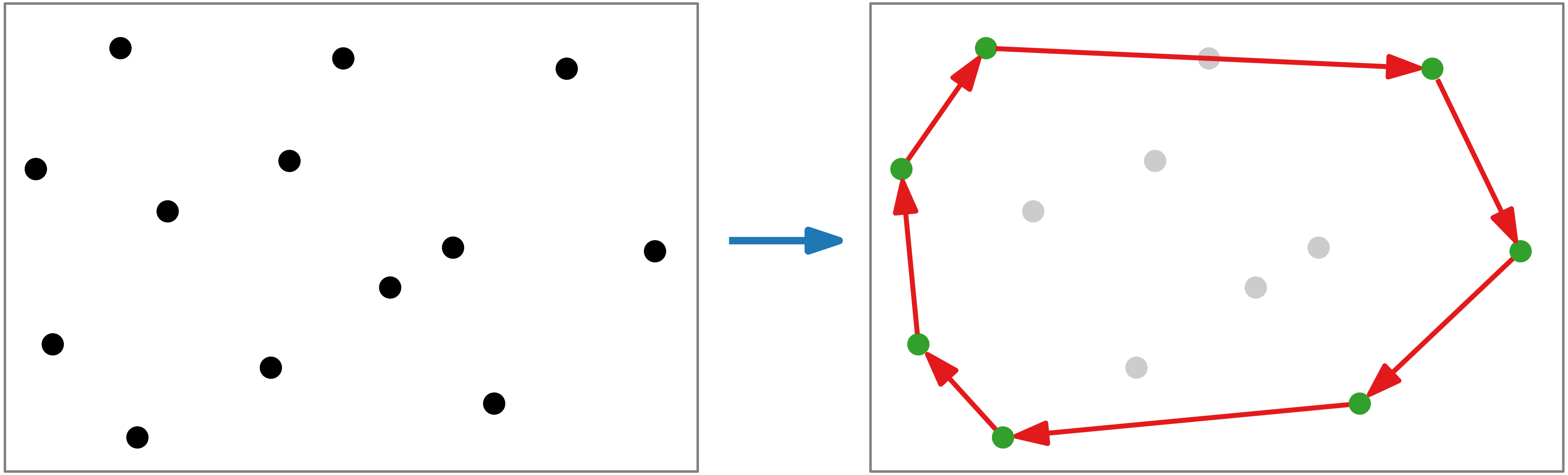

Output: list of vertices of $\mathrm{CH}(S)$ in clockwise order

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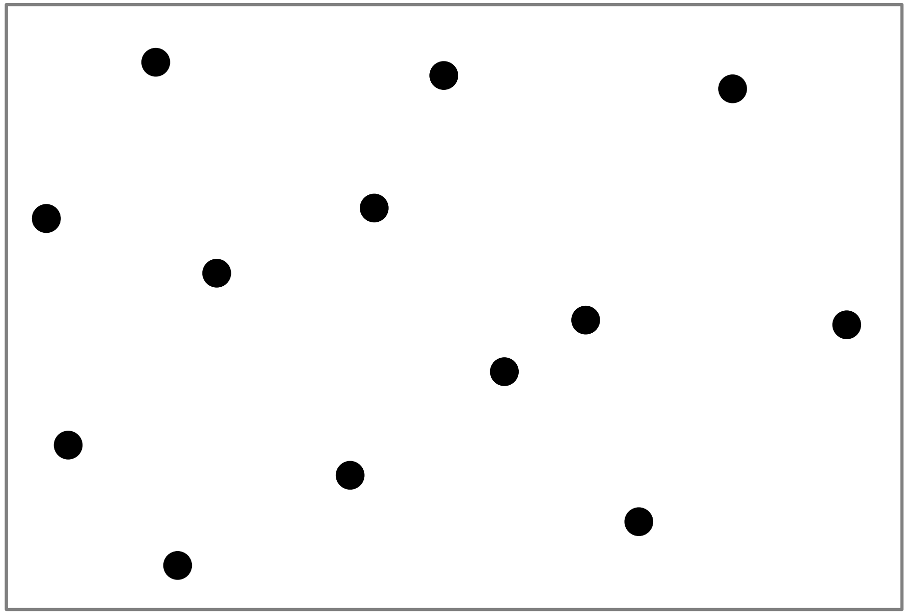

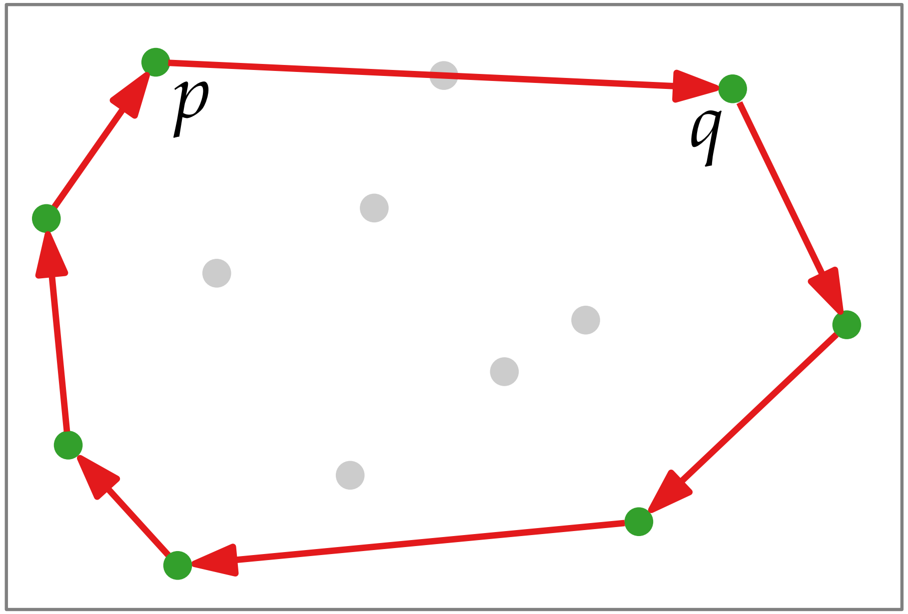

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Observation. (p, q) is an edge of $\mathrm{CH}(S)$ $\Leftrightarrow$

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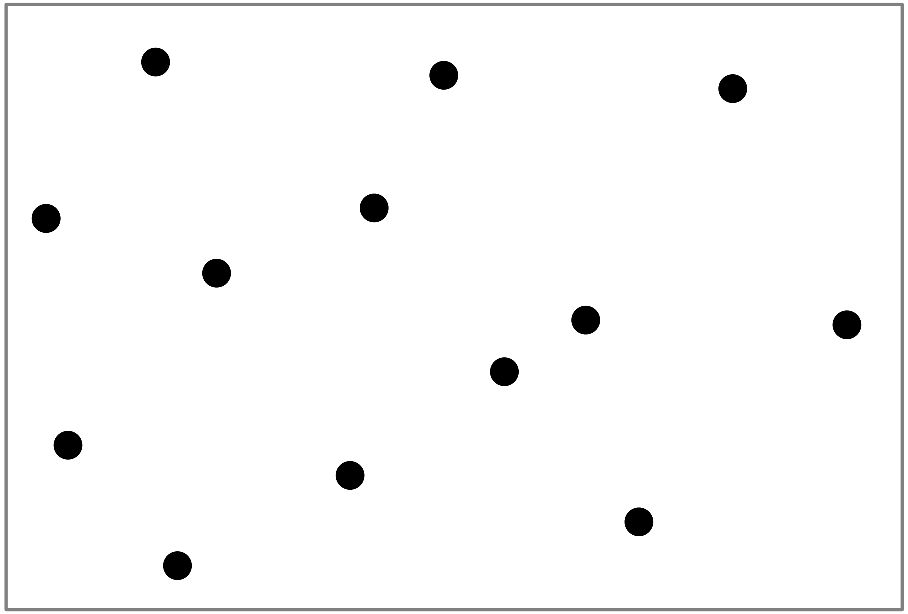

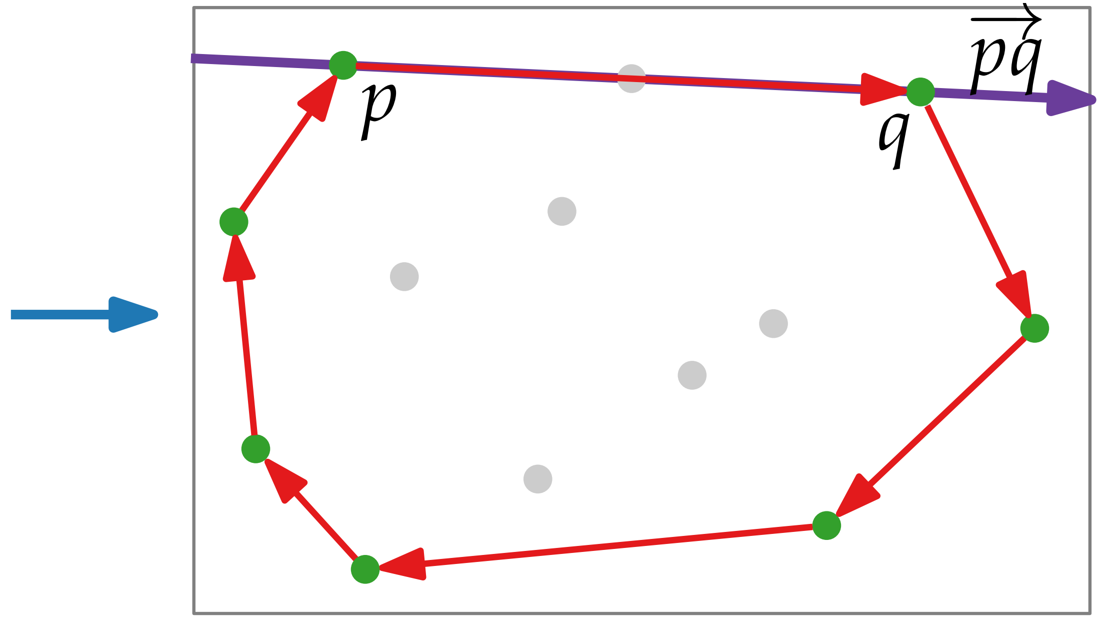

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Observation. (p, q) is an edge of $\mathrm{CH}(S)$ $\Leftrightarrow$
each point in S lies
– strictly to the right of the directed line $\overrightarrow{pq}$ or
– on the line segment $\overline{pq}$

Finally, an Algorithm

FirstConvexHull(S)

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$E \leftarrow \emptyset$

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from E construct sorted list L of vertices of CH(S)

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Finally, an Algorithm

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foreach $(p, q) \in S \times S$ with $p \neq q$ **do**

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FirstConvexHull(S)

$E \leftarrow \emptyset$

foreach $(p, q) \in S \times S$ with $p \neq q$ **do** // test: (p, q) edge of CH(S)?

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```
FirstConvexHull(S)
  E ← ∅
  foreach (p, q) ∈ S × S with p ≠ q do  // test: (p, q) edge of CH(S)?
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      if valid then
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r strictly right of $\overrightarrow{pq}$

$$\Updownarrow$$

$$\begin{vmatrix} x_r & y_r & 1 \\ x_p & y_p & 1 \\ x_q & y_q & 1 \end{vmatrix} < 0$$

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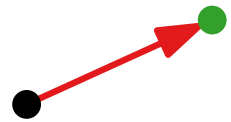

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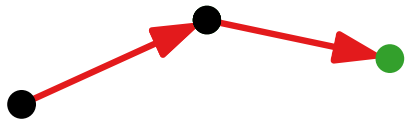

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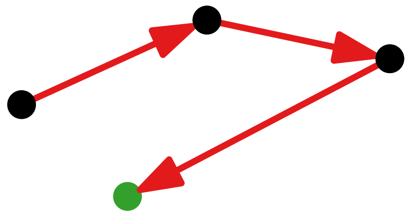

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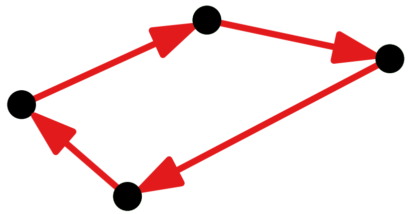

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$\quad\quad$ **if not** (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then** $\Theta(1)$

$\quad\quad\quad$ *valid* $\leftarrow$ *false*

$\quad$ **if** *valid* **then**

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from E construct sorted list L of vertices of CH(S)

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Running Time Analysis

FirstConvexHull(S)

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foreach $(p, q) \in S \times S$ with $p \neq q$ **do**

  valid $\leftarrow$ *true*

  foreach $r \in S$ **do** $n \cdot$

    if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then** $\Theta(1)$

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Running Time Analysis

FirstConvexHull(S)

$E \leftarrow \emptyset$

foreach $(p, q) \in S \times S$ with $p \neq q$ **do** $(n^2 - n) \cdot$

valid $\leftarrow$ *true*

foreach $r \in S$ **do** $n \cdot$

if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then** $\Theta(1)$ $\Theta(n)$

valid $\leftarrow$ *false*

if *valid* **then**

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$\Theta(n)$

$\Theta(n^3)$

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Running Time Analysis

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valid $\leftarrow$ *true*

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valid $\leftarrow$ *false*

$\Theta(n)$

$\Theta(n^3)$

if *valid* **then**

$E \leftarrow E \cup \{(p, q)\}$

from E construct sorted list L of vertices of CH(S) $O(n^2)$

return L

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$\Theta(n^3)$

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$O(n^2)$

Lemma. We can compute the convex hull of n pts in the plane in $\Theta(n^3)$ time.

Discussion

if not (r strictly right of $\overrightarrow{pq}$ **or** $r \in \overline{pq}$) **then**
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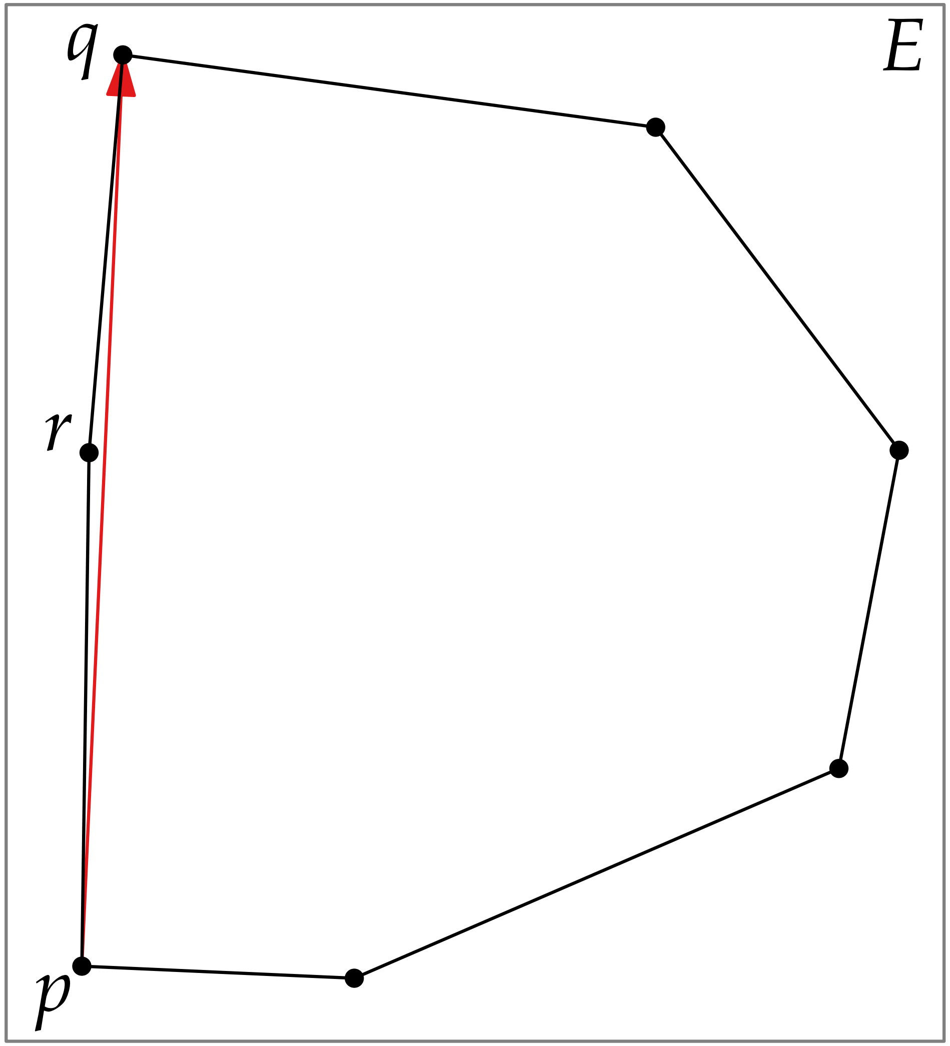

Discussion

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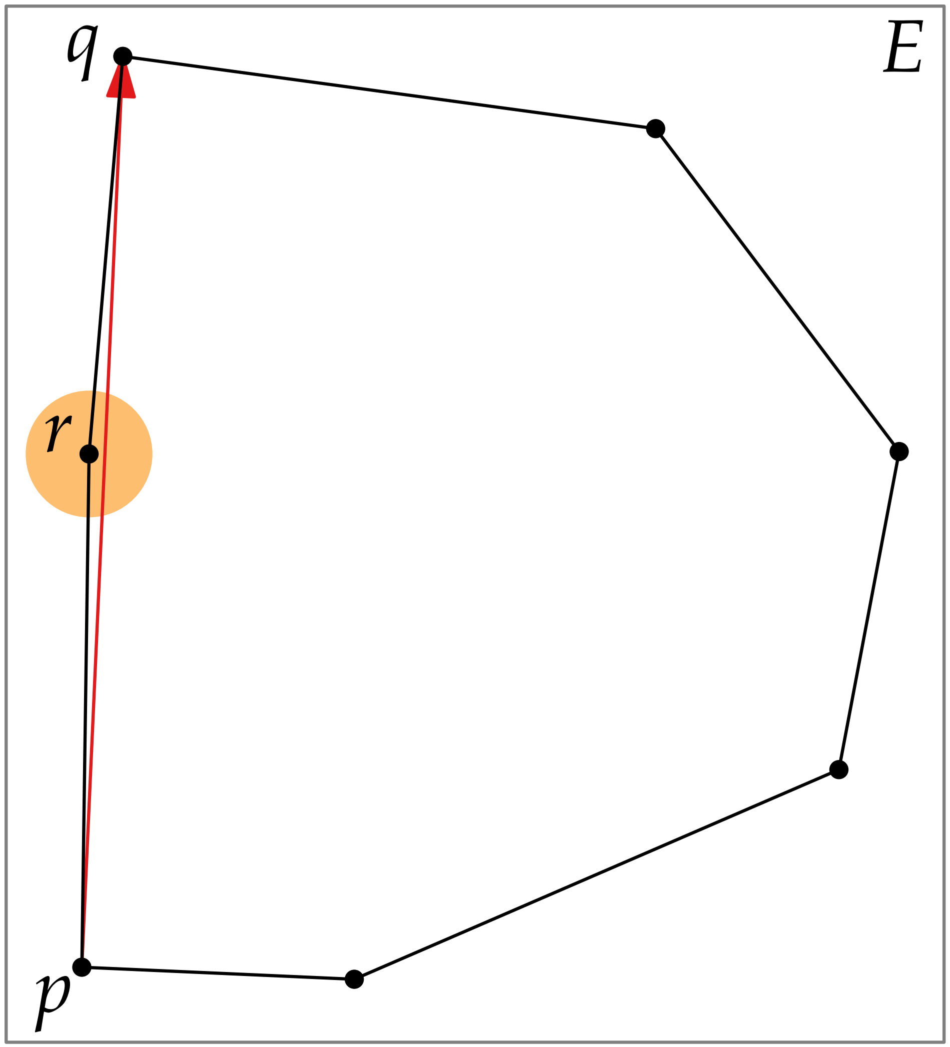

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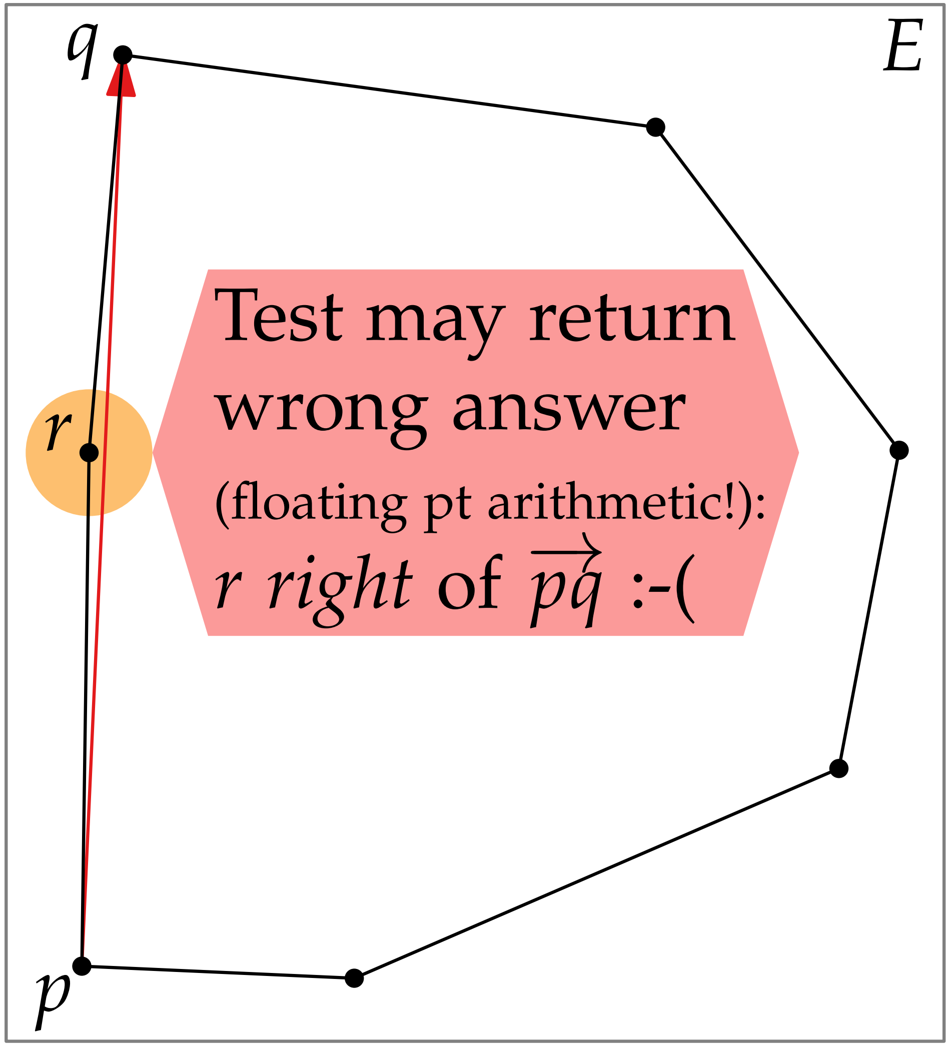

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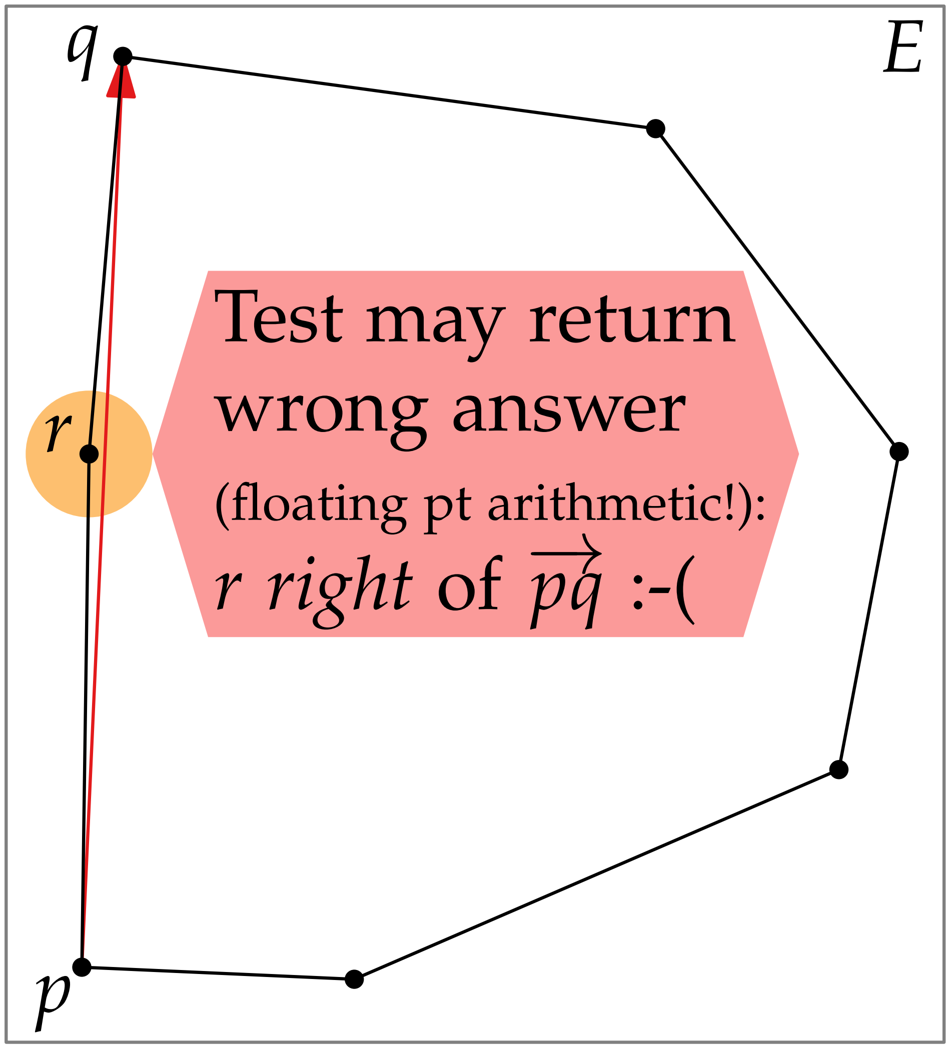

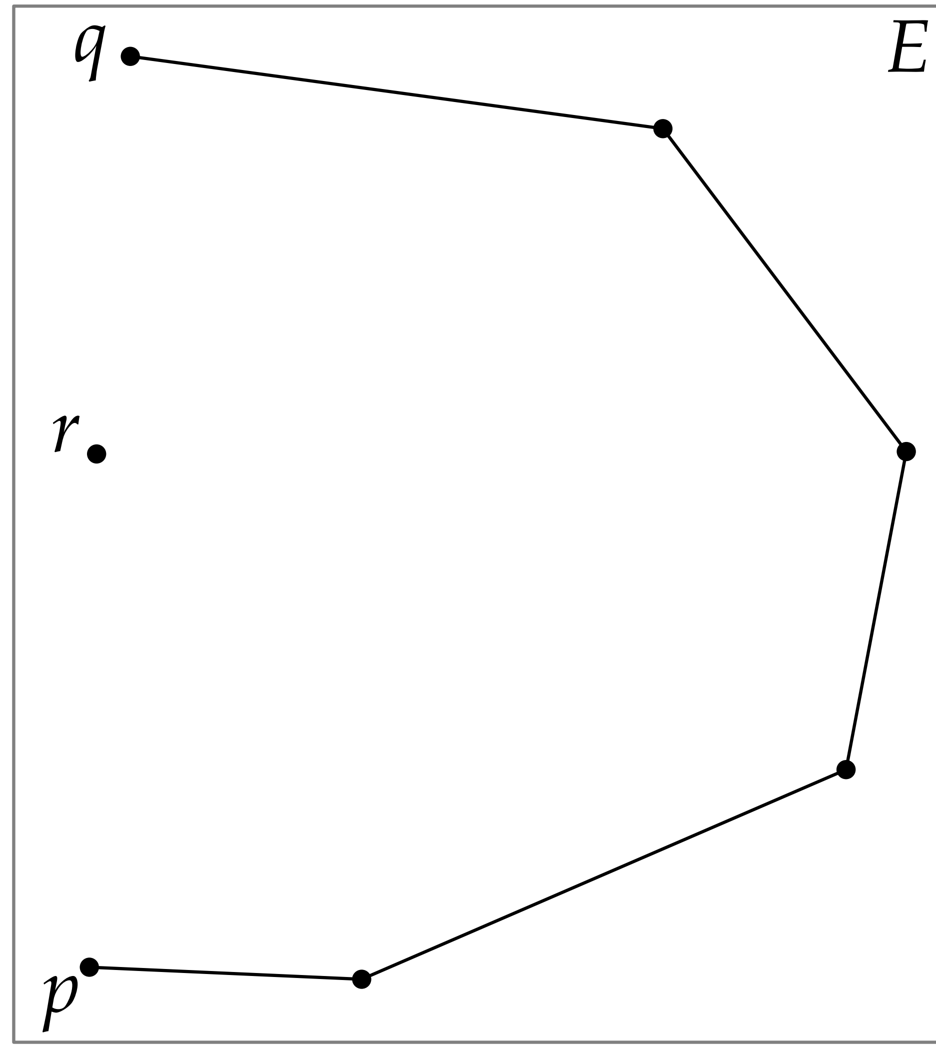

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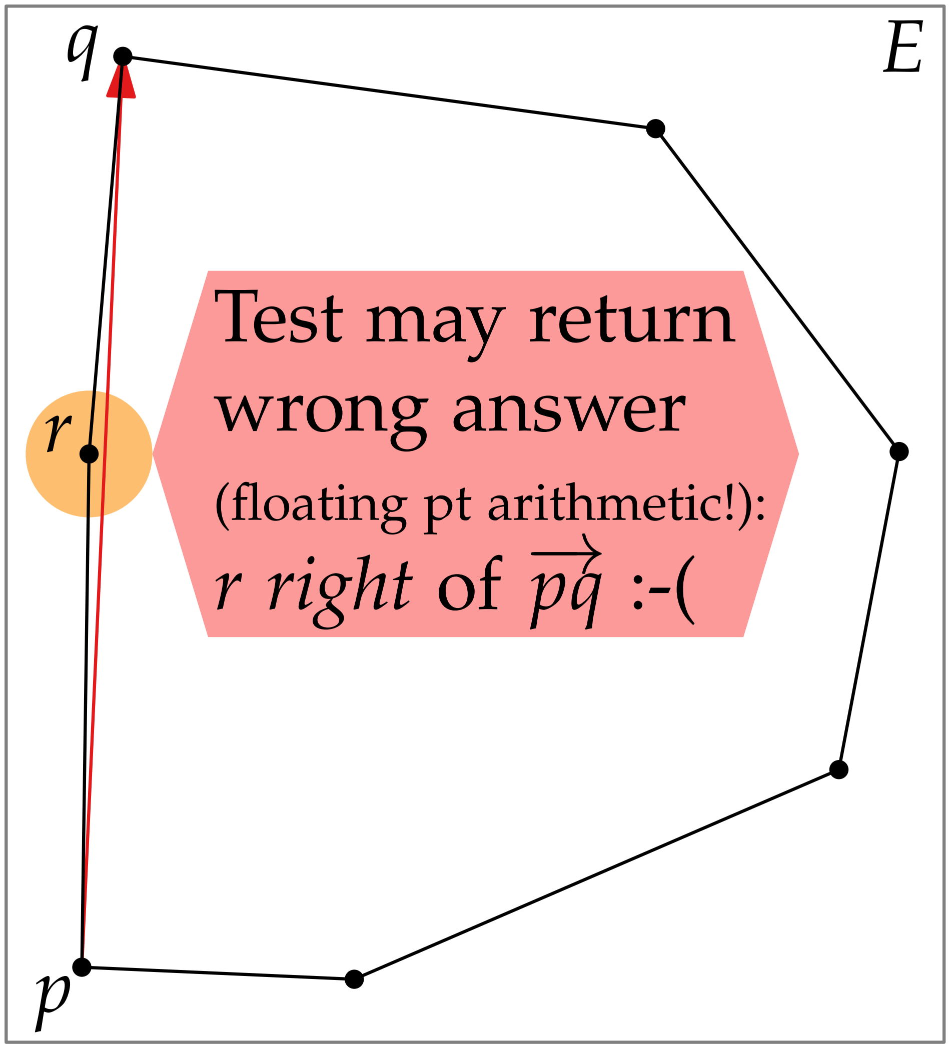

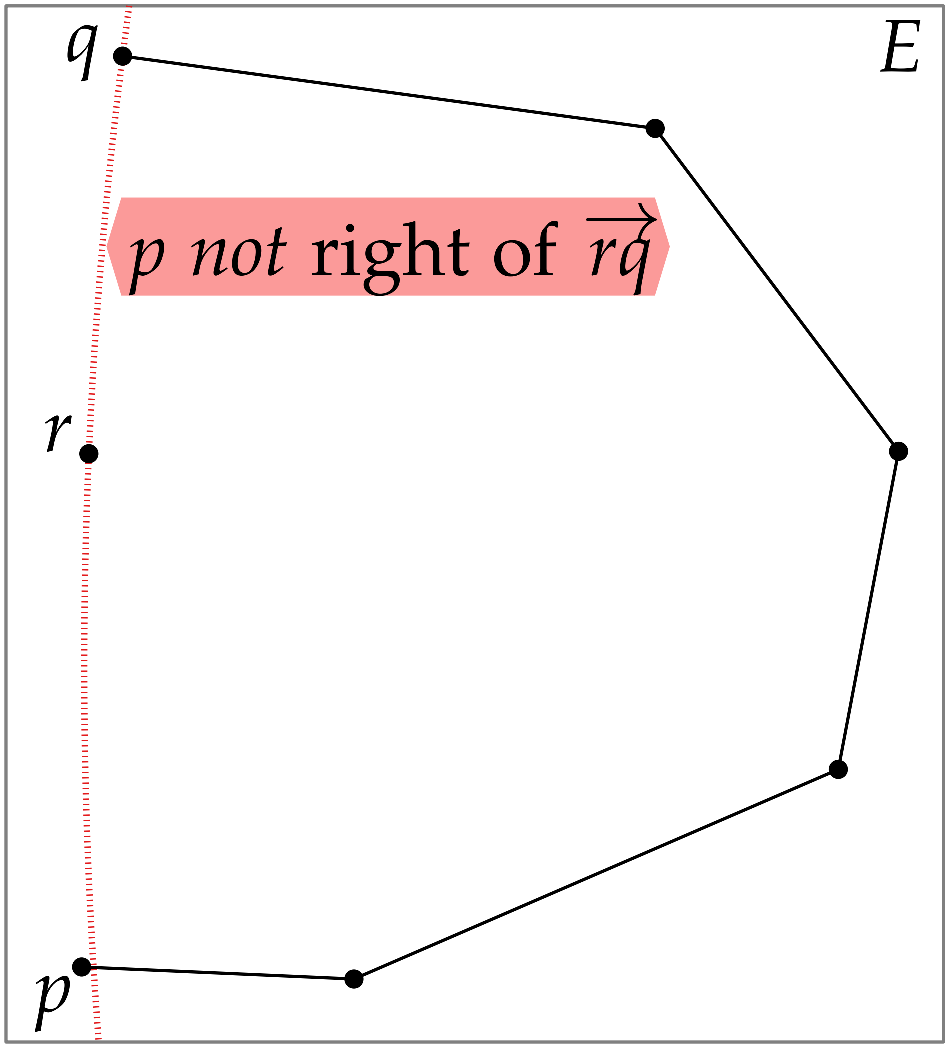

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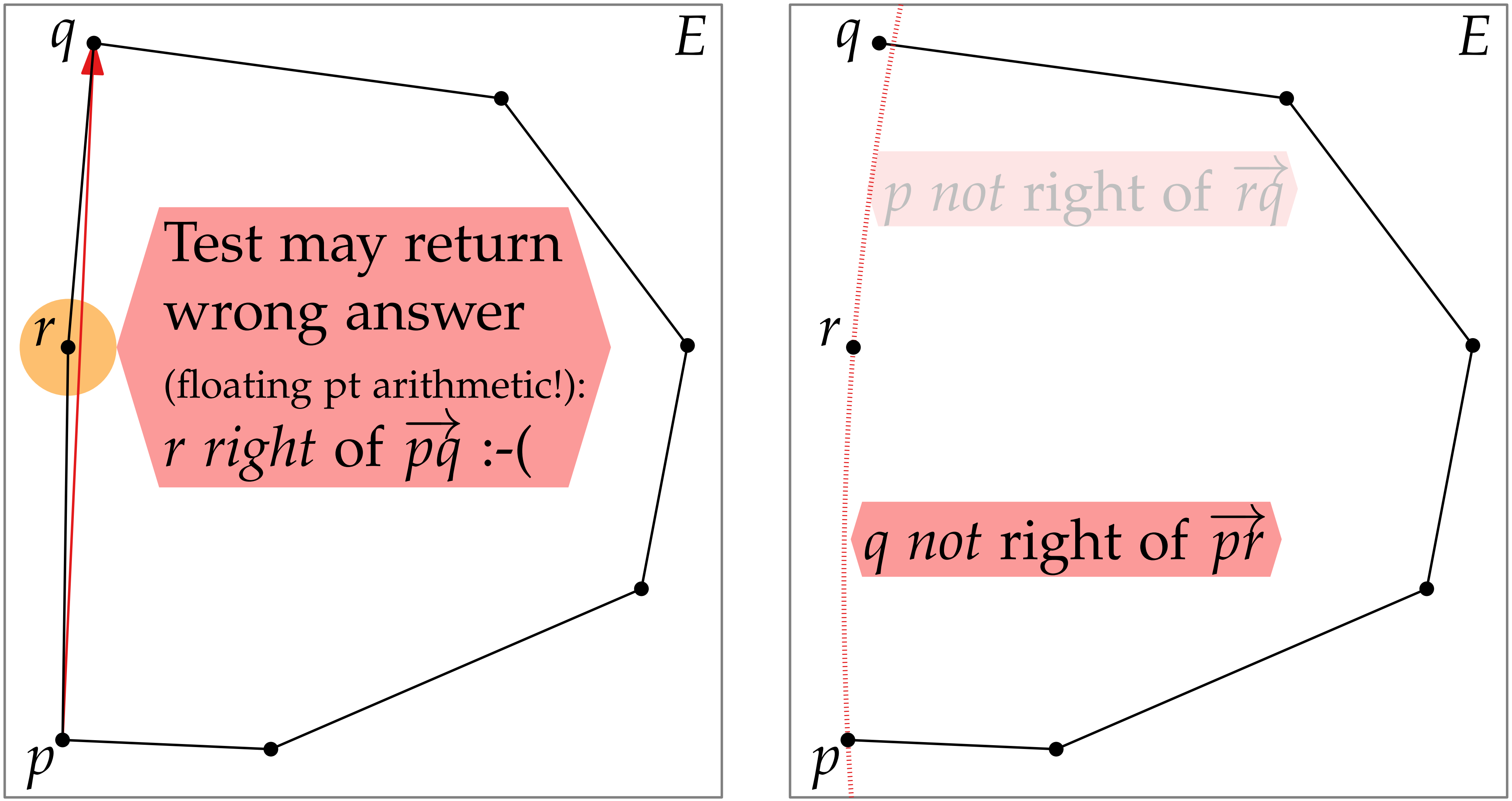

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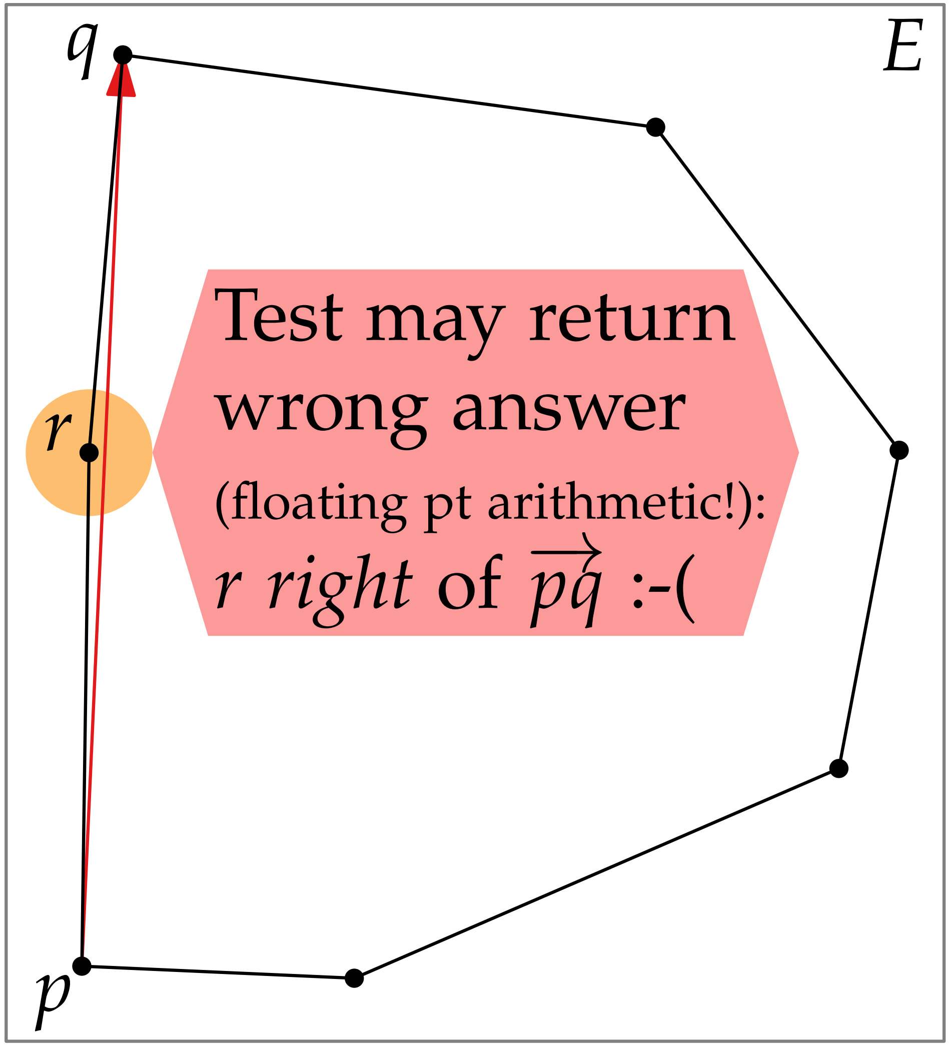

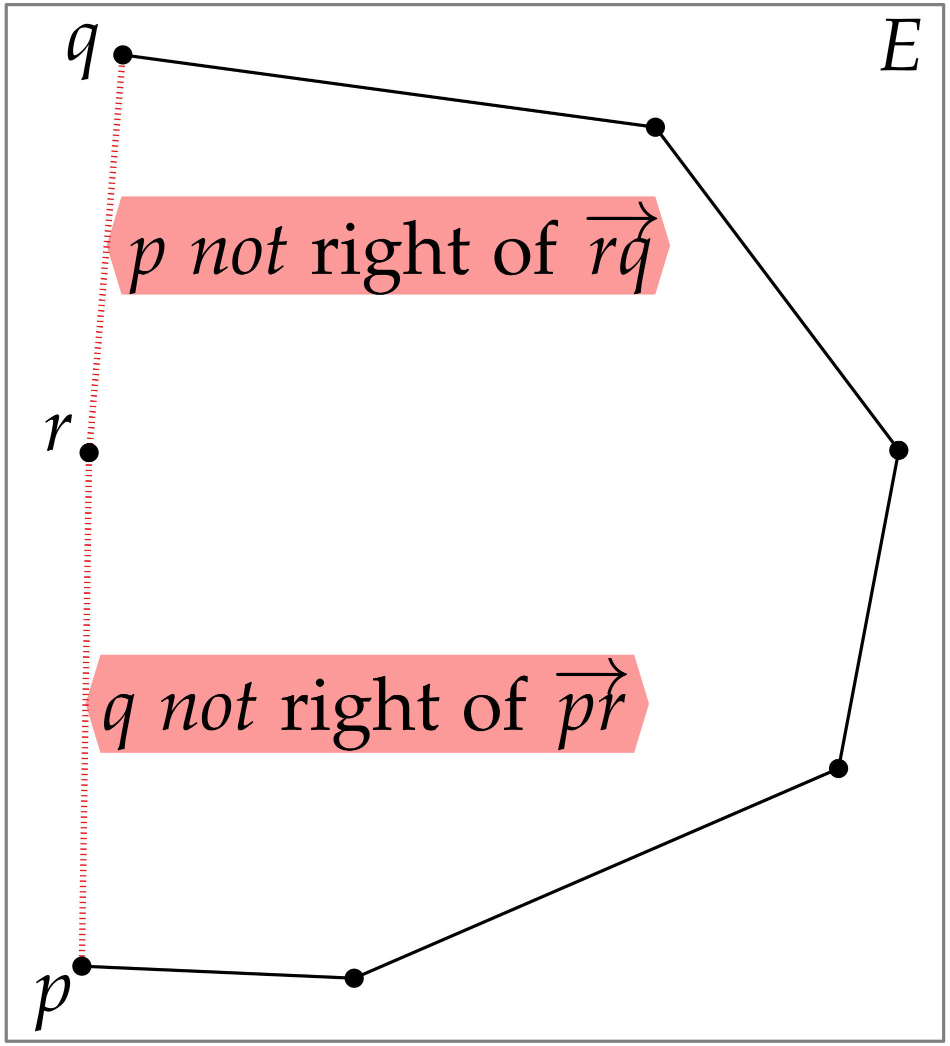

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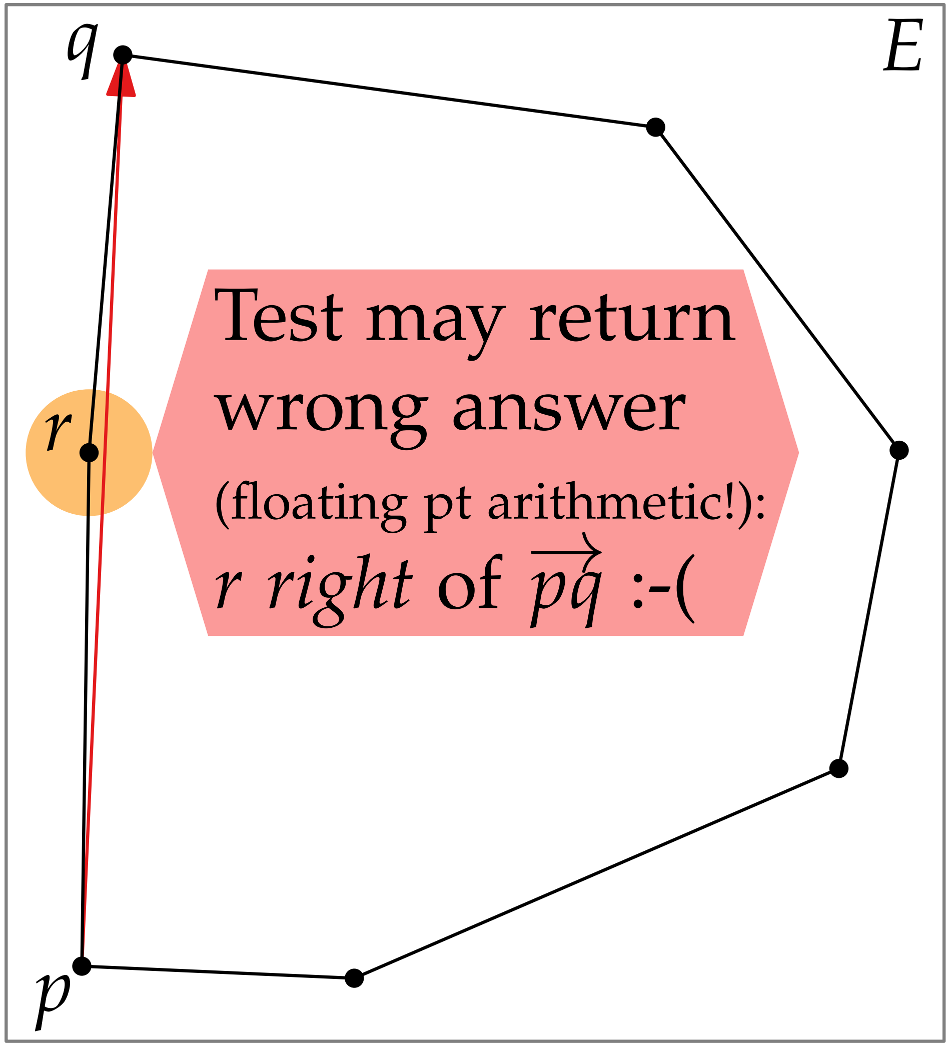
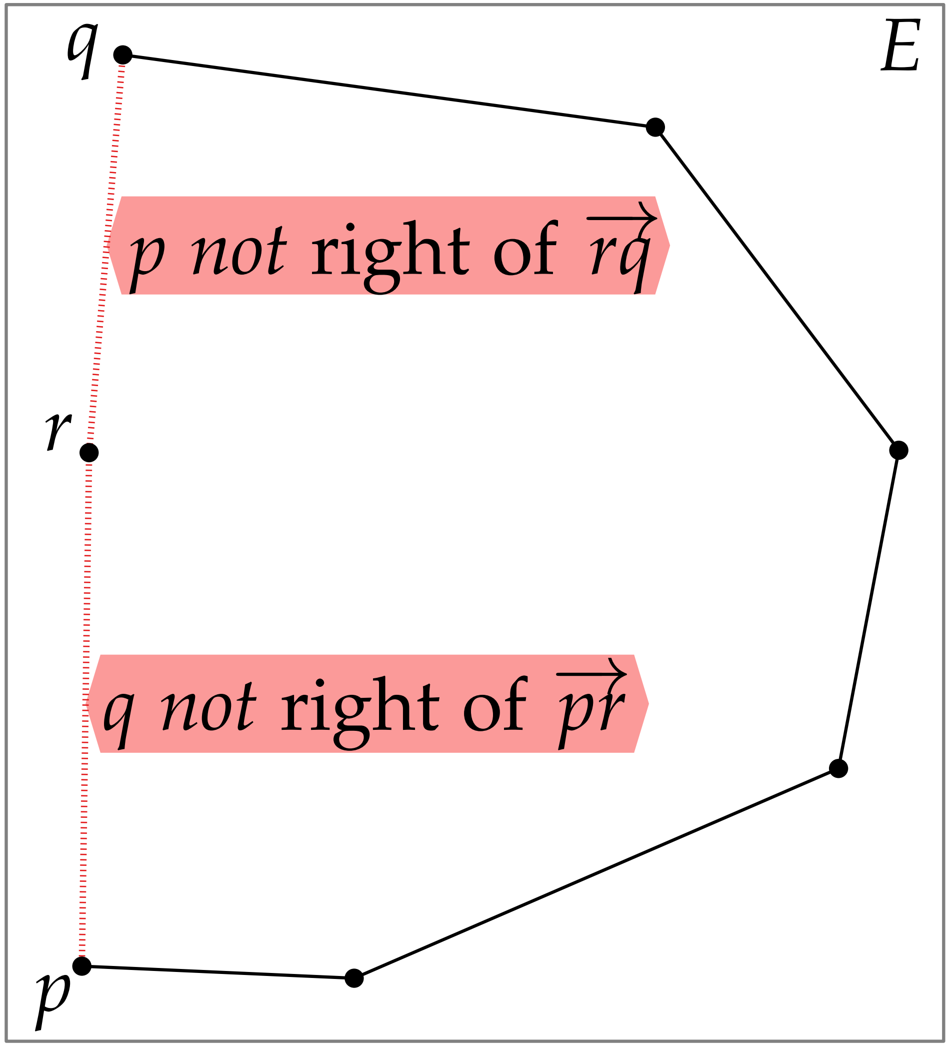




Observation. Algorithm FirstConvexHull is not *robust*.

Computational Geometry

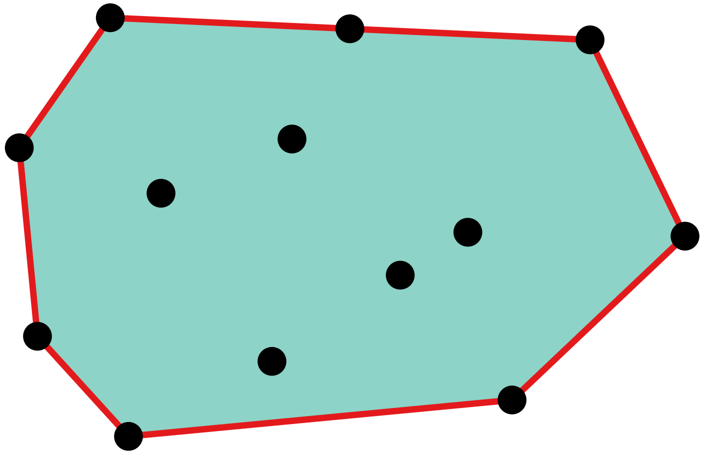
Lecture 1:
Convex Hull or Mixing Things

Part IV:
Graham Scan

Philipp Kindermann

Winter Semester 2020

New Ideas (Graham Scan)



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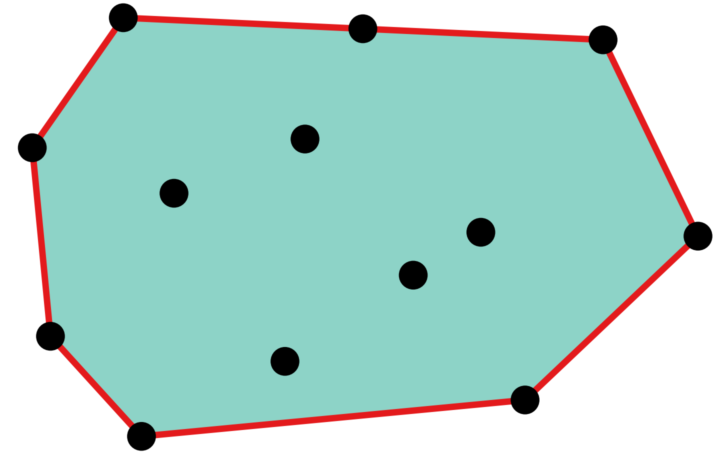

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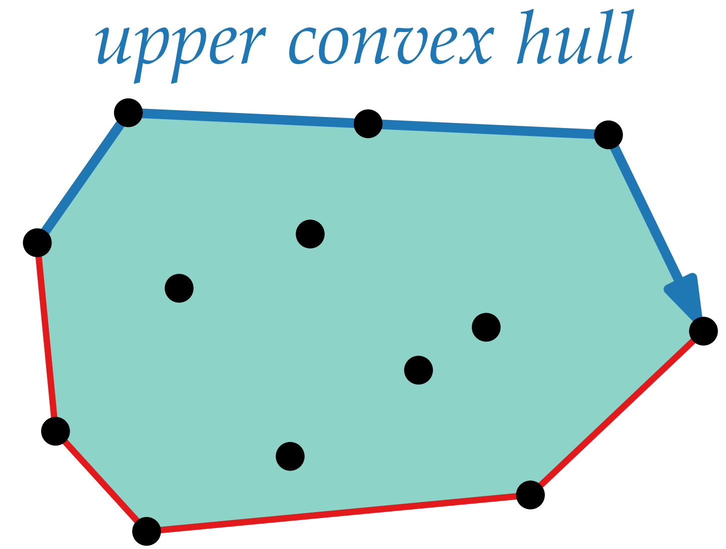

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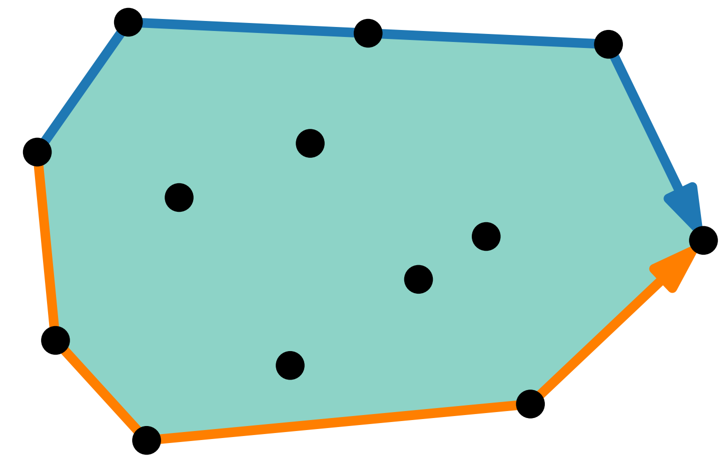

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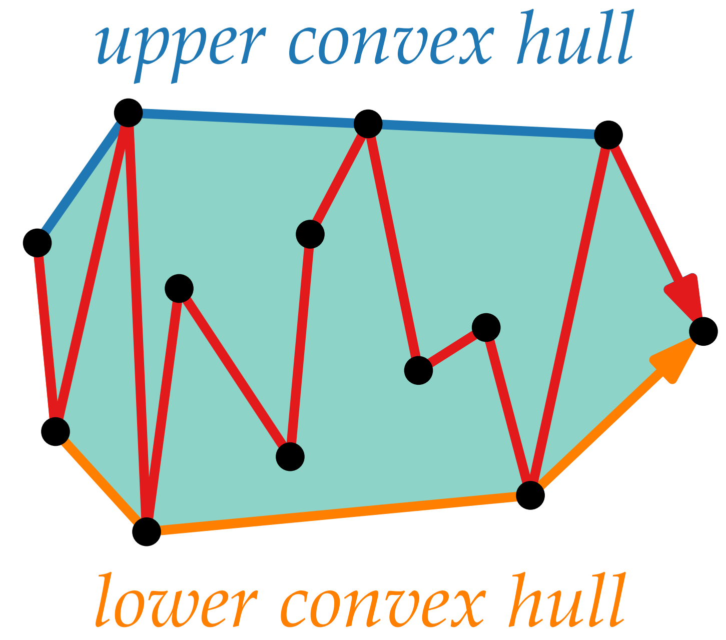

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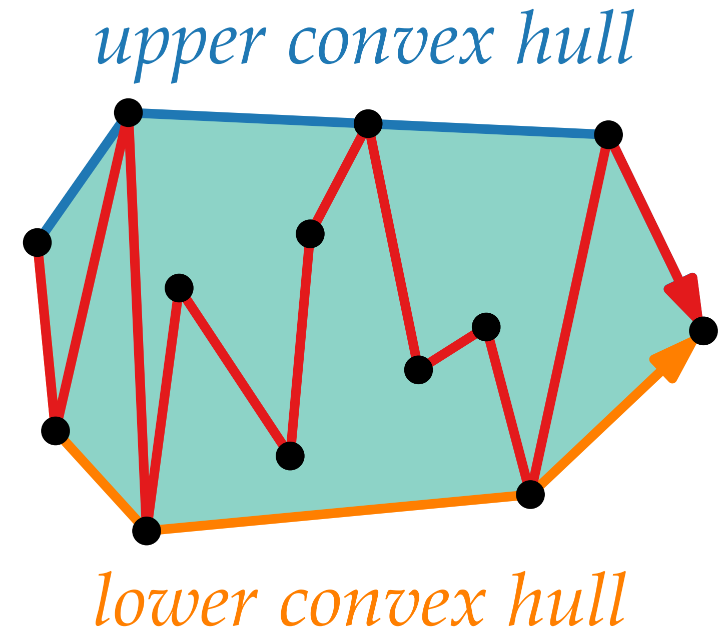

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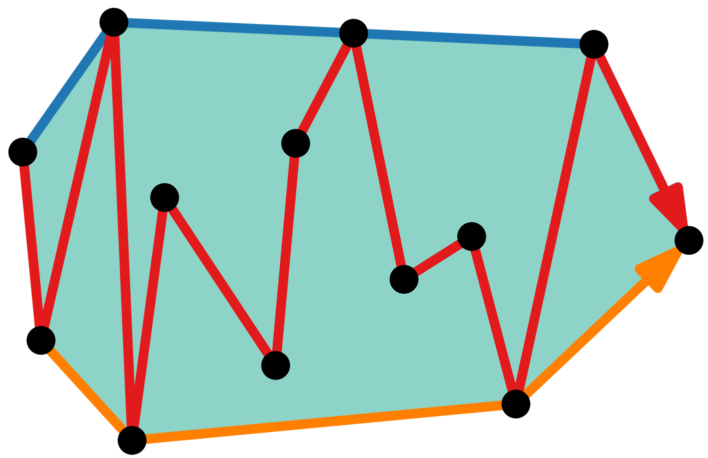

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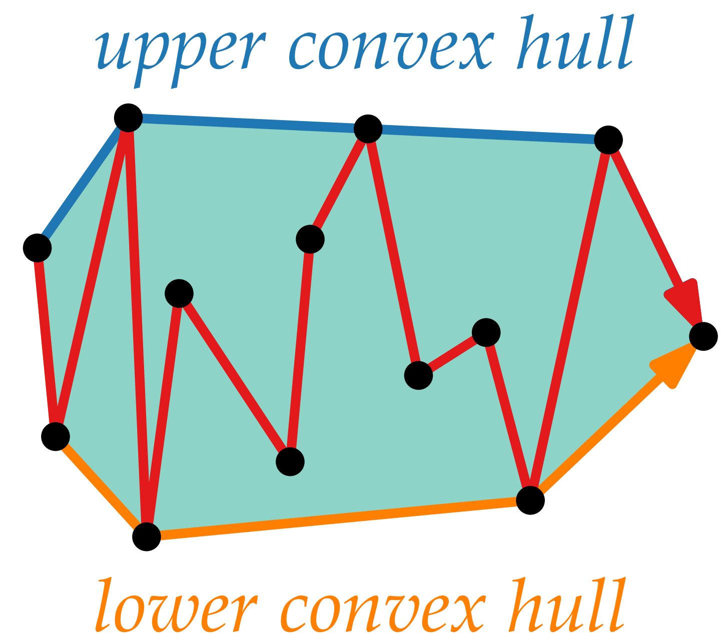

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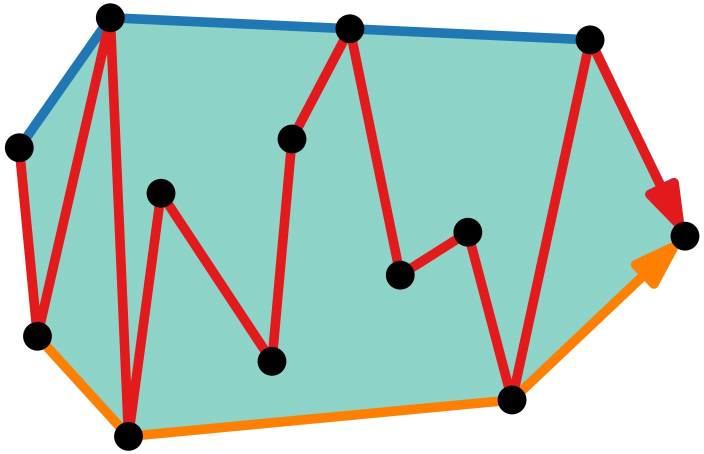

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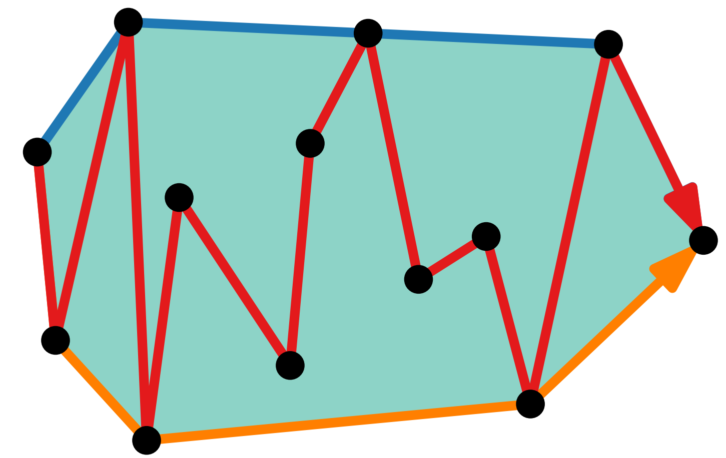

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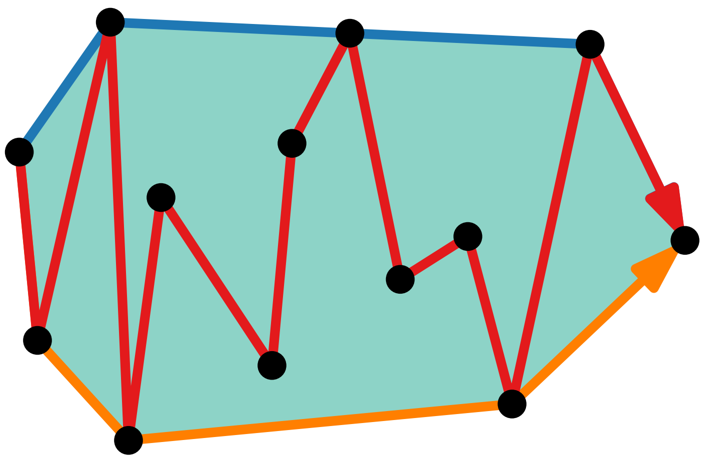

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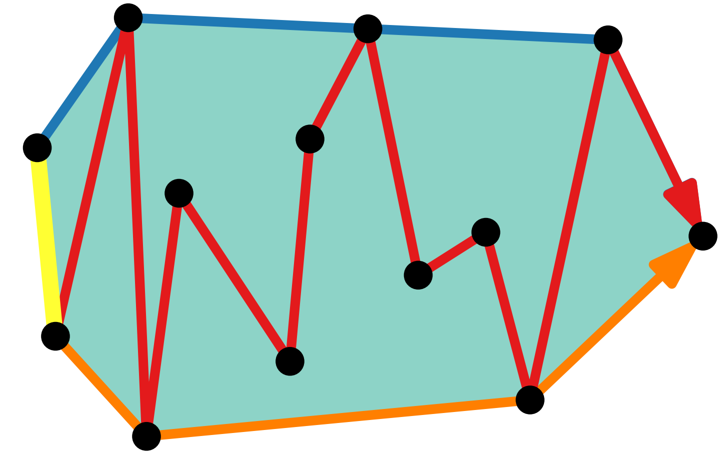

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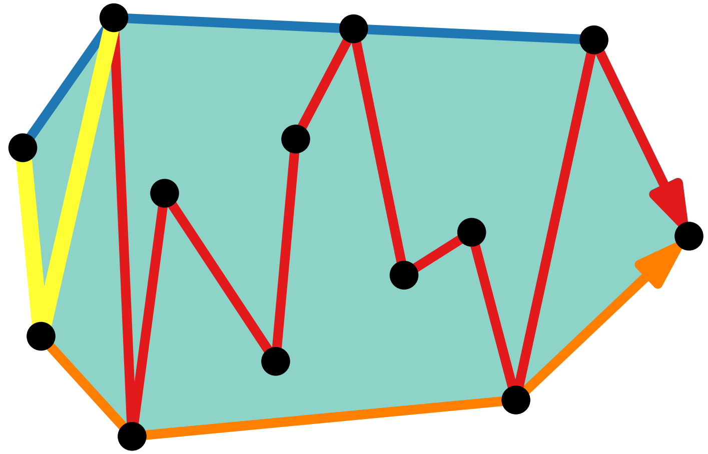

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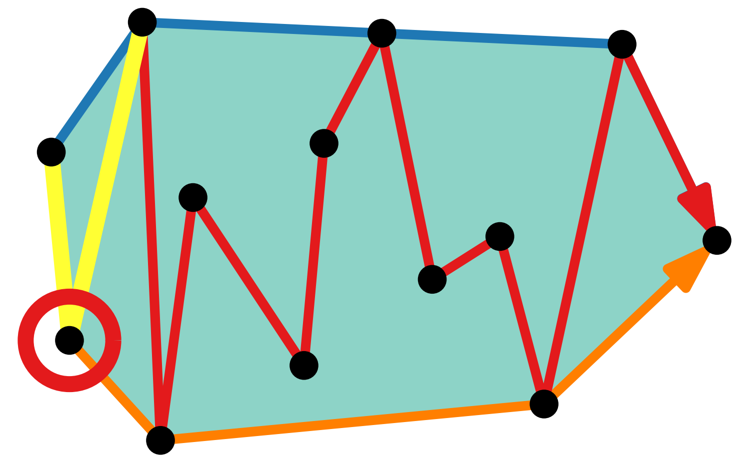

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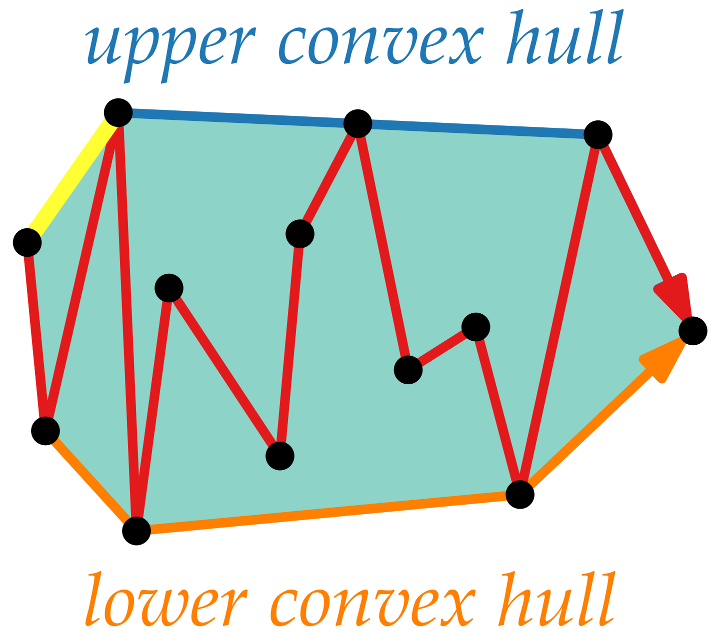

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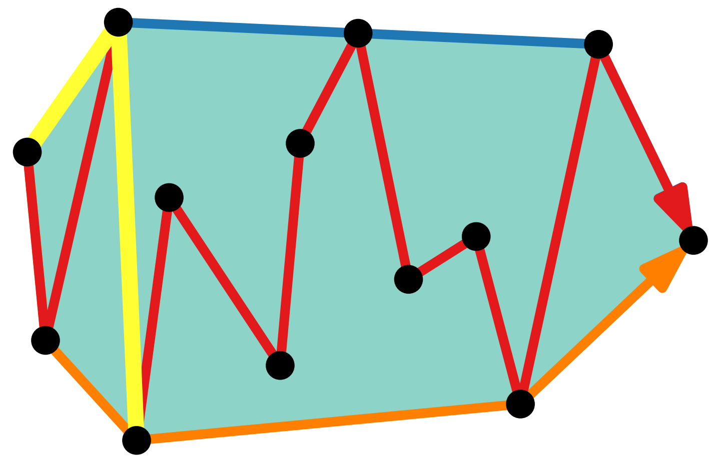

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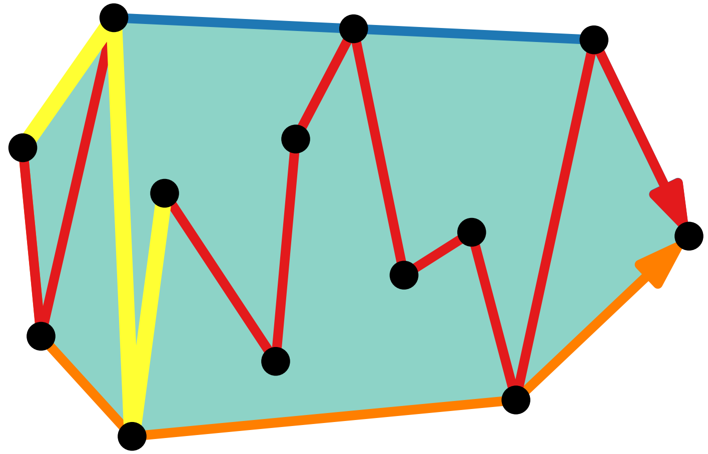

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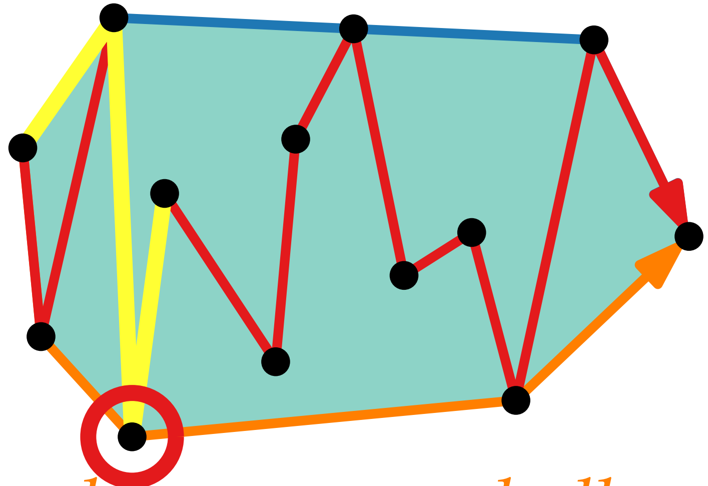

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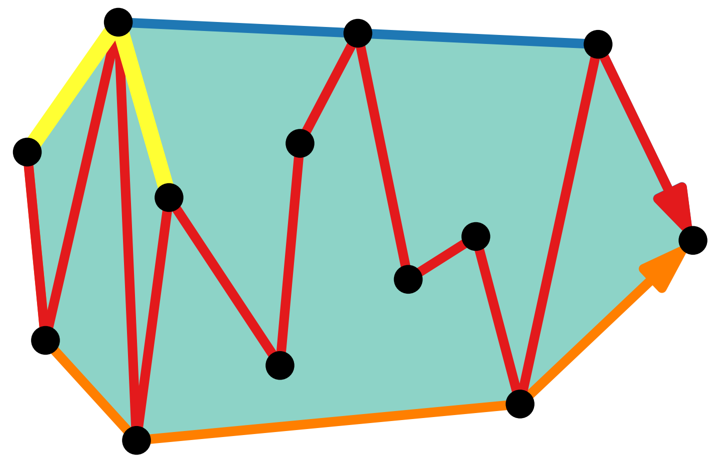

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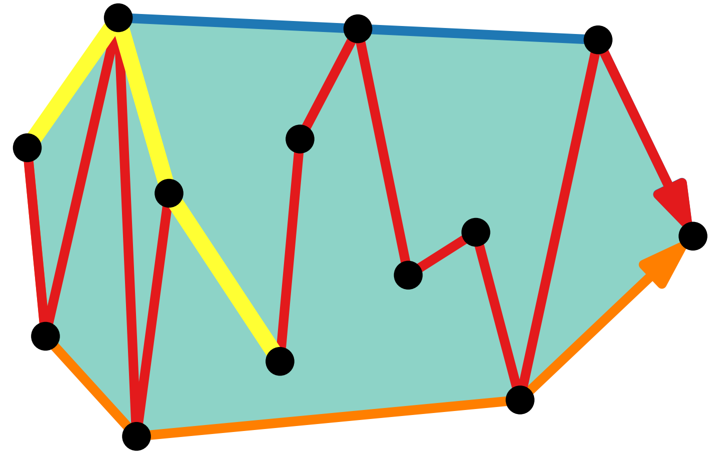

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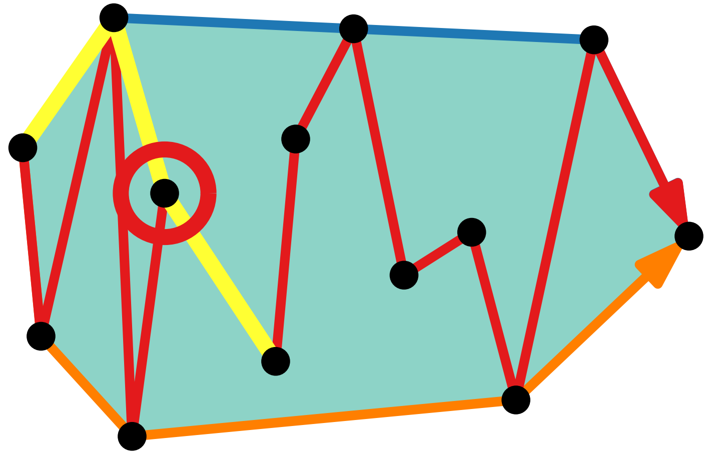

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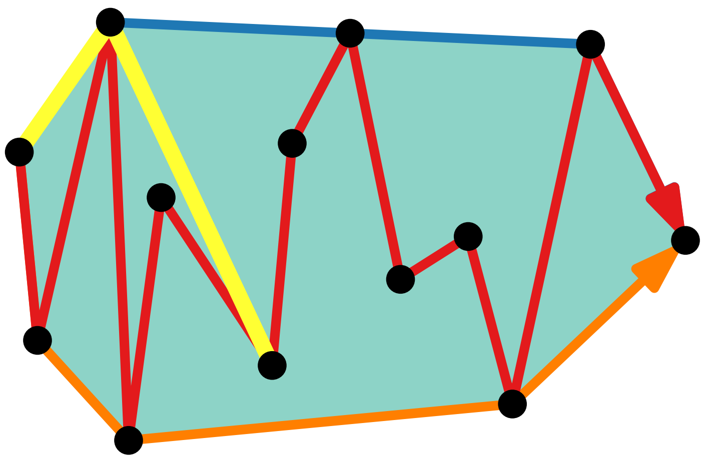

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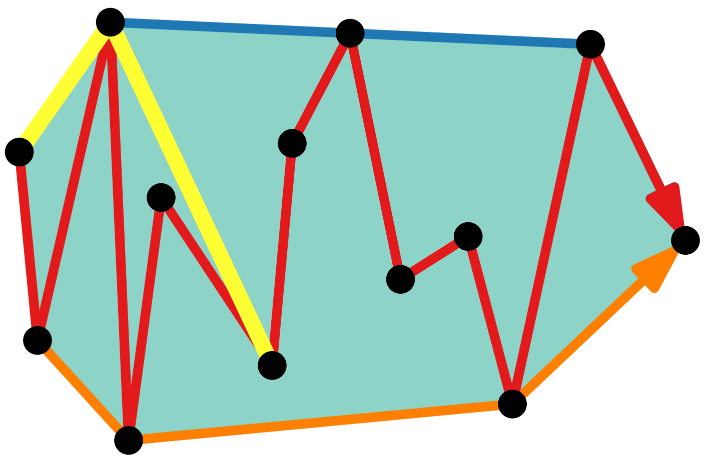

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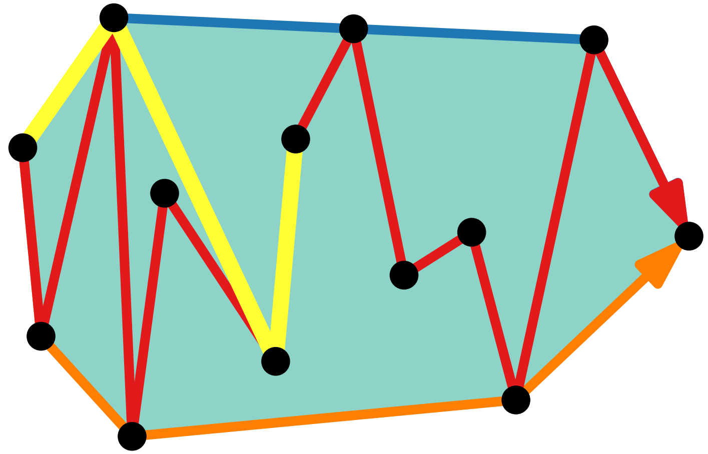

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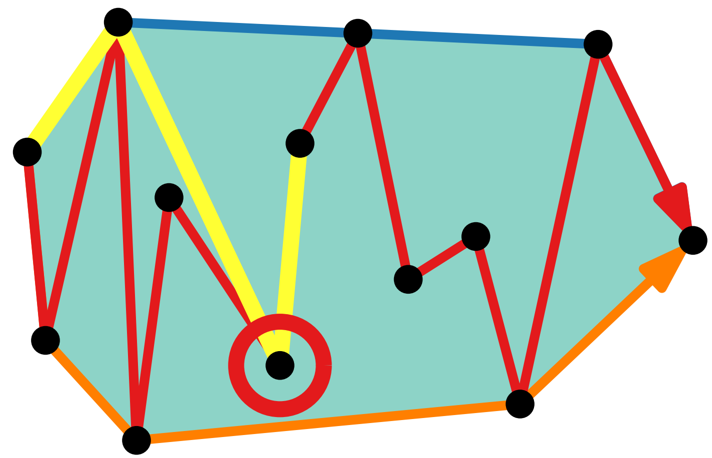

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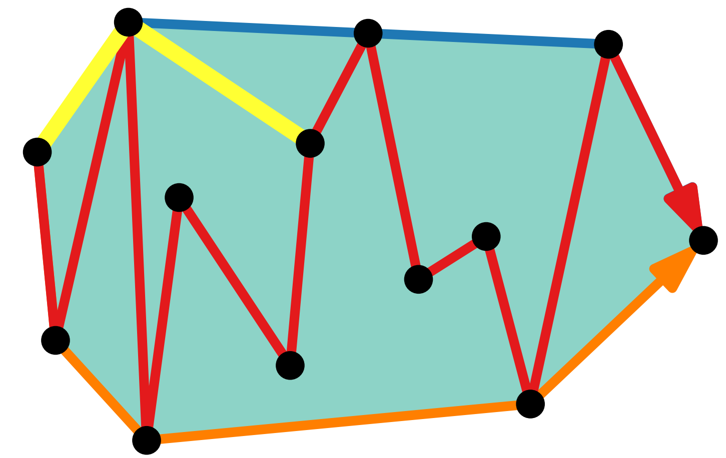

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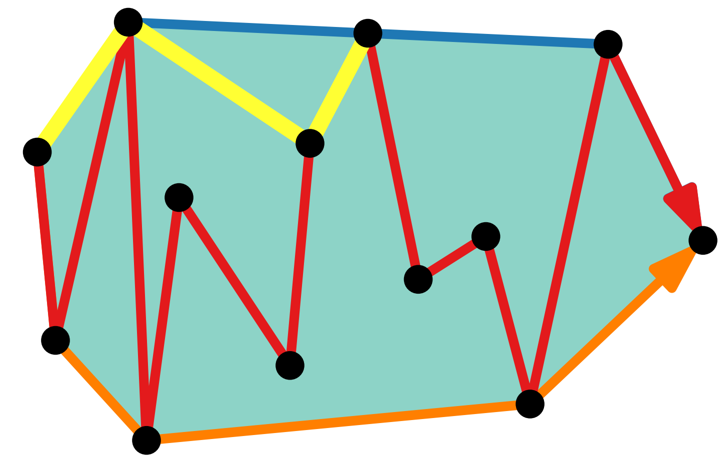

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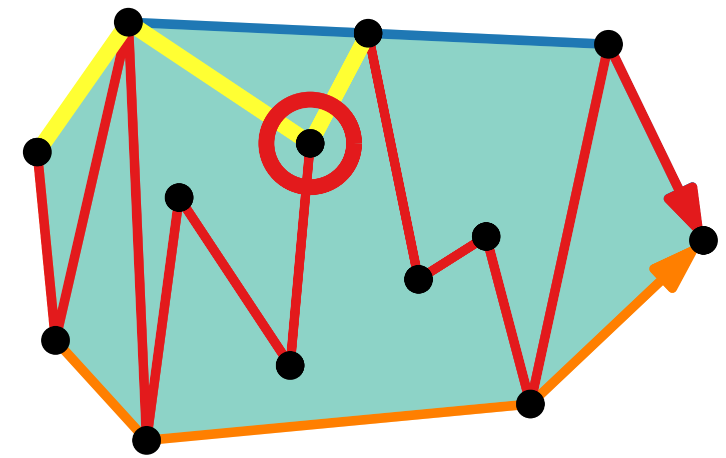

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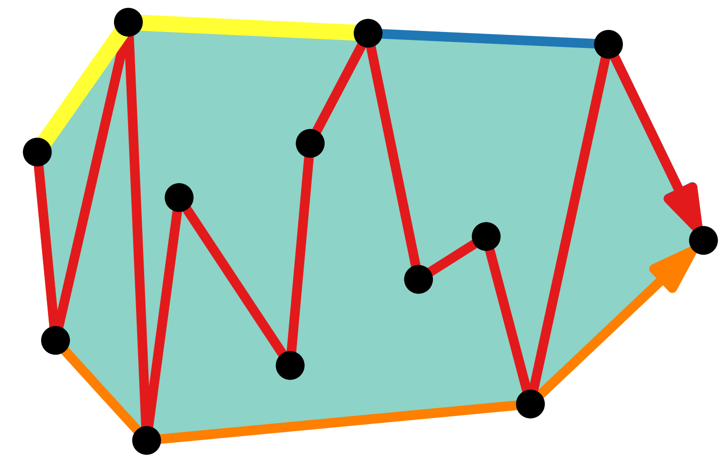

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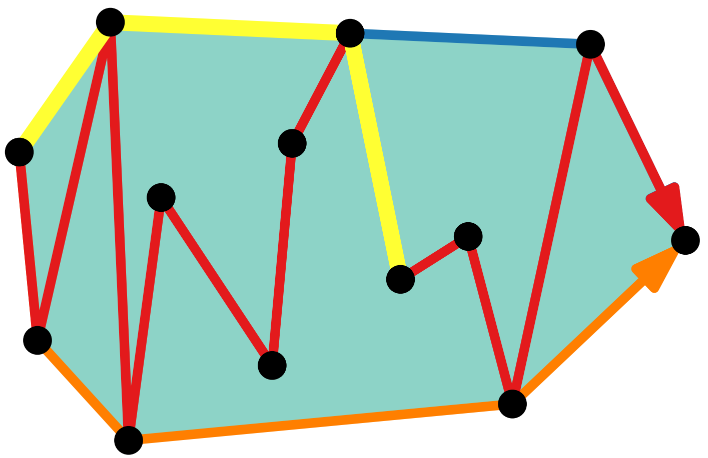

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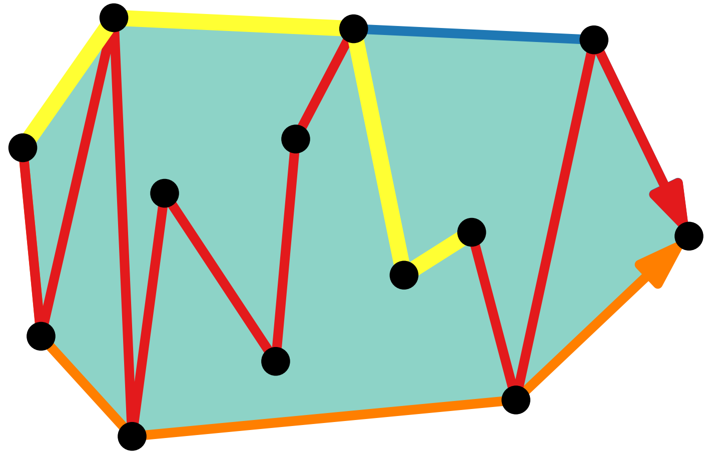

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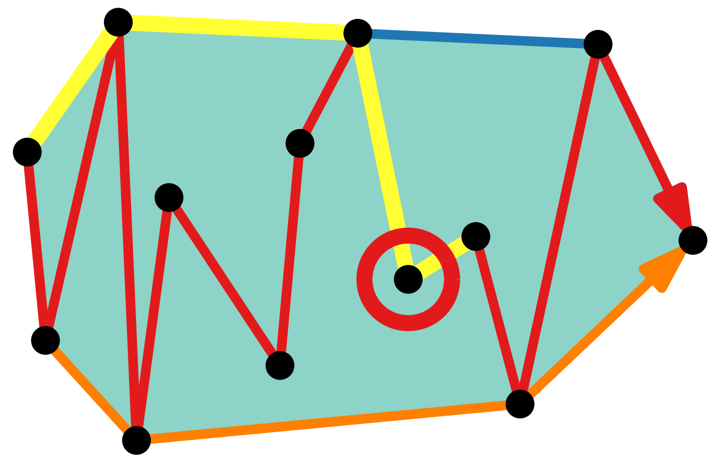

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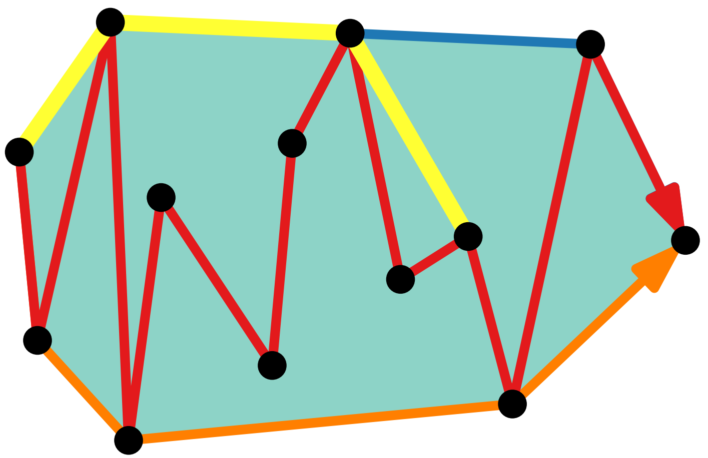

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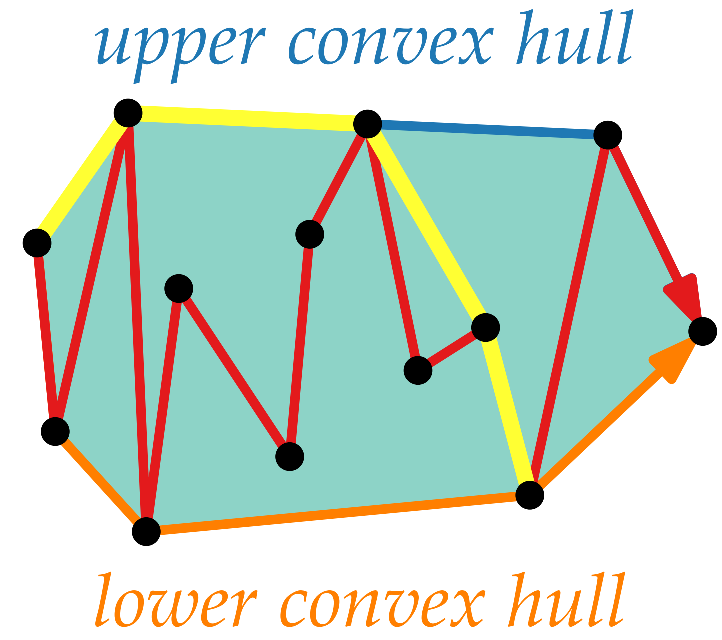

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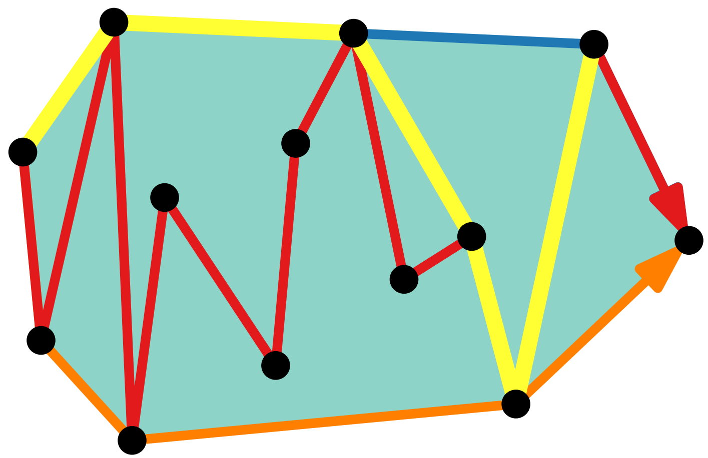

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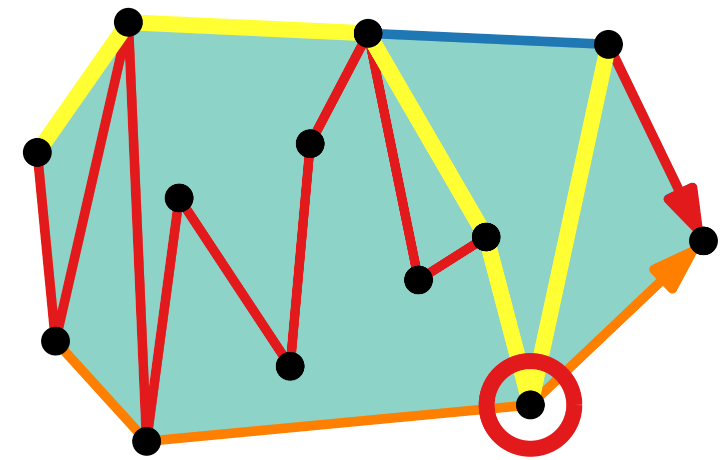

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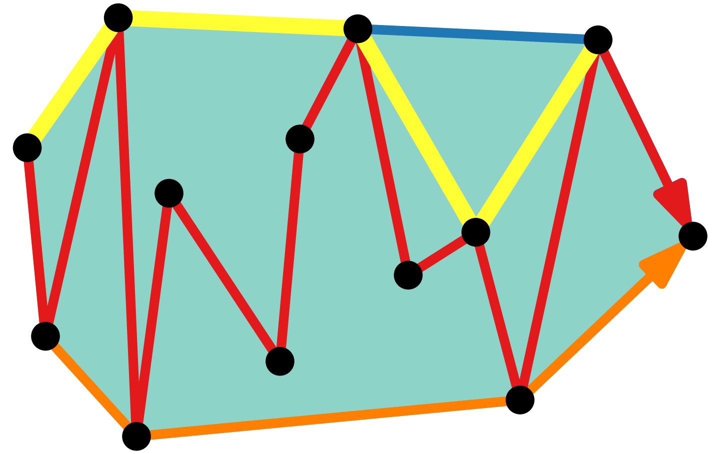

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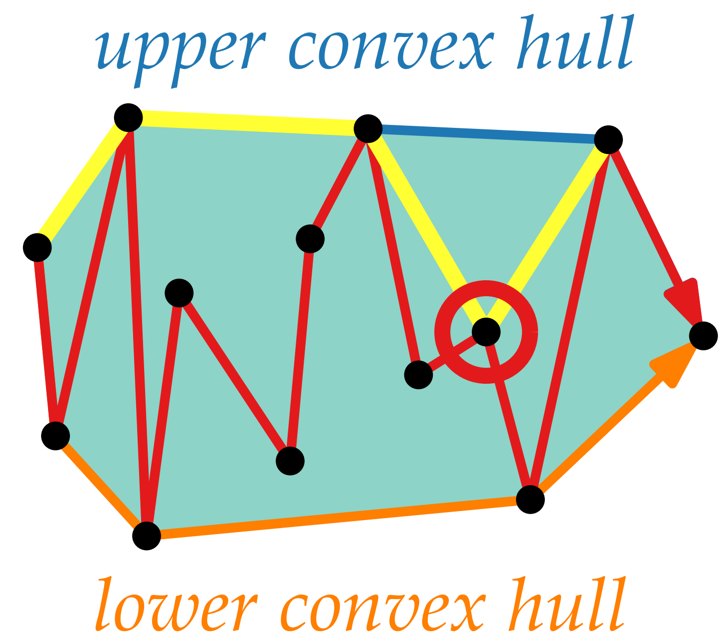

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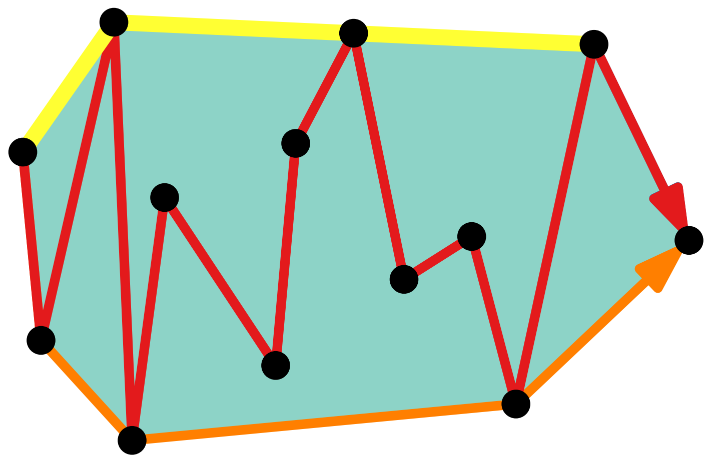

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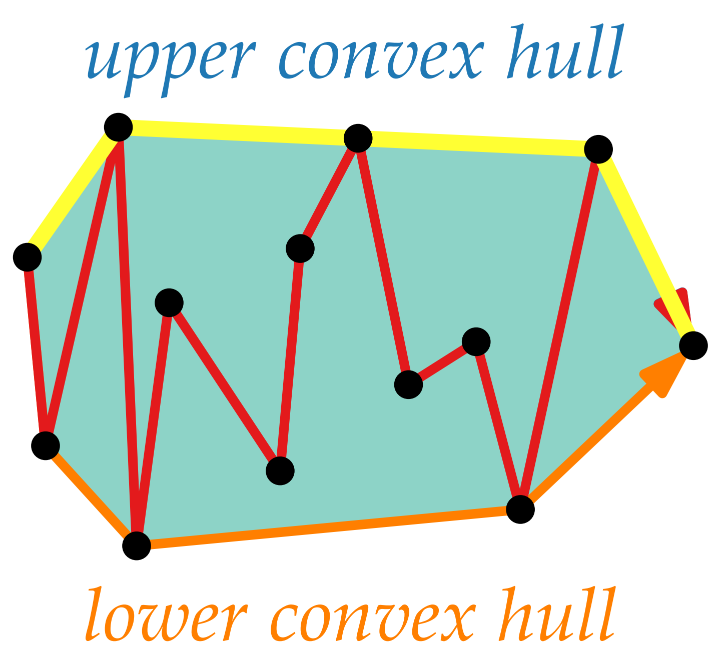

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UpperConvexHull(S: set of pts in the plane)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort S lexicographically $\quad O(n \log n)$

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for $i \leftarrow 3$ **to** n **do** $\quad (n-2)\cdot$

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Running Time Analysis

UpperConvexHull(S: set of pts in the plane)

$\langle p_1, p_2, \ldots, p_n \rangle \leftarrow$ sort S lexicographically — $O(n \log n)$

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for $i \leftarrow 3$ **to** n **do** — $(n-2)\cdot$

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UpperConvexHull(S: set of pts in the plane)

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Amortized analysis:

Running Time Analysis

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– each pt $p_2, \dots, p_{n-1}$ pays 1 € for its potential removal later on

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Theorem. We can compute the convex hull of n pts in the plane in $O(n \log n)$ time – in a robust way.

Computational Geometry

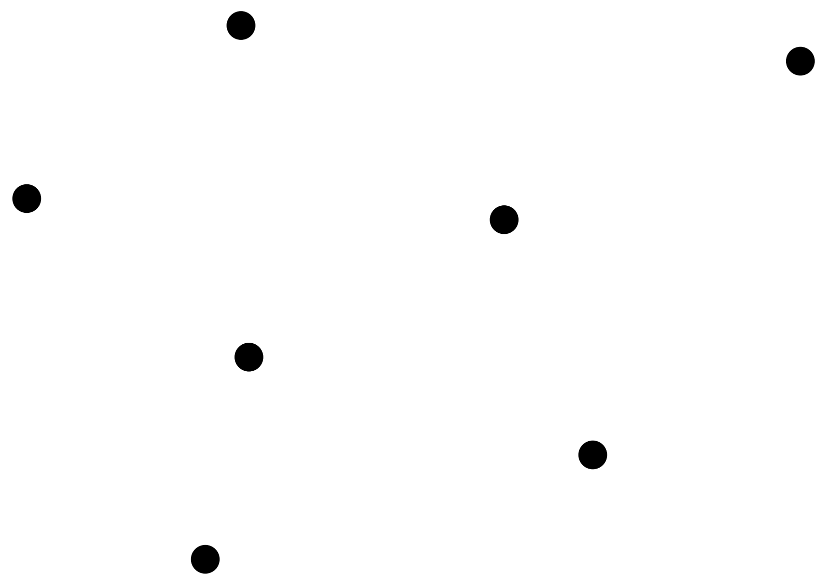
Lecture 1:
Convex Hull or Mixing Things

Part V:
Output-Sensitive Algorithms

Philipp Kindermann Winter Semester 2020

Output-Sensitive Algorithms

- Jarvis' gift-wrapping algorithm (aka Jarvis' march)



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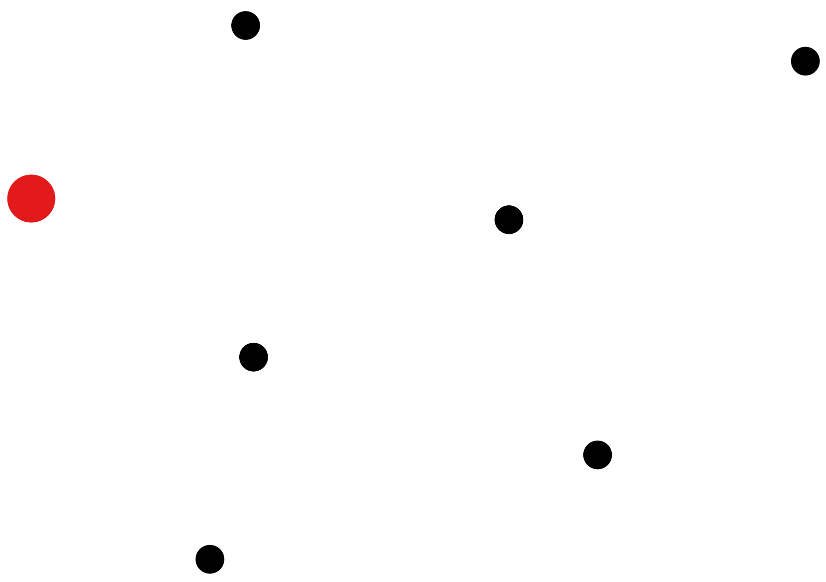

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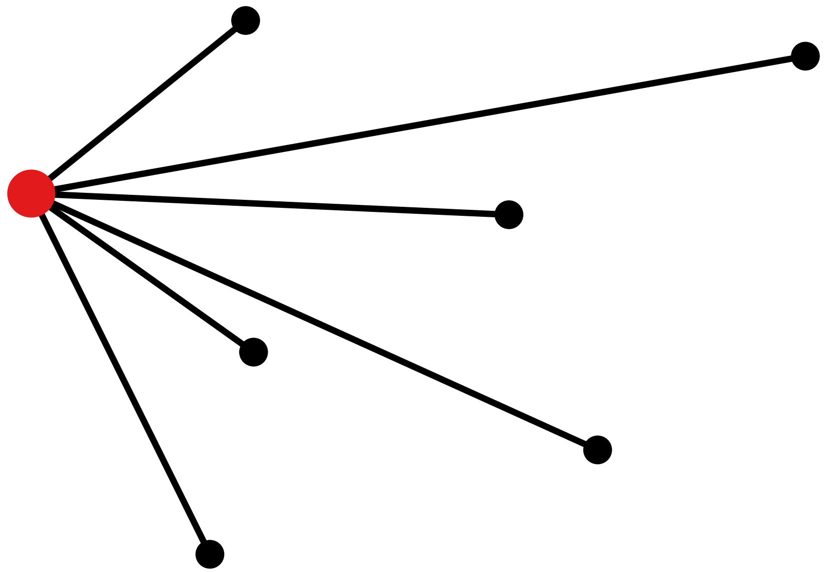

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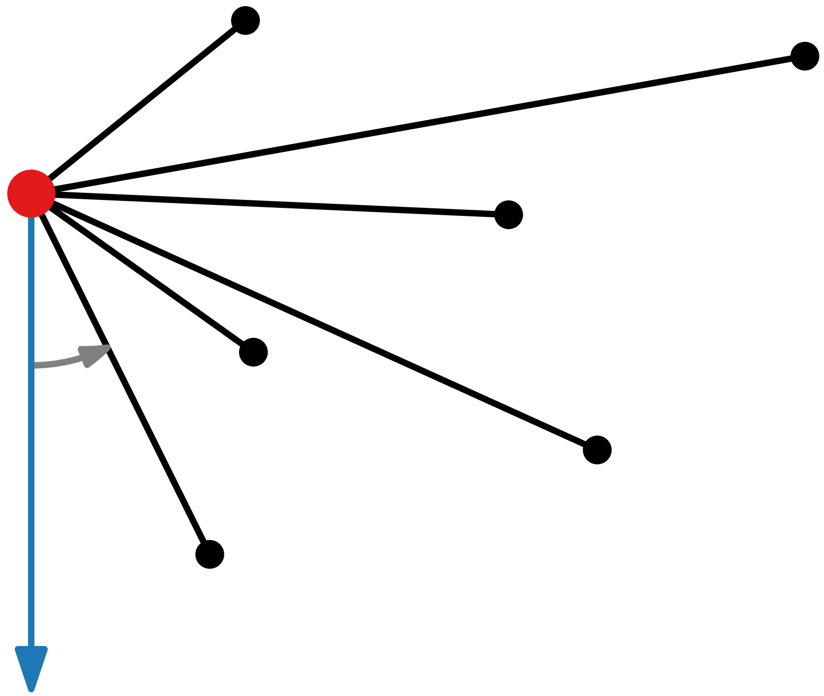

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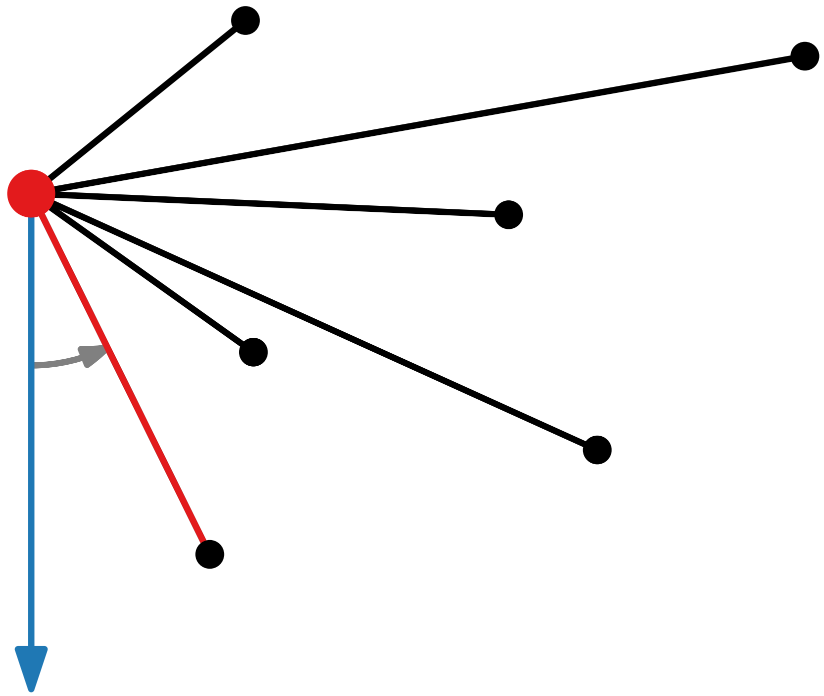

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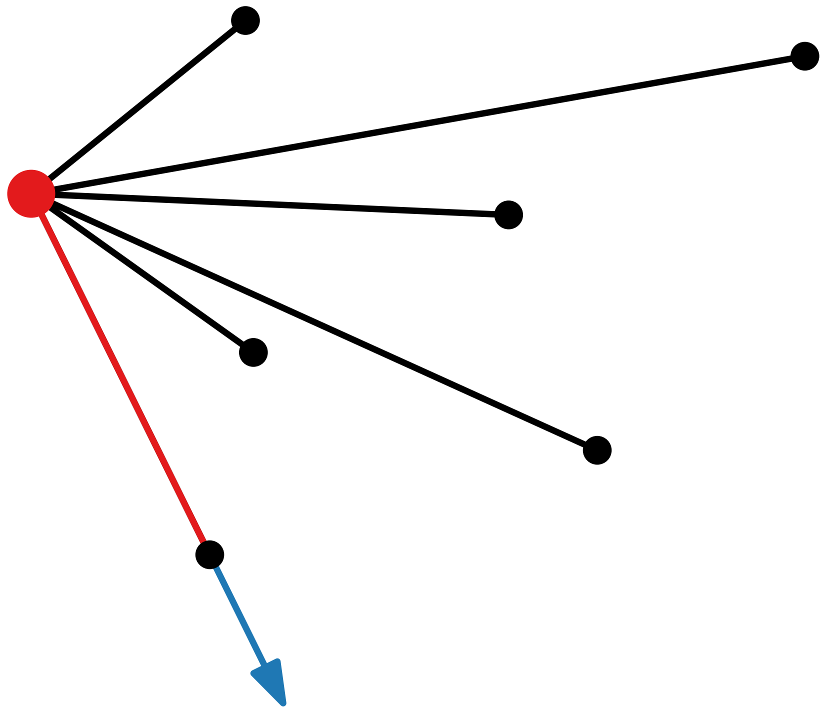

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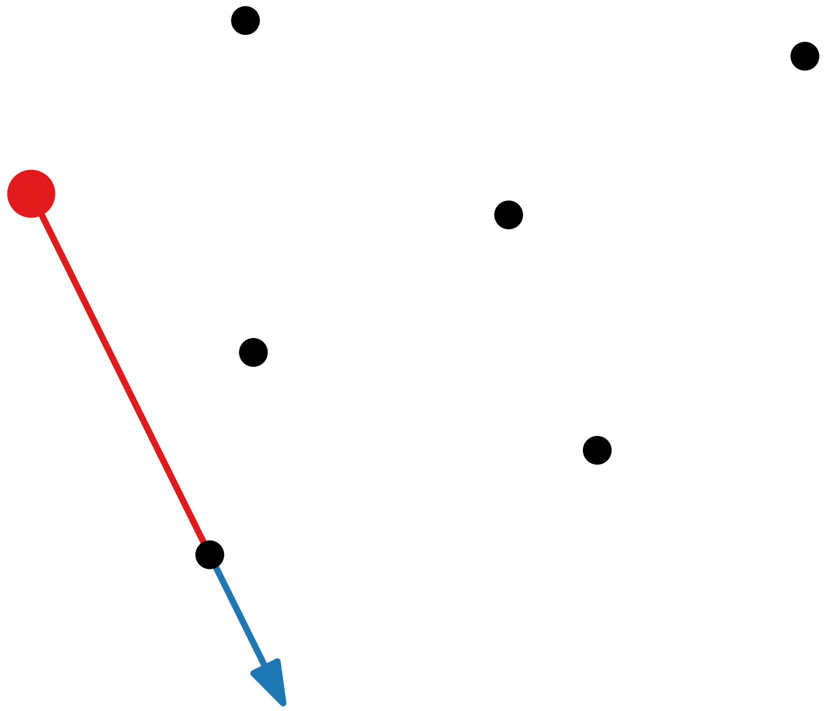

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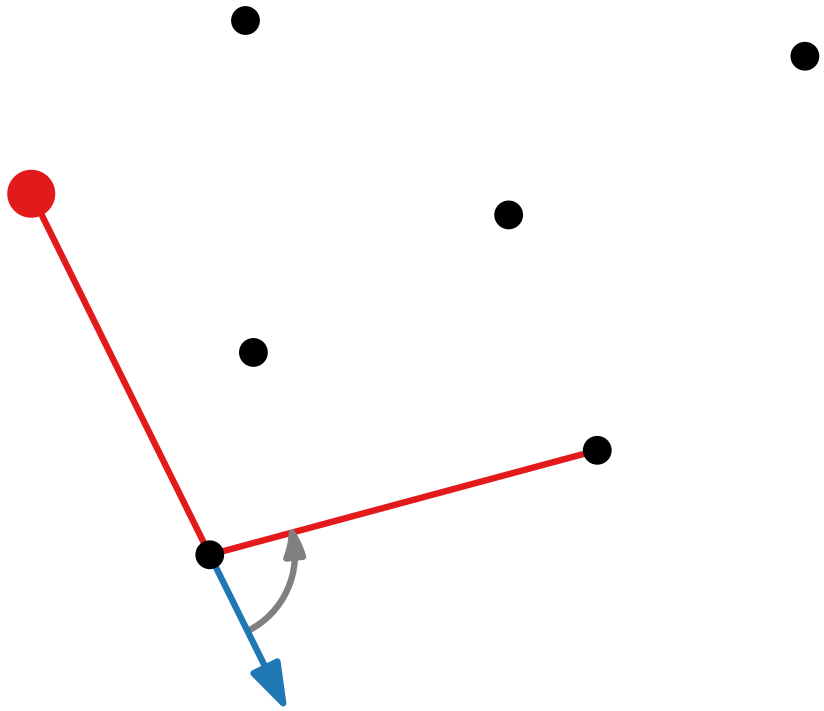

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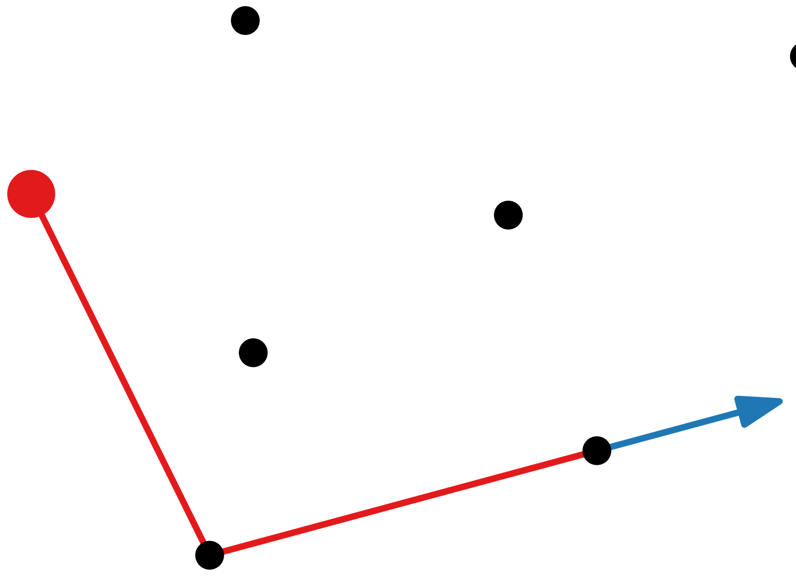

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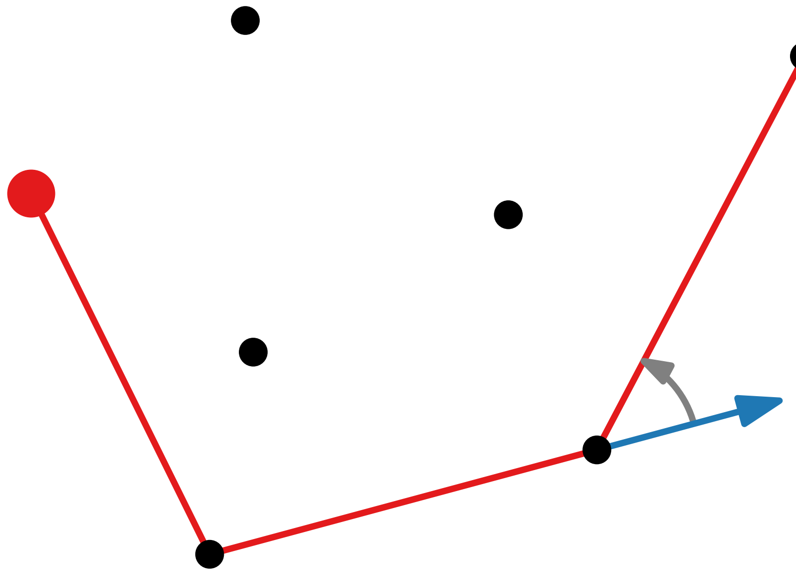

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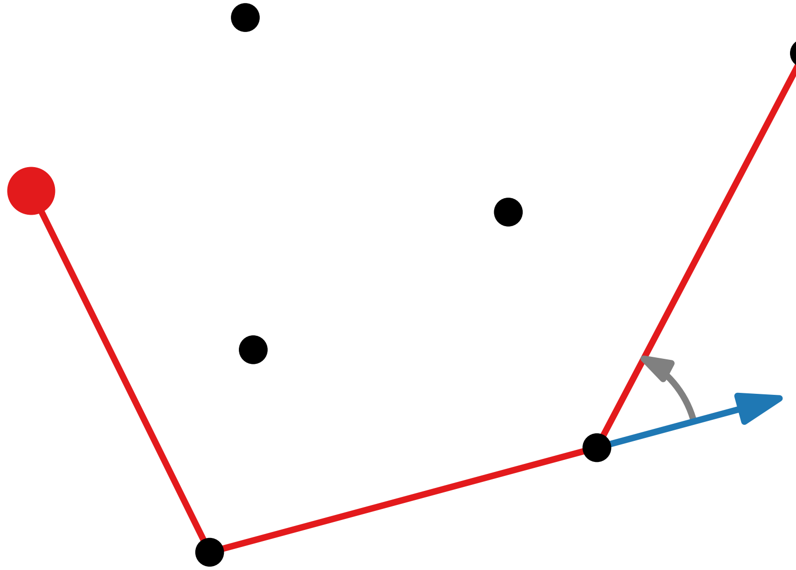
Runtime?



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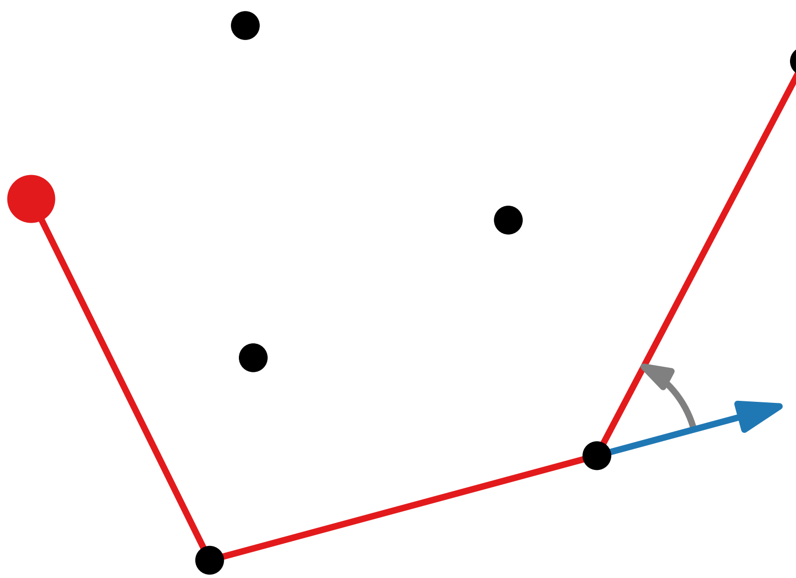
Runtime? $O(n \cdot h)$



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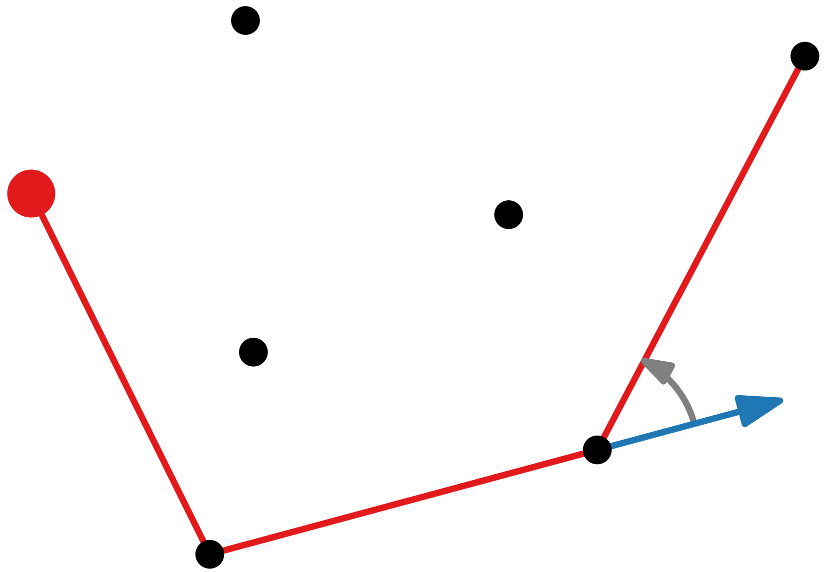

... where $h = |\mathrm{CH}(S)|$ = size of the output

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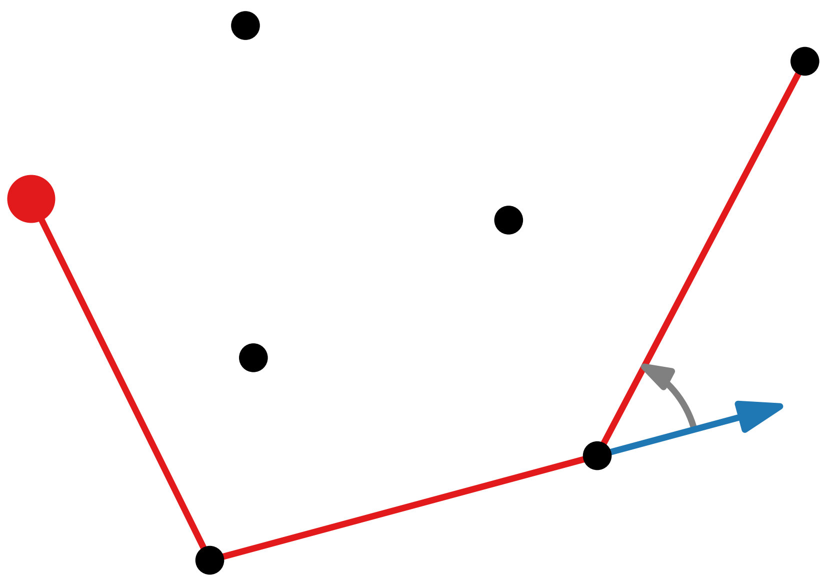

- Chan's exponential-search algorithm $O(n \log h)$

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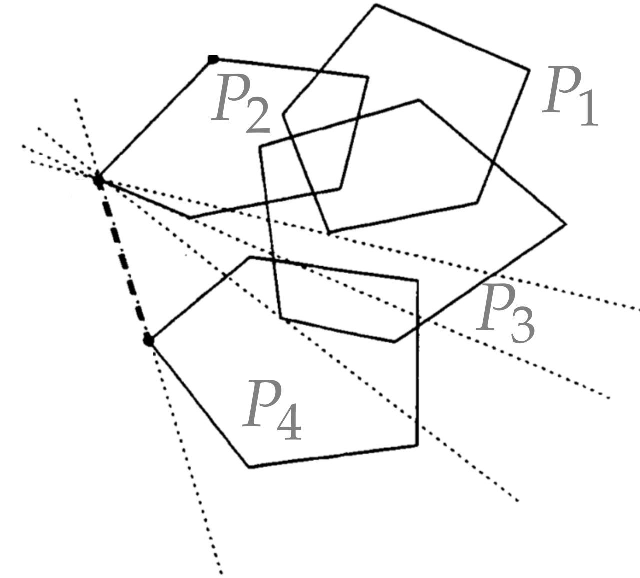
Chan's Algorithm



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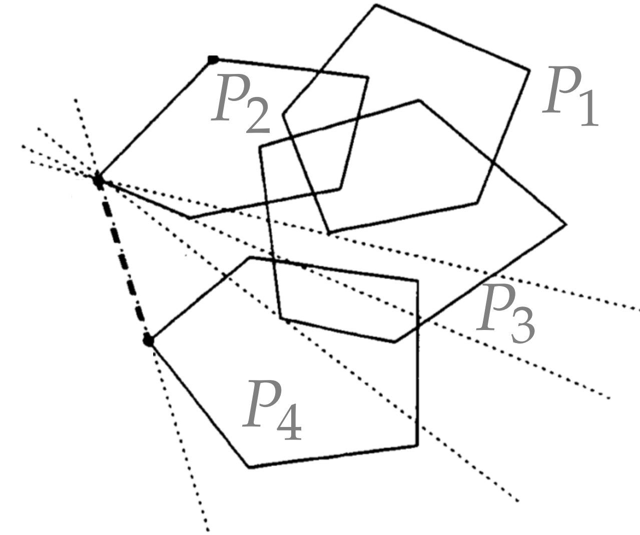
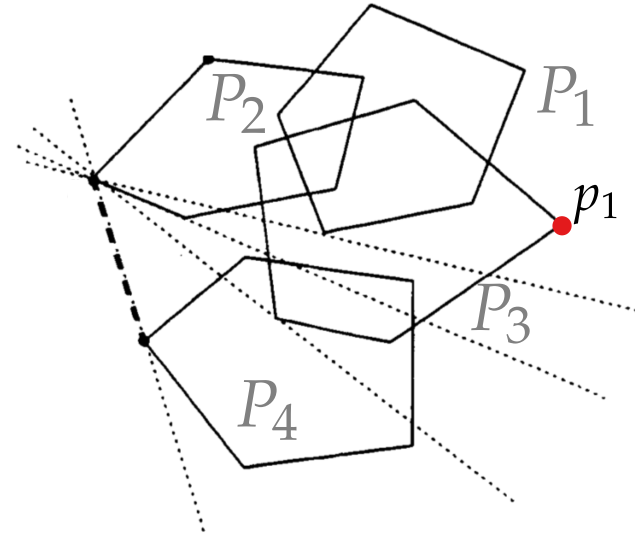


Algorithm Hull2D(P, m, H), where $P \subset E^2$, $3 \leq m \leq n$, and $H \geq 1$

1. partition P into subsets $P_1, \ldots, P_{\lceil n/m \rceil}$ each of size at most m
2. for $i = 1, \ldots, \lceil n/m \rceil$ do
3. compute conv(P_i) by Graham's scan and store its vertices in an array in ccw order [in $O(m \log m)$ time]
4. $p_0 \leftarrow (0, -\infty)$
5. $p_1 \leftarrow$ the rightmost point of P

Chan's Algorithm



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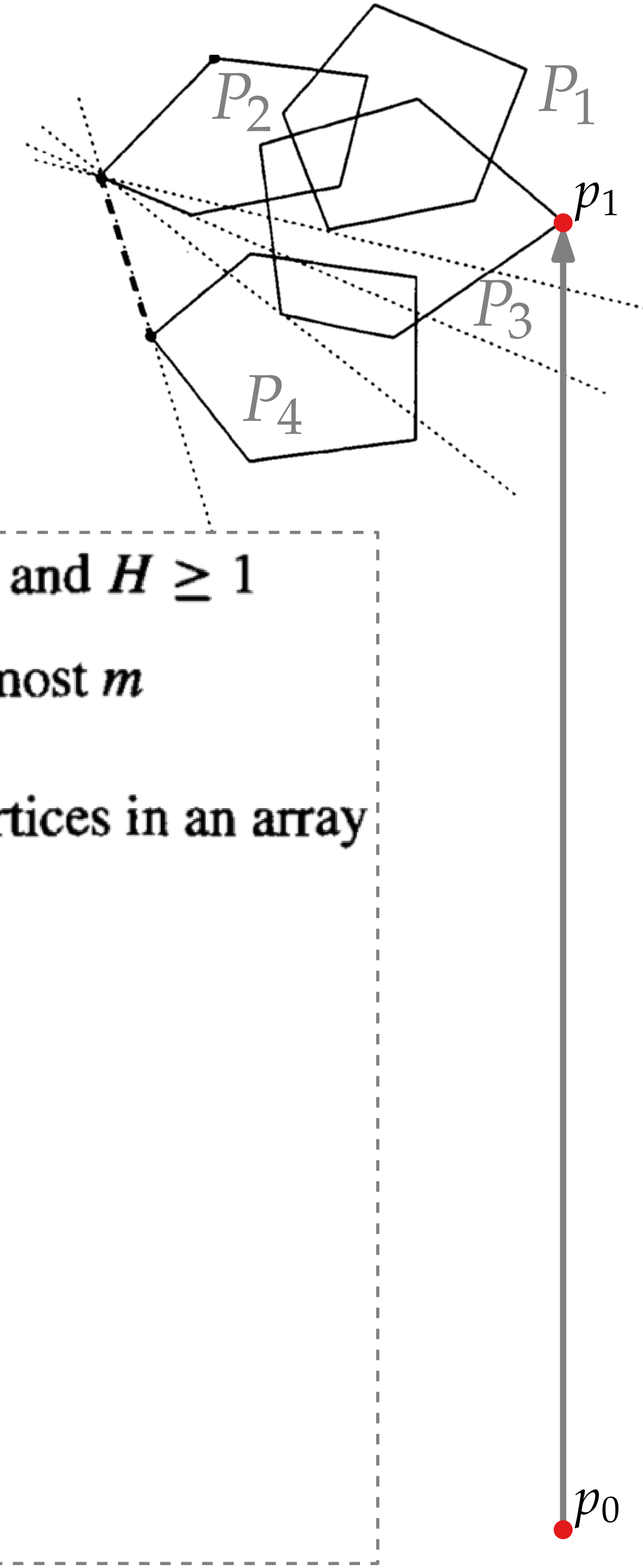

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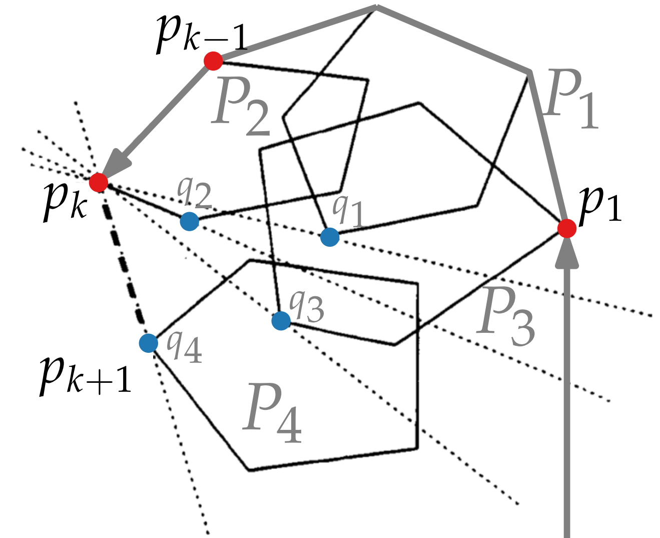
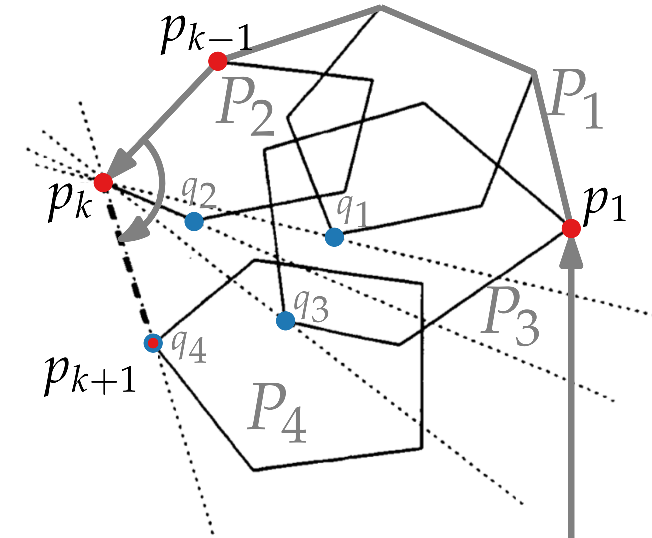
Chan's Algorithm



Algorithm Hull2D(P, m, H), where $P \subset E^2$, $3 \leq m \leq n$, and $H \geq 1$

1. partition P into subsets $P_1, \ldots, P_{\lceil n/m \rceil}$ each of size at most m
2. for $i = 1, \ldots, \lceil n/m \rceil$ do
3.   compute conv(P_i) by Graham's scan and store its vertices in an array in ccw order   [in $O(m \log m)$ time]
4. $p_0 \leftarrow (0, -\infty)$
5. $p_1 \leftarrow$ the rightmost point of P
6. for $k = 1, \ldots, H$ do
7.   for $i = 1, \ldots, \lceil n/m \rceil$ do
8.   compute the point $q_i \in P_i$ that maximizes $\angle p_{k-1} p_k q_i$ $(q_i \neq p_k)$ by performing a binary search on the vertices of conv(P_i)
9. $p_{k+1} \leftarrow$ the point q from $\{q_1, \ldots, q_{\lceil n/m \rceil}\}$ that maximizes $\angle p_{k-1} p_k q$
10.   if $p_{k+1} = p_1$ then return the list $\langle p_1, \ldots, p_k \rangle$
11. return *incomplete*

Chan's Algorithm



Algorithm Hull2D(P, m, H), where $P \subset E^2$, $3 \le m \le n$, and $H \ge 1$

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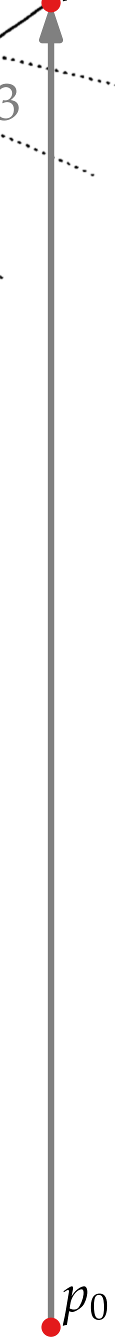
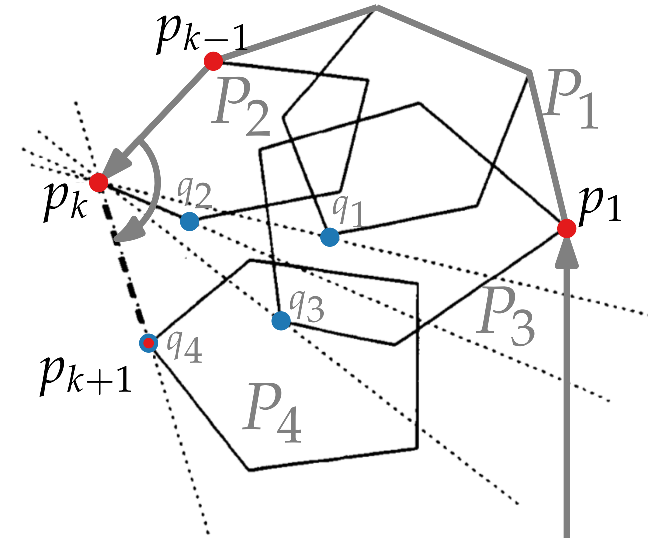
  if $p_{k+1} = p_1$ then return the list $\langle p_1, \ldots, p_k \rangle$
return *incomplete*



Chan's Algorithm



Algorithm Hull2D(P), where $P \subset E^2$

1. for $t = 1, 2, \ldots$ do
2. $\quad L \leftarrow$ Hull2D(P, m, H), where $m = H = \min\{2^{2^t}, n\}$
3. $\quad$ if $L \neq$ *incomplete* then return L

Algorithm Hull2D(P, m, H), where $P \subset E^2$, $3 \leq m \leq n$, and $H \geq 1$

1. partition P into subsets $P_1, \ldots, P_{\lceil n/m \rceil}$ each of size at most m
2. for $i = 1, \ldots, \lceil n/m \rceil$ do
3. $\quad$ compute conv(P_i) by Graham's scan and store its vertices in an array in ccw order $\quad$ [in $O(m \log m)$ time]
4. $p_0 \leftarrow (0, -\infty)$
5. $p_1 \leftarrow$ the rightmost point of P
6. for $k = 1, \ldots, H$ do
7. $\quad$ for $i = 1, \ldots, \lceil n/m \rceil$ do
8. $\quad\quad$ compute the point $q_i \in P_i$ that maximizes $\angle p_{k-1} p_k q_i$ $(q_i \neq p_k)$ by performing a binary search on the vertices of conv(P_i)
9. $\quad p_{k+1} \leftarrow$ the point q from $\{q_1, \ldots, q_{\lceil n/m \rceil}\}$ that maximizes $\angle p_{k-1} p_k q$
10. $\quad$ if $p_{k+1} = p_1$ then return the list $\langle p_1, \ldots, p_k \rangle$
11. return *incomplete*

Chan's Algorithm

[Text copied on October 17, 2017 from: https://en.wikipedia.org/wiki/Chan's_algorithm]

Initially, we assume that the value of h is known and make a parameter $m = h$. This assumption is not realistic, but we remove it later. The algorithm starts by arbitrarily partitioning P into at most $1 + \frac{n}{m}$ subsets Q with at most m points each. Then, it computes the convex hull of each subset Q using an $O(n \log n)$ algorithm – **Graham's scan**. Note that, as there are $O(n/m)$ subsets of $O(m)$ points each, this phase takes $O(n/m) \cdot O(m \log m) = O(n \log m)$ time.

The second phase consists of executing the **Jarvis' march** algorithm algorithm and using the precomputed convex hulls to speed up the execution. At each step in Jarvis's march, we have a point p_i in the convex hull, and need to find a point $p_{i+1} = f(p_i, P)$ such that all other points of P are to the right of the line $p_i p_{i+1}$. If we know the convex hull of a set Q of m points, then we can compute $f(p_i, Q)$ in $O(\log m)$ time, by using binary search. We can compute $f(p_i, Q)$ for all the $O(n/m)$ subsets Q in $O(n/m \log m)$ time. Then, we can determine $f(p_i, P)$ using the same technique as normally used in Jarvis's march, but only considering the points that are $f(p_i, Q)$ for some subset Q. As Jarvis's march repeats this process $O(h)$ times, the second phase also takes $O(n \log m)$ time, and therefore $O(n \log h)$ time if $m = h$.

By running the two phases described above, we can compute the convex hull of n points in $O(n \log h)$ time, assuming that we know the value of h. If we make $m < h$, we can abort the execution after $m + 1$ steps, therefore spending only $O(n \log m)$ time (but not computing the convex hull). We can initially set m as a small constant (we use 2 for our analysis, but in practice numbers around 5 may work better), and increase the value of m until $m > h$, in which case we obtain the convex hull as a result.

If we increase the value of m too slowly, we may need to repeat the steps mentioned before too many times, and the execution time will be large. On the other hand, if we increase the value of m too quickly, we risk making m much larger than h, also increasing the execution time. Similar to strategy used by Chazelle and Matoušek's algorithm, Chan's algorithm squares the value of m at each iteration, and makes sure that m is never larger than n. In other words, at iteration t (starting at 1), we have $m = \min\left(n, 2^{2^t}\right)$. The total running time of the algorithm is

$$\sum_{t=1}^{\lceil \log \log h \rceil} O\left(n \log(2^{2^t})\right) = O(n) \sum_{t=1}^{\lceil \log \log h \rceil} O(2^t) = O\left(n \cdot 2^{1+\lceil \log \log h \rceil}\right) = O(n \log h).$$

To generalize this construction for the 3-dimensional case, an $O(n \log n)$ algorithm to compute the 3-dimensional convex hull should be used instead of Graham scan, and a 3-dimensional version of Jarvis's march needs to be used. The time complexity remains $O(n \log h)$.