Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

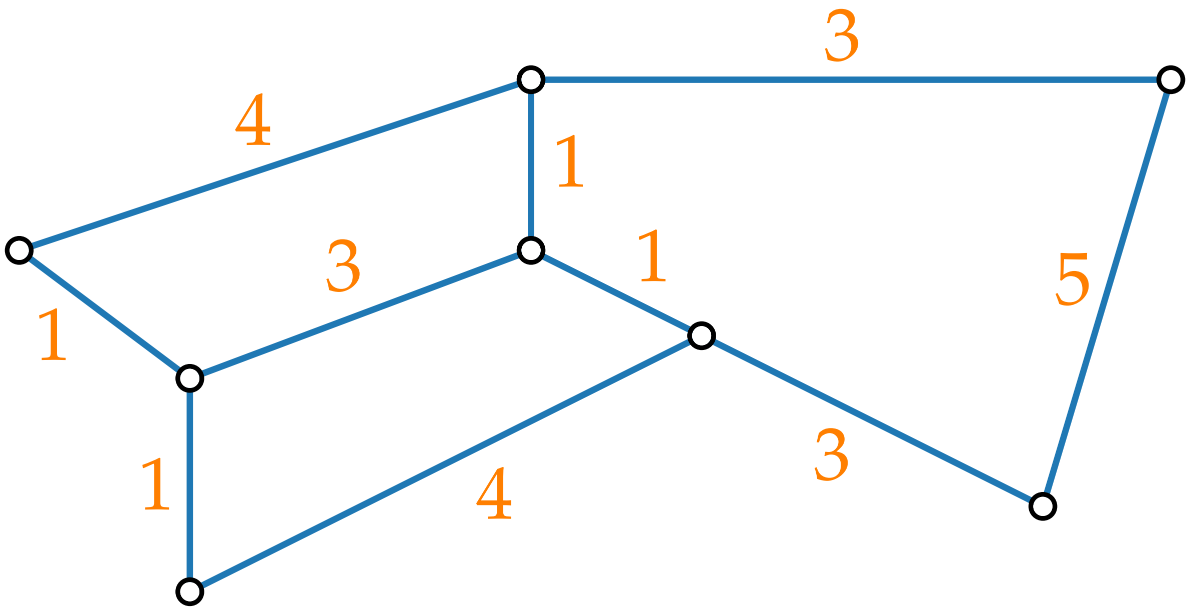
Part I:
SteinerForest

Philipp Kindermann

Summer Semester 2020

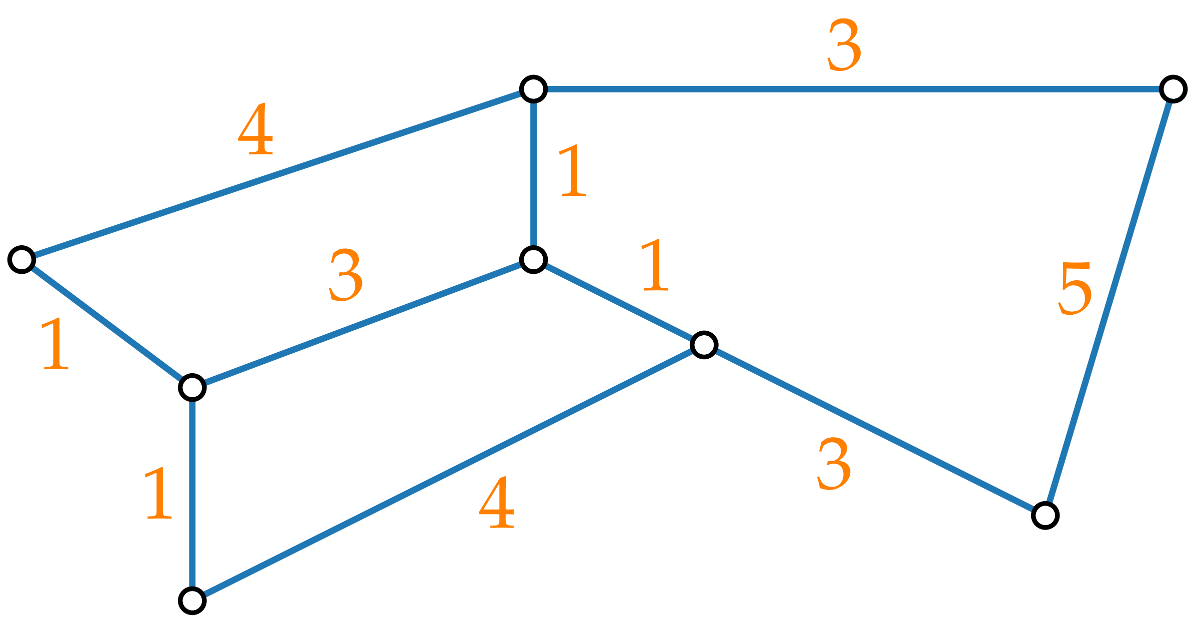
SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$



SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of k pairs of vertices



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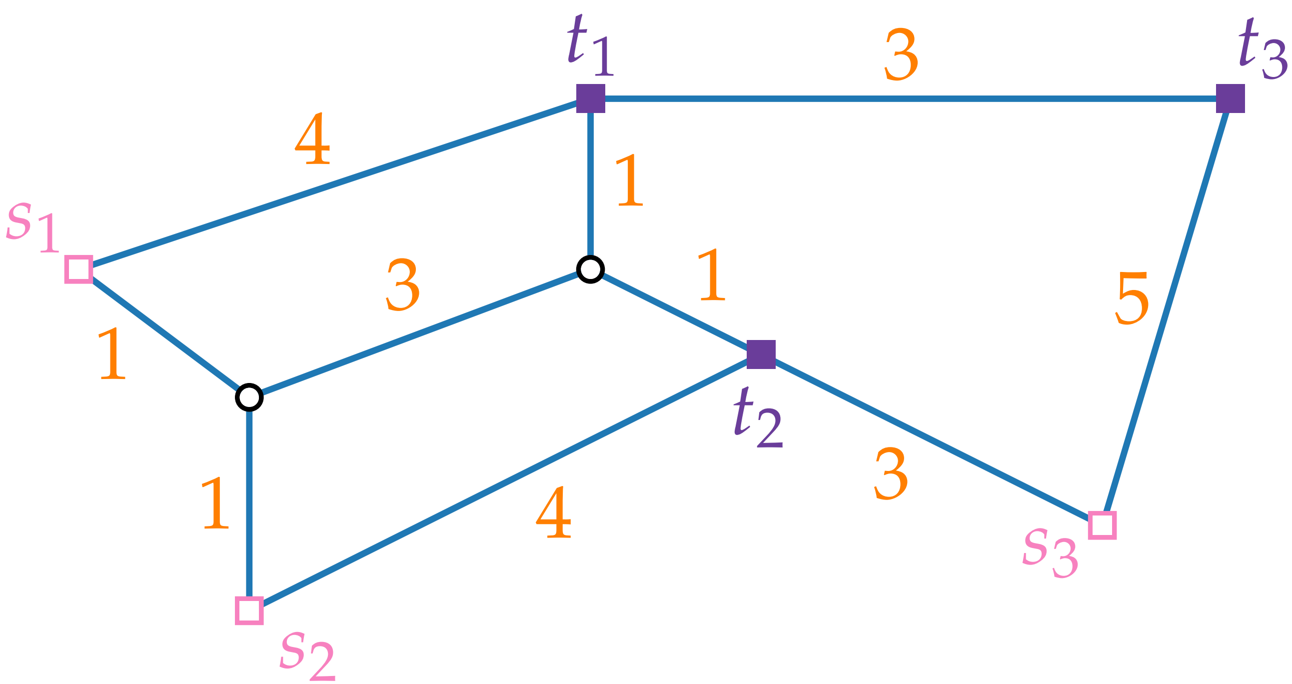

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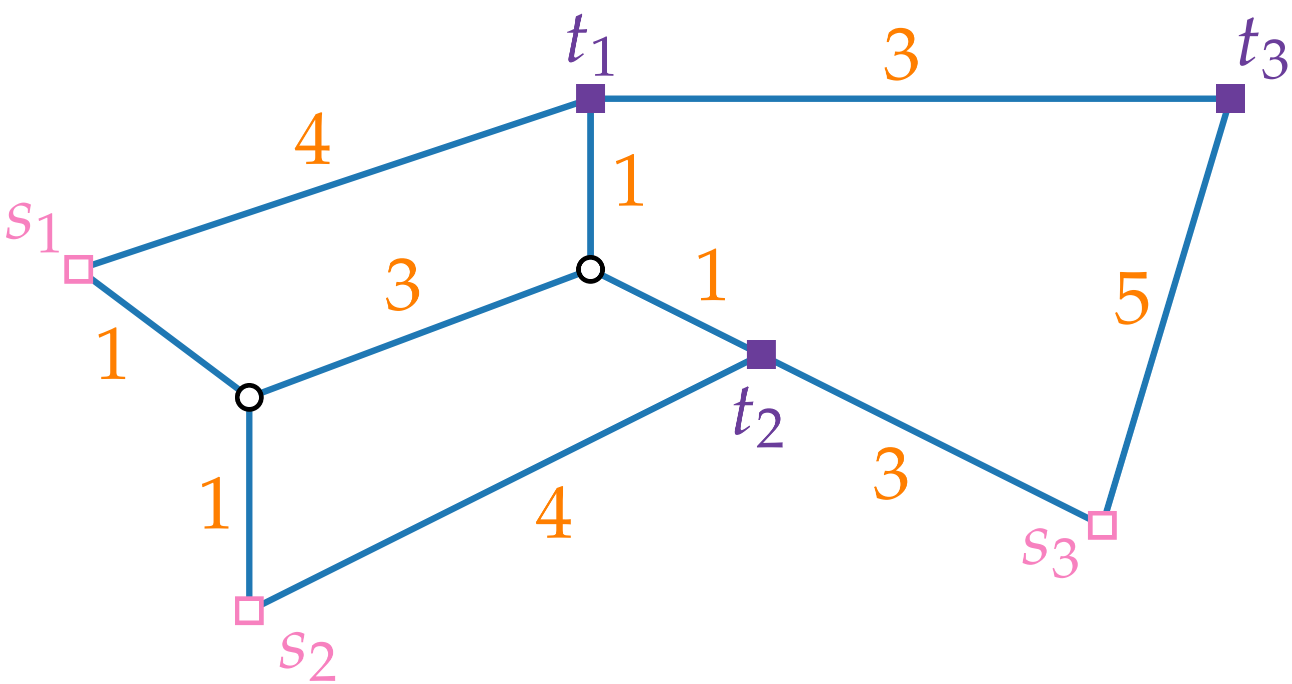
Task: Find an edge set $F \subseteq E$ with min. total cost $c(F)$



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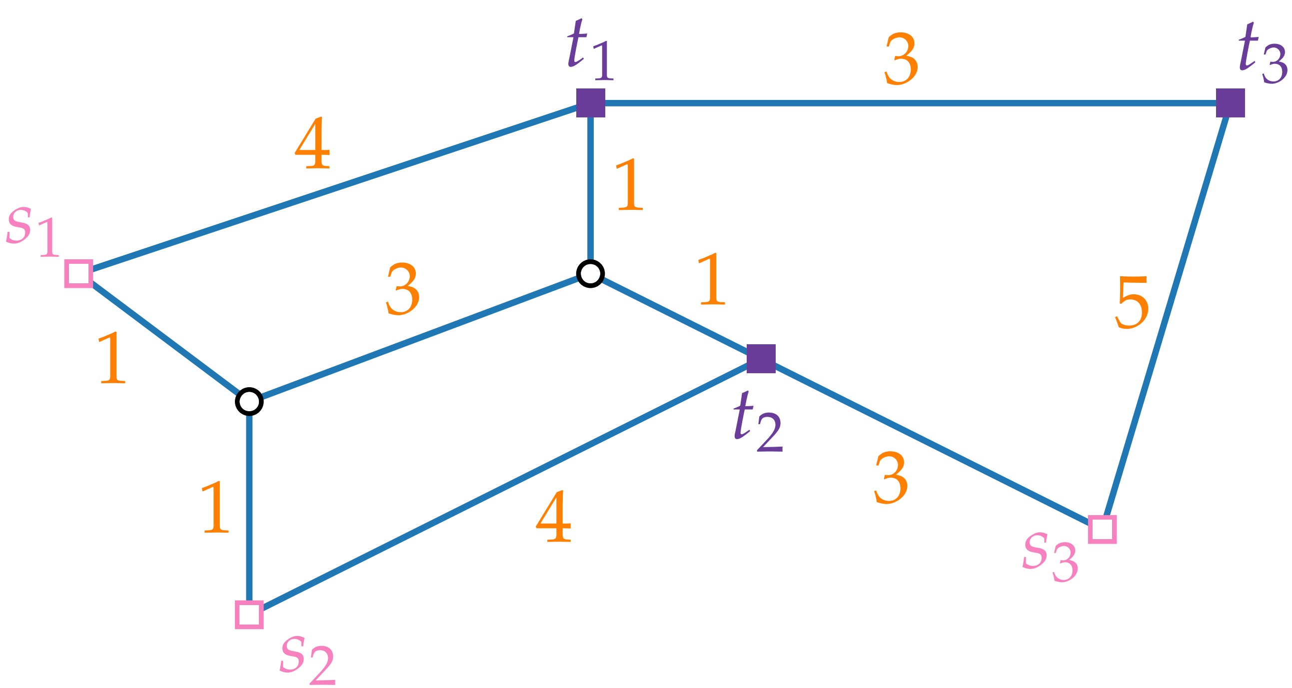
Task: Find an edge set $F \subseteq E$ with min. total cost $c(F)$ such that in the subgraph (V, F) each pair (s_i, t_i), $i = 1, \dots, k$ is connected.



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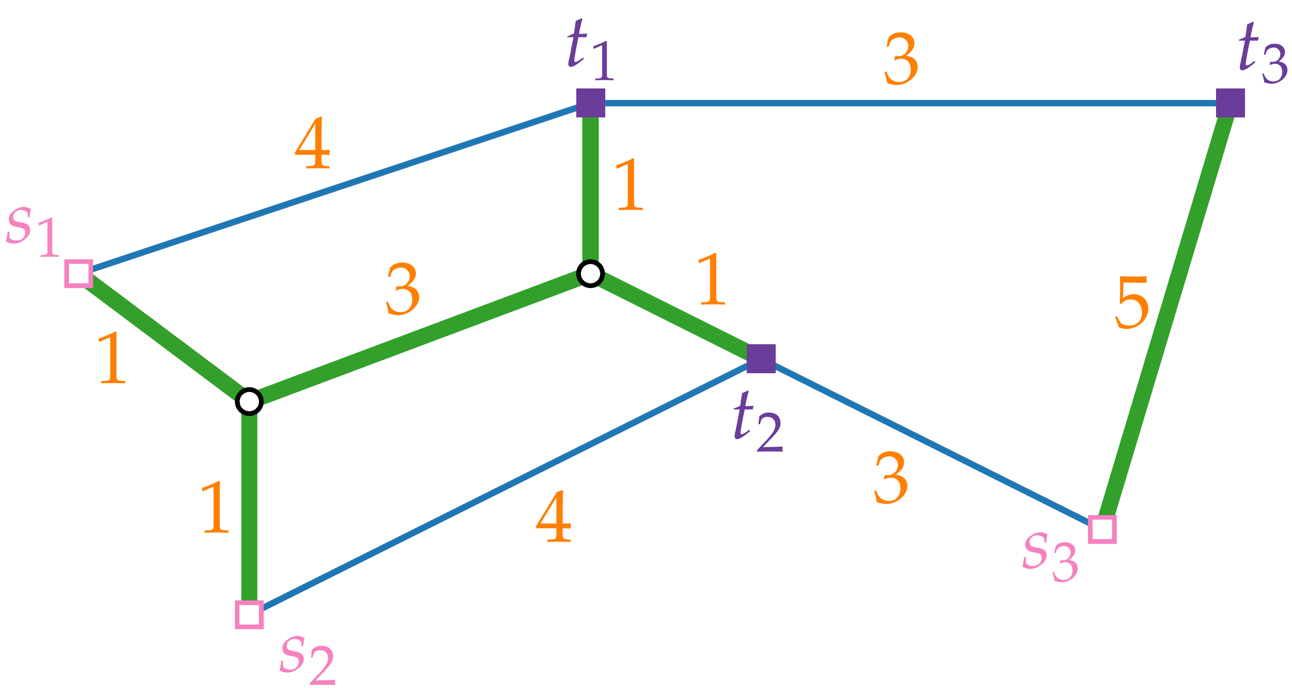

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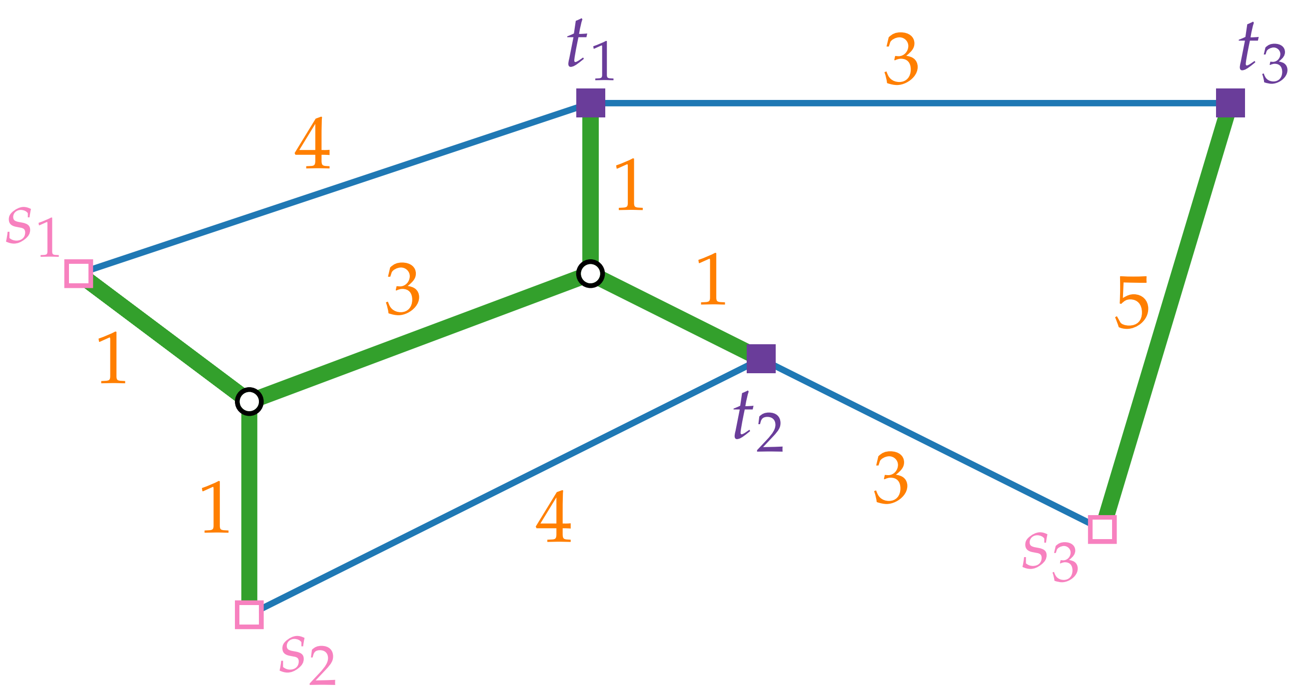

Special cases?

SteinerForest

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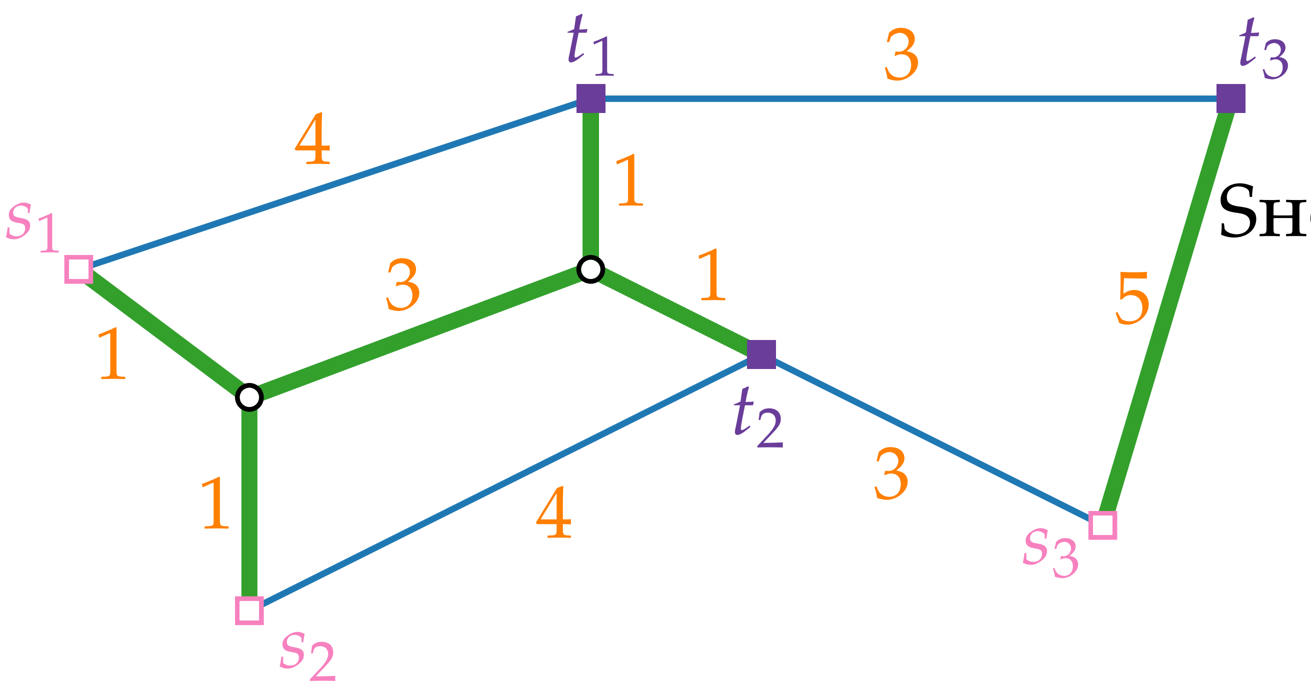

Special cases?

ShortestPath ($R = \{s, t\}$)

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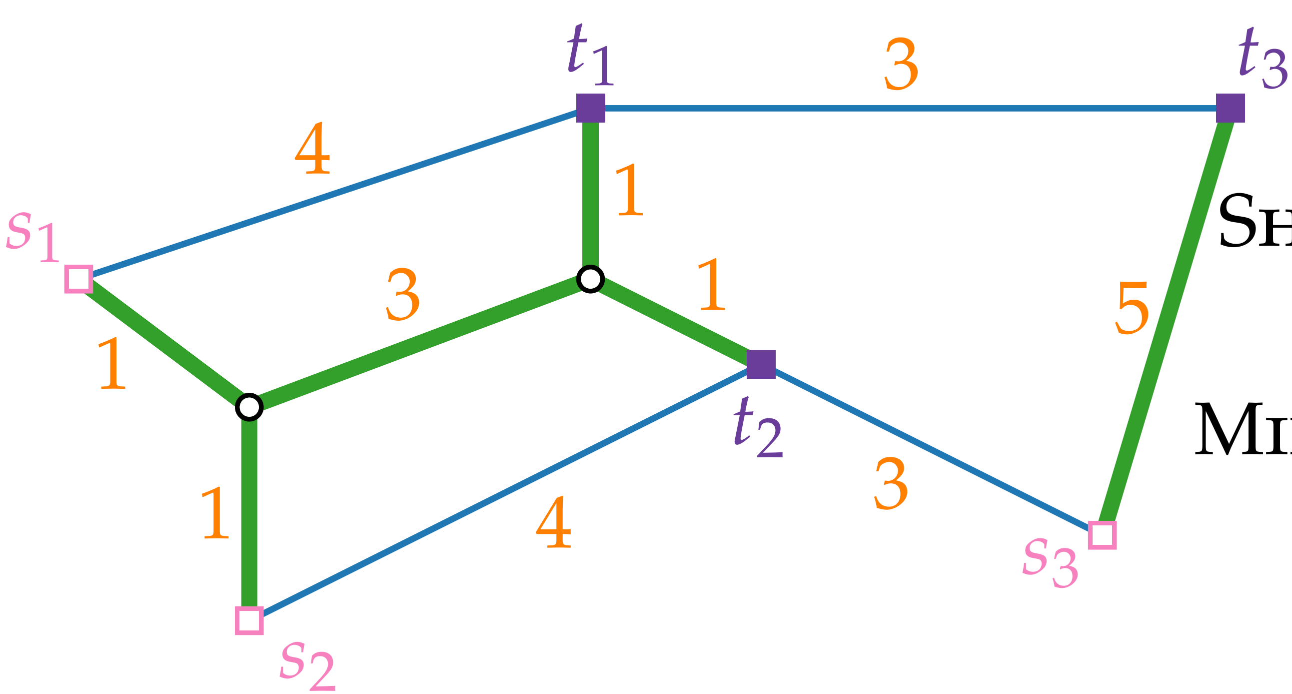

Special cases?

ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

SteinerForest

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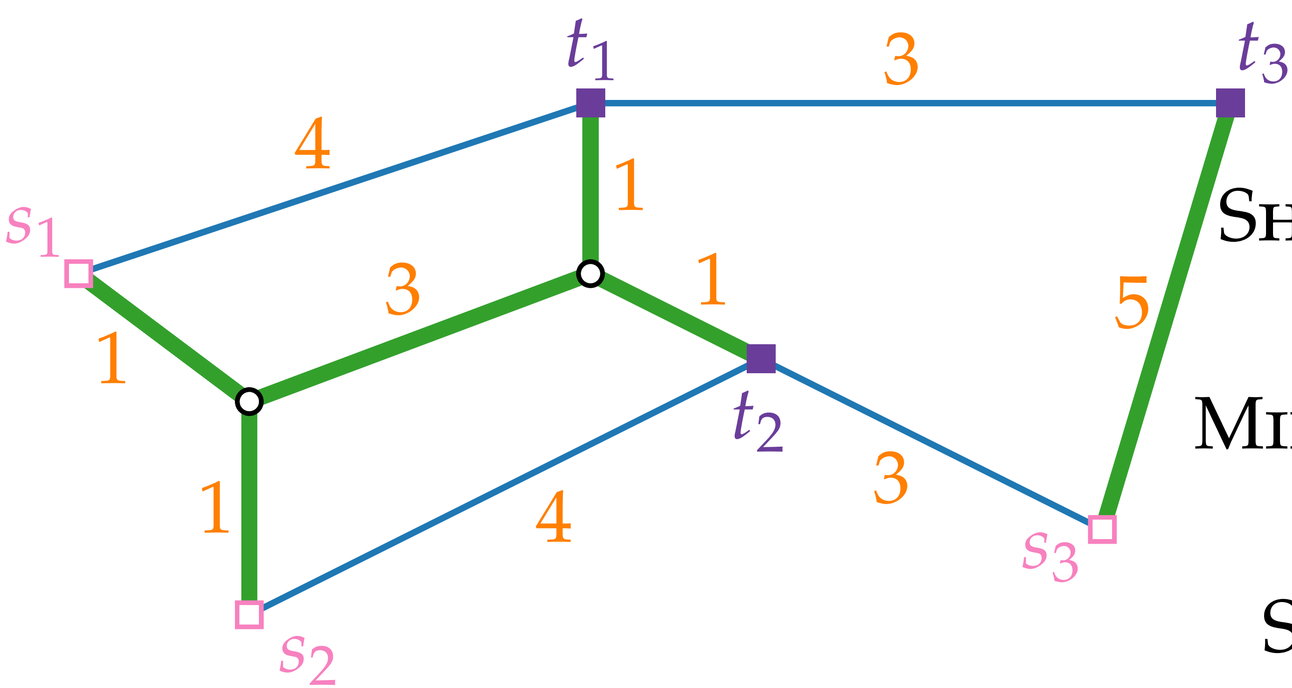


Special cases?

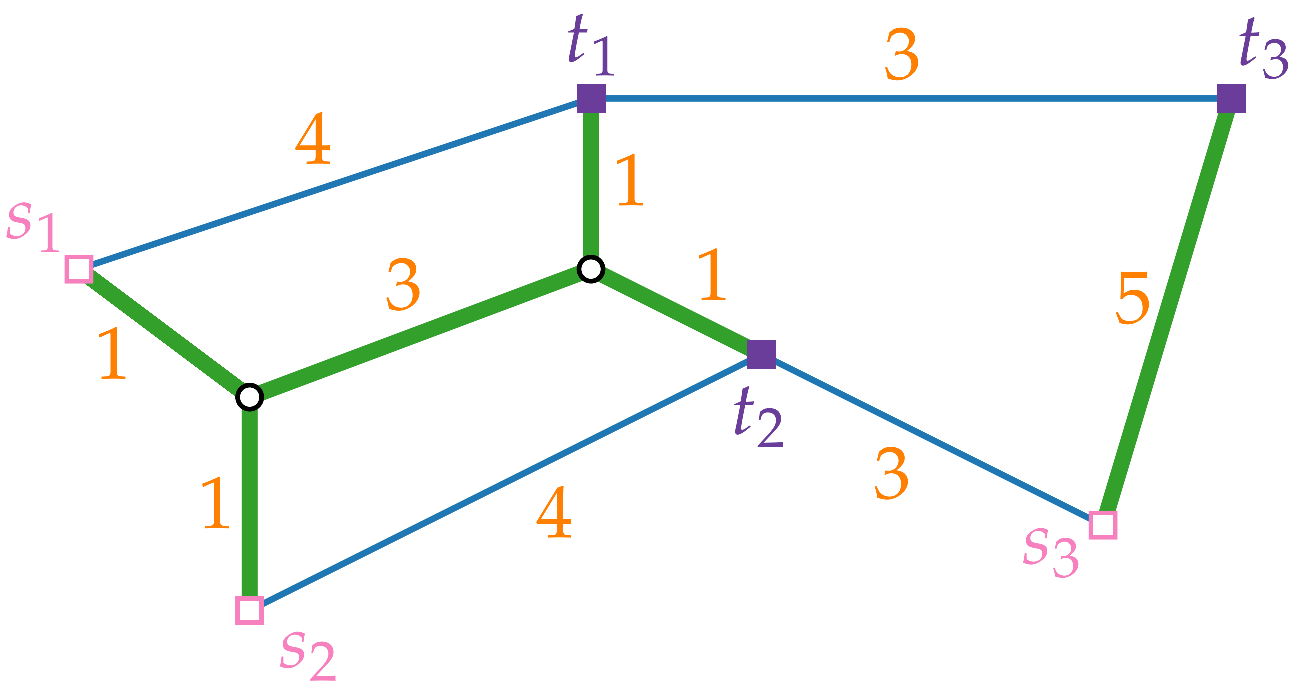
ShortestPath $(R = \{s, t\})$

MinSpanningTree $(R = E)$

SteinerTree $(R = T \times T)$

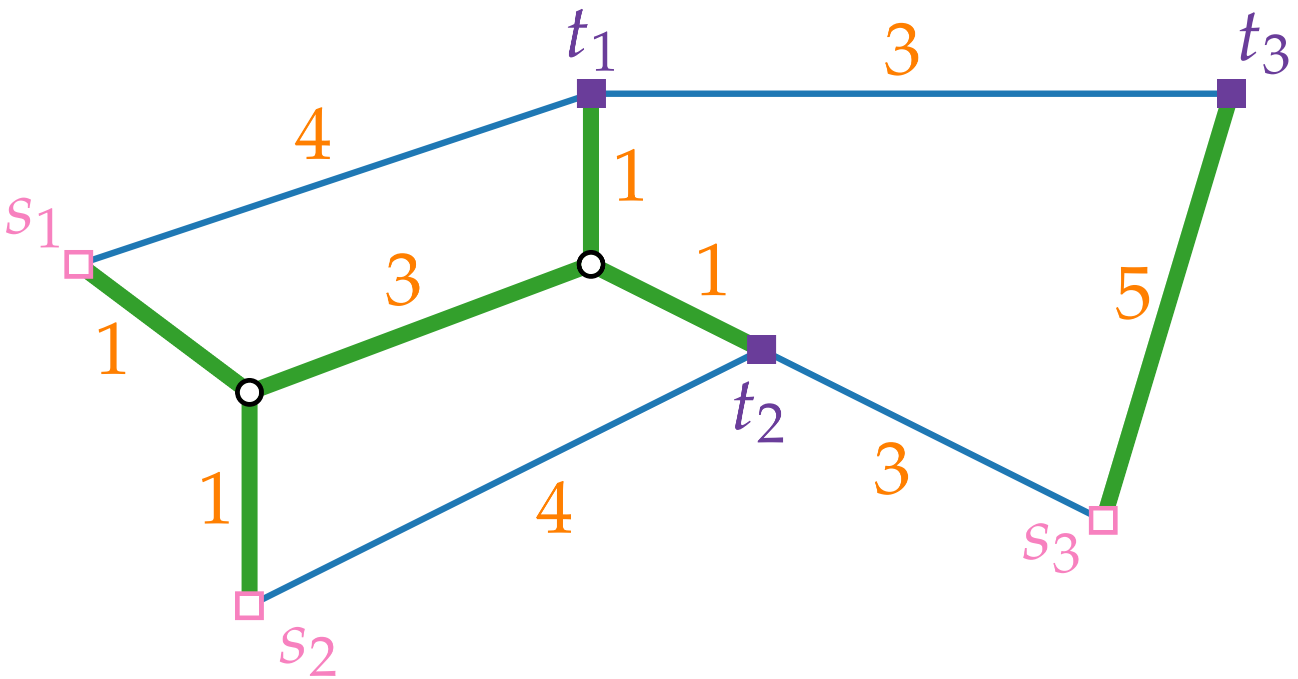
Approaches?

- Merge k shortest s_i-t_i-paths



Approaches?

- Merge k shortest s_i-t_i-paths
- SteinerTree on the set of terminals



Approaches?

- Merge k shortest s_i-t_i-paths
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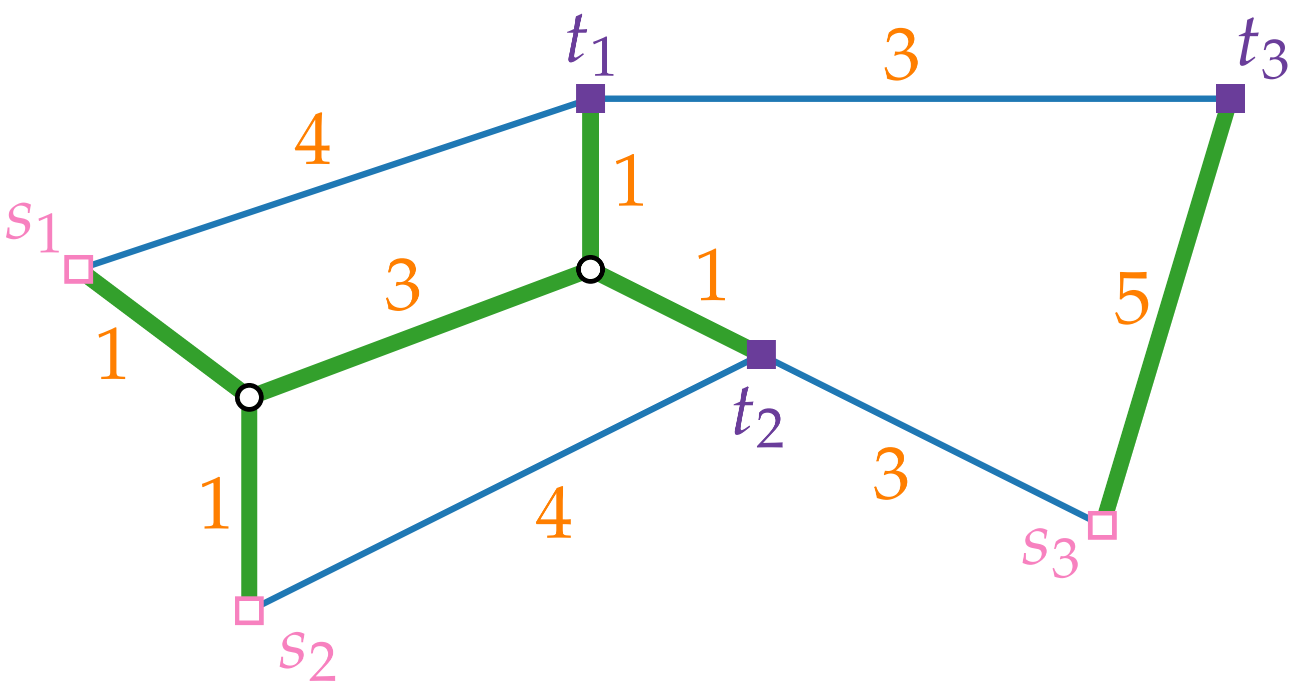
Above approaches perform poorly :-(



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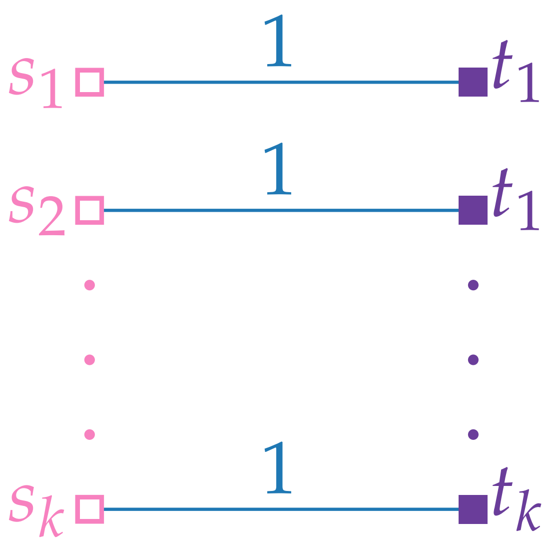

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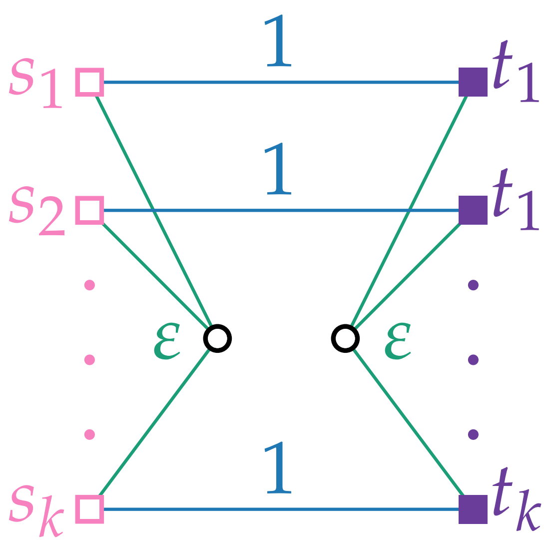

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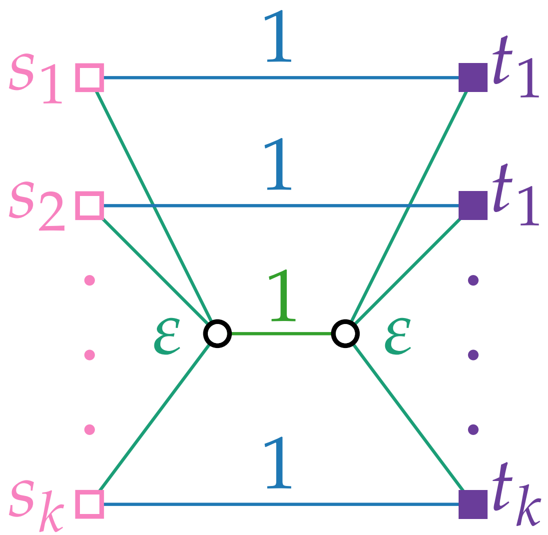

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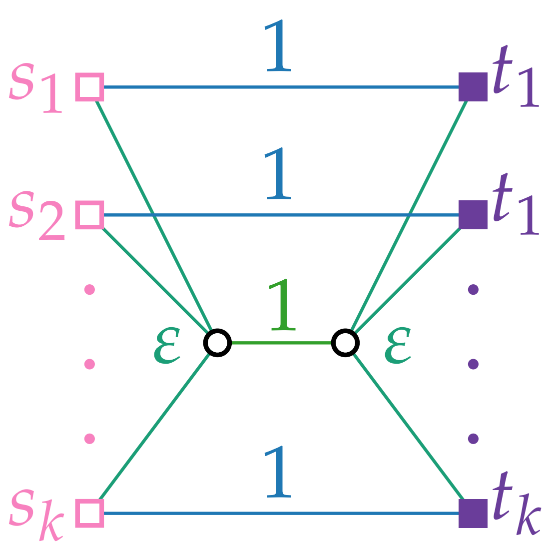

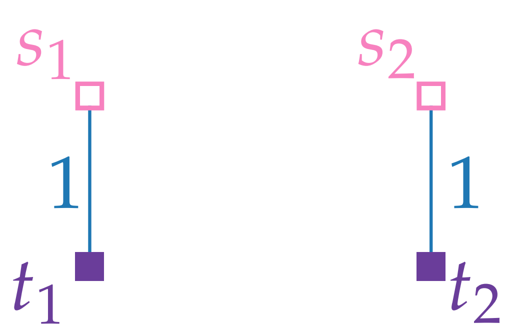

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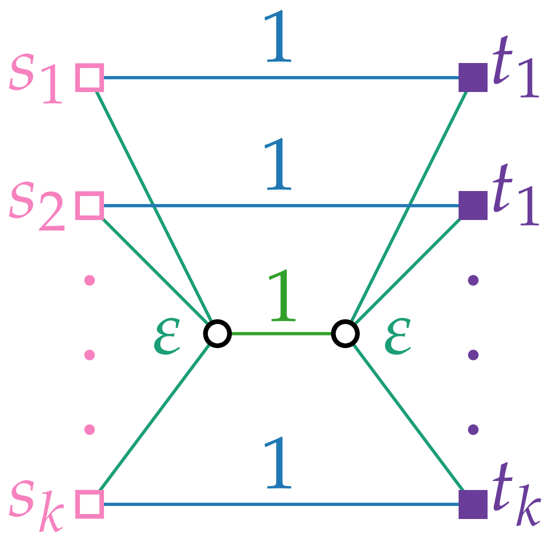

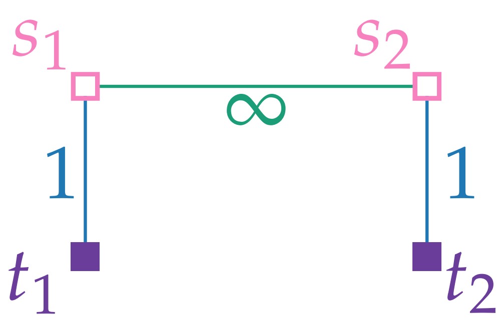

Approaches?

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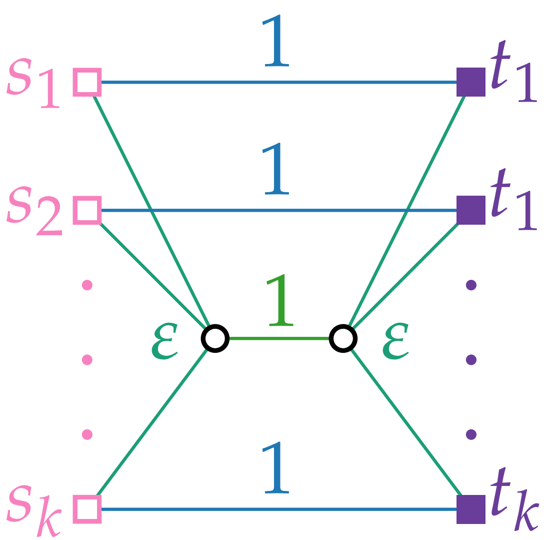
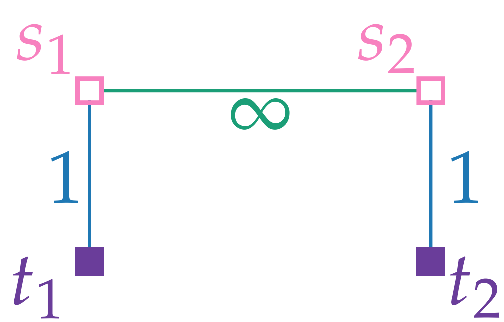
Above approaches perform poorly :-(

Difficulty: which terminals belong to the same tree of the forest?





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Part II:
Primal and Dual LP

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Summer Semester 2020

An ILP

minimize

subject to

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$$x_e \in \{0, 1\} \qquad e \in E$$

An ILP

minimize $\sum_{e \in E} c_e x_e$

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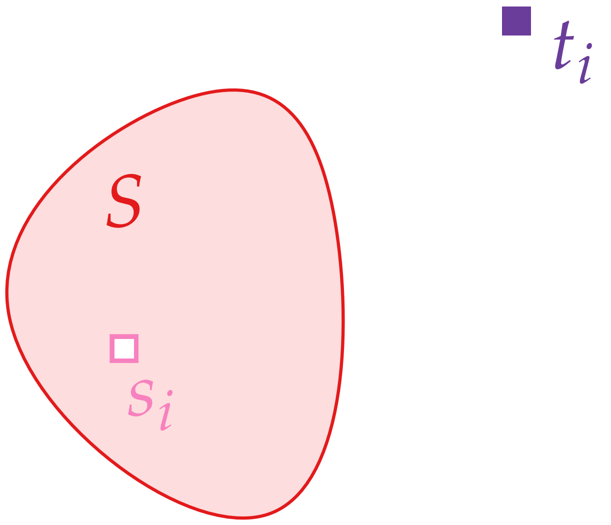

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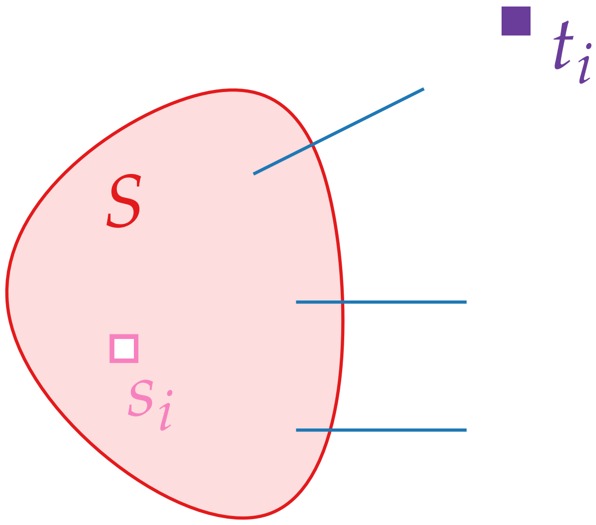

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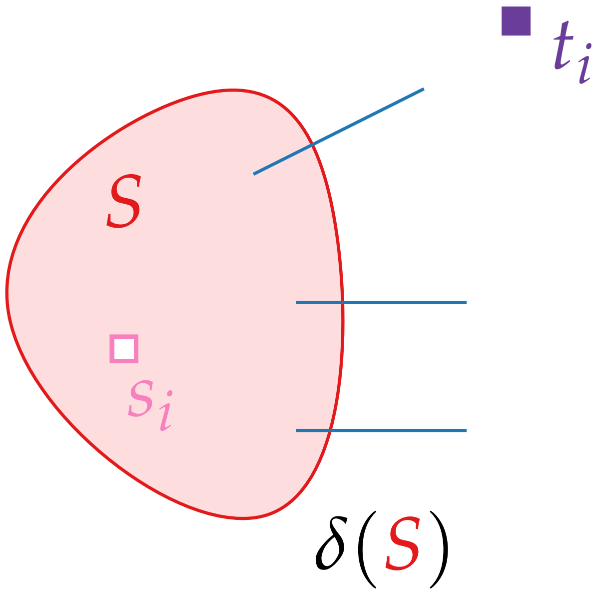

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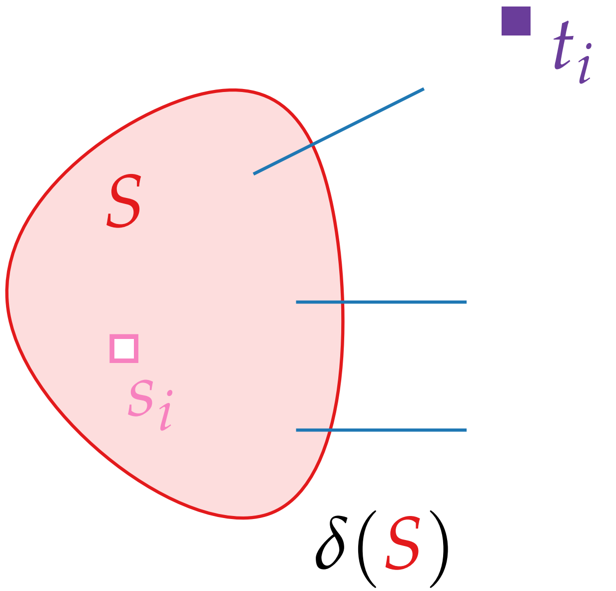
$$\delta(S) := \{(u,v) \in E : u \in S \text{ and } v \notin S\}$$



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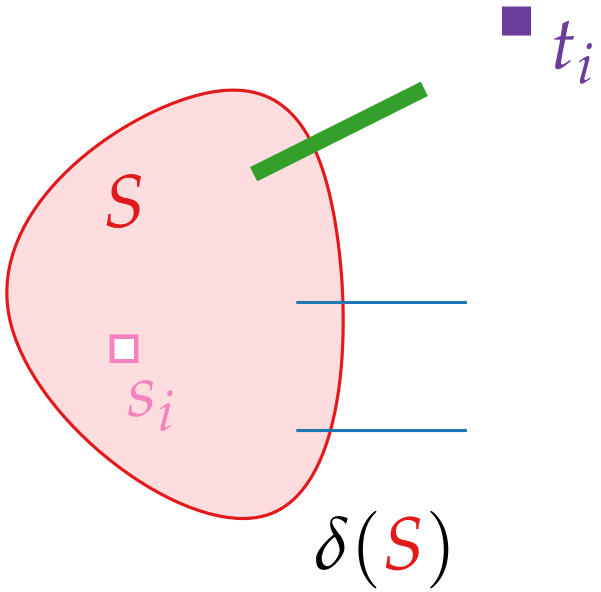

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$$
\begin{array}{ll}
\textbf{minimize} & \displaystyle\sum_{e \in E} c_e x_e \\
\textbf{subject to} & \displaystyle\sum_{e \in \delta(S)} x_e \geq 1 \\
 & x_e \in \{0,1\} \qquad\qquad e \in E
\end{array}
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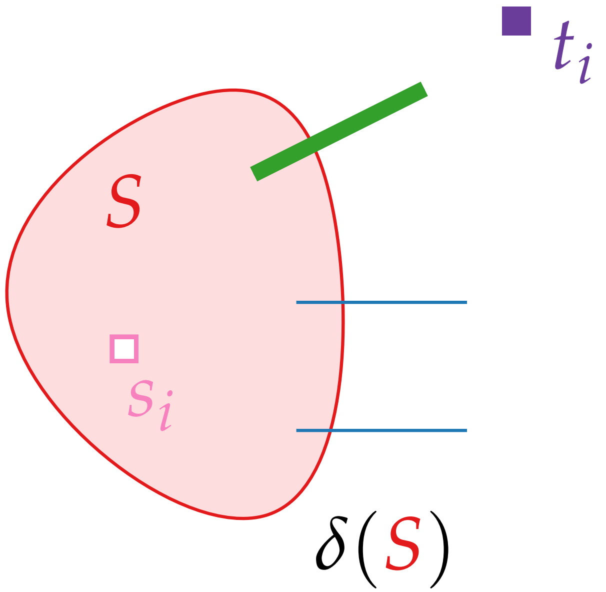

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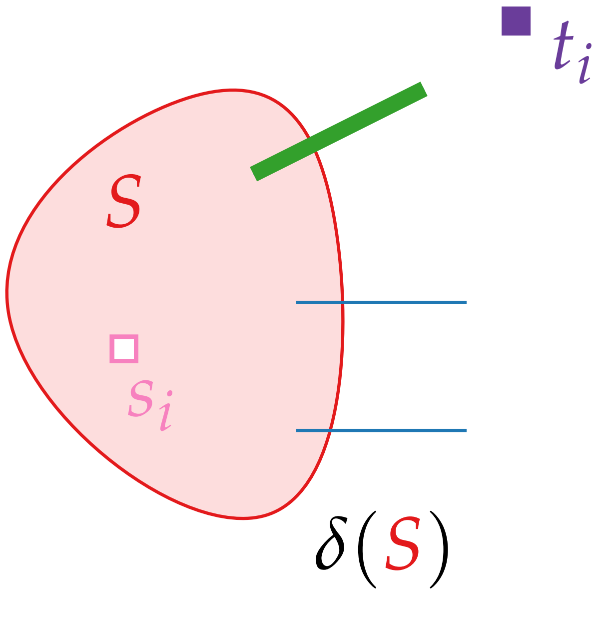

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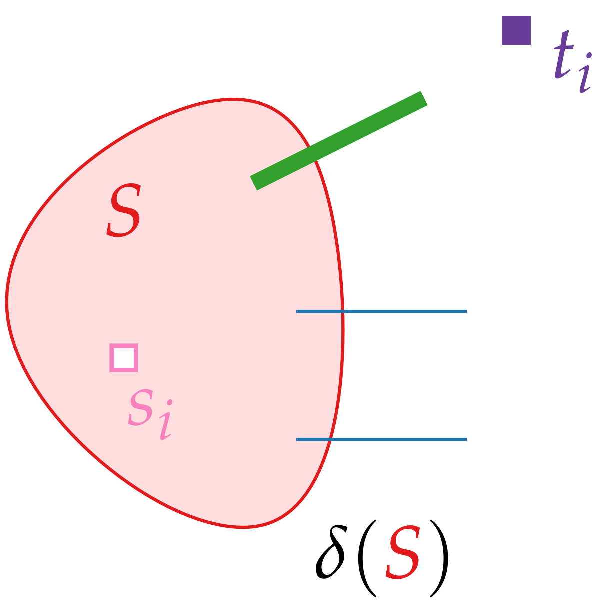

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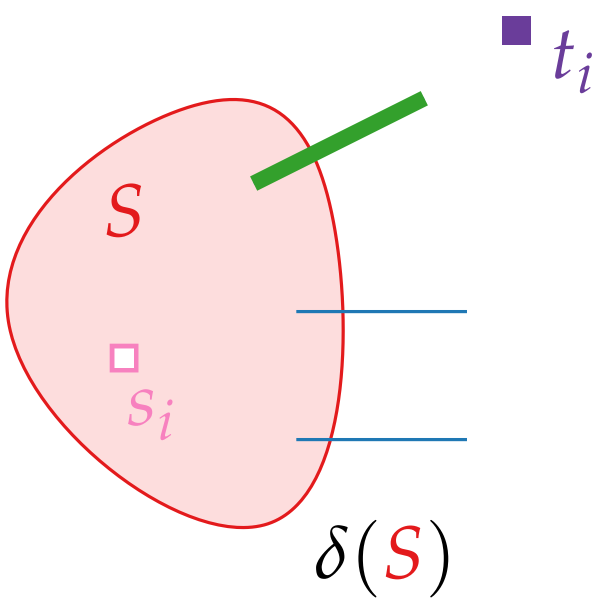
$\rightsquigarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

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Intuition for the Dual

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The graph is a network of **bridges**, spanning the **moats**.

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$$\begin{aligned} \textbf{maximize} \quad & \sum_{\substack{S \in \mathcal{S}_i \\ i=1,\dots,k}} y_S \\ \textbf{subject to} \quad & \sum_{S:\, e \in \delta(S)} y_S \le c_e && e \in E \\ & y_S \ge 0 && S \in \mathcal{S}_i, i = 1, \dots, k \end{aligned}$$

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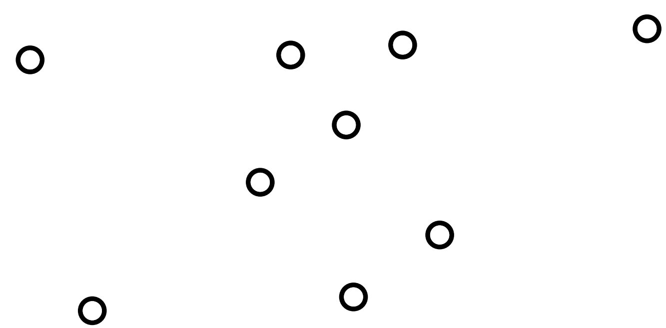

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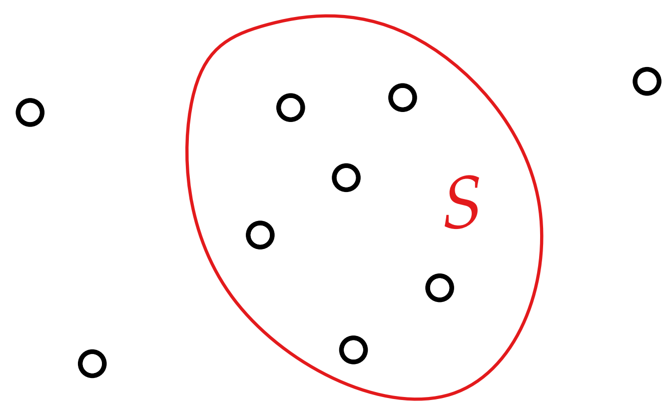

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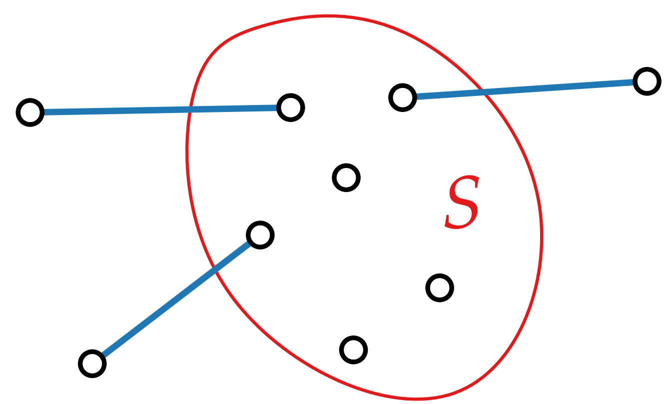

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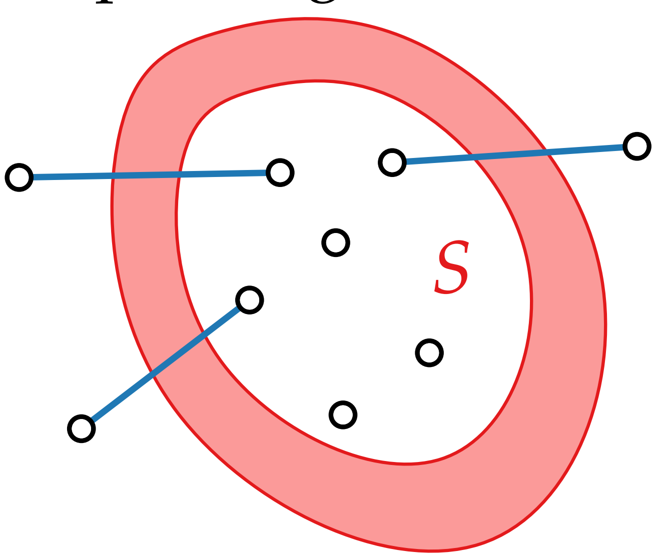

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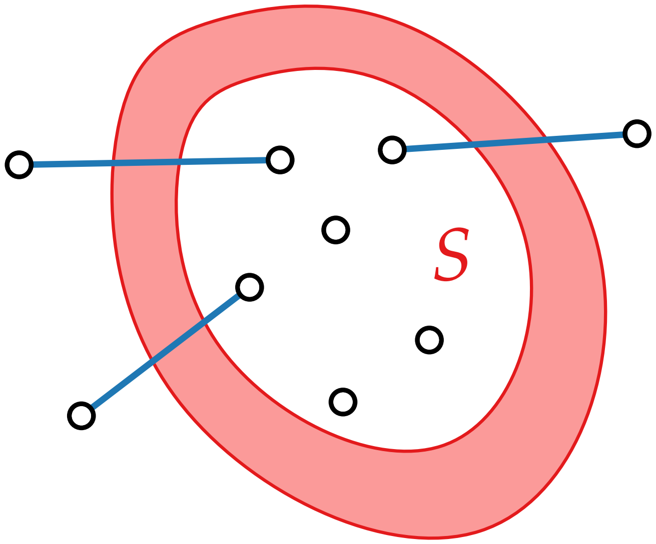

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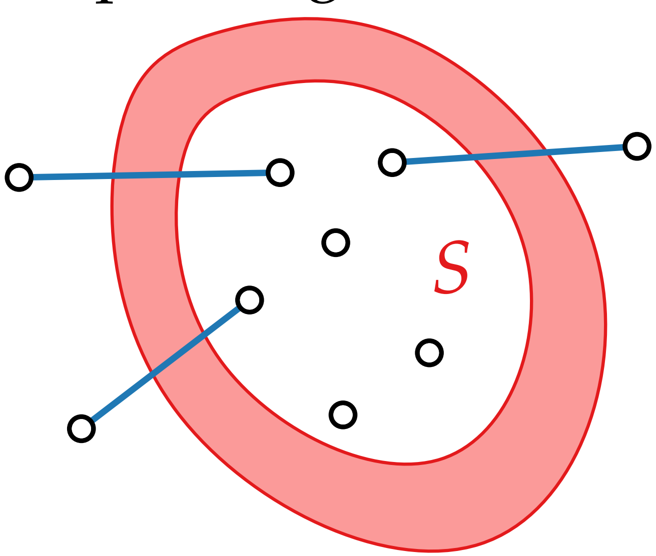

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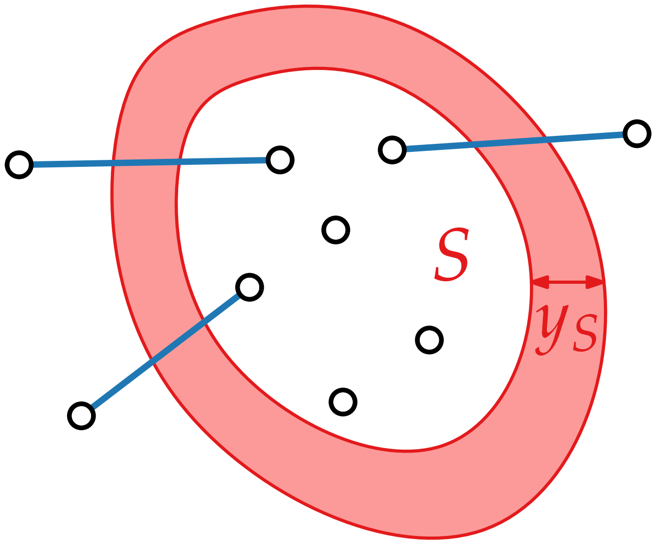

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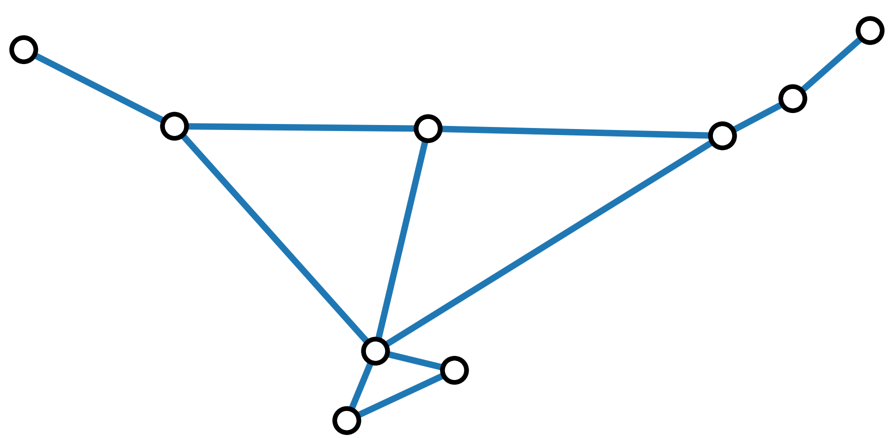

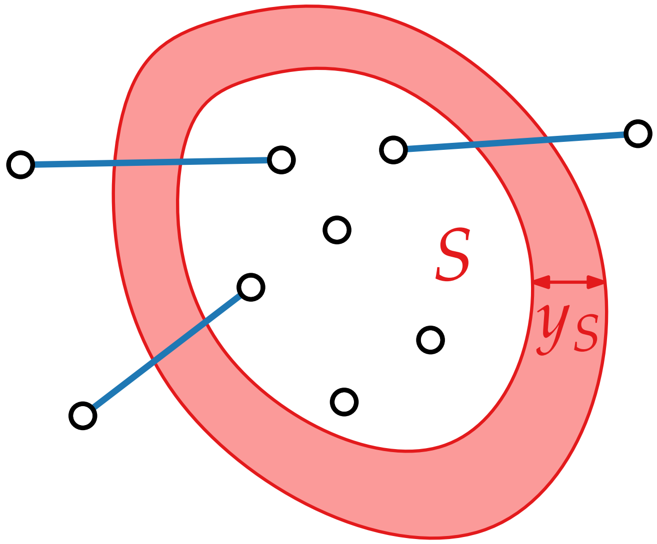

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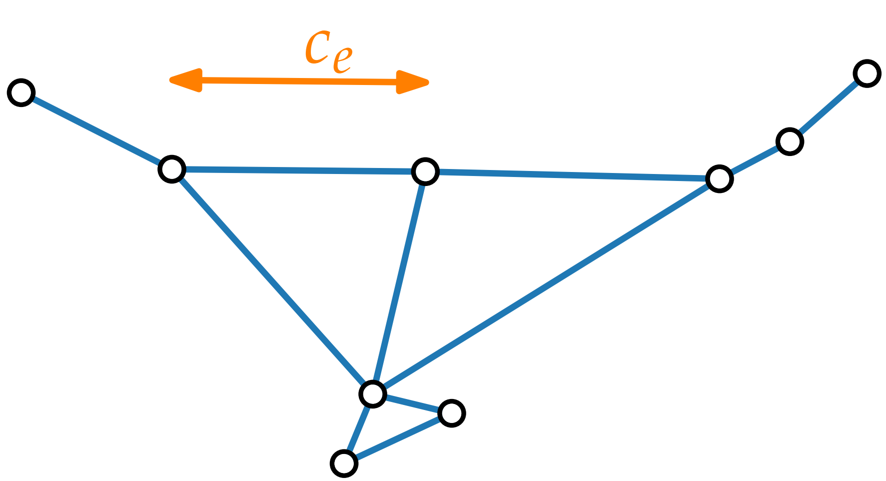

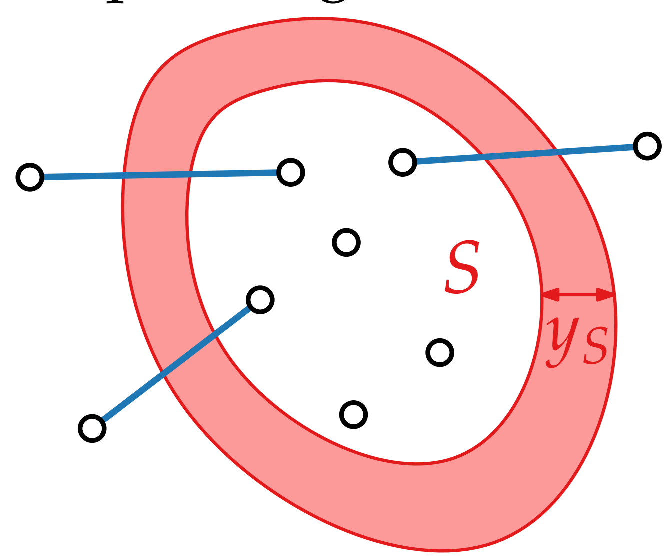

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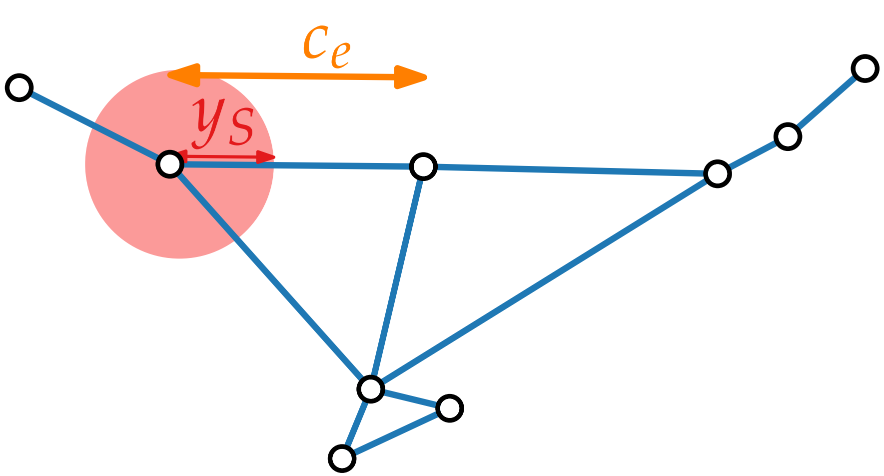

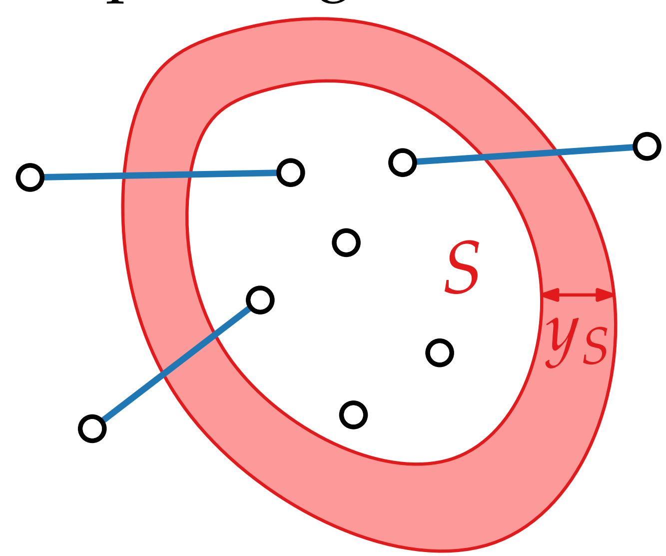

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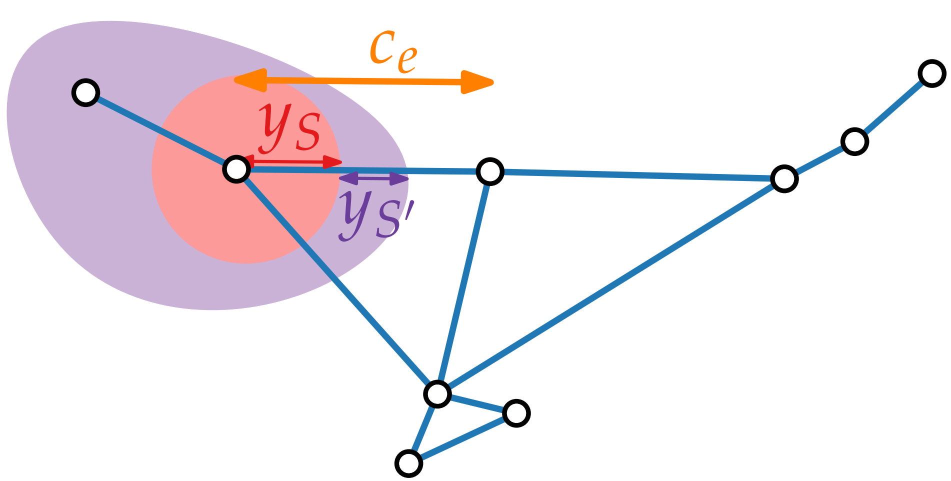

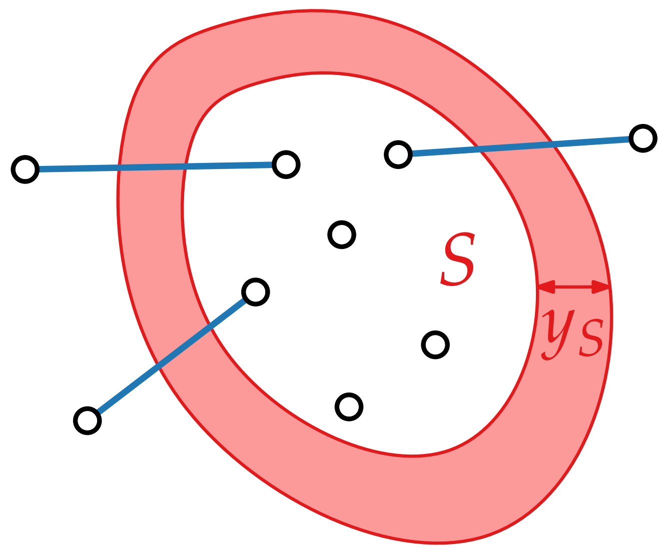

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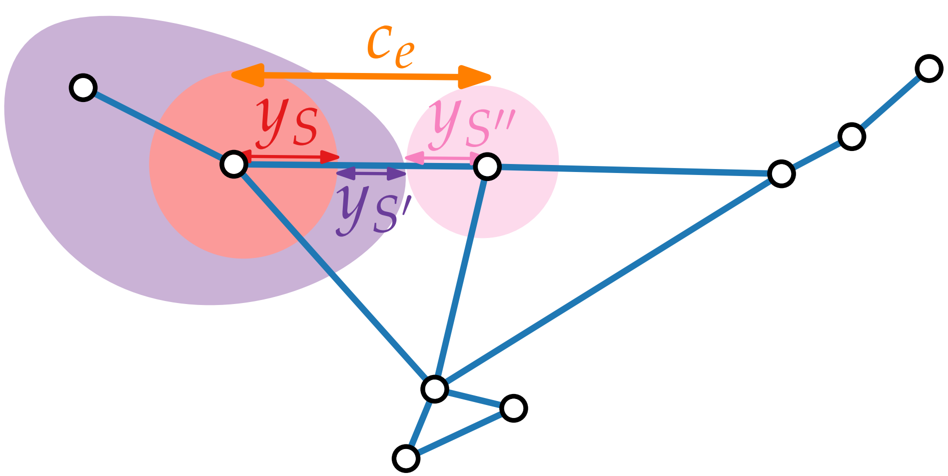

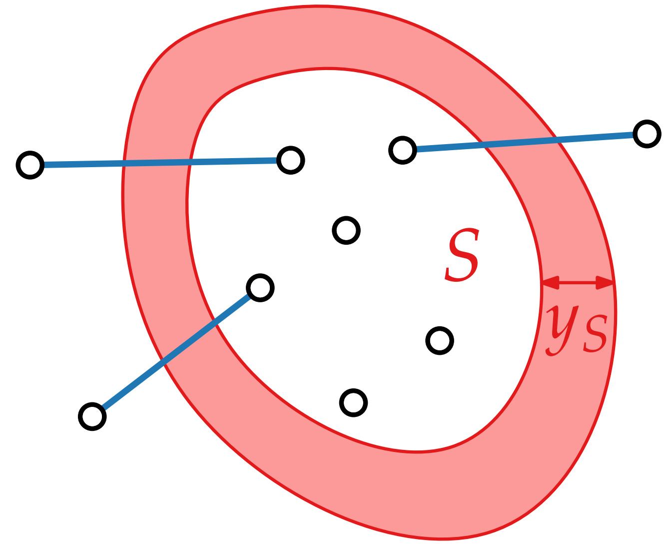

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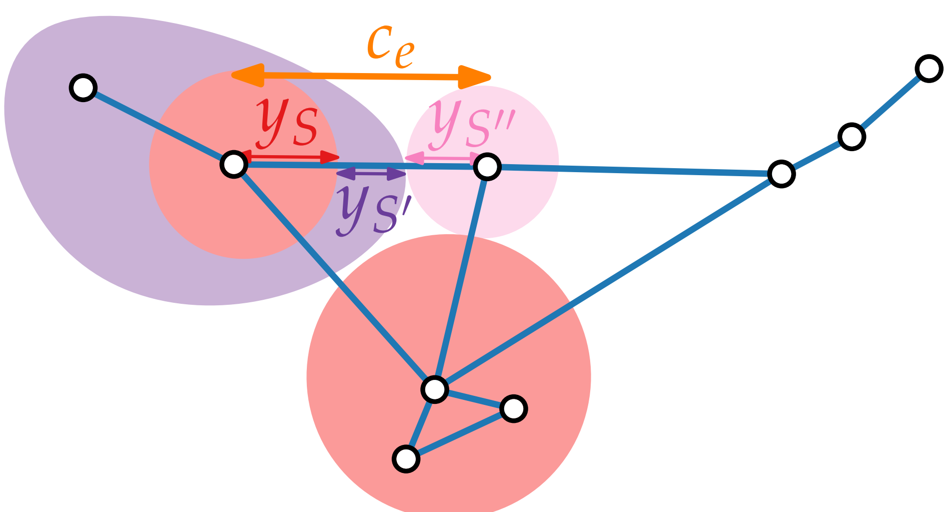

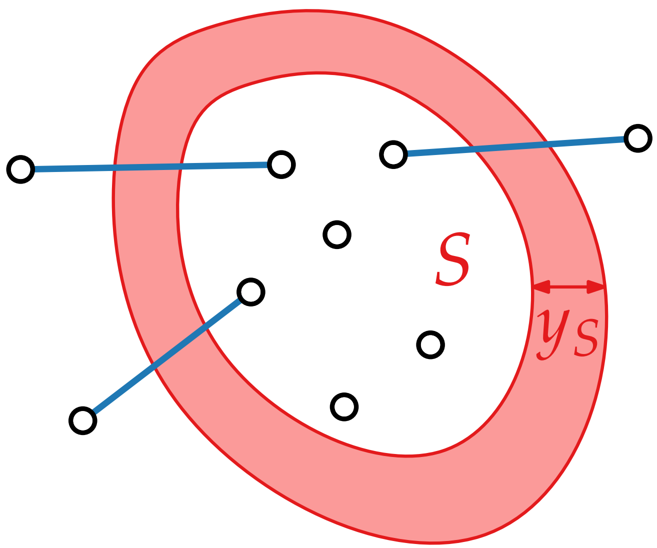

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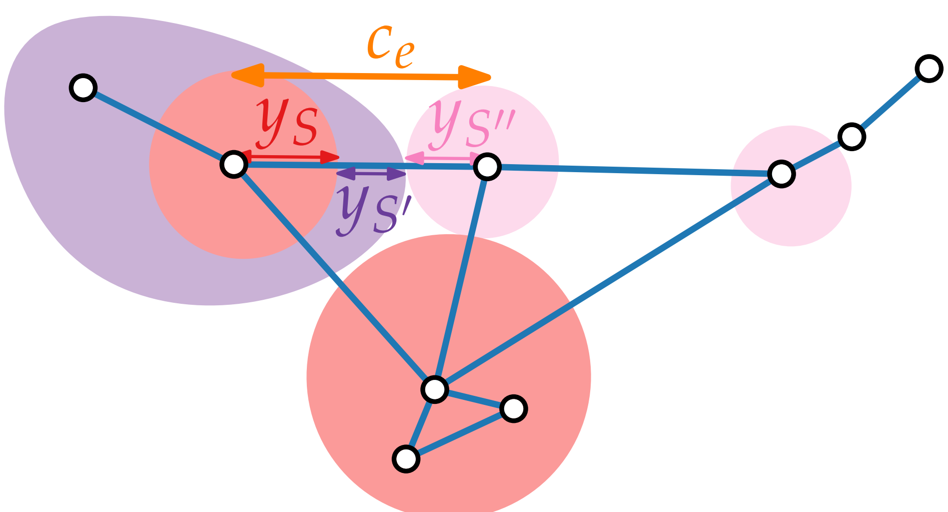

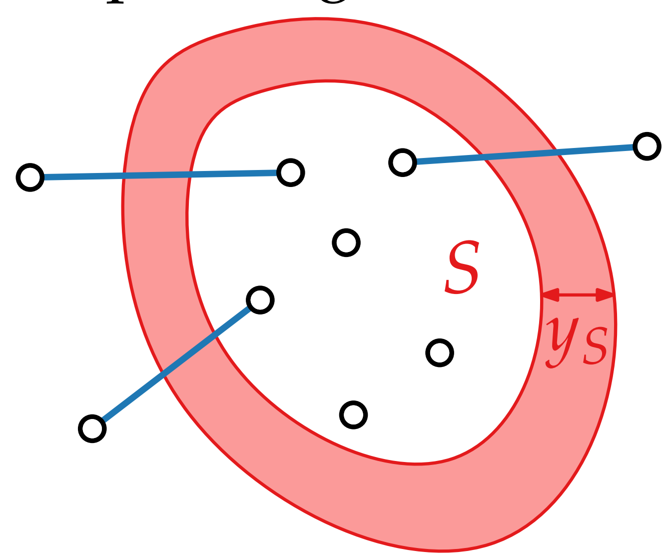

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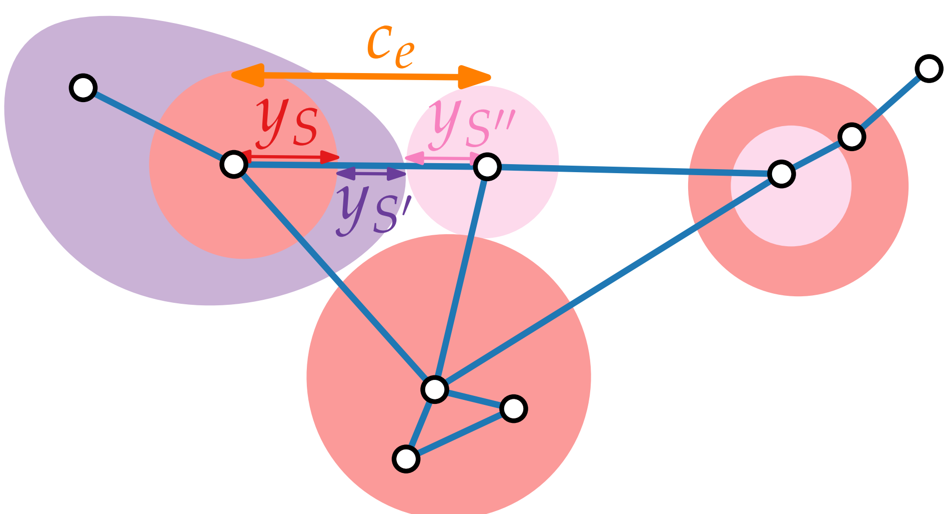

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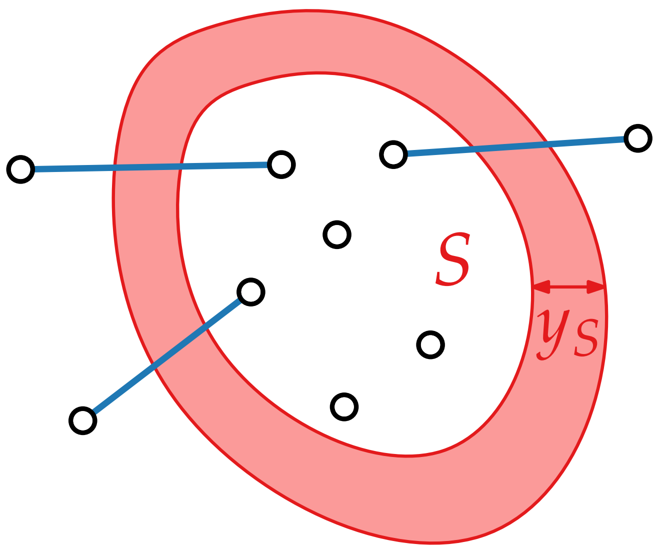

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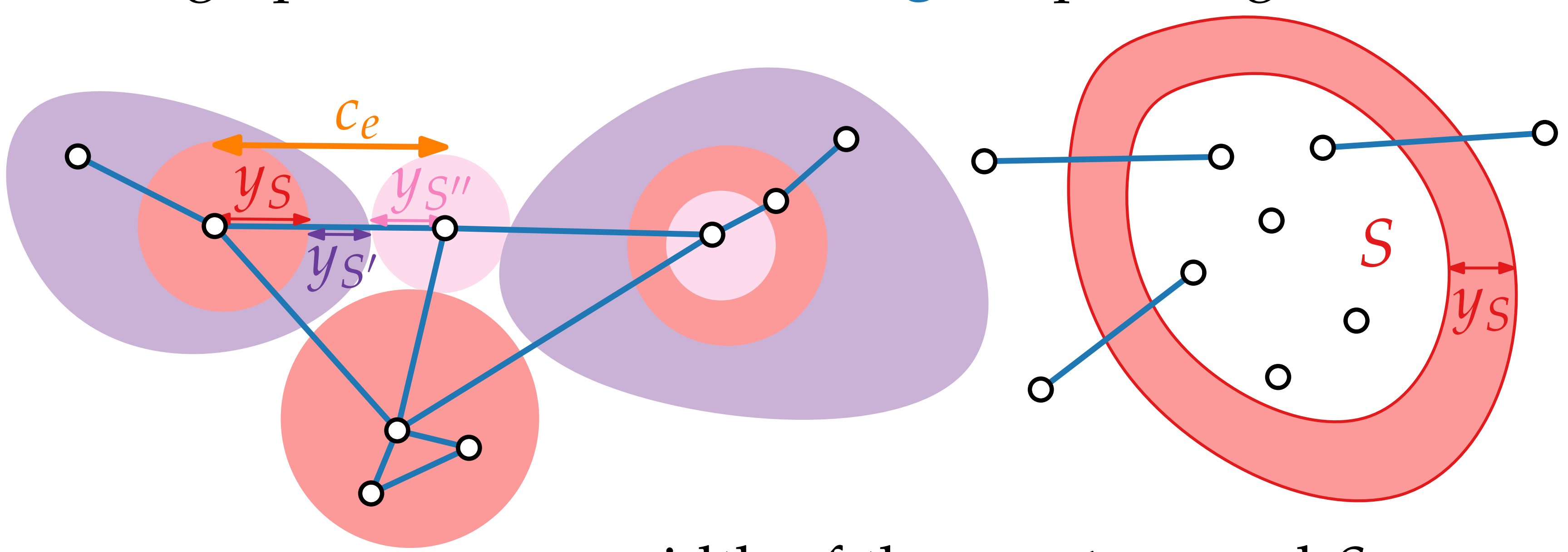

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part III:
A First Primal-Dual Approach

Philipp Kindermann

Summer Semester 2020

Complementary Slackness (Rep.)

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

A First Primal-Dual Approach

Complementary slackness: $x_e > 0 \quad \Rightarrow$

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How do we iteratively improve the Dual-Solution?

$\rightsquigarrow$ increase y_C! (until some edge in $\delta(C)$ becomes critical)

A First Primal-Dual Approach

PrimalDualSteinerForestNaive(G, c, R)

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$y \leftarrow 0, F \leftarrow \emptyset$

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Running Time?

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Running Time?

Trick: Handle all y_S with $y_S = 0$ implicitly

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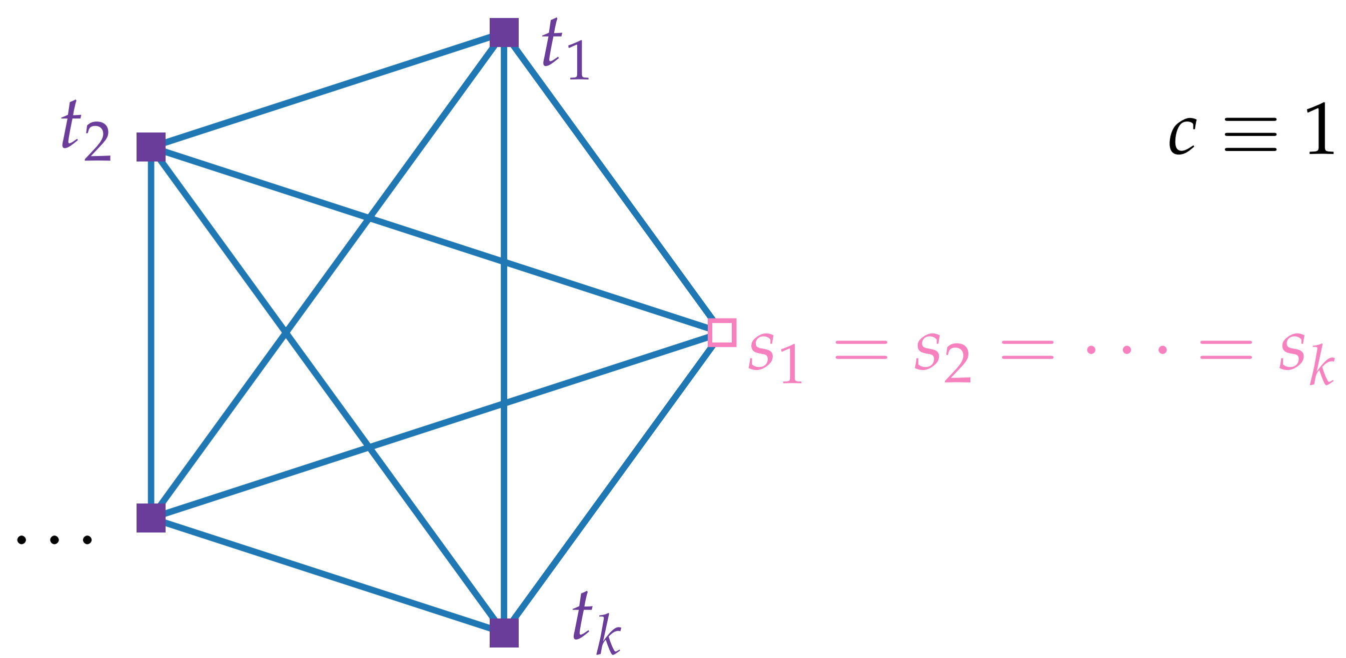

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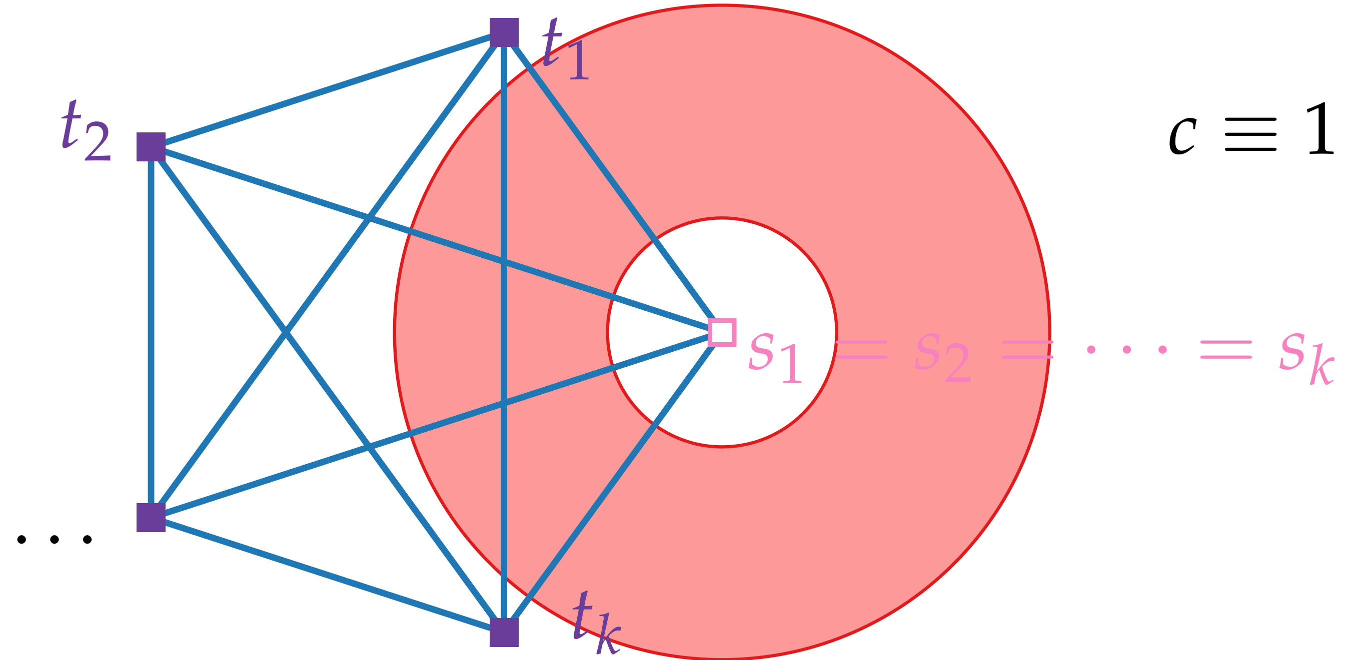

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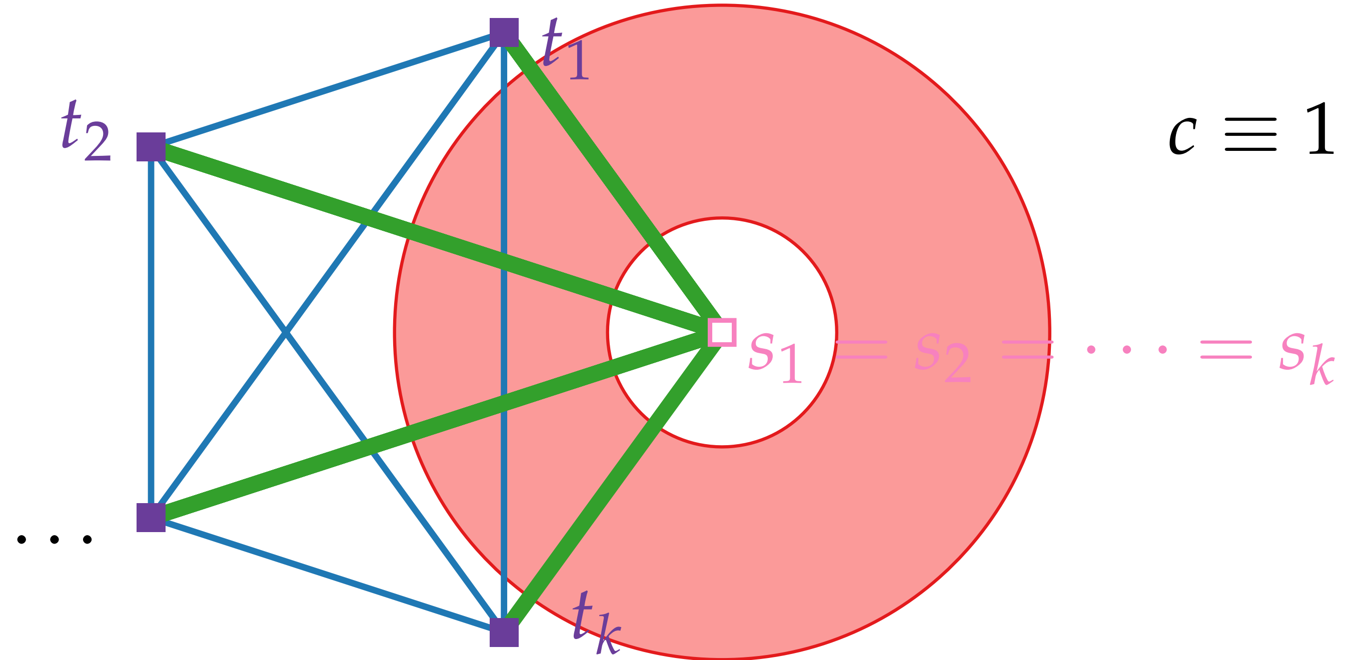

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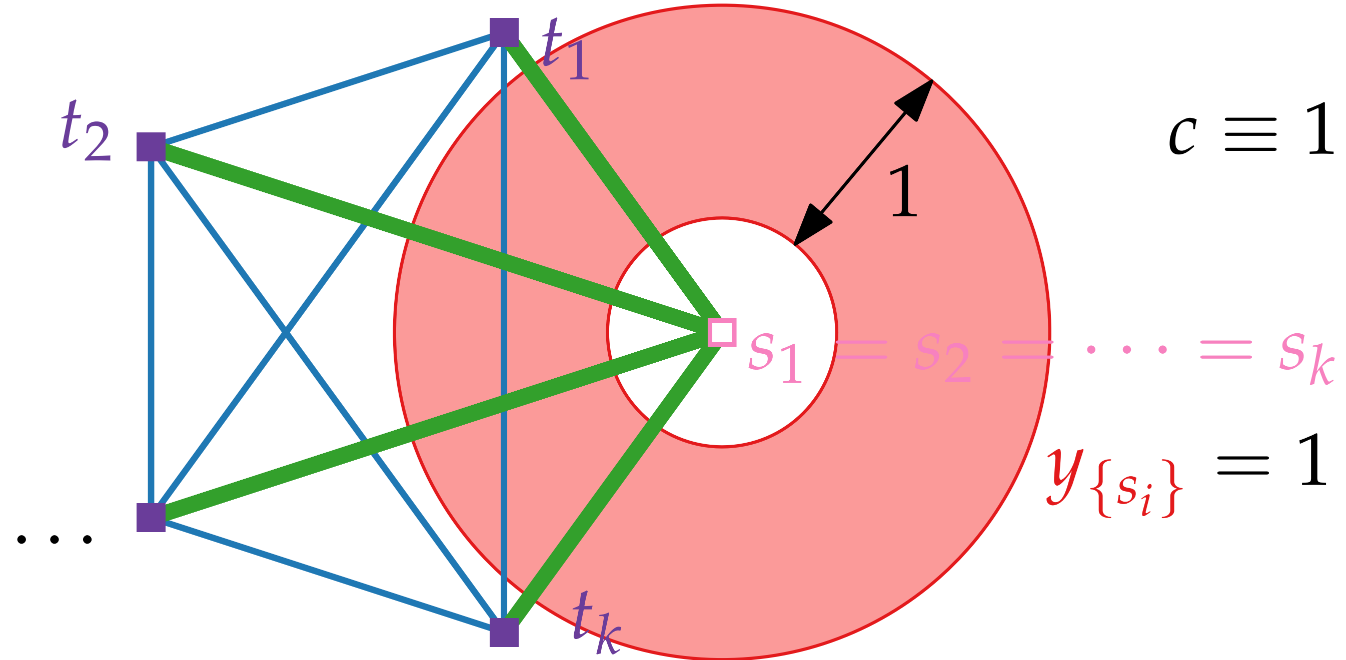

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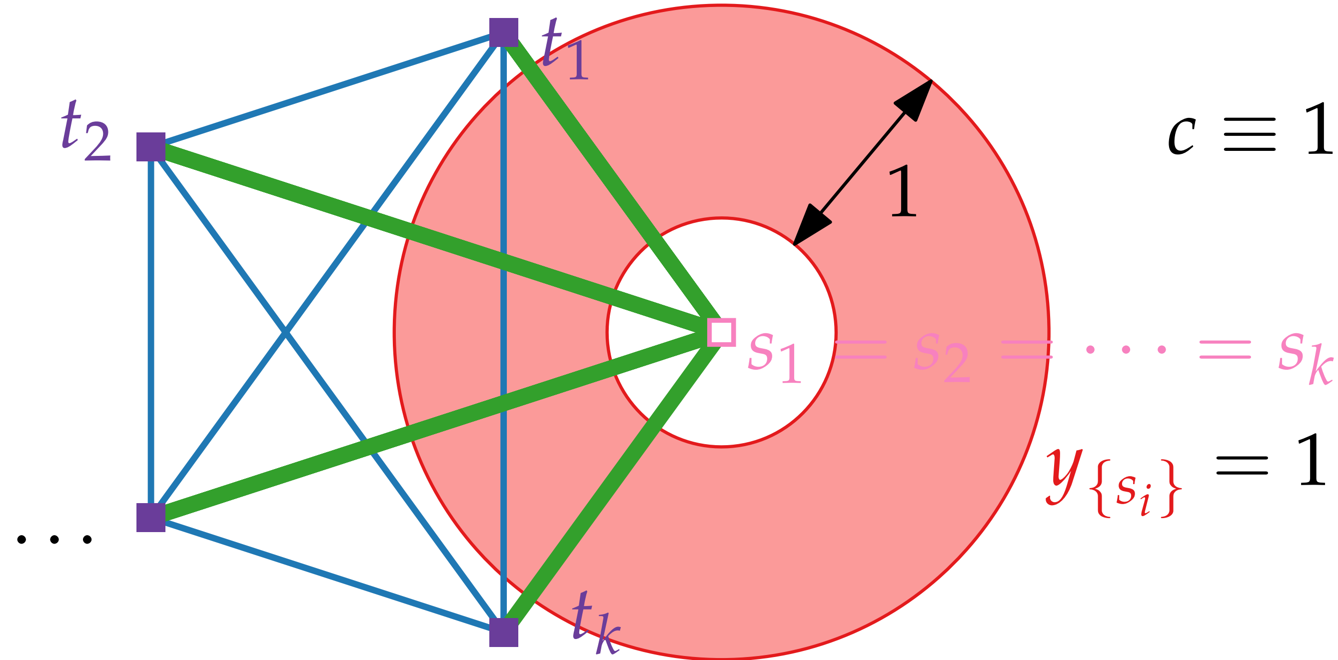

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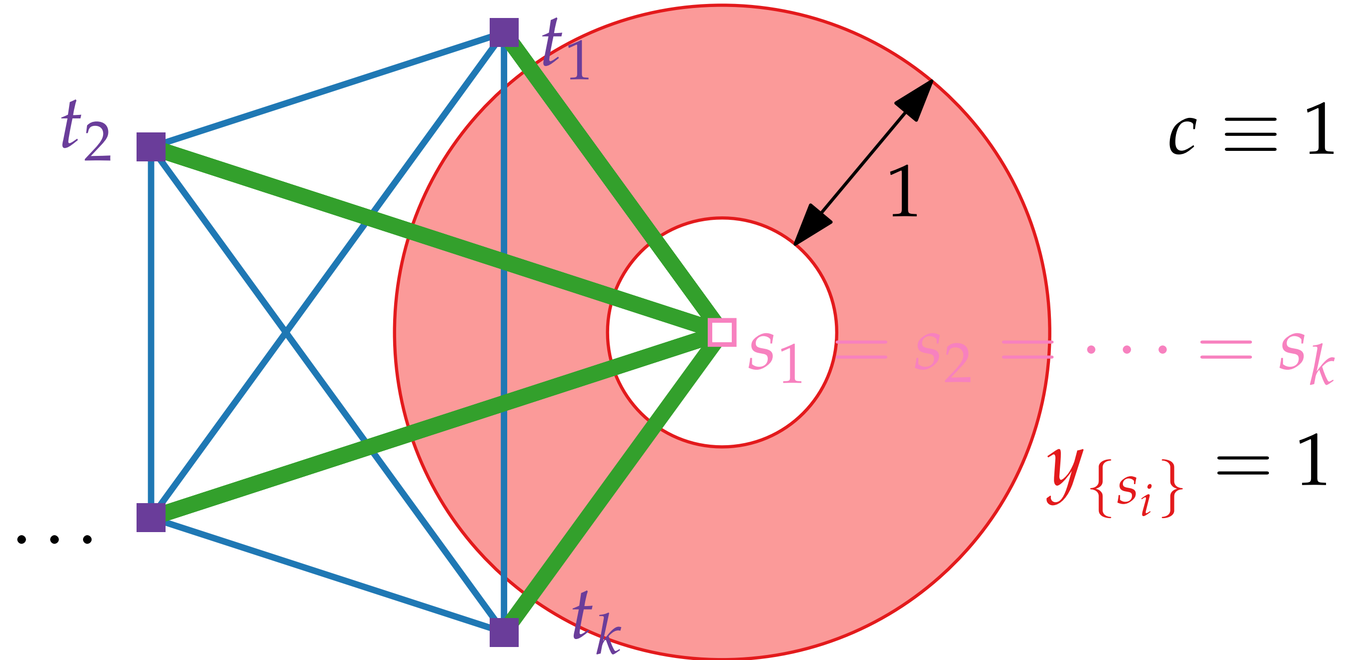
There are examples with $|\delta(S) \cap F| = k$ for each $y_S > 0$:

But: Average degree of component is 2!

$\Rightarrow$ Increase y_C for all components C simultaneously!



Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part IV:
Primal-Dual with Synchronized Increases

Philipp Kindermann

Summer Semester 2020

Primal-Dual with Synchronized Increases

PrimalDualSteinerForest(G, c, R)

$y \leftarrow 0, F \leftarrow \emptyset, \ell \leftarrow 0$
while some $(s_i, t_i) \in R$ not connected in (V, F) **do**
  $\ell \leftarrow \ell + 1$

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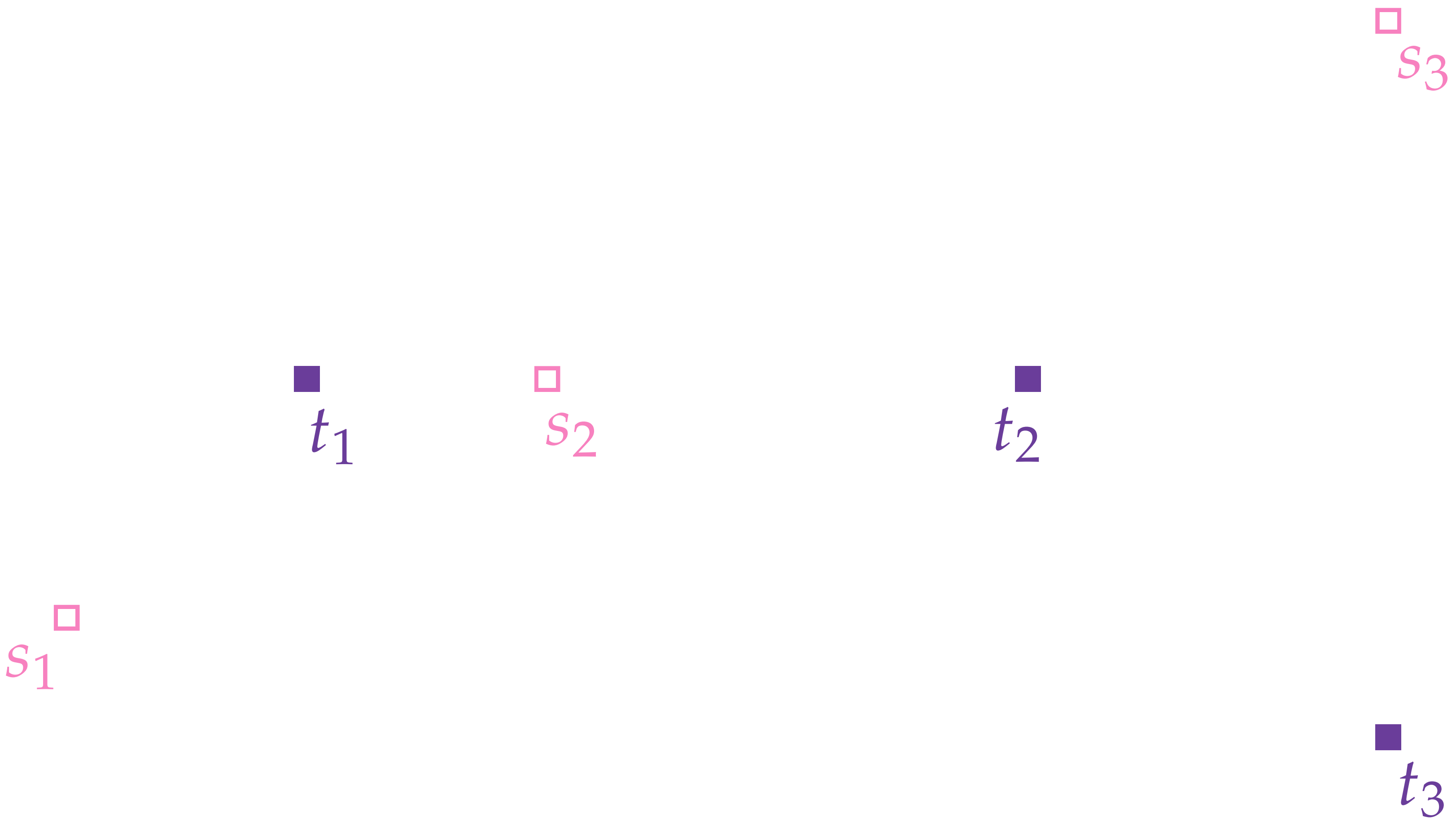
Illustration

$G = K_6$ with Euclidean edge costs



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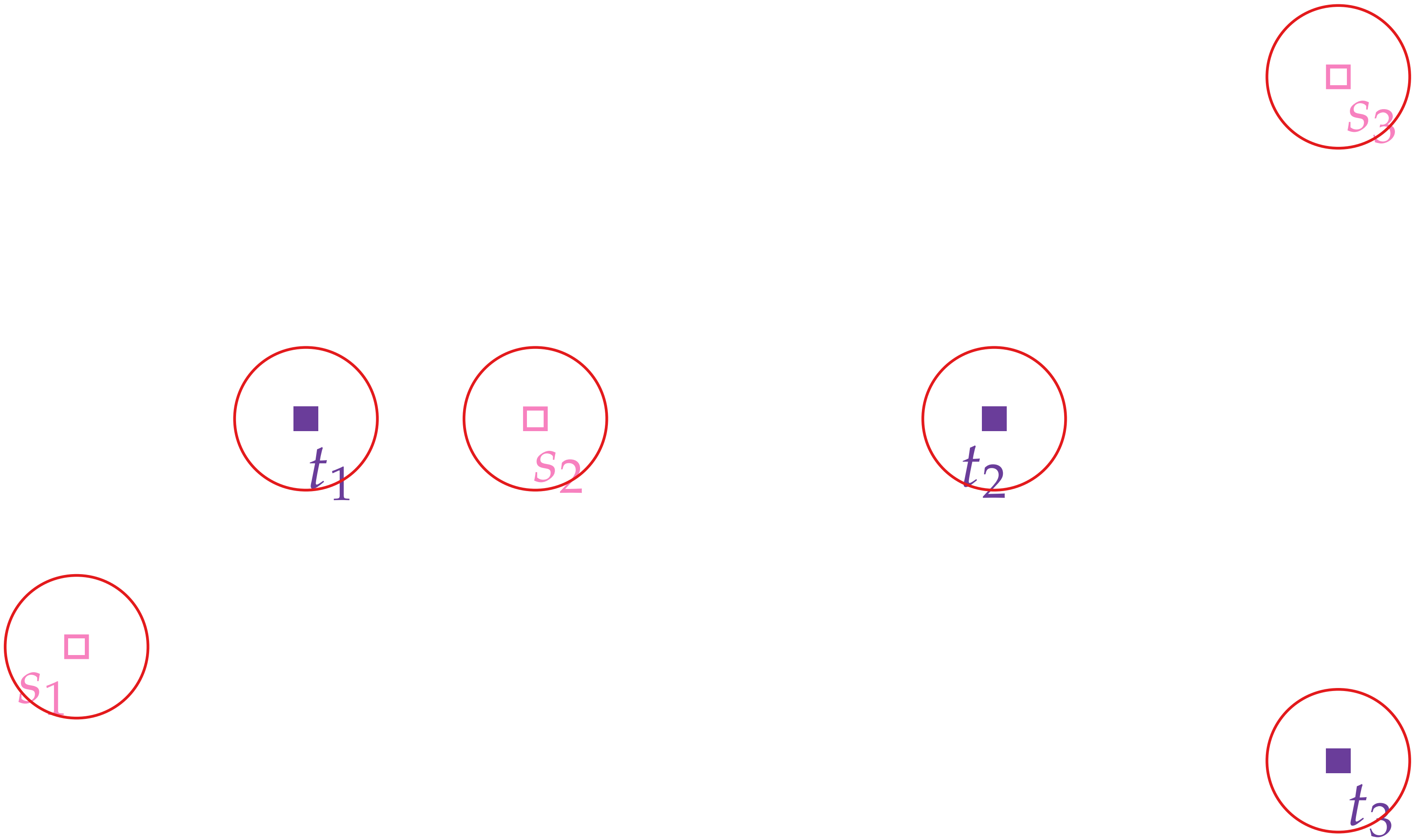

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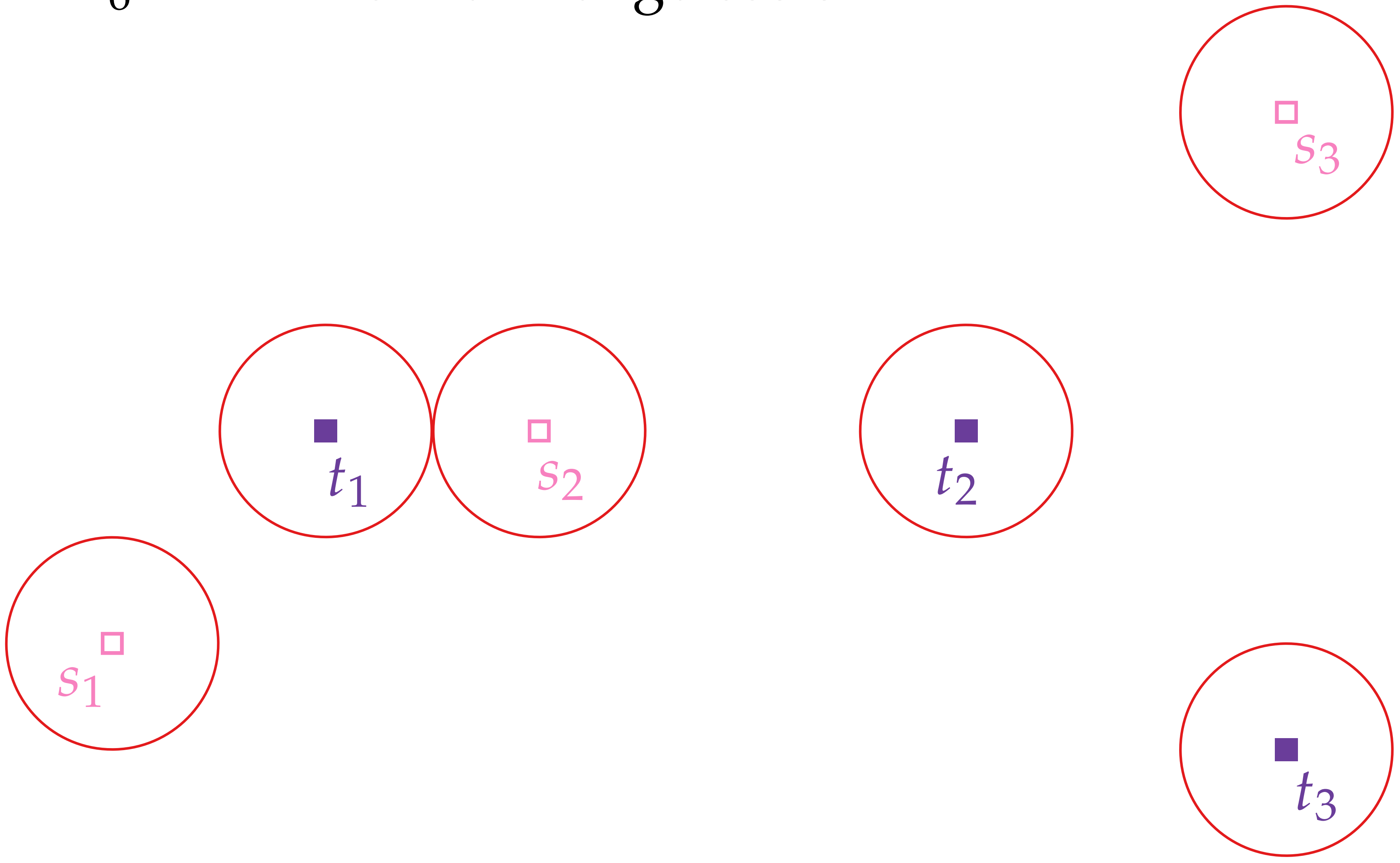

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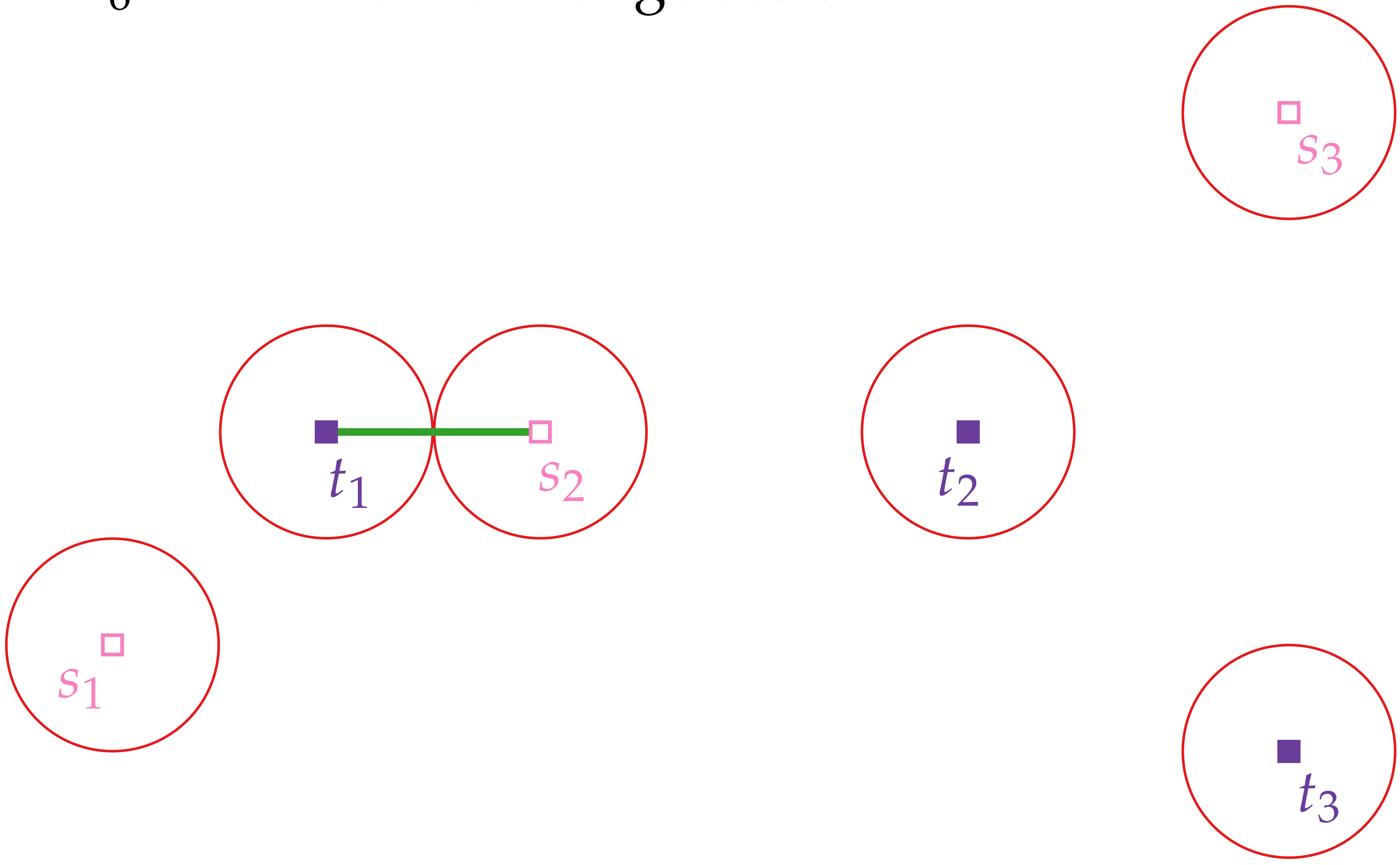

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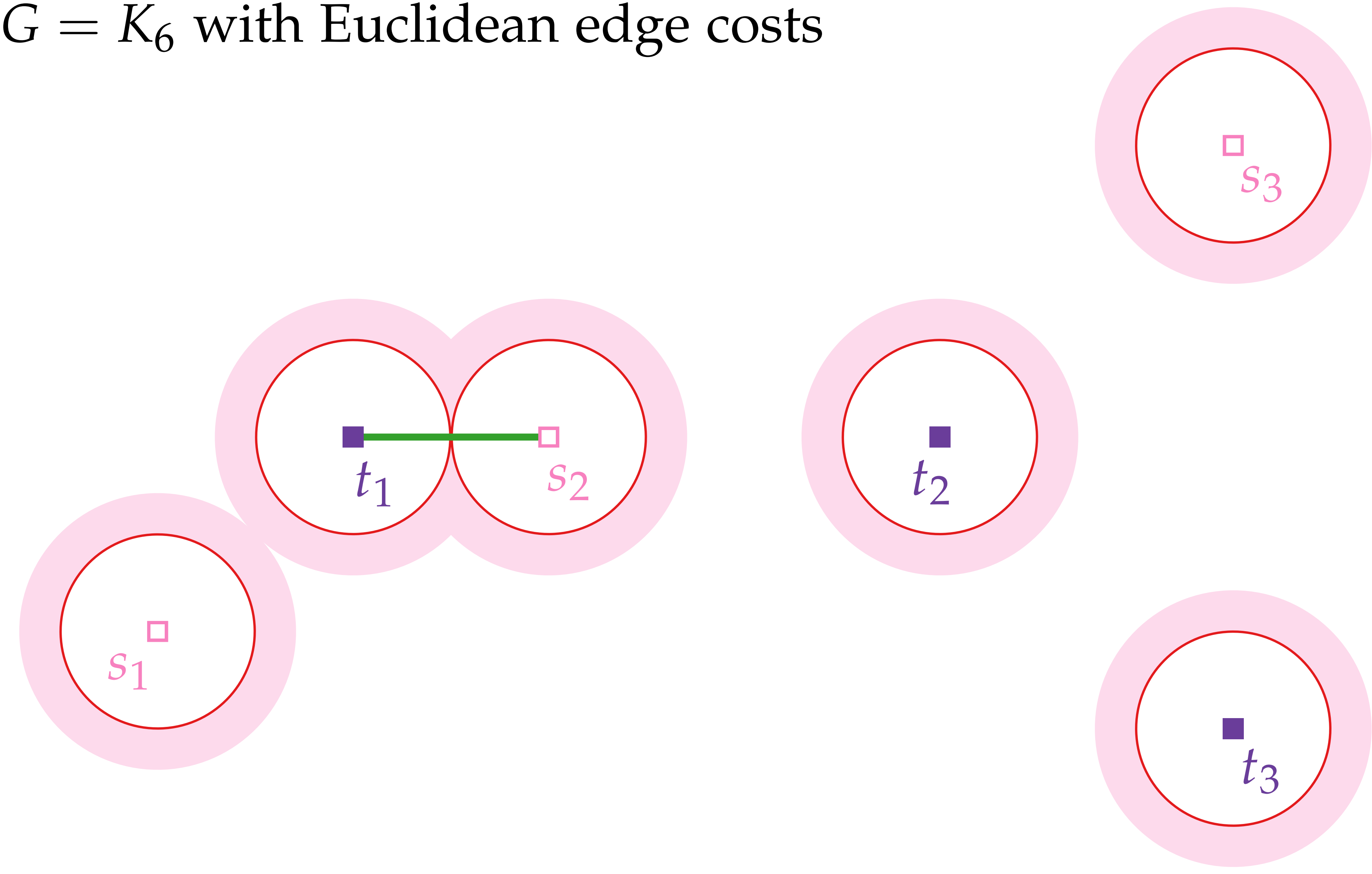

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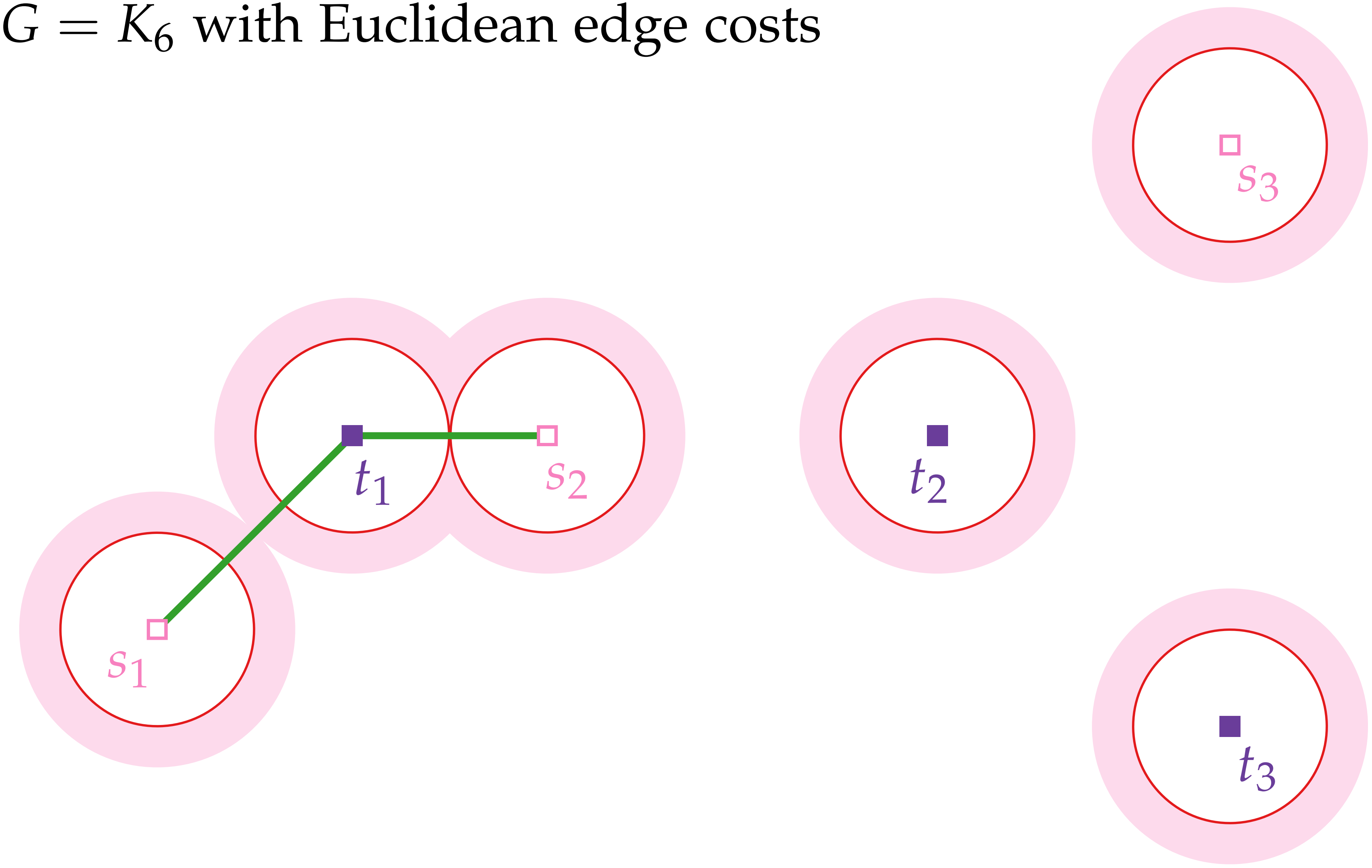

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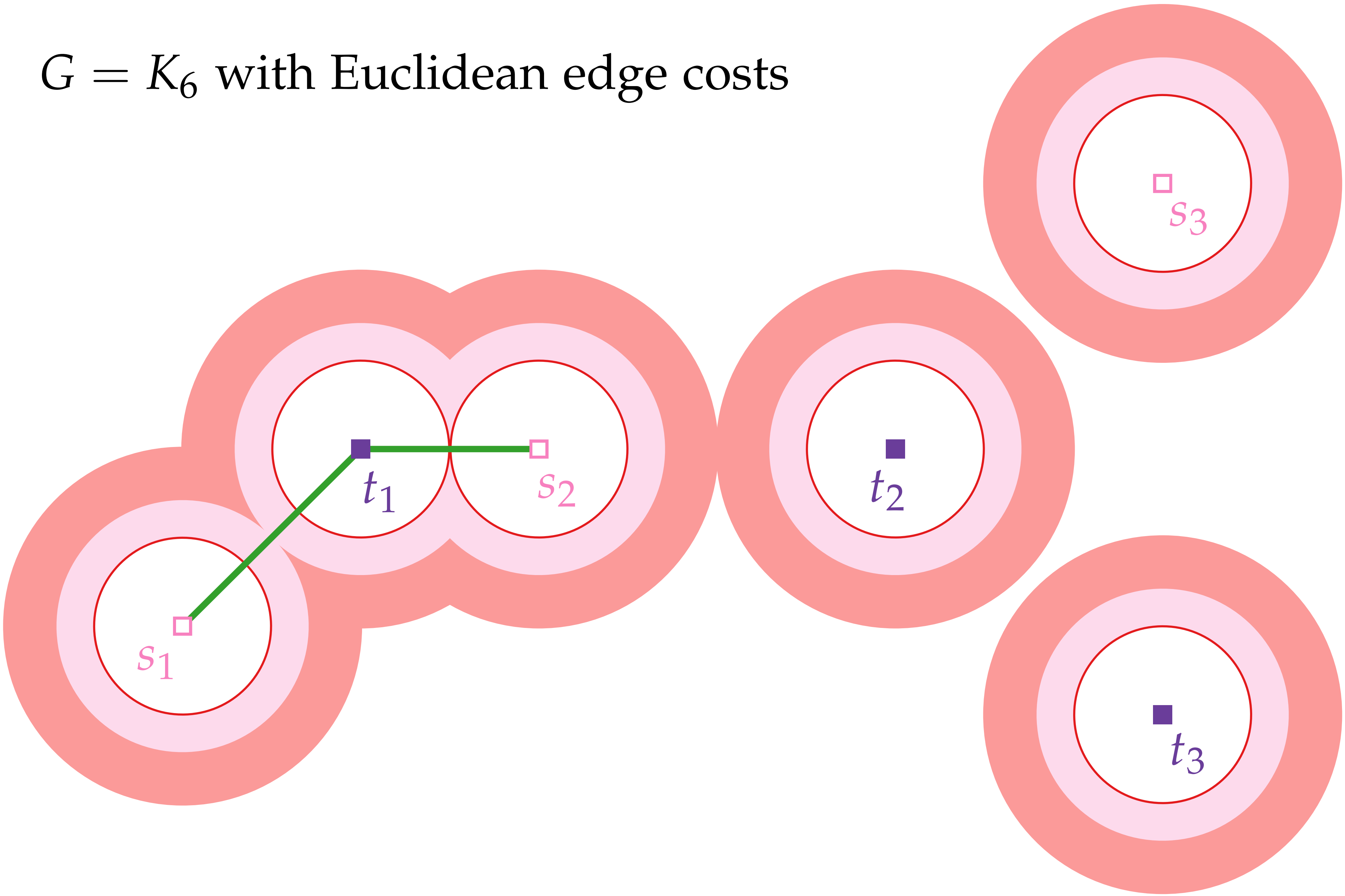

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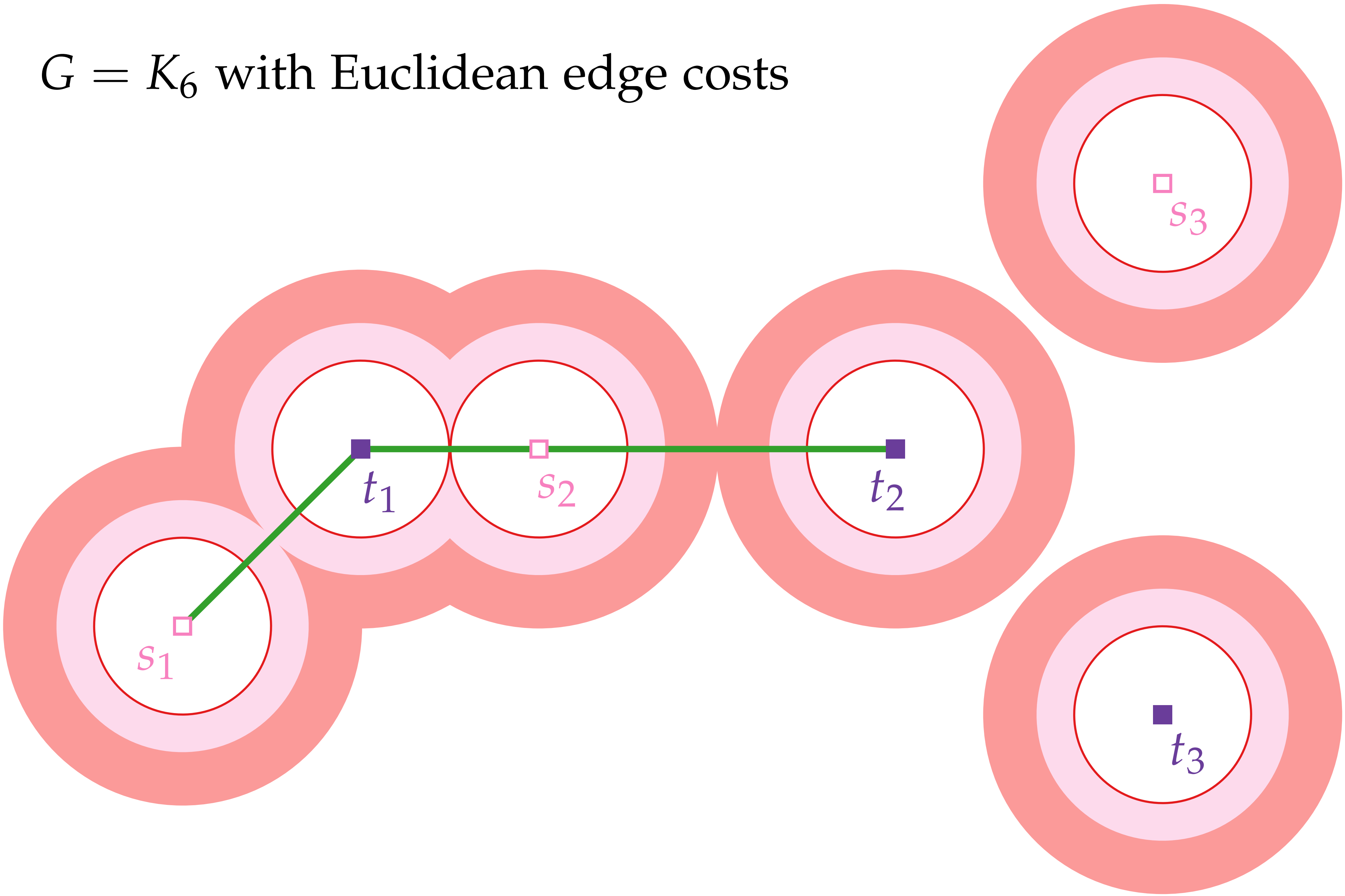

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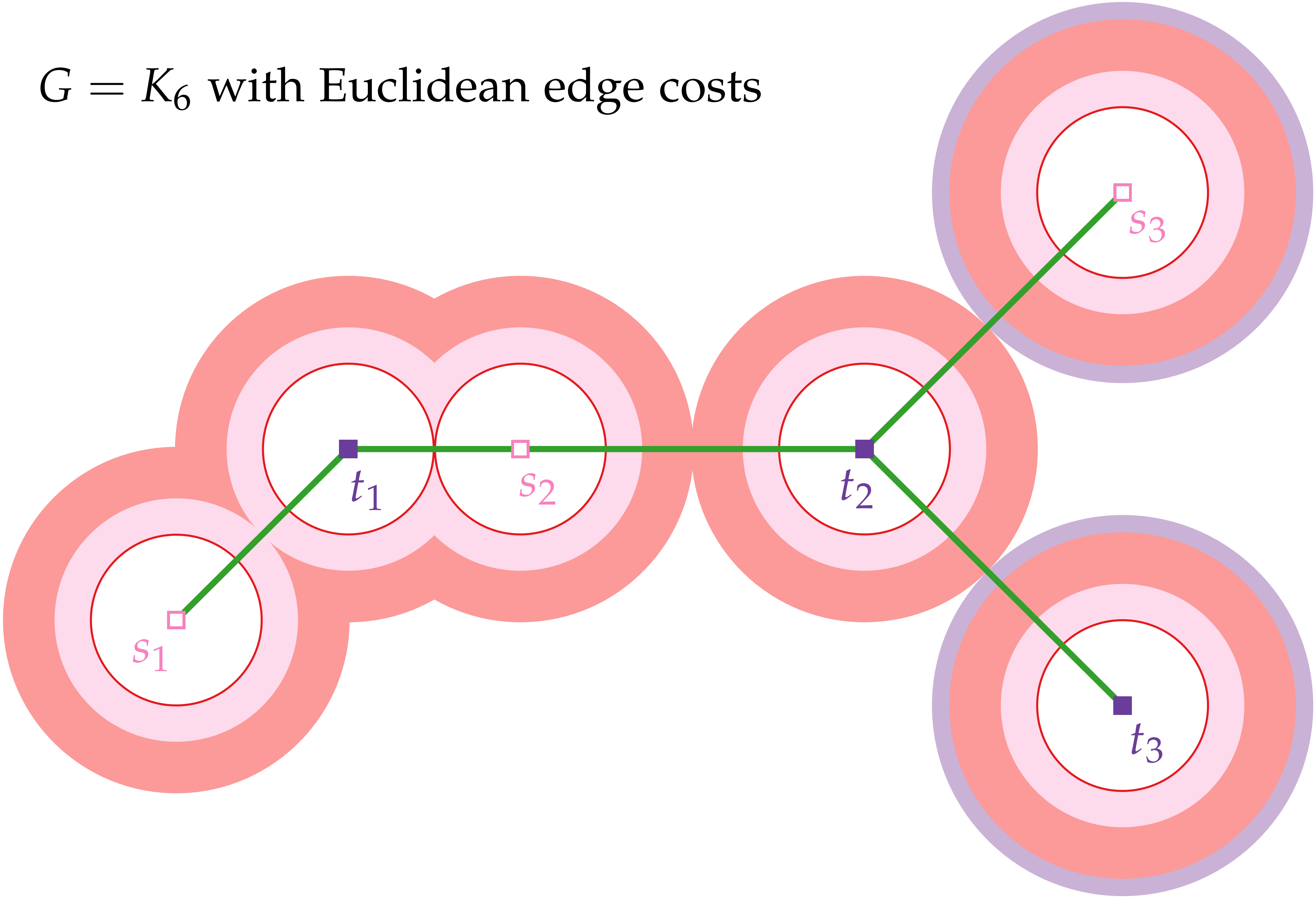

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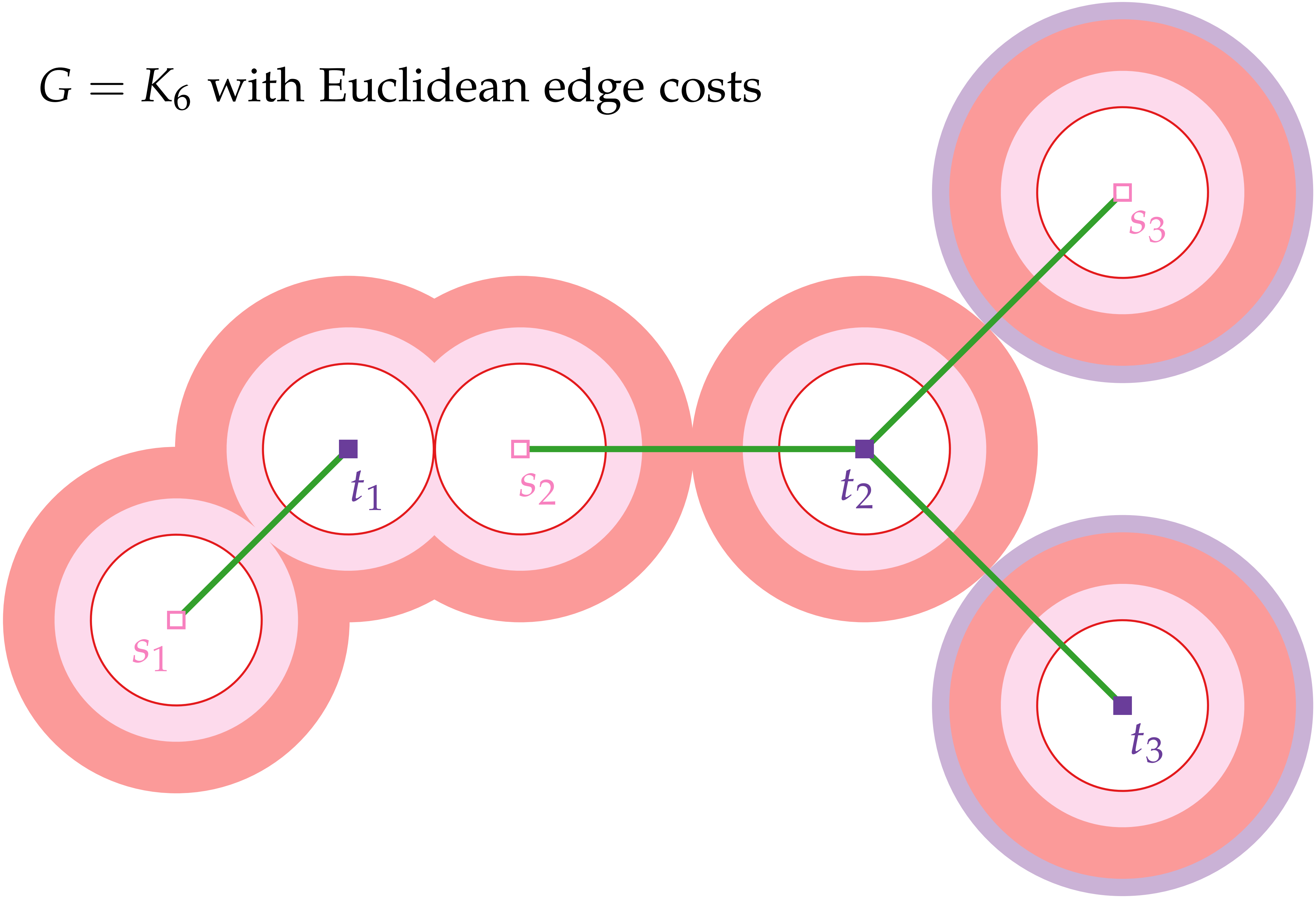

Approximation Algorithms

Lecture 12:
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Part V:
Structure Lemma

Philipp Kindermann

Summer Semester 2020

Structure Lemma

Lemma. For each $\mathcal{C}$ of an iteration of the algorithm:

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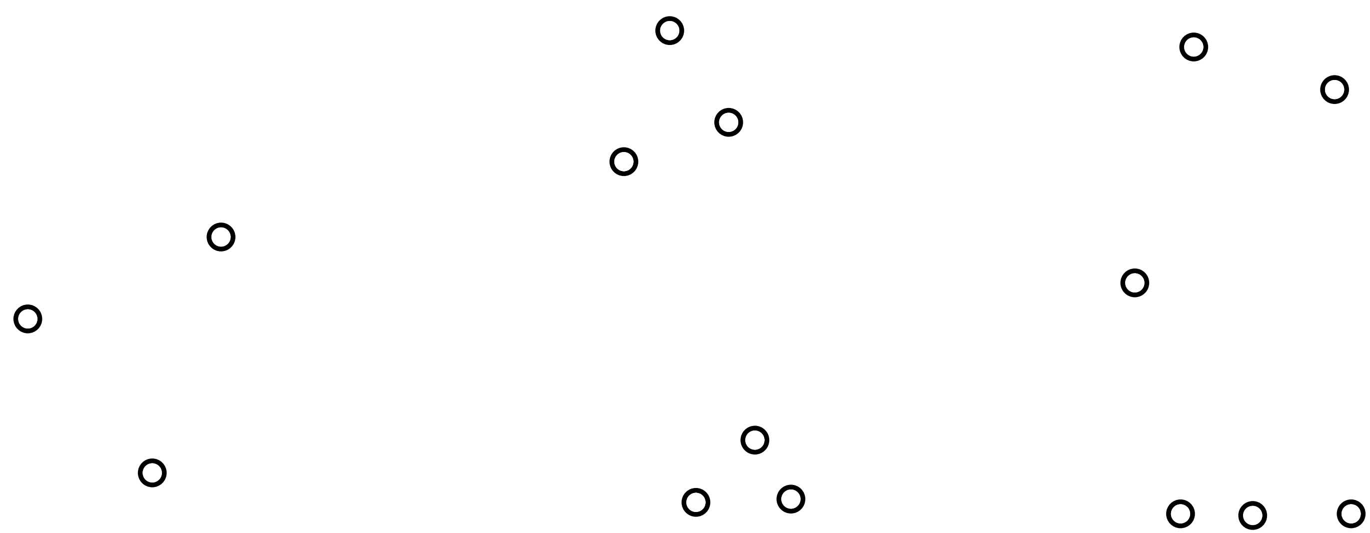

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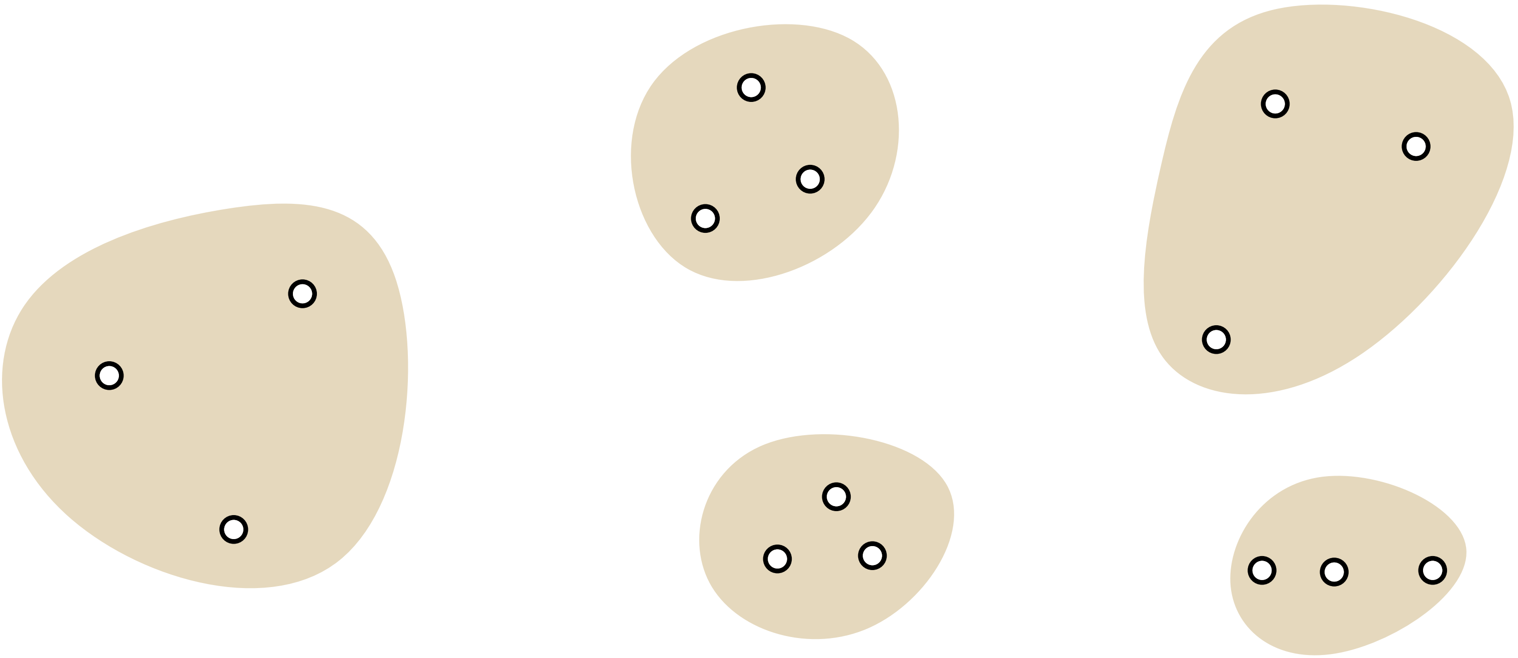

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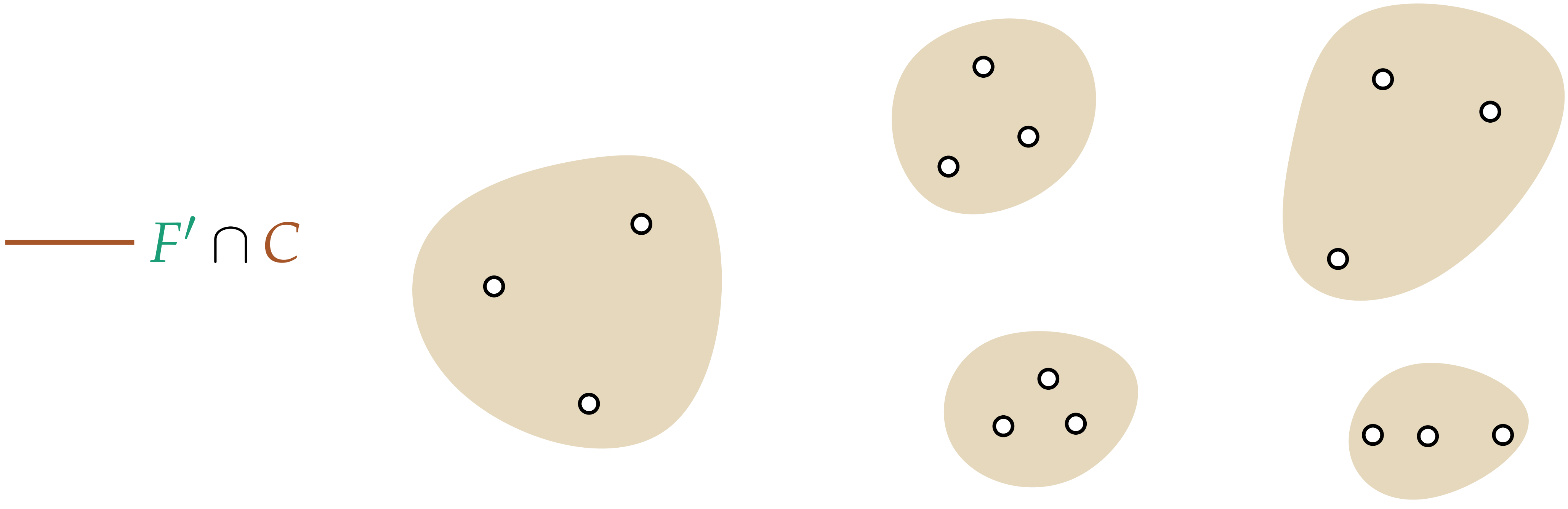

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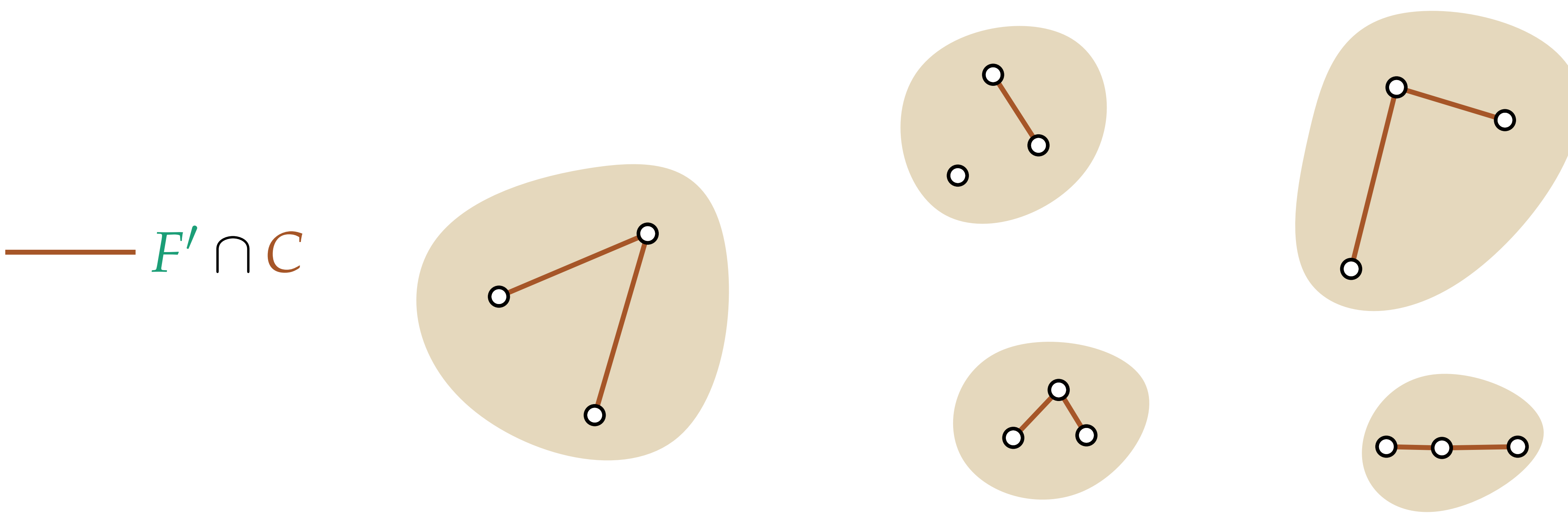

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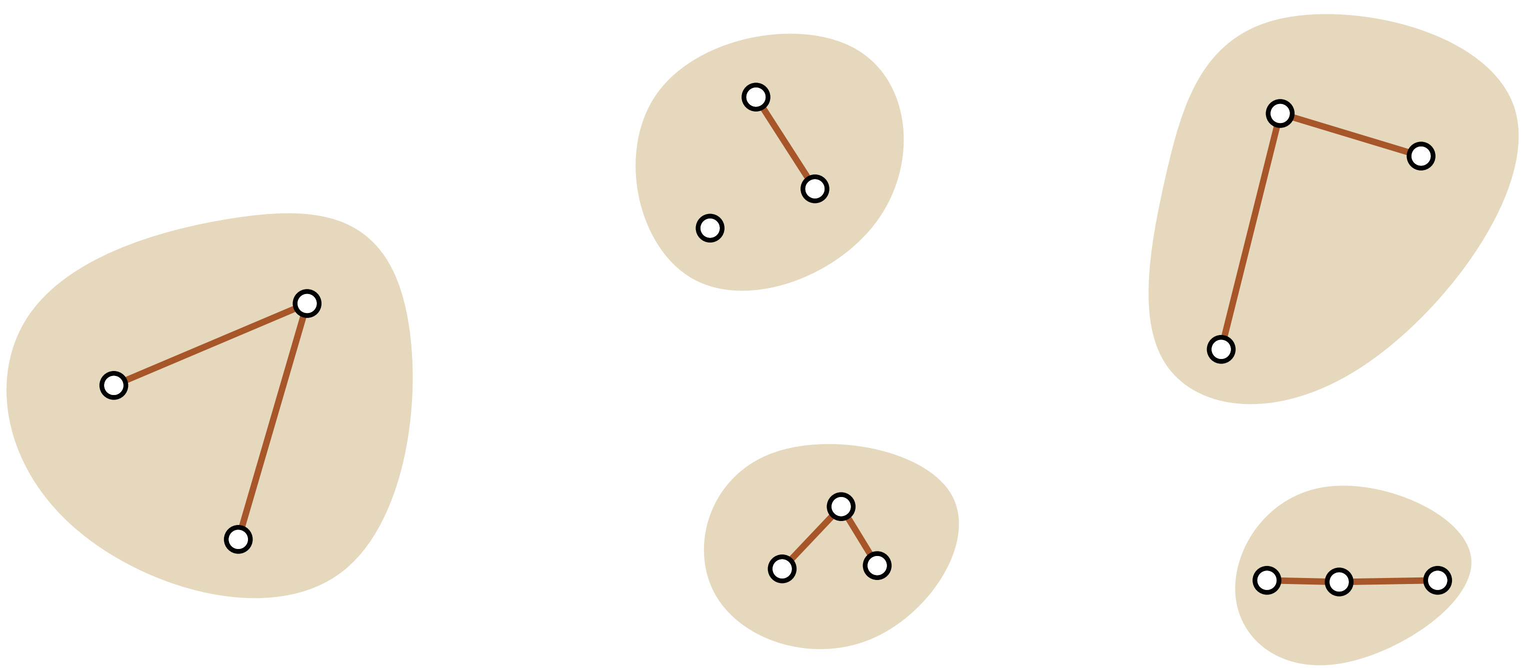

$F' \cap C$

$F - F'$

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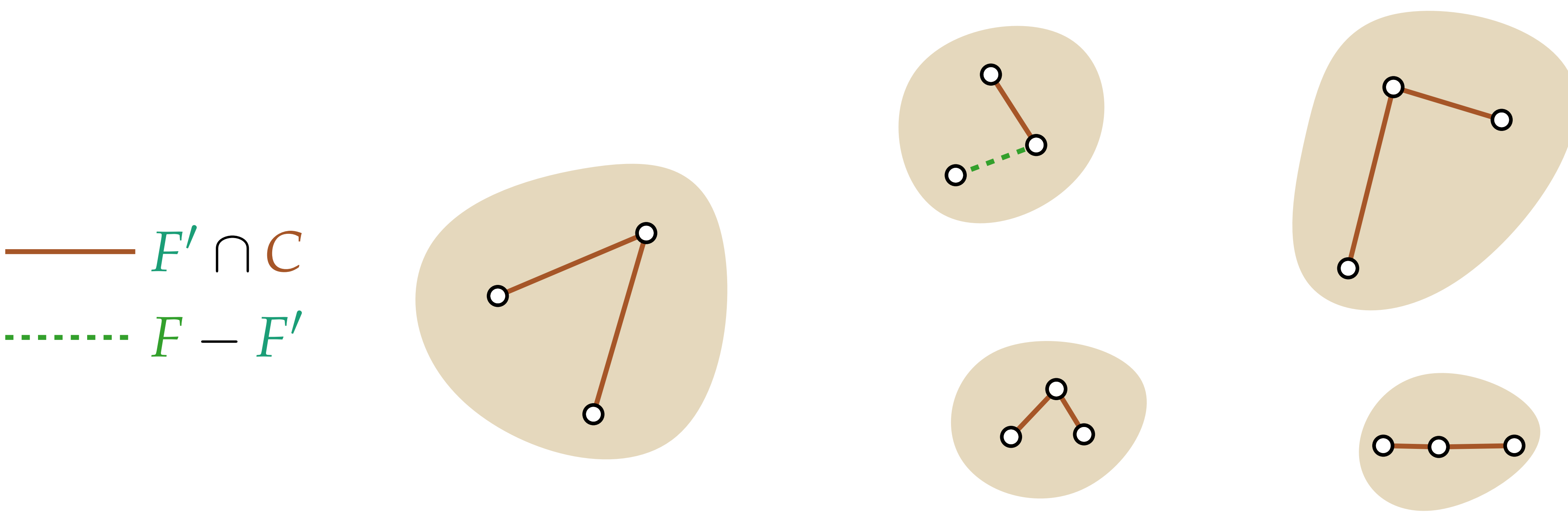

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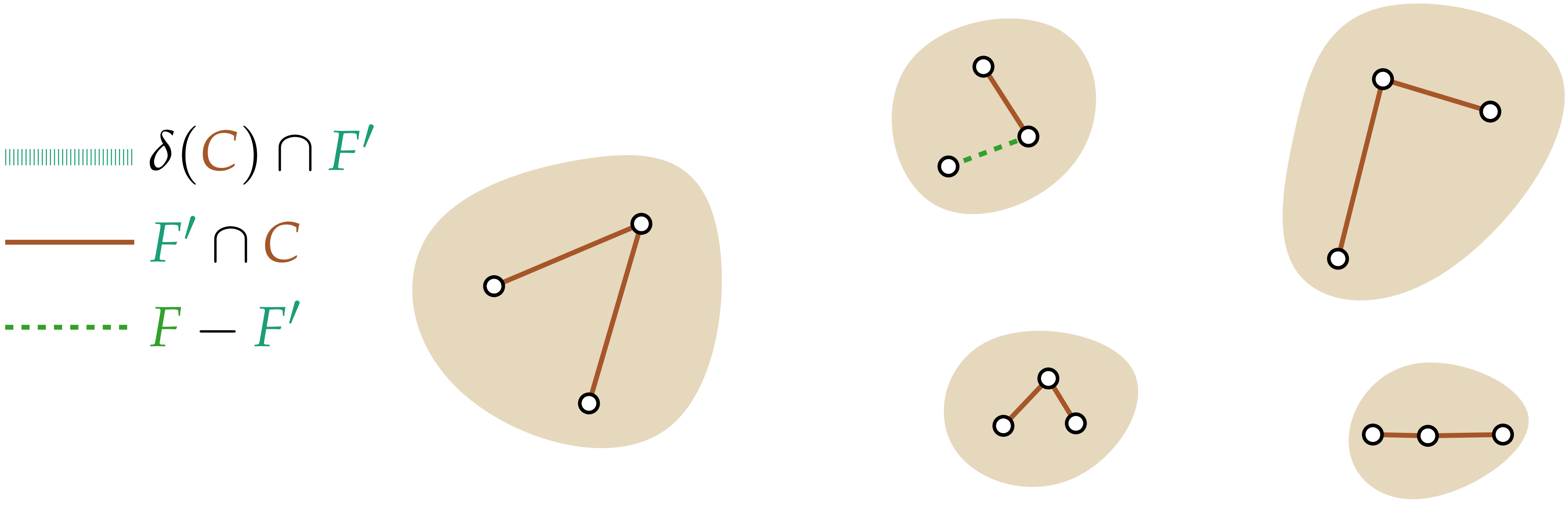

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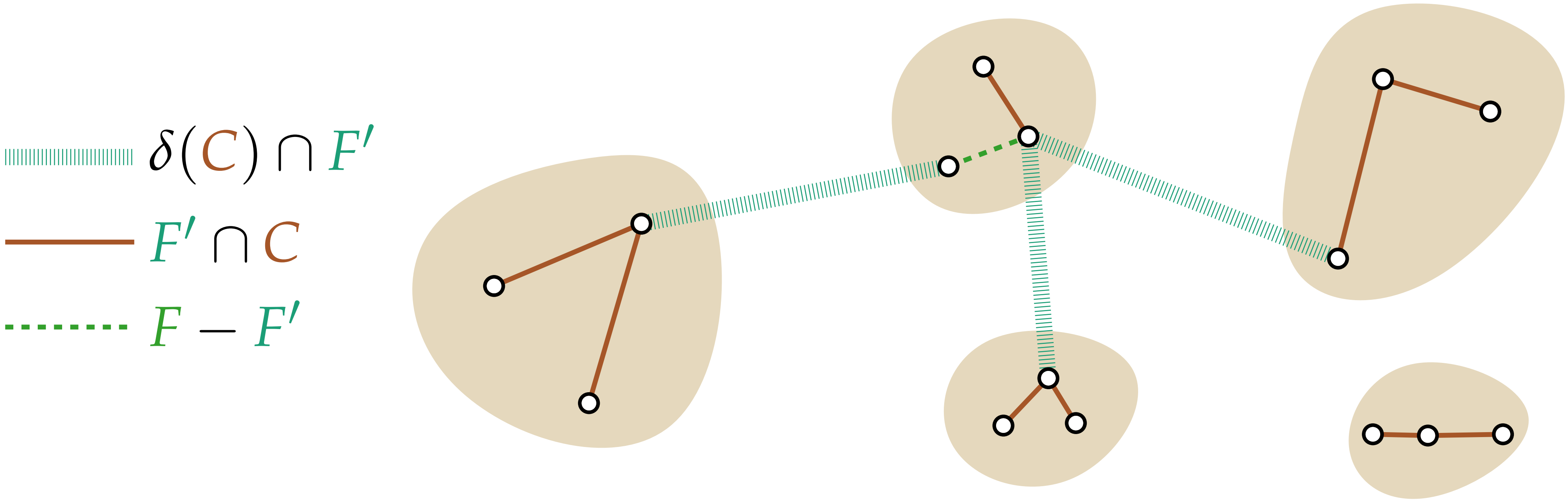

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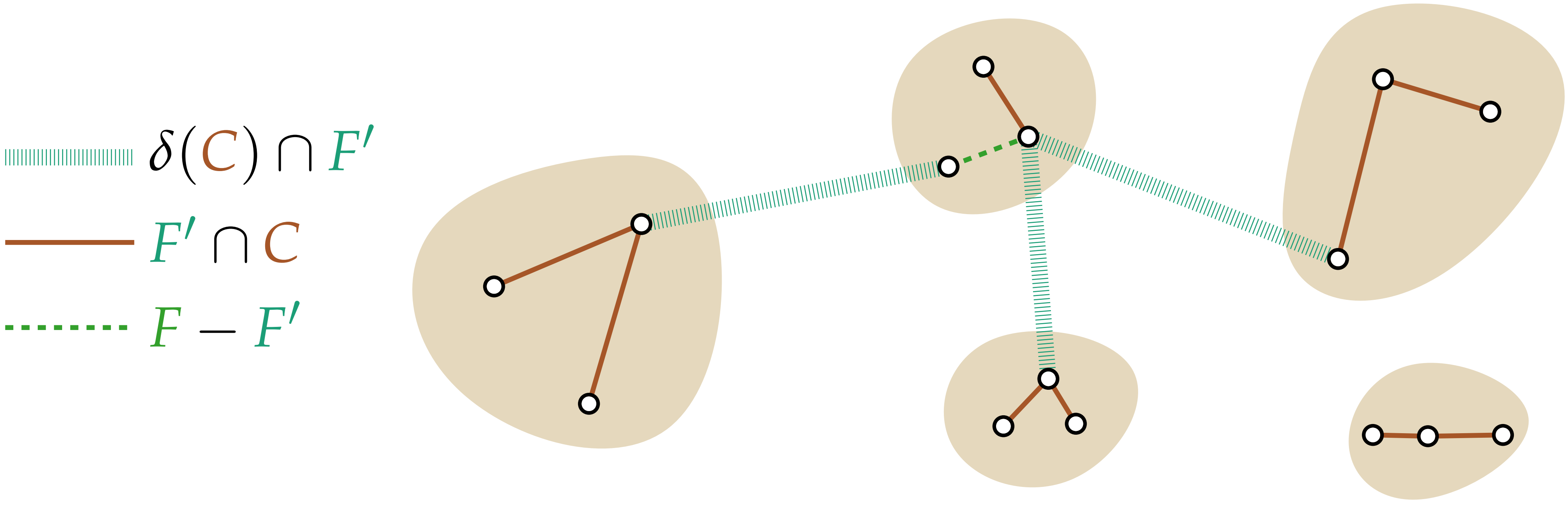

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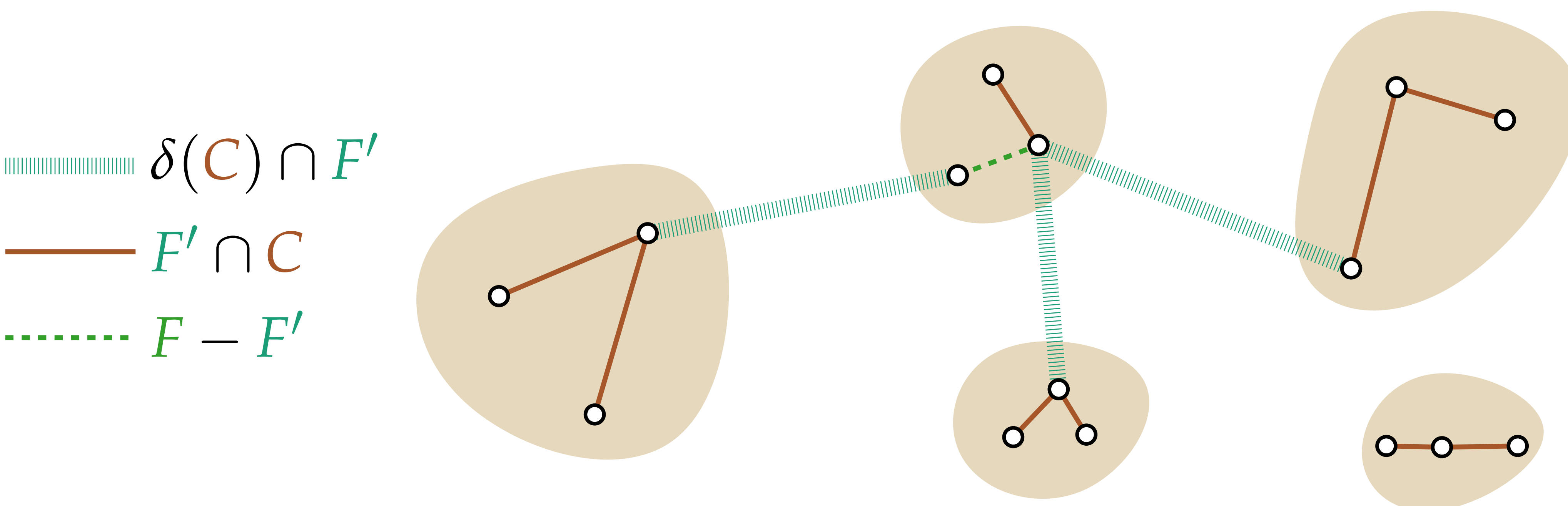

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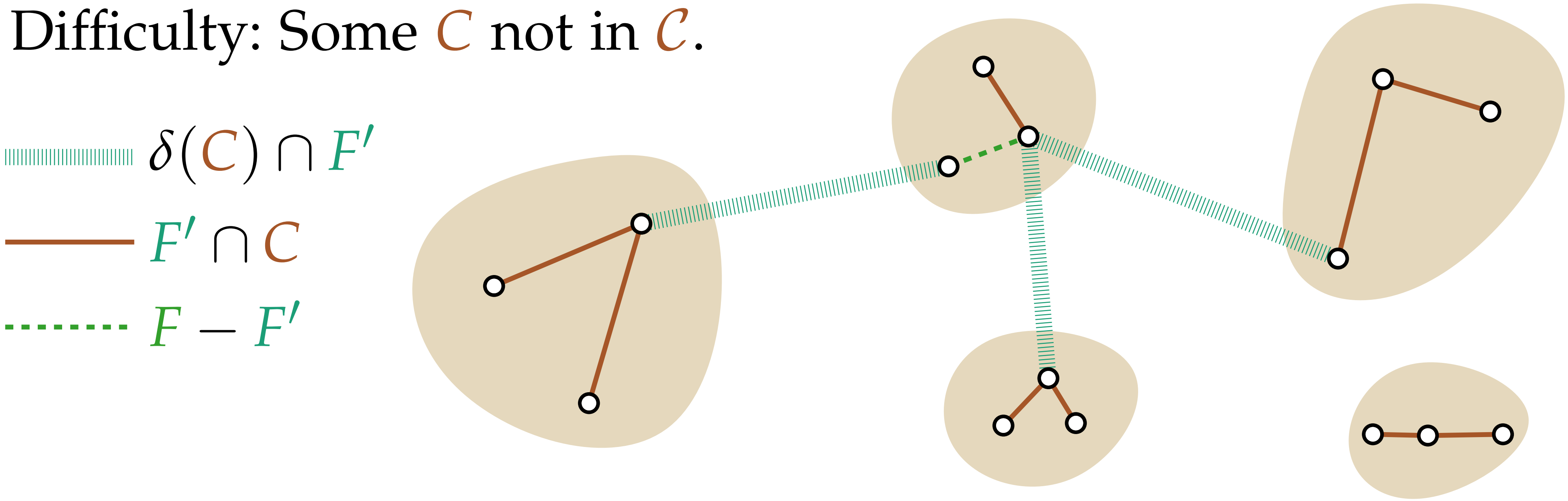
Difficulty: Some C not in $\mathcal{C}$.



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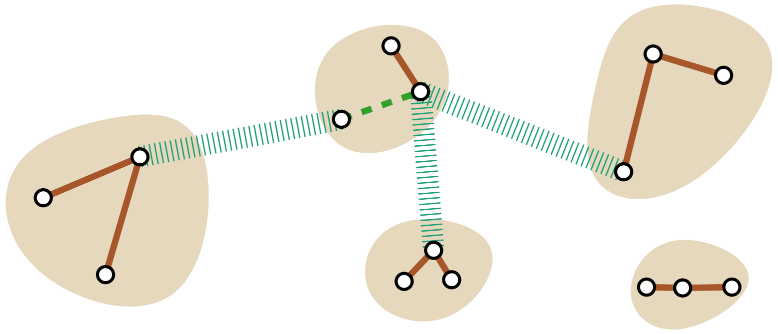

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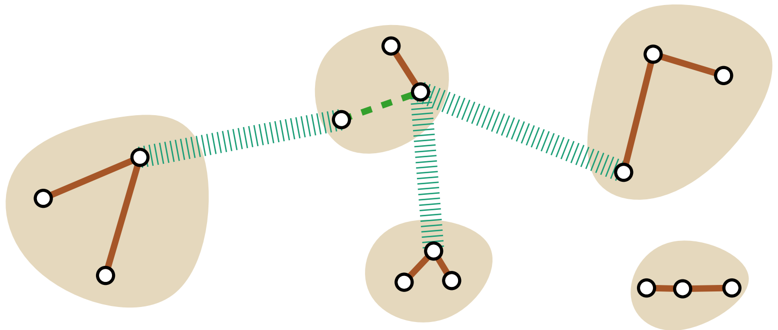

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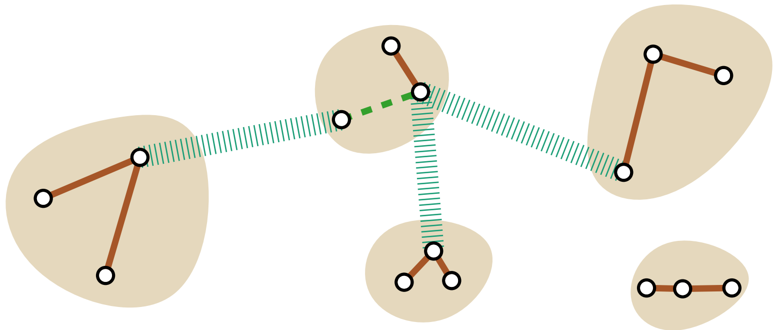

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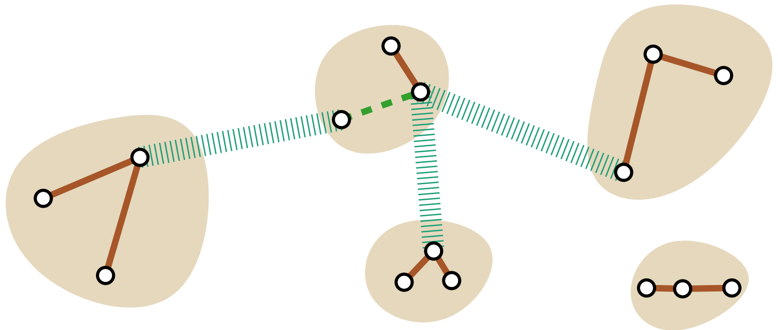

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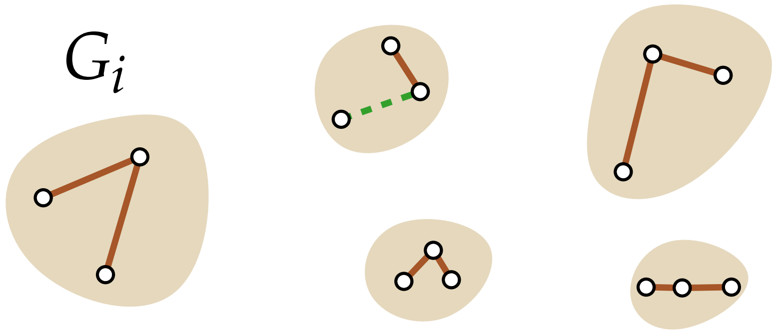

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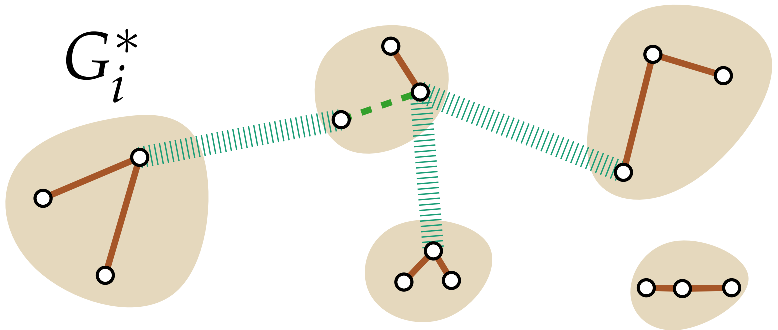

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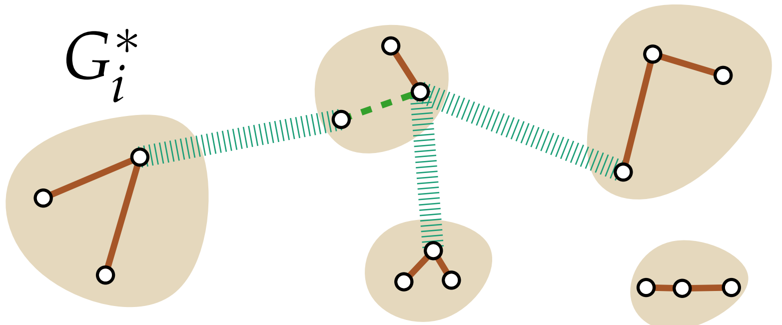

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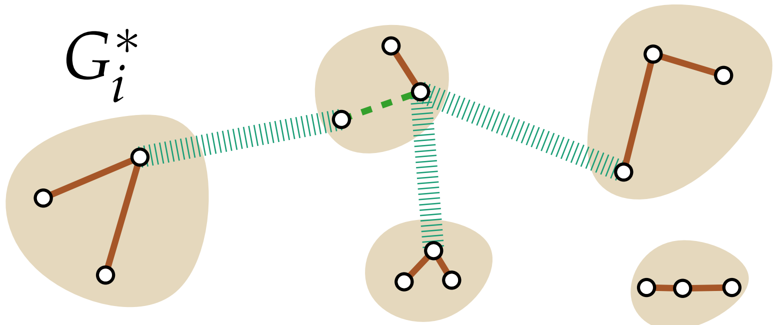

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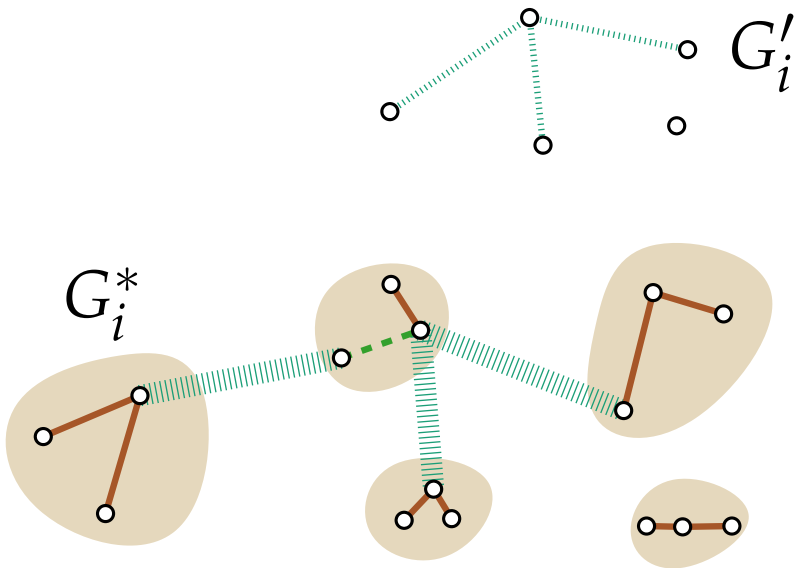

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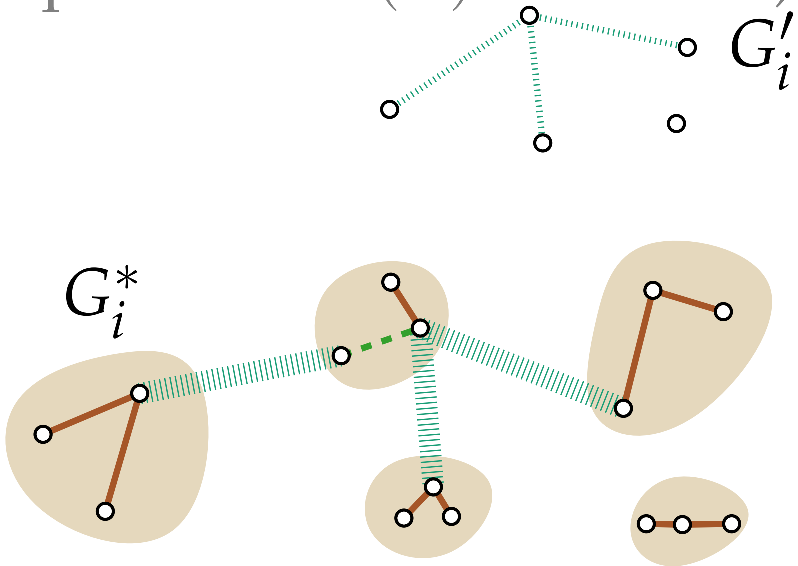

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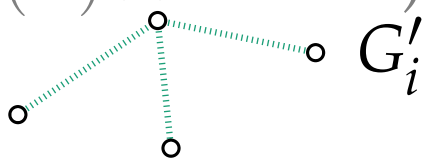

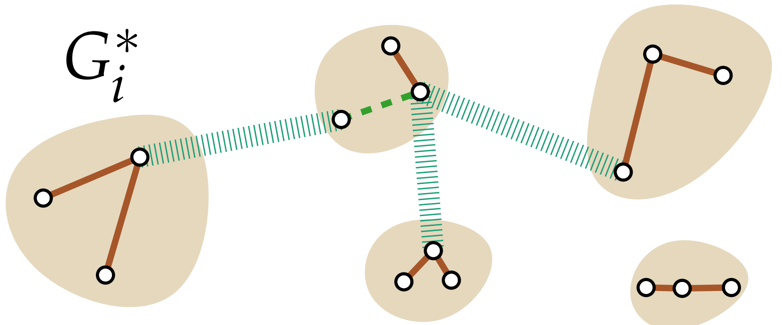

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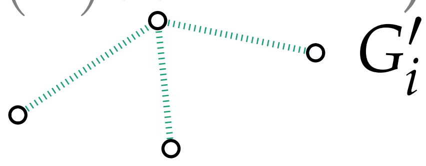

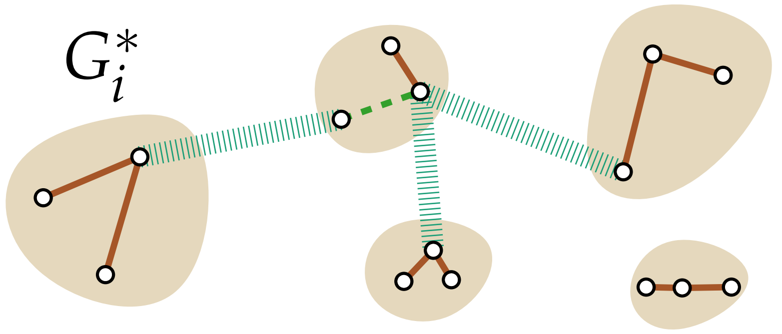

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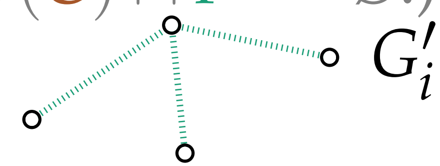

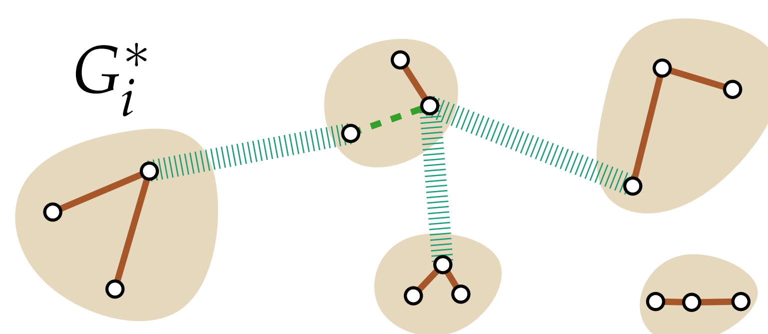

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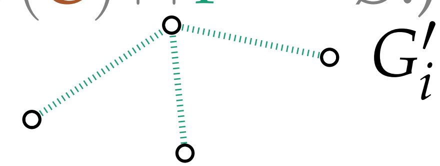

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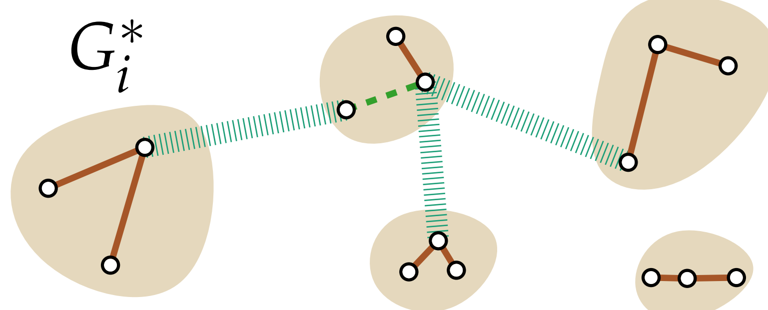

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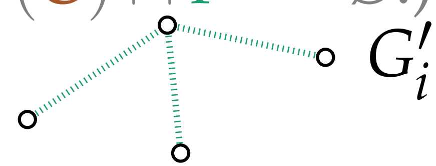

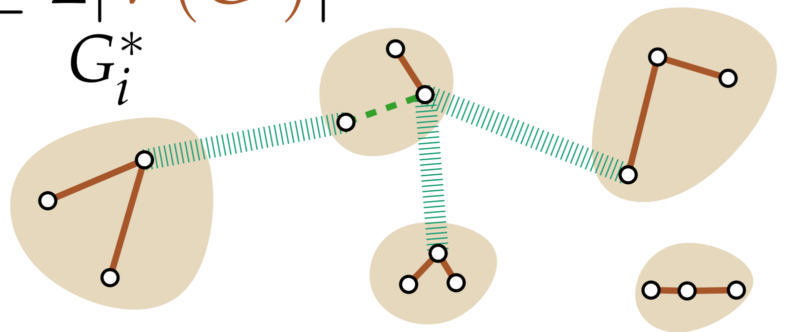

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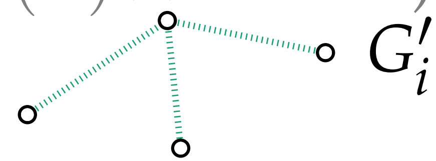

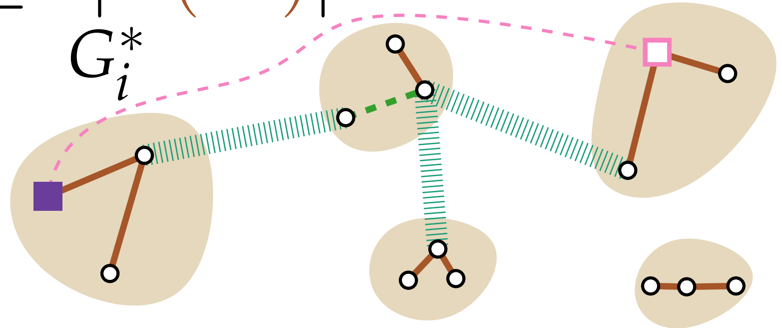

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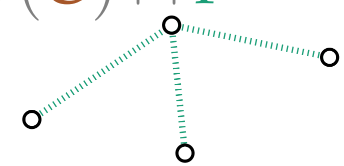

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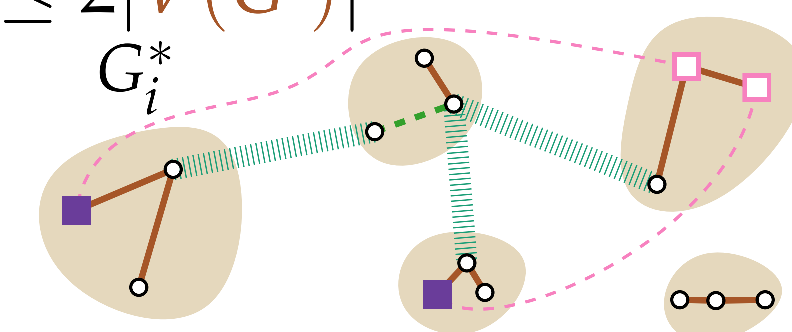

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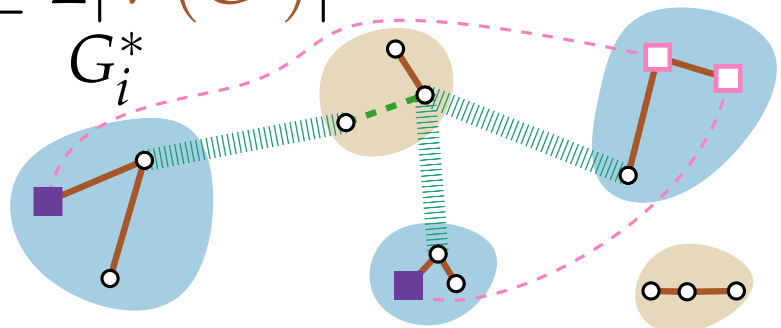

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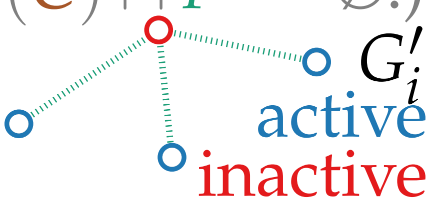

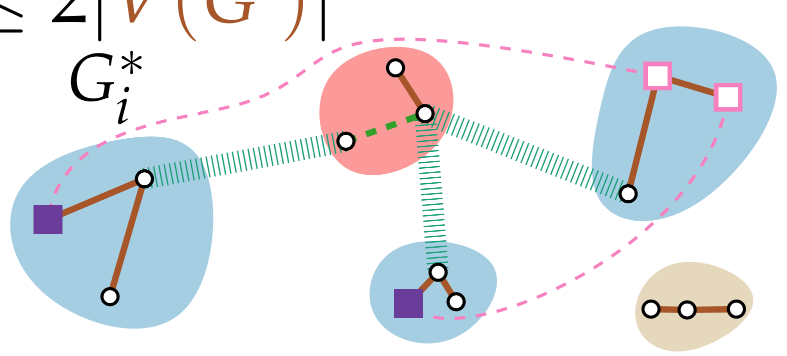

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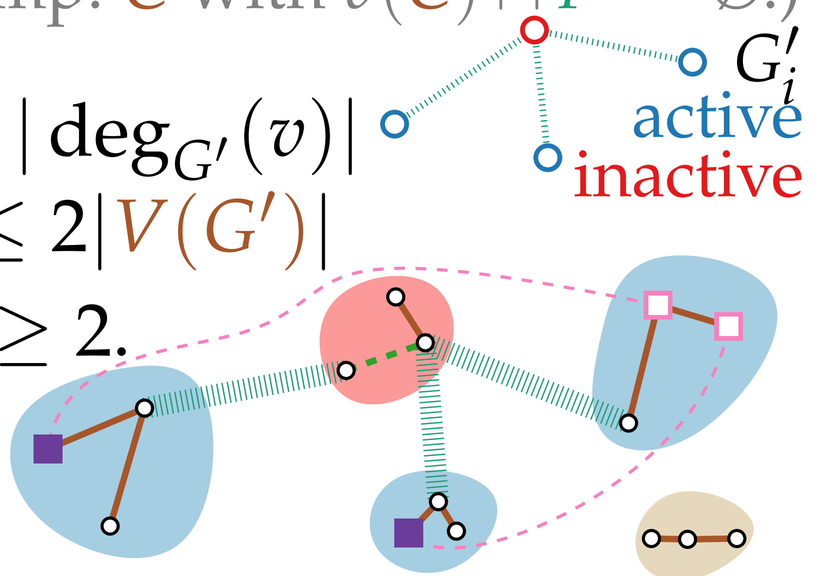

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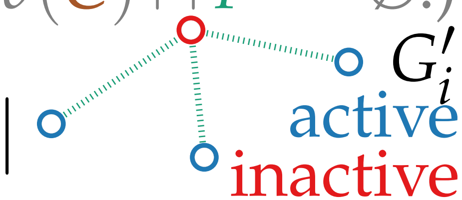

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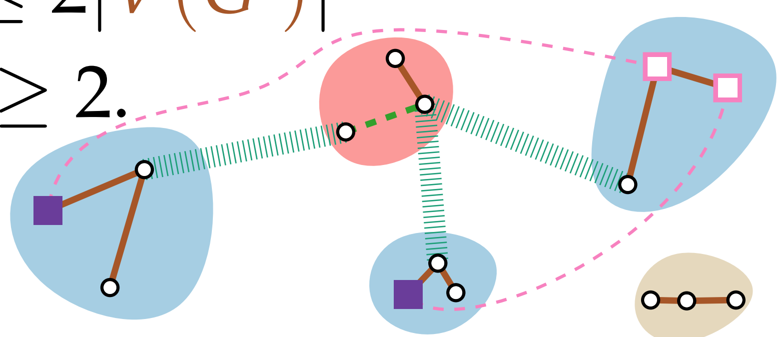

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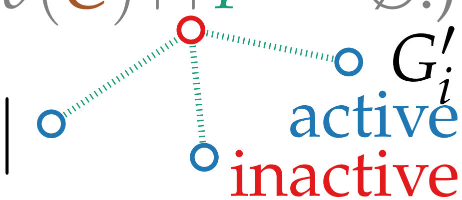

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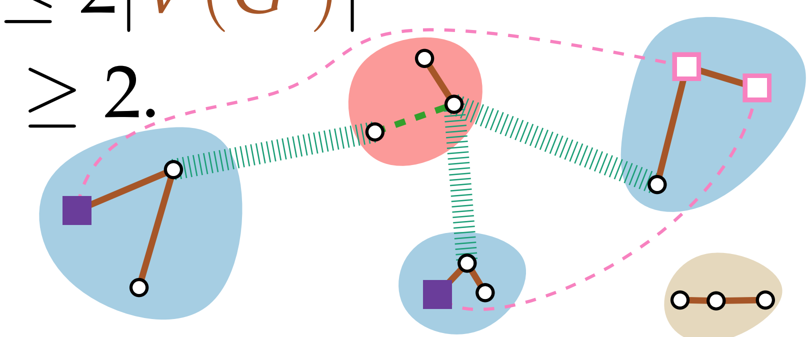

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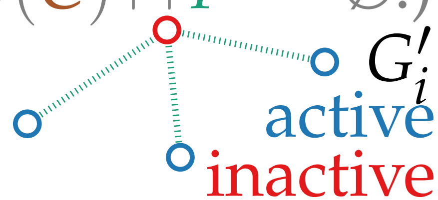

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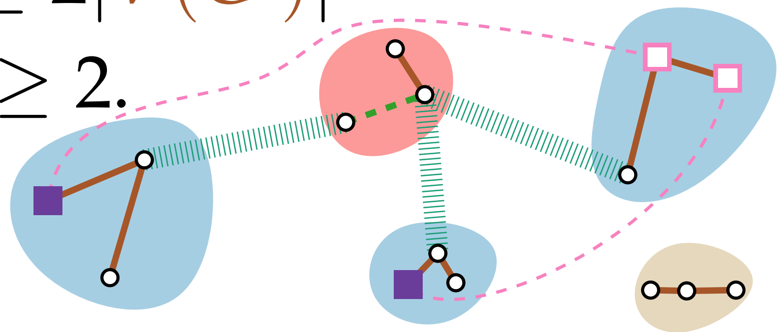

Approximation Algorithms

Lecture 12:
SteinerForest via Primal-Dual

Part VI:
Analysis

Philipp Kindermann

Summer Semester 2020

Analysis

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Proof.

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As before

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From that, the claim of the theorem follows.

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Thus, by the Structure Lemma, $(*)$ also holds after the active iteration. $\square$

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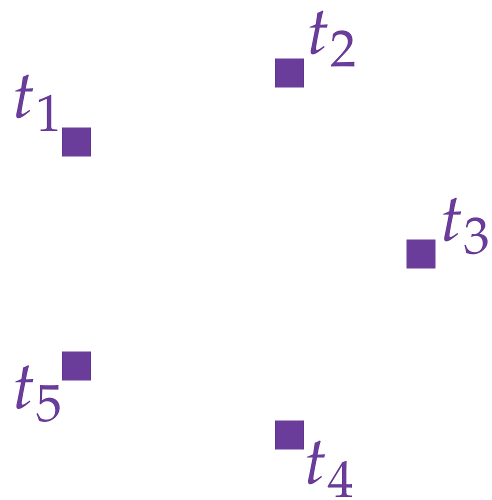

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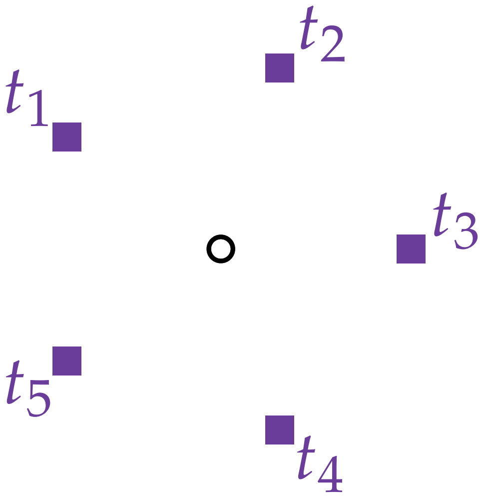

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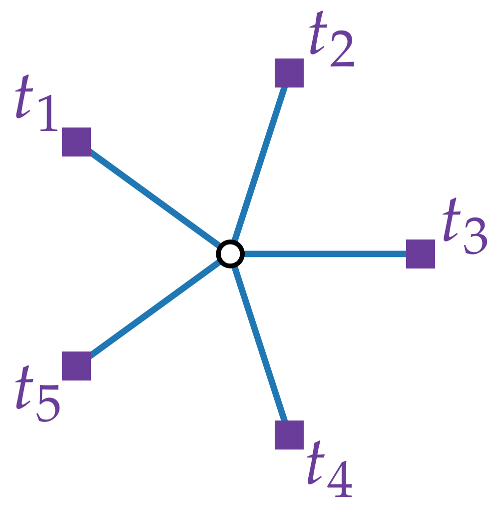

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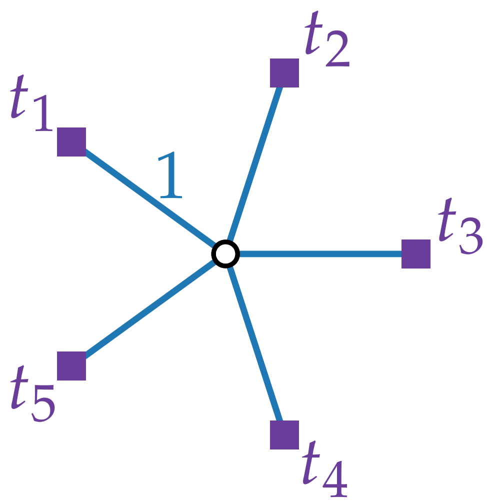

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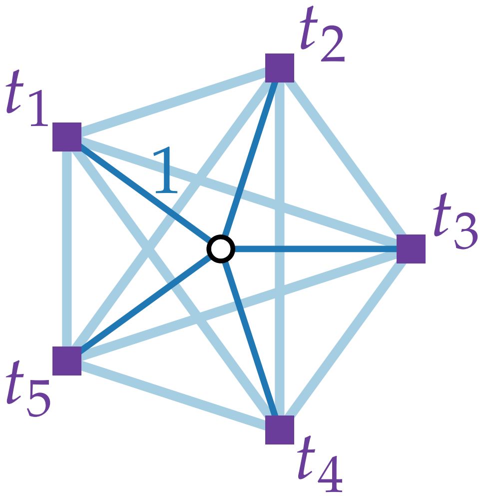
Theorem. The Primal-Dual algorithm with synchronized increases gives a 2-approximation for SteinerForest.

Analysis tight?



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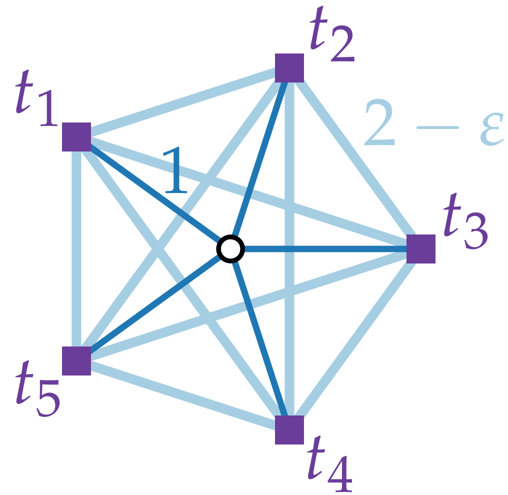

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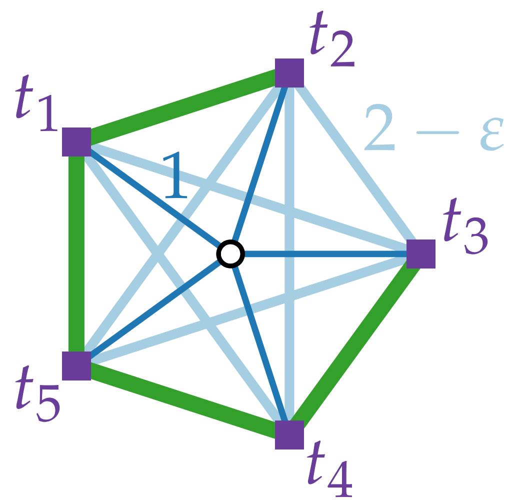

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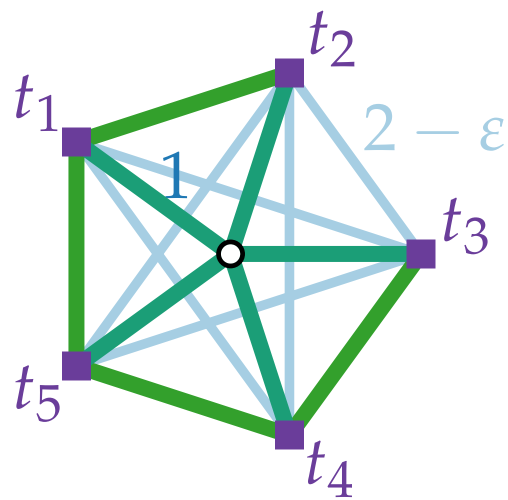

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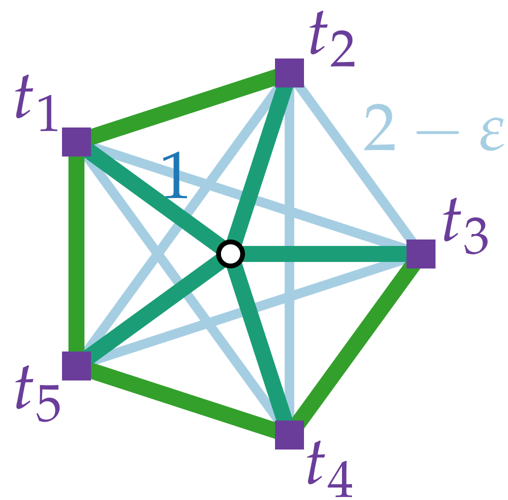

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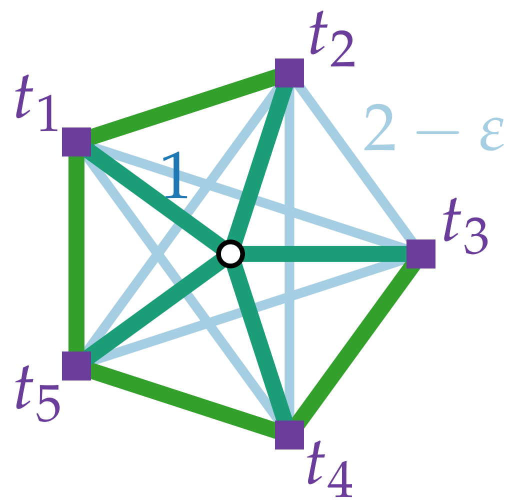

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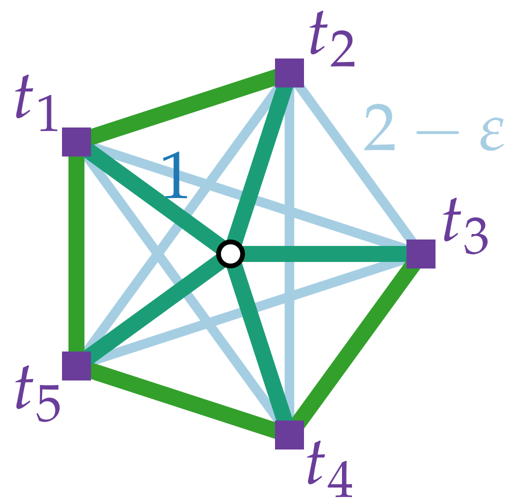

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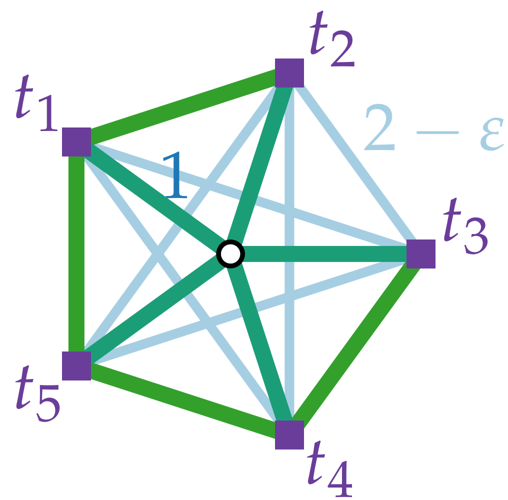

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SteinerForest cannot be approximated within factor 1.0074 (unless P=NP) [Thimm '03]