# Approximation Algorithms

## Lecture 5:
## LP-based Approximation Algorithms for SetCover

### Part I:
### LP-based Approximation Techniques

Philipp Kindermann

Summer Semester 2020

# I) LP-Rounding


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Approximation factor $ALG/OPT_{\Pi} \leq ALG/OPT_{relax}$

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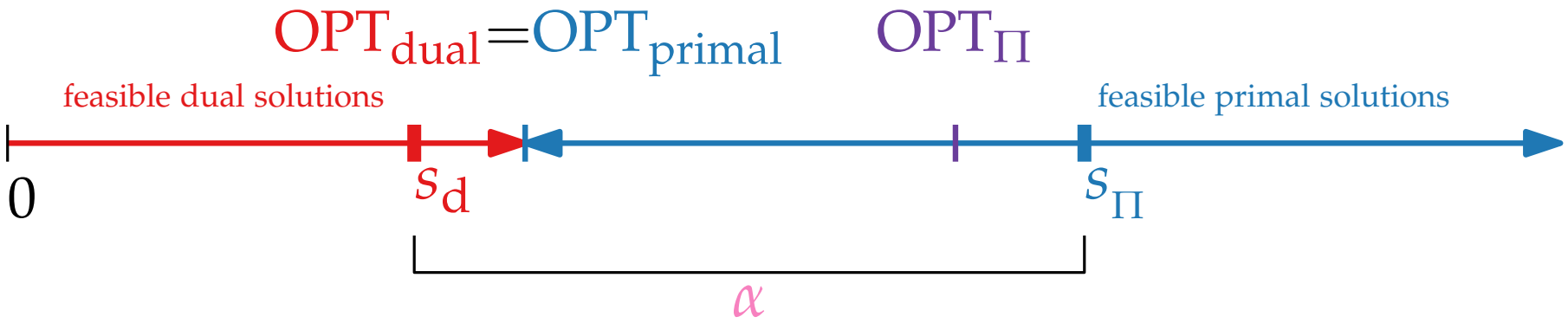


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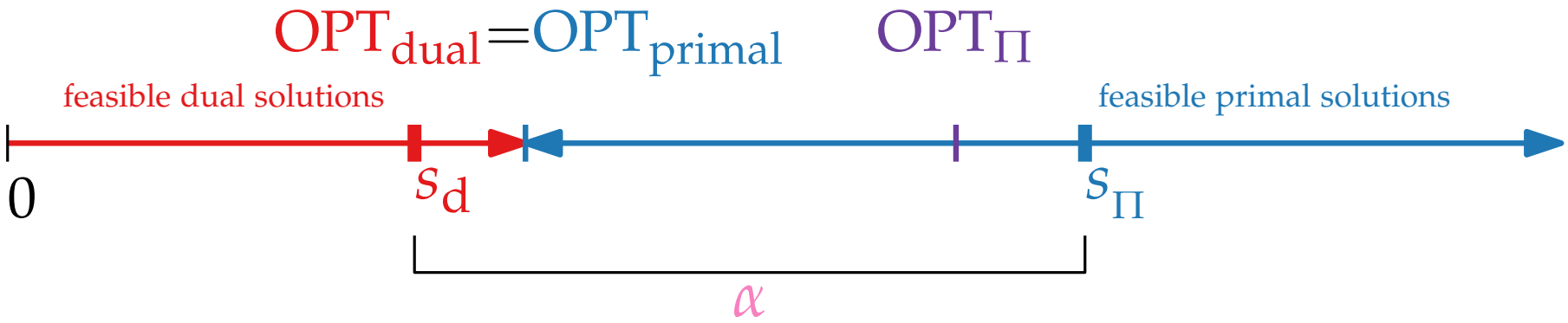


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Advantage: don't need LP-"machinery"; possibly faster, more flexible.

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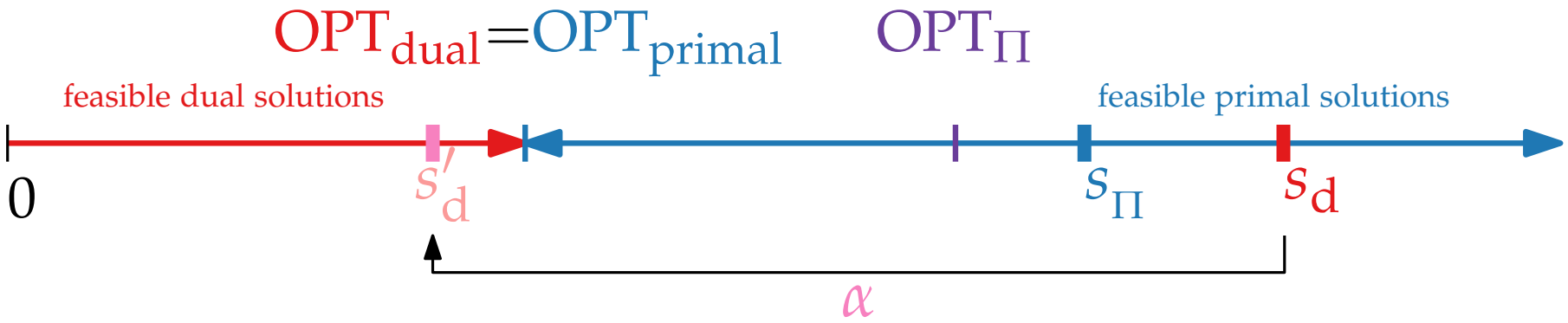


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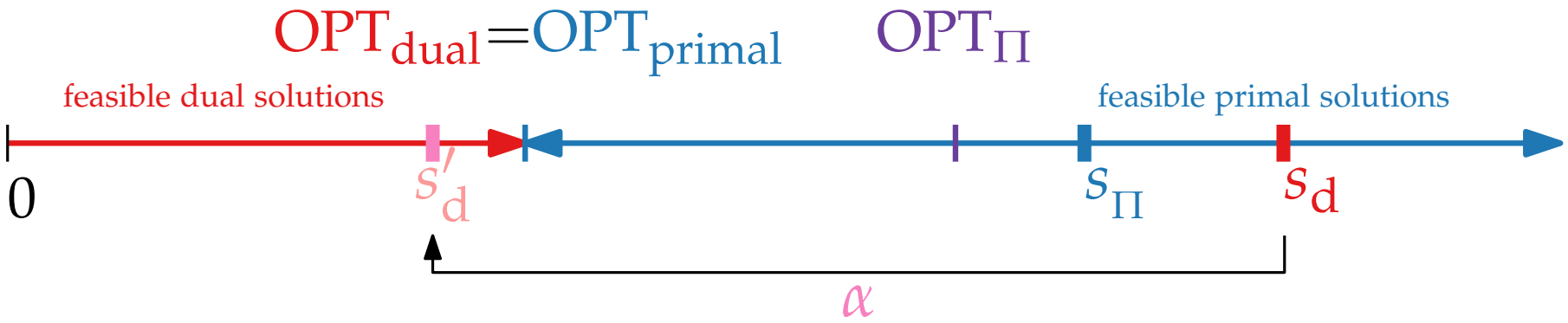


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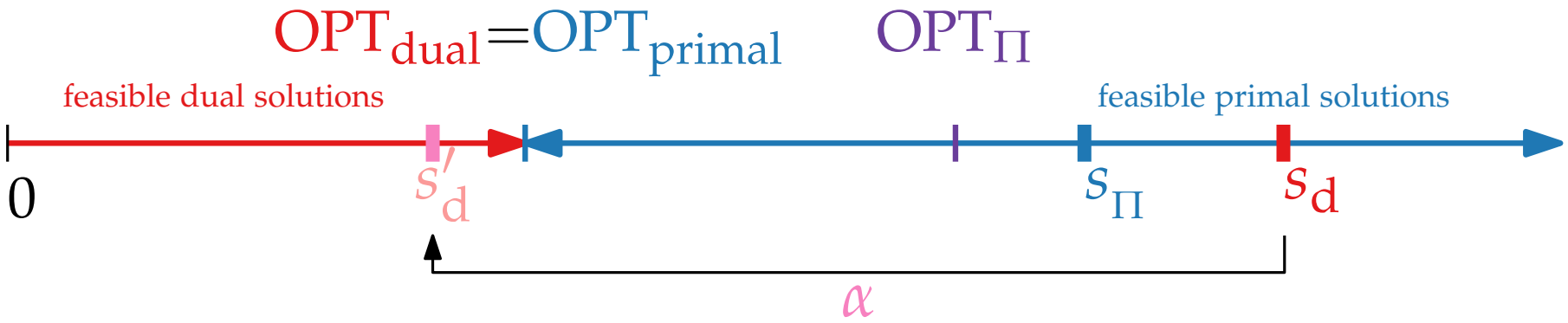


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Scale the dual variables $\rightsquigarrow$ feasible dual solution $s'_{\mathrm{d}}$.

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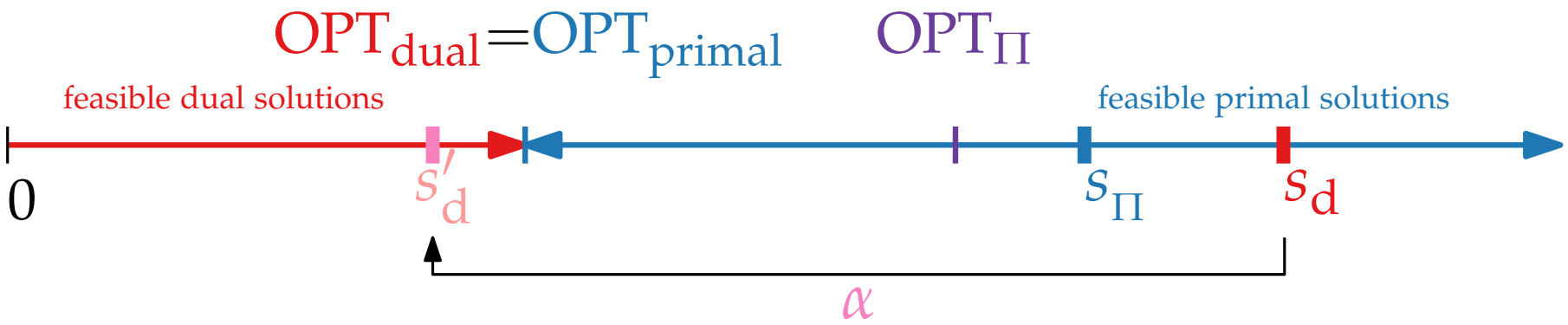


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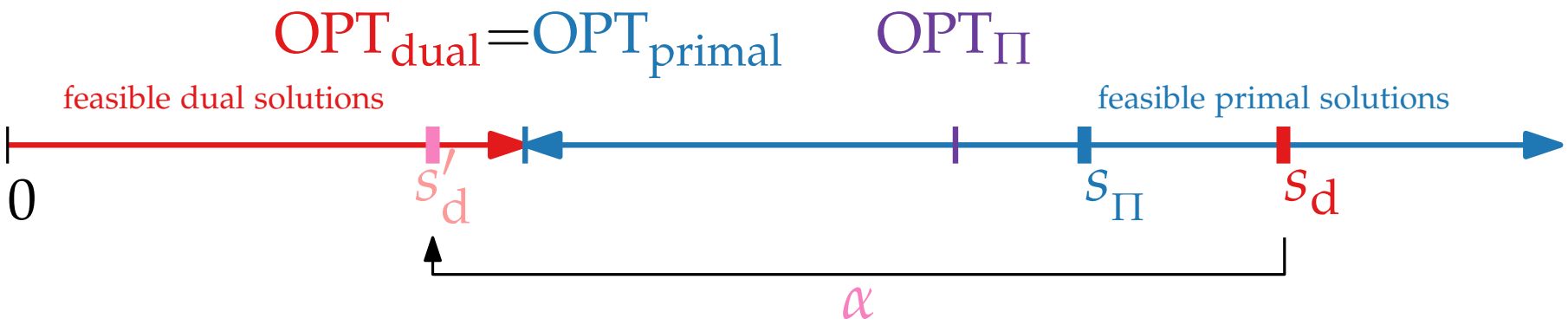


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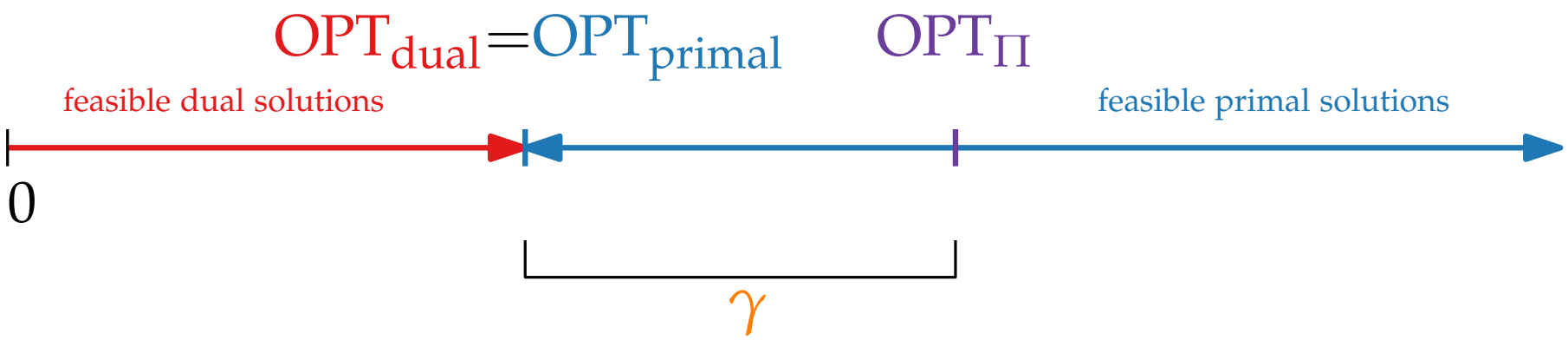
Scale the dual variables $\rightsquigarrow$ feasible dual solution $s'_d$.

$\Rightarrow \text{obj}(s_{\Pi})/\alpha \leq \text{obj}(s_d)/\alpha = \text{obj}(s'_d) \leq OPT_{dual} \leq OPT_{\Pi}$

$\Rightarrow$ Scaling factor $\alpha$ is approximation factor.

# Integrality Gap

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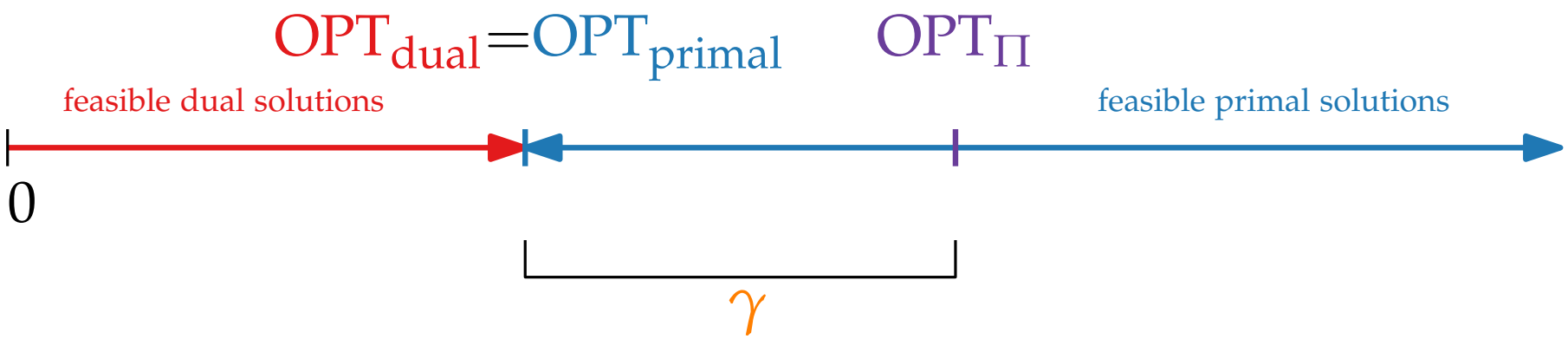


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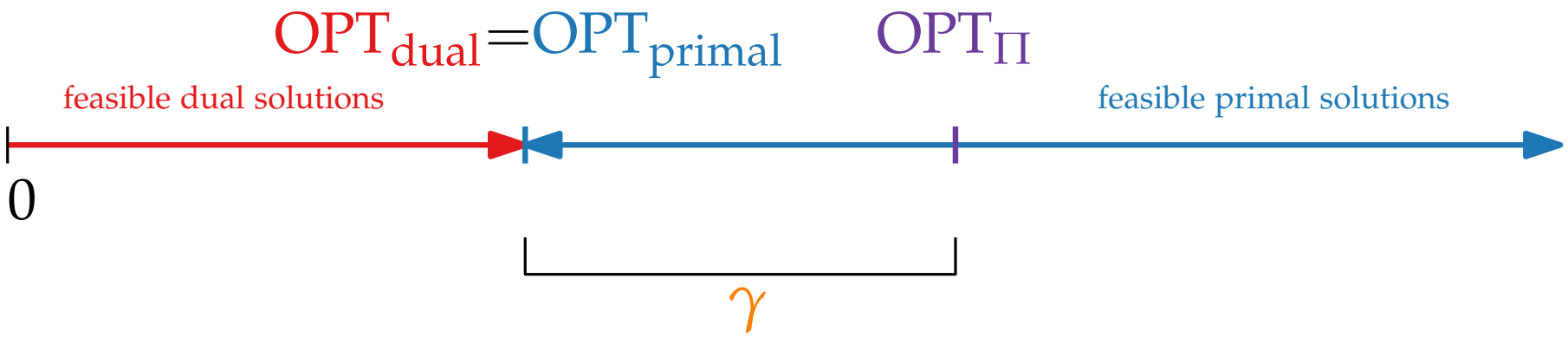


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$$\gamma = \sup_{I} \frac{\mathrm{OPT}_{\Pi}(I)}{\mathrm{OPT}_{\mathrm{primal}}(}$$

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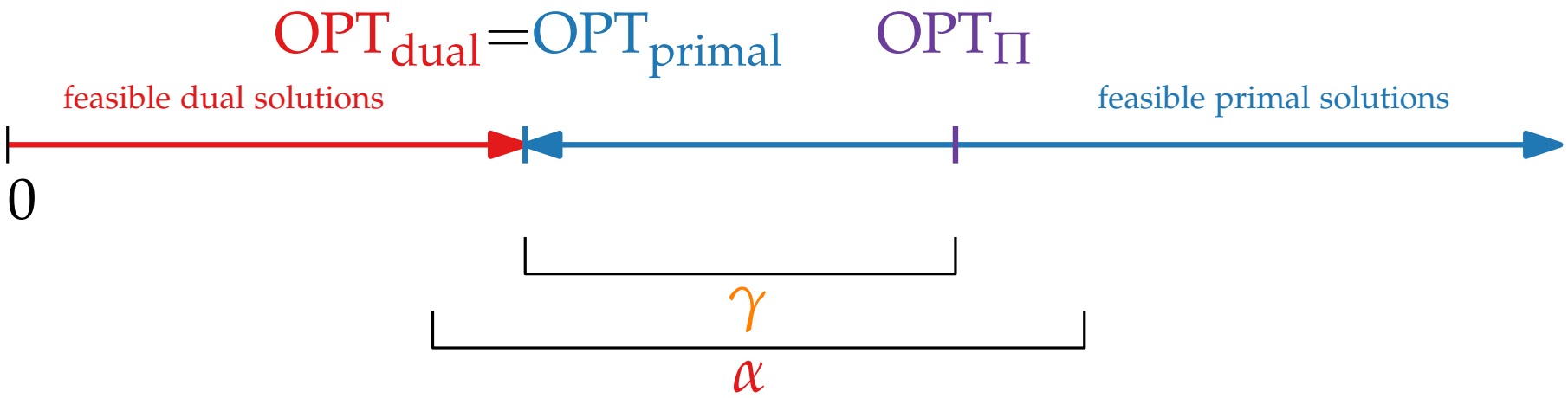


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### Part II:
### SetCover as an ILP

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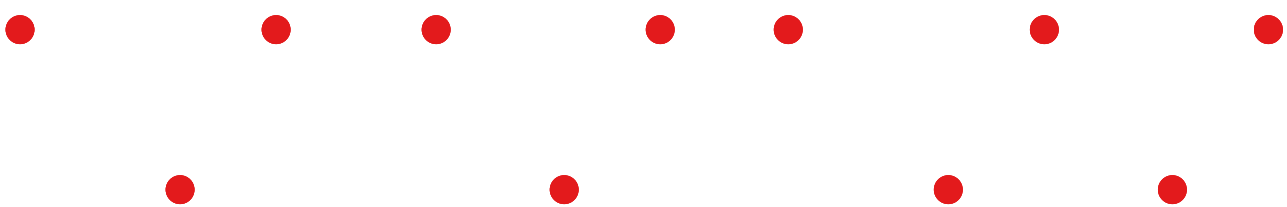
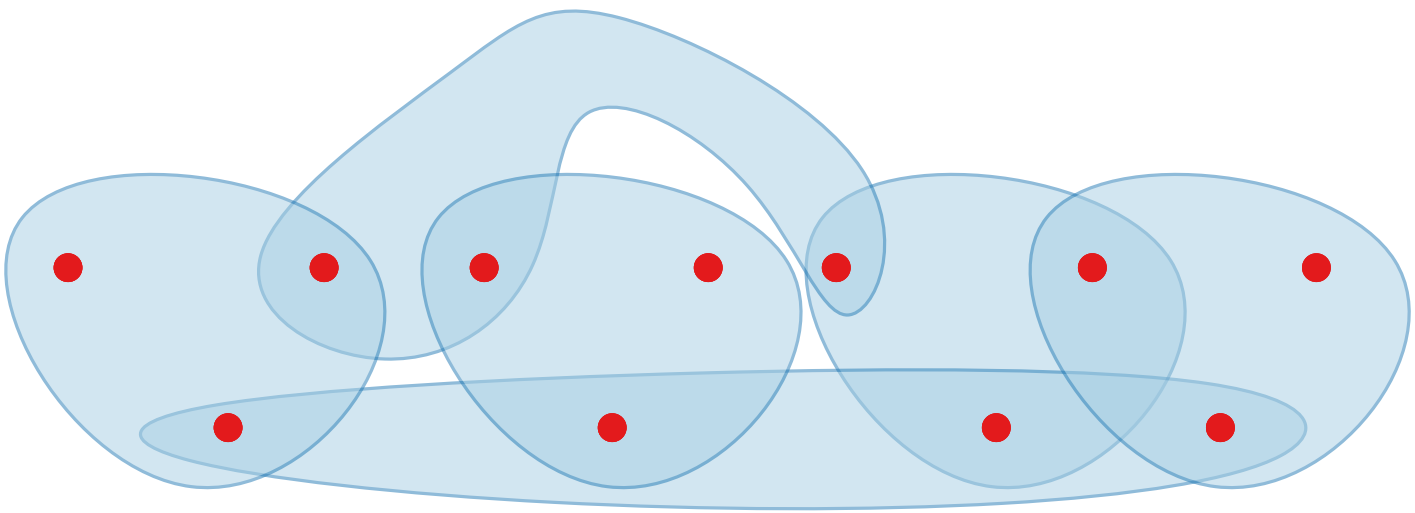
# SetCover as an ILP

Ground set $U$



# SetCover as an ILP

Ground set $U$
Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$



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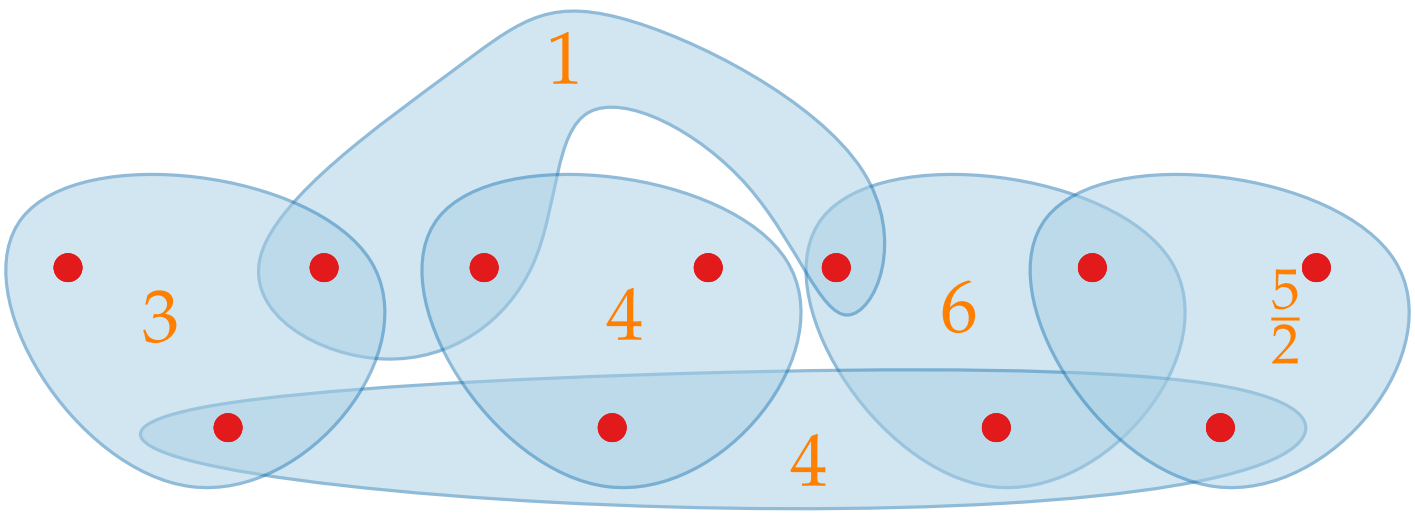

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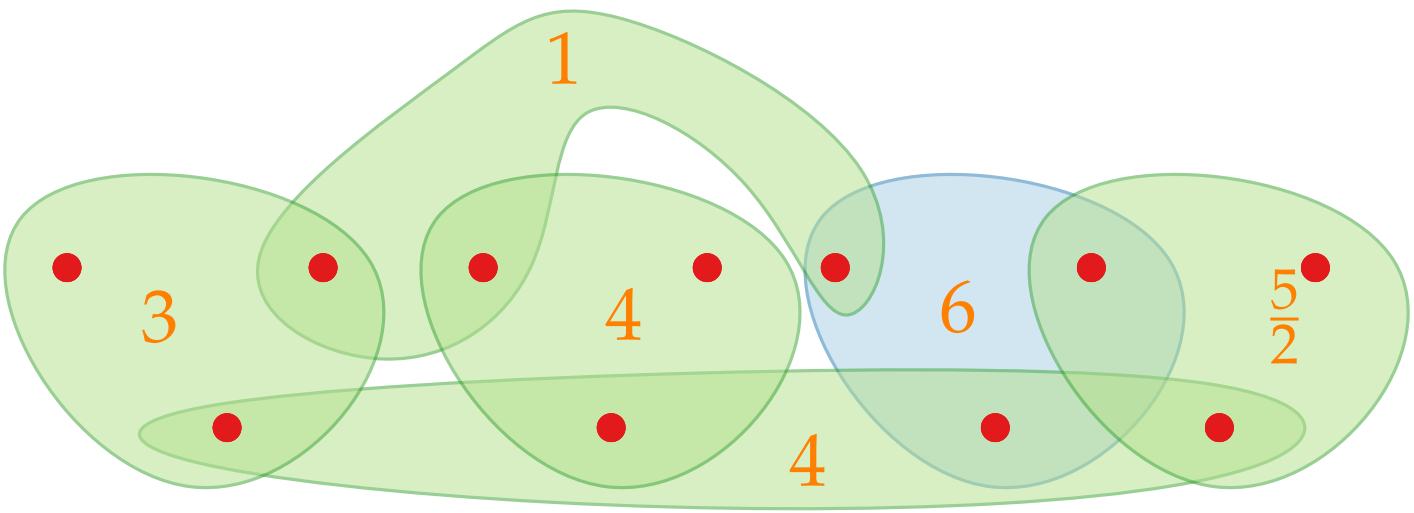


Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of $U$ with minimum cost.

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**minimize**

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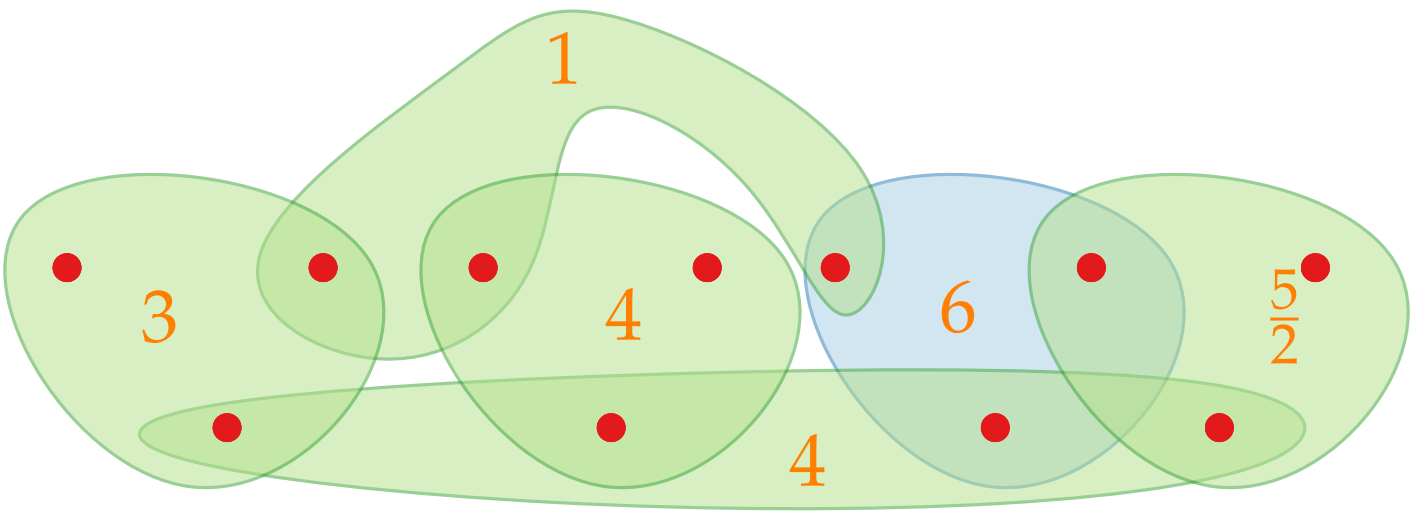


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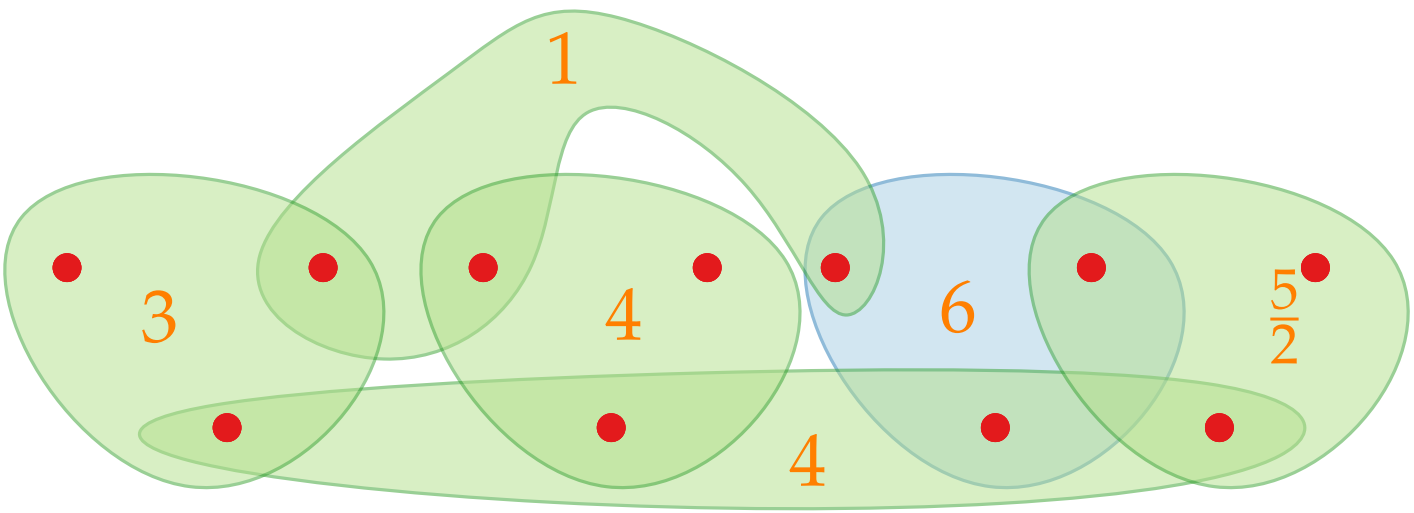


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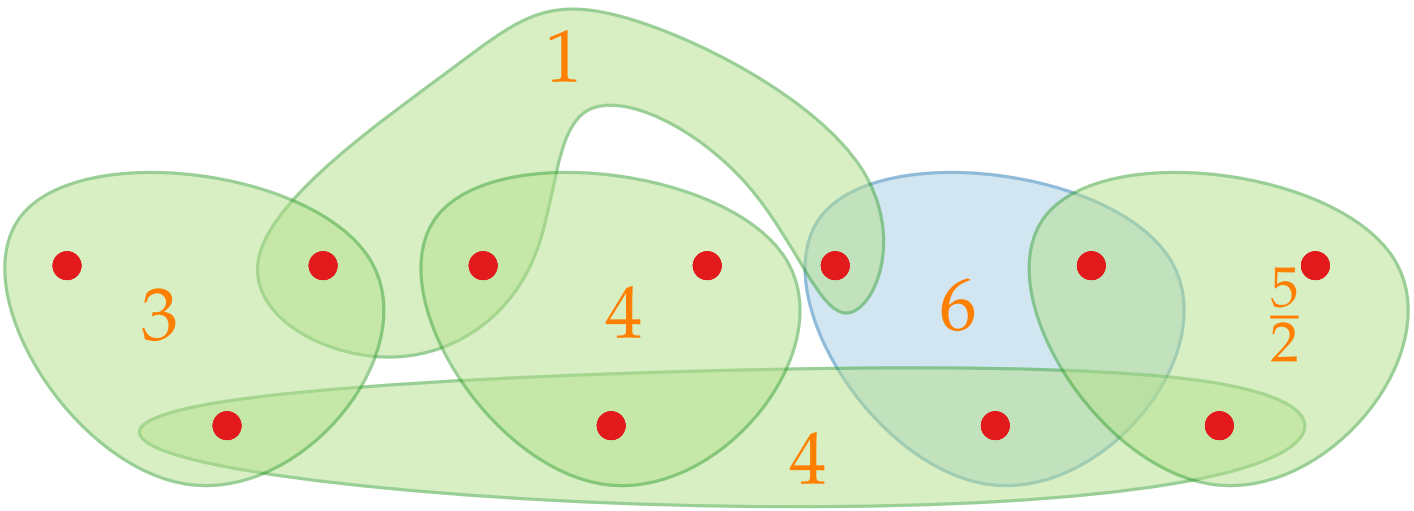


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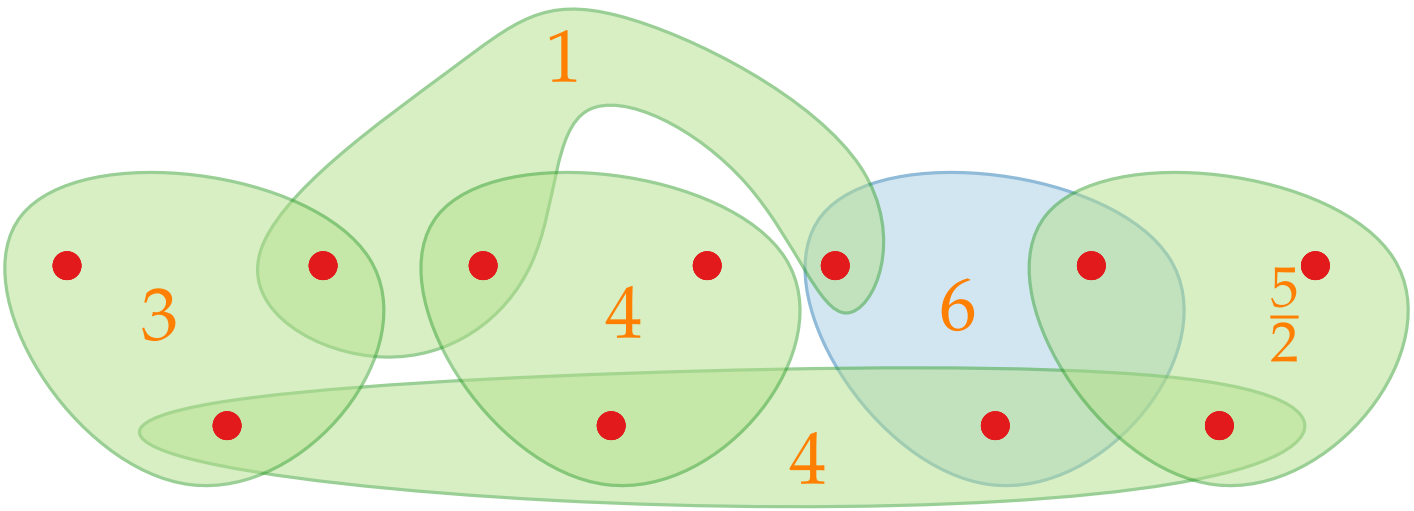


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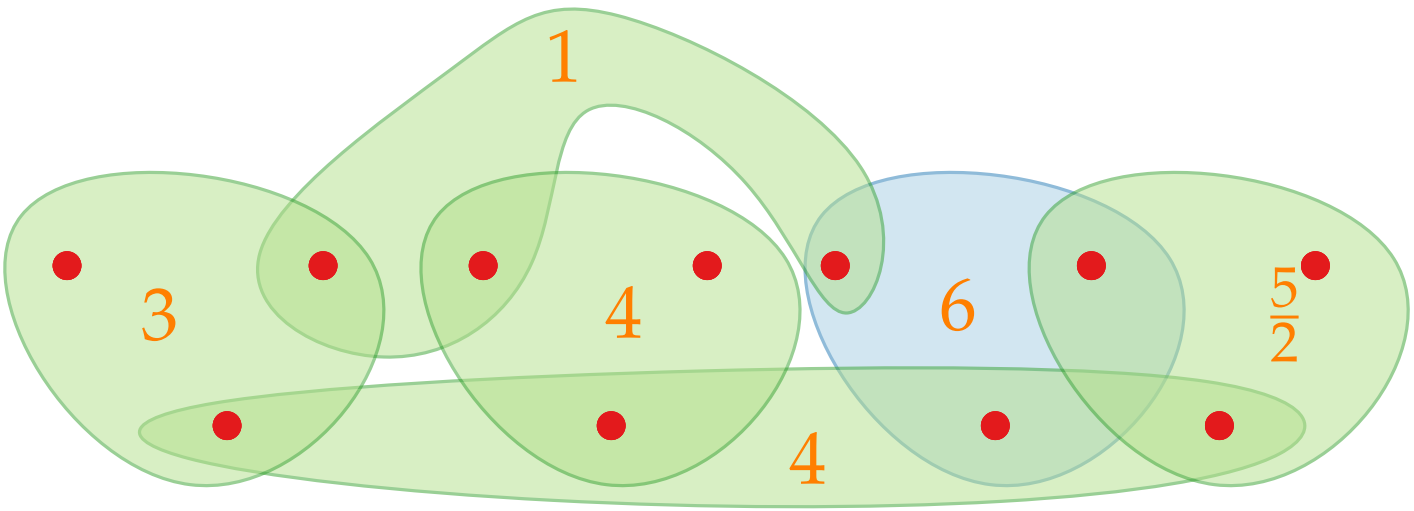


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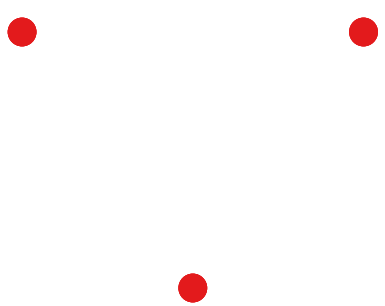

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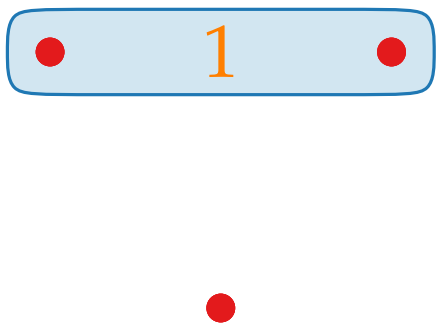

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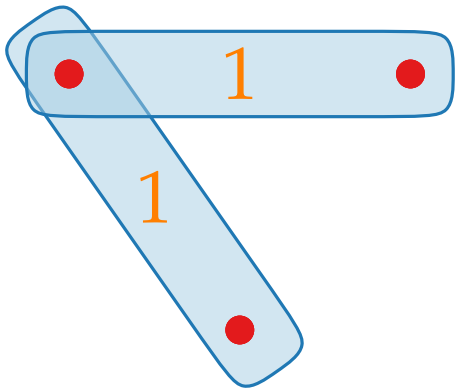

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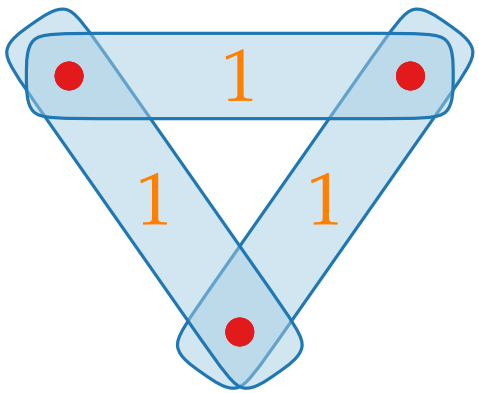

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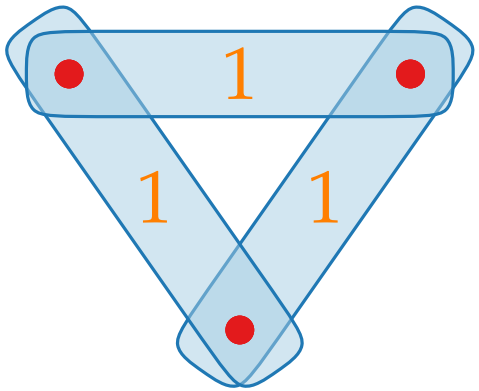


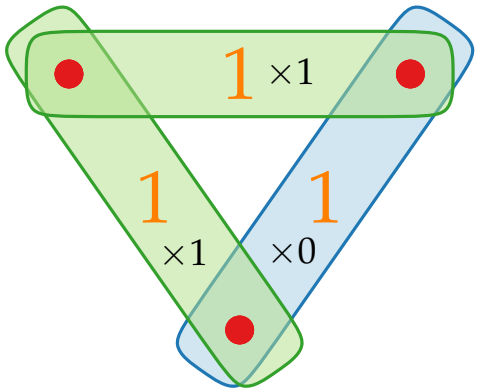


integer: 2

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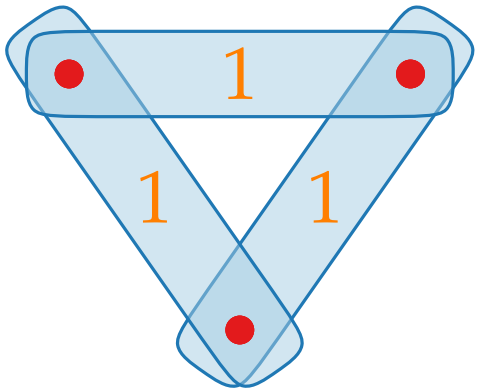


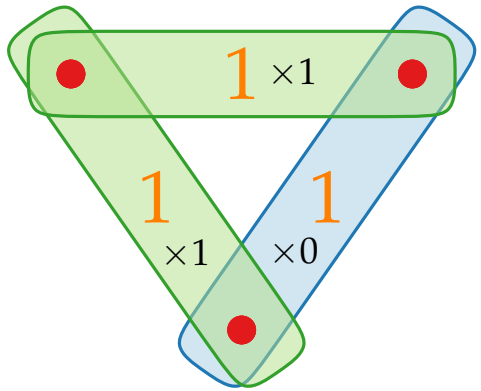


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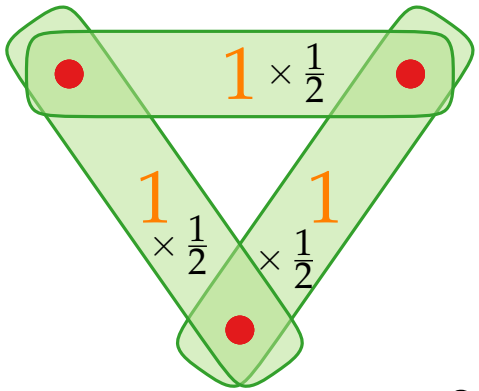


fractional: $\frac{3}{2}$

# SetCover – Dual LP

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& x_S \geq 0 & S \in \mathcal{S}
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# SetCover – Dual LP

**minimize** $\sum_{S \in \mathcal{S}} c_S x_S$

**subject to** $\sum_{S \ni u} x_S \geq 1 \qquad u \in U$

$x_S \geq 0 \qquad S \in \mathcal{S}$

**maximize**

**subject to**

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$x_S \geq 0 \qquad S \in \mathcal{S}$

**maximize**

**subject to**

$y_u \geq 0 \qquad u \in U$

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$$\begin{aligned} \textbf{maximize} \quad & \sum_{u \in U} y_u \\ \textbf{subject to} \quad & \\ & y_u \geq 0 \qquad u \in U \end{aligned}$$

# SetCover – Dual LP

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

$$\begin{aligned} \textbf{maximize} \quad & \sum_{u \in U} y_u \\ \textbf{subject to} \quad & \sum_{u \in S} y_u \leq c_S \qquad S \in \mathcal{S} \\ & y_u \geq 0 \qquad u \in U \end{aligned}$$

# Approximation Algorithms

## Lecture 5:
## LP-based Approximation Algorithms for SetCover

### Part III:
### SetCover via LP-Rounding

Philipp Kindermann

Summer Semester 2020

# LP-Rounding: Approach I

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

LP-Rounding-One($U, \mathcal{S}, c$)

# LP-Rounding: Approach I

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

LP-Rounding-One($U, \mathcal{S}, c$)

Compute optimal solution $x$ for LP-Relaxation.
Round each $x_S$ with $x_S > 0$ to 1.

# LP-Rounding: Approach I

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

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Generates a valid solution

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Compute optimal solution $x$ for LP-Relaxation.
Round each $x_S$ with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large

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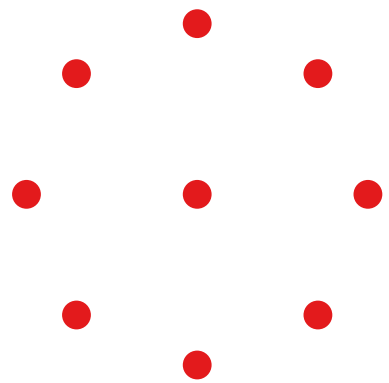
Compute optimal solution $x$ for LP-Relaxation.
Round each $x_S$ with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large



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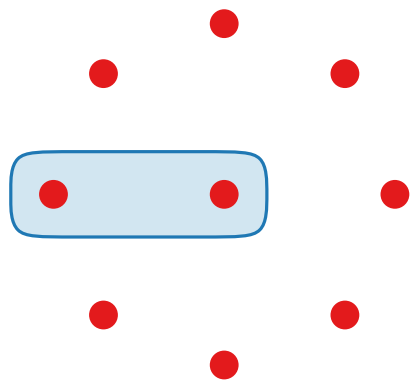

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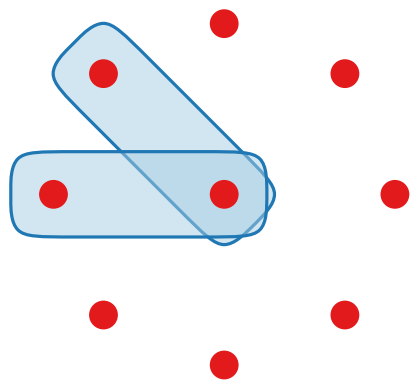

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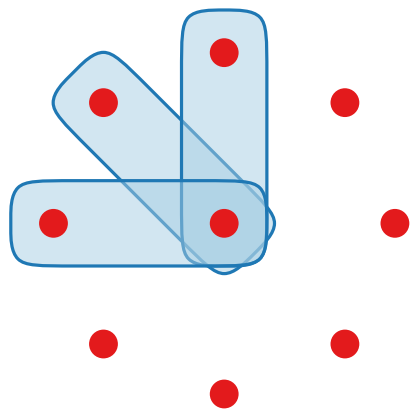

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Generates a valid solution

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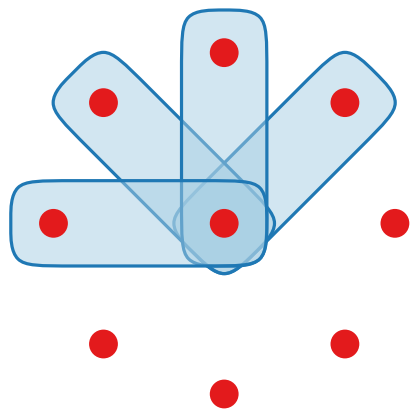

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LP-Rounding-One($U, \mathcal{S}, c$)

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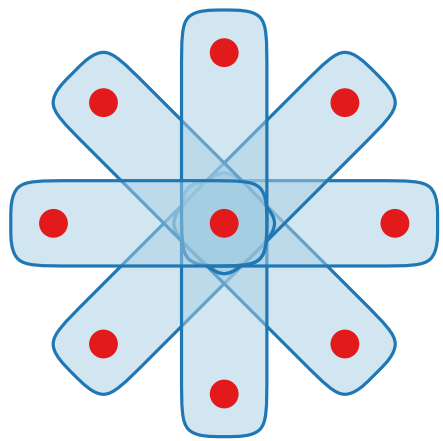
Round each $x_S$ with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large



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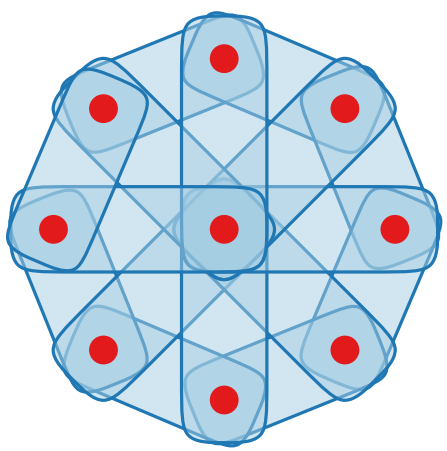
Round each $x_S$ with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large



# LP-Rounding: Approach I

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Round each $x_S$ with $x_S > 0$ to 1.

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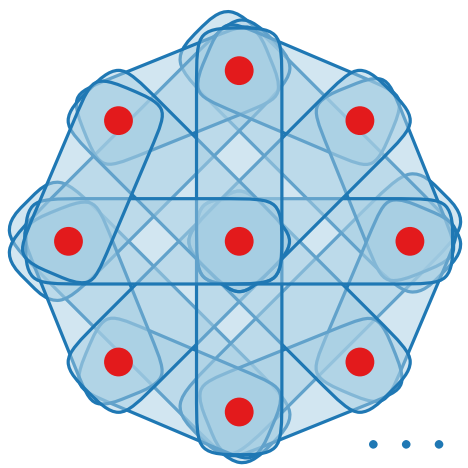


# LP-Rounding: Approach I

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

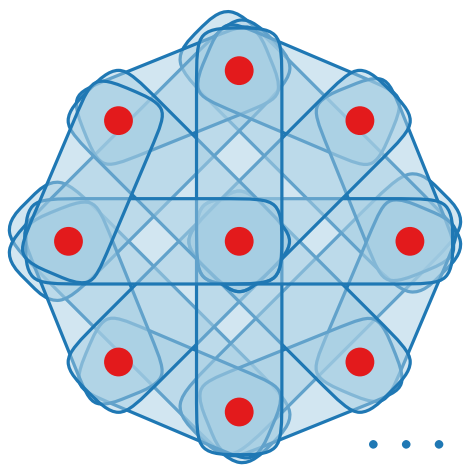
LP-Rounding-One($U, \mathcal{S}, c$)

Compute optimal solution $x$ for LP-Relaxation.
Round each $x_S$ with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large

Use frequency $h$



# LP-Rounding: Approach II

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution $x$ for LP-Relaxation.
Round each $x_S$ with $x_S \geq 1/h$ to 1; remaining to 0.

Let $h$ be the frequency of (number of sets containing) the most frequent element.

# LP-Rounding: Approach II

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution $x$ for LP-Relaxation.
Round each $x_S$ with $x_S \geq 1/h$ to 1; remaining to 0.

Let $h$ be the frequency of (number of sets containing) the most frequent element.

**Theorem.** LP-Rounding-Two is a factor-$h$-approximation algorithm for SetCover.

# Approximation Algorithms

## Lecture 5:
## LP-based Approximation Algorithms for SetCover

### Part IV:
### SetCover via Primal-Dual Schema

Philipp Kindermann

Summer Semester 2020

# Complementary Slackness

| | | | |
|---|---|---|---|
| **minimize** | $c^\top x$ | | |
| **subject to** | $Ax$ | $\geq$ | $b$ |
| | $x$ | $\geq$ | $0$ |

| | | | |
|---|---|---|---|
| **maximize** | $b^\top y$ | | |
| **subject to** | $A^\top y$ | $\leq$ | $c$ |
| | $y$ | $\geq$ | $0$ |

**Theorem.** Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then $x$ and $y$ are optimal if and only if the following conditions are met:

**Primal CS**:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

**Dual CS**:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

# Relaxing Complementary Slackness

| | | | |
|---|---|---|---|
| **minimize** | $c^\top x$ | | |
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For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

# Relaxing Complementary Slackness

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| **maximize** | $b^\top y$ | | |
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~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or ~~$\sum_{i=1}^{m} a_{ij} y_i = c_j$~~

$$c_j / \alpha \leq \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

**Dual CS**:

For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

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| **minimize** | $c^\top x$ | | |
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| | | | |
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~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or ~~$\sum_{i=1}^{m} a_{ij} y_i = c_j$~~

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~~**Dual CS**~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: either $y_i = 0$ or ~~$\sum_{j=1}^{n} a_{ij} x_j = b_i$~~

$$b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

# Relaxing Complementary Slackness

| | | | |
|---|---|---|---|
| **minimize** | $c^\intercal x$ | | |
| **subject to** | $Ax$ | $\geq$ | $b$ |
| | $x$ | $\geq$ | $0$ |

| | | | |
|---|---|---|---|
| **maximize** | $b^\intercal y$ | | |
| **subject to** | $A^\intercal y$ | $\leq$ | $c$ |
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~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or ~~$\sum_{i=1}^{m} a_{ij} y_i = c_j$~~

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~~**Dual CS**~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: either $y_i = 0$ or ~~$\sum_{j=1}^{n} a_{ij} x_j = b_i$~~

$$b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i \quad \Rightarrow \sum_{j=1}^{n} c_j x_j \leq \alpha\beta \sum_{i=1}^{m} b_i y_i \leq \alpha\beta \cdot \mathrm{OPT}_{\mathrm{LP}}$$

# Primal-Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

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Maintain that the primal solution is integer valued.

# Primal-Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

"Improve" the feasibility of the primal solution...

...and simultaneously the obj. value of the dual solution.

Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer valued.

The feasibility of the primal solution and relaxed CS condition provide an approximation ratio.

# Relaxed CS for SetCover

**minimize** $\sum_{S \in \mathcal{S}} c_S x_S$

**subject to** $\sum_{S \ni u} x_S \geq 1 \qquad u \in U$

$x_S \geq 0 \qquad S \in \mathcal{S}$

**maximize** $\sum_{u \in U} y_u$

**subject to** $\sum_{u \in S} y_u \leq c_S \qquad S \in \mathcal{S}$

$y_u \geq 0 \qquad u \in U$

# Relaxed CS for SetCover

| | | |
|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | |
| **subject to** | $\sum_{S \ni u} x_S \geq 1$ | $u \in U$ |
| | $x_S \geq 0$ | $S \in \mathcal{S}$ |

| | | |
|---|---|---|
| **maximize** | $\sum_{u \in U} y_u$ | |
| **subject to** | $\sum_{u \in S} y_u \leq c_S$ | $S \in \mathcal{S}$ |
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**(Unrelaxed) primal CS:**

# Relaxed CS for SetCover

| | | |
|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | |
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|---|---|---|
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**(Unrelaxed) primal CS**: $x_S \neq 0 \Rightarrow$

# Relaxed CS for SetCover

| | | | | | |
|---|---|---|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | **maximize** | $\sum_{u \in U} y_u$ | |
| **subject to** | $\sum_{S \ni u} x_S \geq 1$ | $u \in U$ | **subject to** | $\sum_{u \in S} y_u \leq c_S$ | $S \in \mathcal{S}$ |
| | $x_S \geq 0$ | $S \in \mathcal{S}$ | | $y_u \geq 0$ | $u \in U$ |

**(Unrelaxed) primal CS**: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

# Relaxed CS for SetCover

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| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | |
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critical set

**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

# Relaxed CS for SetCover

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**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

critical set

only chooses critical sets

# Relaxed CS for SetCover

| | | | | | |
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| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | **maximize** | $\sum_{u \in U} y_u$ | |
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**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

critical set

only chooses critical sets

**Relaxed dual CS:**

# Relaxed CS for SetCover

| | | | |
|---|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | |
| **subject to** | $\sum_{S \ni u} x_S \geq 1$ | $u \in U$ | |
| | $x_S \geq 0$ | $S \in \mathcal{S}$ | |

| | | | |
|---|---|---|---|
| **maximize** | $\sum_{u \in U} y_u$ | | |
| **subject to** | $\sum_{u \in S} y_u \leq c_S$ | $S \in \mathcal{S}$ | |
| | $y_u \geq 0$ | $u \in U$ | |

**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

critical set

only chooses critical sets

**Relaxed dual CS:** $y_u \neq 0 \Rightarrow$

# Relaxed CS for SetCover

| | | | | | |
|---|---|---|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | **maximize** | $\sum_{u \in U} y_u$ | |
| **subject to** | $\sum_{S \ni u} x_S \geq 1$ | $u \in U$ | **subject to** | $\sum_{u \in S} y_u \leq c_S$ | $S \in \mathcal{S}$ |
| | $x_S \geq 0$ | $S \in \mathcal{S}$ | | $y_u \geq 0$ | $u \in U$ |

critical set

**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

**Relaxed dual CS:** $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq h$ .

# Relaxed CS for SetCover

| | | | | | |
|---|---|---|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | **maximize** | $\sum_{u \in U} y_u$ | |
| **subject to** | $\sum_{S \ni u} x_S \geq 1$ | $u \in U$ | **subject to** | $\sum_{u \in S} y_u \leq c_S$ | $S \in \mathcal{S}$ |
| | $x_S \geq 0$ | $S \in \mathcal{S}$ | | $y_u \geq 0$ | $u \in U$ |

critical set

**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

**Relaxed dual CS:** $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq h \cdot 1$

# Relaxed CS for SetCover

| | | | | | |
|---|---|---|---|---|---|
| **minimize** | $\sum_{S \in \mathcal{S}} c_S x_S$ | | **maximize** | $\sum_{u \in U} y_u$ | |
| **subject to** | $\sum_{S \ni u} x_S \geq 1$ | $u \in U$ | **subject to** | $\sum_{u \in S} y_u \leq c_S$ | $S \in \mathcal{S}$ |
| | $x_S \geq 0$ | $S \in \mathcal{S}$ | | $y_u \geq 0$ | $u \in U$ |

critical set

**(Unrelaxed) primal CS:** $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

trivial for binary $x$

**Relaxed dual CS:** $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq h \cdot 1$

# Primal-Dual-Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0, y \leftarrow 0$

**repeat**

|

|

|

**until** all elements are covered.

**return** $x$

# Primal-Dual-Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0, y \leftarrow 0$

**repeat**

  Select an uncovered element $u$.

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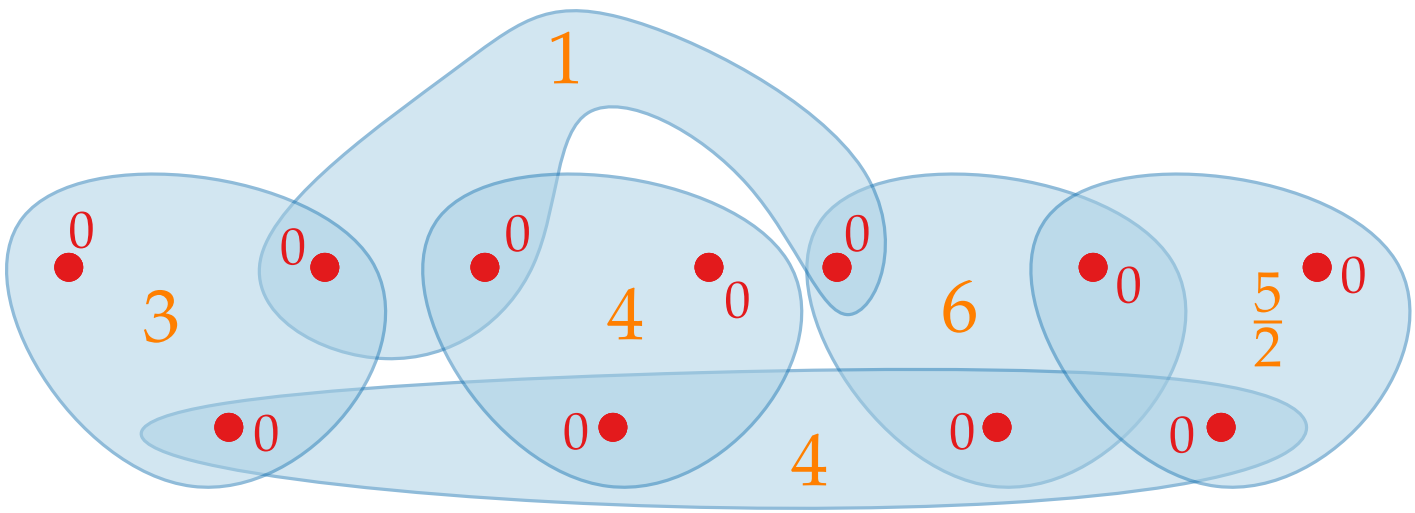

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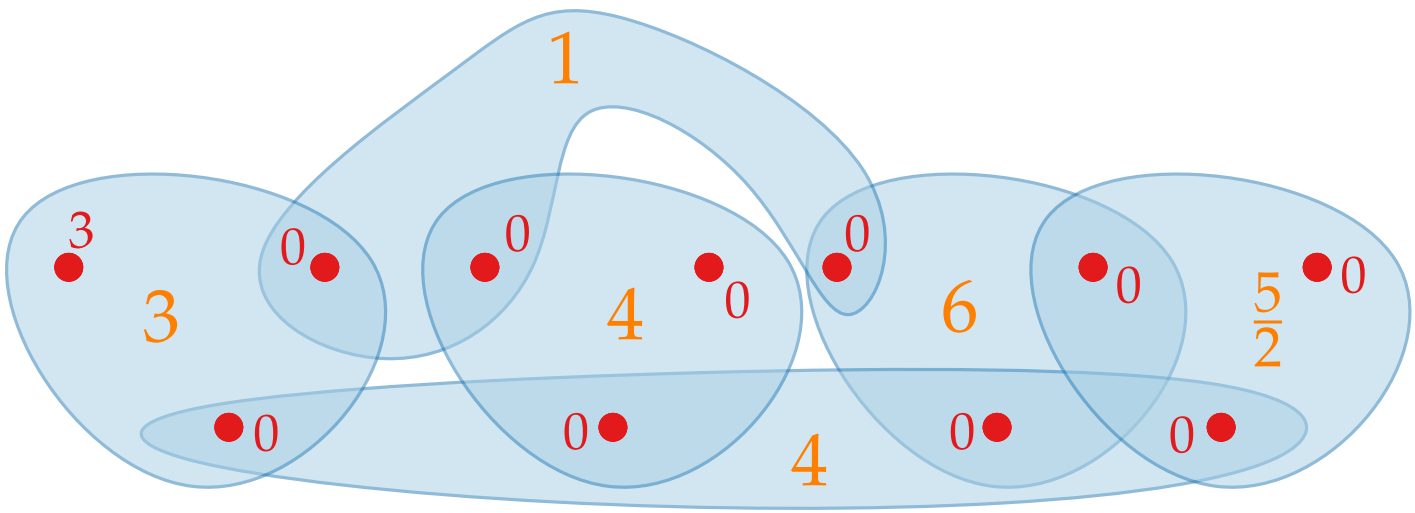

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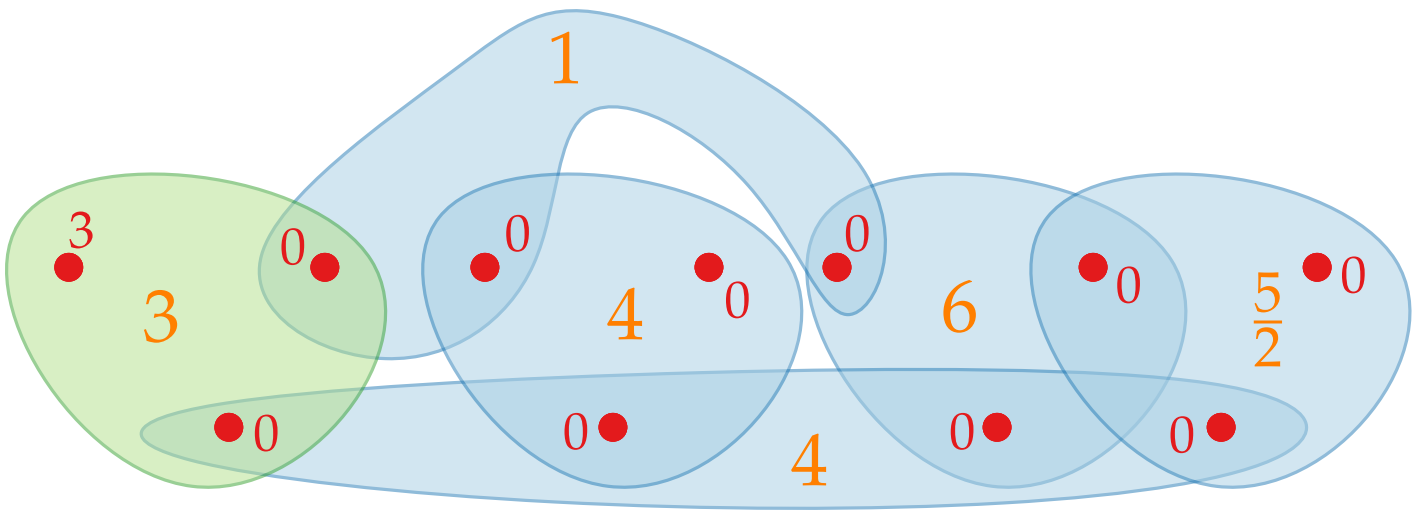

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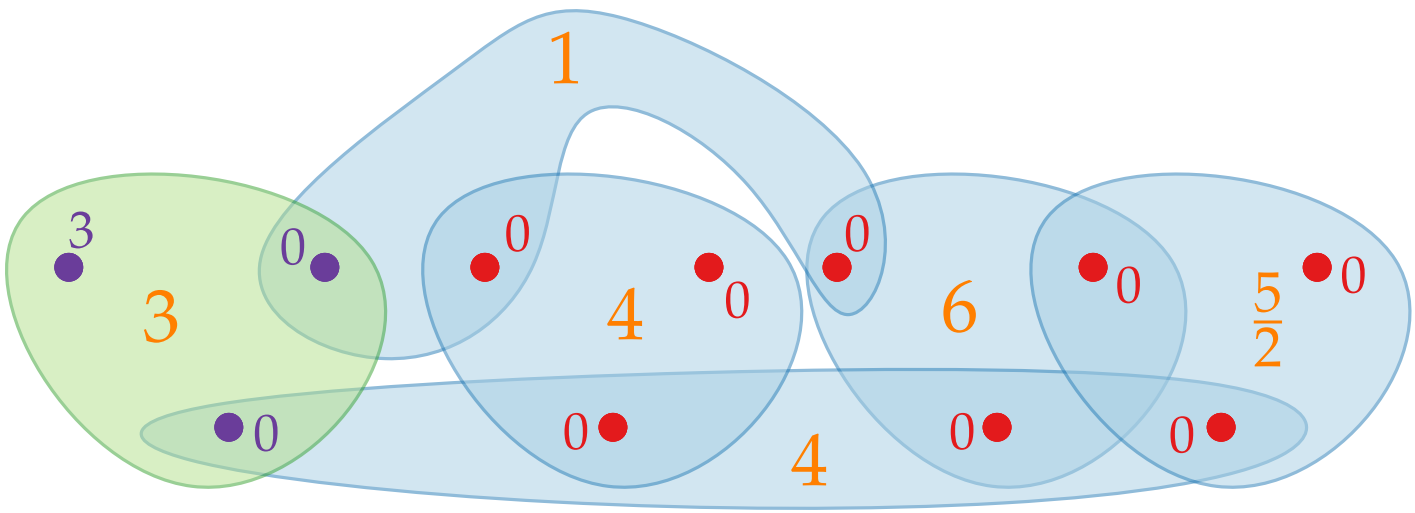

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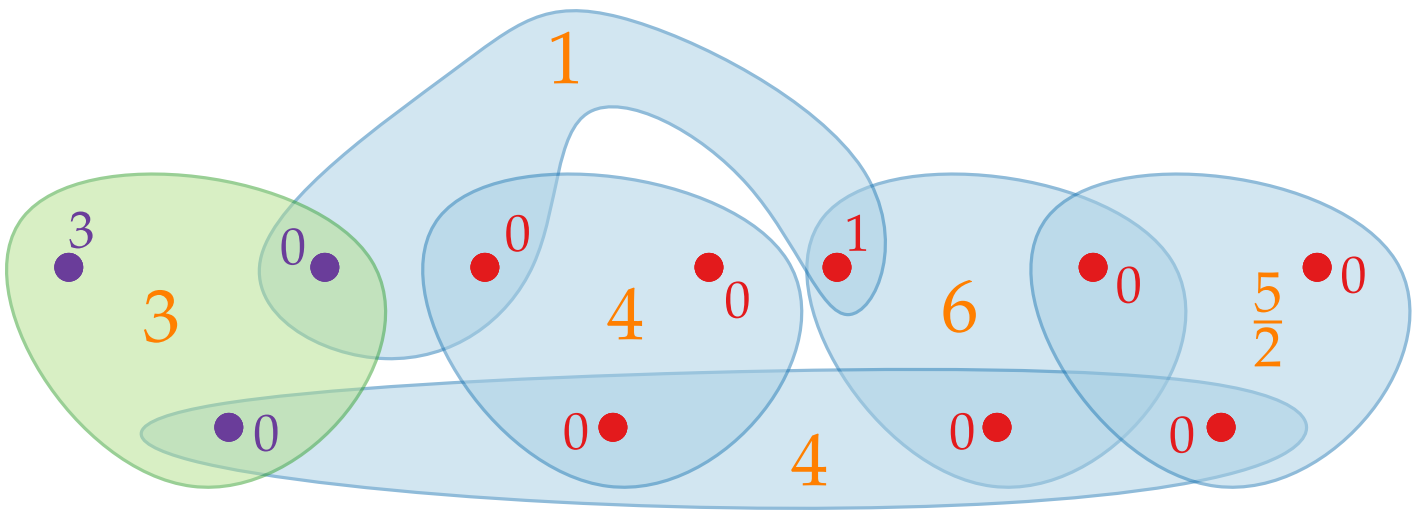

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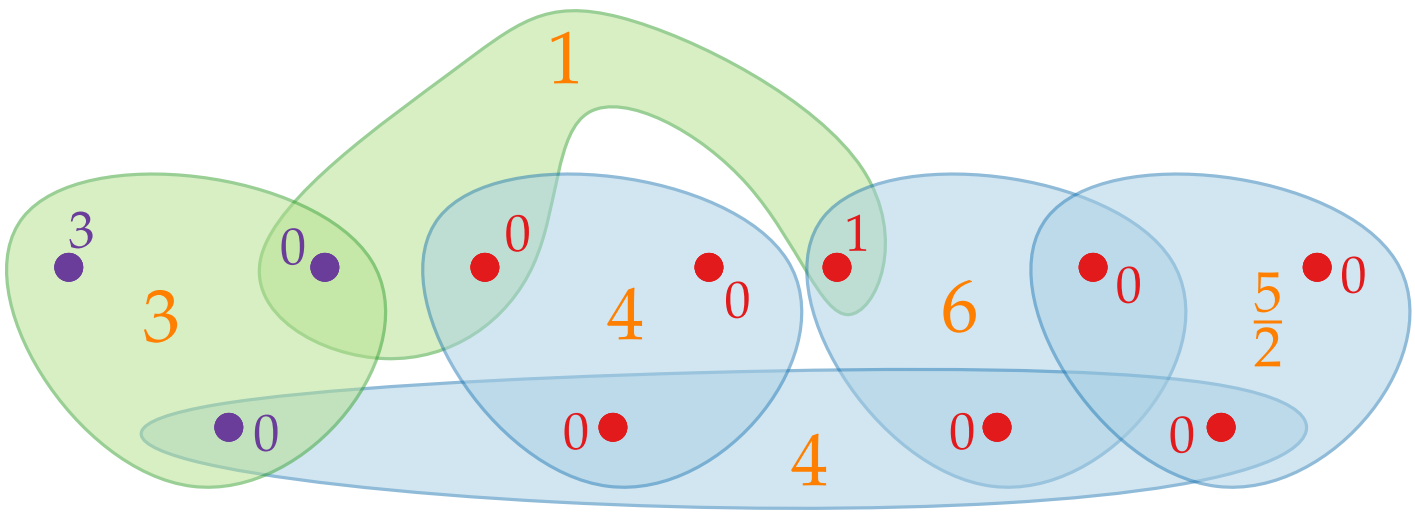

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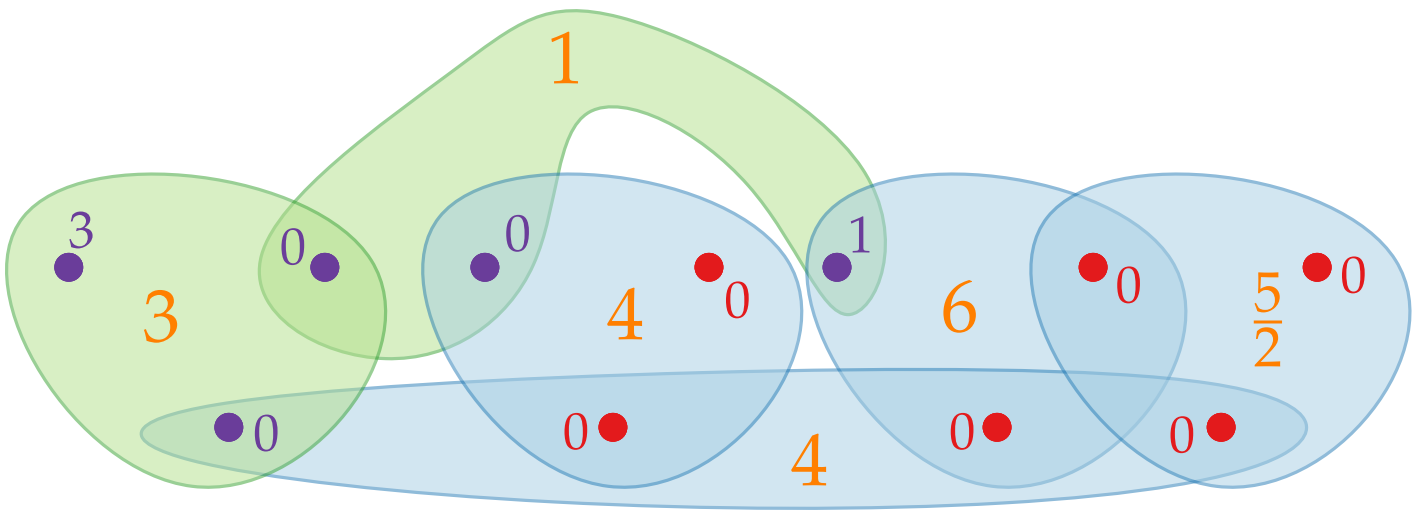

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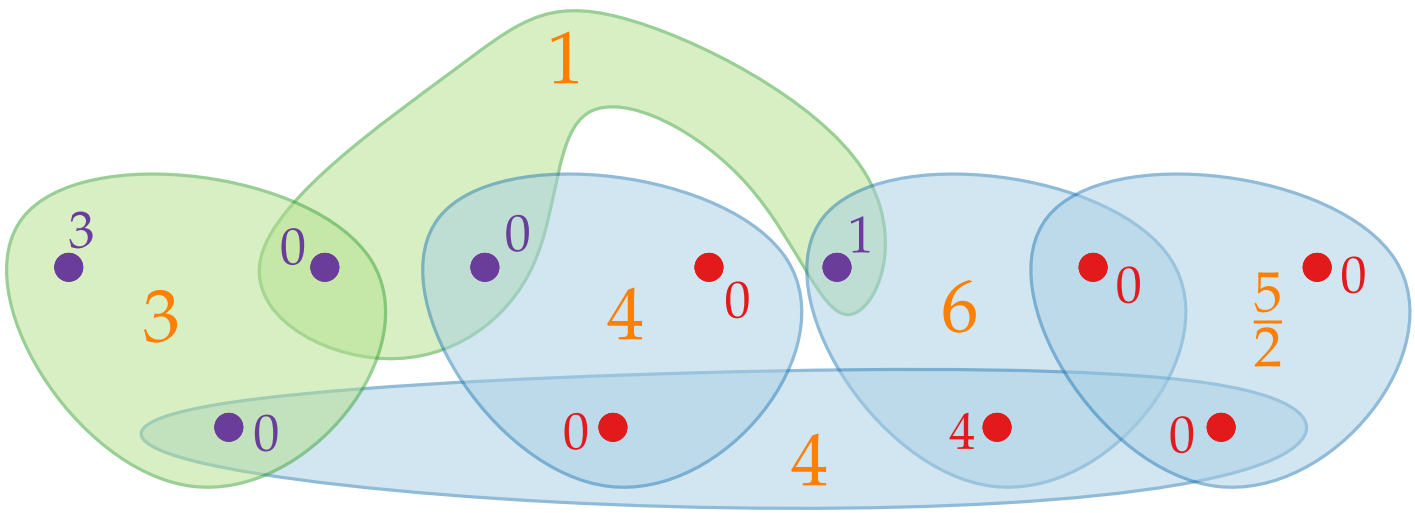

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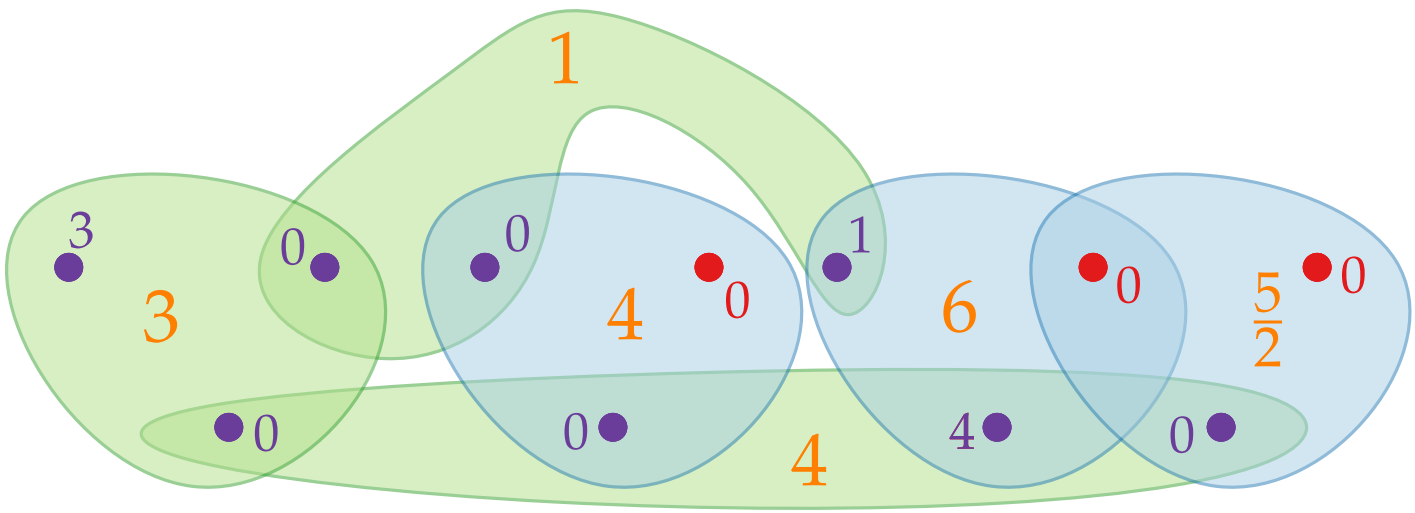

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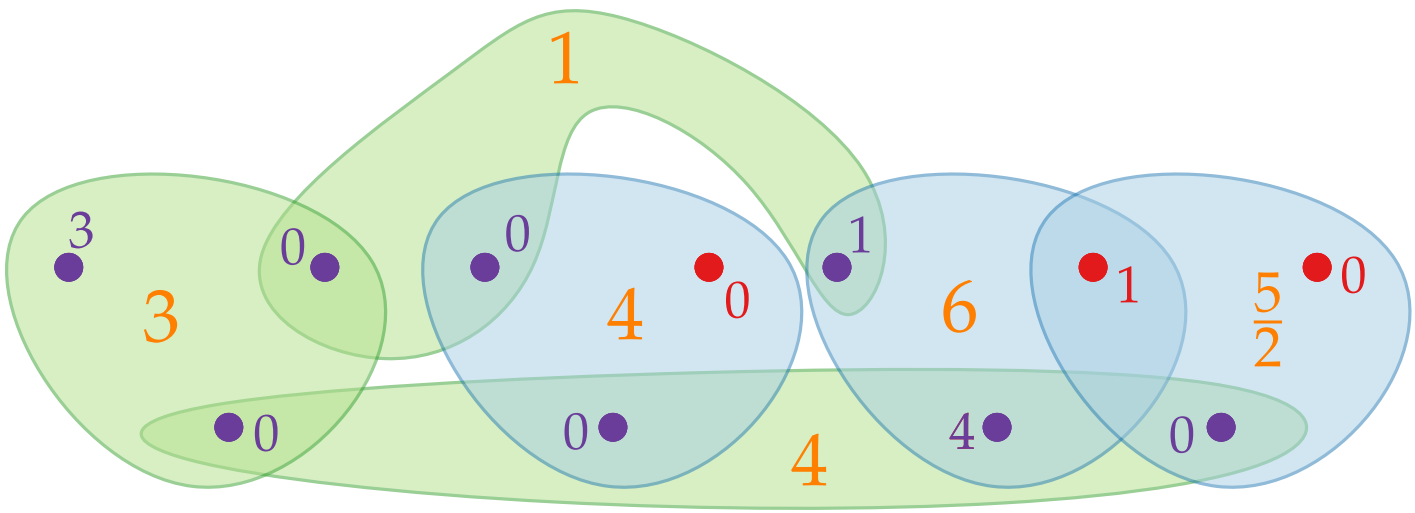

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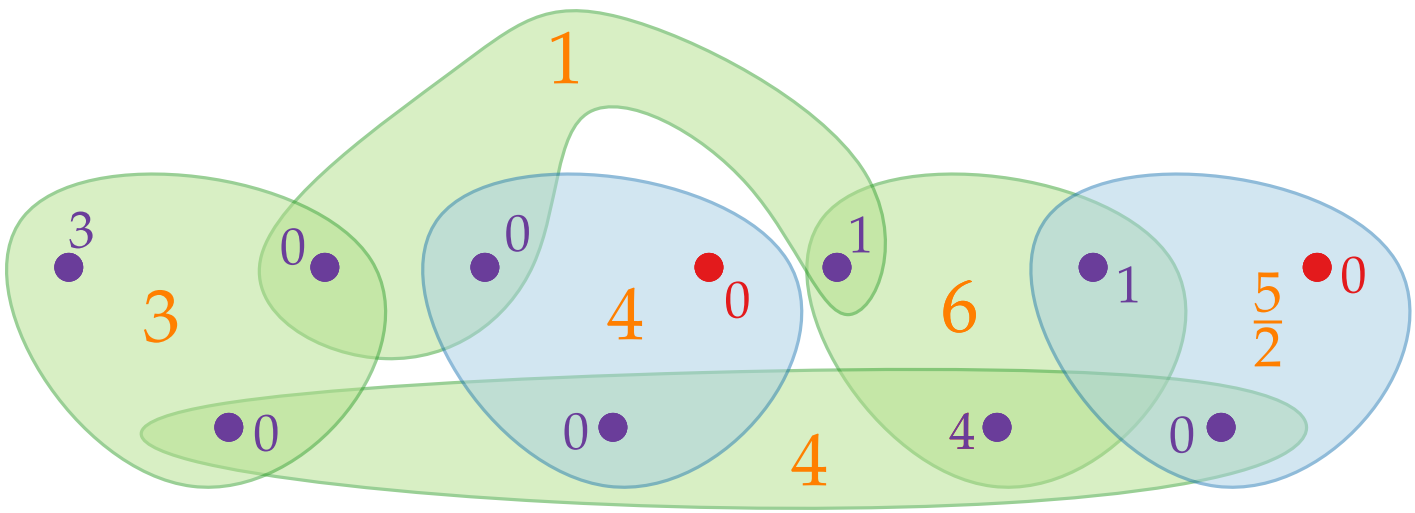

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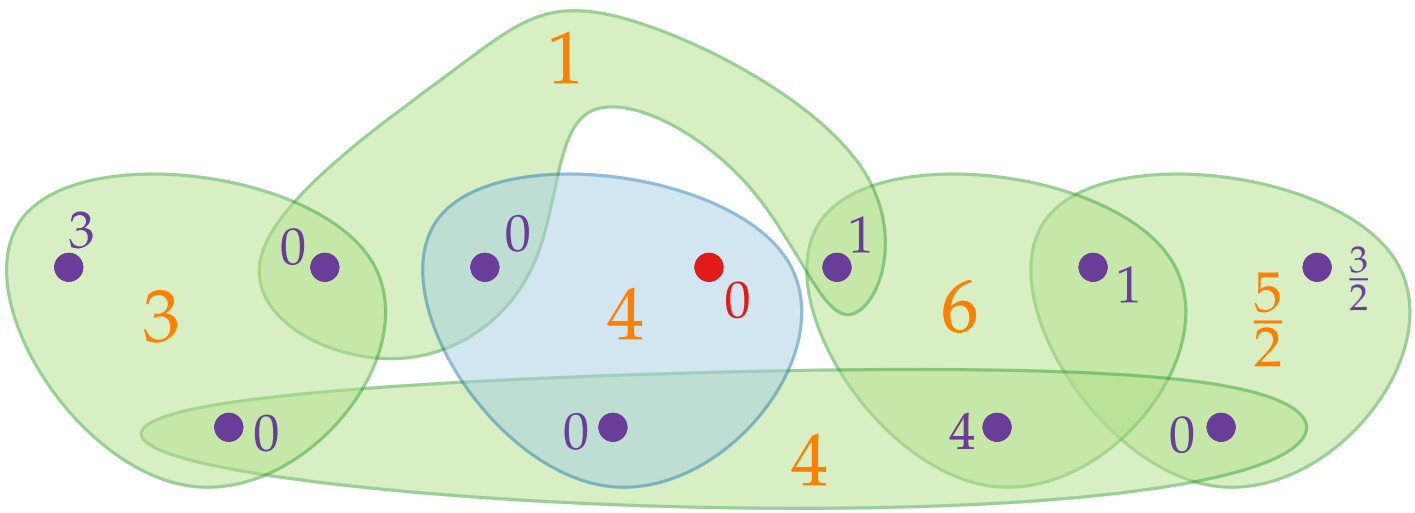

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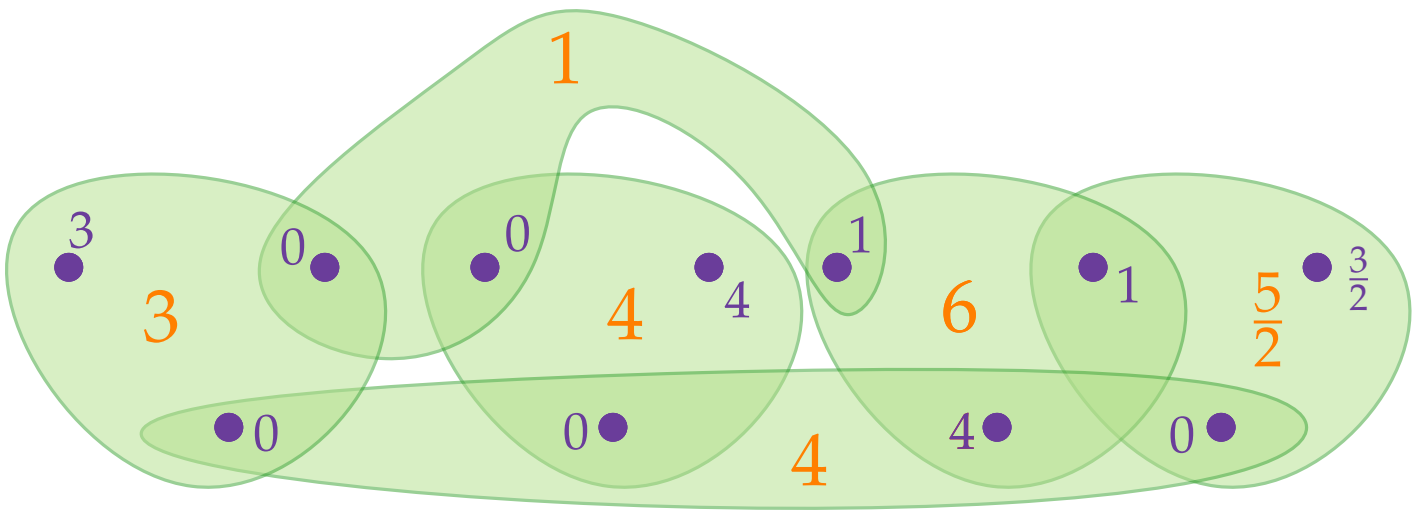

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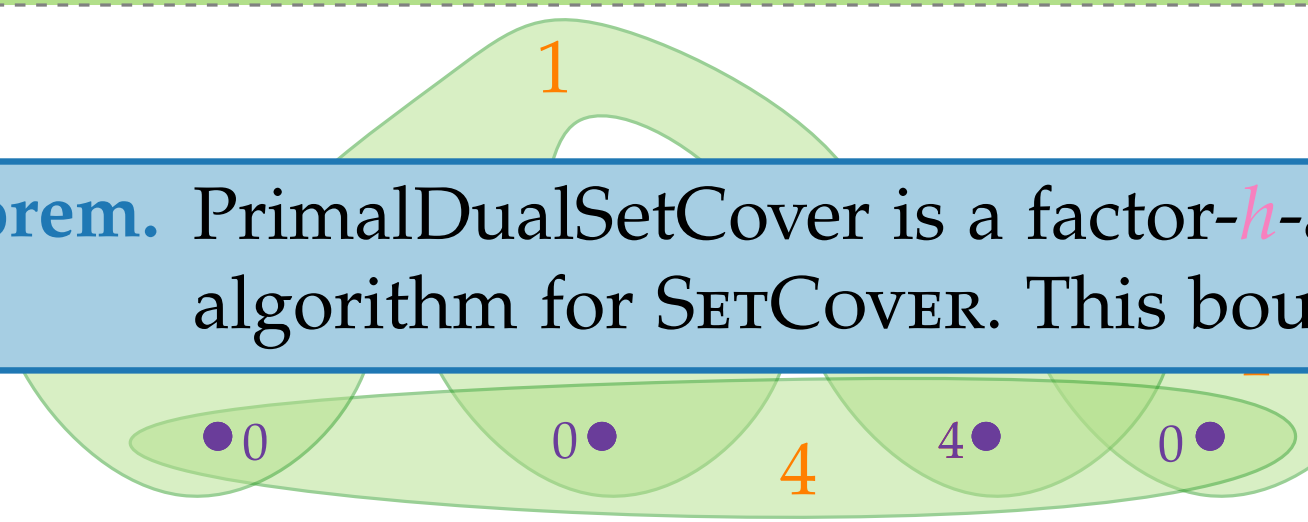
**Theorem.** PrimalDualSetCover is a factor-$h$-approximation algorithm for SetCover. This bound is tight.



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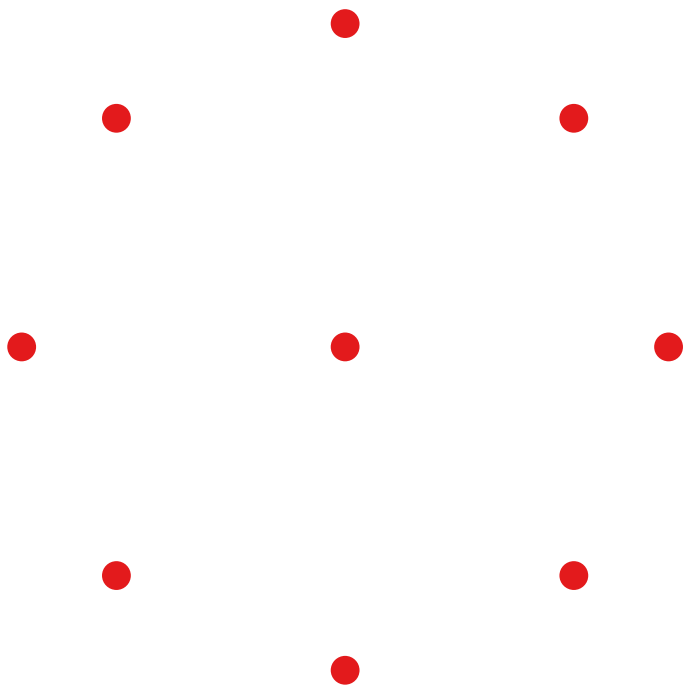

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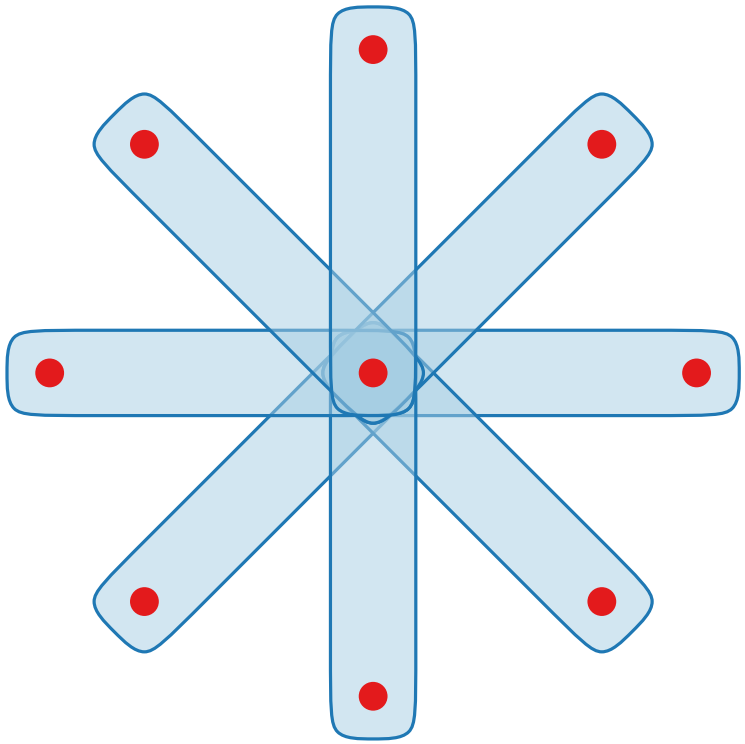

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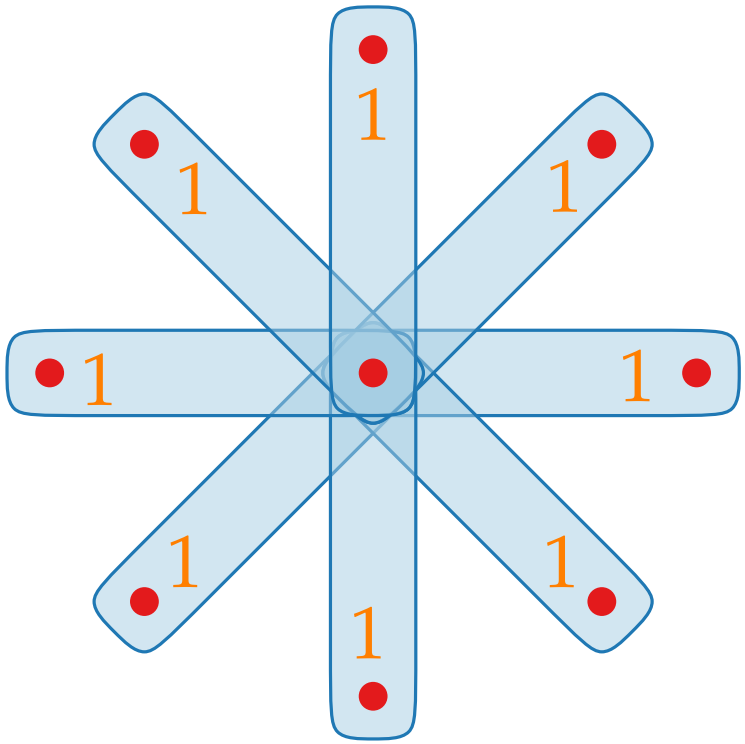

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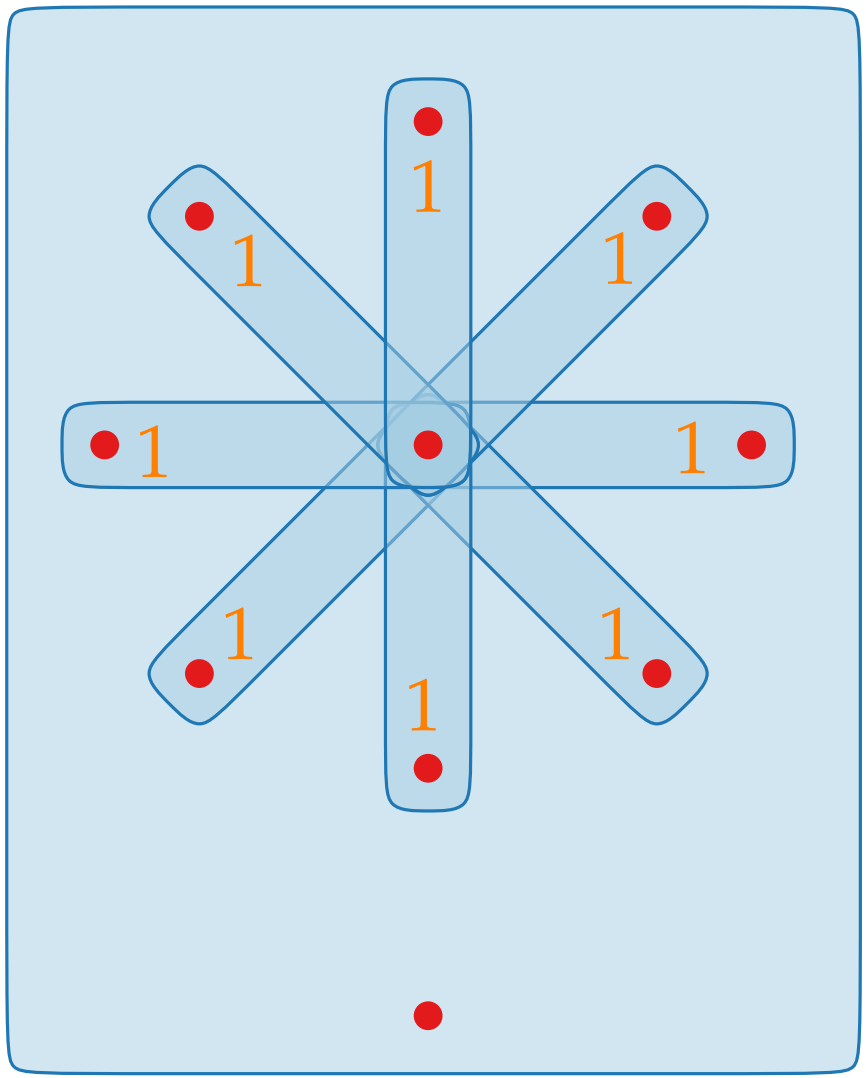

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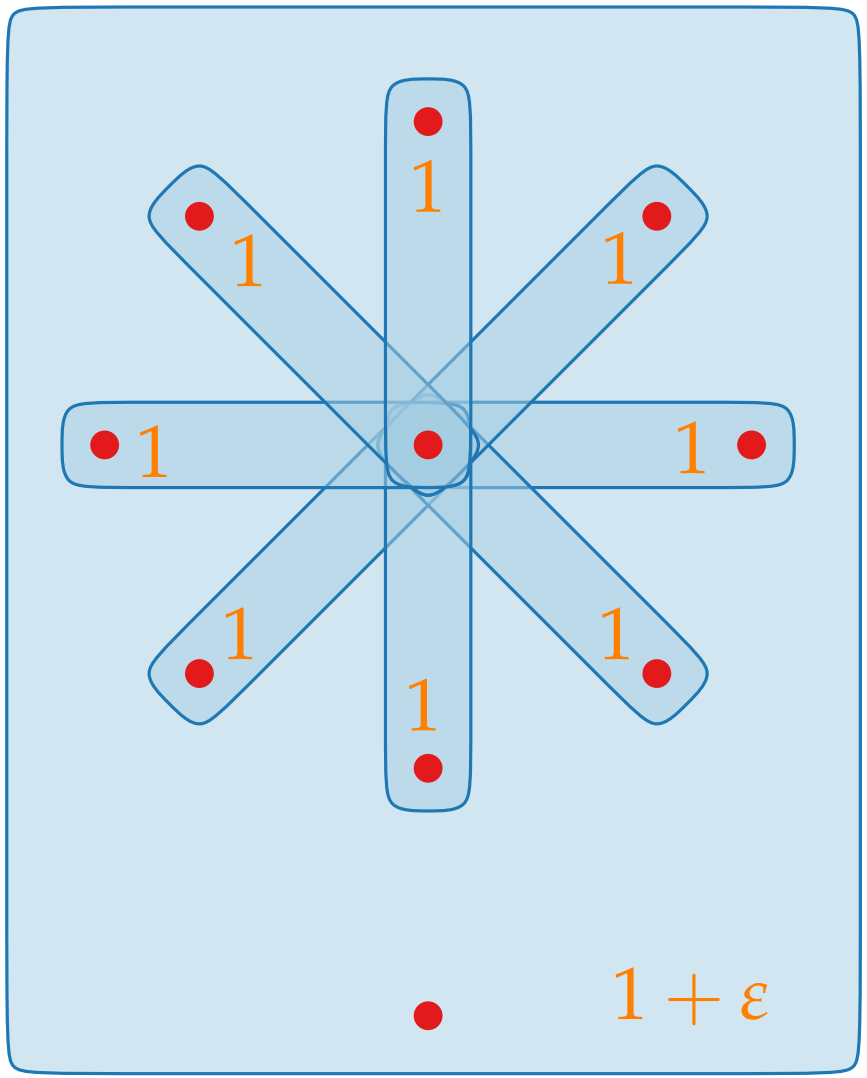

# Approximation Algorithms

## Lecture 5:
## LP-based Approximation Algorithms for SetCover

### Part V:
### SetCover via Dual Fitting

Philipp Kindermann

Summer Semester 2020

# Dual Fitting for SetCover

Combinatorial (greedy) algorithm (see Lecture 02):

```
GreedySetCover(U, S, c)
    C ← ∅
    S' ← ∅
    while C ≠ U do
        S ← Set from S that minimizes c(S) / |S \ C|
        foreach u ∈ S \ C do
            price(u) ← c(S) / |S \ C|
        C ← C ∪ S
        S' ← S' ∪ {S}
    return S'                                  // Cover of U
```

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Reminder: $\sum_{u \in U} \text{price}(u)$ ...

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Reminder: $\sum_{u \in U} \text{price}(u)$ completely pays for $\mathcal{S}'$.

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**Observation.** For each $u \in U$, $\text{price}(u)$ is a dual variable

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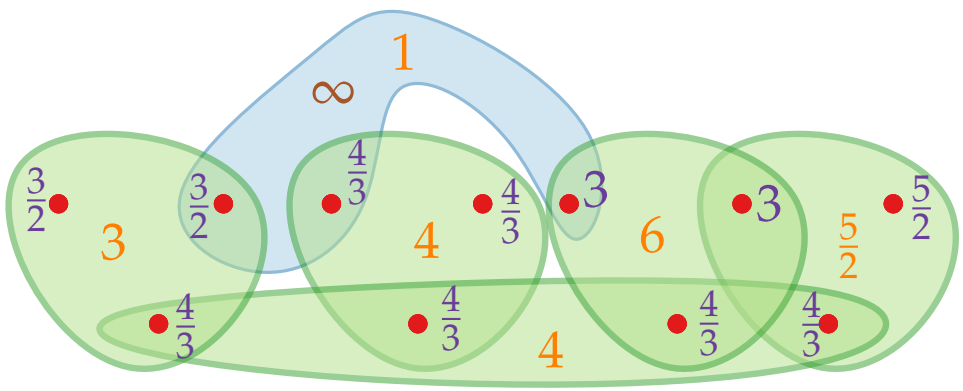


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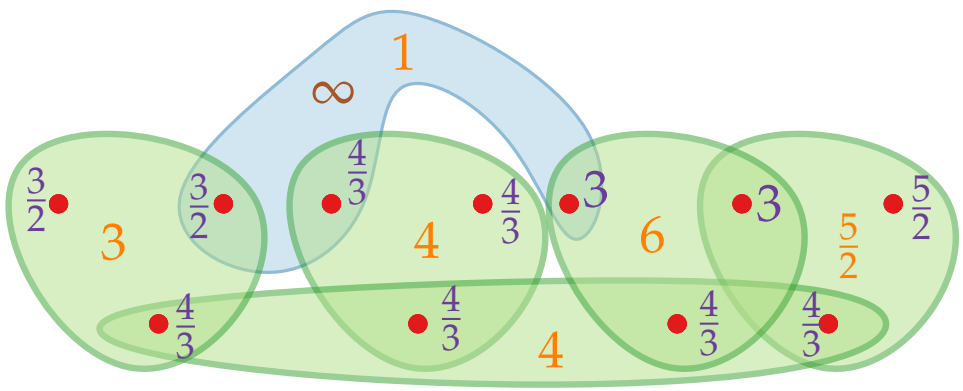


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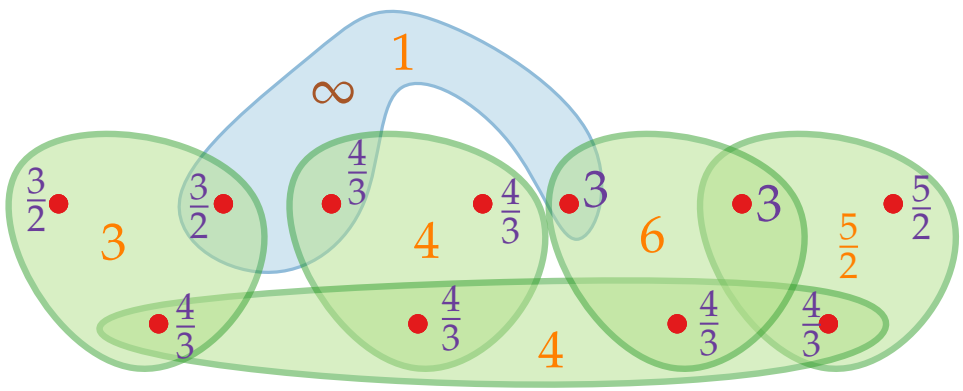
HA: Construct instance where some $S$ are "overpacked" by factor $\approx H_{|S|}$.

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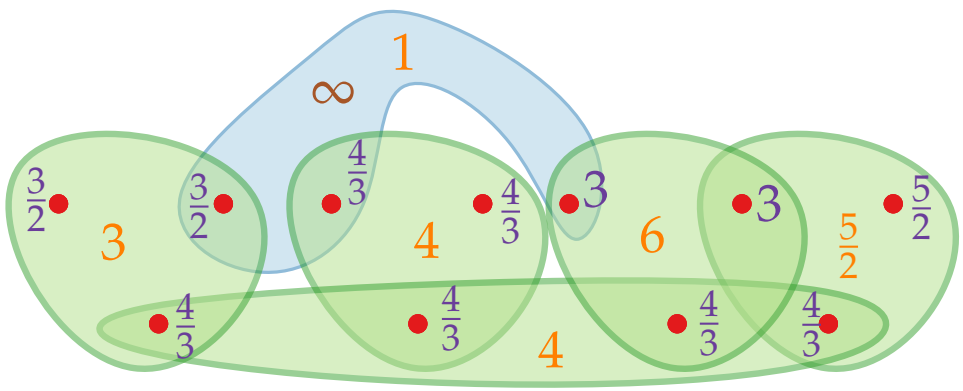
Scale duale variables such that no set is overpacked.

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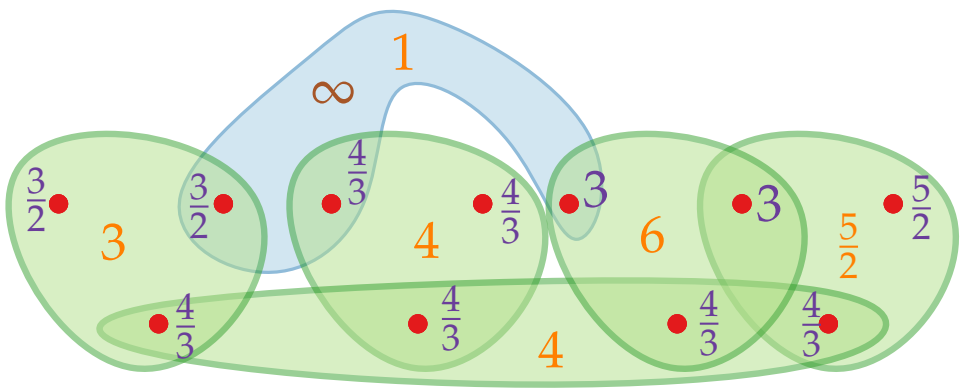


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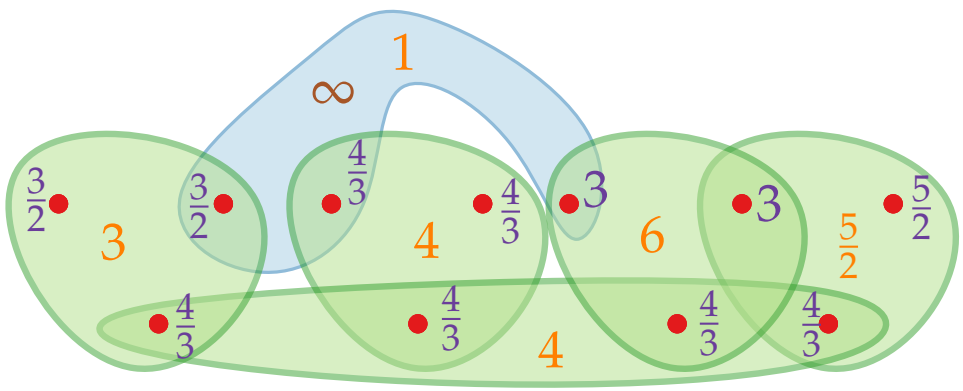


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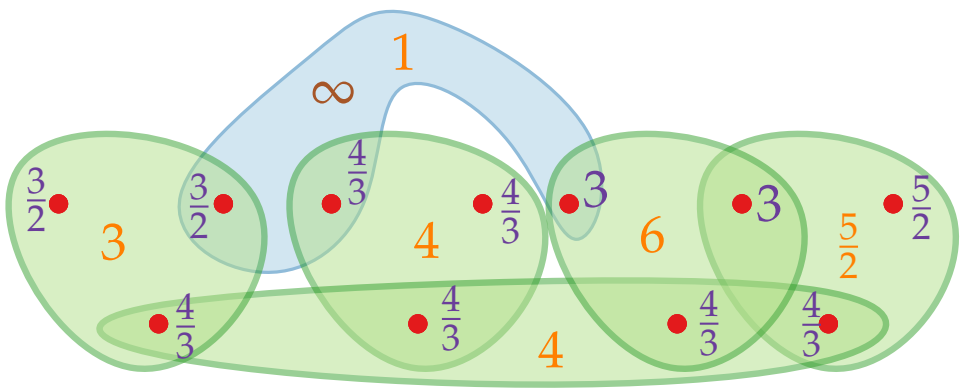


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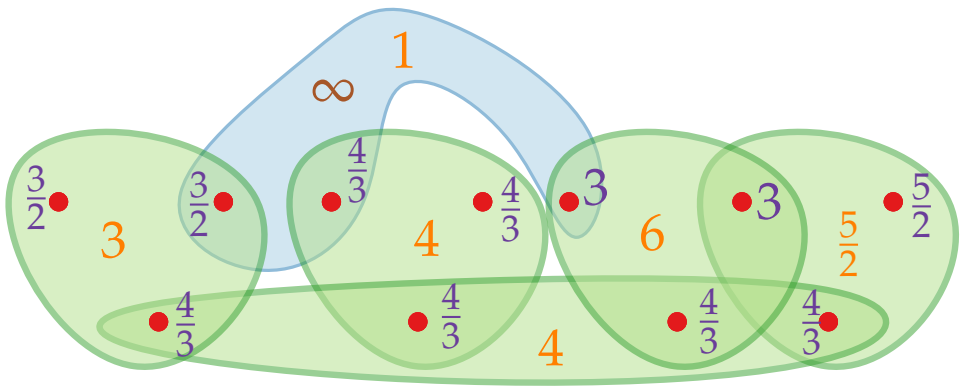


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