Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part I:
Metric-k-Center

Philipp Kindermann

Summer Semester 2020

Metric-k-Center

Given: A graph $G = (V, E)$

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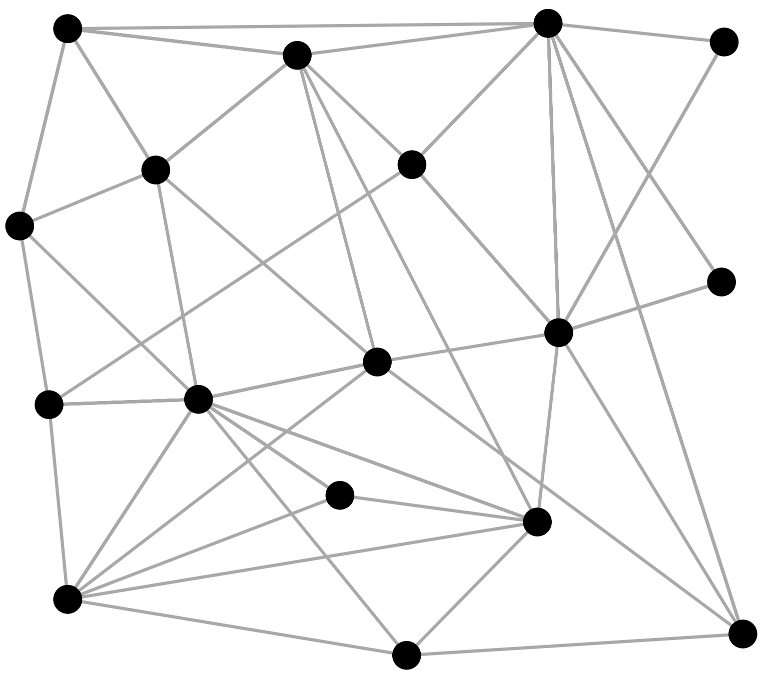

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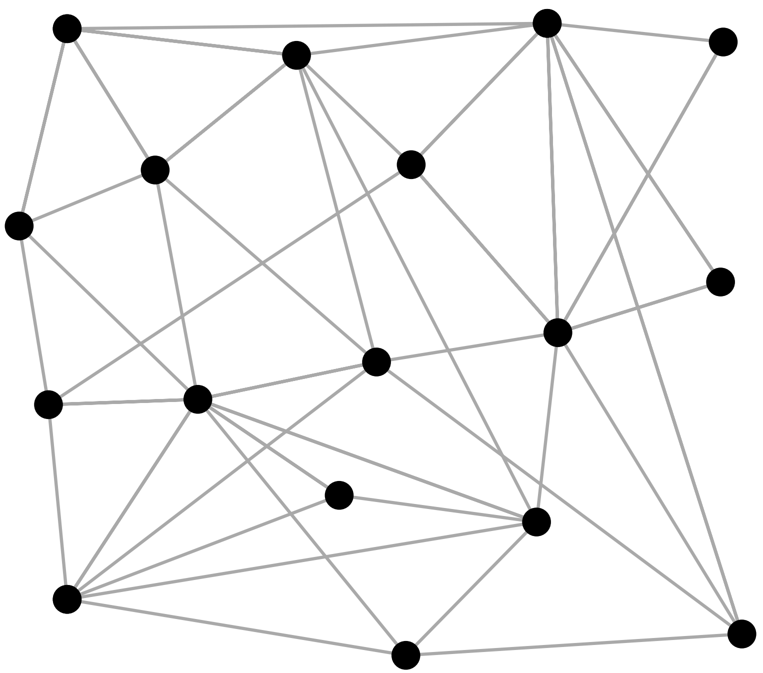

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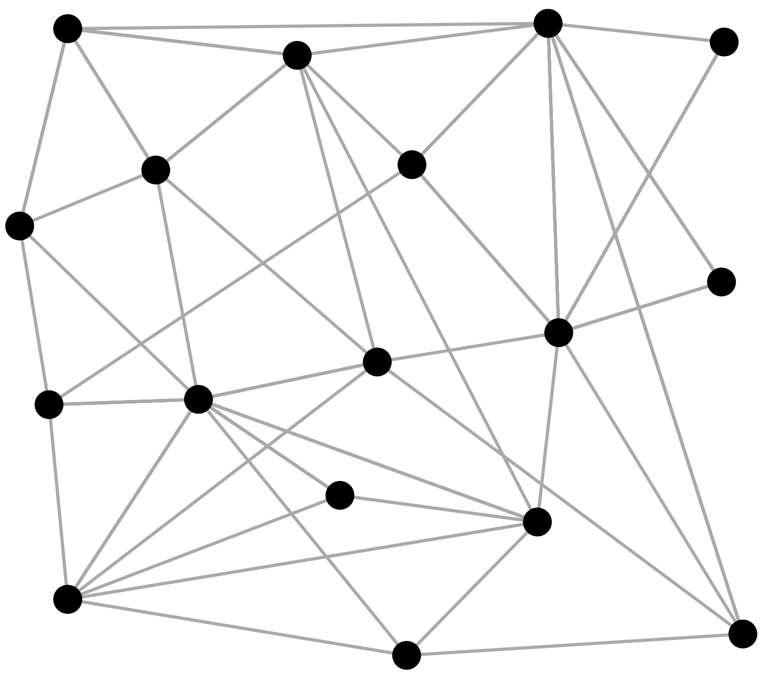

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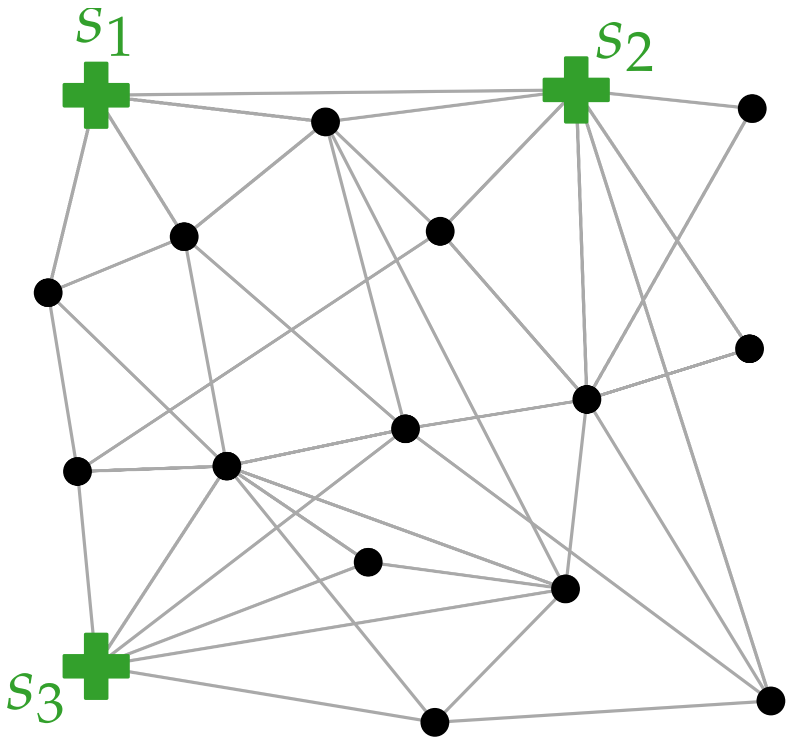

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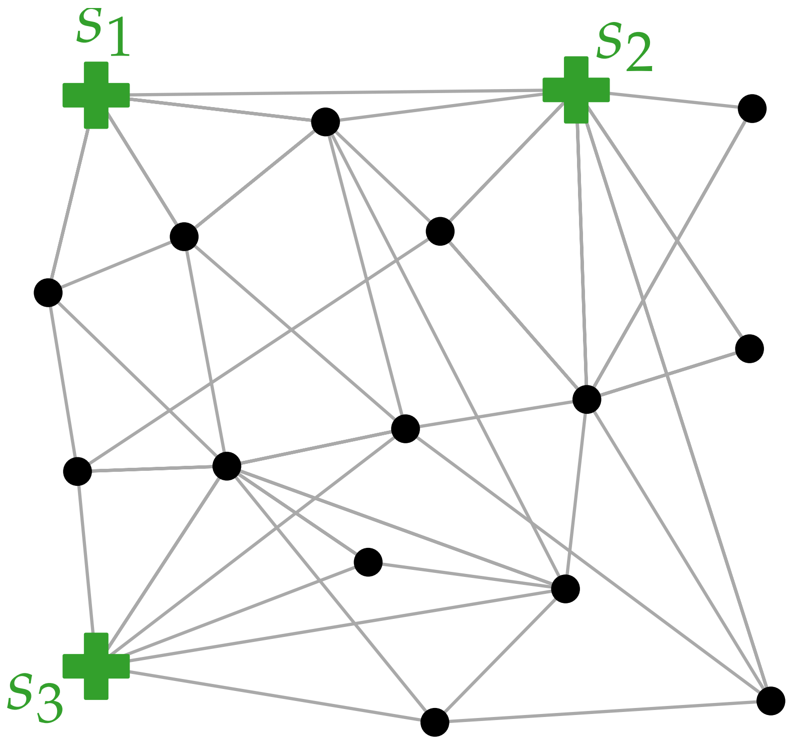
For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.



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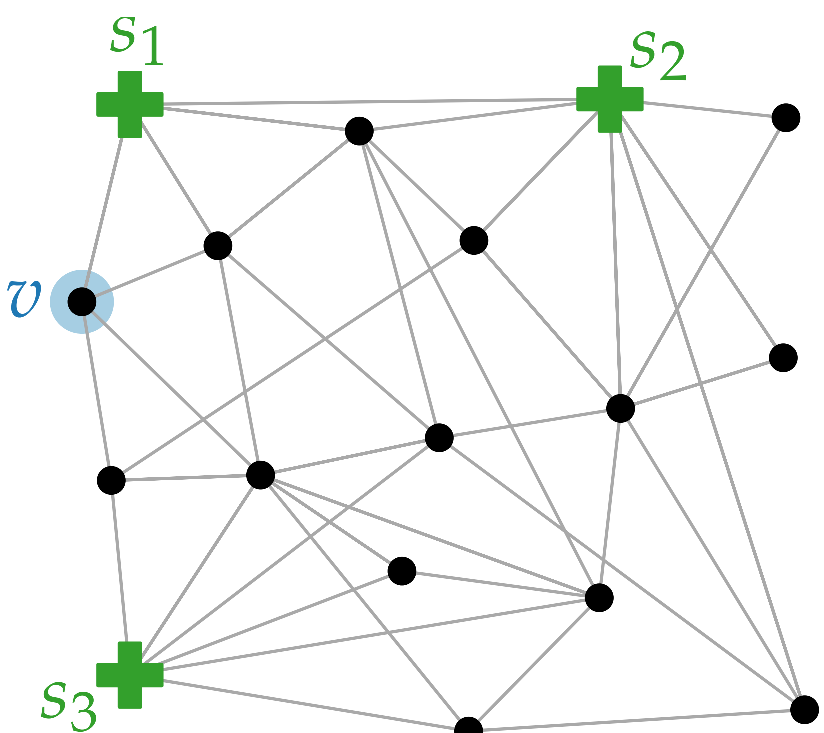

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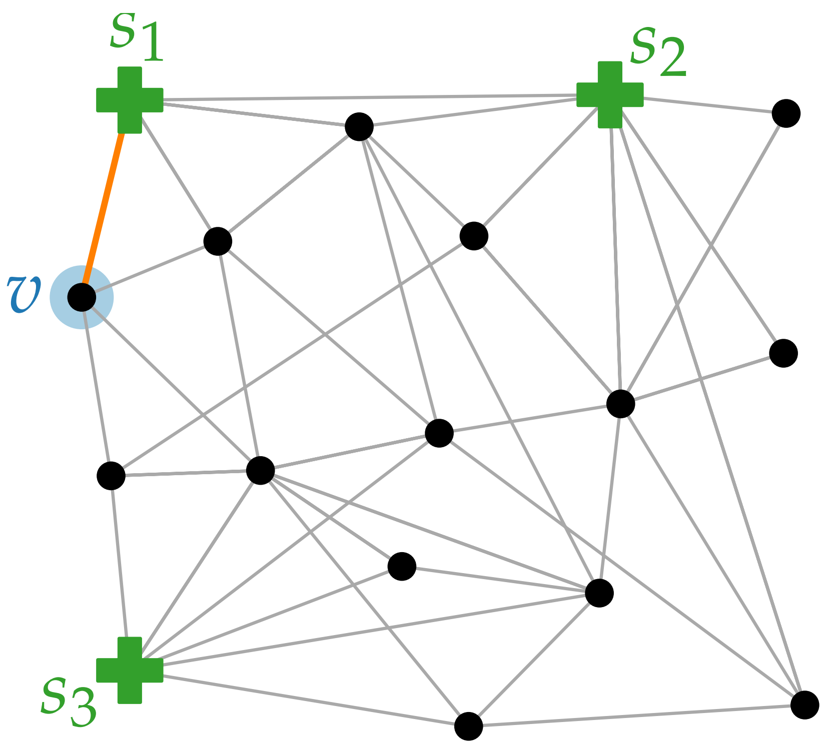

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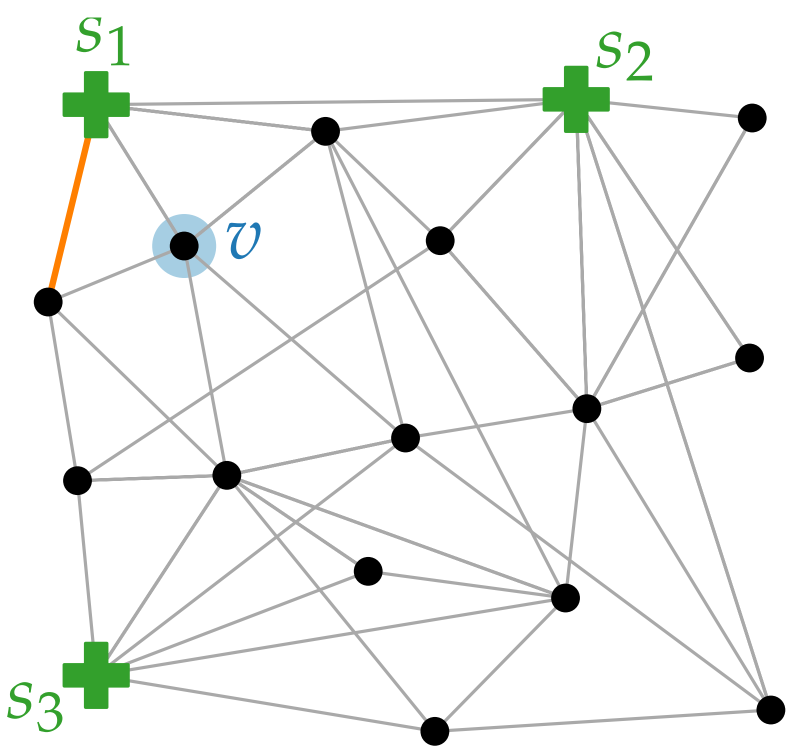

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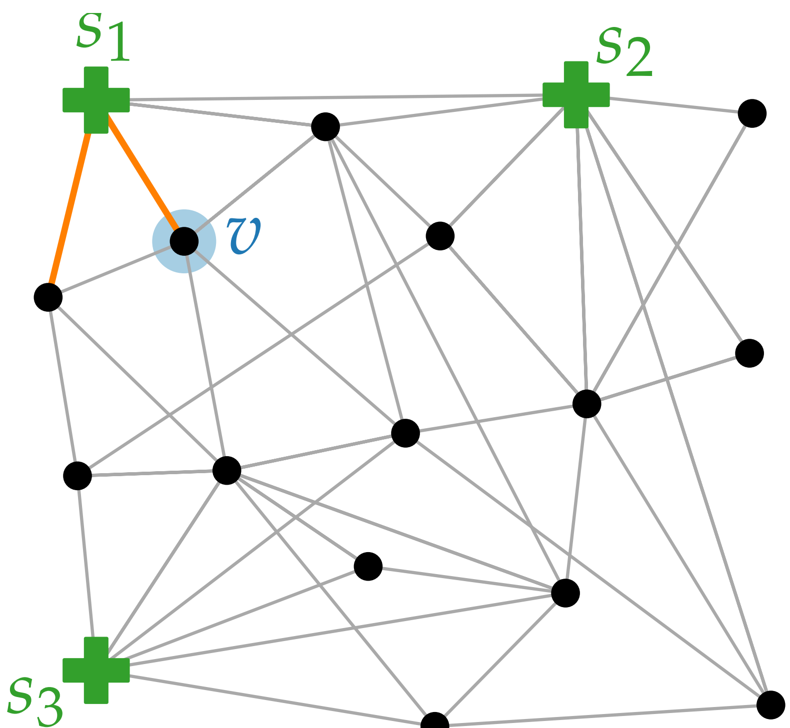

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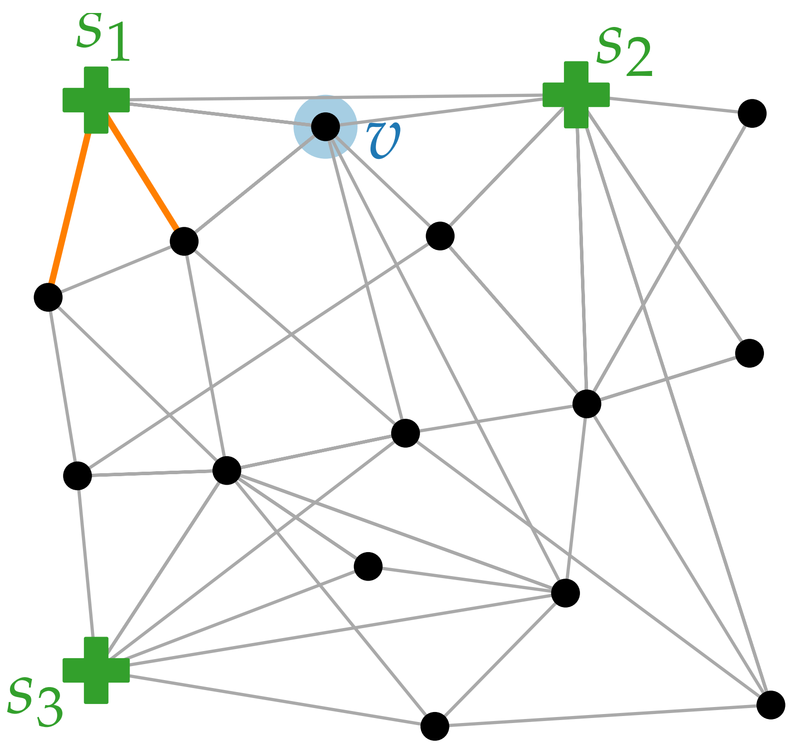

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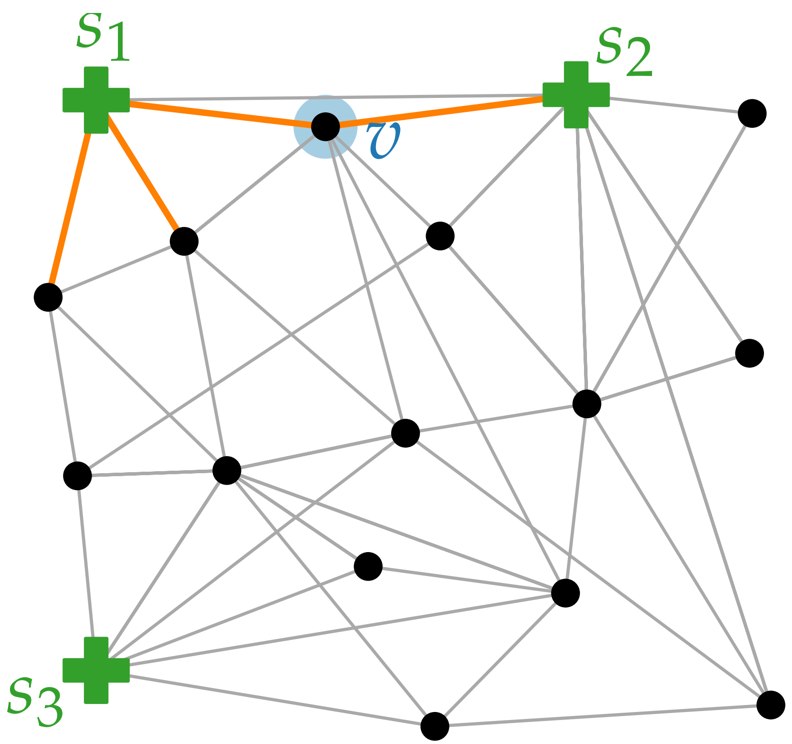

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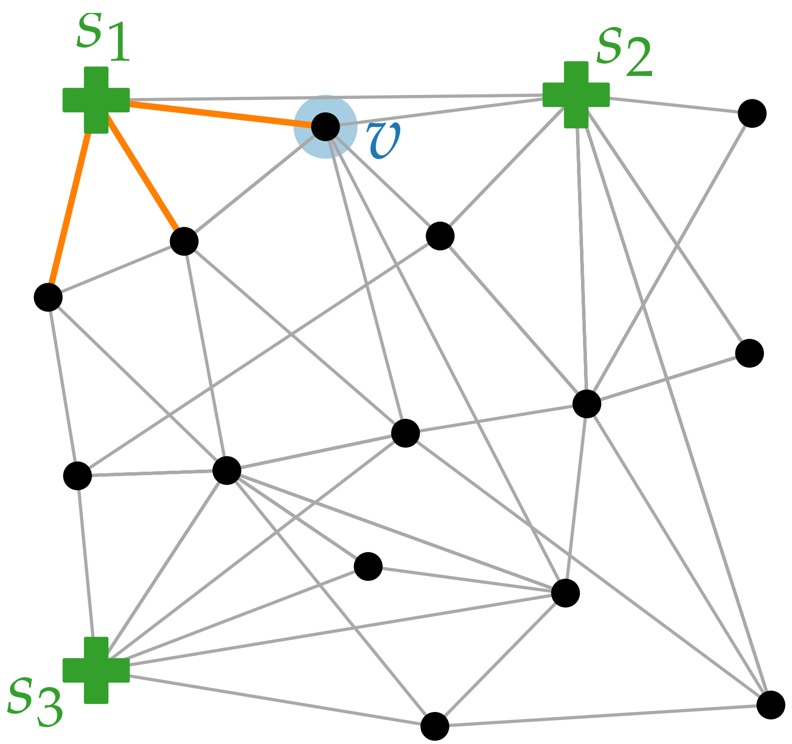

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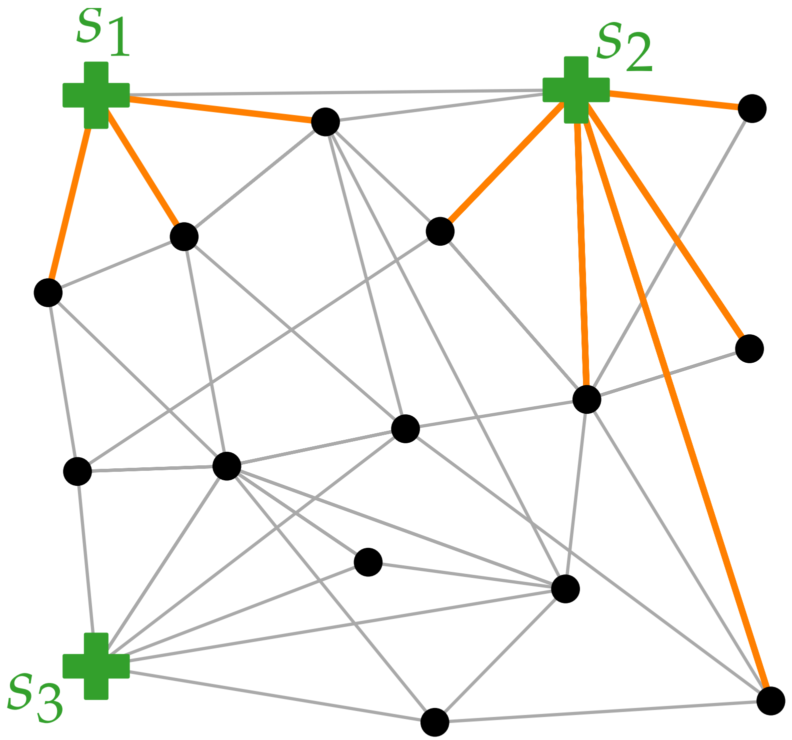

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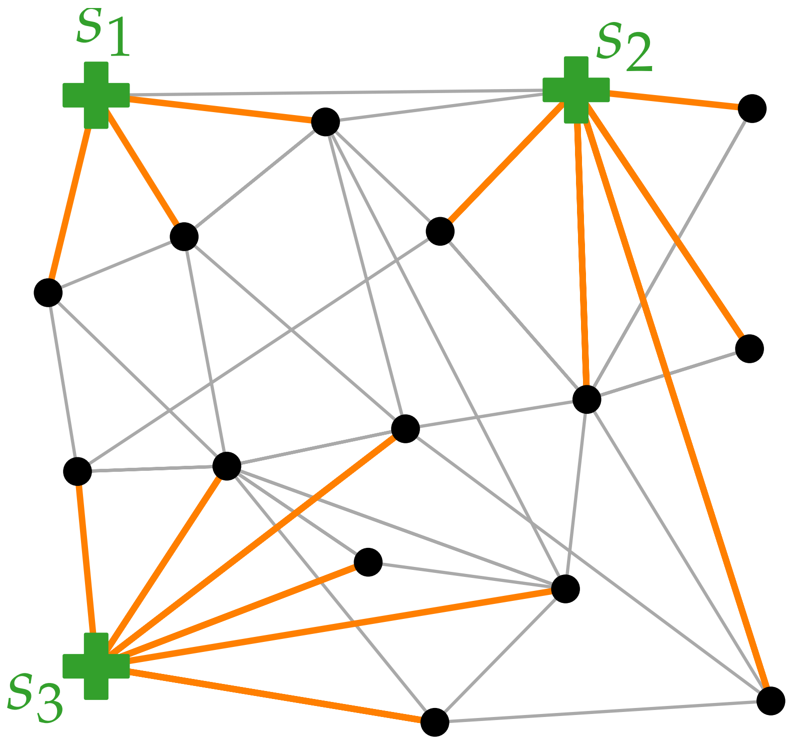

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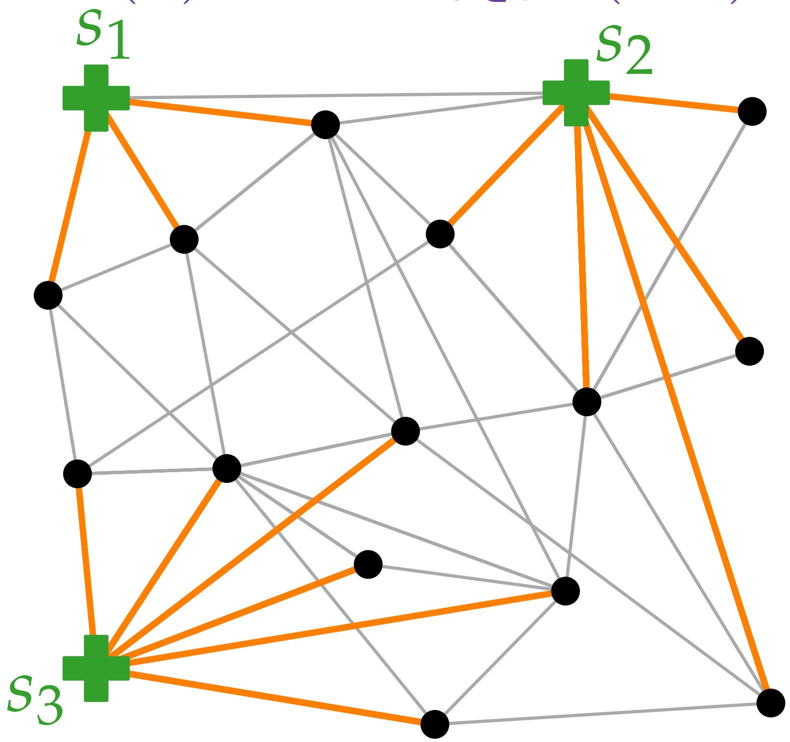

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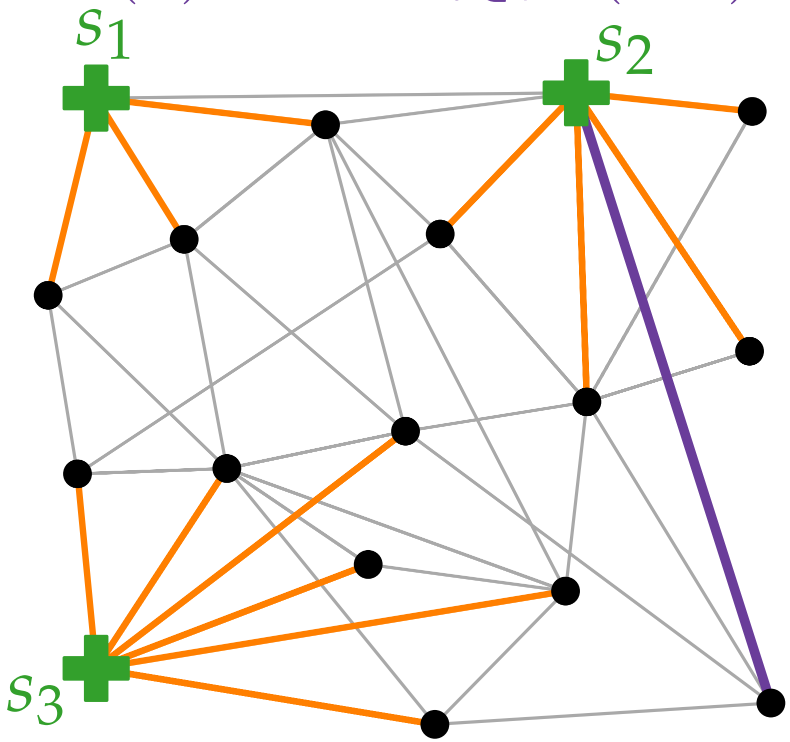

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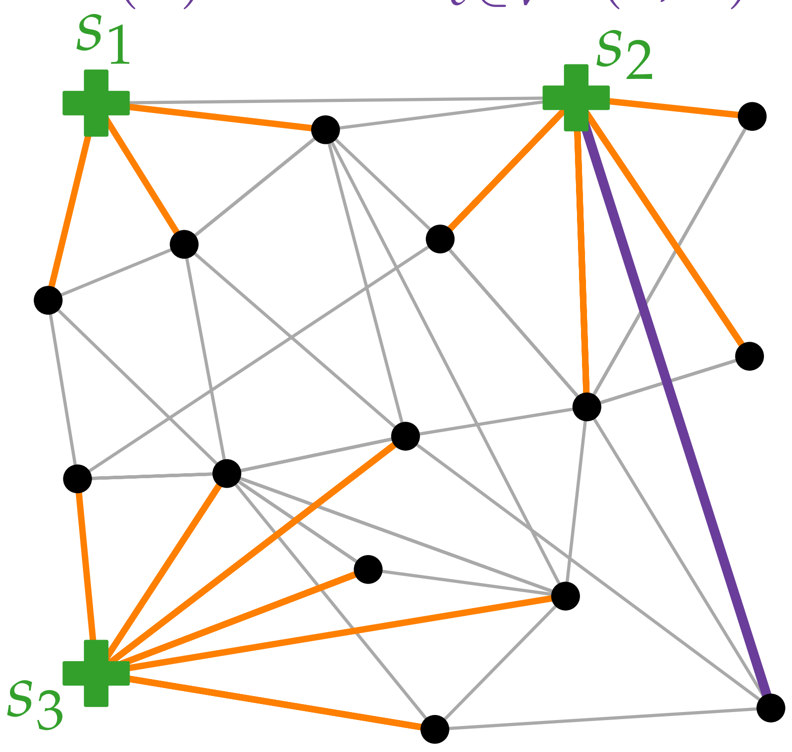

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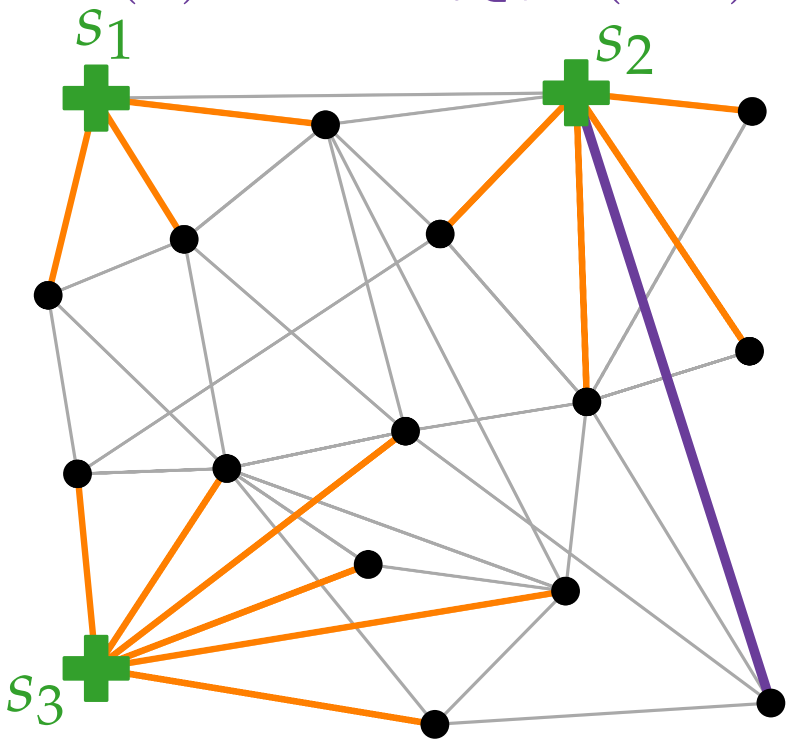

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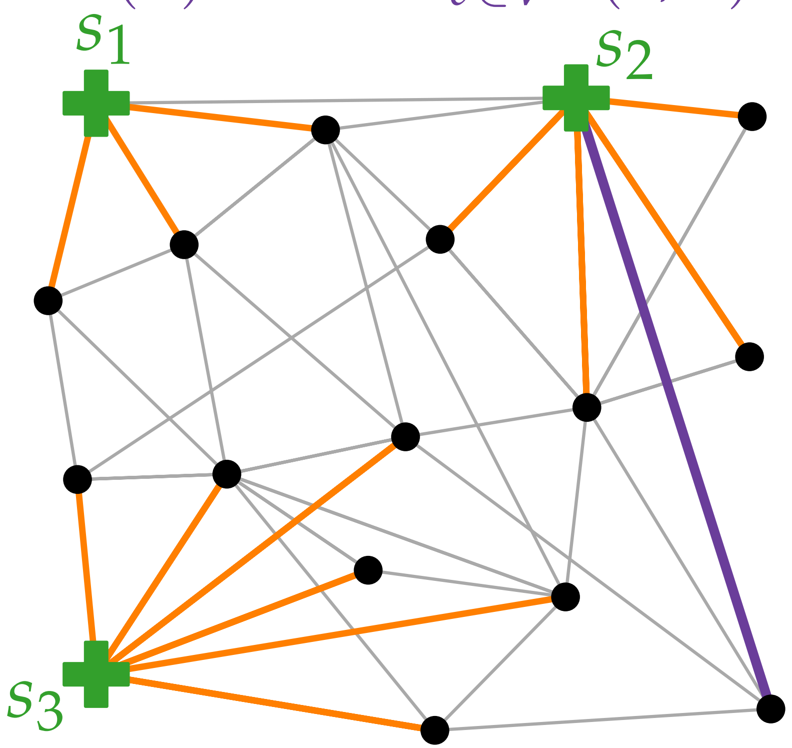

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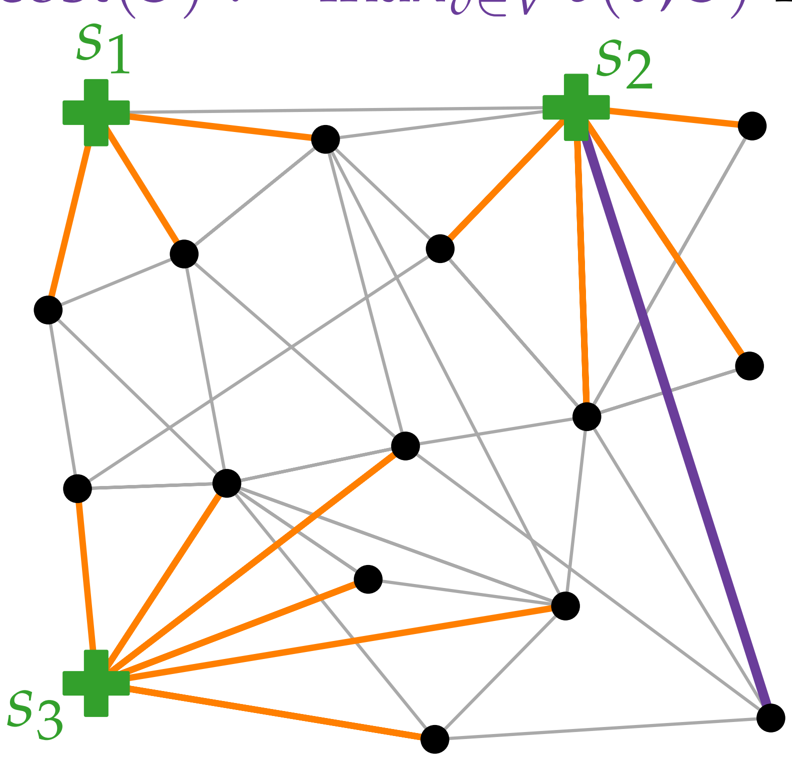

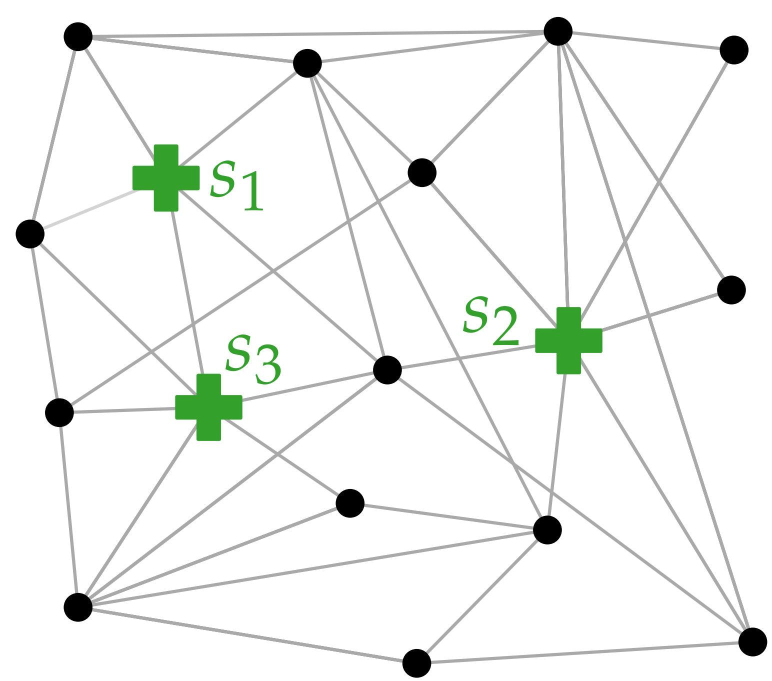

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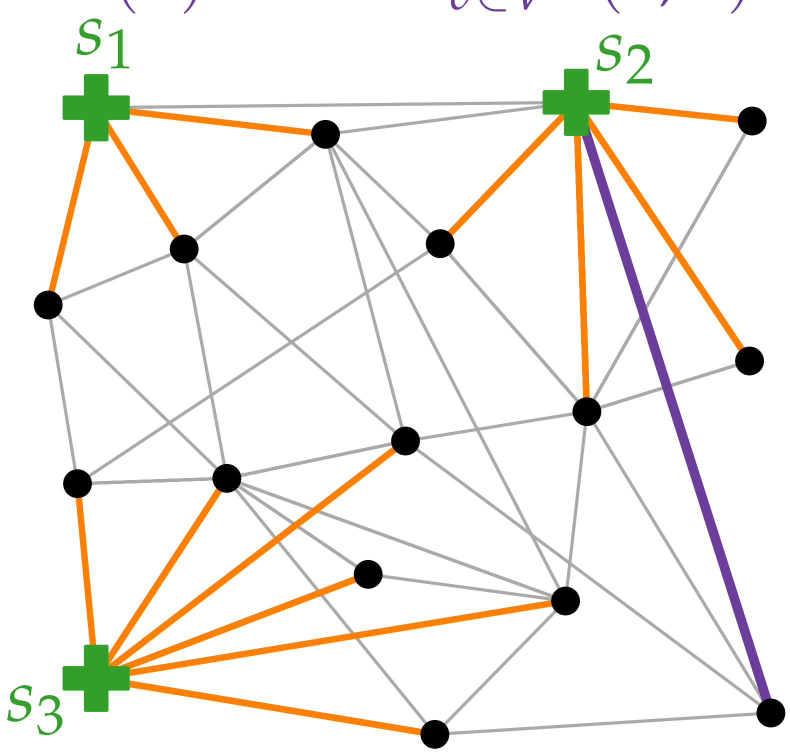

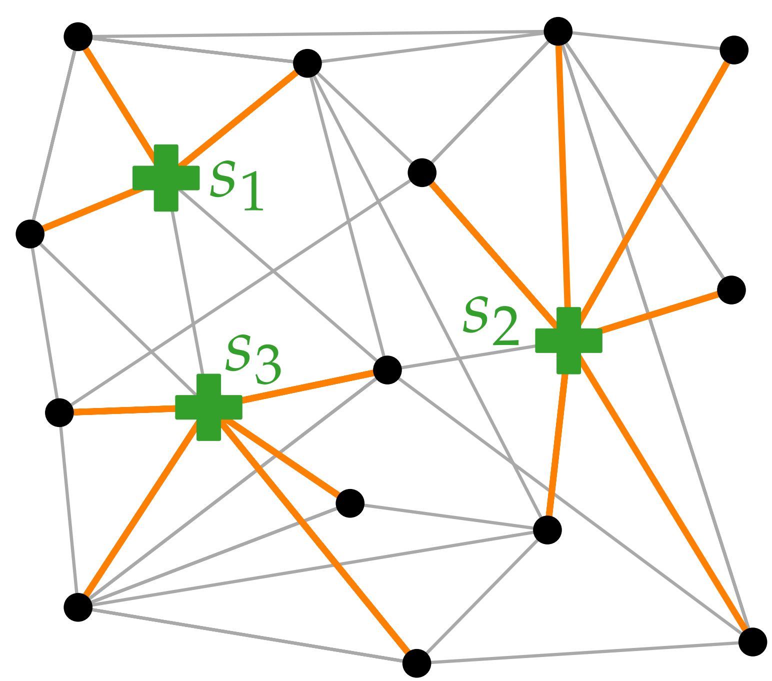

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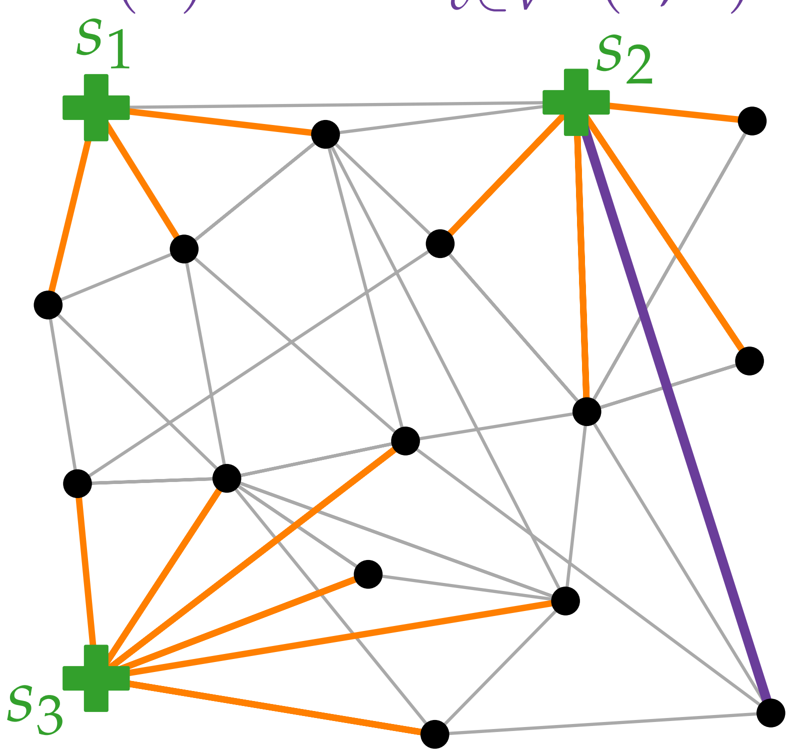

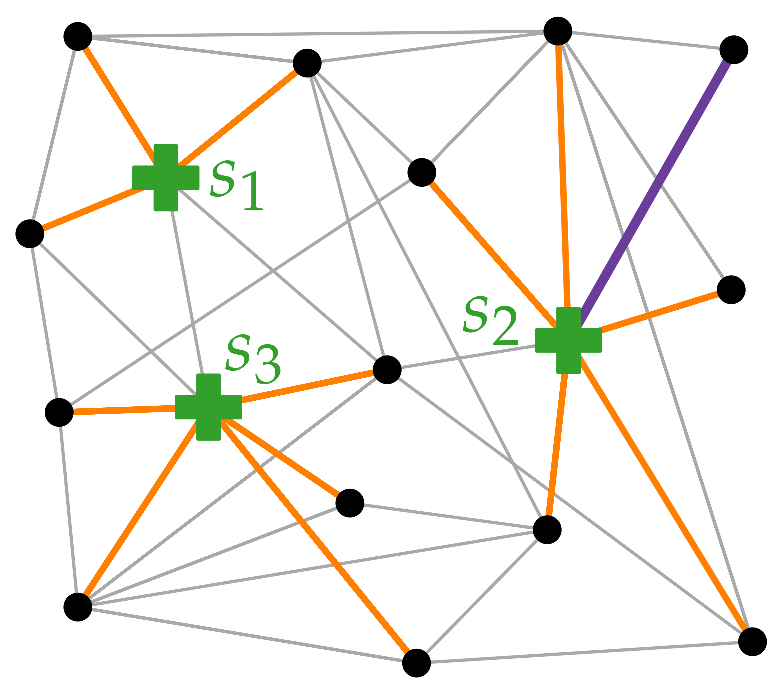

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part II:
Parametric Pruning

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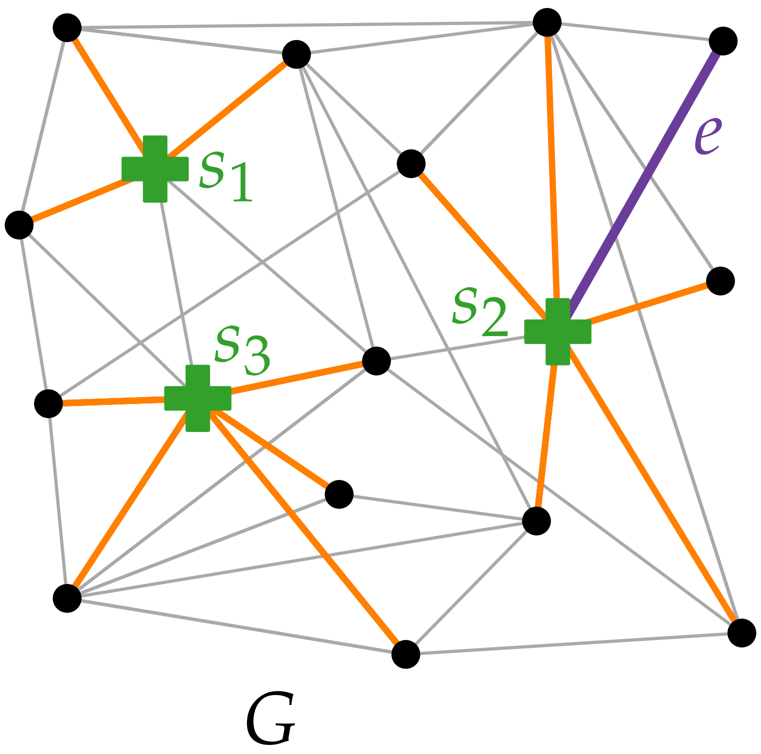
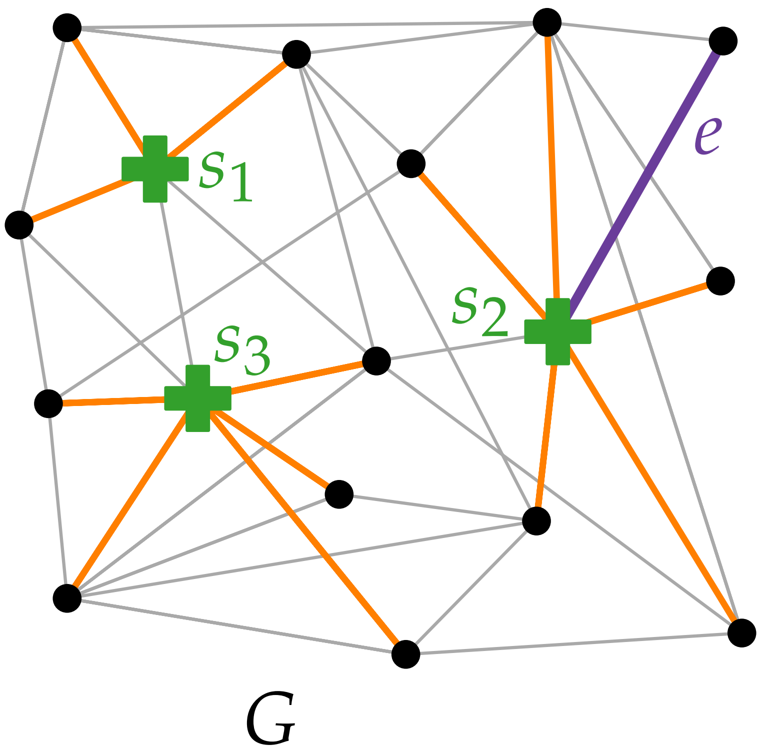
Parametric Pruning



Parametric Pruning

Let $E = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \ldots \leq c(e_m)$.



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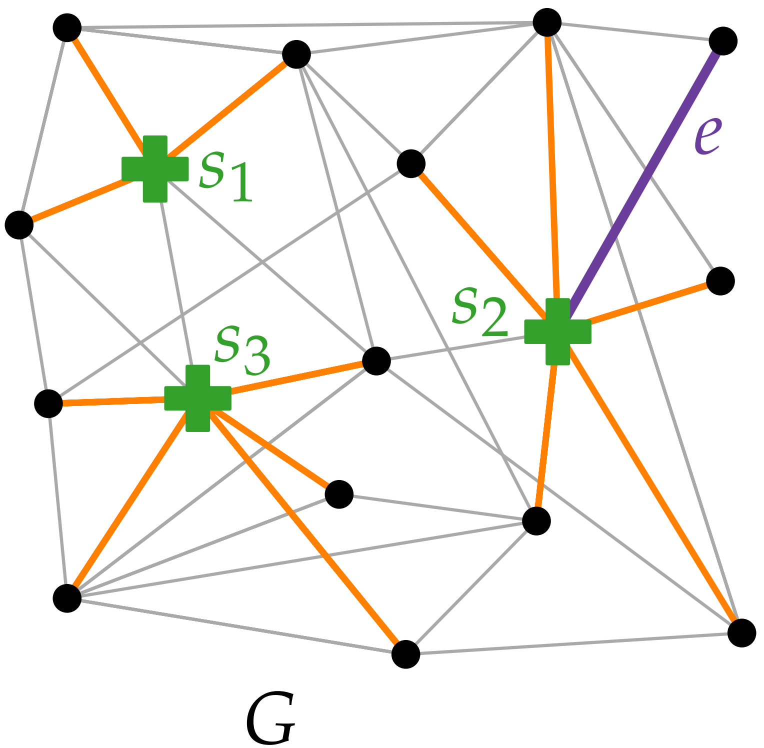
Suppose we know that $\text{OPT} = c(e_j)$.



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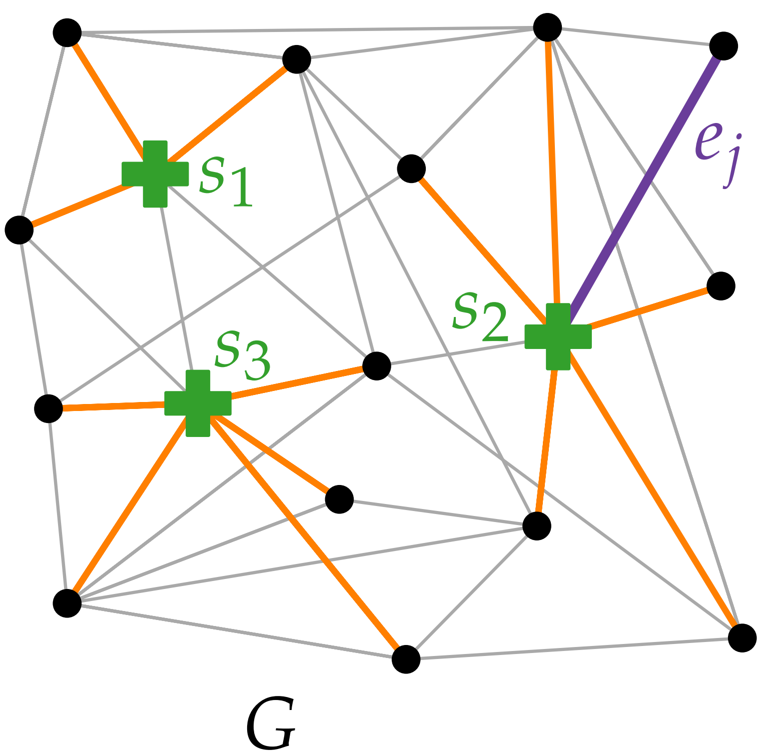

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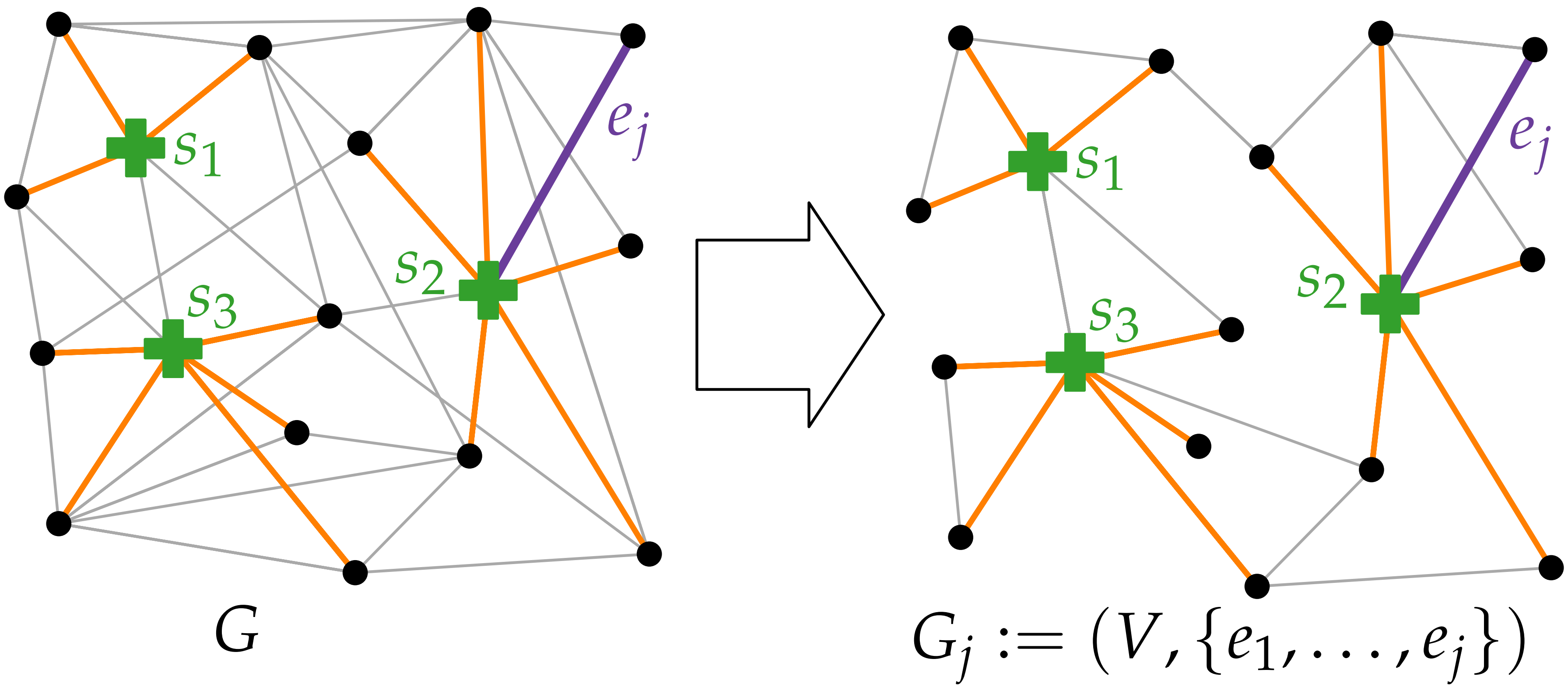

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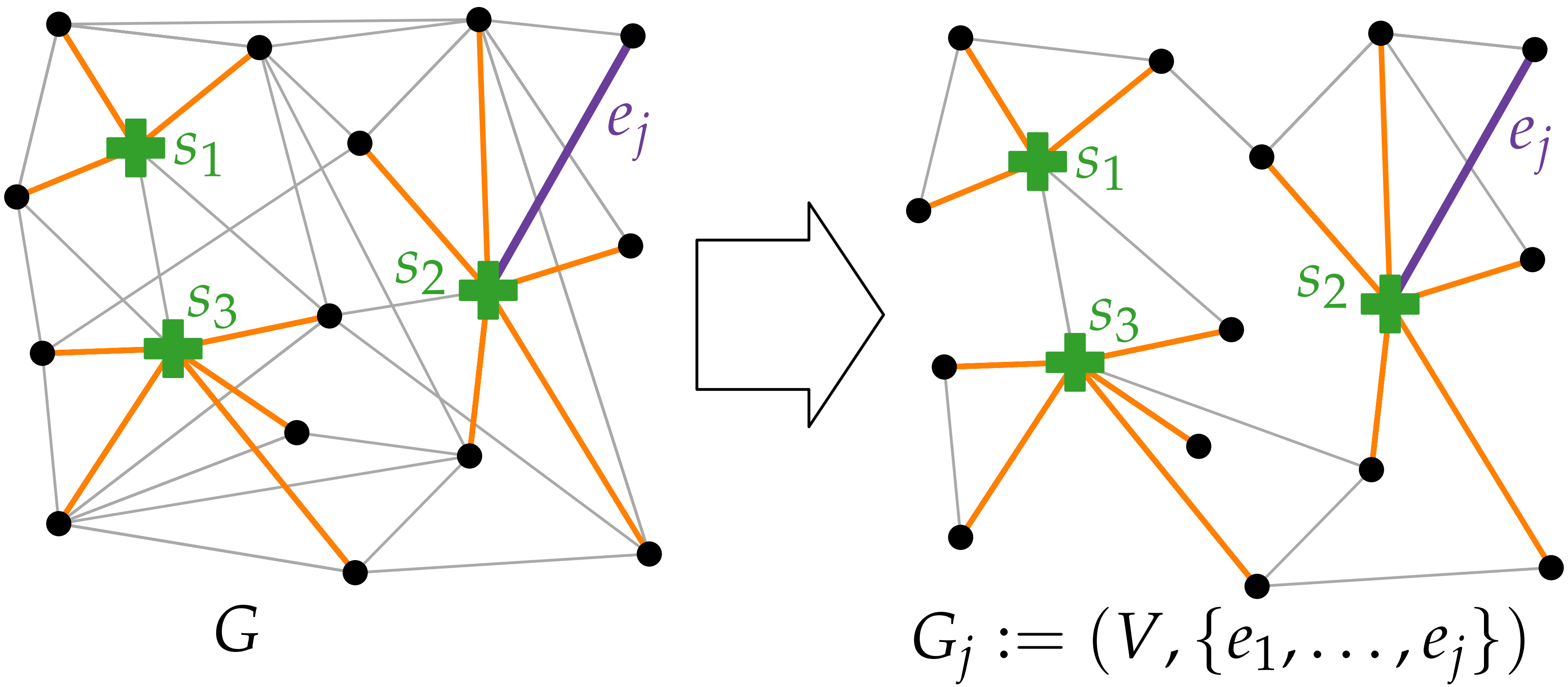

...try each G_j.

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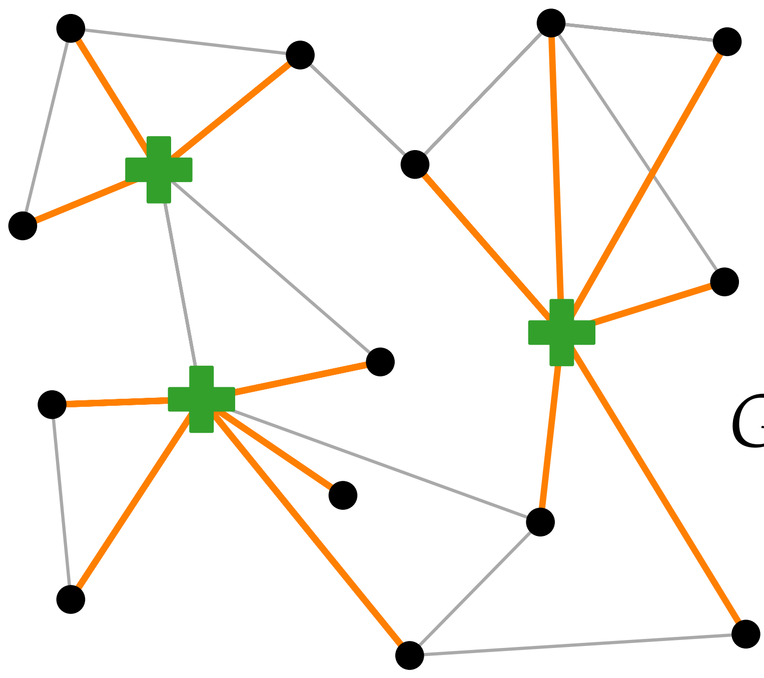
Def.



$$G_j := (V, \{e_1, \ldots, e_j\})$$

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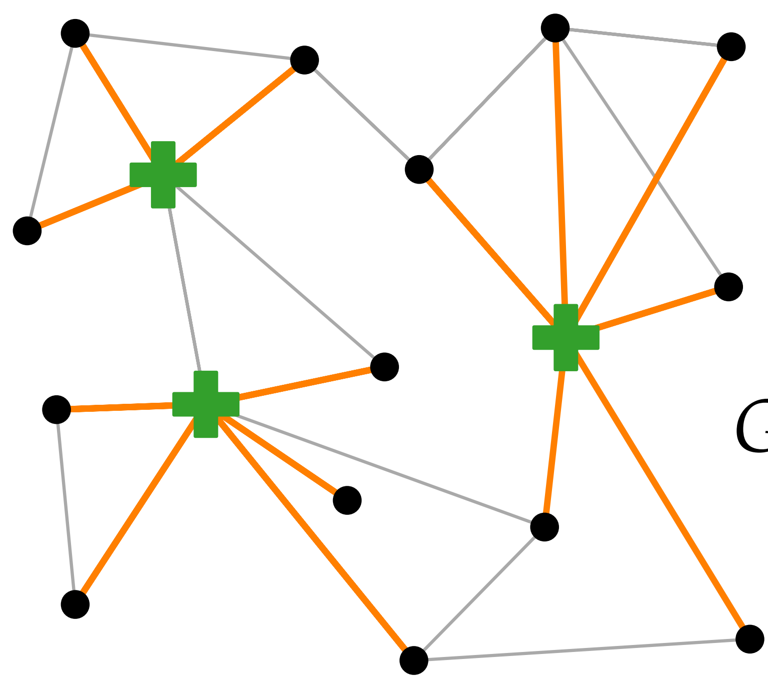
Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D.



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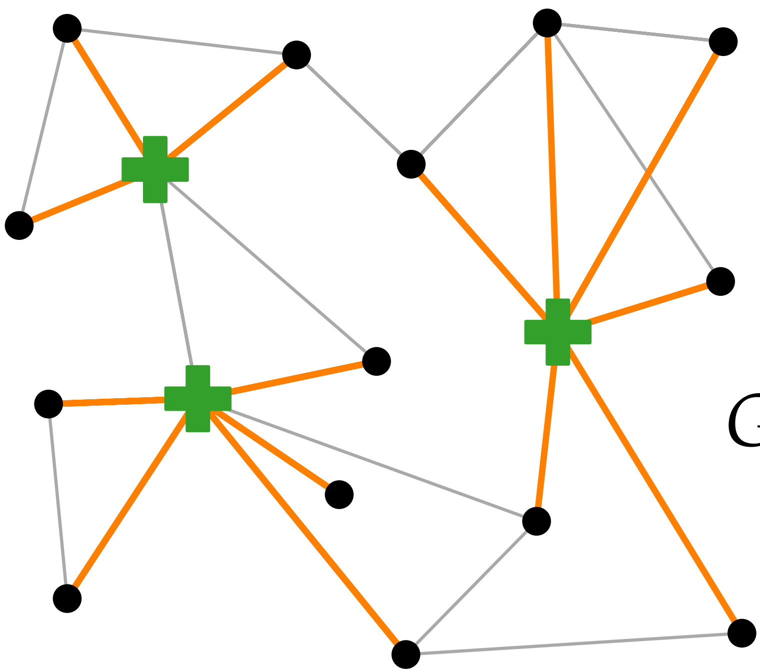
Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D. The cardinality of a smallest dominating set in H is denoted by $\text{dom}(H)$.



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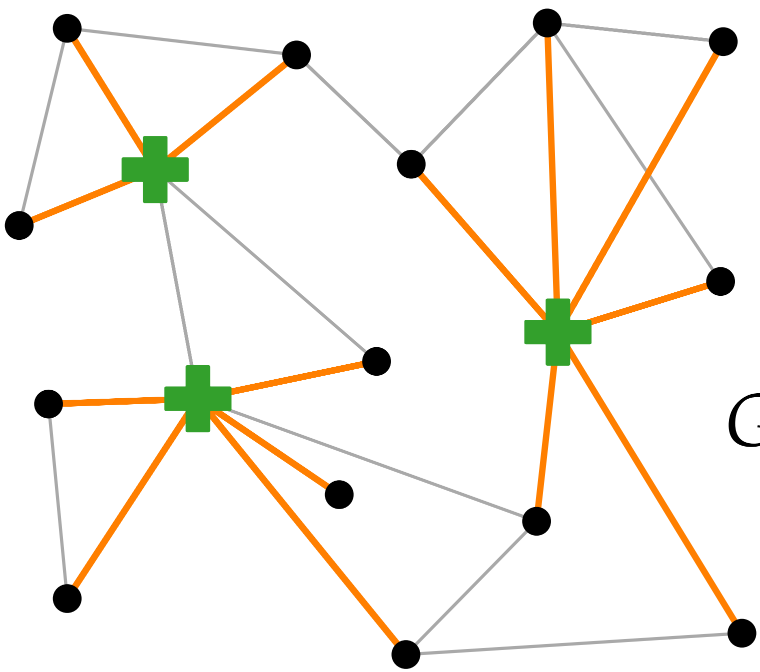

$$\mathrm{dom}(G_j) \leq k$$

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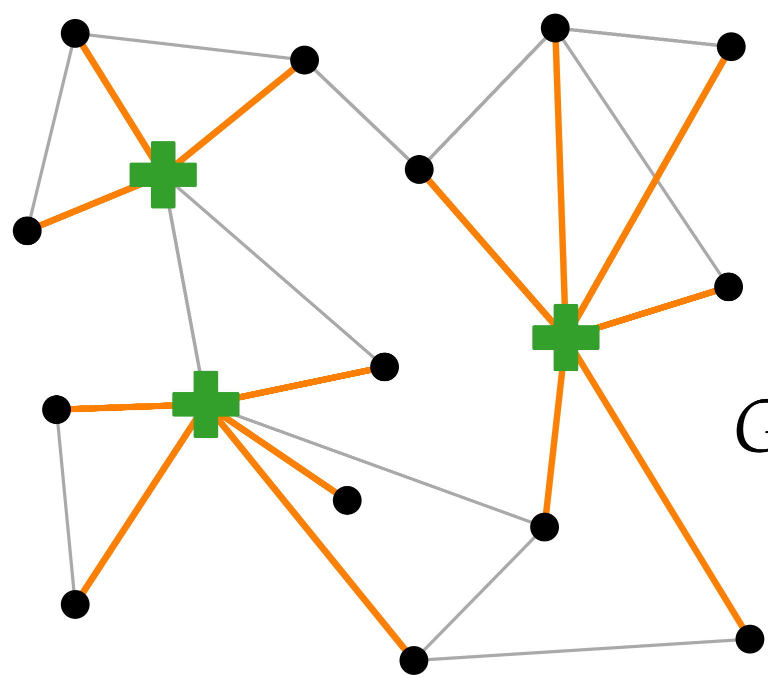

$$\mathrm{dom}(G_j) \leq k$$

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. . . but computing $\mathrm{dom}(H)$ is NP-hard.

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part III:
Square of a Graph

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Square of a Graph

Idea: Find a small dominating set in a "coarsened" G_j

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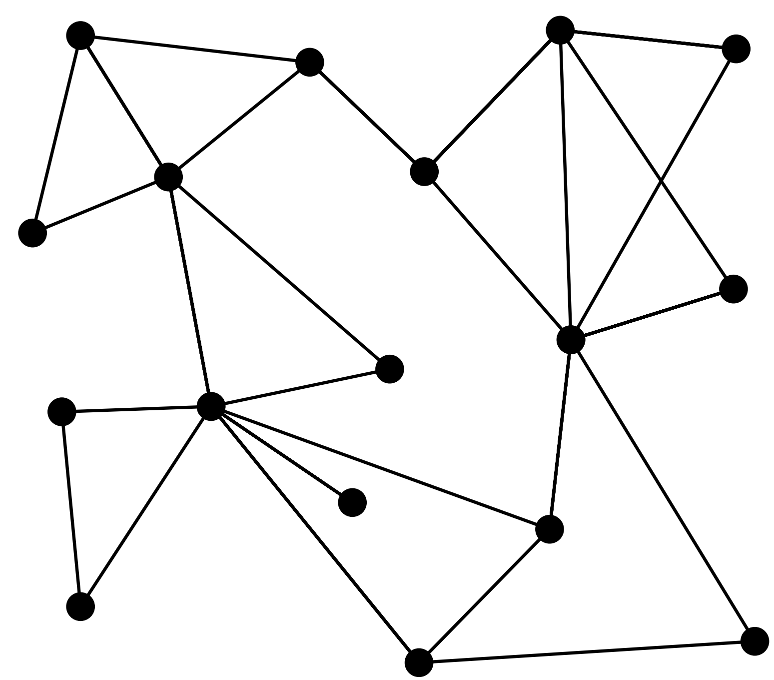

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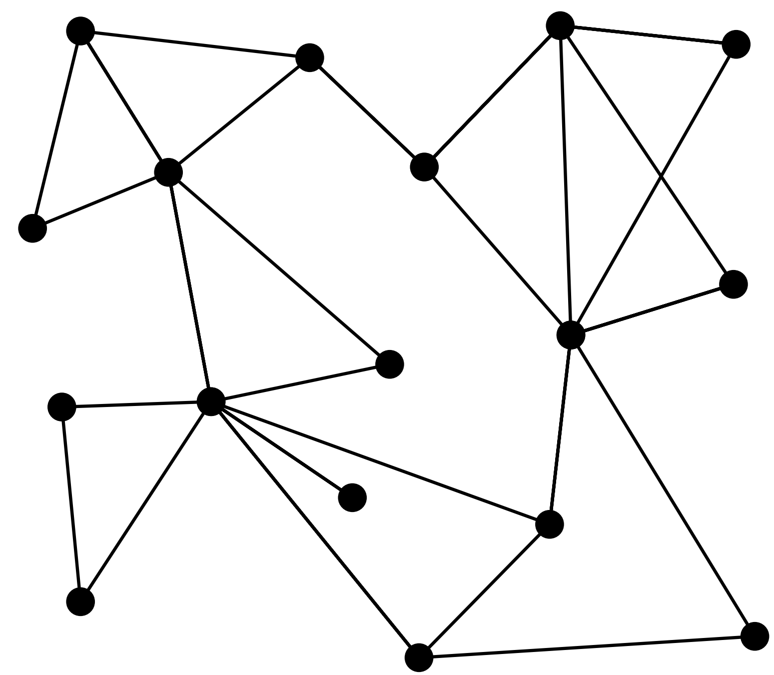
Idea: Find a small dominating set in a "coarsened" G_j

Def. The **square** H^2 of a graph H has the same vertex set as H. Additionally, two vertices $u \neq v$ are adjacent in H^2 when they are within distance **two** in H.



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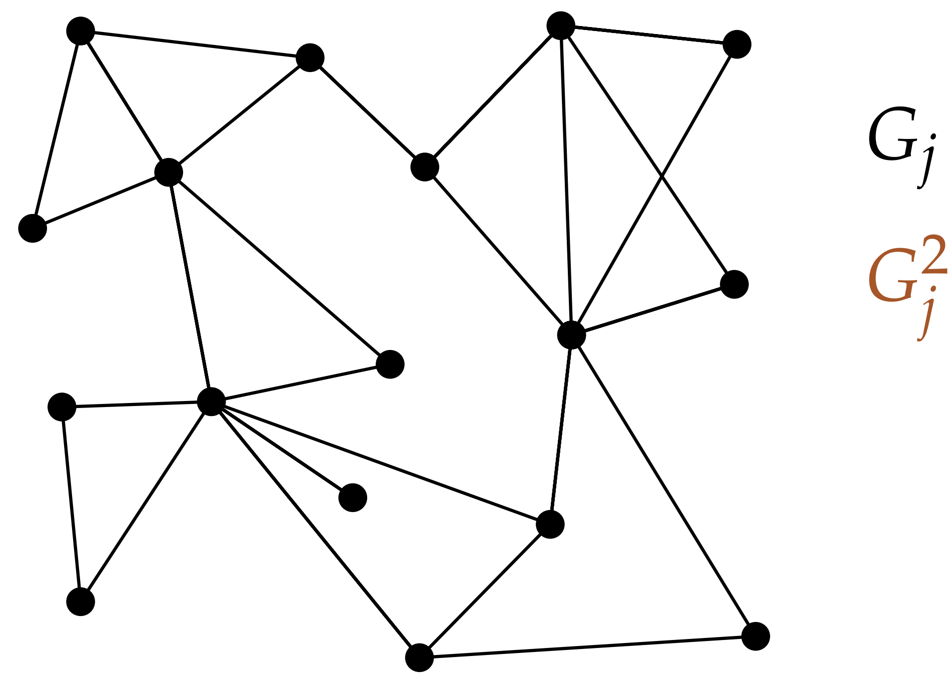

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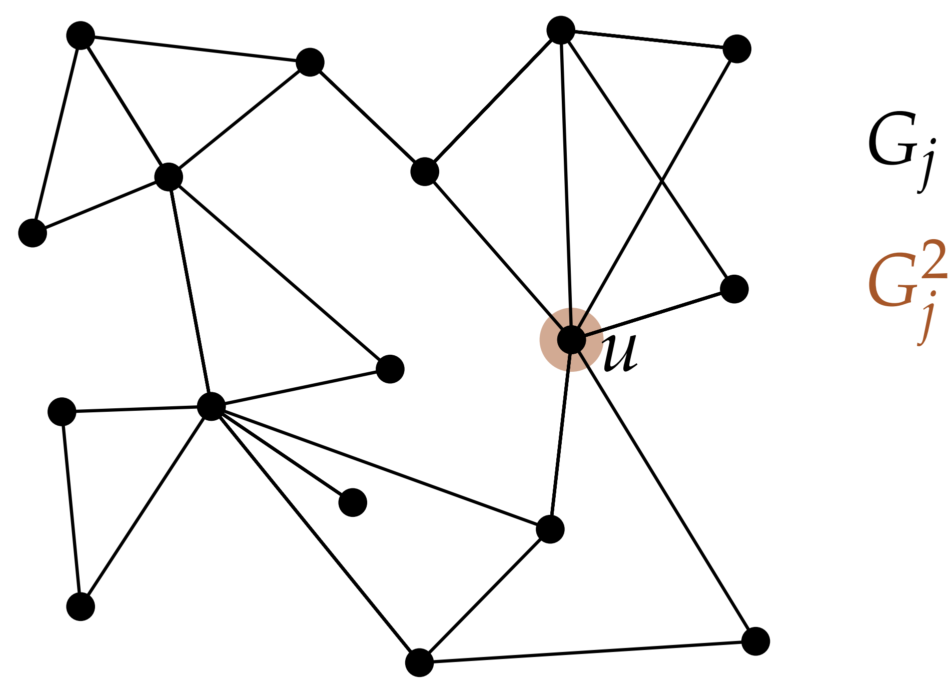

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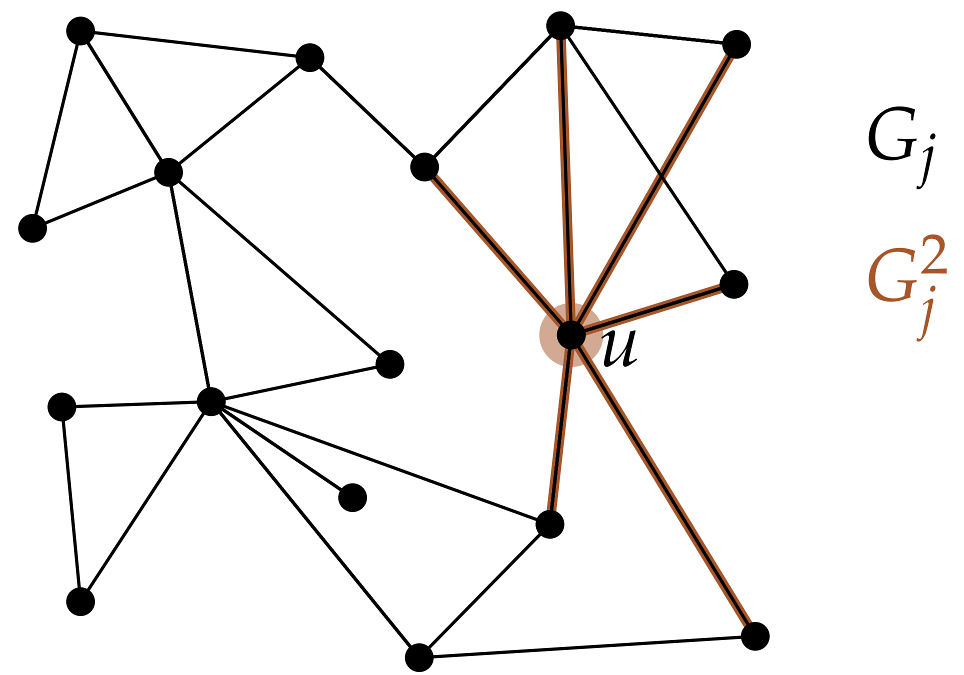

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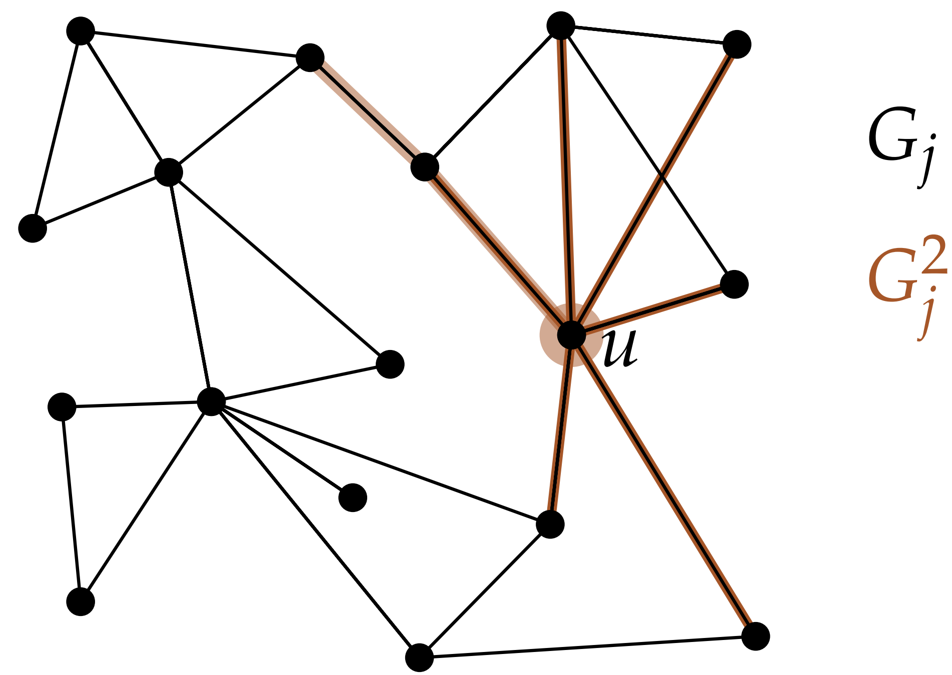

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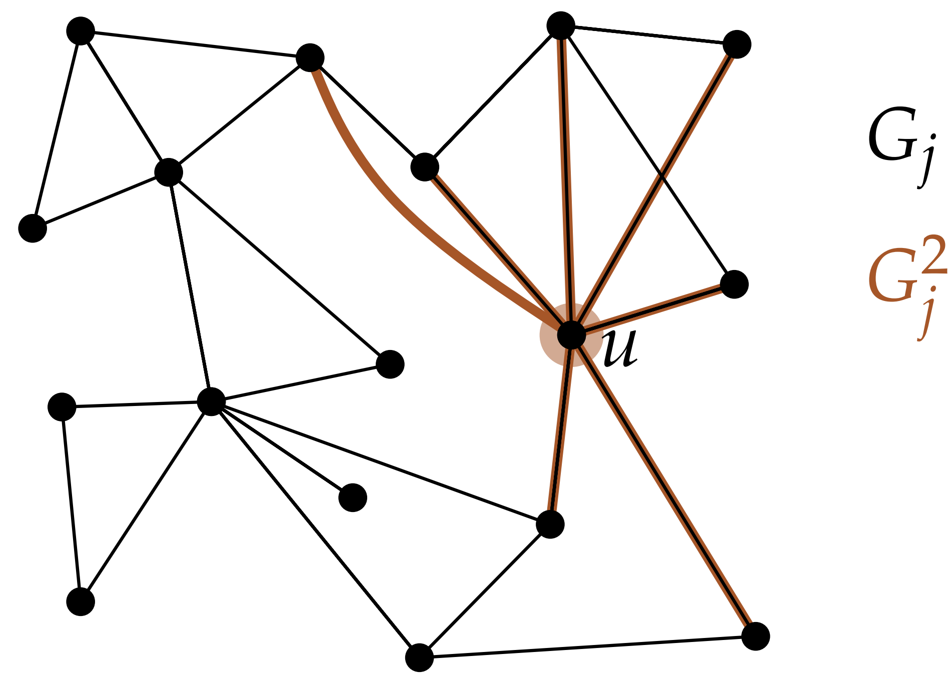

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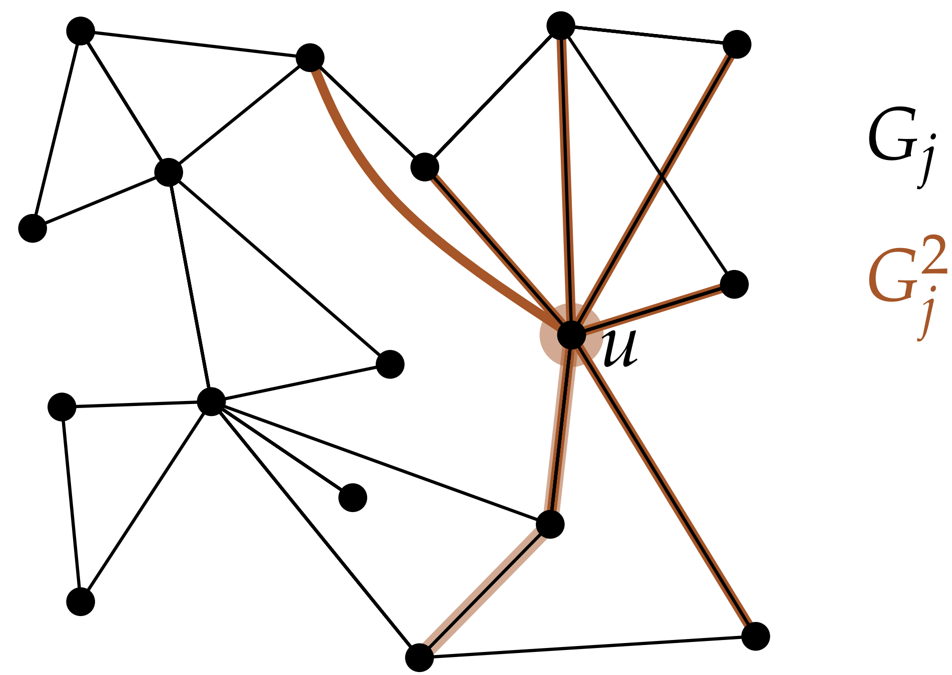

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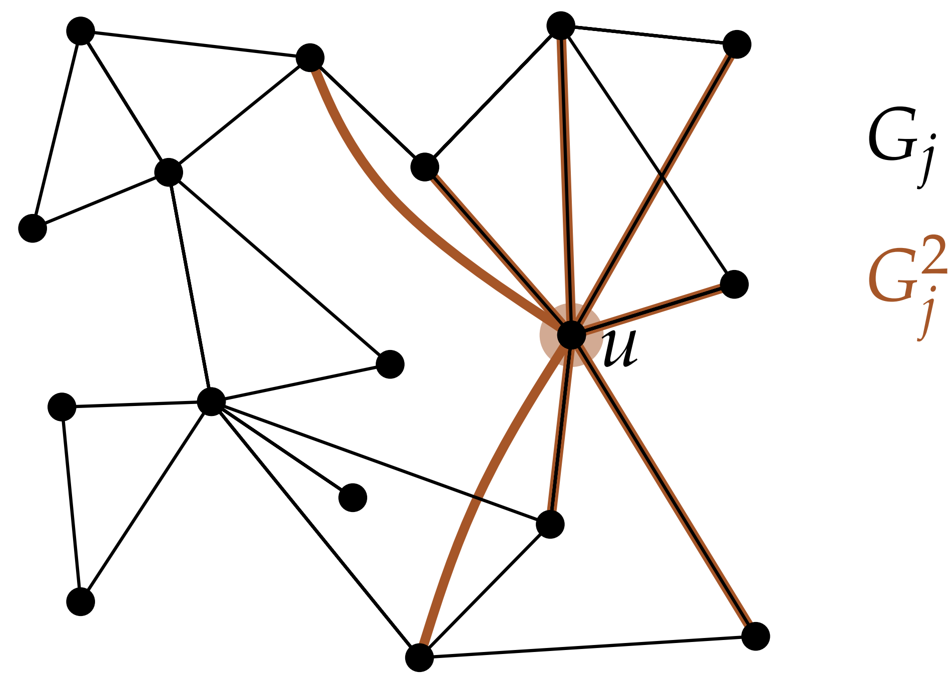

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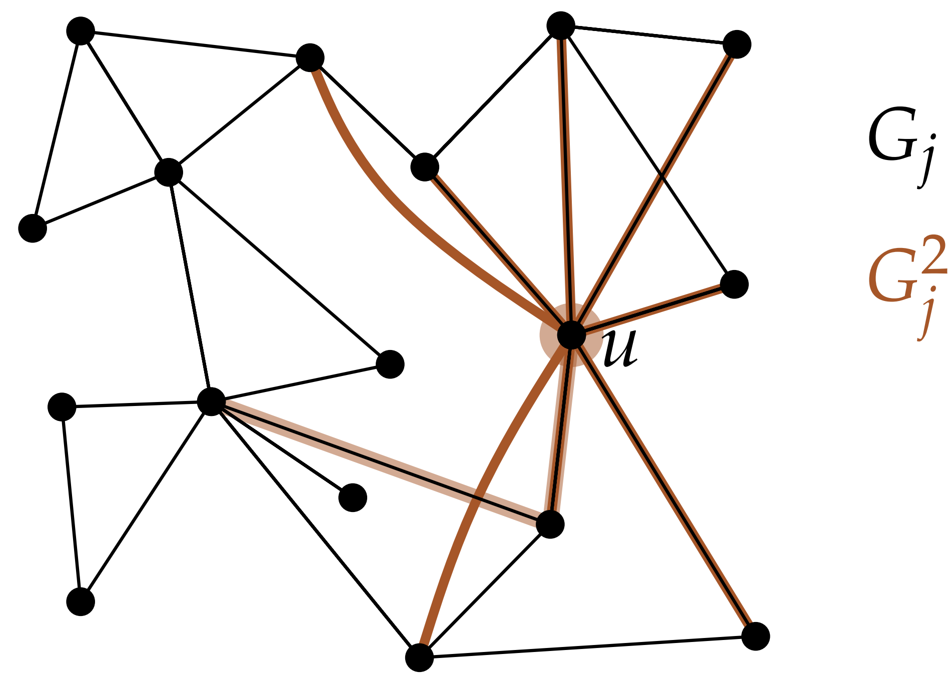

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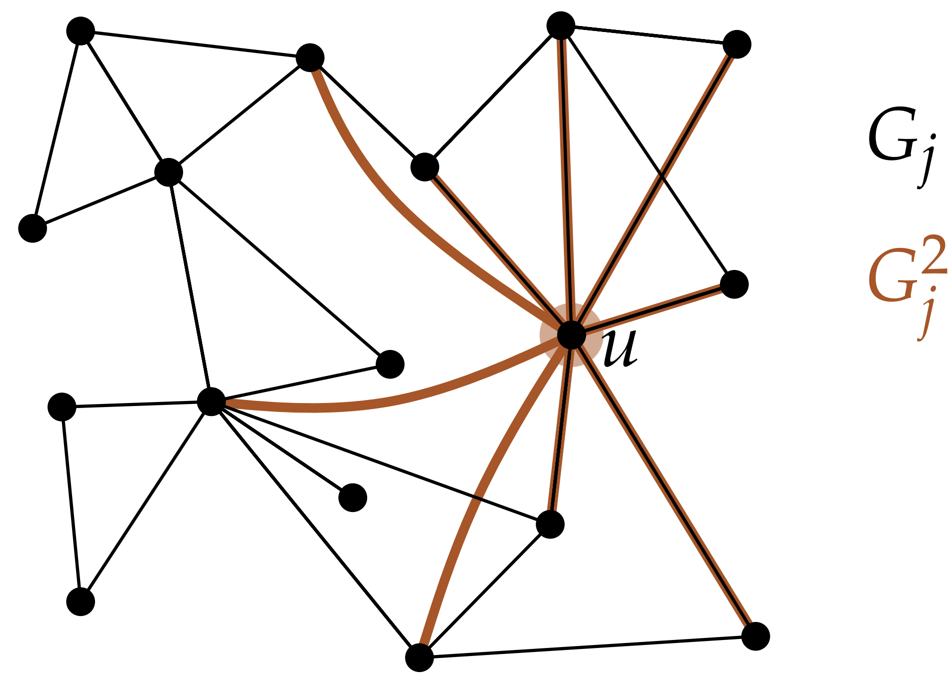

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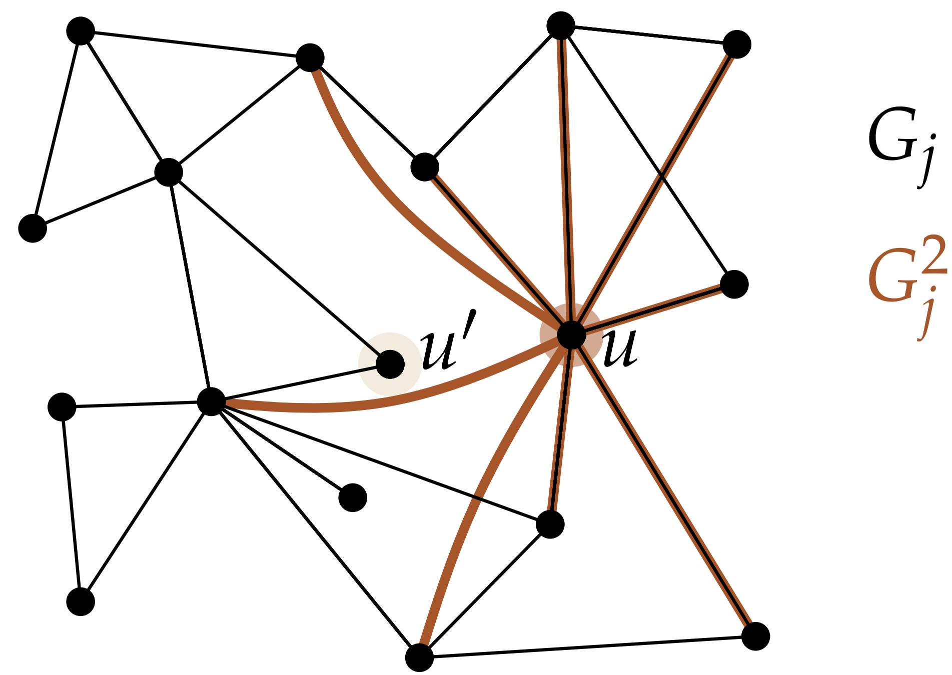

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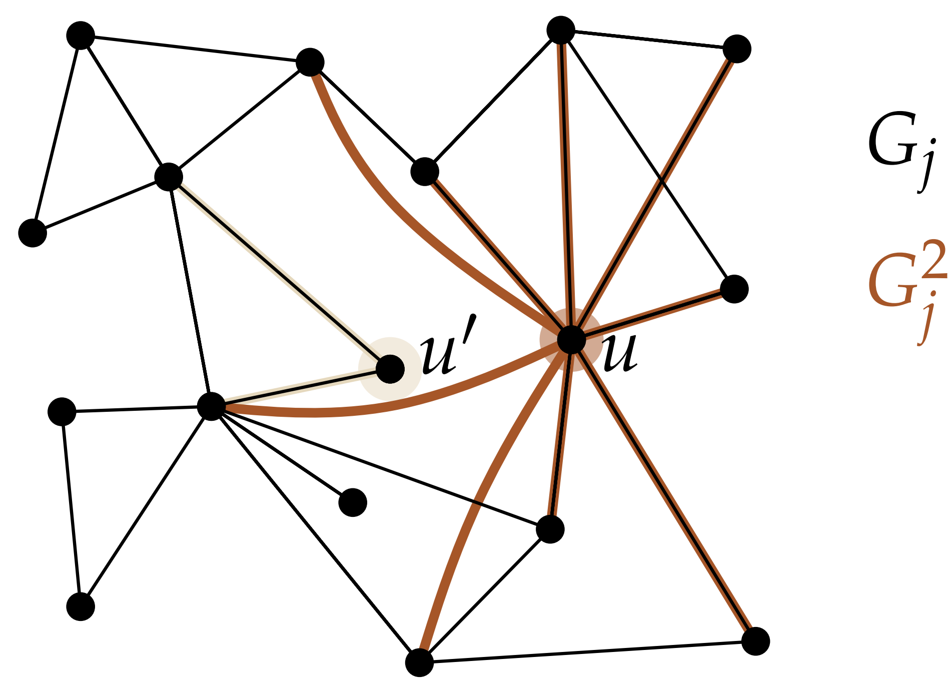

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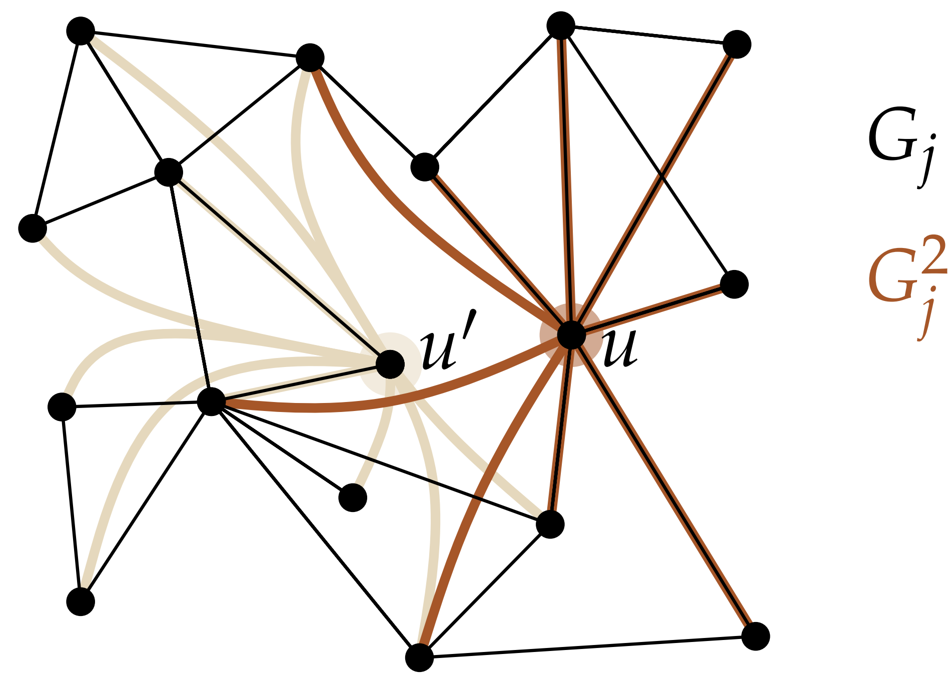

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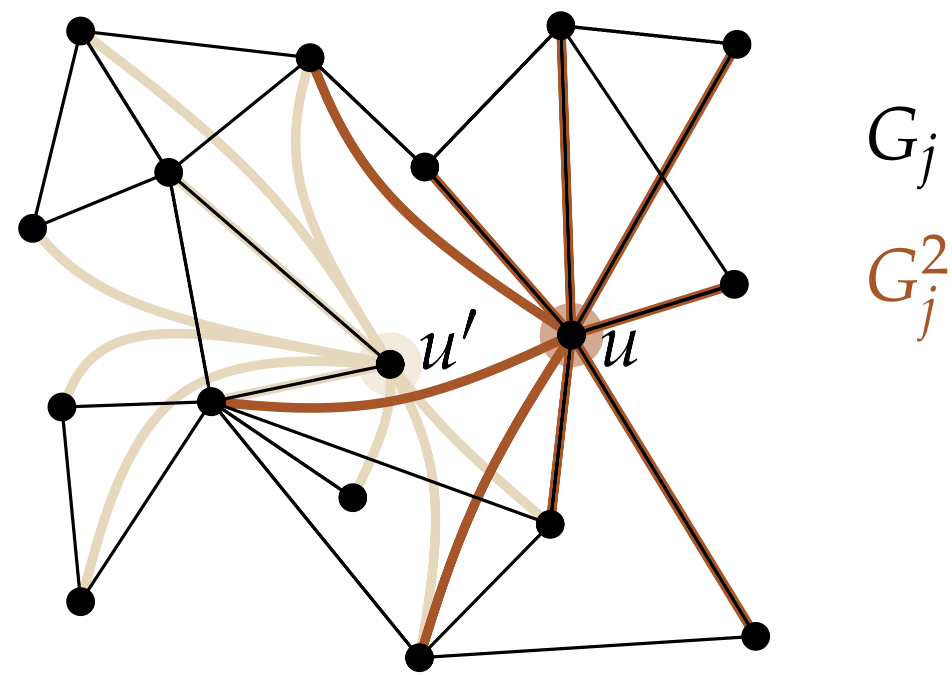

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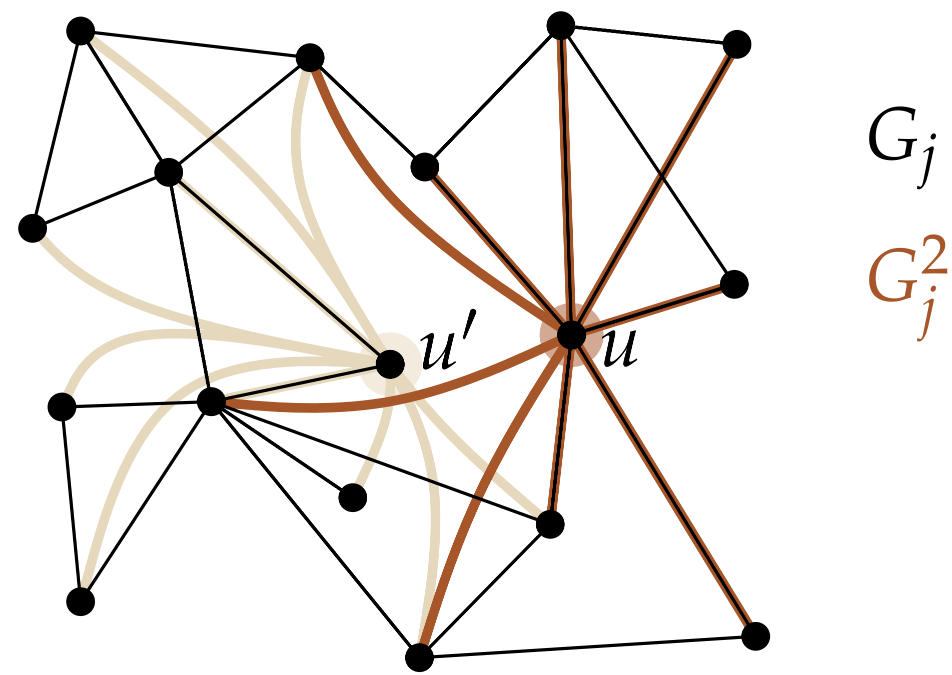

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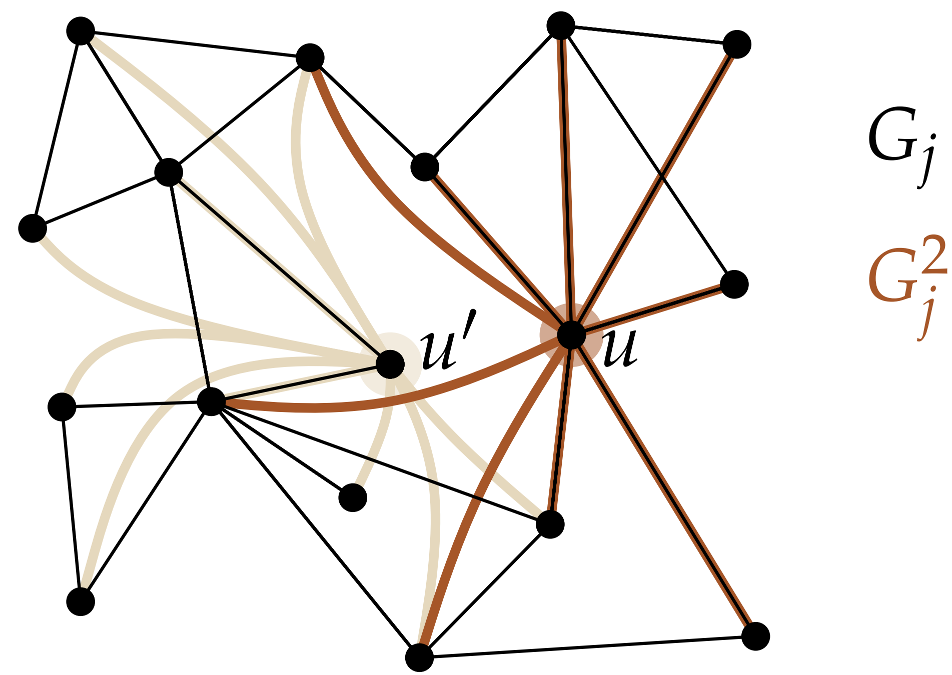

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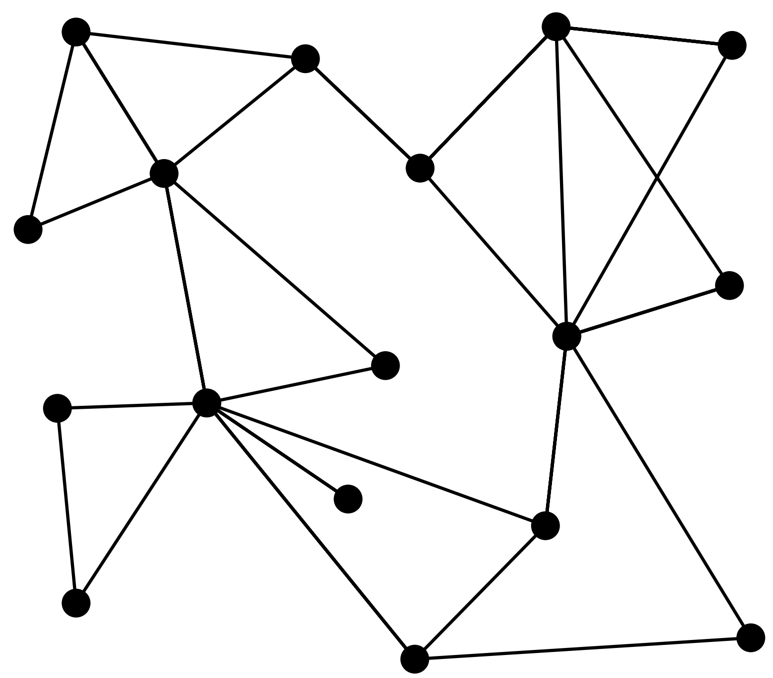

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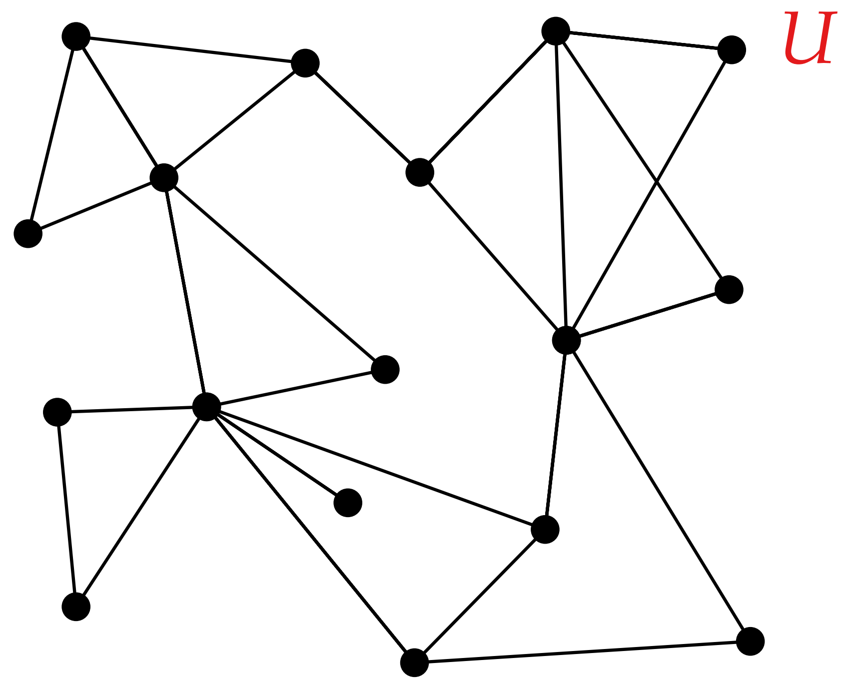

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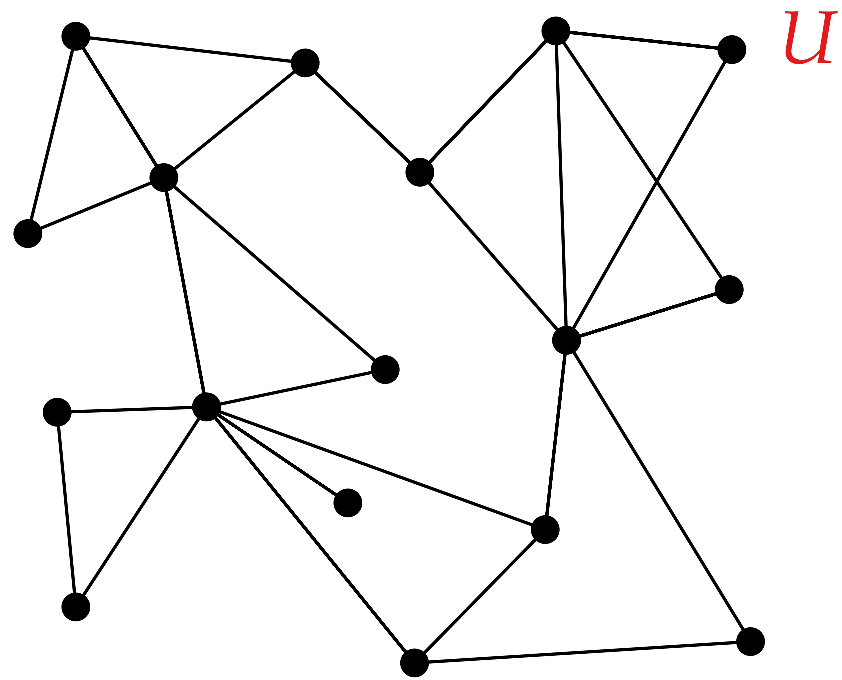

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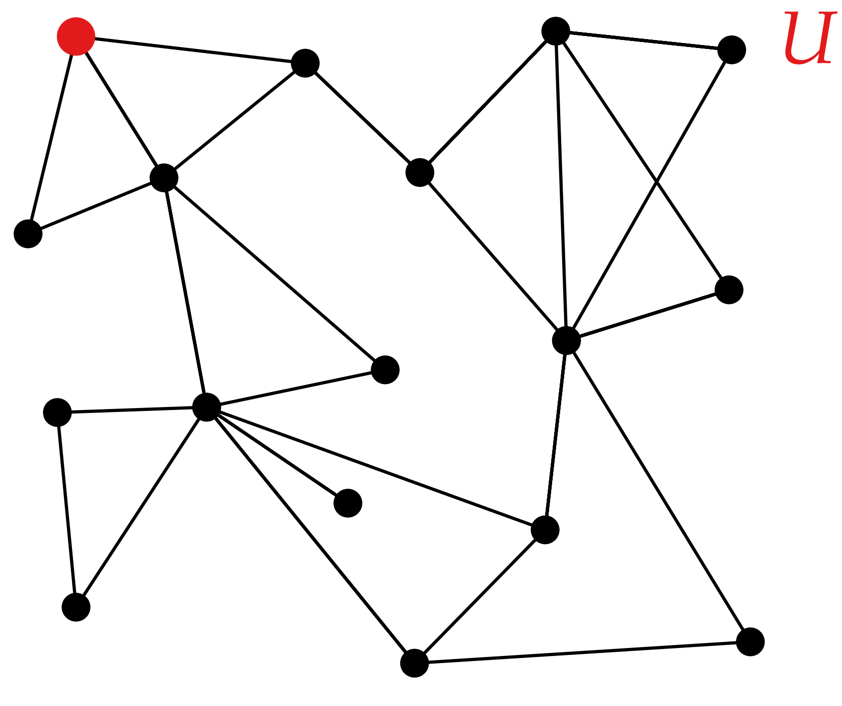

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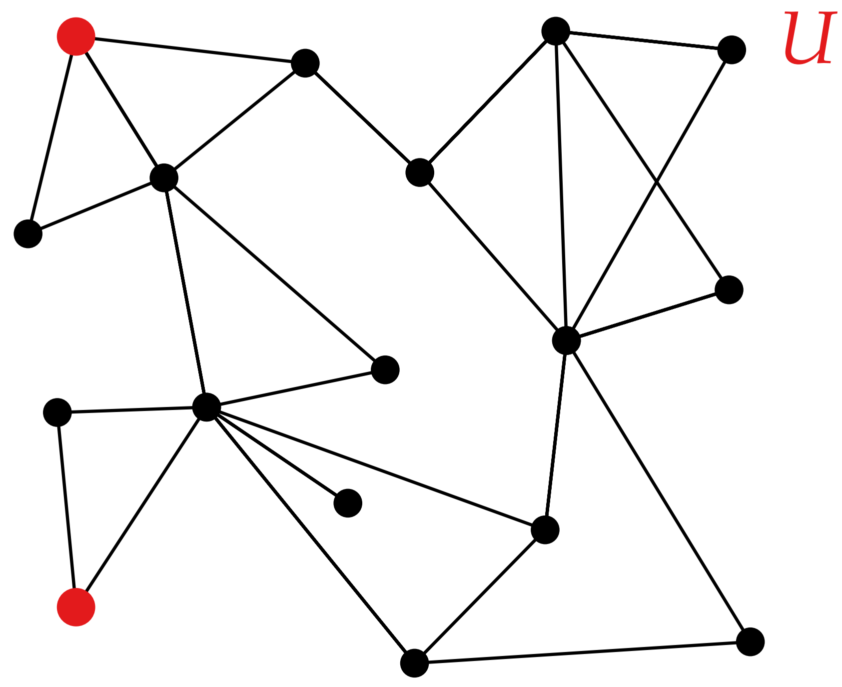

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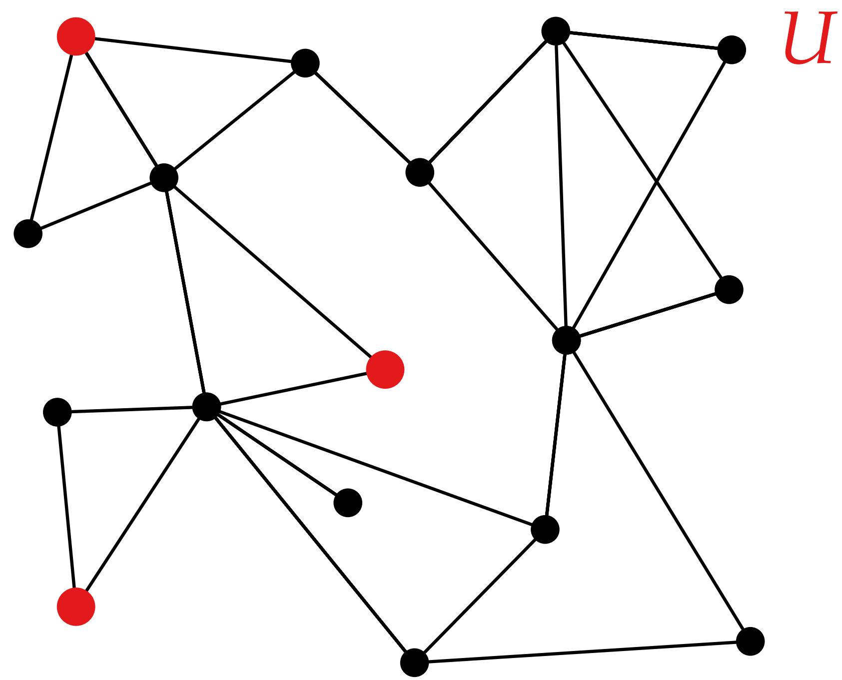

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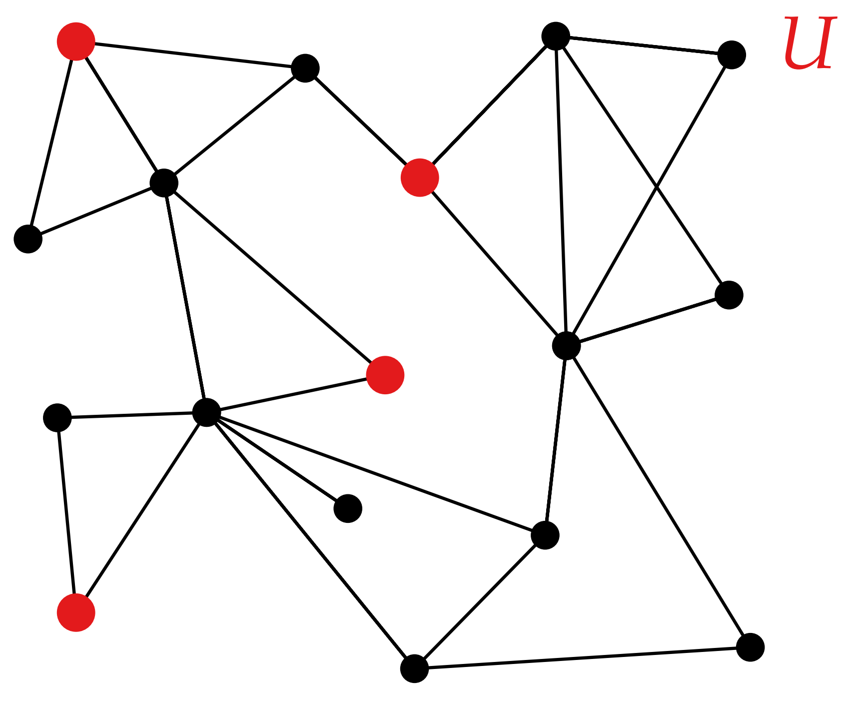

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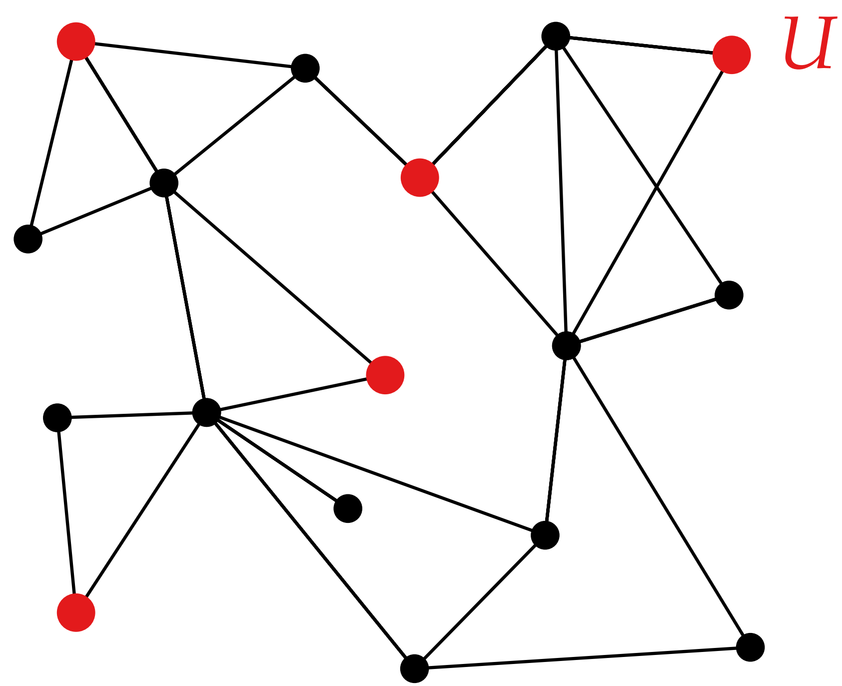

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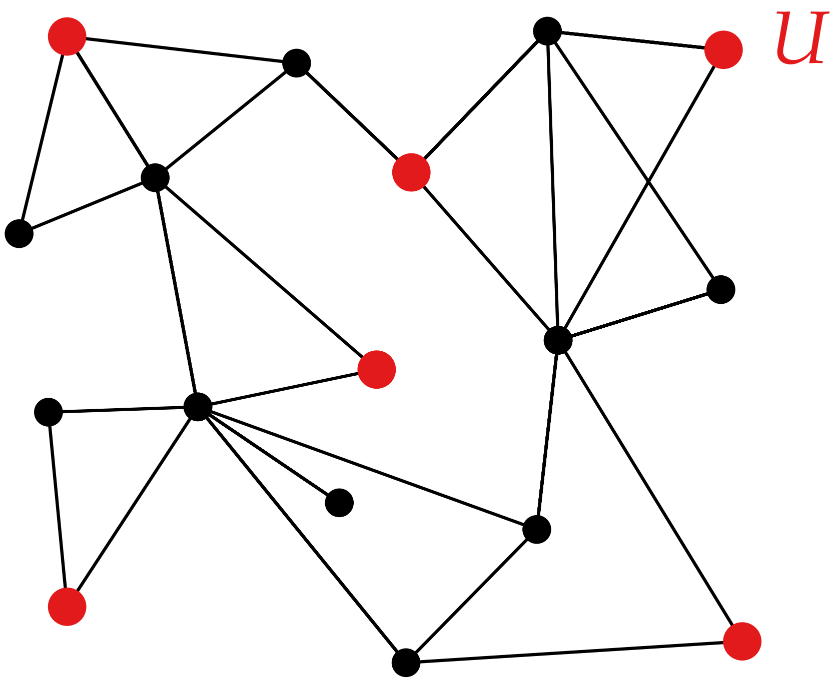

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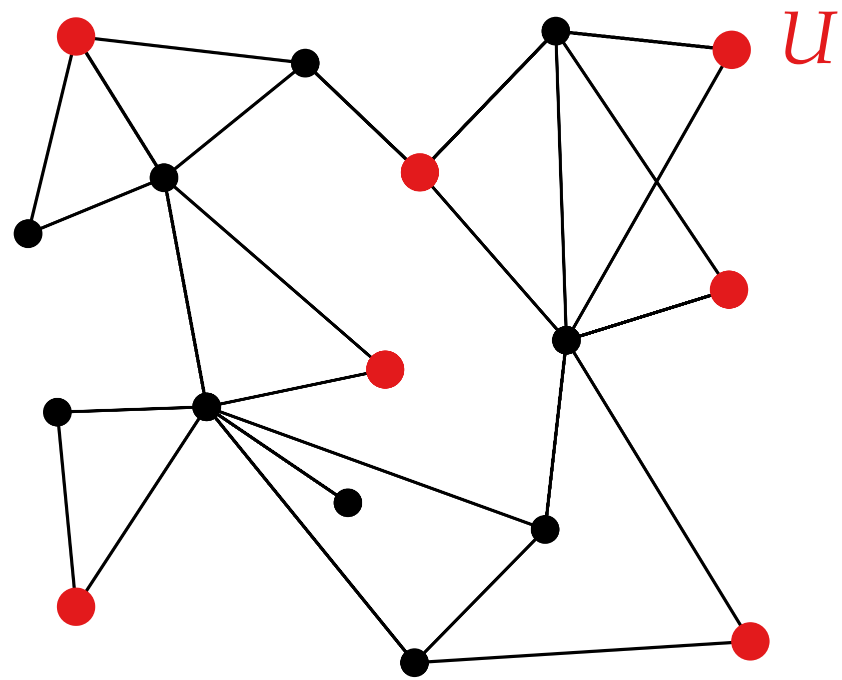

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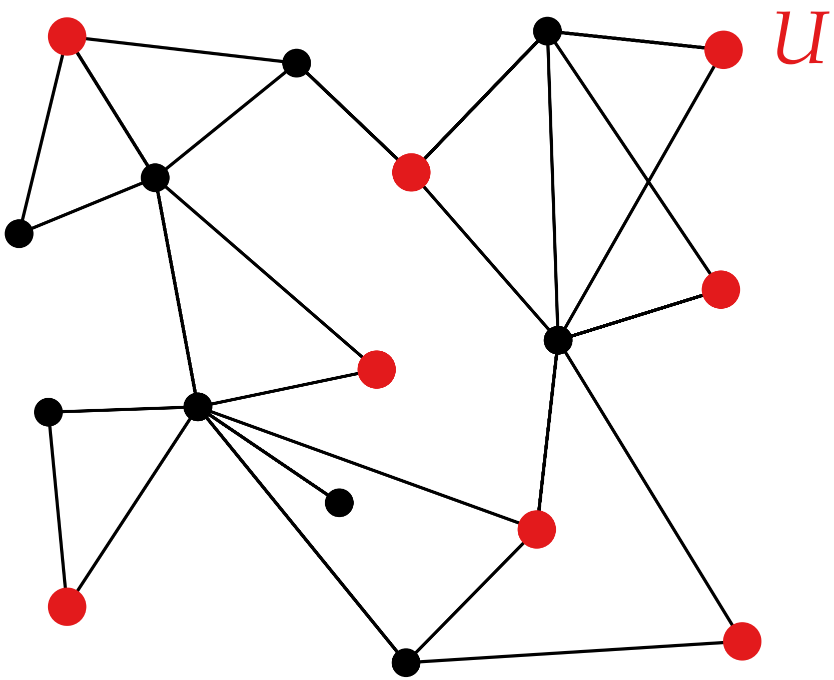

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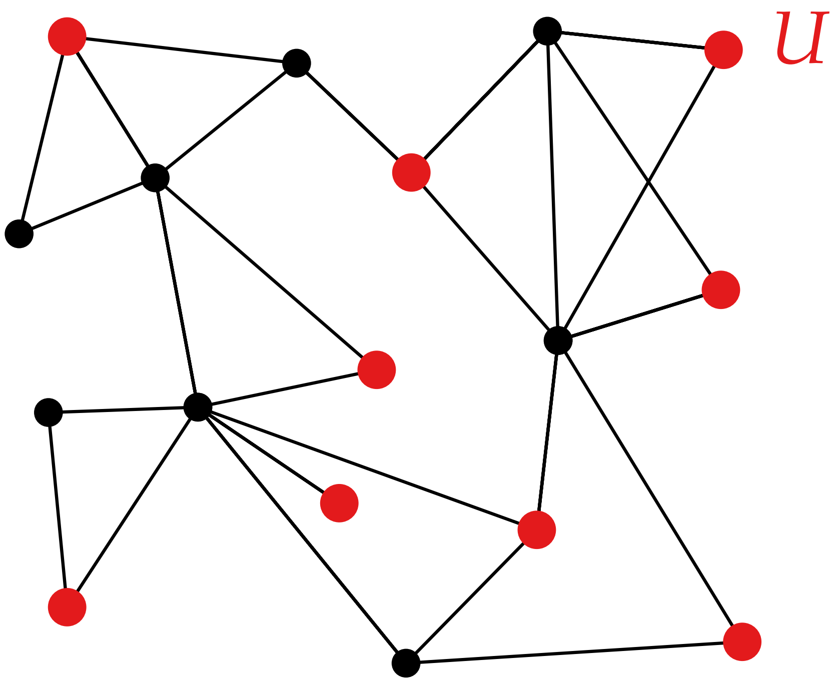

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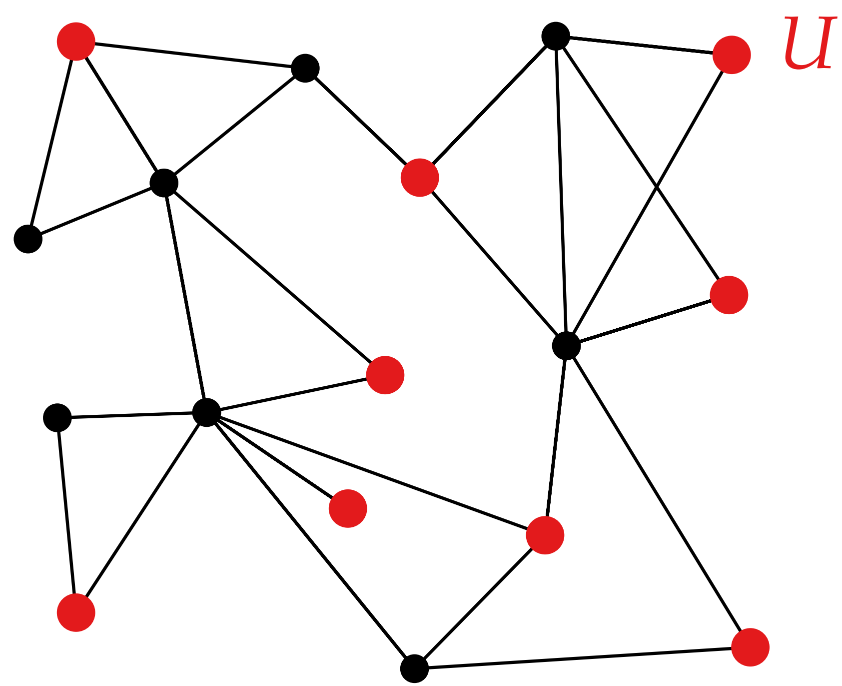

Obs. Maximal independent sets are dominating sets :-)

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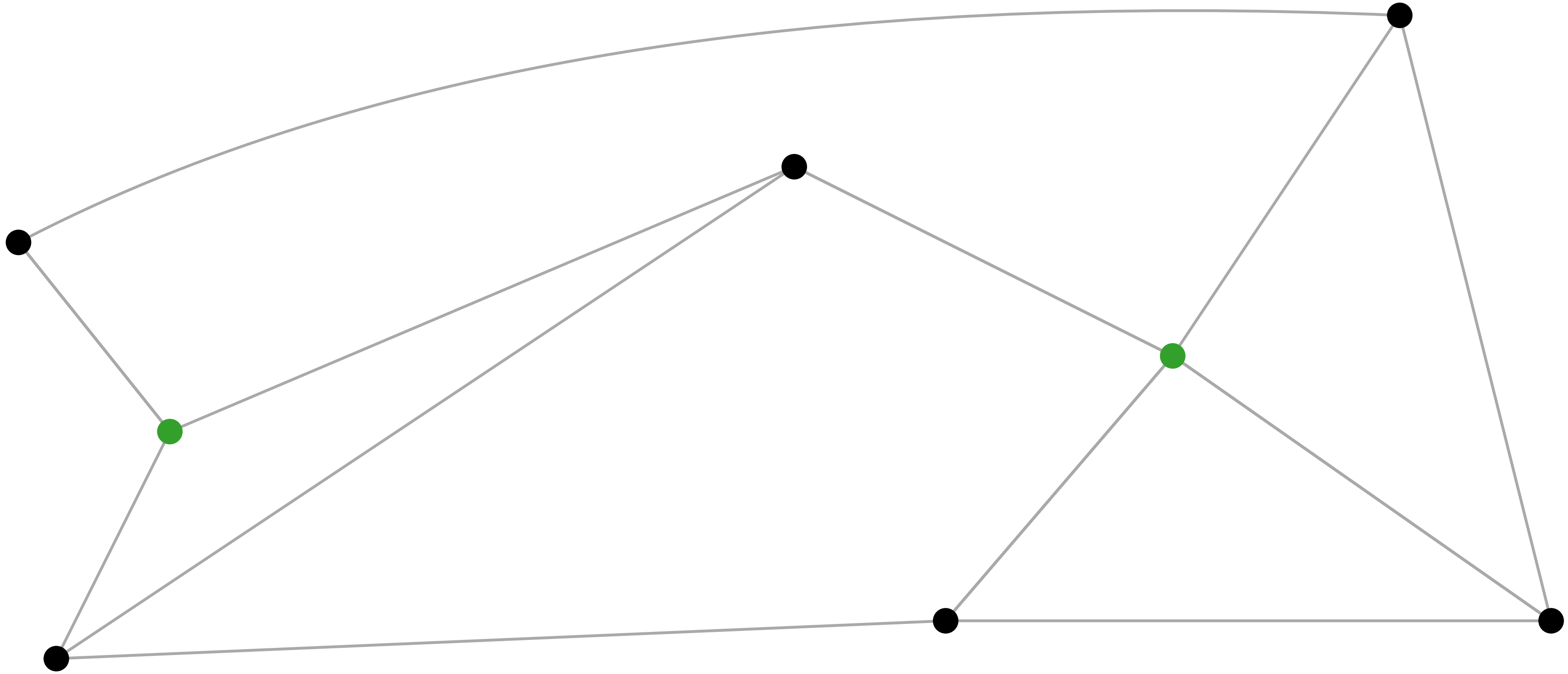

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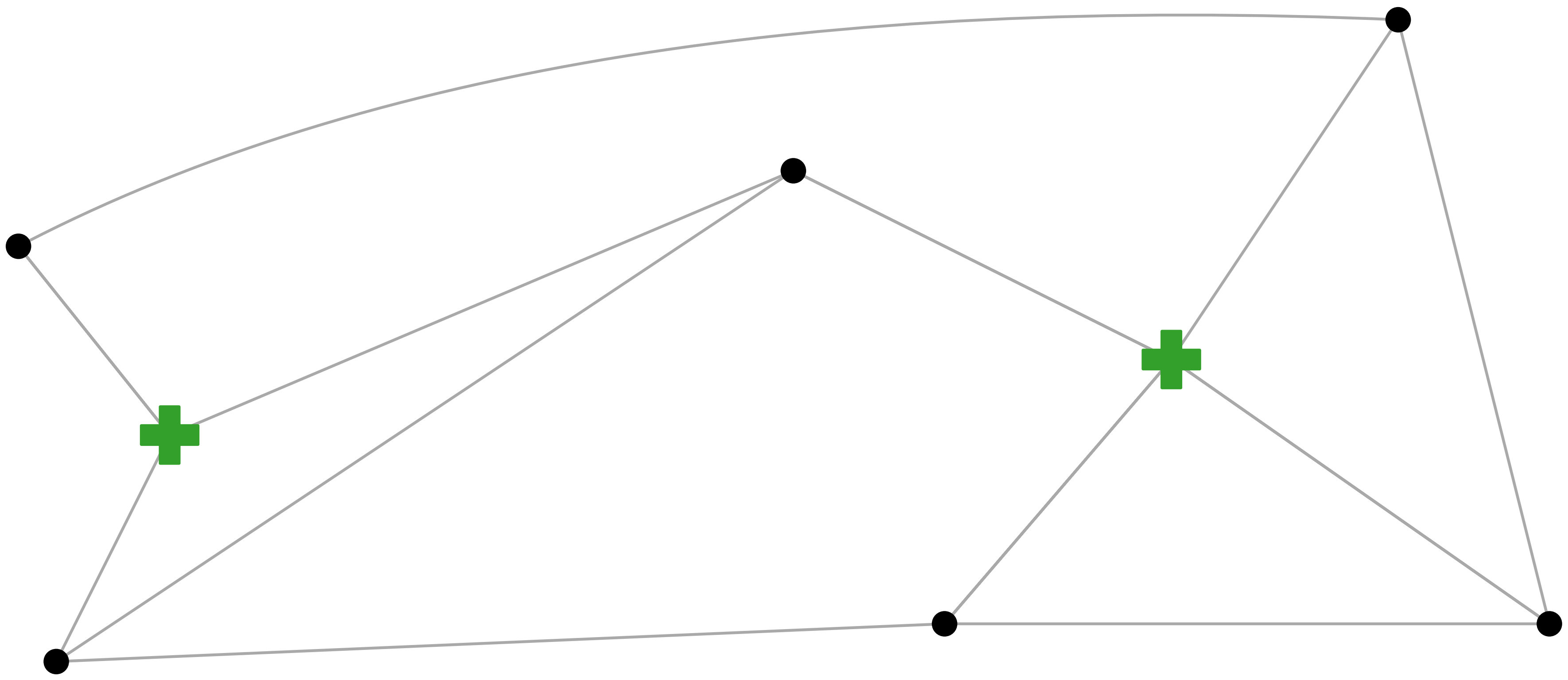

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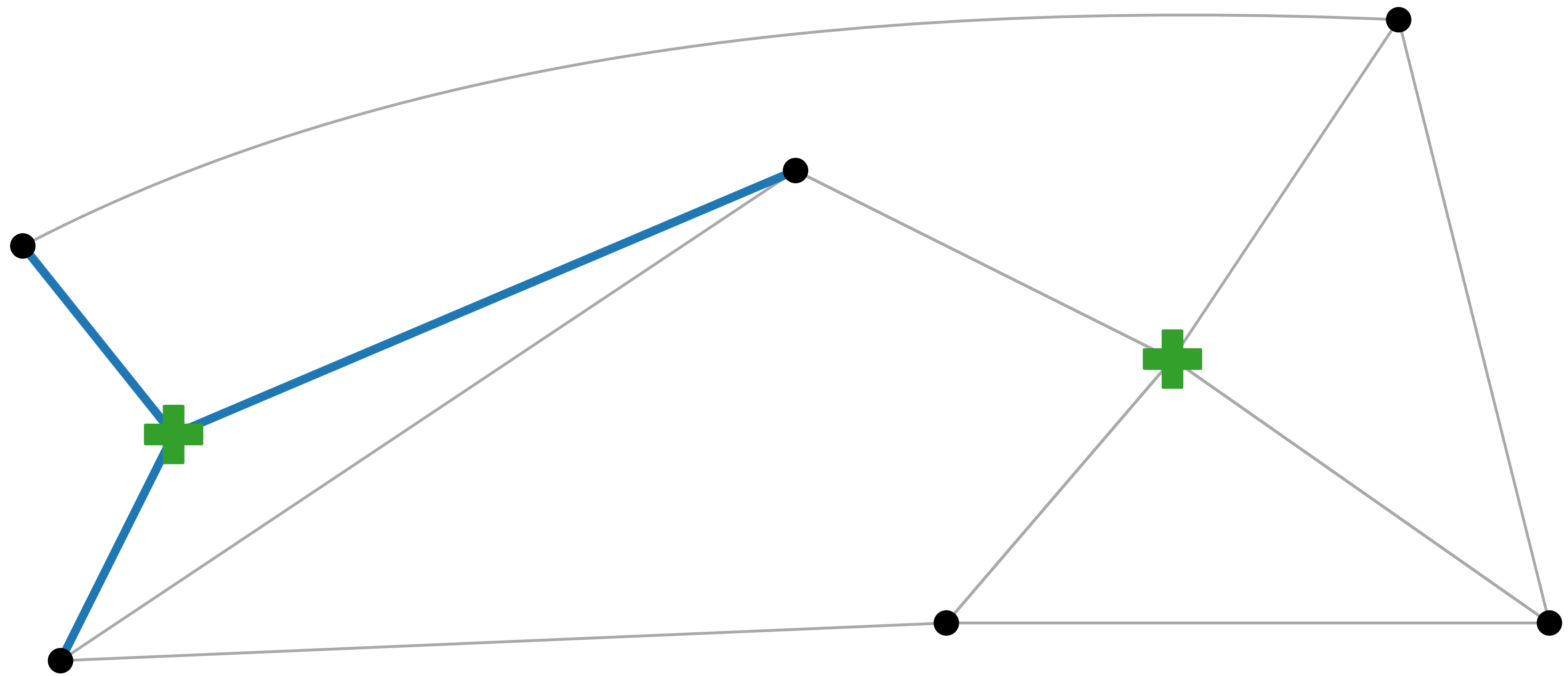

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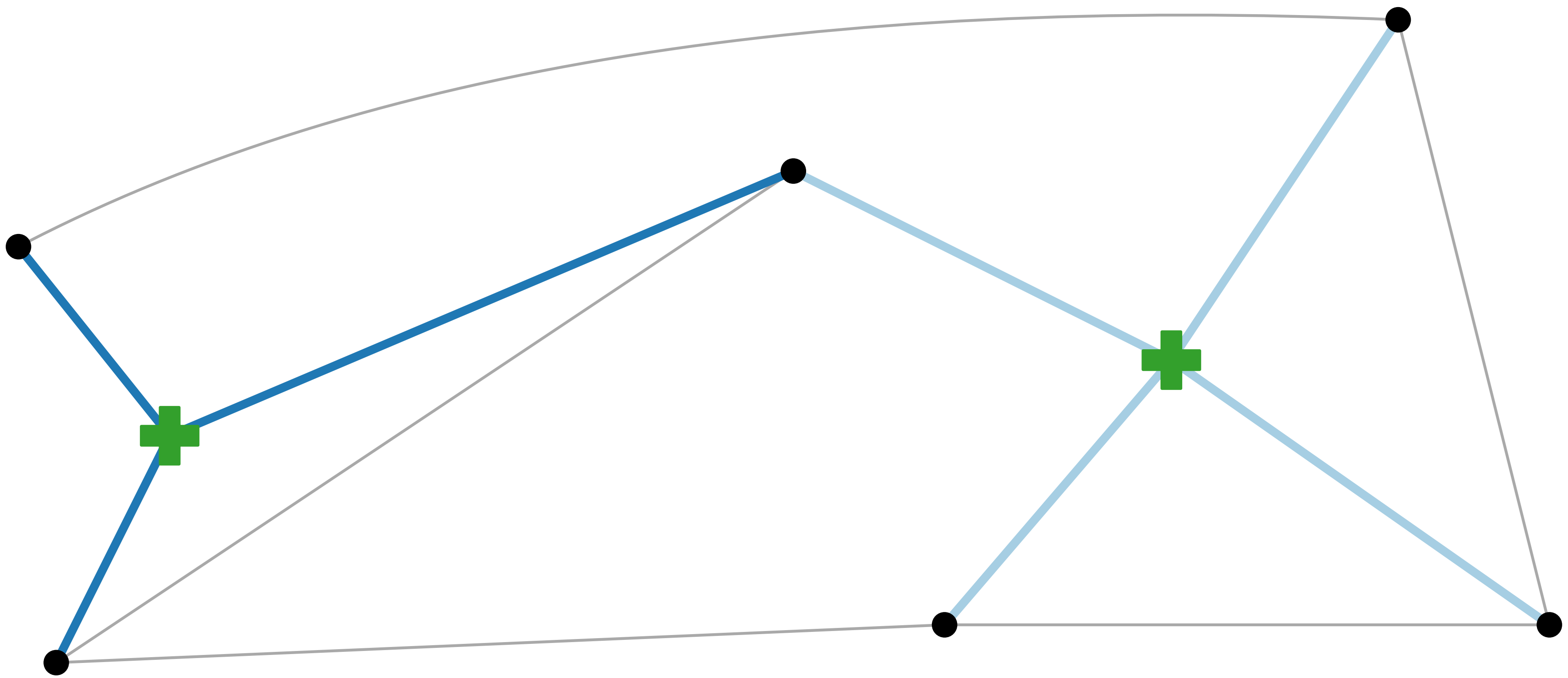

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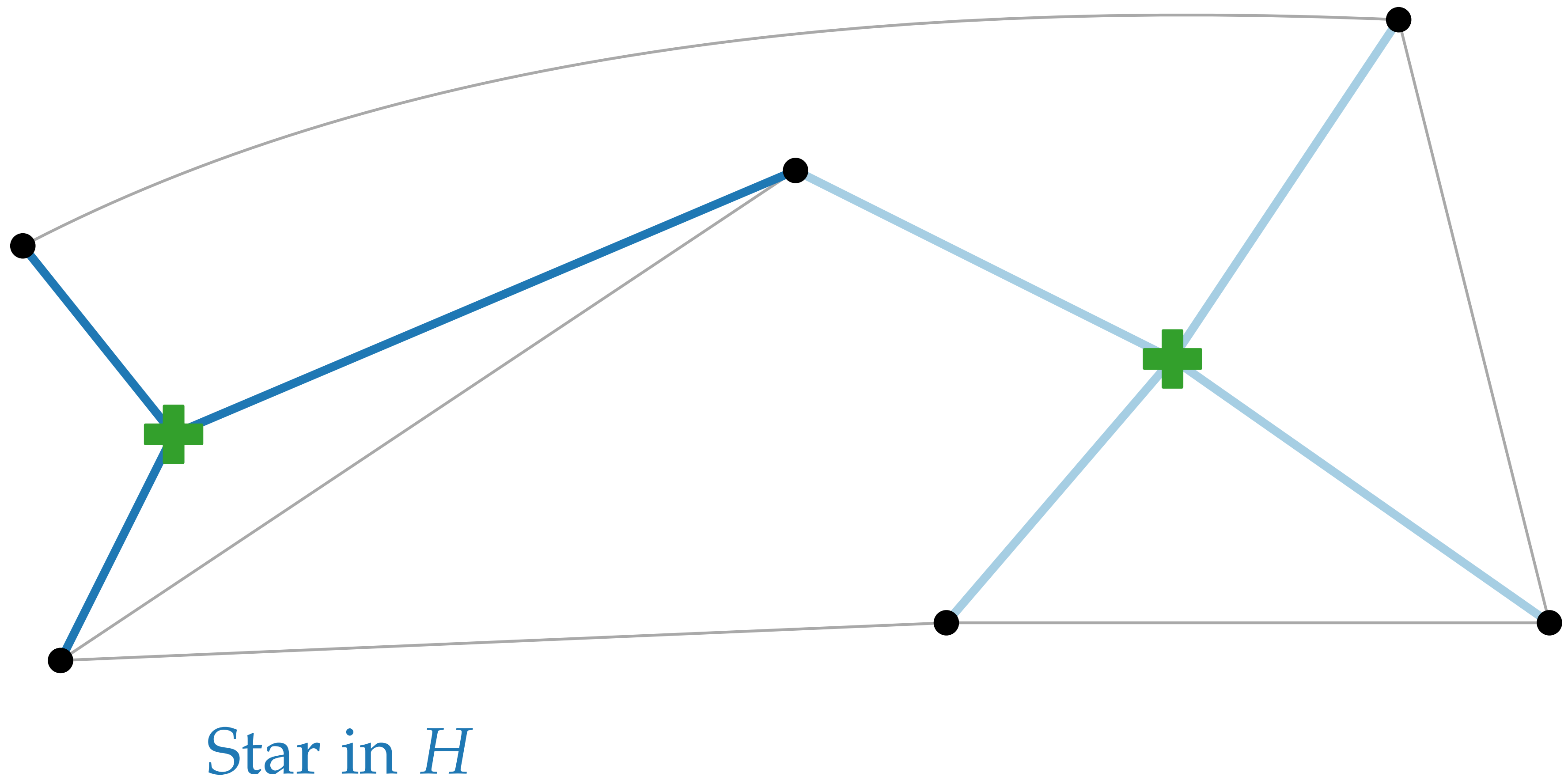

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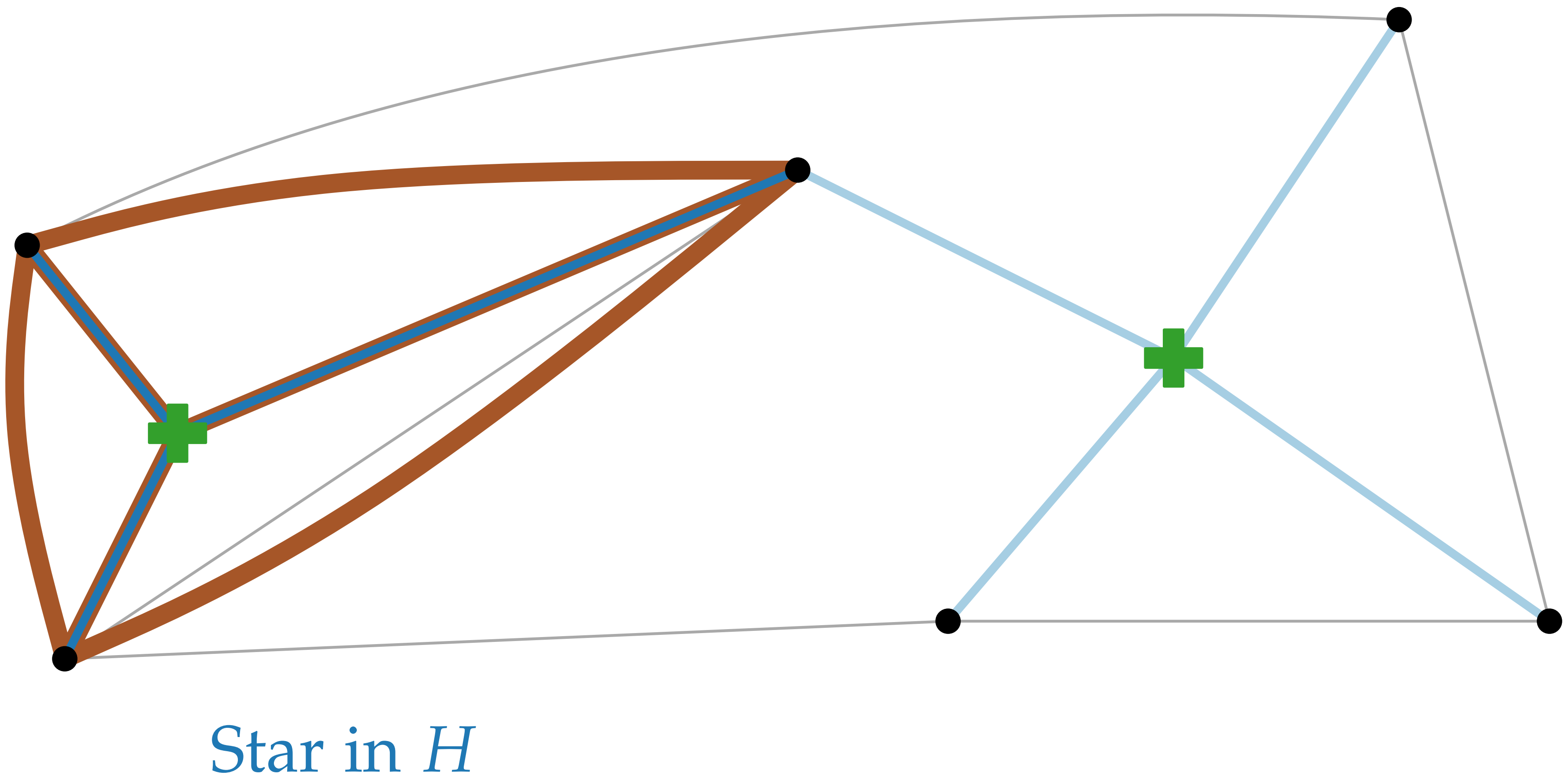

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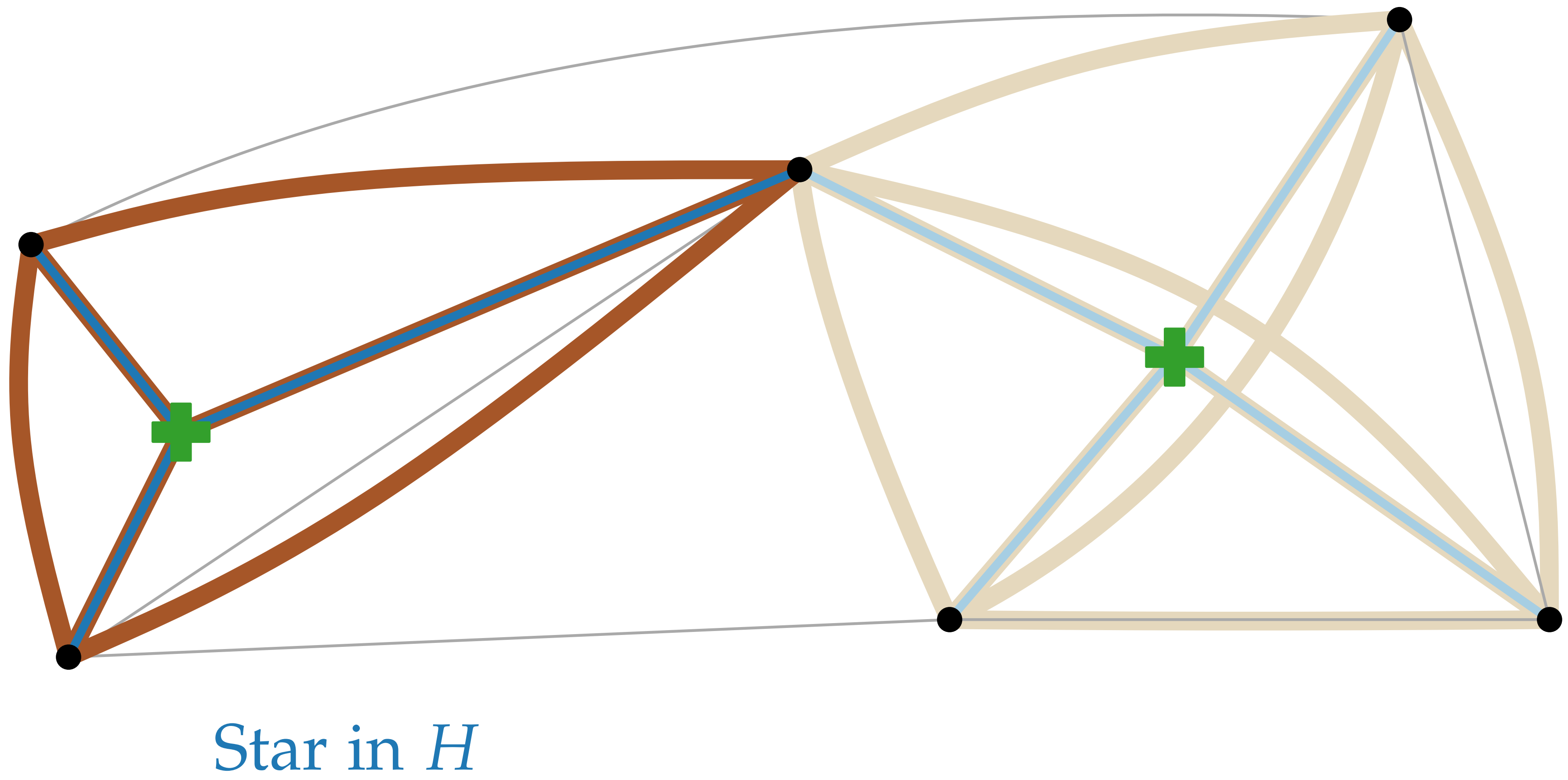

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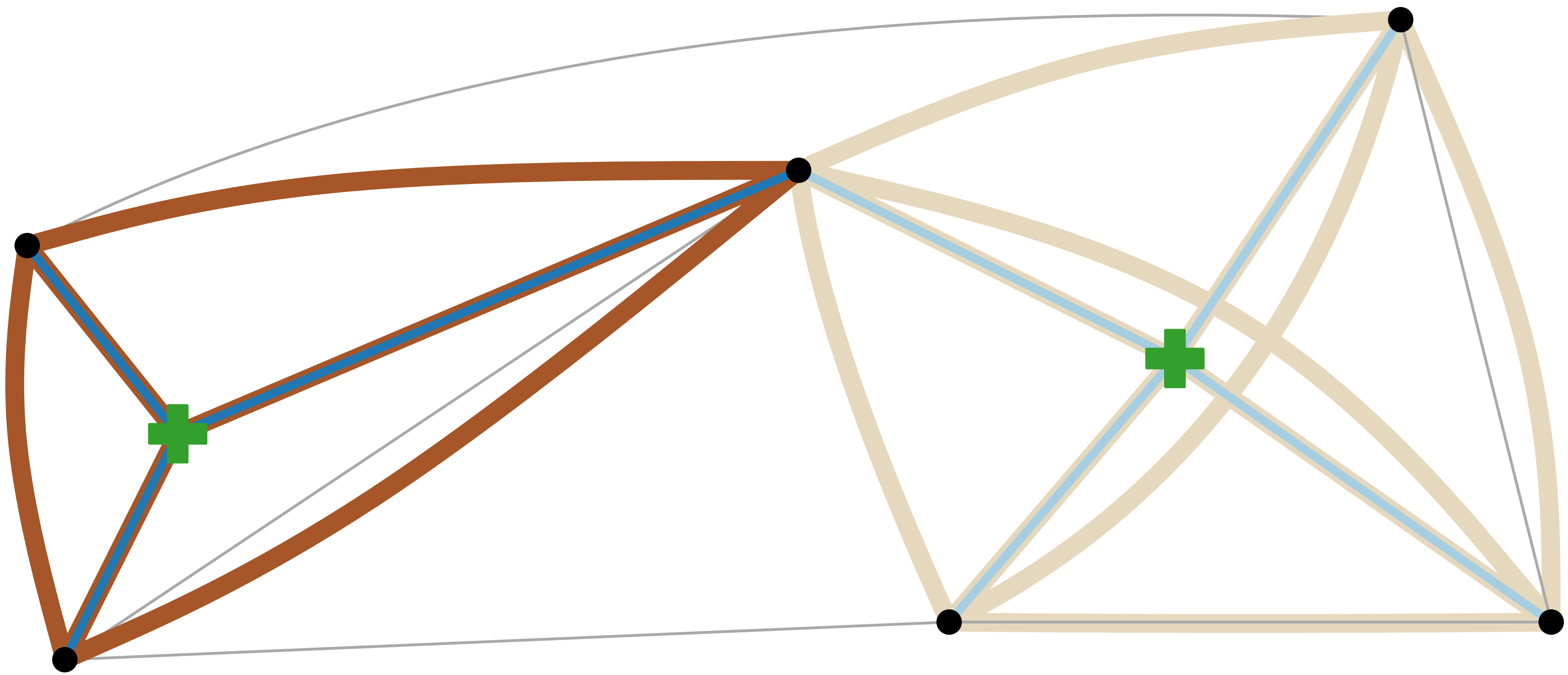

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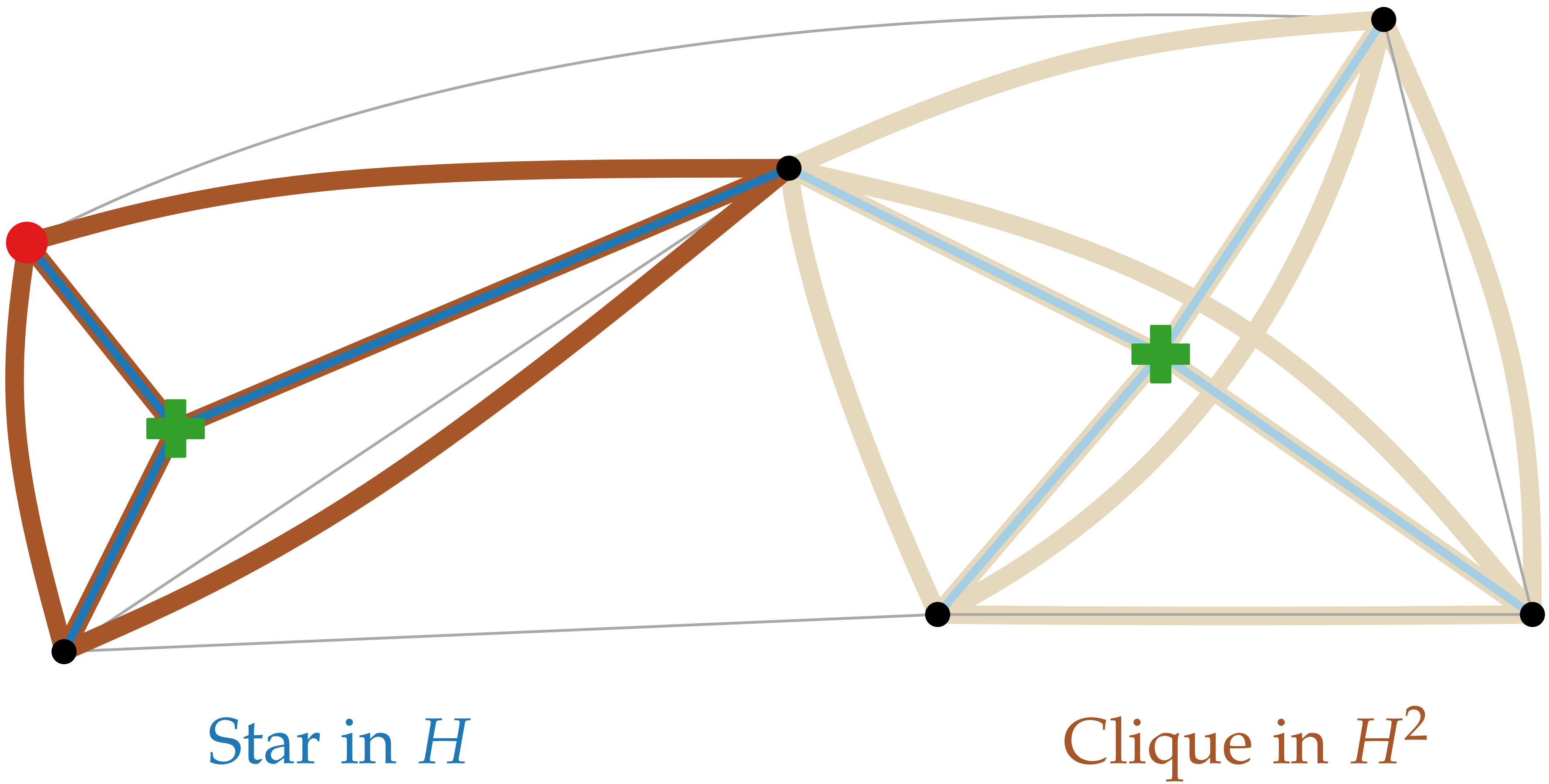

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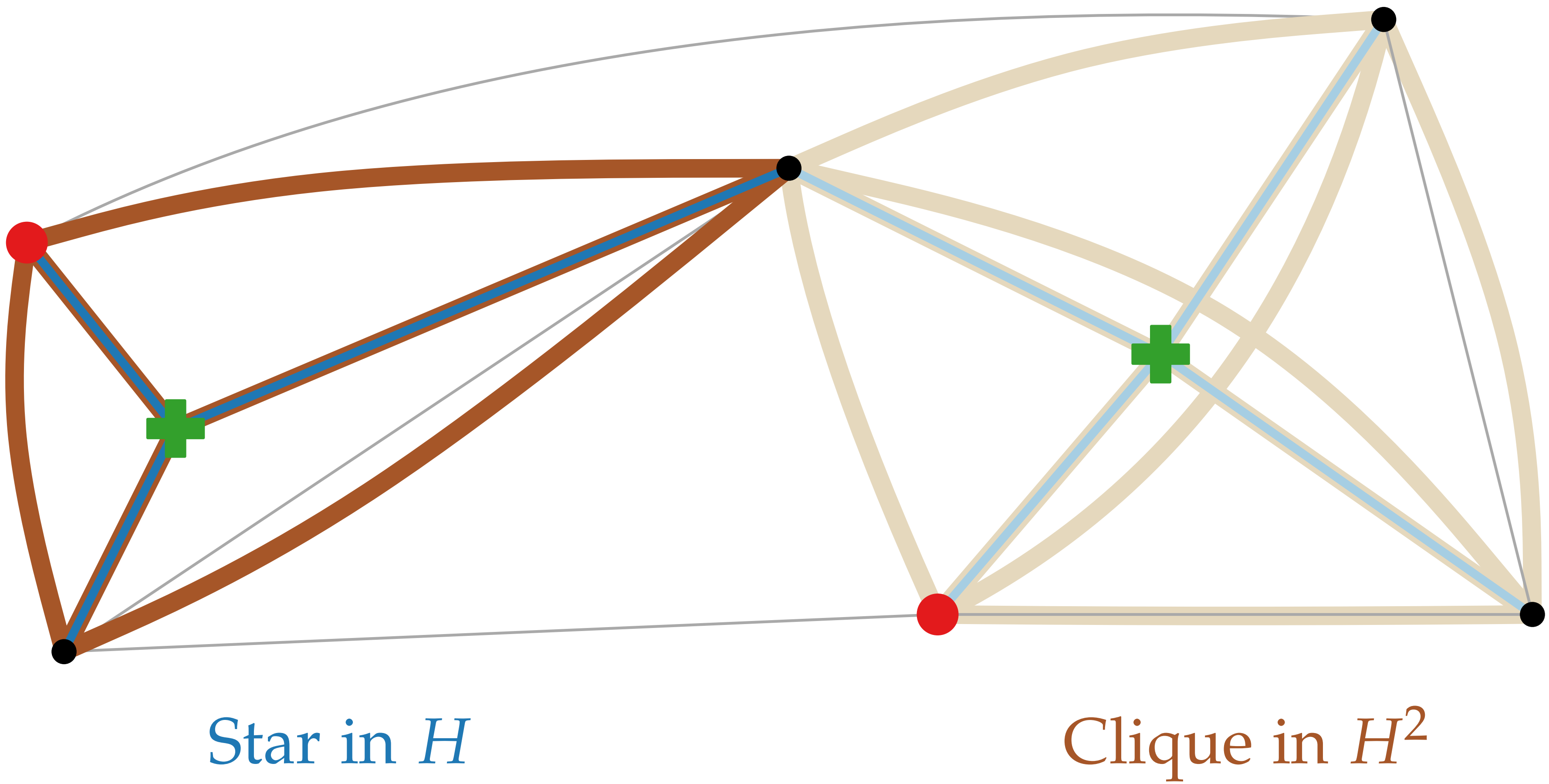

Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part IV:
Factor-2-Approximation for Metric-k-Center

Philipp Kindermann

Summer Semester 2020

Factor-2-Approx for Metric-k-Center

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Lemma. For j provided by the algorithm, we have $c(e_j) \leq \mathrm{OPT}$.

Theorem. The above algorithm is a factor-2-approximation algorithm for Metric-k-Center problem.

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What about a tight example?

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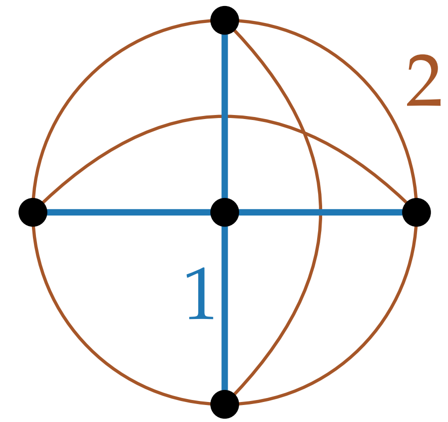

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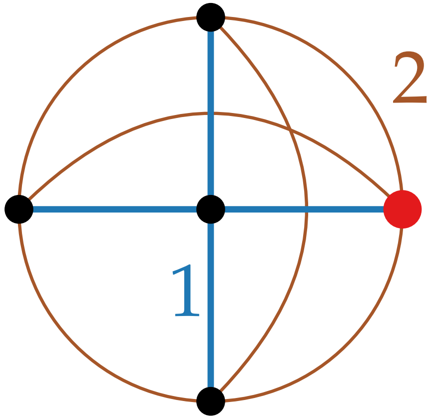

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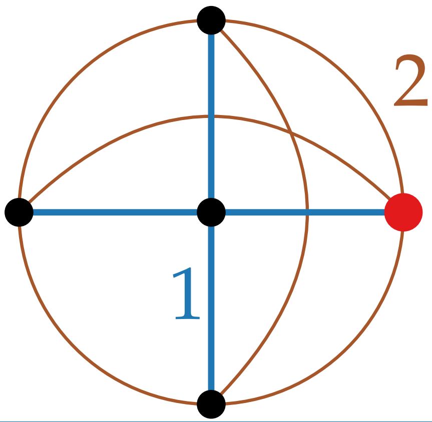

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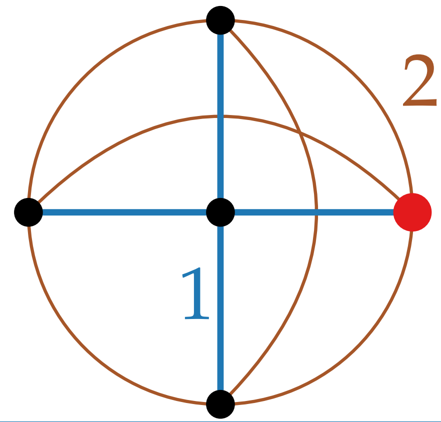

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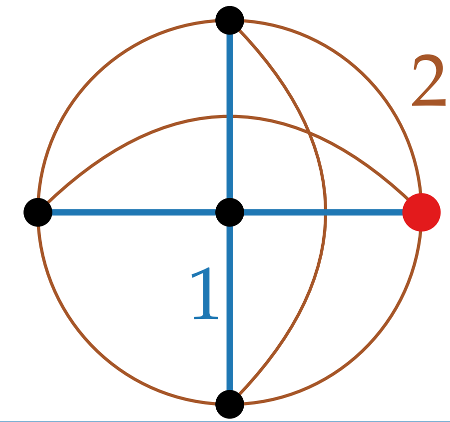

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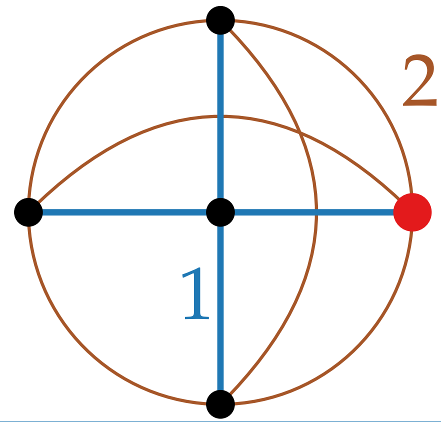

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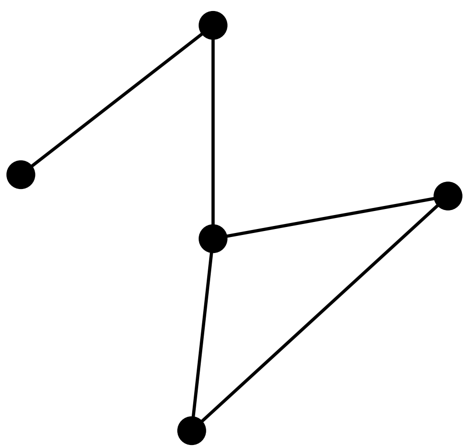

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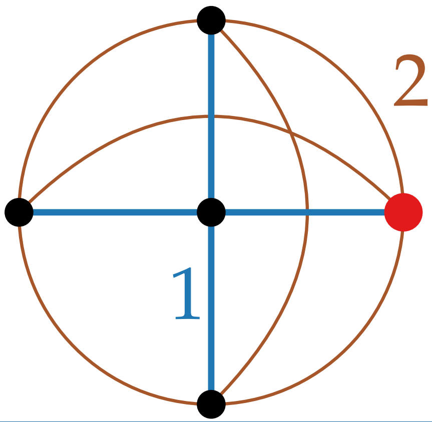

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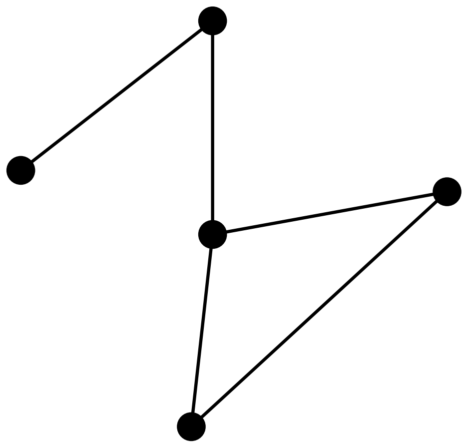

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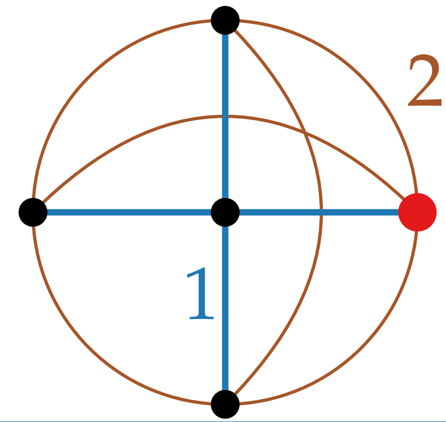

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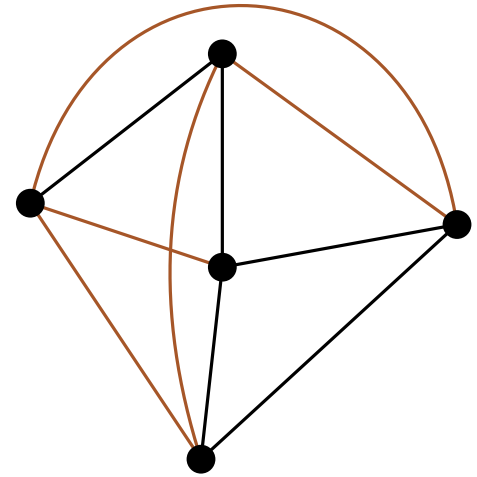

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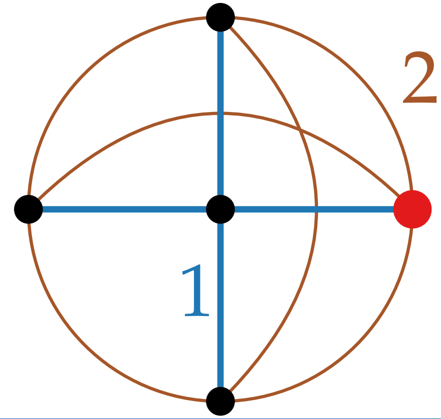

Theorem. Assuming P $\neq$ NP, there is no factor-$(2-\varepsilon)$ approximation algorithm for the metric k-Center problem, for any $\varepsilon > 0$.

Proof. Reduce from dominating set to metric k-Center.
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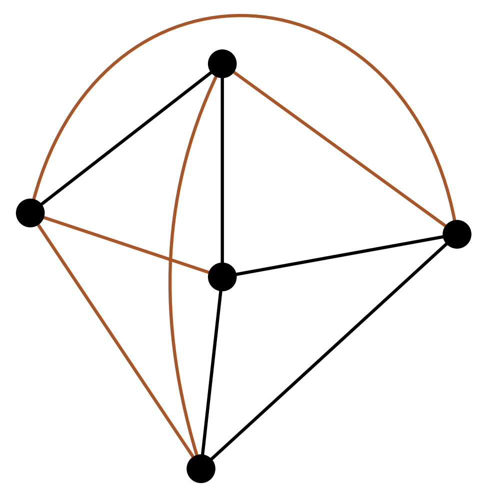

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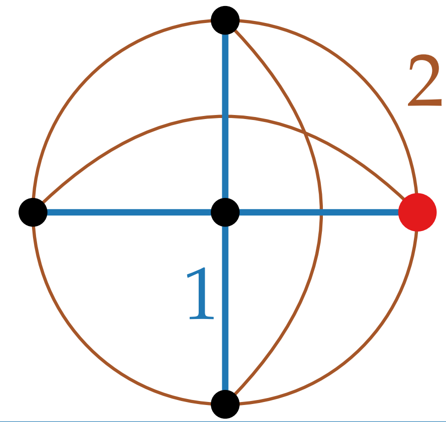

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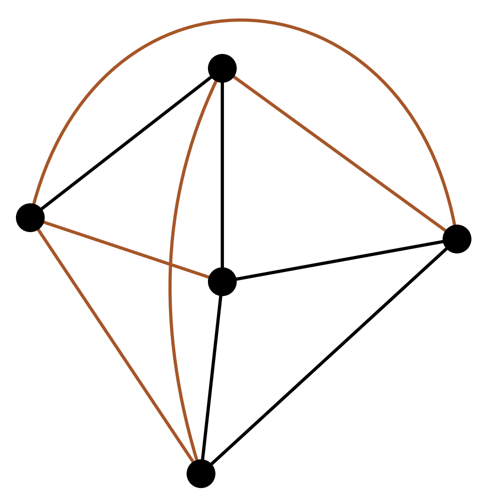

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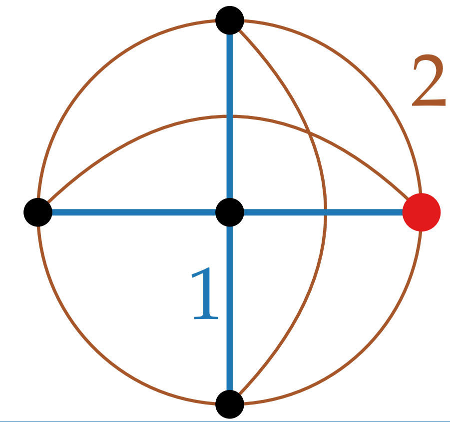

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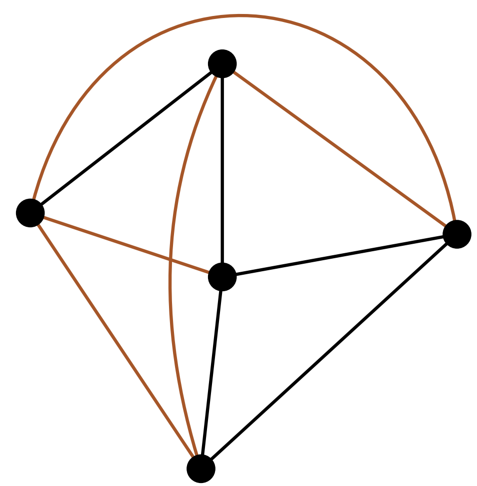

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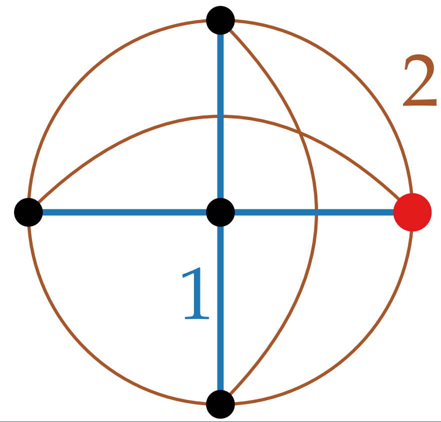

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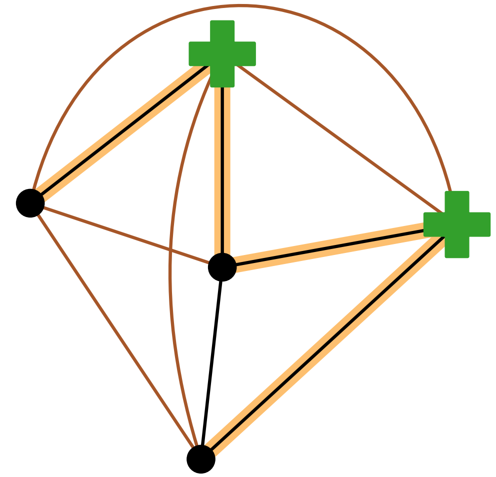

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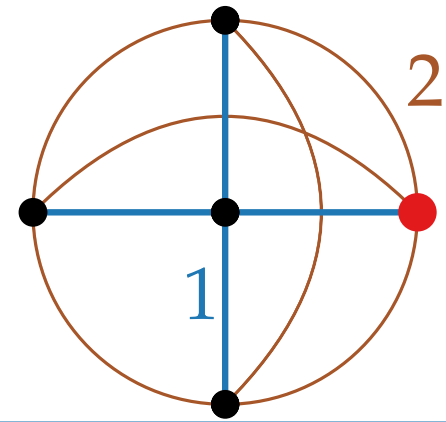

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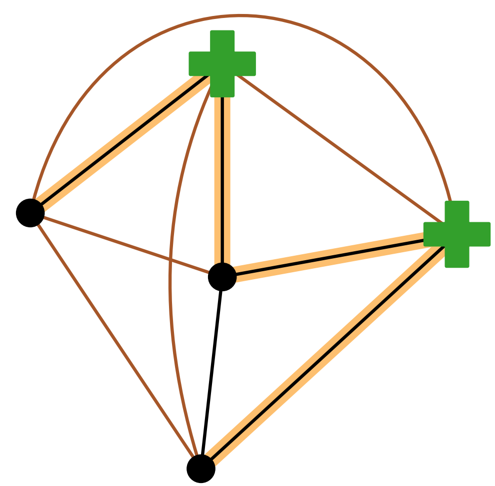

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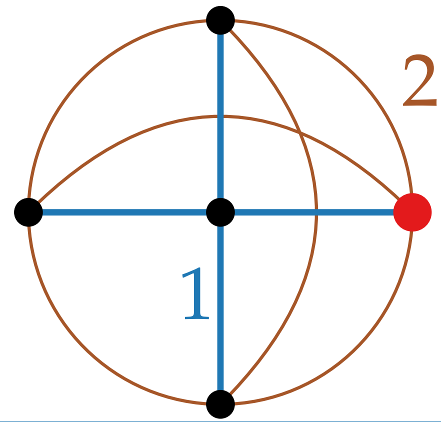

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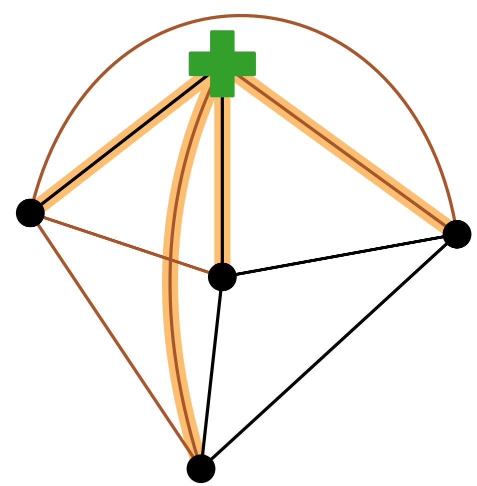

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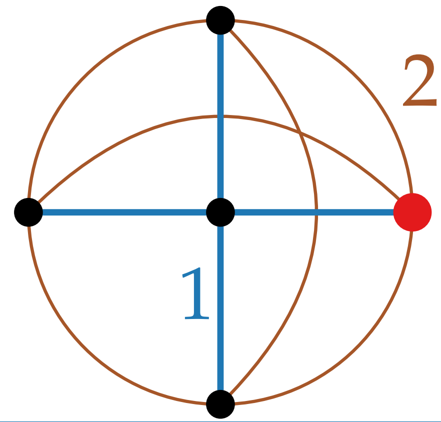

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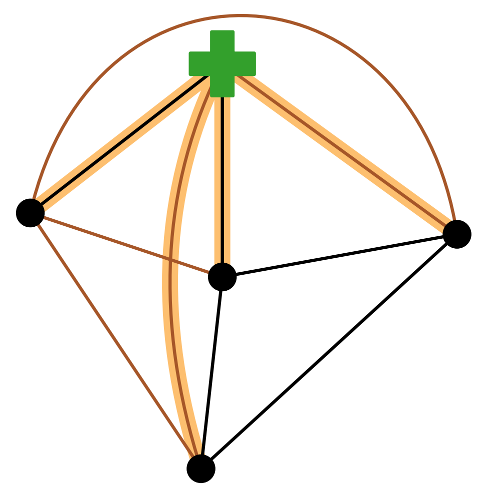

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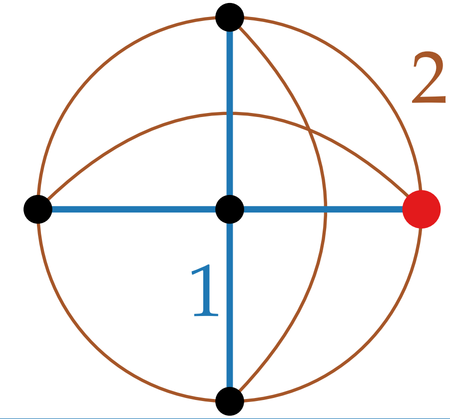

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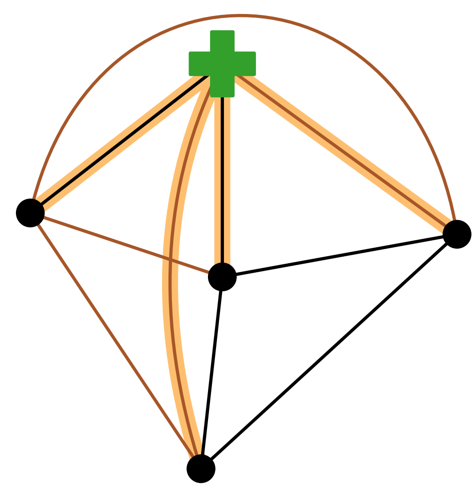
$\triangle$-inequality holds



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Approximation Algorithms

Lecture 6: k-Center via Parametric Pruning

Part V: Metric-Weighted-Center

Philipp Kindermann

Summer Semester 2020

Metric-k-Center

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to the a vertex in S.

Find: A k-element vertex set S, such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

METRIC-k-CENTER

WEIGHTED

Given: A complete graph $G = (V, E)$ with metric edge costs $c \colon E \to \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

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METRIC-~~k~~-CENTER

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Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and ~~a natural number $k \leq |V|$.~~ , vertex weights $w\colon V \to \mathbb{Q}_{\geq 0}$ and a weight limit $W \in \mathbb{Q}_+$

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vertex set S of weight at most W

Find: A ~~k-element vertex set S~~, such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Algorithm for the Weighted Version

Algorithm Metric- -CENTER

Sort the edges of G by cost : $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

  Construct G_j^2

  Find a maximal independent set U_j in G_j^2

  if $|U_j| \leq k$ **then**

    return U_j

Algorithm for the Weighted Version

Algorithm Metric-Weighted-Center

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  - what about the weights?
- **if** $|U_j| \leq k$ **then**
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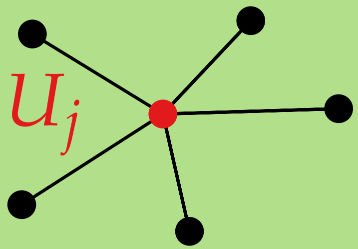
$u \in U_j$



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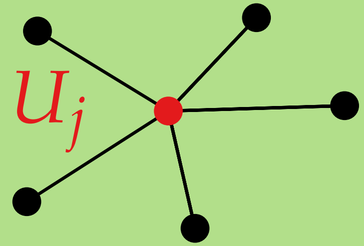

$s_j(u) :=$ lightest node in $N_{G_j}(u) \cup \{u\}$

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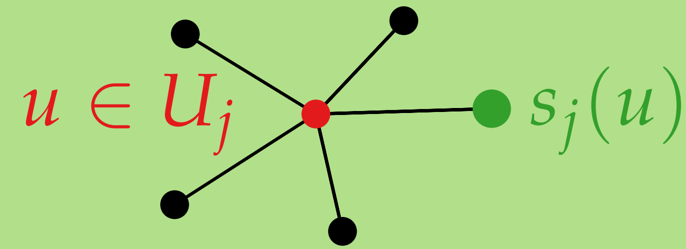

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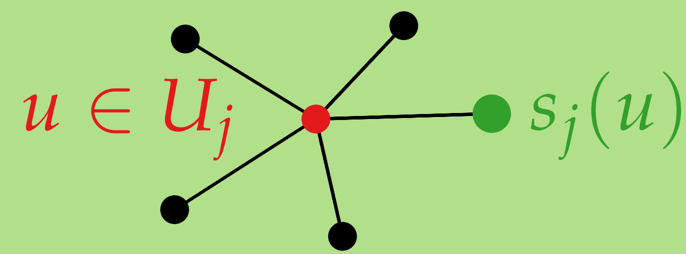

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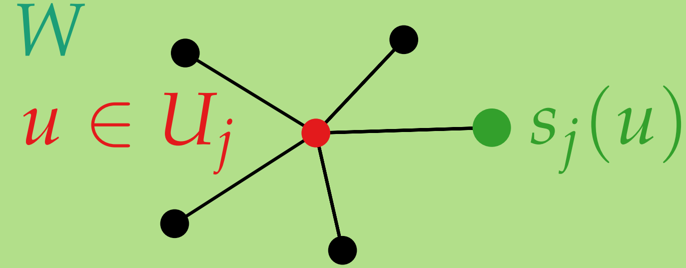

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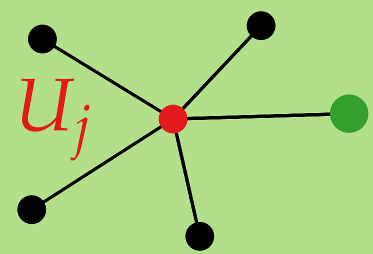

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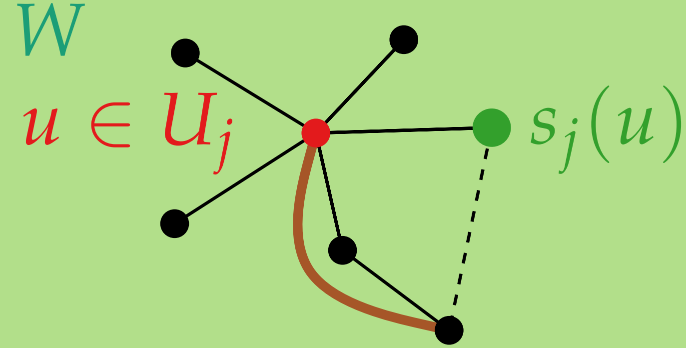

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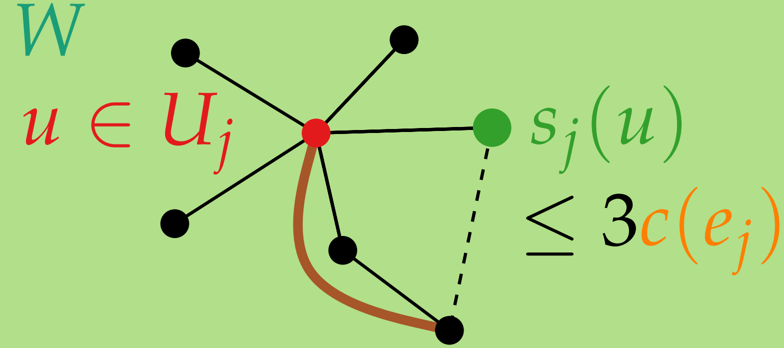

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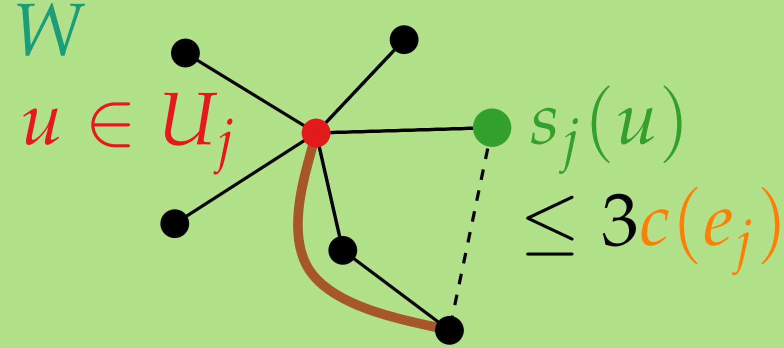

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Theorem. The above is a factor-3-approximation algorithm for Metric-Weighted-Center.

Tight Example... ?

Here, we need to have a weight limit W, and edge costs satisfying the triangle inequality.

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Consider $W = 3$

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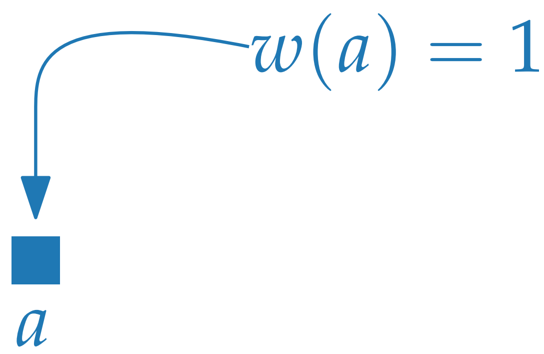

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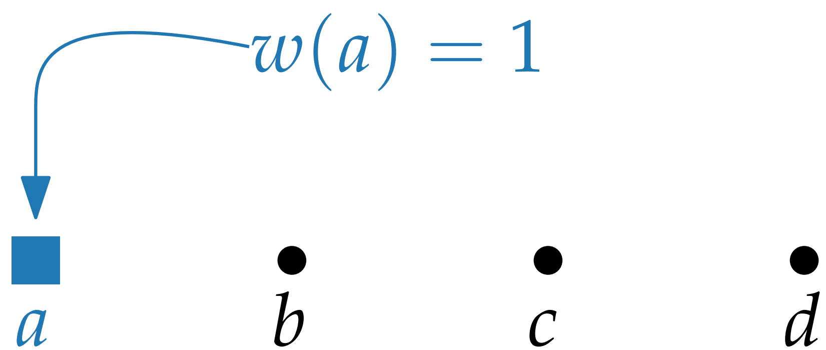

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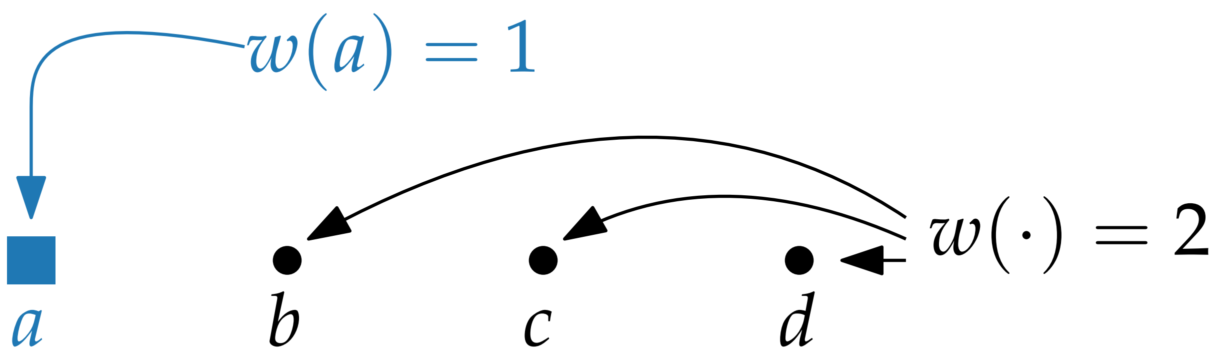

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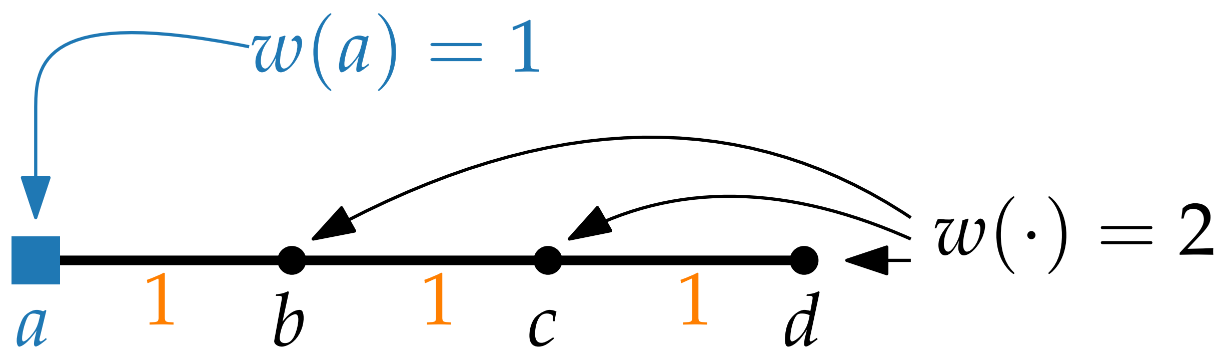

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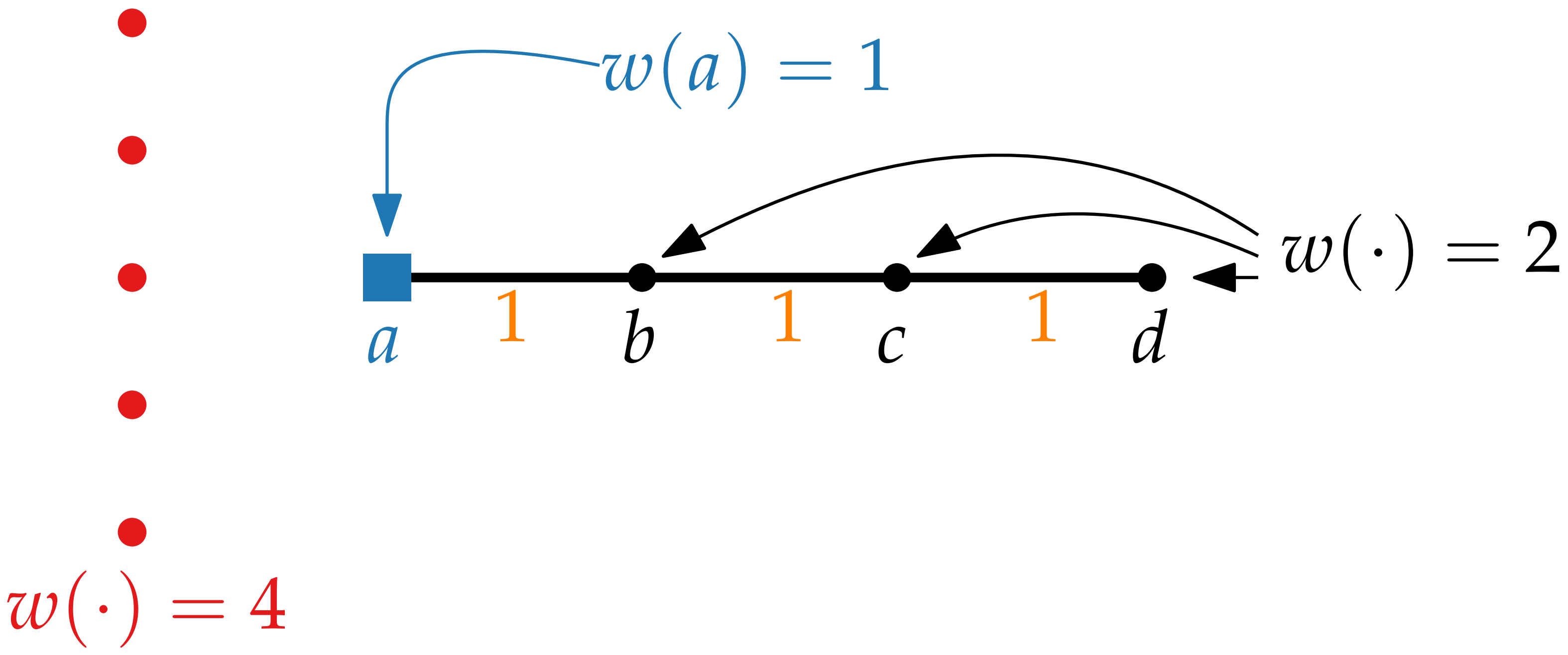

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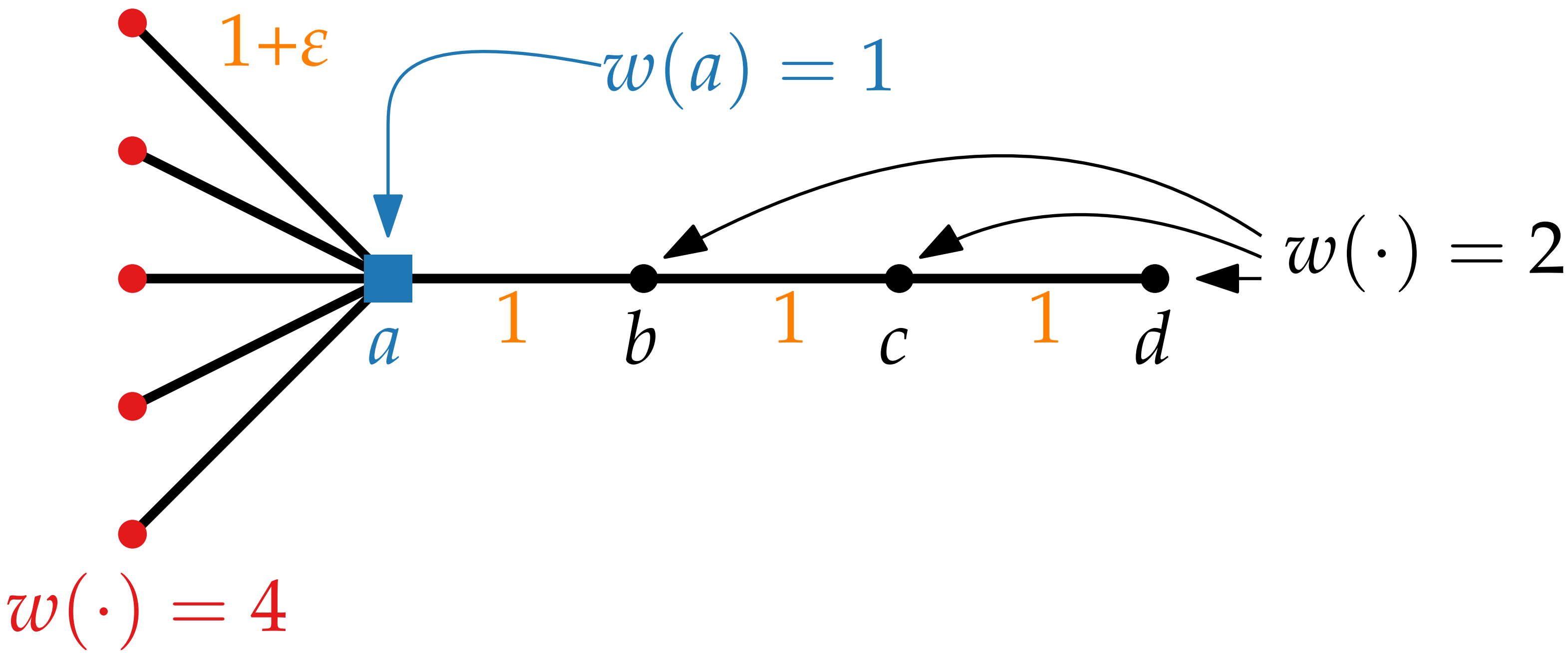

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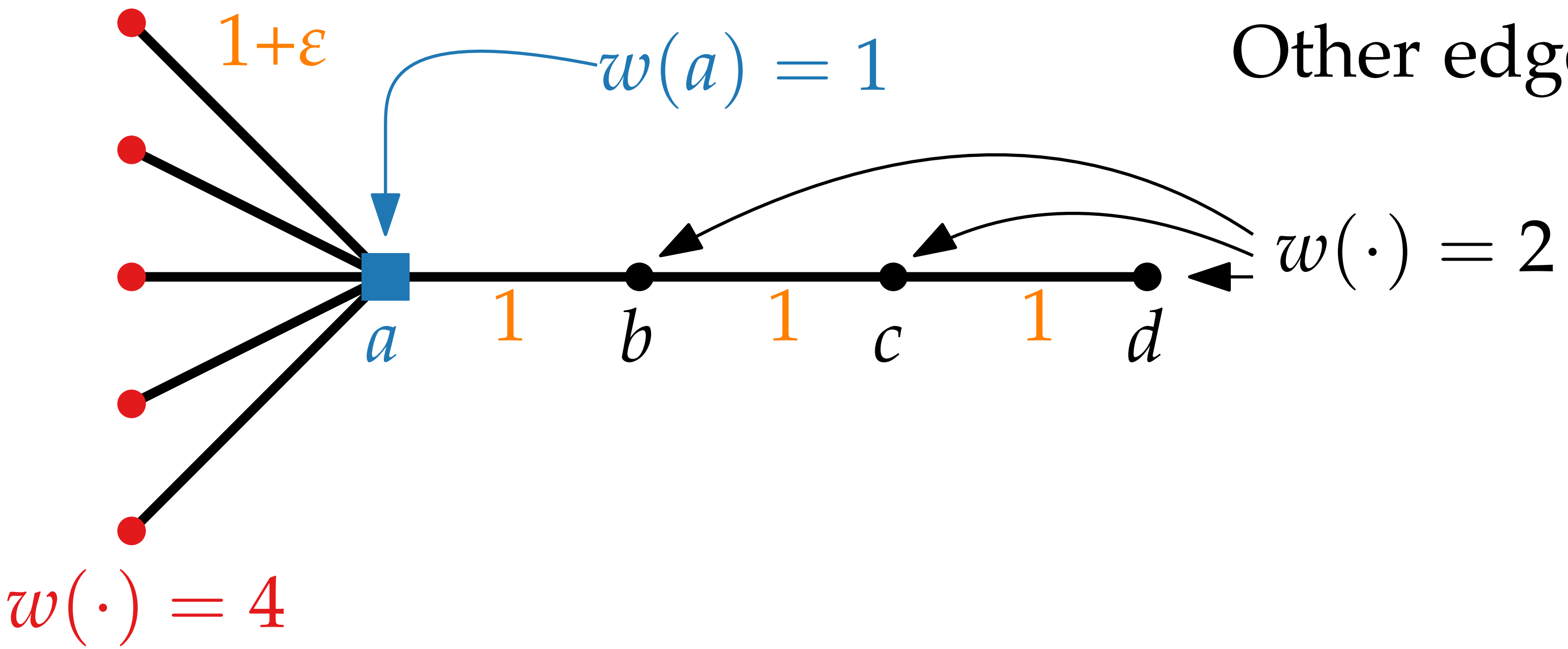
Consider $W = 3$

Other edge costs?



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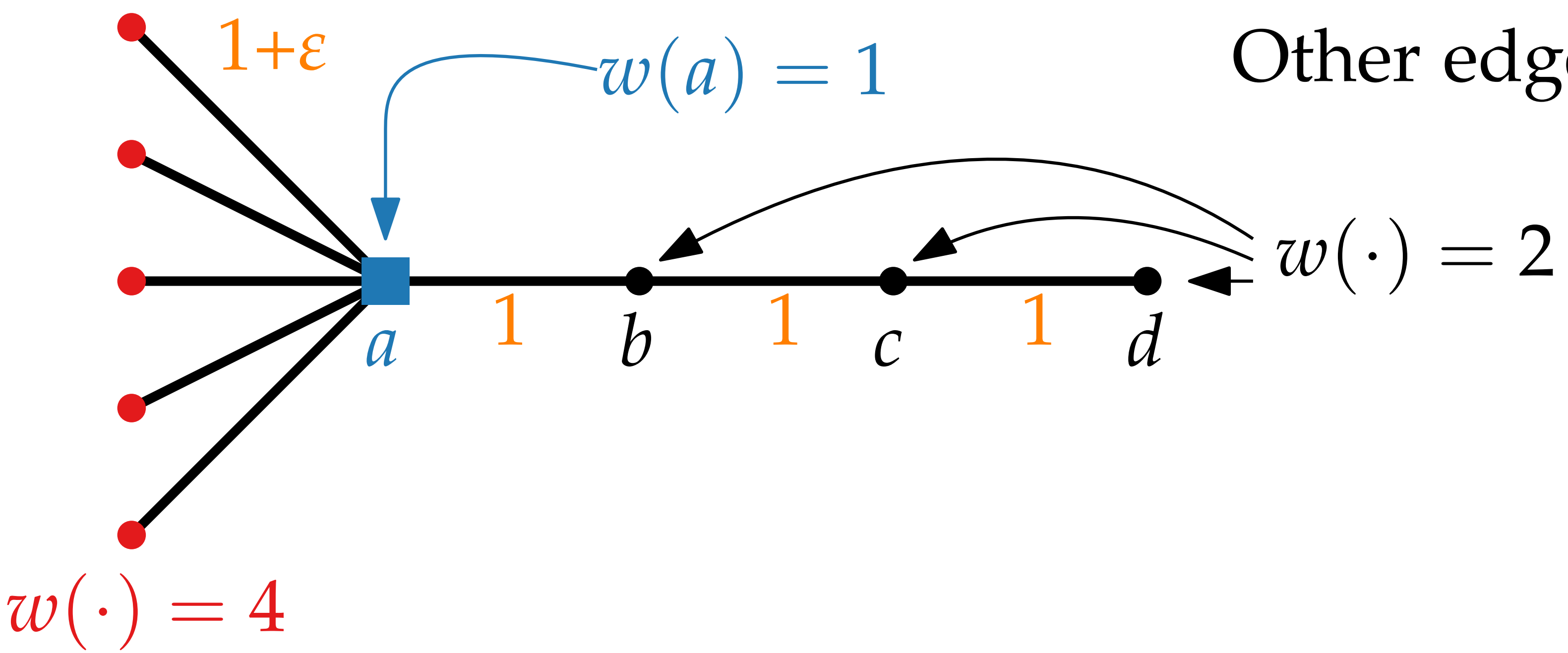
Consider $W = 3$

Other edge costs? 2



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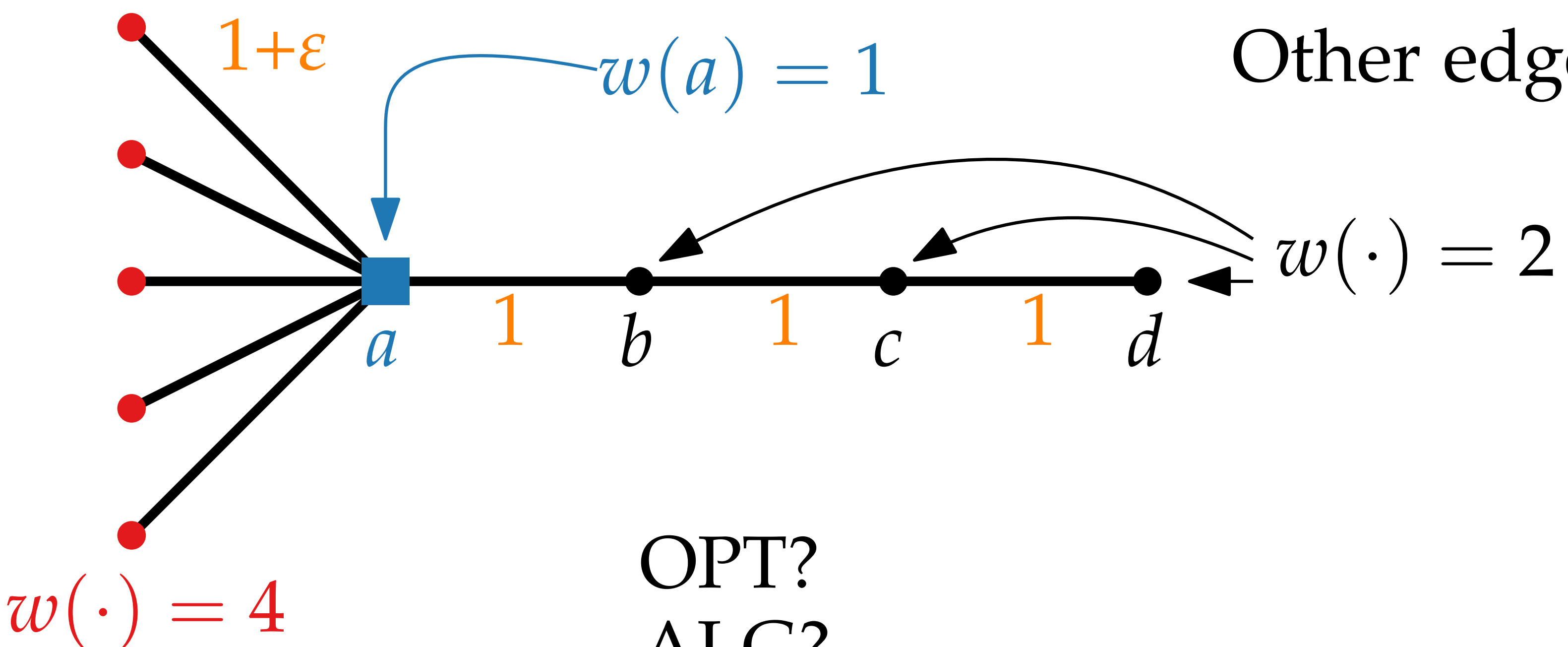
Here, we need to have a weight limit W, and edge costs satisfying the triangle inequality.

Consider $W = 3$

Other edge costs? 2



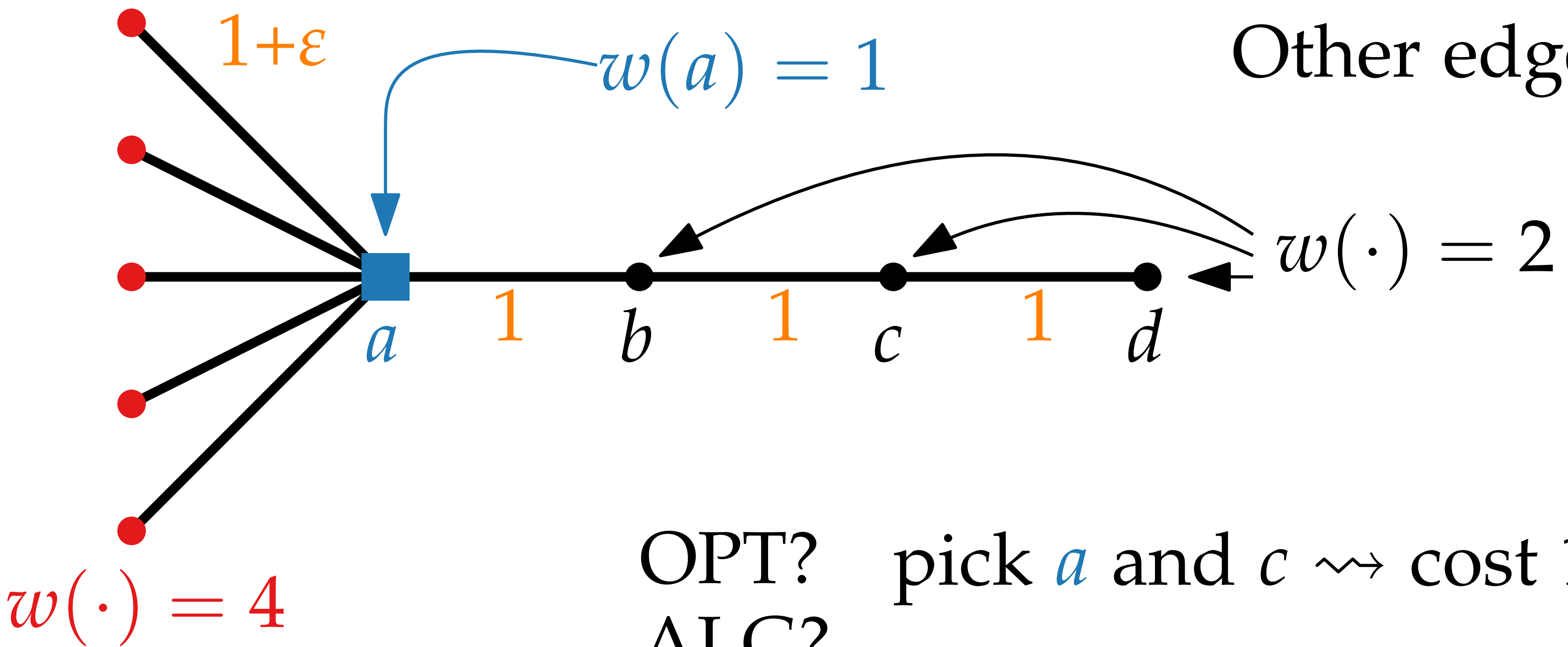
OPT?
ALG?

Tight Example... ?

Here, we need to have a weight limit W, and edge costs satisfying the triangle inequality.

Consider $W = 3$

Other edge costs? 2



OPT? pick a and $c \rightsquigarrow$ cost $1 + \varepsilon$

ALG?

Tight Example... ?

Here, we need to have a weight limit W, and edge costs satisfying the triangle inequality.

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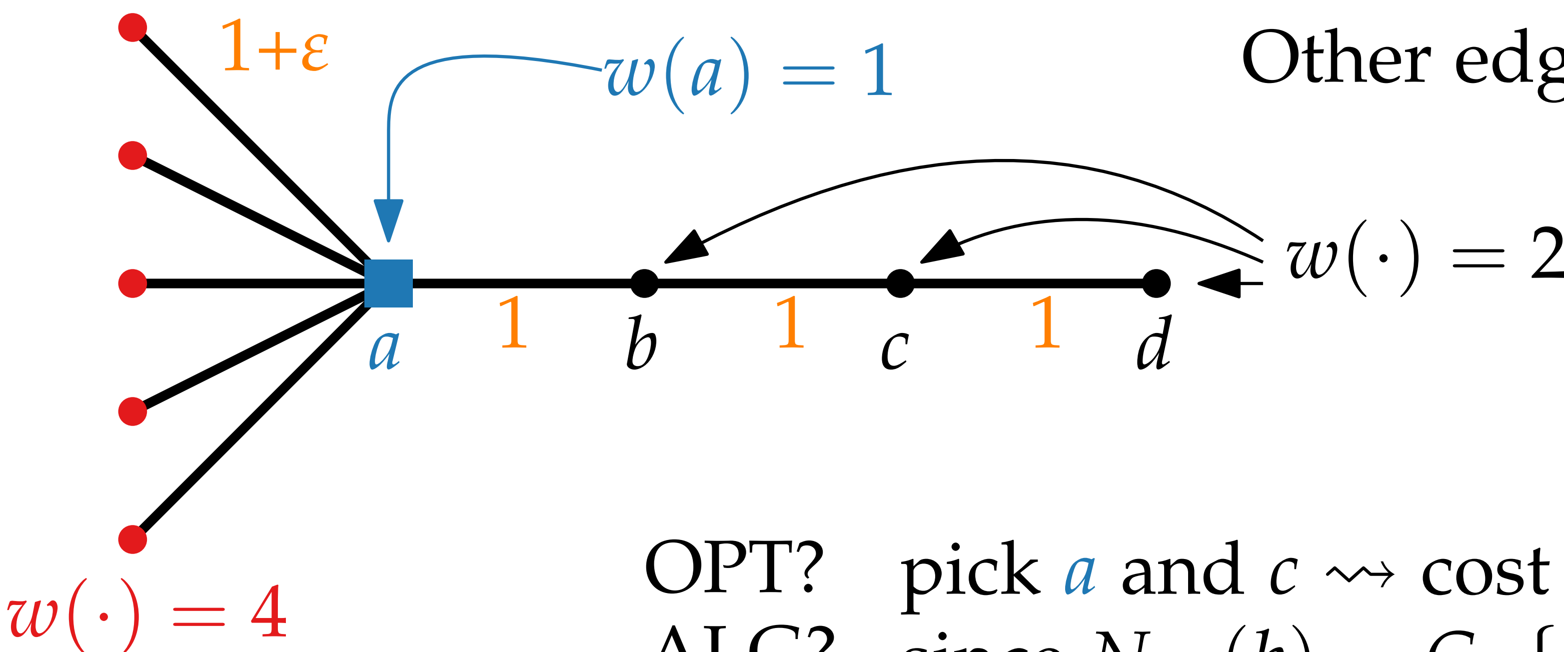

OPT? pick a and c $\rightsquigarrow$ cost $1 + \varepsilon$

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in G^2

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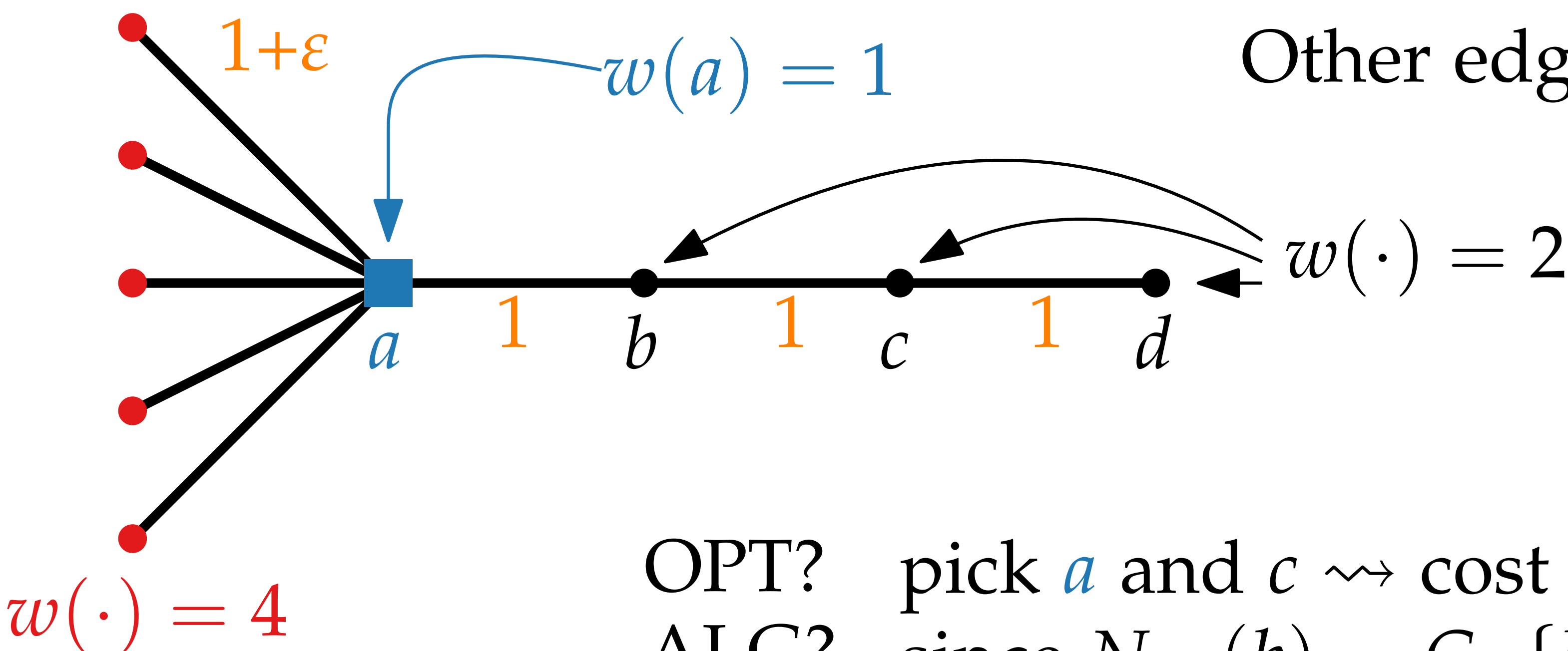

OPT? pick a and c $\rightsquigarrow$ cost $1 + \varepsilon$

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in G^2

Thus, alg. picks a $\rightsquigarrow$ cost 3

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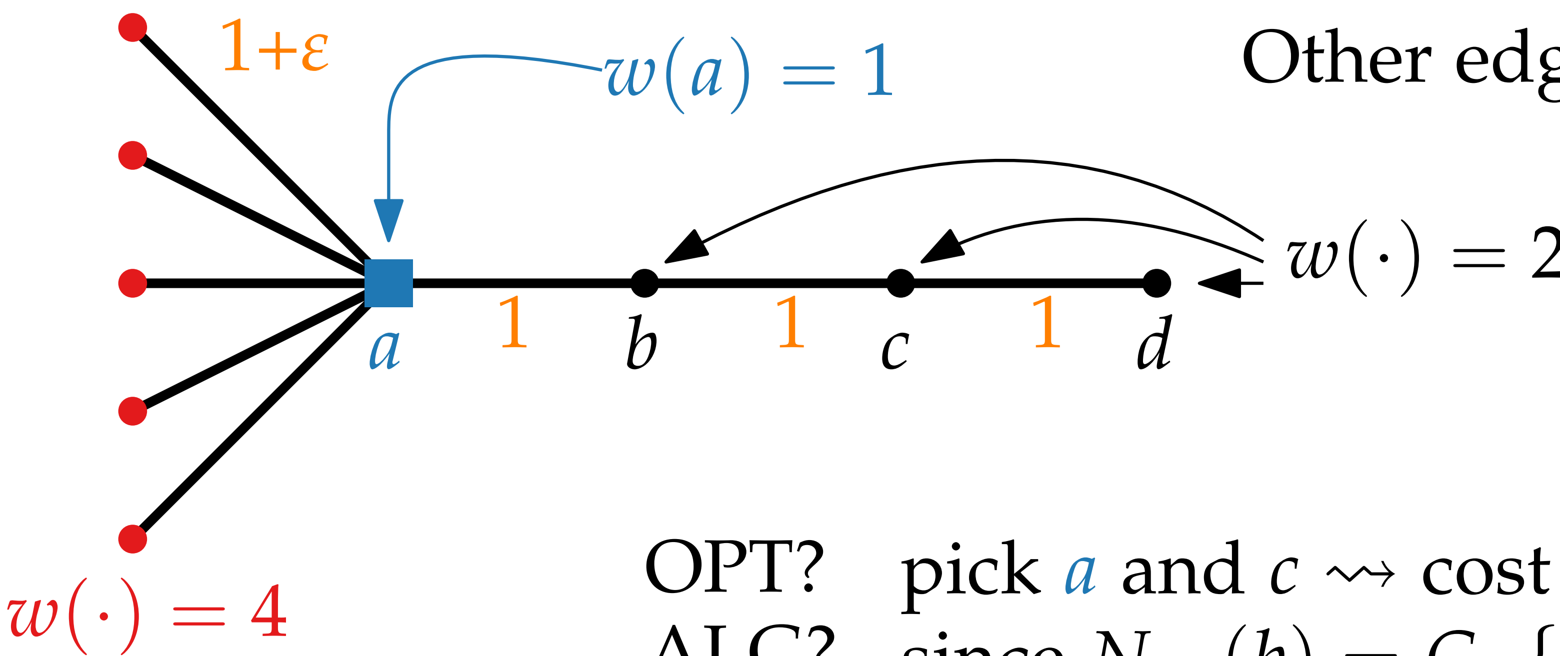
~~Consider $W = 3$~~

Other edge costs? 2



OPT? pick a and $c \rightsquigarrow$ cost $1 + \varepsilon$

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in G^2

Thus, alg. picks $a \rightsquigarrow$ cost 3

How can we generalize this to larger W?

Tight Example... ?

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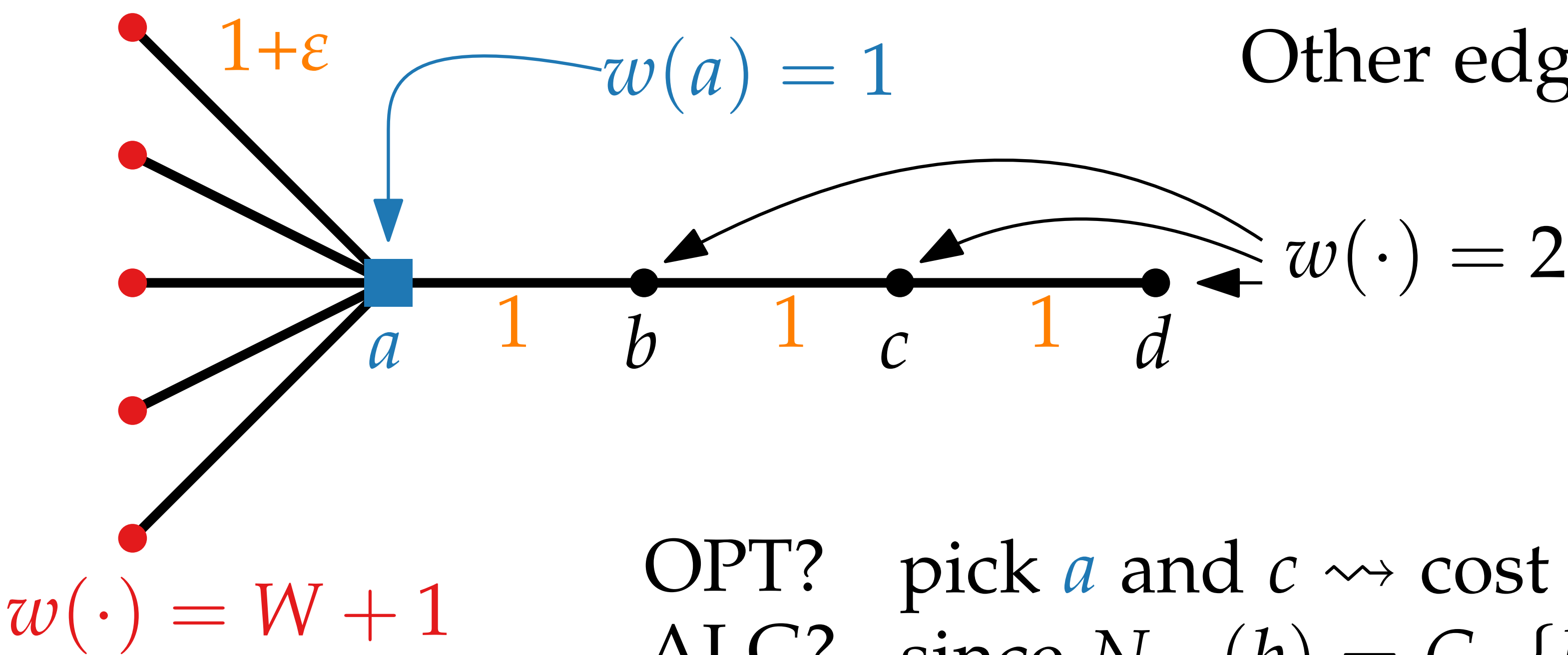
Other edge costs? 2



OPT? pick a and c $\rightsquigarrow$ cost $1 + \varepsilon$

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in G^2

Thus, alg. picks a $\rightsquigarrow$ cost 3

How can we generalize this to larger W?