Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part I:
Maximum Satisfiability (MaxSat)

Philipp Kindermann

Summer Semester 2020

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Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part II:
A Simple Randomized Algorithm

Philipp Kindermann

Summer Semester 2020

A Simple Randomized Algorithm

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Thus, $E[W] \geq 1/2 \sum_{j=1}^{m} w_j \geq$

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Theorem. Independently setting each variable to 1 (`true`) with probability $1/2$ provides an expected $1/2$-approximation for MaxSat.

Proof.

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Let l_j be length of C_j. $\Rightarrow \Pr[C_j \text{ satisfied}] = 1 - (1/2)^{l_j} \geq 1/2$.

Thus, $E[W] \geq 1/2 \sum_{j=1}^{m} w_j \geq \text{OPT}/2$. $\square$

Approximation Algorithms

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Part III:
Derandomization by Conditional Expectation

Philipp Kindermann

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Theorem. The previous algorithm can be derandomized, i.e., there is a deterministic $1/2$-approximation algorithm for MaxSat.

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Then (similar to the base case):

$$\big(E[W | x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 0] \\ + E[W | x_1 = b_1, \ldots, x_i = b_i, x_{i+1} = 1]\big)/2$$

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly to $E[W | x_1 = b_1, \ldots, x_i = b_i]$.

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly w_j to $E[W | x_1 = b_1, \ldots, x_i = b_i]$.

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If C_j is already satisfied, then it contributes exactly w_j to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

If C_j is not yet satisfied and contains k unassigned variables, then it contributes exactly to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

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The conditional expectation is simply the sum of the contributions from each clause. $\square$

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Standard procedure with which many randomized algorithms can be derandomized.

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Global optimization?

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Part IV:
Randomized Rounding

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An ILP

maximize

subject to

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

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$$y_i \in \{0, 1\}, \qquad \text{for } i = 1, \ldots, n$$

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An ILP

maximize $\sum_{j=1}^{m} w_j z_j$

subject to

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$$\begin{aligned} \textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\ \textbf{subject to} \quad & \sum_{i \in P_j} \qquad + \sum_{i \in N_j} & \text{for } j = 1, \ldots, m \\ & y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\ & z_j \in \{0, 1\}, & \text{for } j = 1, \ldots, m \end{aligned}$$

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$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} && \text{for } j = 1, \ldots, m \\
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\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) && \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, && \text{for } i = 1, \ldots, n \\
& z_j \in \{0, 1\}, && \text{for } j = 1, \ldots, m
\end{aligned}
$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \quad \text{for } j = 1, \ldots, m.$

An ILP

$$\begin{aligned}
&\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j \\
&\textbf{subject to} \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq \qquad \text{for } j = 1, \ldots, m \\
&\qquad y_i \in \{0, 1\}, \qquad \text{for } i = 1, \ldots, n \\
&\qquad z_j \in \{0, 1\}, \qquad \text{for } j = 1, \ldots, m
\end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$\begin{aligned} \textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\ \textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad && \text{for } j = 1, \ldots, m \\ & y_i \in \{0, 1\}, && \text{for } i = 1, \ldots, n \\ & z_j \in \{0, 1\}, && \text{for } j = 1, \ldots, m \end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j$$

$$\textbf{subject to} \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \text{for } j = 1, \ldots, m$$

$$y_i \in \{0, 1\}, \quad \text{for } i = 1, \ldots, n$$

~~$z_j \in \{0, 1\}$~~, $0 \leq z_j \leq 1$ for $j = 1, \ldots, m$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \boxed{\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i)} \geq z_j && \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, && \text{for } i = 1, \ldots, n \\
& \cancel{z_j \in \{0, 1\}}, \; 0 \leq z_j \leq 1 && \text{for } j = 1, \ldots, m
\end{aligned}
$$

That suffices because [boxed expression] $\in \mathbb{Z}$.

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

... and its Relaxation

$$\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j && \text{for } j = 1, \ldots, m \\
& 0 \leq y_i \leq 1, && \text{for } i = 1, \ldots, n \\
& 0 \leq z_j \leq 1, && \text{for } j = 1, \ldots, m
\end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a ( )-approximation for MaxSat.

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Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$-approximation for MaxSat.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$-approximation for MaxSat.

$\underbrace{1 - 1/e}_{\approx 0.63}$

Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part V: Randomized Rounding – Proof

Philipp Kindermann

Summer Semester 2020

Mathematical Toolkit

Let f be function that is concave on $[0, 1]$

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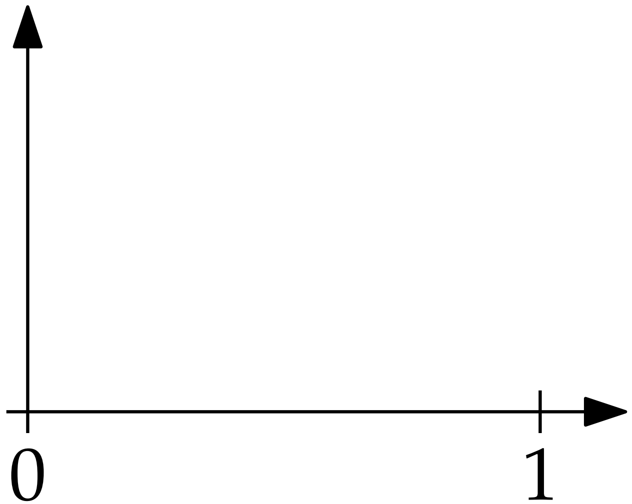

Mathematical Toolkit

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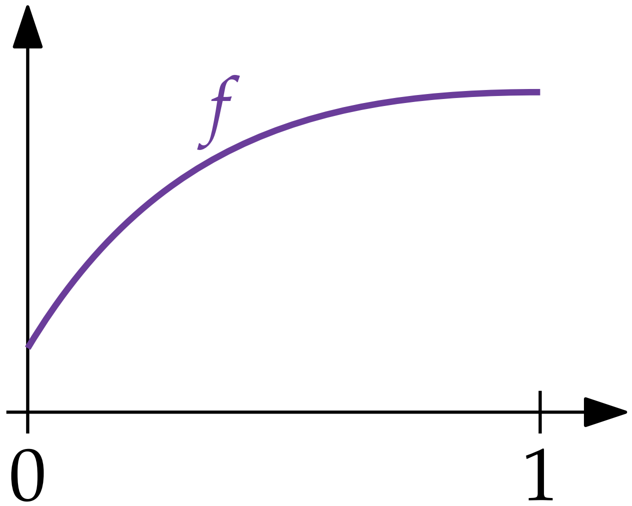
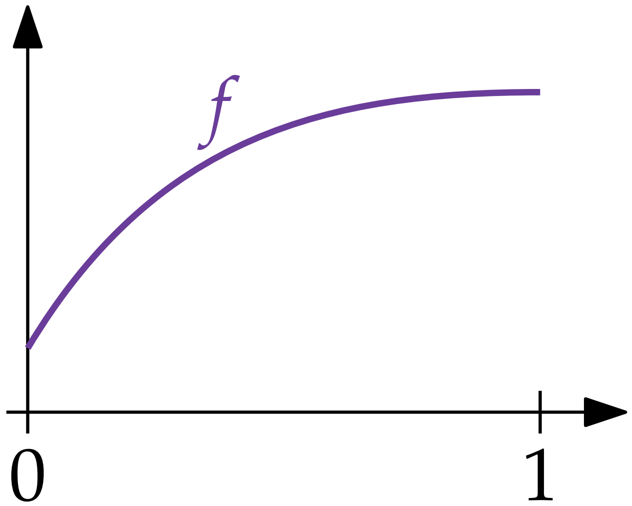


Mathematical Toolkit

Let f be function that is concave on $[0,1]$ (i.e. $f''(x) \leq 0$ on $[0,1]$) with $f(0) = a$



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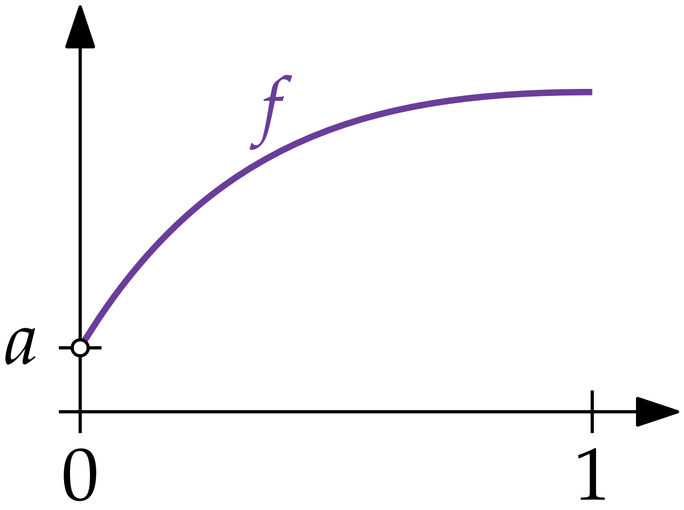

Mathematical Toolkit

Let f be function that is concave on $[0, 1]$
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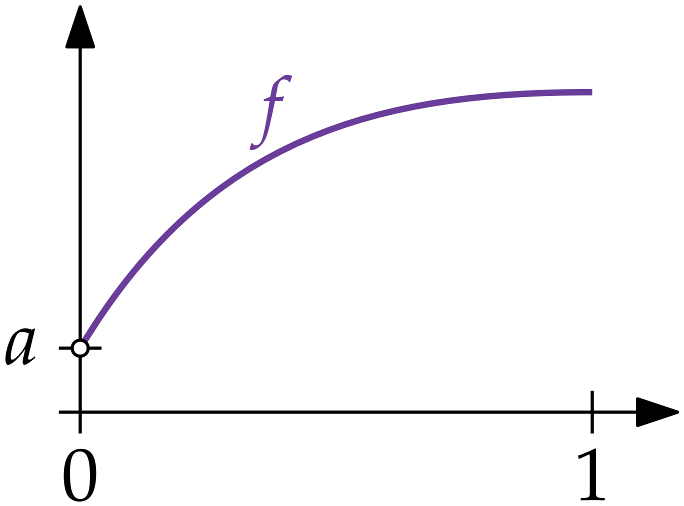

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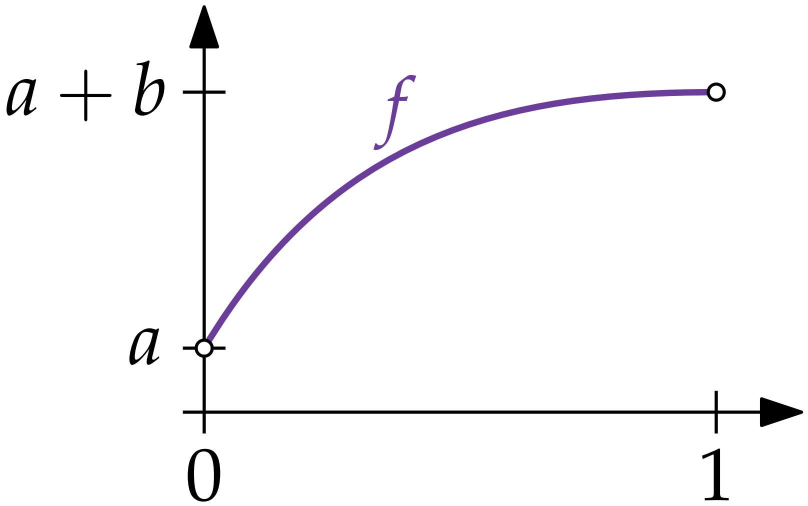

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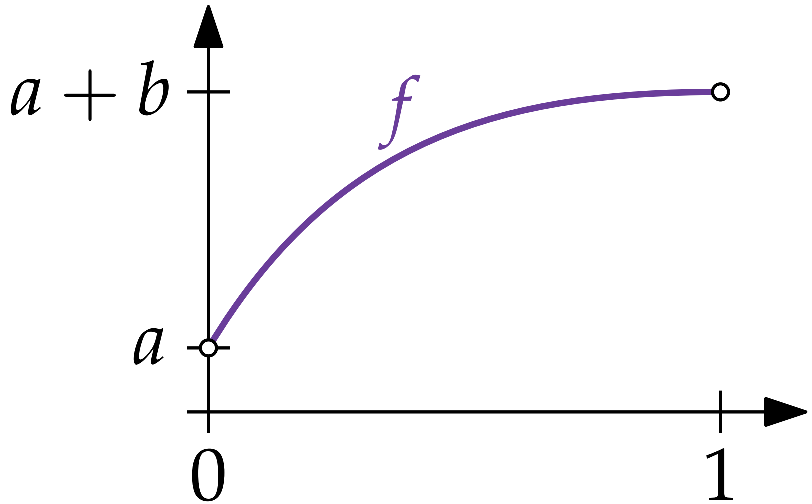
$\Rightarrow f(x) \geq bx + a$ for $x \in [0, 1]$.



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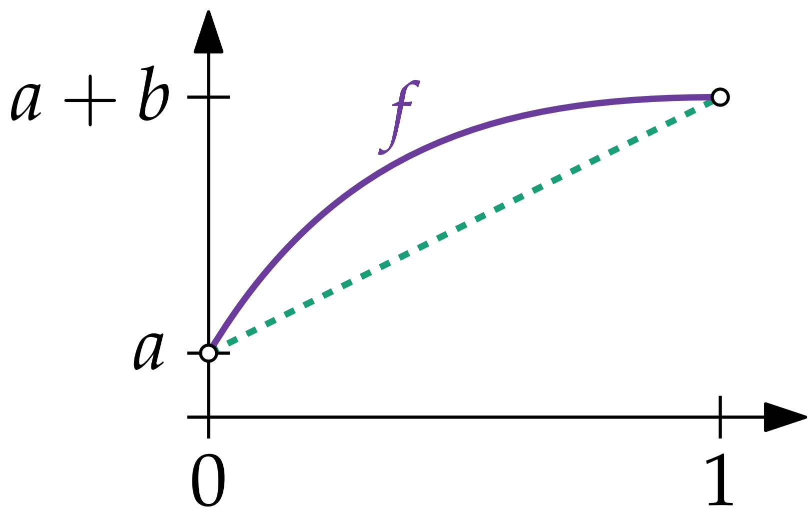

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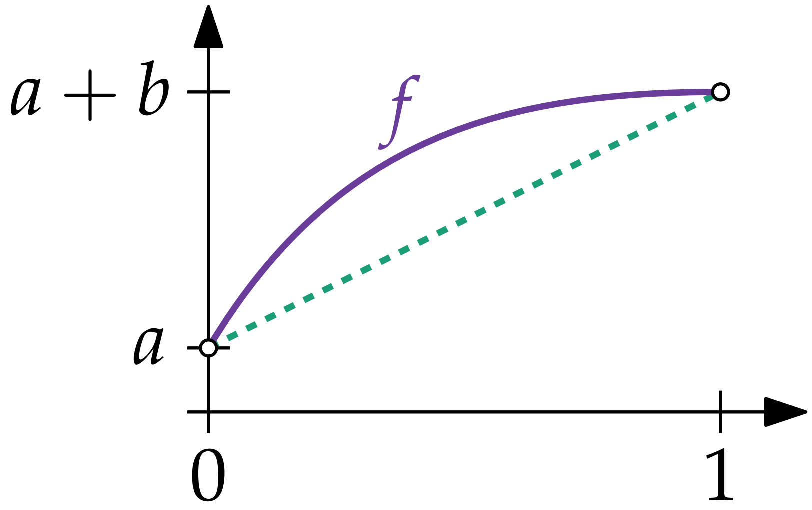

Arithmetic-Geometric Mean Inequality (AGMI):

Mathematical Toolkit

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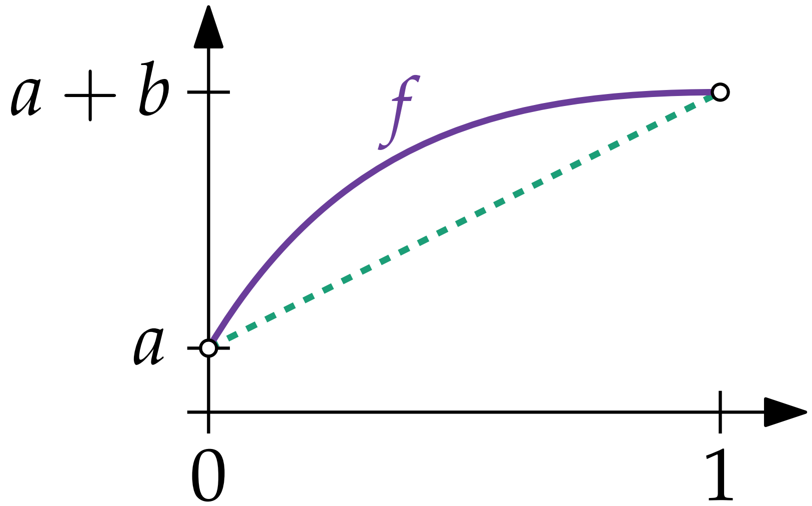

Arithmetic-Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

Mathematical Toolkit

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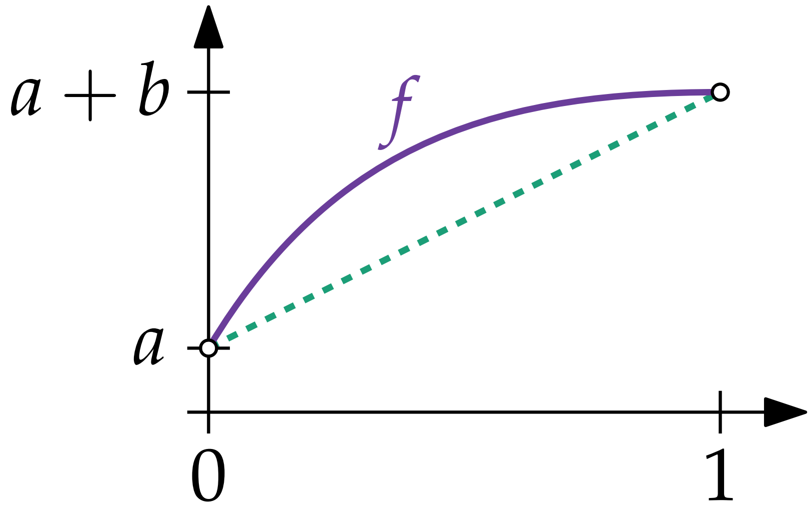

Arithmetic-Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq$$

Mathematical Toolkit

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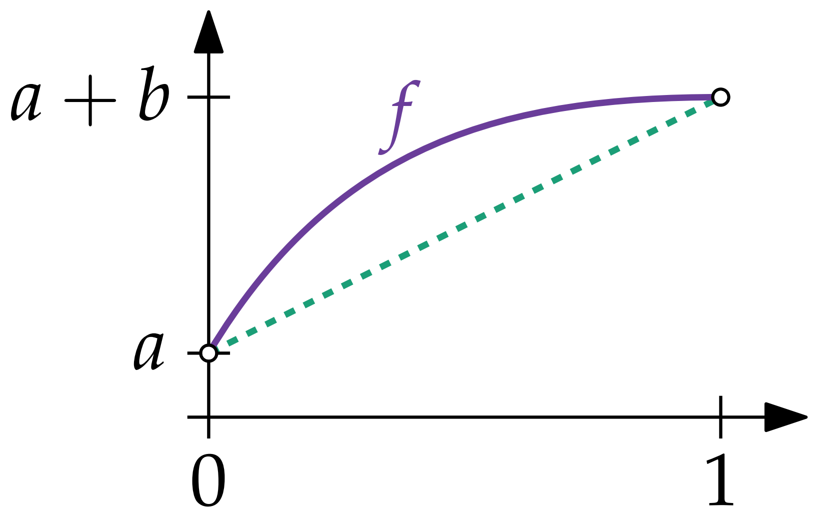

Arithmetic-Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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$$\leq$$

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Consider a fixed clause C_j of length l_j. Then we have:

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AGMI

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AGMI

$$\leq \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^*\right)$$

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$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

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$$= \left[1 - \frac{1}{l_j} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right]^{l_j}$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

$$\Pr[C_j \text{ not sat.}] = \prod_{i\in P_j}(1-y_i^*)\prod_{i\in N_j} y_i^*$$

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by LP constraints

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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$$\left(\prod_{i=1}^{k} a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^{k} a_i \right)$$

AGMI

$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

$$= \left[1 - \frac{1}{l_j} \underbrace{\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right)}_{\geq z_j^* \text{ by LP constraints}} \right]^{l_j}$$

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AGMI

$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

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$$\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

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The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

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$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0,1]$.

Thus

$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$

$$\geq$$

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The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

Thus

$$\begin{aligned} \Pr[C_j \text{ satisfied}] &\geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0) \\ &\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^* \\ &\geq \end{aligned}$$

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The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0,1]$.

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$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$

$$\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^*$$

$$\geq$$

$1 + x \leq e^x$

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$x = -\frac{1}{l_j}$

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$x = -\frac{1}{l_j} \Rightarrow 1 - \frac{1}{l_j} \geq e^{-1/l_j}$

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Thus

$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$
$$\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^*$$
$$\geq \left(1 - \frac{1}{e}\right) z_j^*$$

$1 + x \leq e^x$

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Randomized Rounding (Proof)

Therefore

$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

$$\geq$$

Randomized Rounding (Proof)

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$$=$$

Randomized Rounding (Proof)

Therefore

$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

$$\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*} \quad \text{(LP target function)}$$

$$=$$

Randomized Rounding (Proof)

Therefore

$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

$$\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*} \quad \text{(LP target function)}$$

$$= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}}$$

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Randomized Rounding (Proof)

Therefore

$$
\begin{aligned}
E[W] &= \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j \\
&\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^{m} w_j z_j^*} \quad \text{(LP target function)} \\
&= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}} \\
&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}
\end{aligned}
$$

$\square$

Randomized Rounding (Proof)

Therefore

$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

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$$= \left(1 - \frac{1}{e}\right) \mathrm{OPT}_{\mathrm{LP}}$$

$$\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT} \qquad \square$$

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part VI:
Combining the Algorithms

Philipp Kindermann

Summer Semester 2020

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a -approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument. With probability $1/2$ choose the solution of the first algorithm, otherwise the solution of the second algorithm.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument. With probability $1/2$ choose the solution of the first algorithm, otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above algorithm.

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

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$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right) z_j^*}_{\text{LP-Rounding}} \qquad\qquad\qquad\right].$$

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The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)z_j^*}_{\text{LP-Rounding}}+\underbrace{\left(1-2^{-l_j}\right)}_{\text{rand. Alg.}}\right].$$

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The probability that clause C_j is satisfied is at least:

$$\underbrace{\frac{1}{2}\left[\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)}_{\text{LP-Rounding}} + \underbrace{\left(1-2^{-l_j}\right)\right]}_{\text{rand. Alg.}} z_j^*.$$

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We claim that this is at least $3/4 \cdot z_j^*$.

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For $l_j = 1, 2$, a simple calculation gives exactly $3/4 \cdot z_j^*$.

For $l_j \geq 3$, $1-(1-1/l_j)^{l_j} \geq (1-1/e)$ and $1-2^{-l_j} \geq \frac{7}{8}$.

Take the better of the two solutions!

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$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right] z_j^* \approx$$

□

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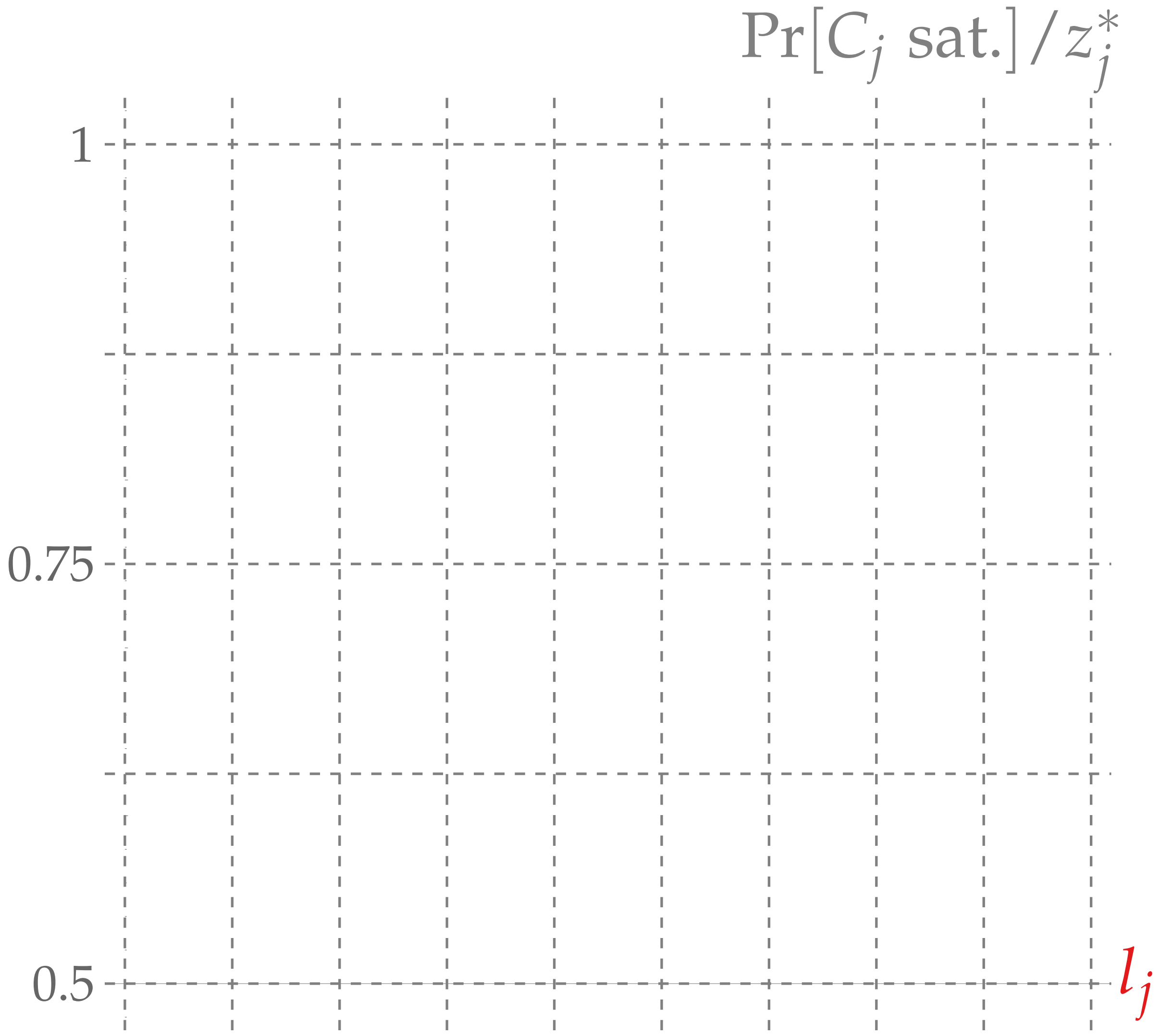
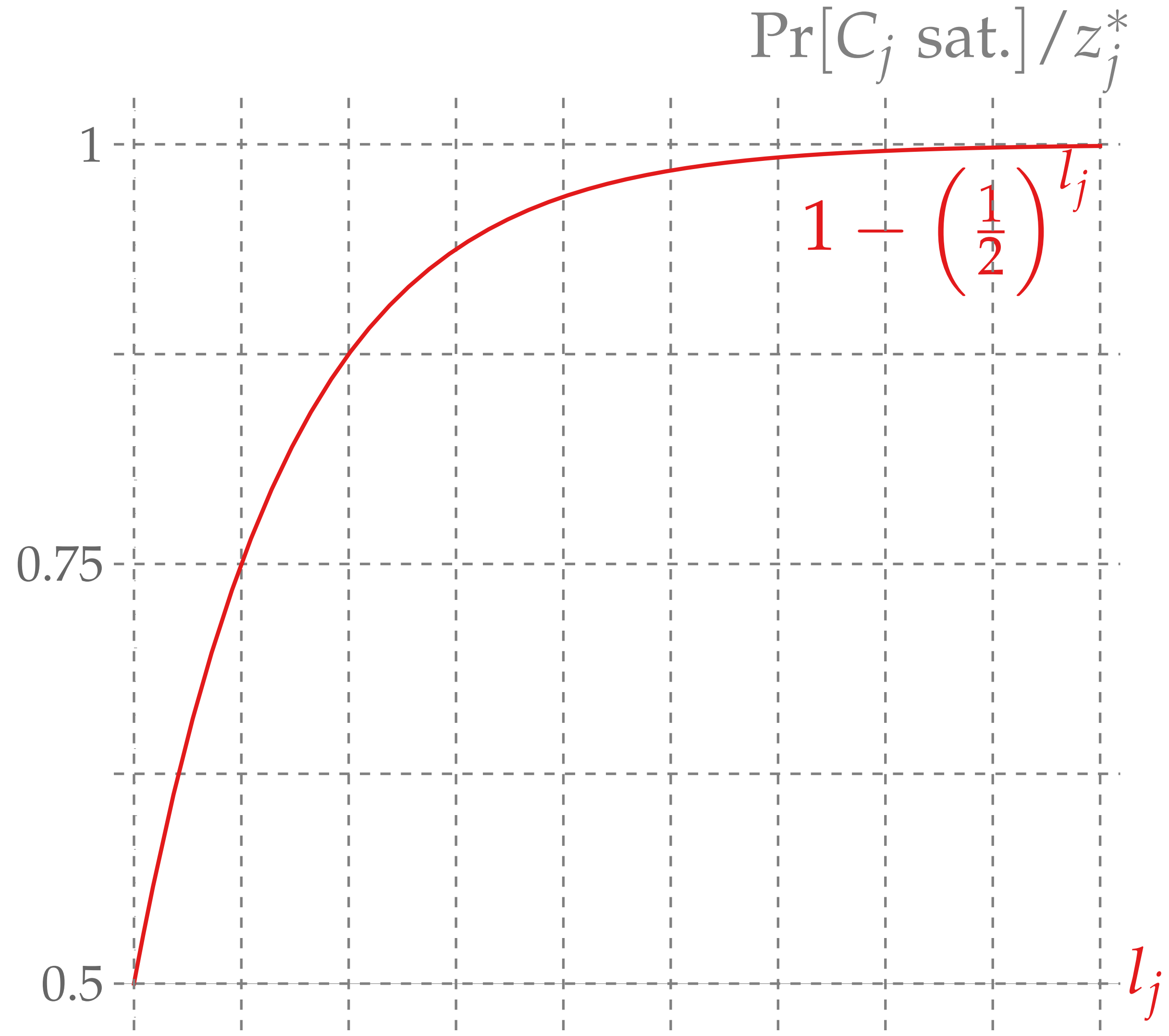
Visualization and Derandomization

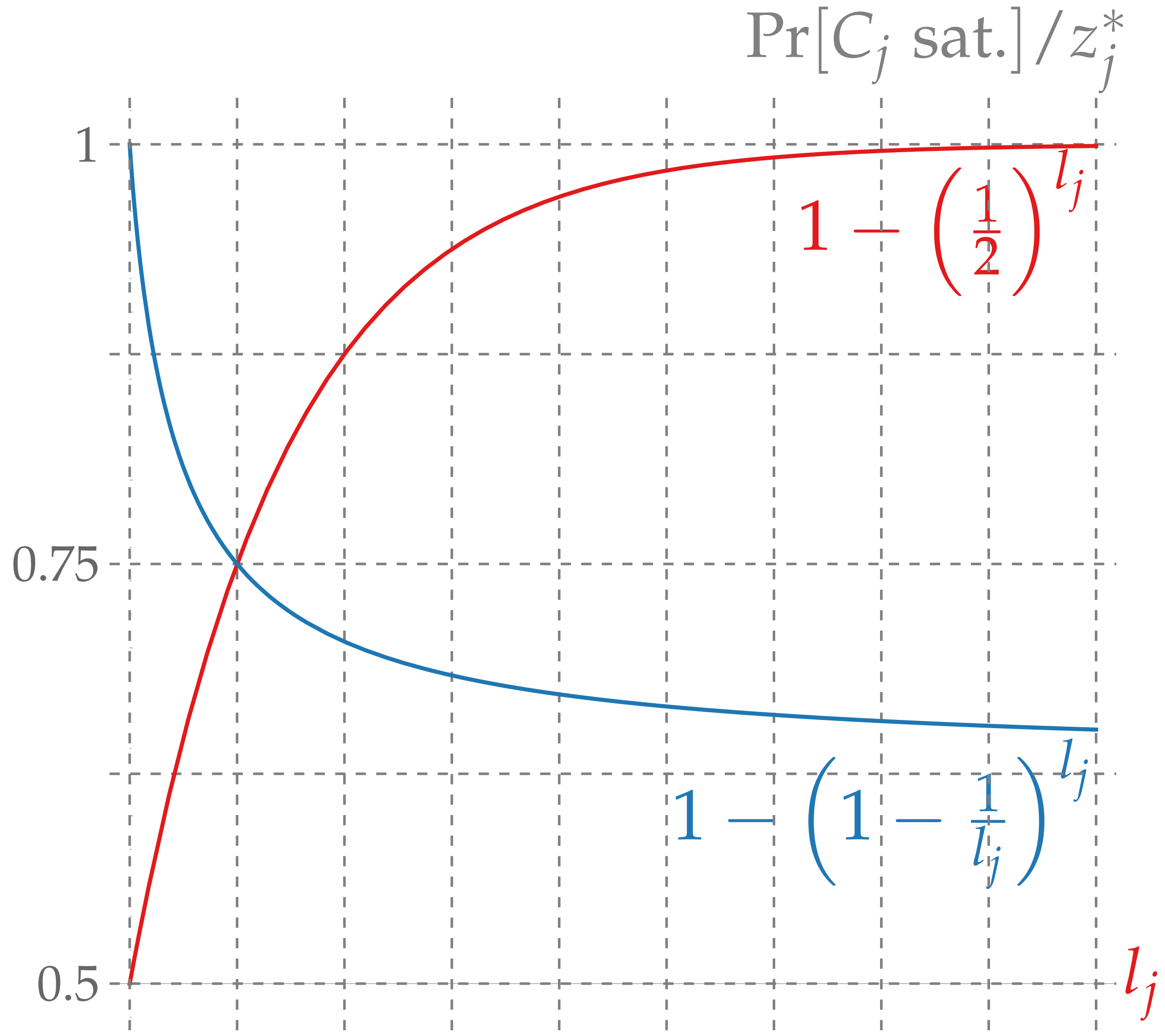
Visualization and Derandomization



Visualization and Derandomization

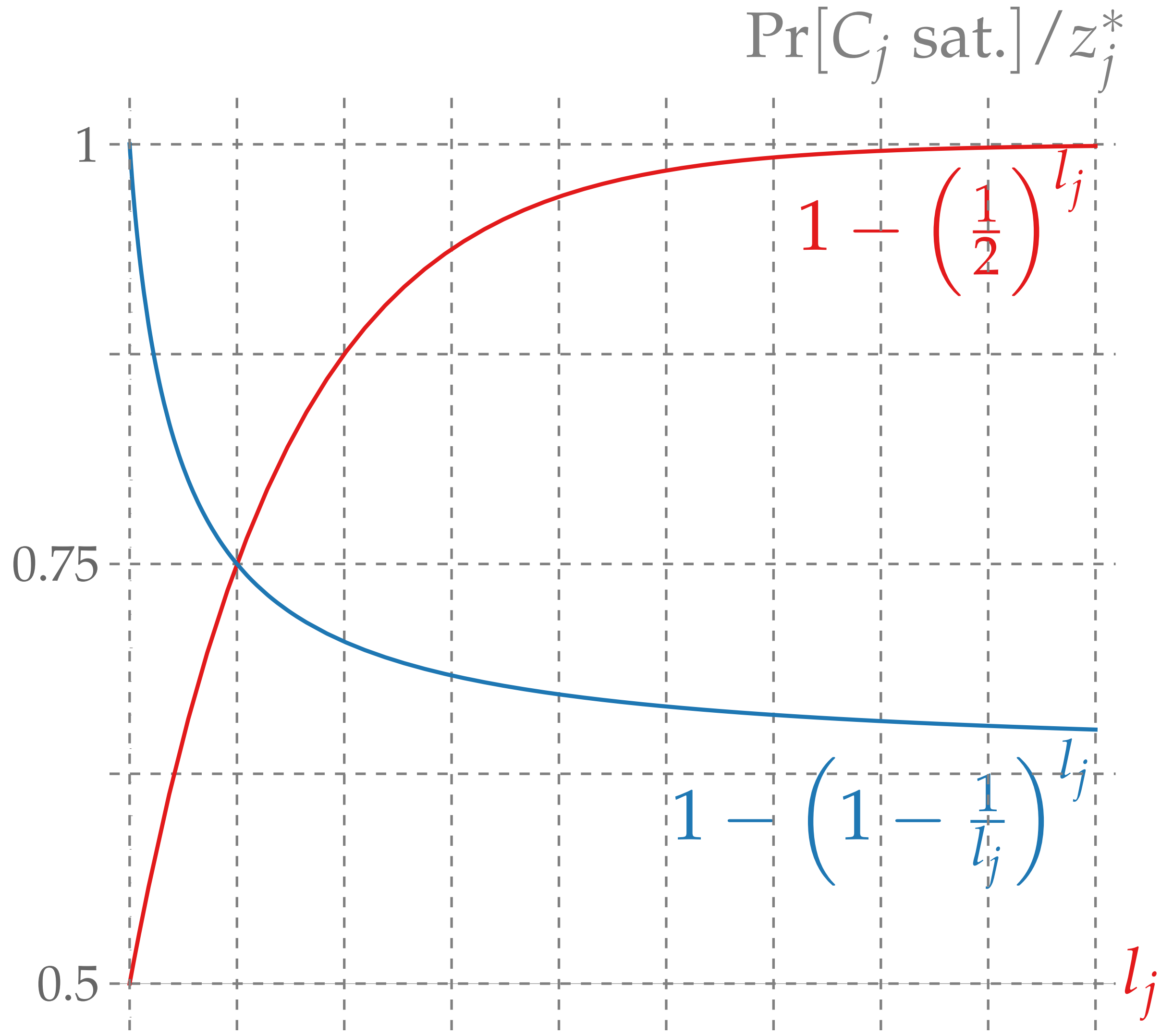


Visualization and Derandomization



Visualization and Derandomization

– Randomized alg. is better for large values of l_j.



Visualization and Derandomization

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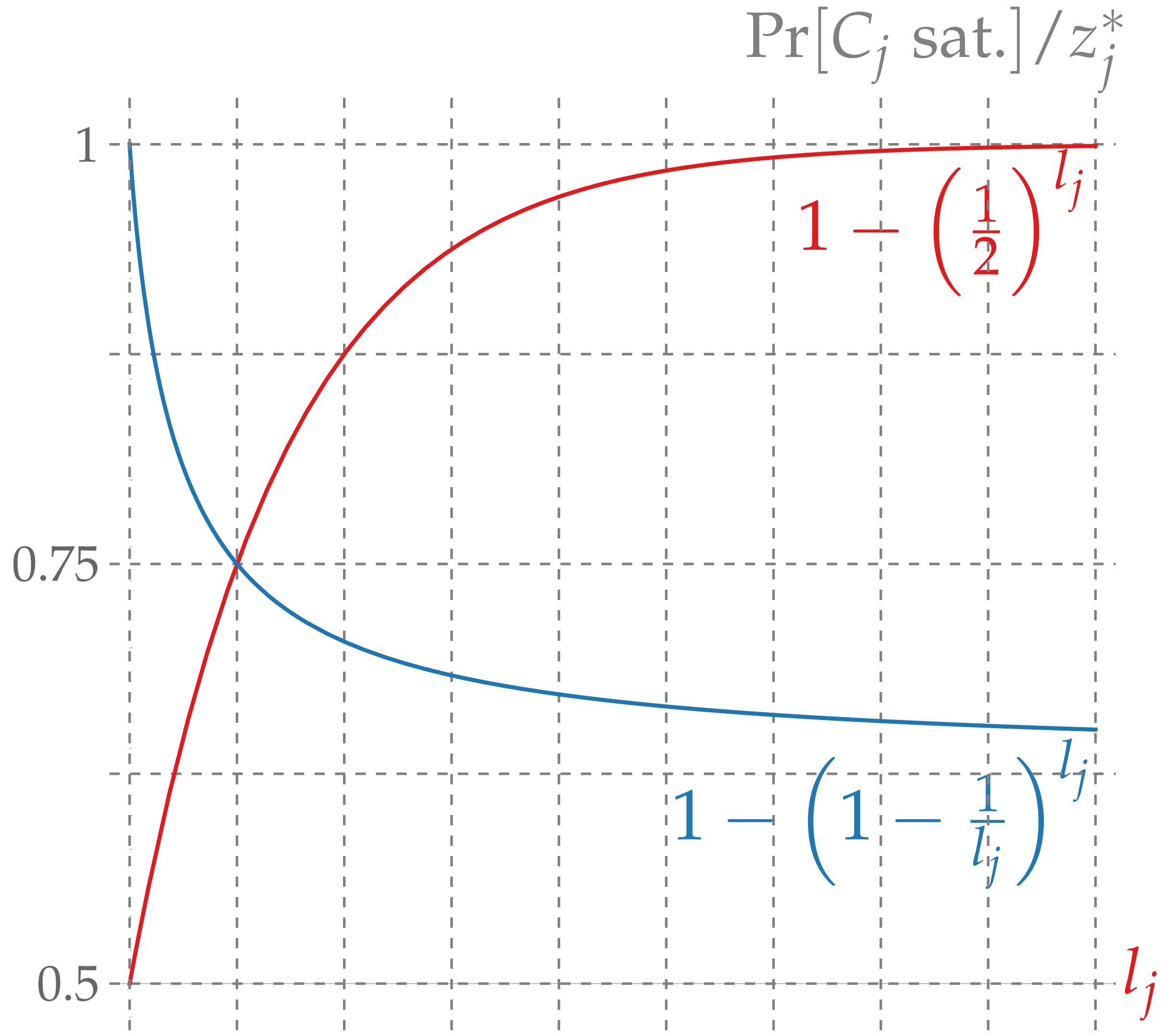
- Randomized LP-rounding is better for small values of l_j



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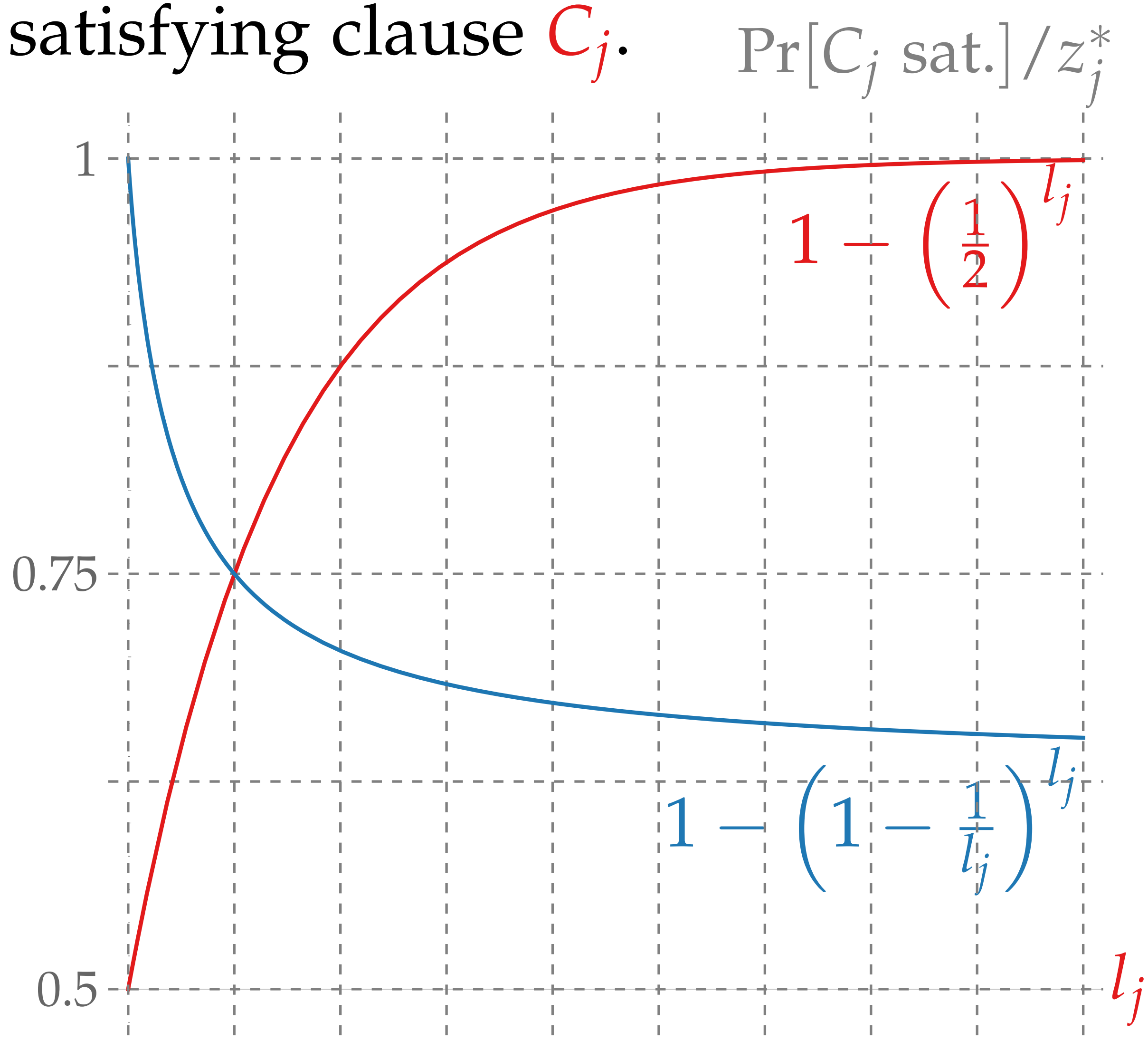
$\Rightarrow$ higher probability of satisfying clause C_j.



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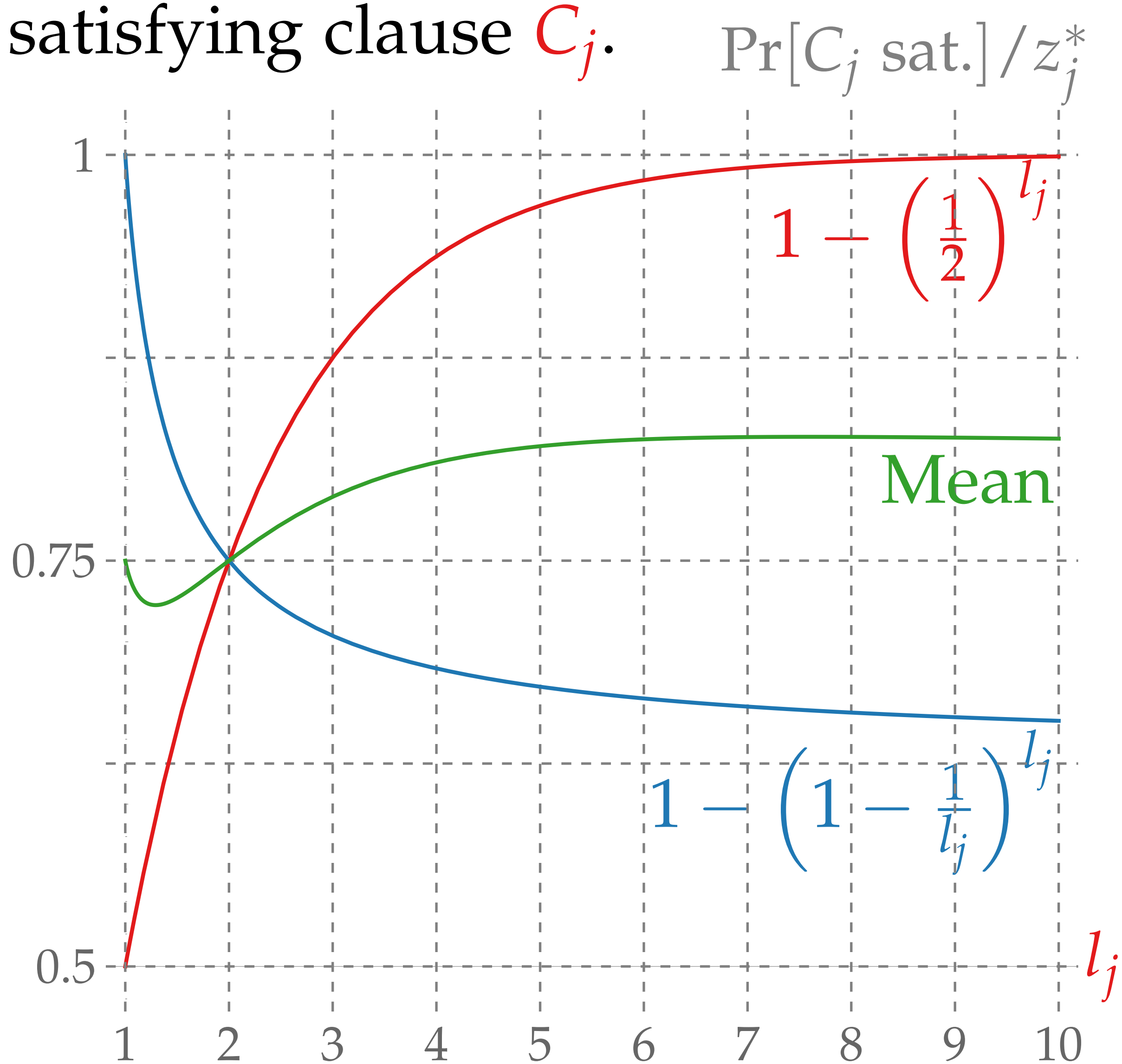

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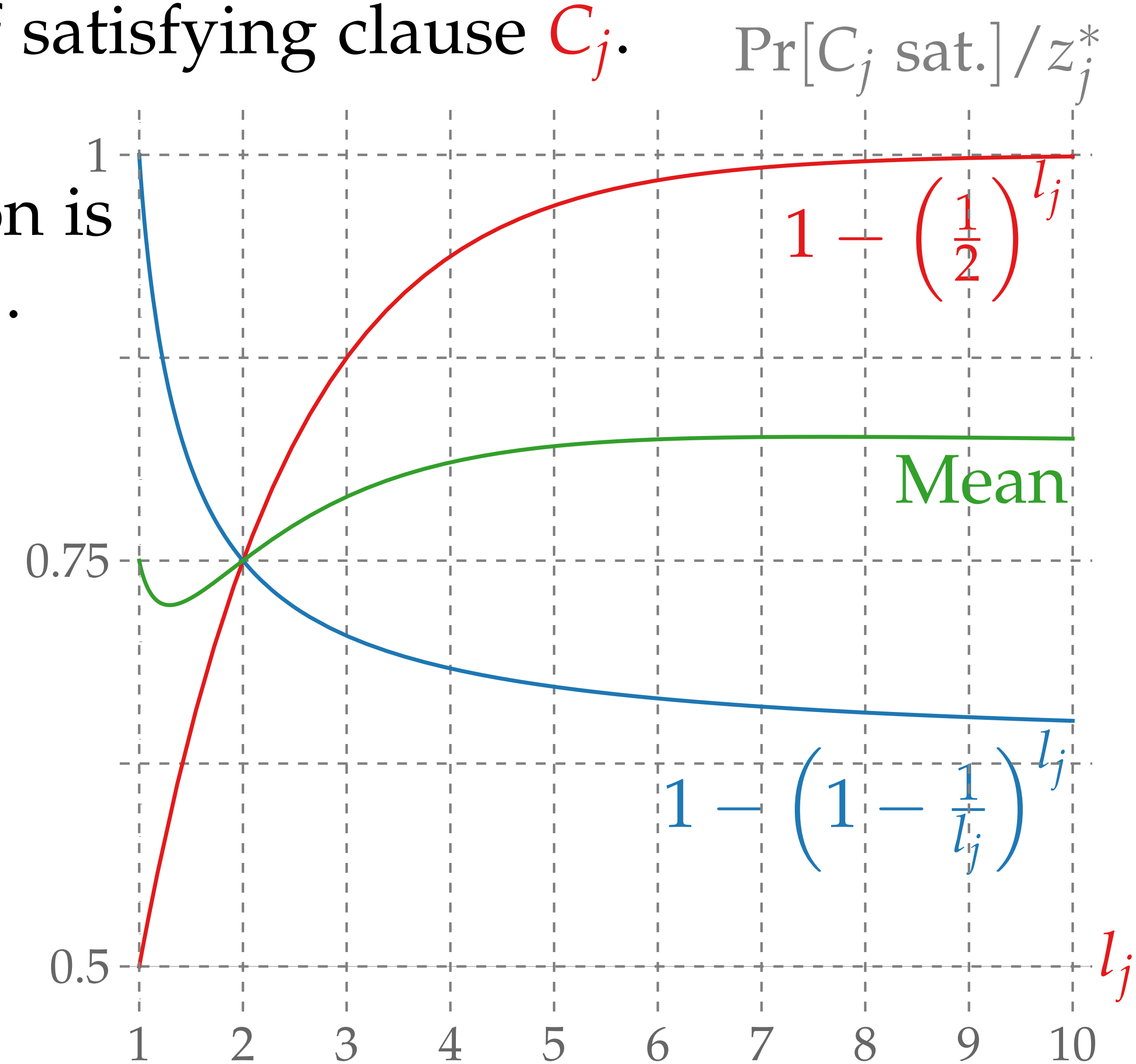
Mean of the two solution is at least $3/4$ for *integer* l_j.



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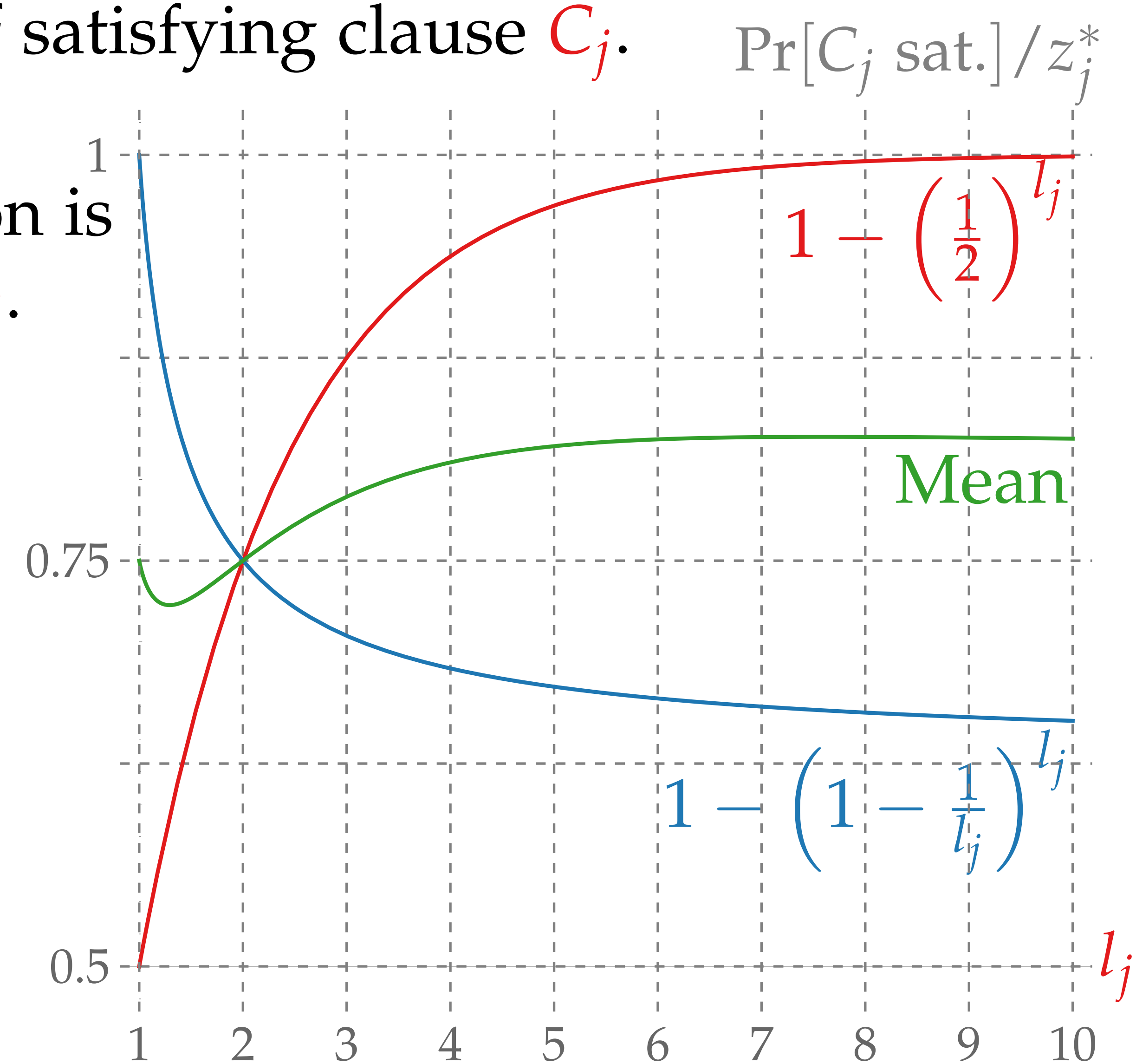
Maximum is at least as large as the mean.



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Maximum is at least as large as the mean.

This algorithm can also be derandomized by conditional expectation.