Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part I: Knapsack

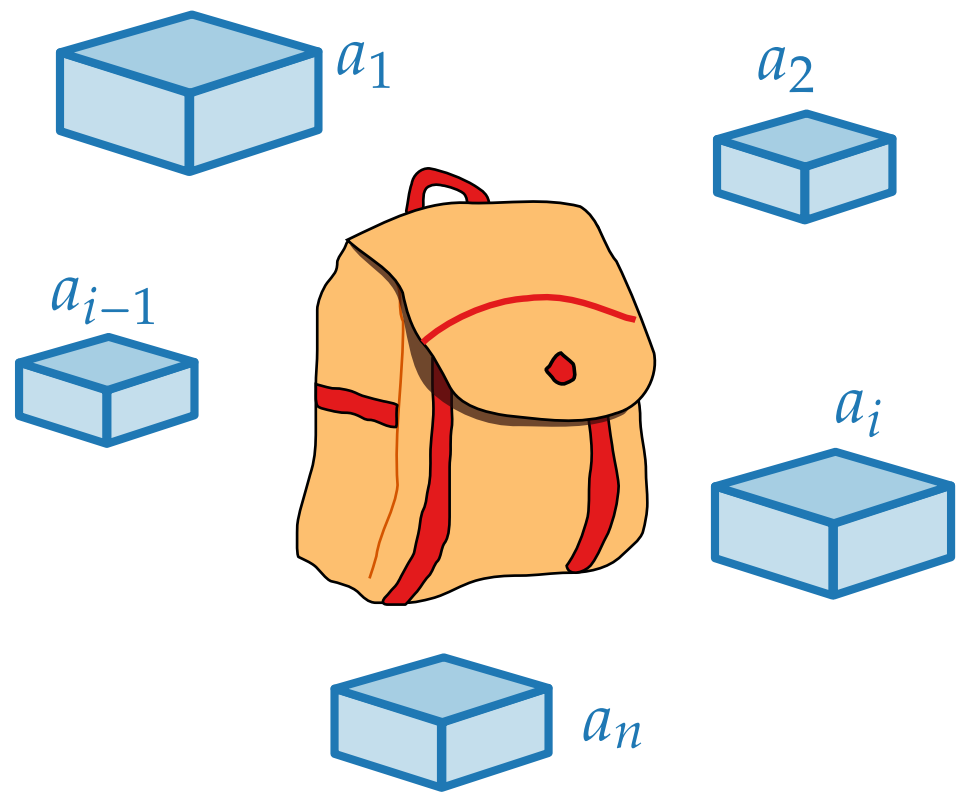
Philipp Kindermann

Summer Semester 2020

Knapsack

Given:

- A set $S = \{a_1, \ldots, a_n\}$ of **objects**.



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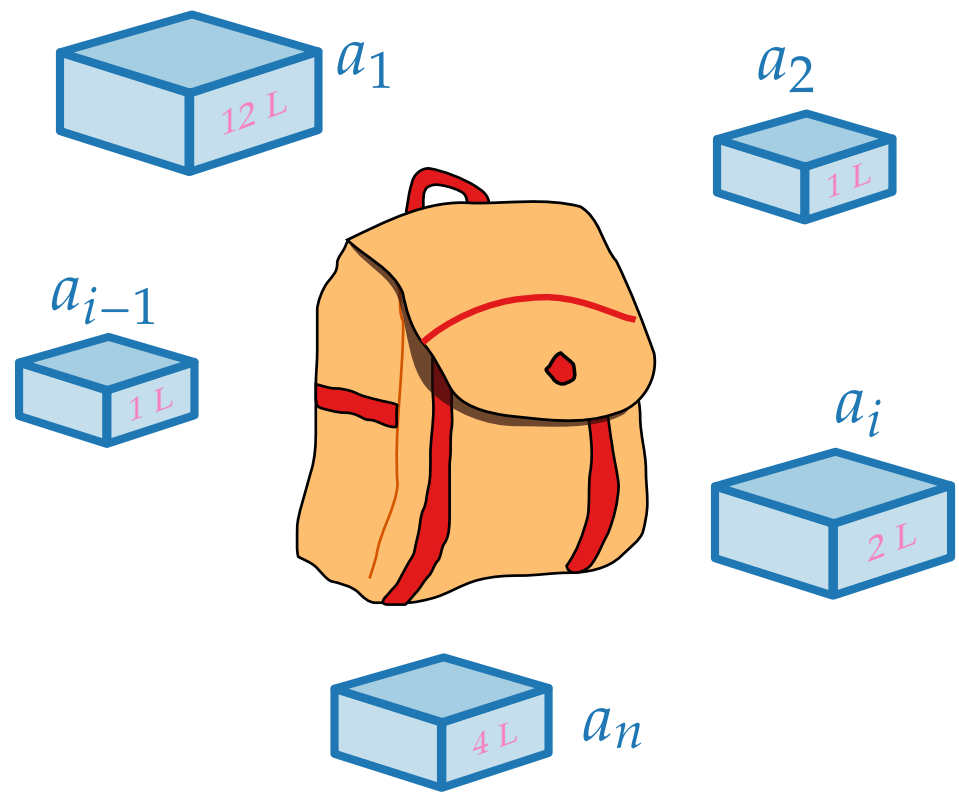

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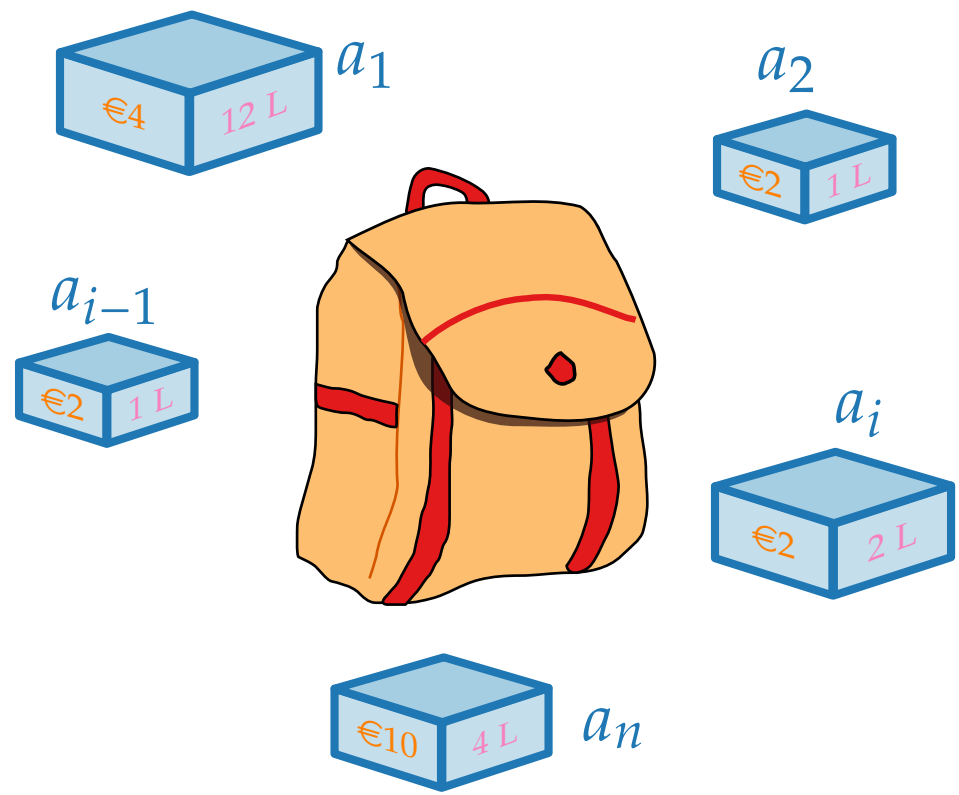

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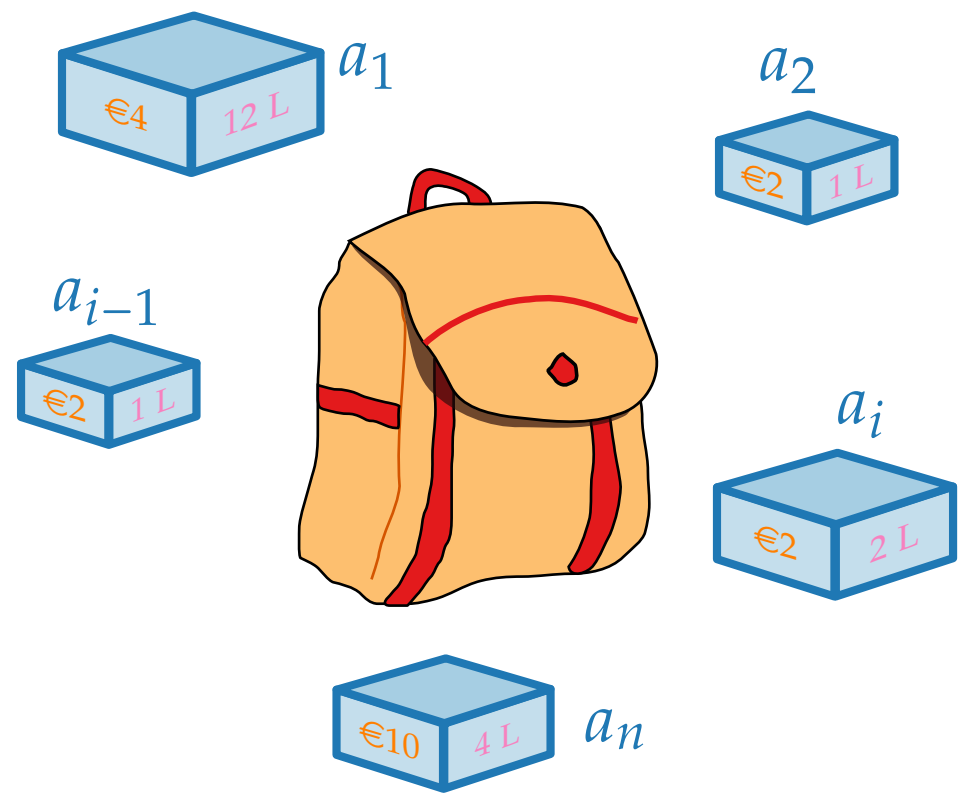

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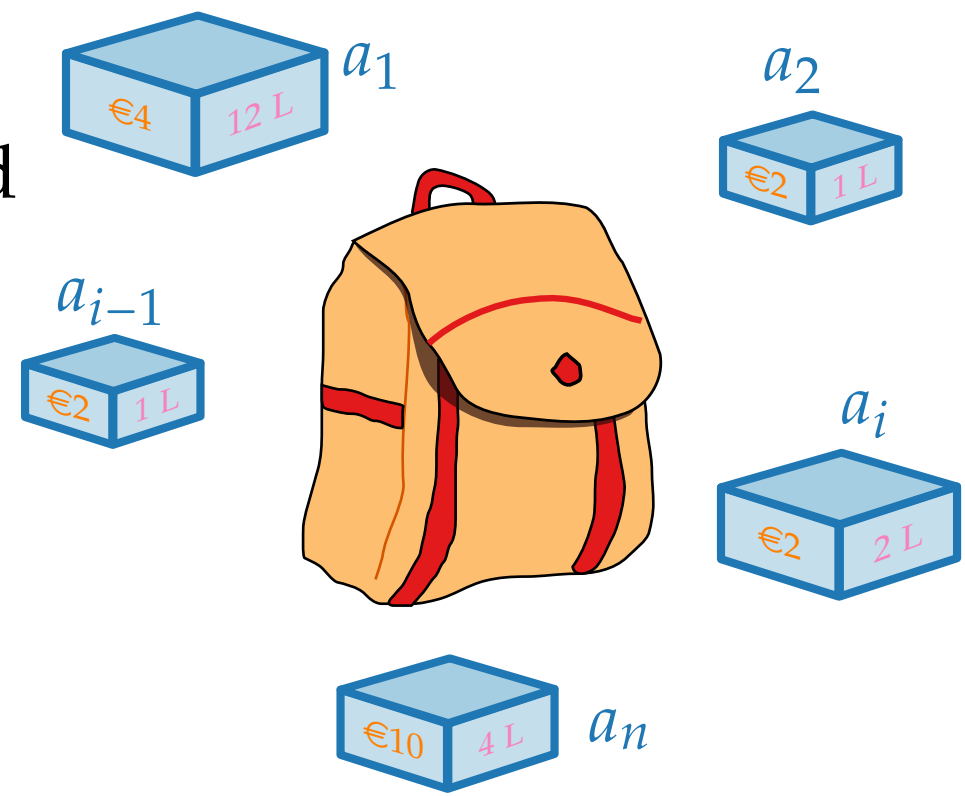
Task:

Find a subset of objects whose **total size** is at most B and whose **total profit** is maximum.



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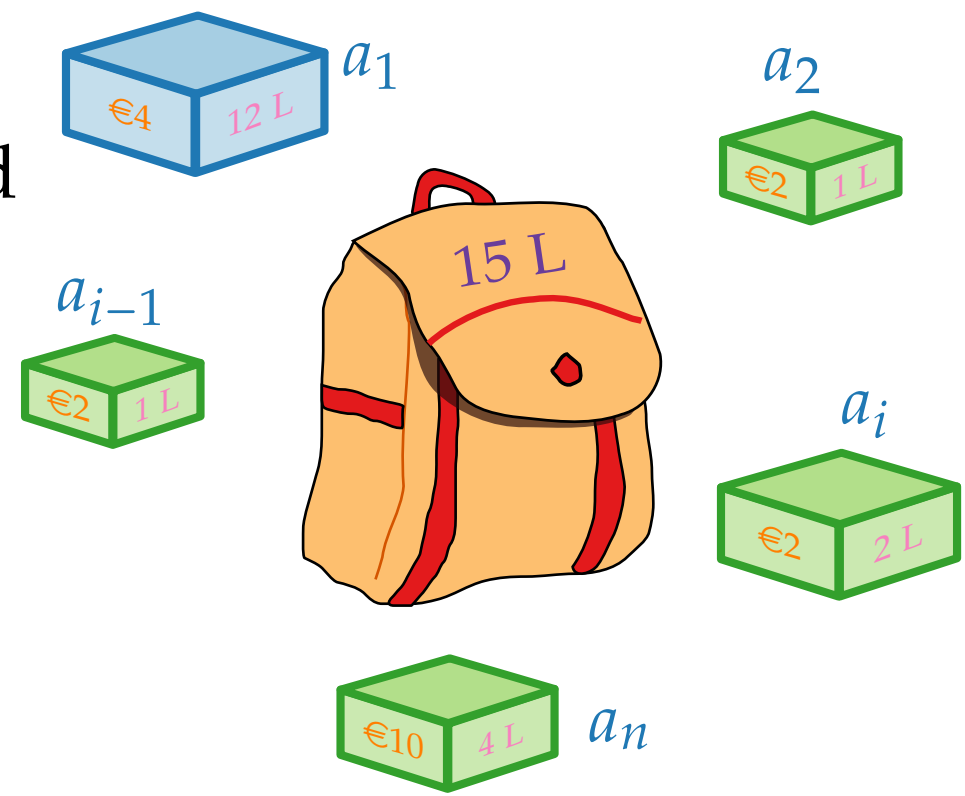

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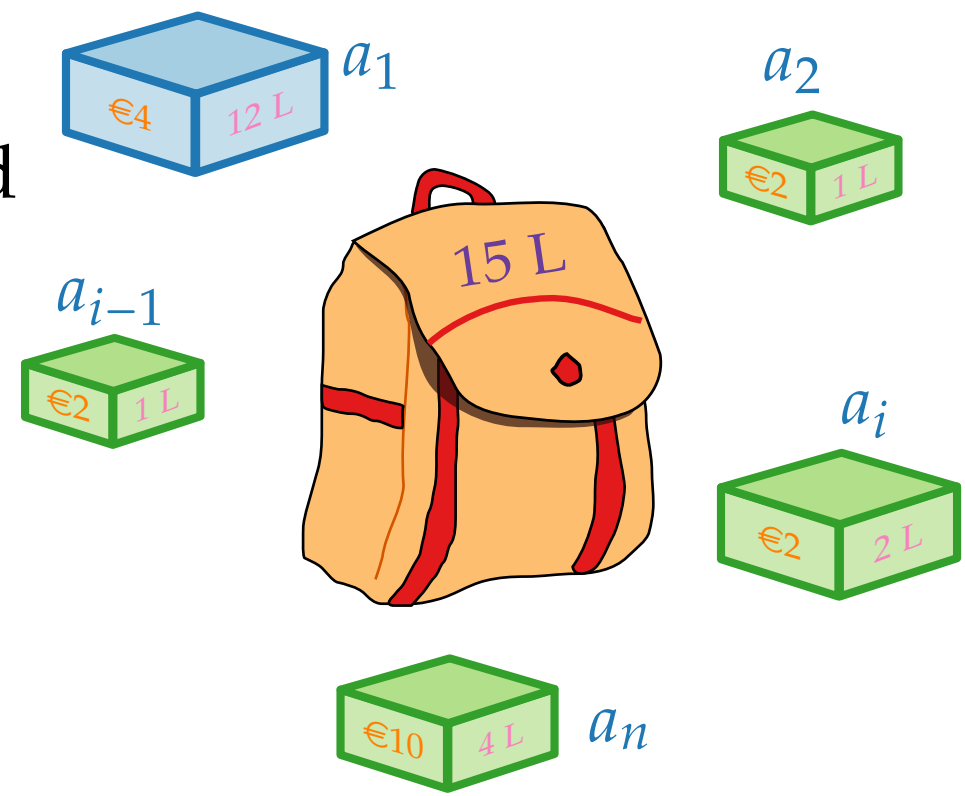
NP-hard



Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part II: Pseudo-Polynomial Algorithms and Strong NP-Hardness

Philipp Kindermann

Summer Semester 2020

Pseudo-Polynomial Algorithms

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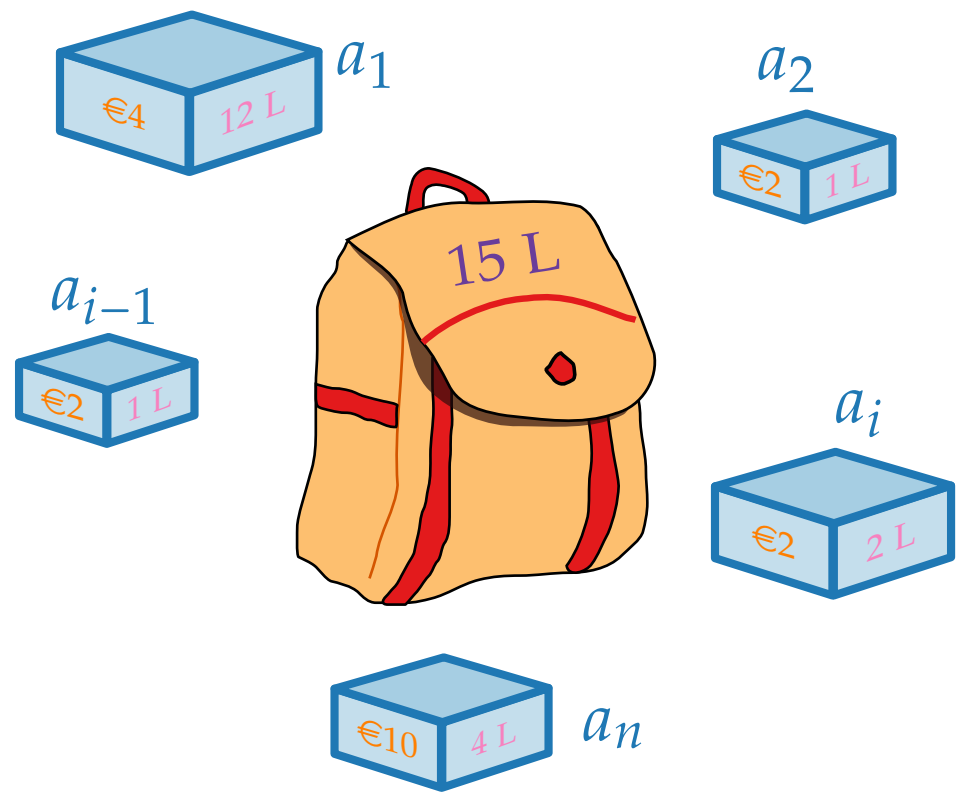

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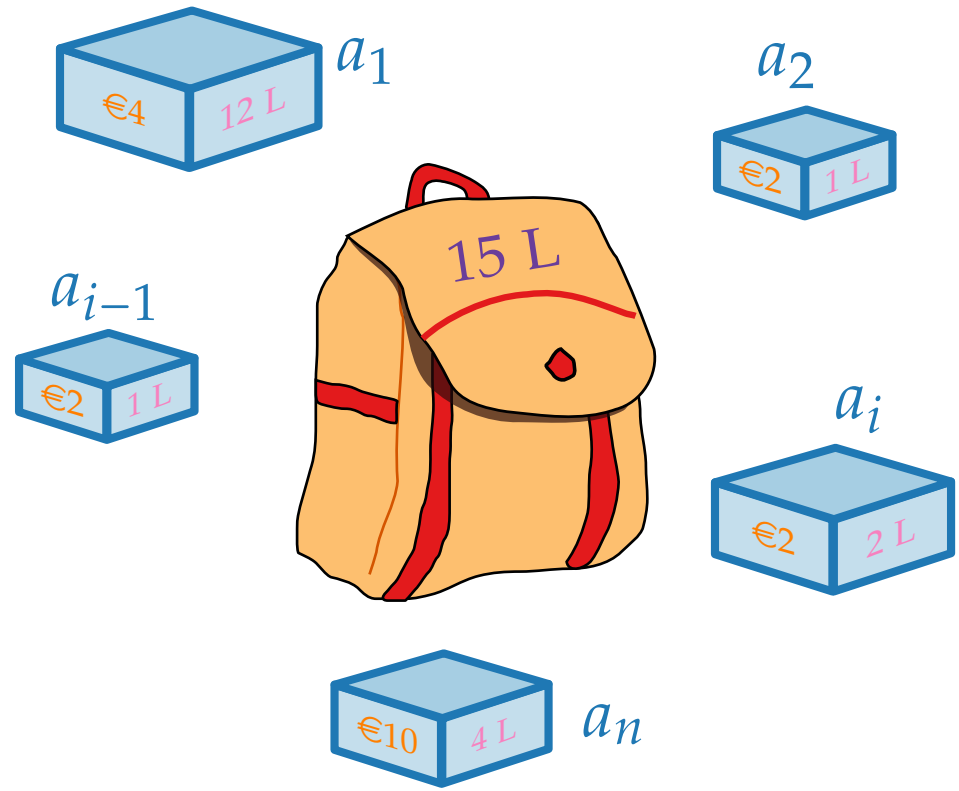

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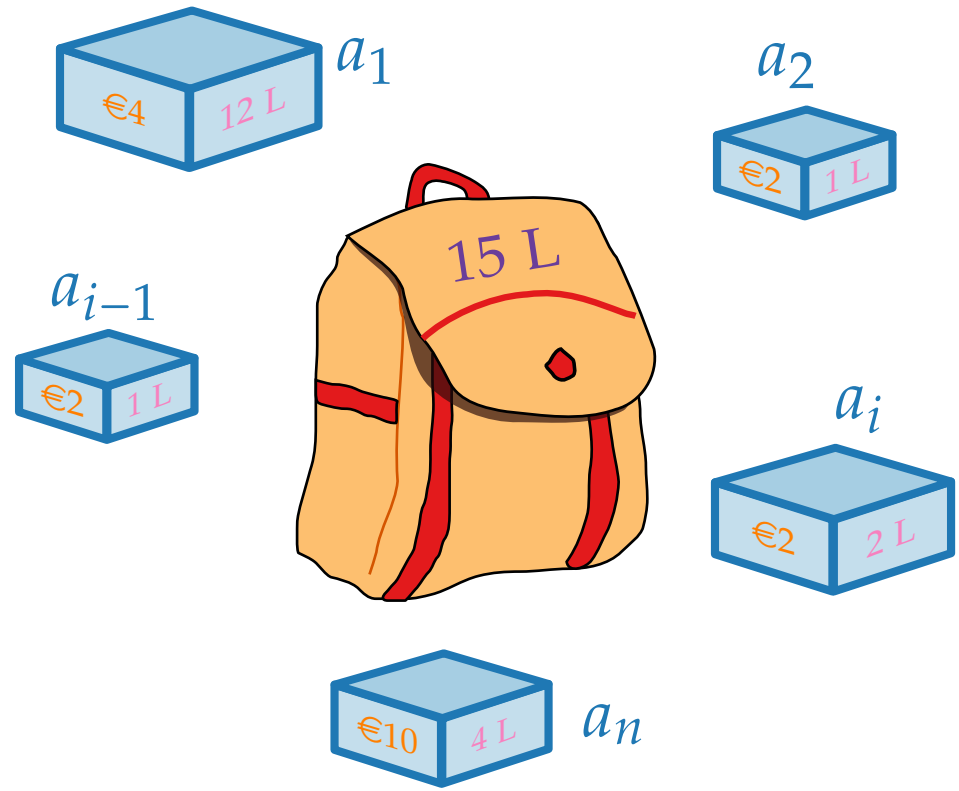

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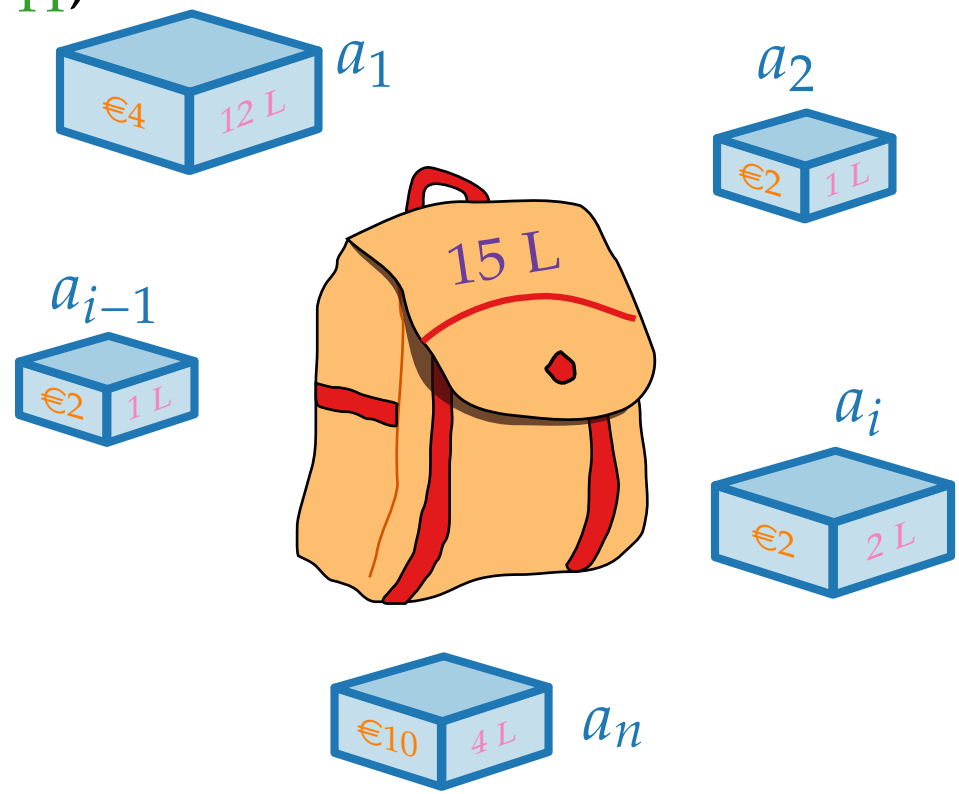

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The running time of a **pseudo-polynomial algorithm** is polynomial in $|I_u|$.

The running time of a pseudo-polynomial algorithm may not be polynomial in $|I|$.

Strong NP-Hardness

An optimization problem is called **strongly NP-hard** if it remains NP-hard with unary numbers.

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An optimization problem is called **strongly NP-hard** if it remains NP-hard with unary numbers.

An optimization problem is called **weakly NP-hard** if it is NP-hard with binary numbers but has a pseudo-polynomial algorithm.

Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless P = NP.

Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

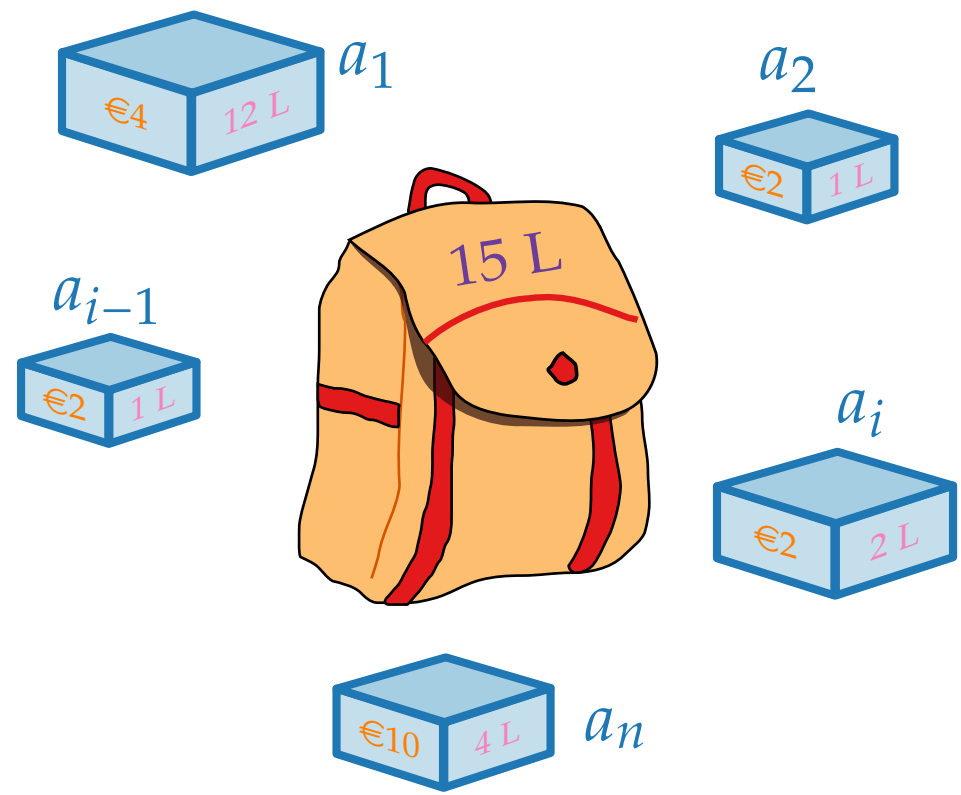
Part III: Pseudo-Polynomial Algorithm for Knapsack

Philipp Kindermann

Summer Semester 2020

Pseudo-Polynomial Alg. for Knapsack

Let $P := \max_i \text{profit}(a_i)$



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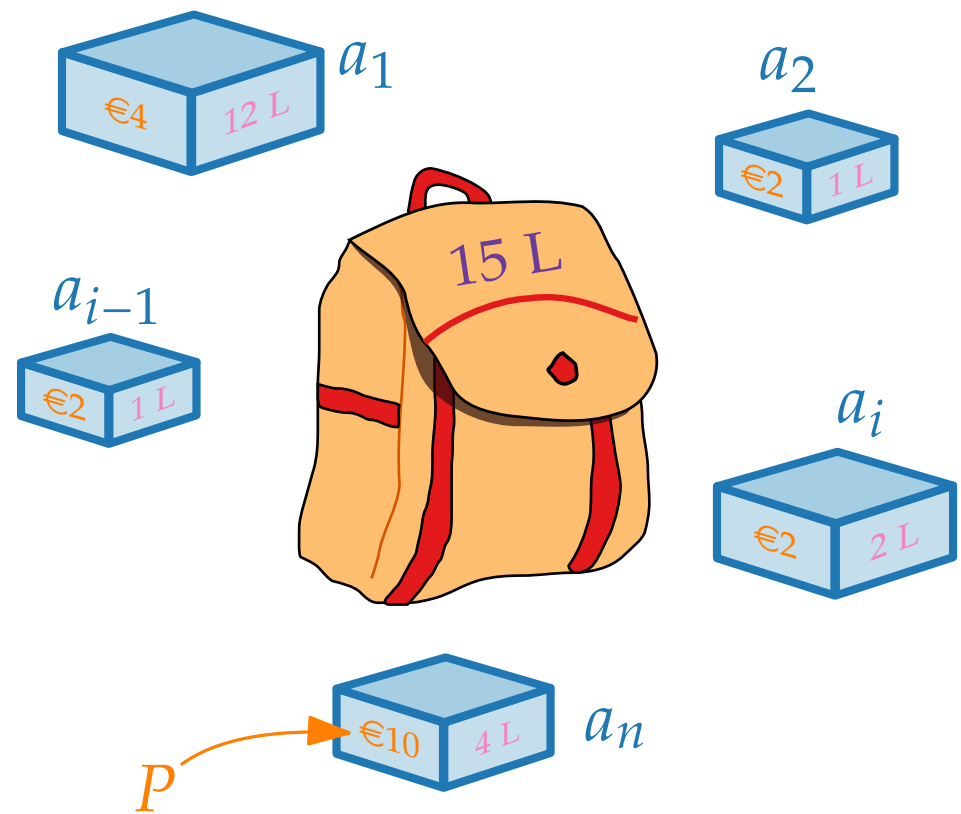

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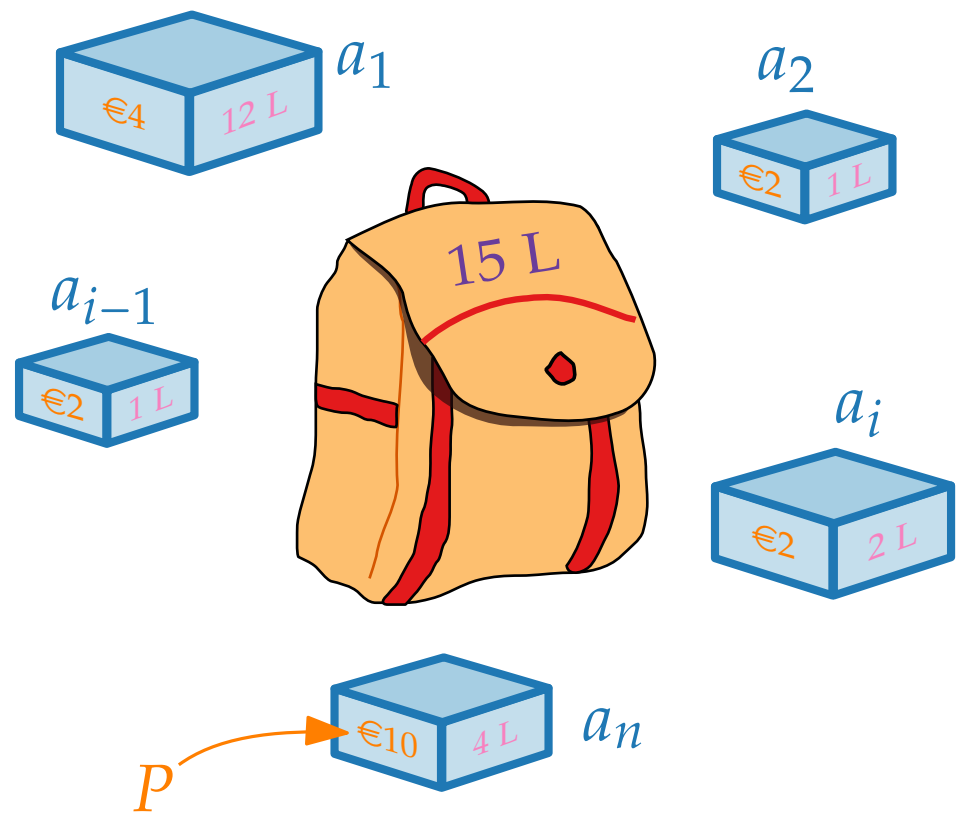

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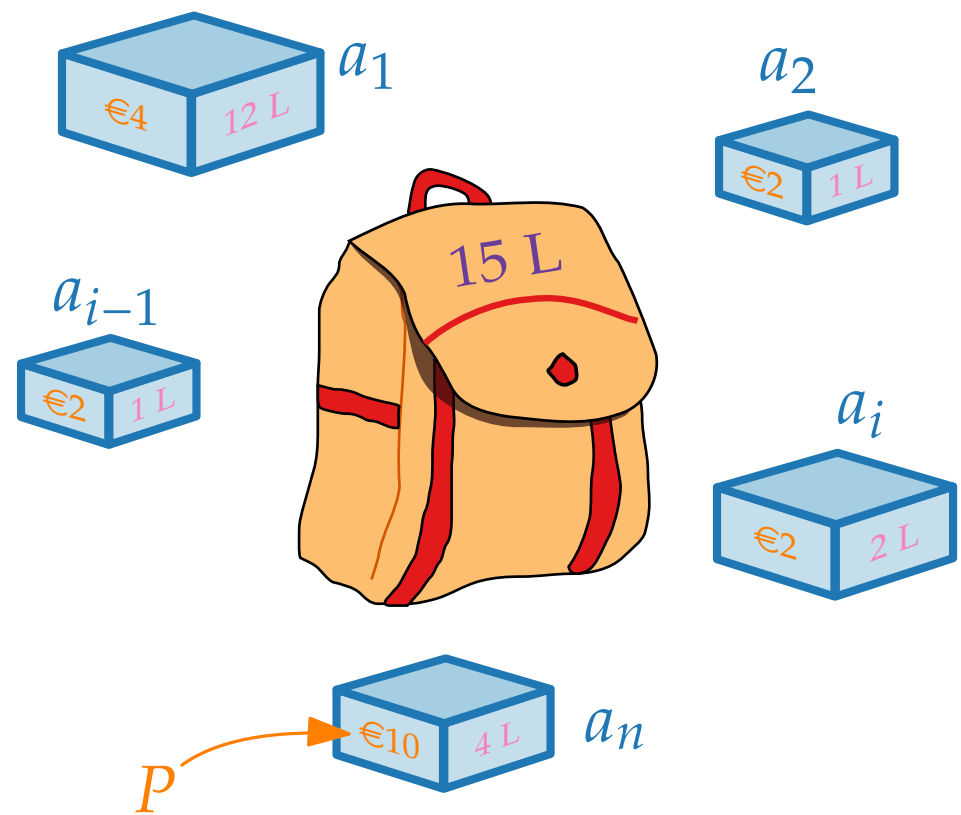
Let $P := \max_i \text{profit}(a_i) \quad \Rightarrow \quad P \leq \text{OPT} \leq nP$



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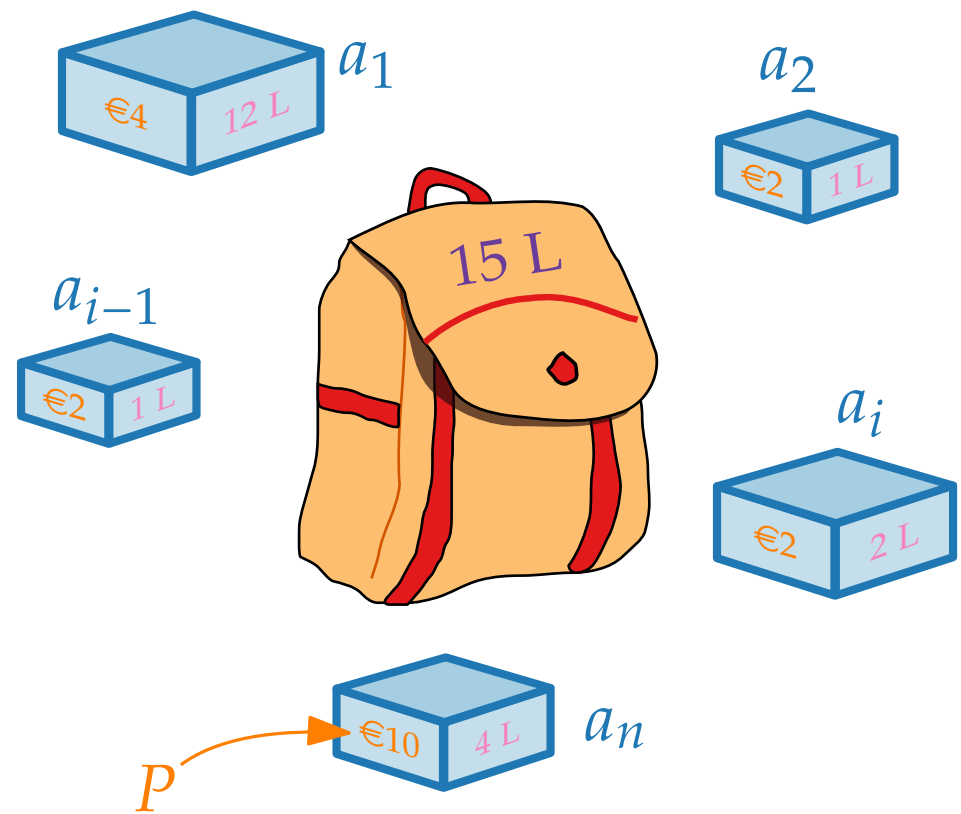

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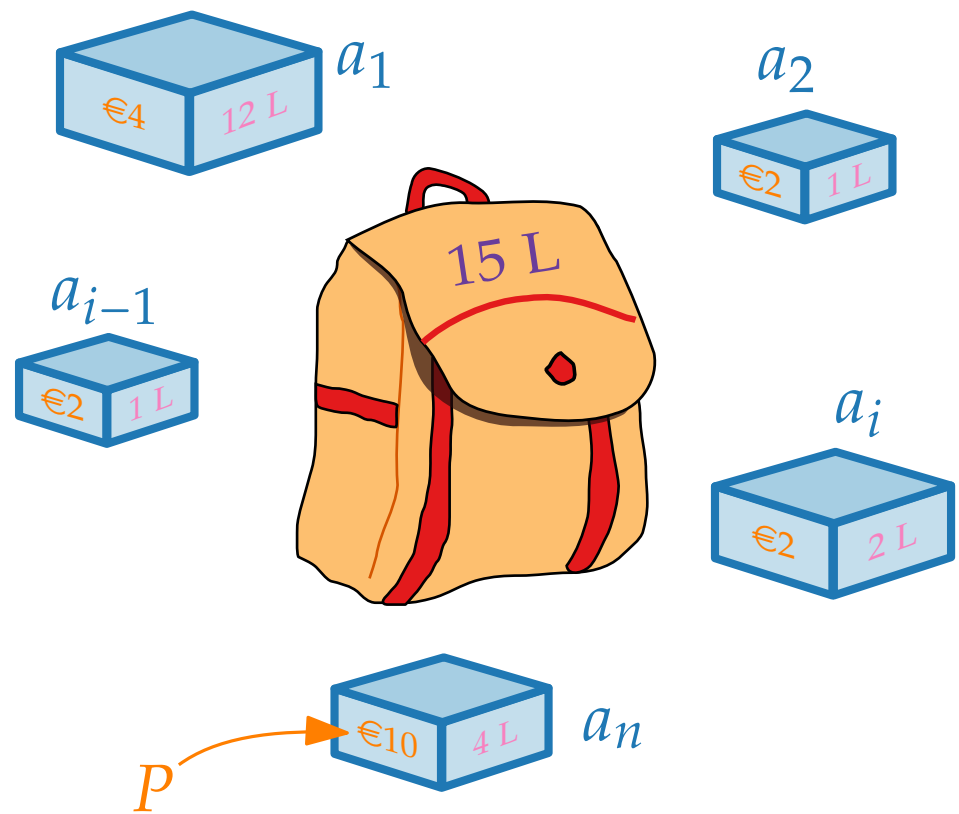

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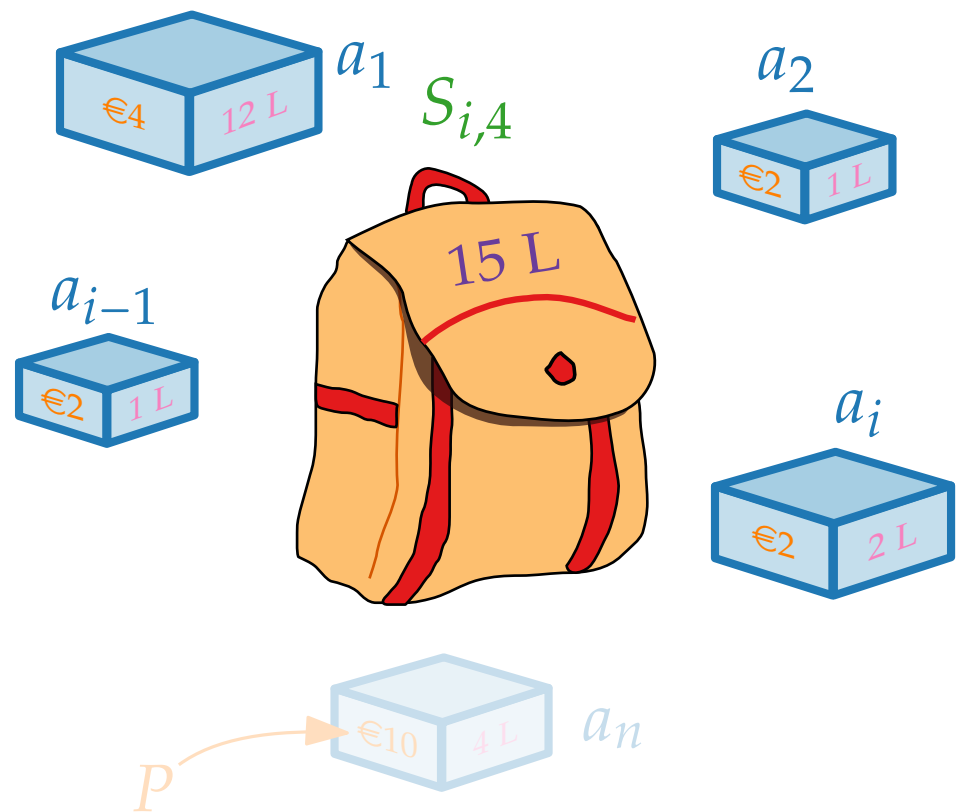

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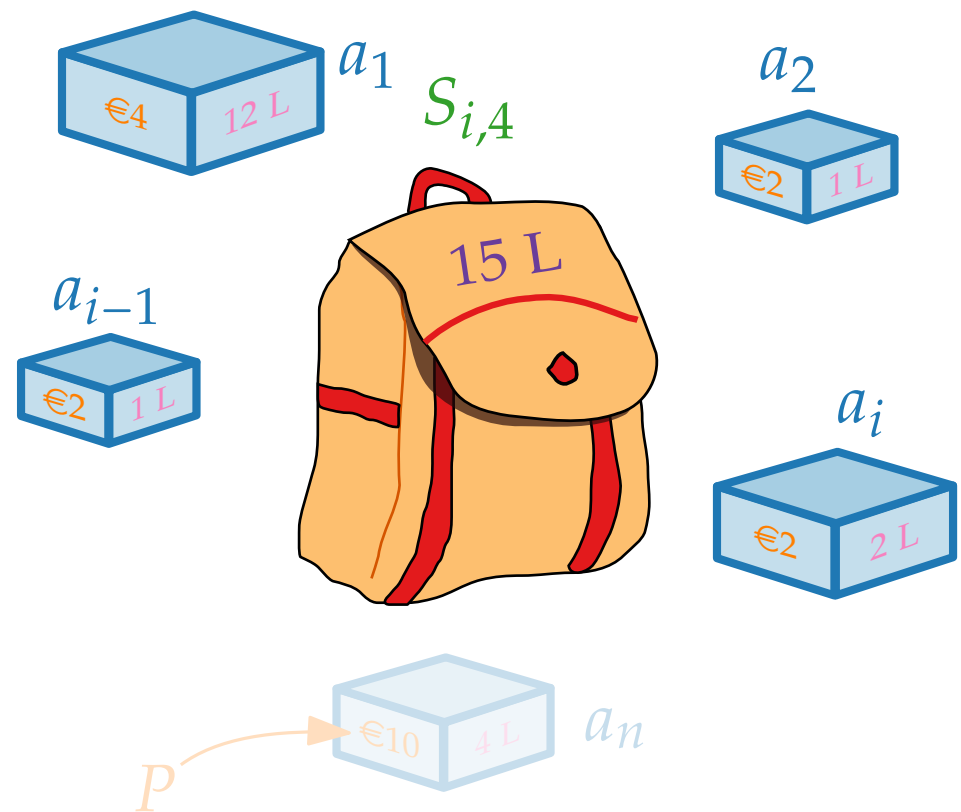

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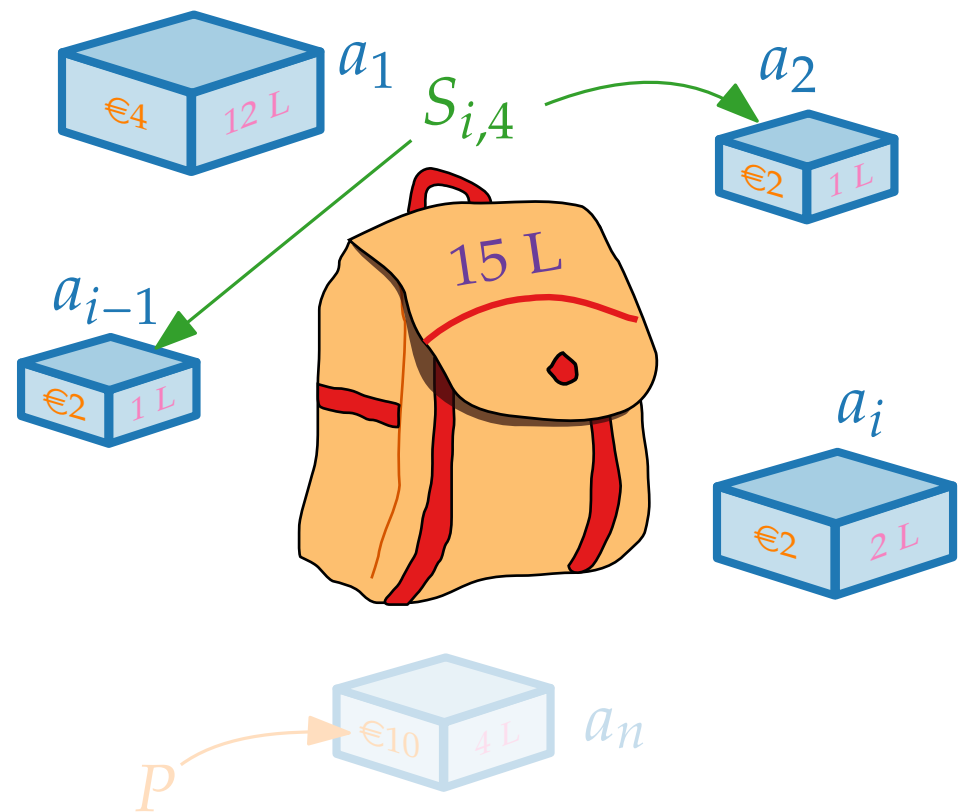

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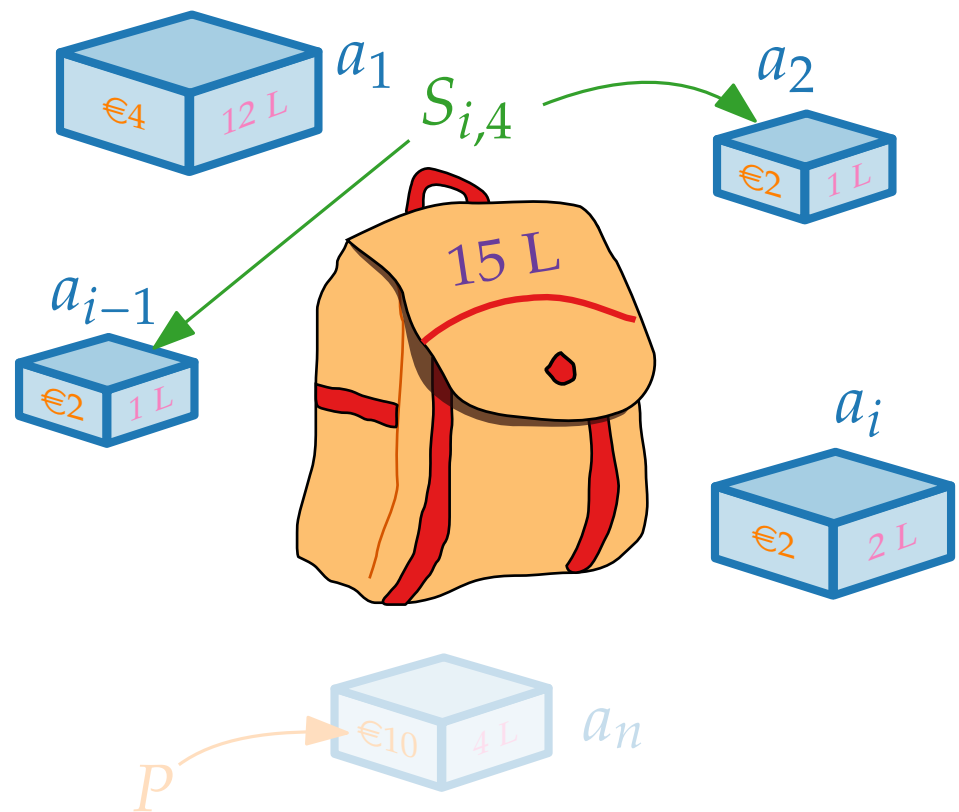

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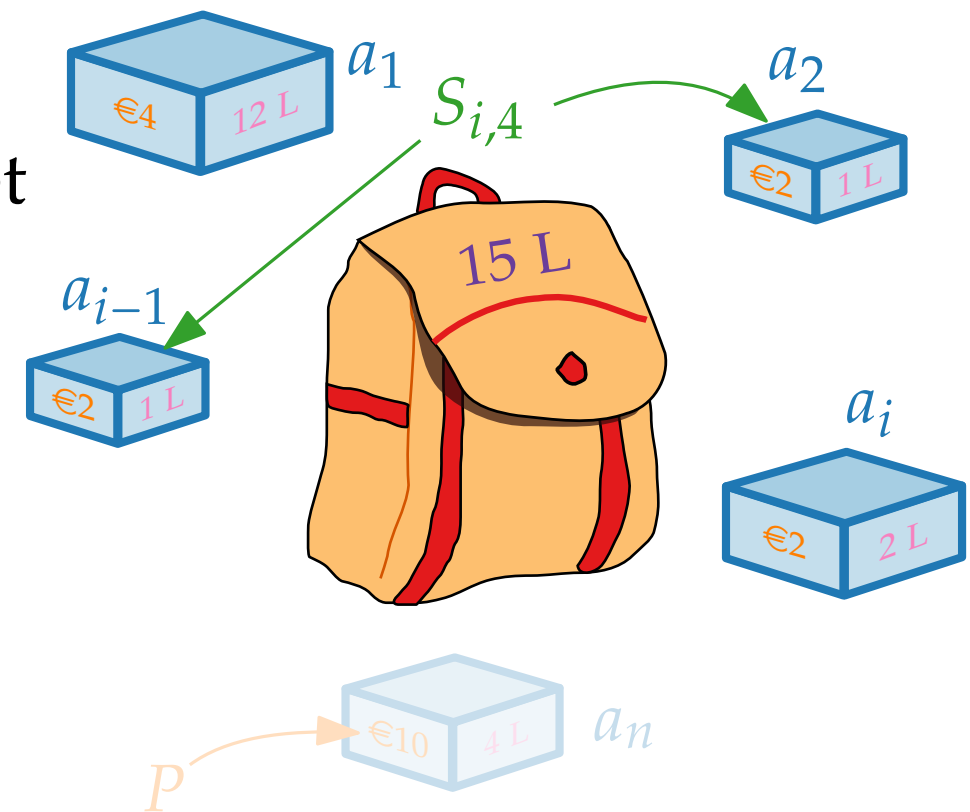

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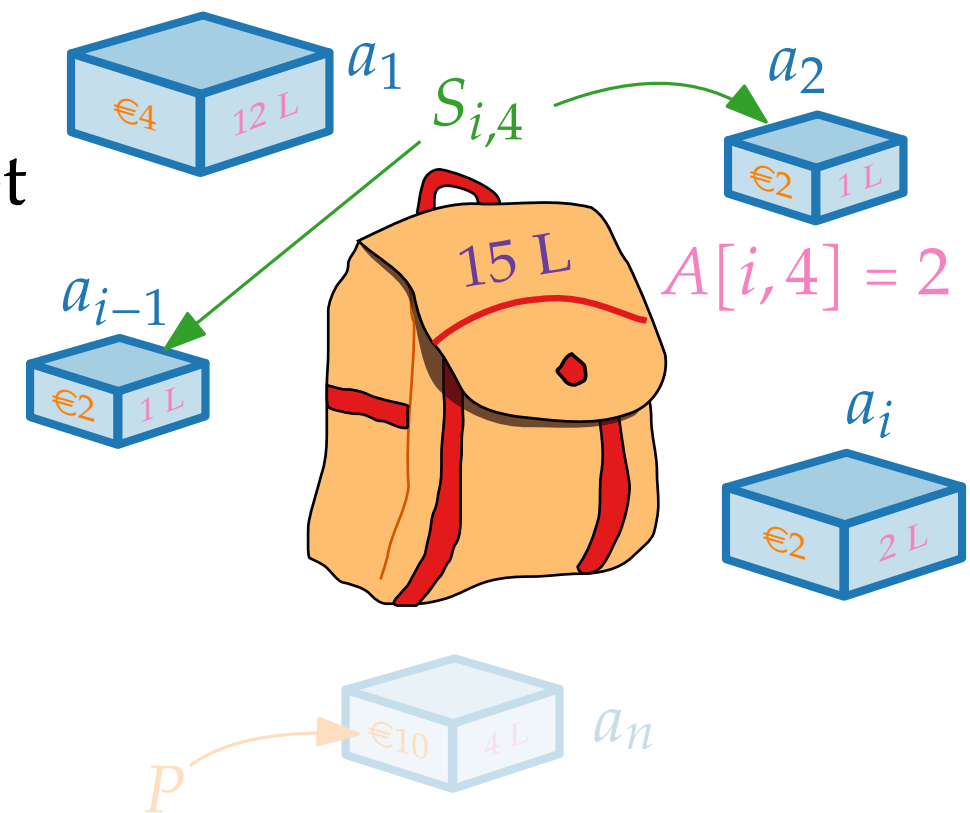

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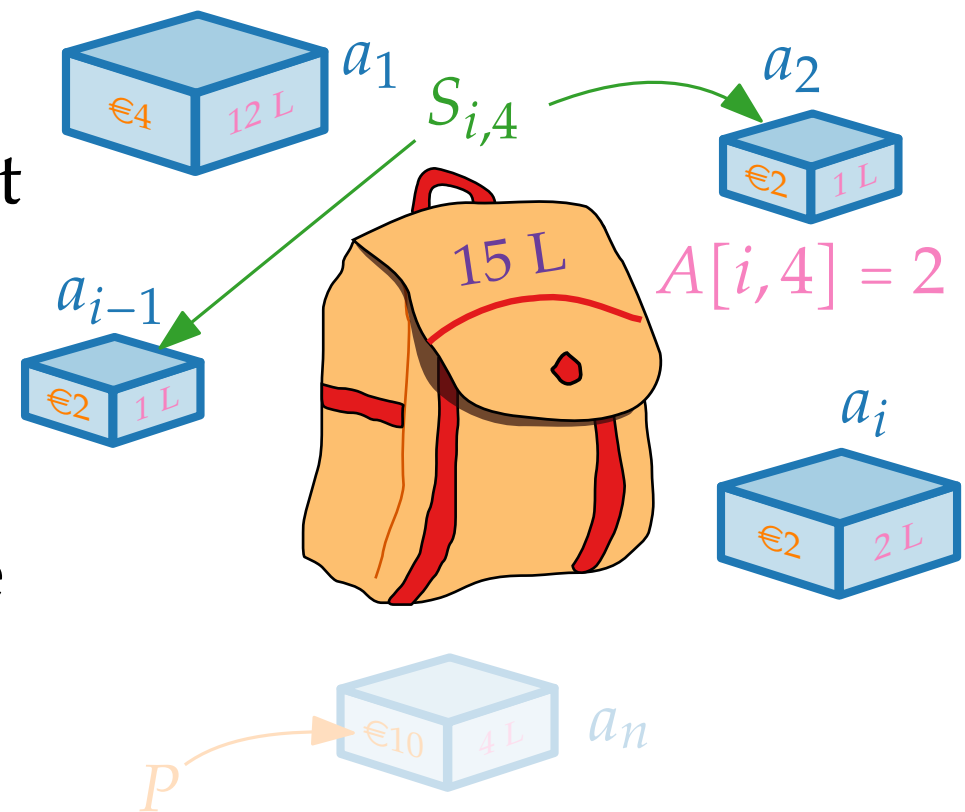

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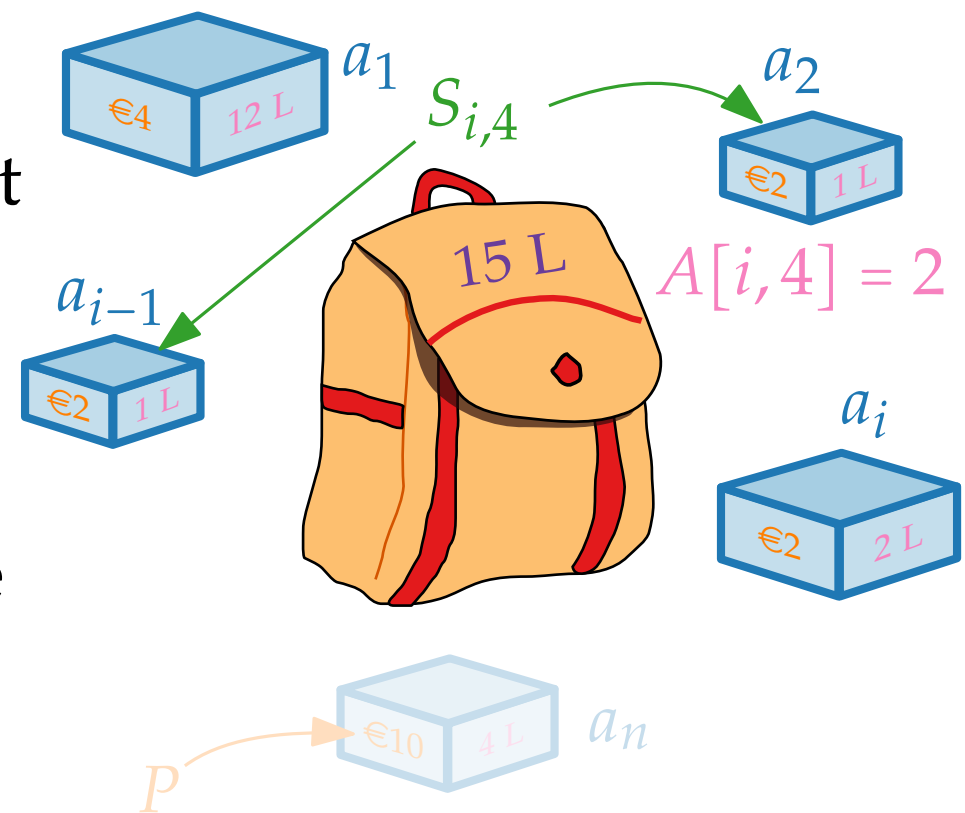
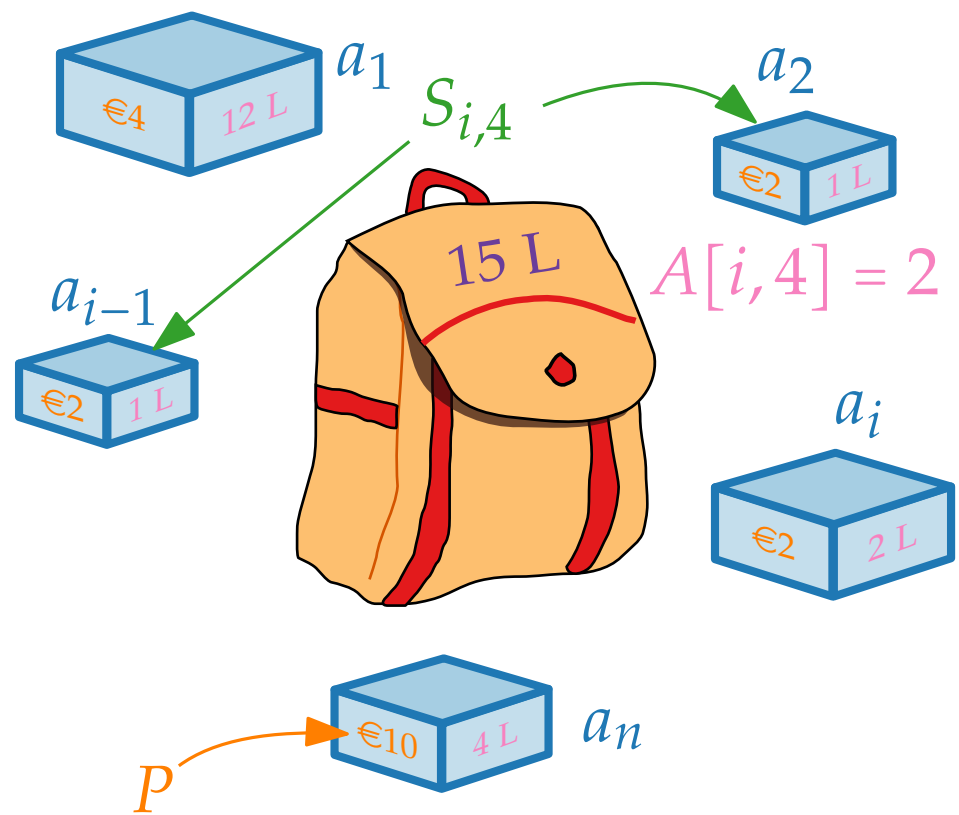


If all $A[i,p]$ are known, then we can compute
$\text{OPT} = \max\{\, p \mid A[n,p] \le B \,\}$.

Pseudo-Polynomial Alg. for Knapsack

$A[1,p]$ can be computed for all $p \in \{0,\ldots,nP\}$.



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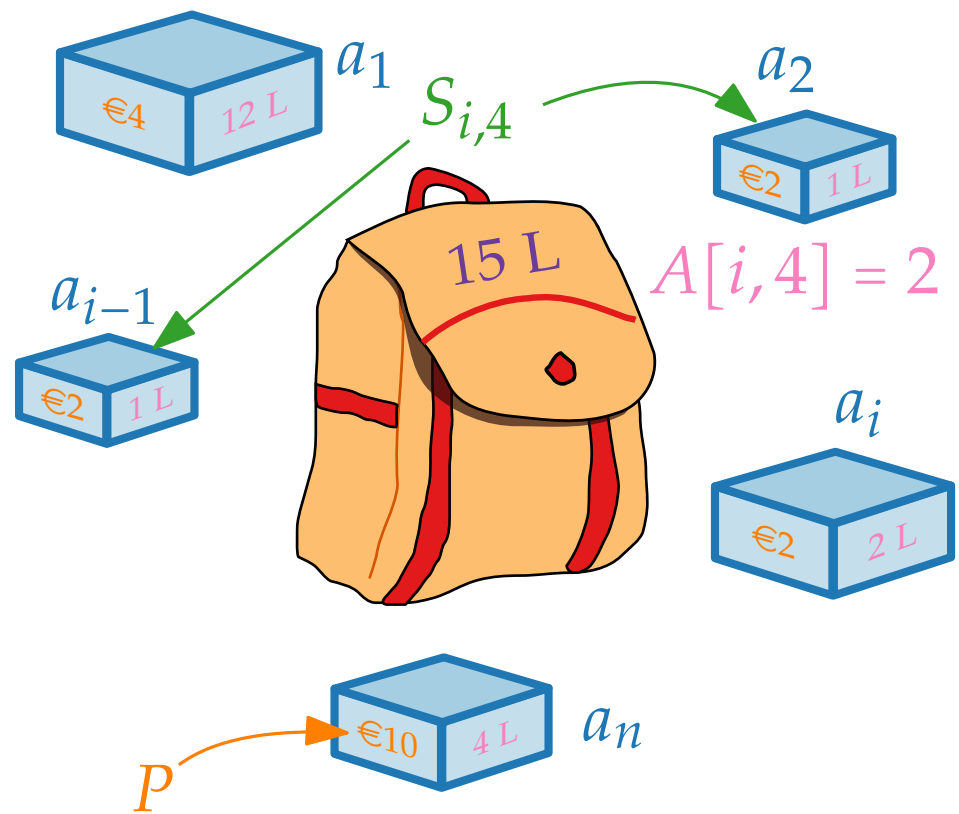

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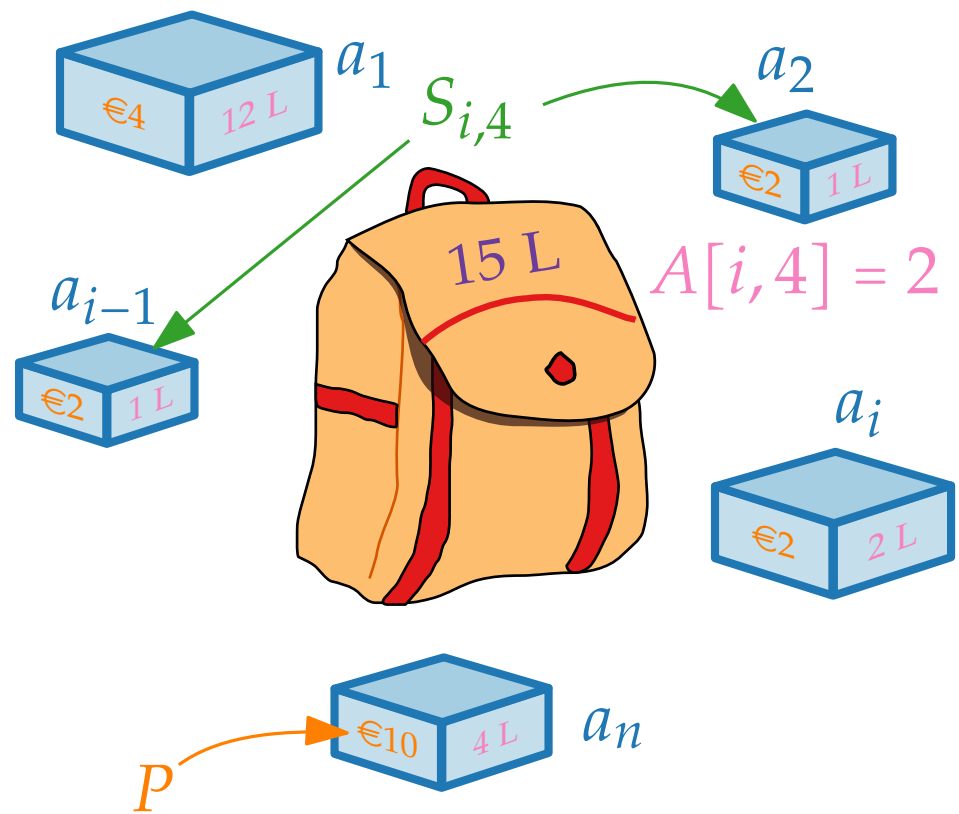

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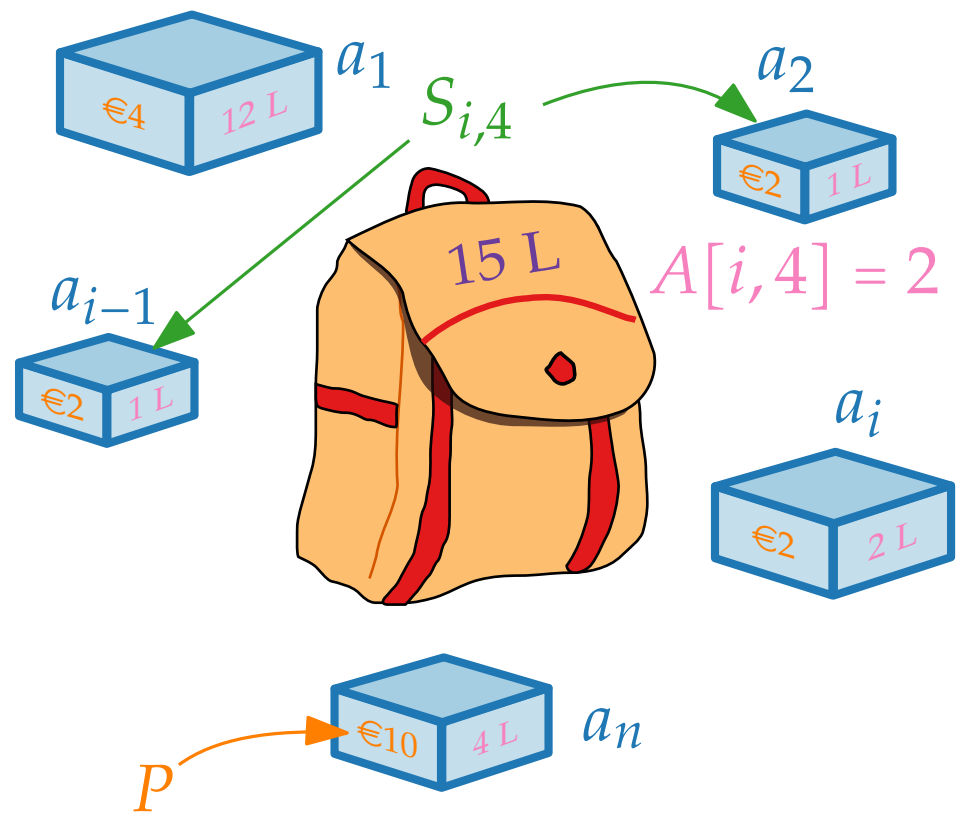

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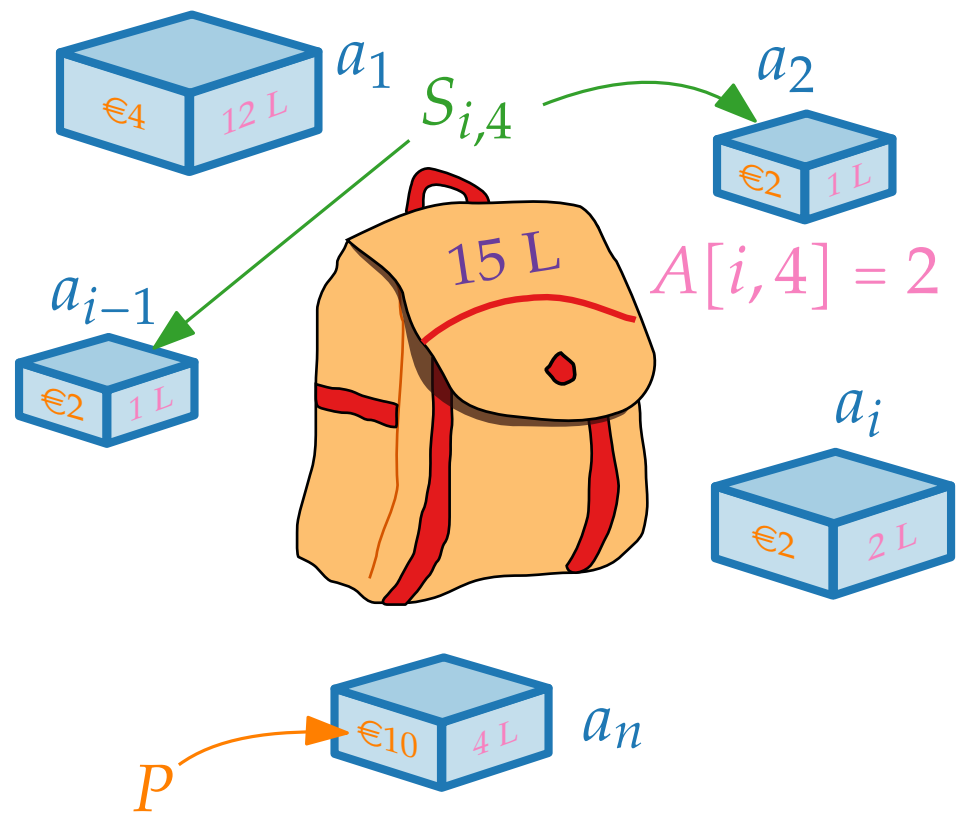

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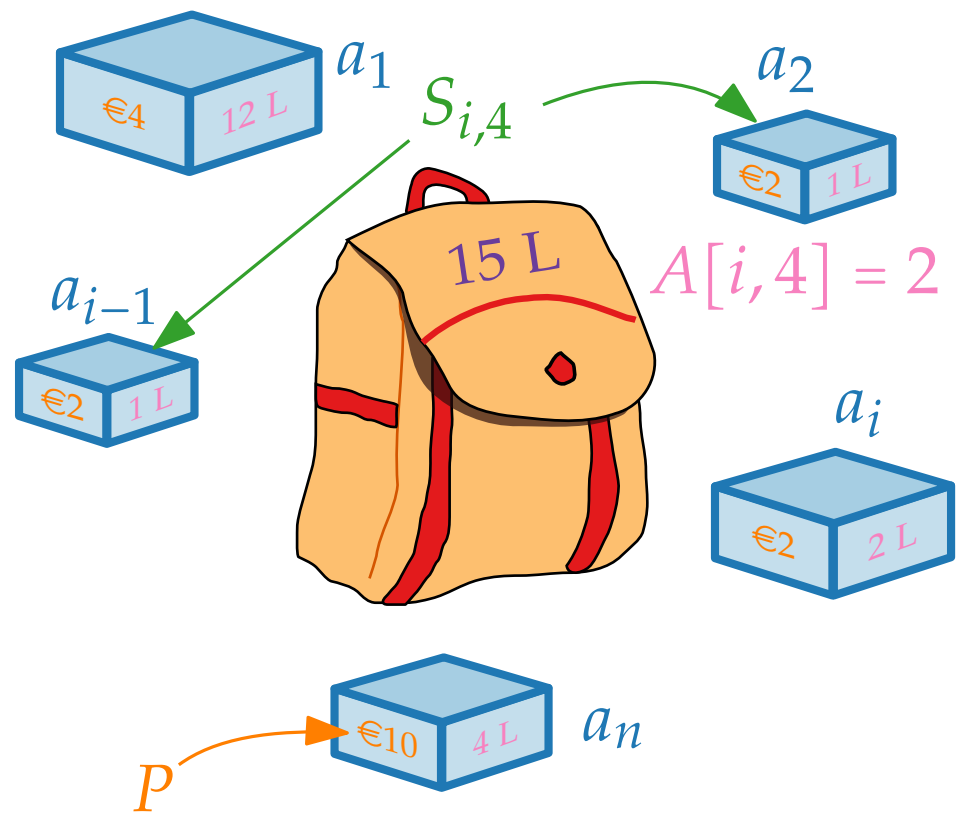

Pseudo-Polynomial Alg. for Knapsack

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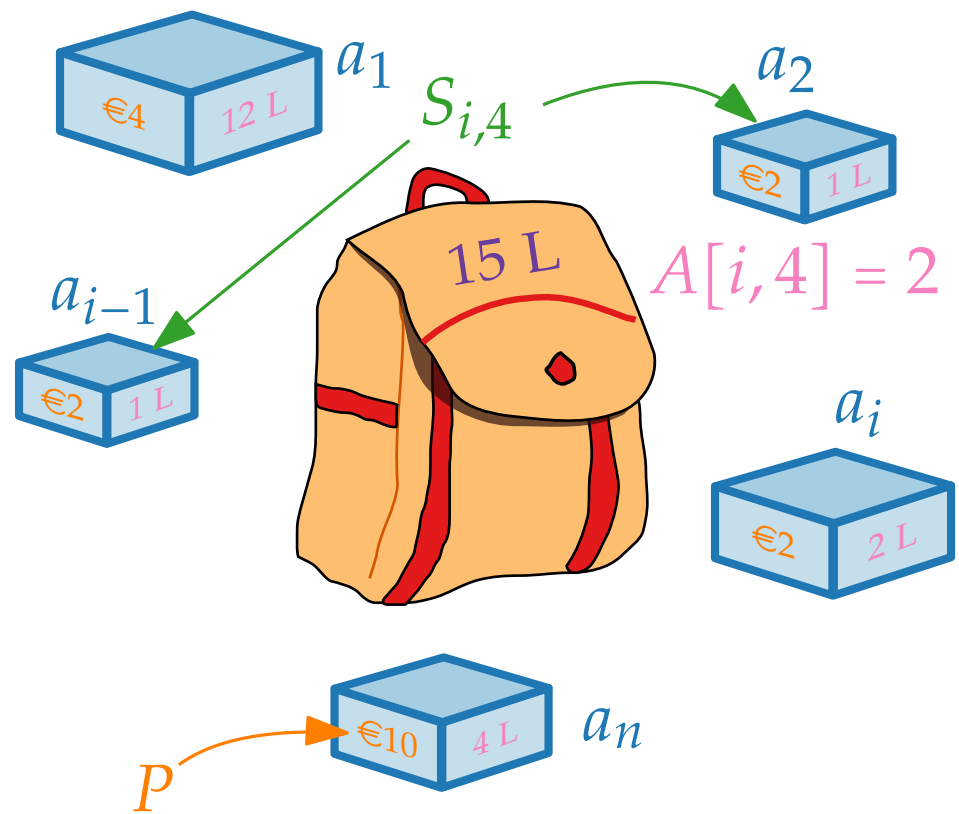

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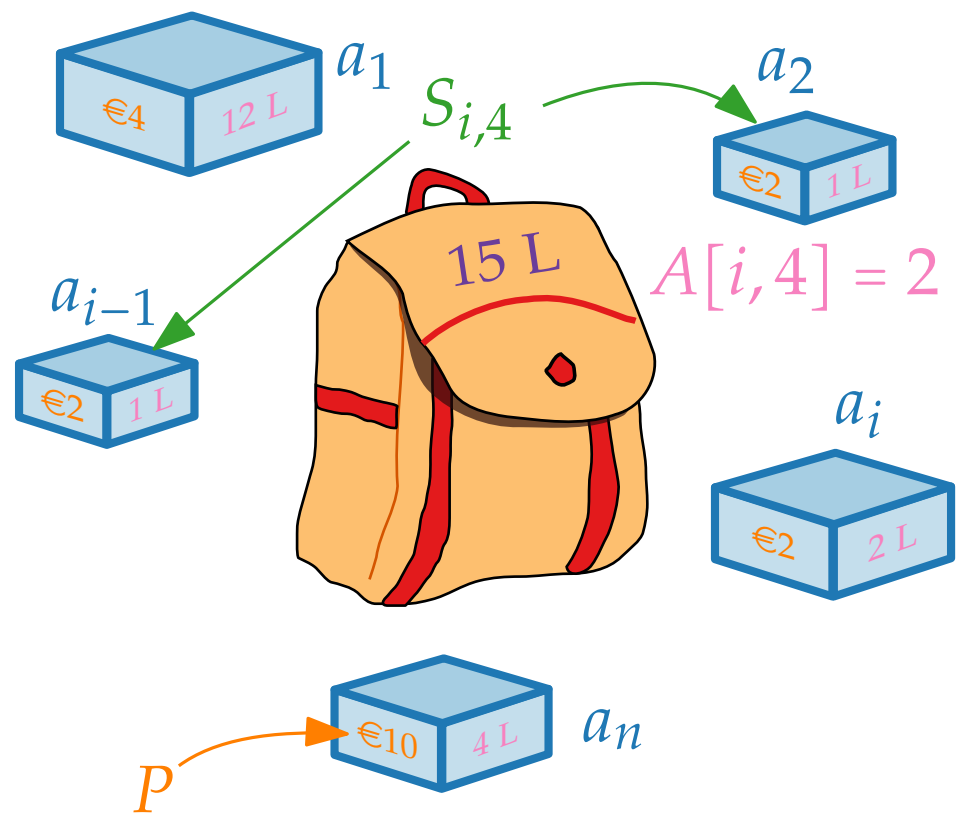

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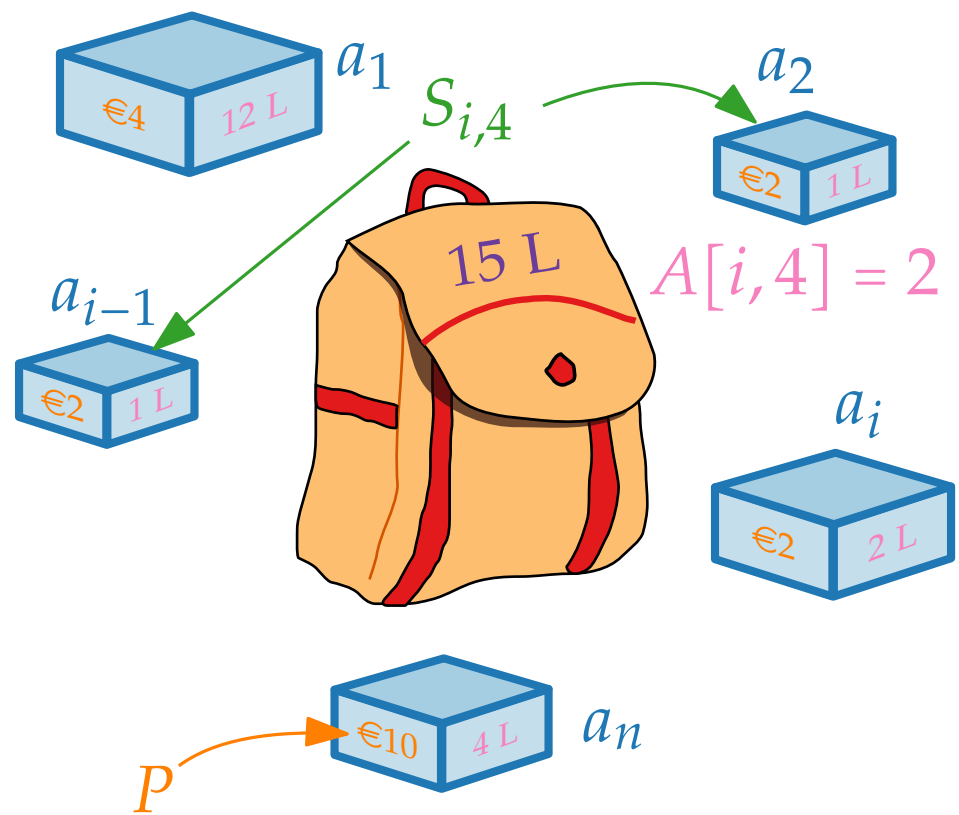

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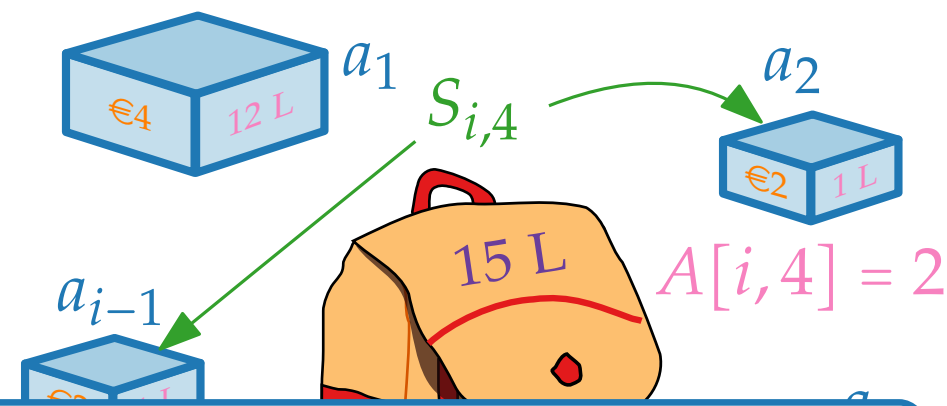
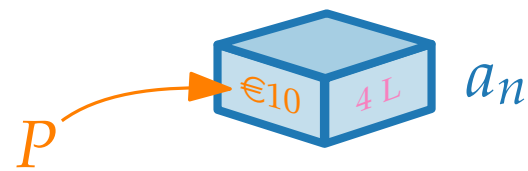


Theorem. KNAPSACK can be solved optimally in pseudo-polynomial time $O(n^2P)$.



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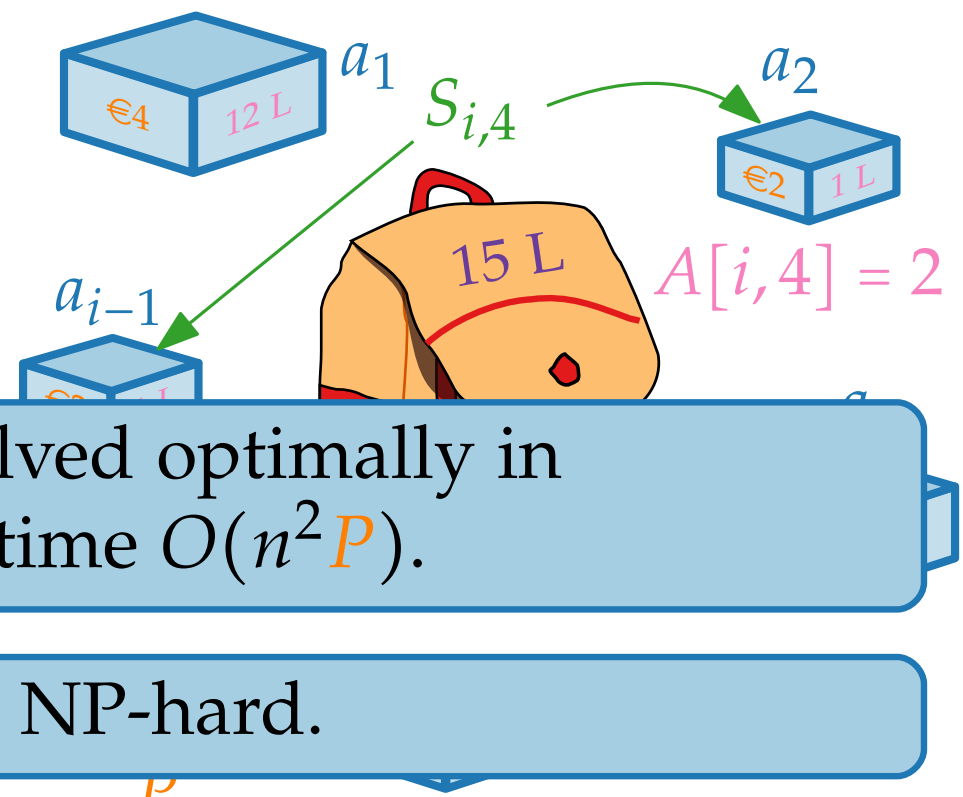

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Corollary. KNAPSACK is weakly NP-hard.

Pseudo-Polynomial Alg. for KNAPSACK

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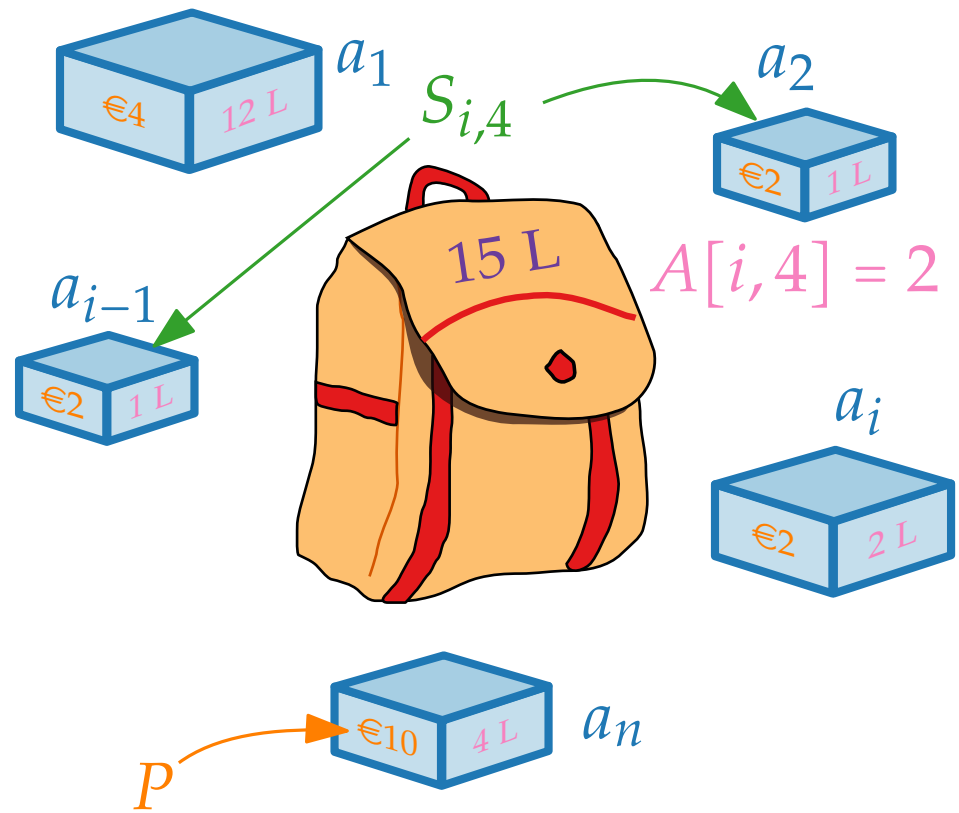

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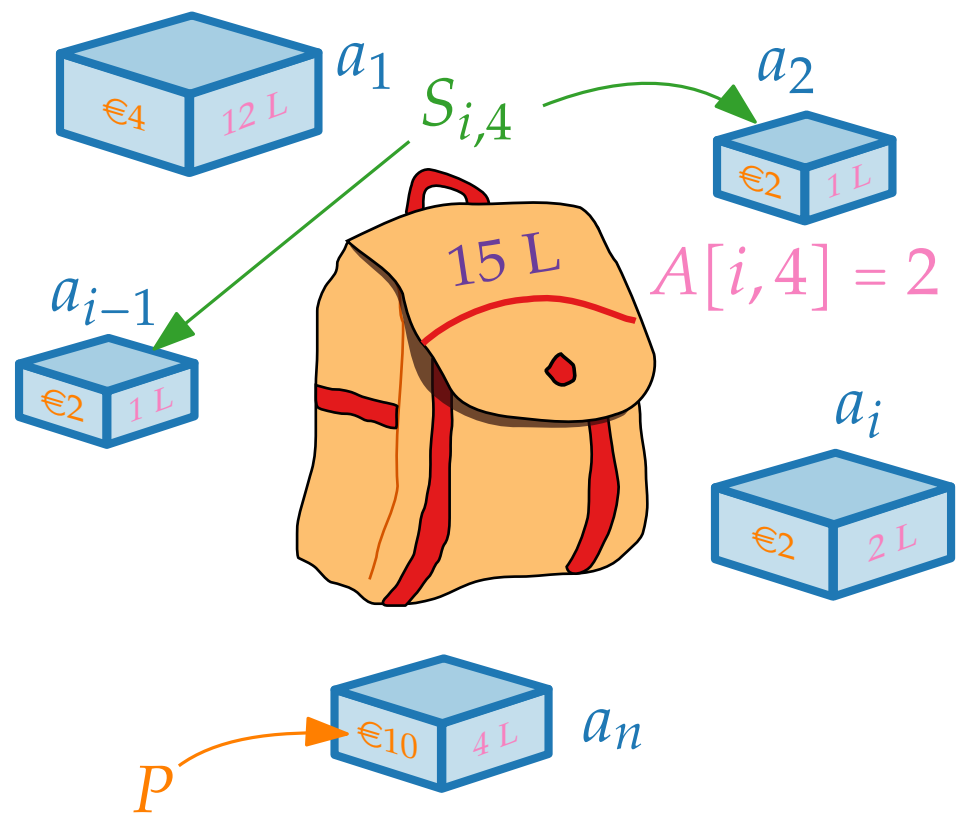
Examples. $P = n^5$



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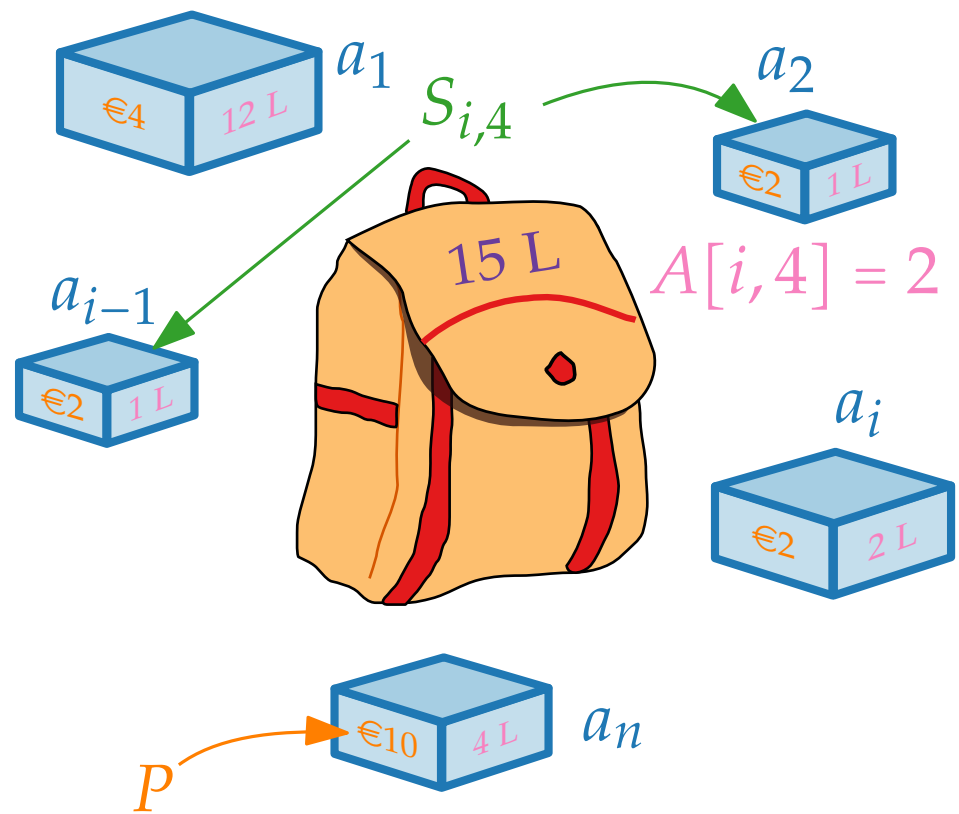
Examples. $P = n^5$ $\Rightarrow$ running time $O(n^7)$



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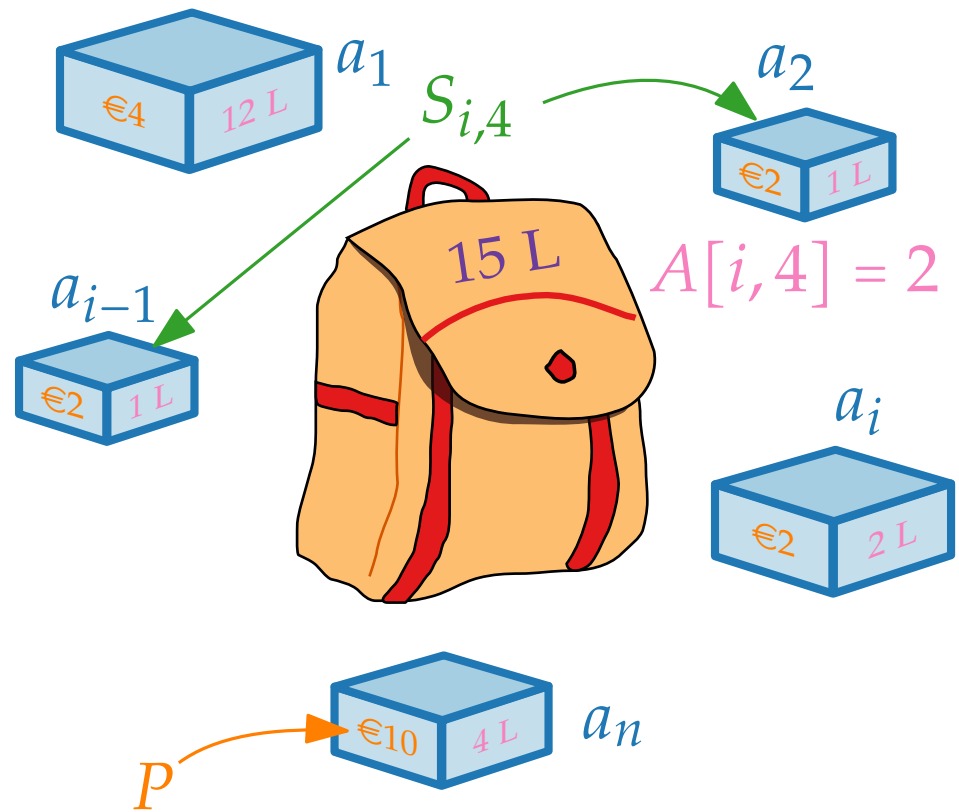

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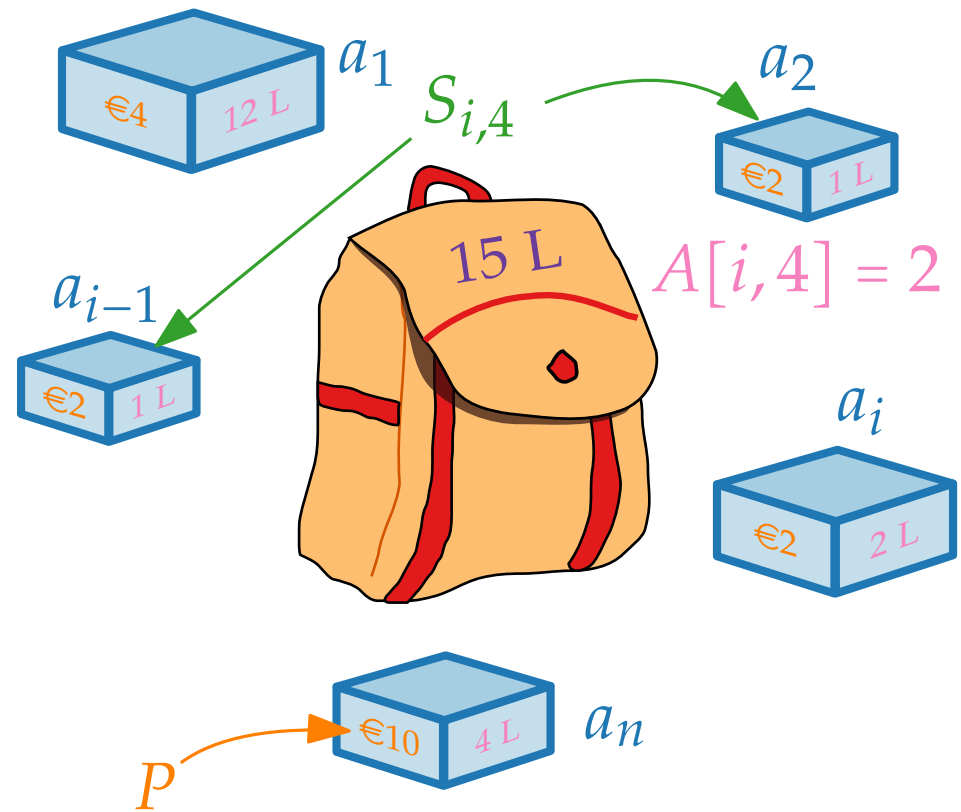

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Observe. Running time $O(n^2 P)$ poly in n if P poly in n.

Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part IV: Approximation Schemes

Philipp Kindermann

Summer Semester 2020

Approximation Schemes

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Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part V: FPTAS for Knapsack

Philipp Kindermann

Summer Semester 2020

FPTAS for Knapsack via Scaling

FPTAS idea: **Scale** profits to polynomial size (as required by the error parameter ε)...

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Lemma. $\text{profit}(S') \geq (1-\varepsilon) \cdot \text{OPT}$.

Proof. Let $\text{OPT} = \{o_1, \ldots, o_k\}$.

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Theorem. KnapsackScaling is an FPTAS for Knapsack with running time $O(n^3/\varepsilon)$.

Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part VI: Connections

Philipp Kindermann

Summer Semester 2020

FPTAS and Pseudo-Poly. Algorithms

Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem

FPTAS and Pseudo-Poly. Algorithms

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Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless P = NP.

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Corollary. Let Π be an NP-hard optimization problem that fulfils the restrictions above. If Π is strongly NP-hard, then there is no FPTAS for Π (unless P = NP).

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Have to scale with product of denominators to become integral.