Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

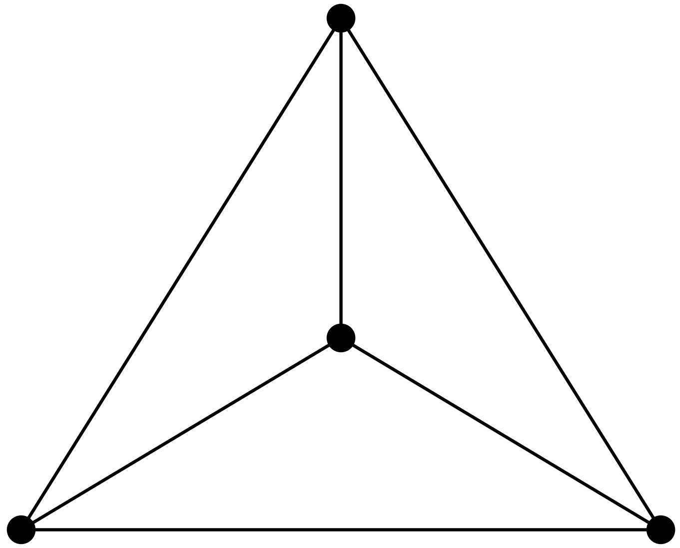
Part I: SteinerTree

Philipp Kindermann

Summer Semester 2020

SteinerTree

Given: A graph $G = (V, E)$



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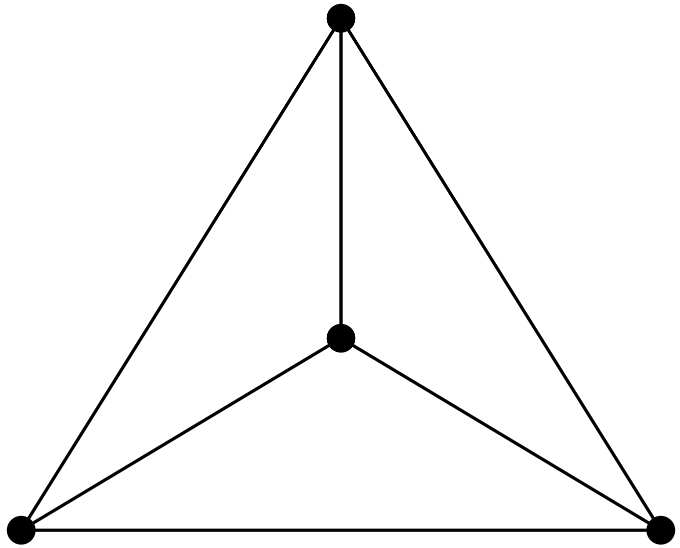

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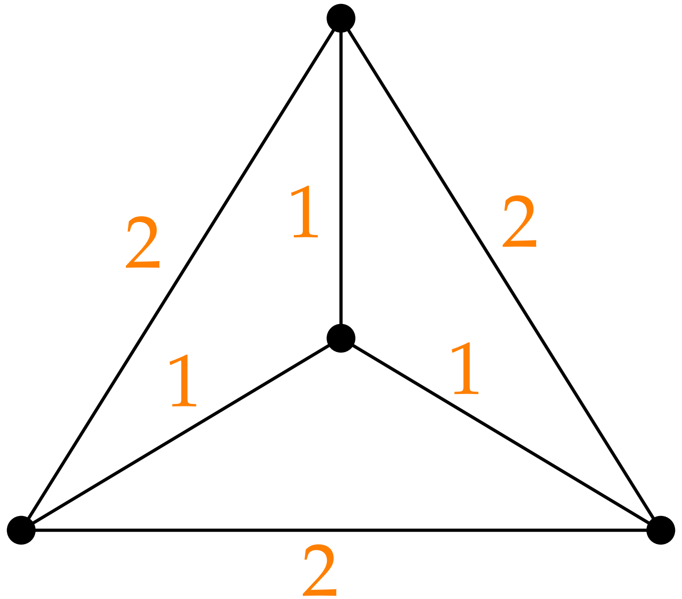

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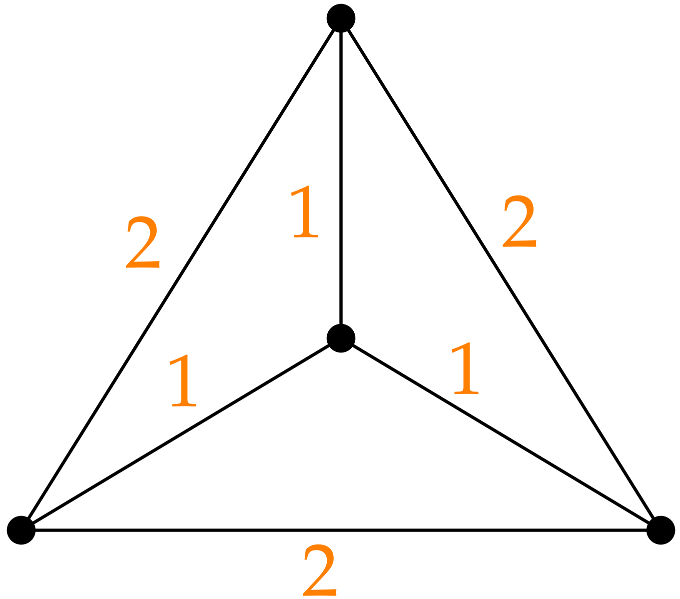

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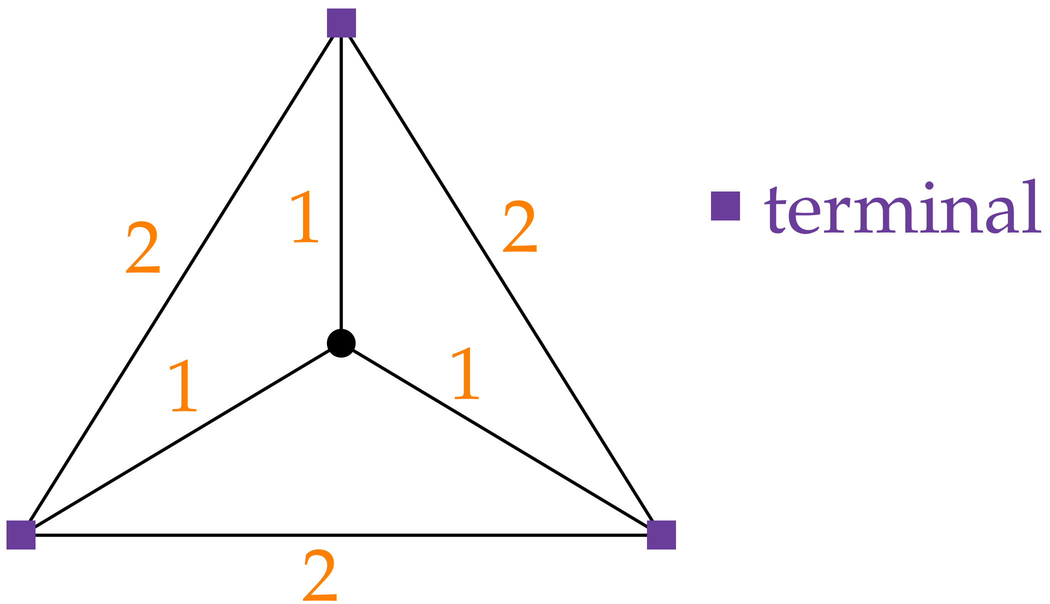

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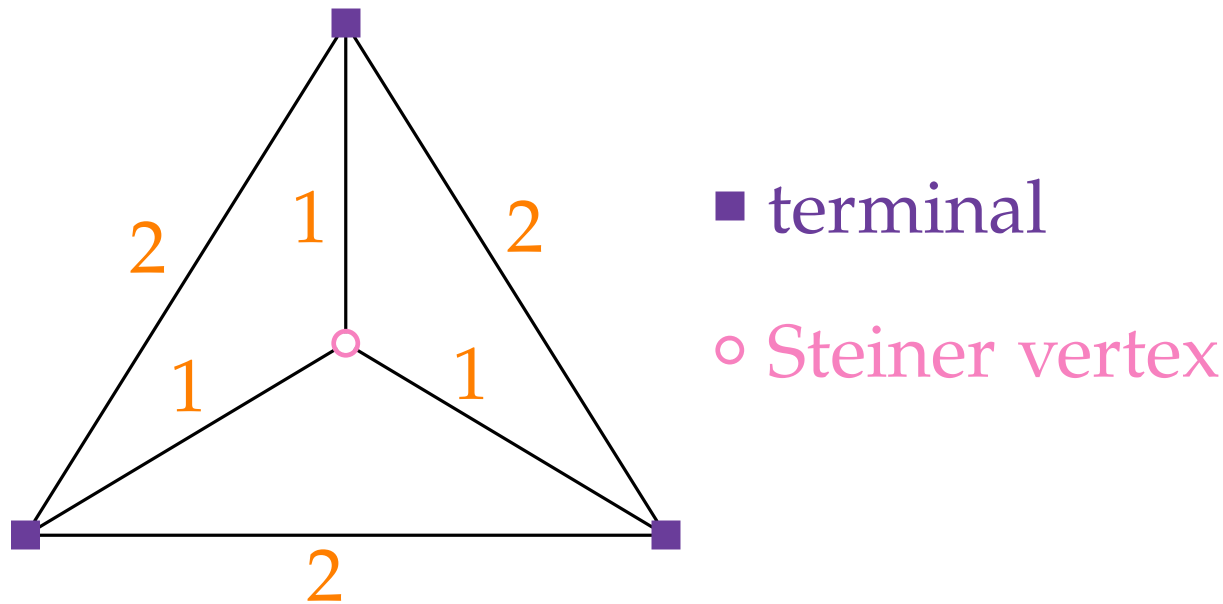

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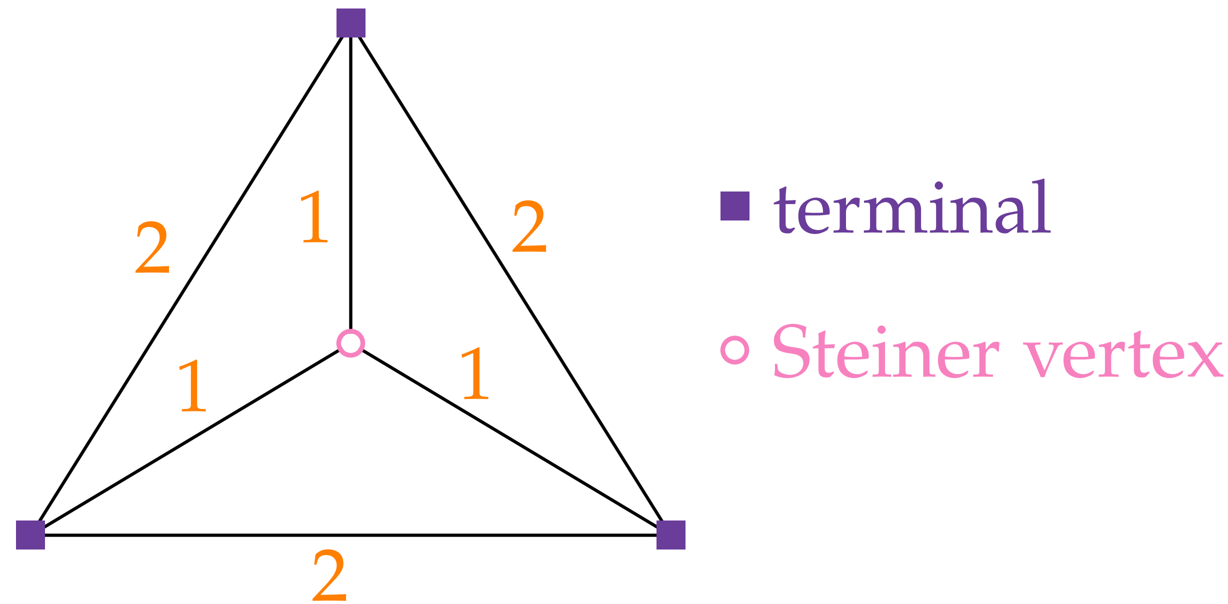

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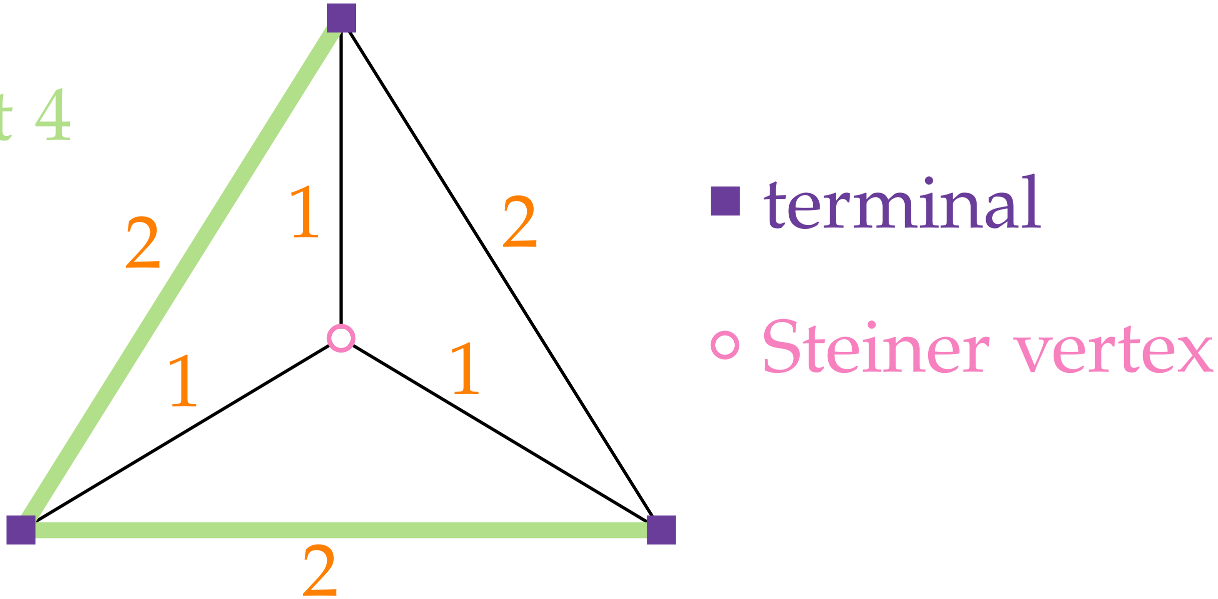

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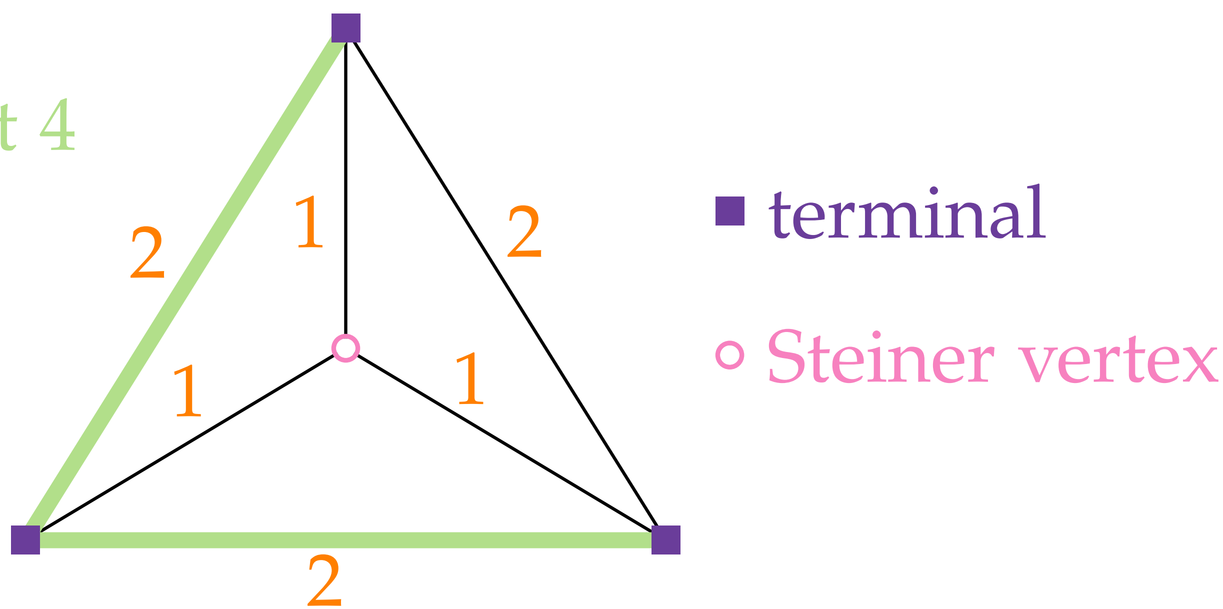

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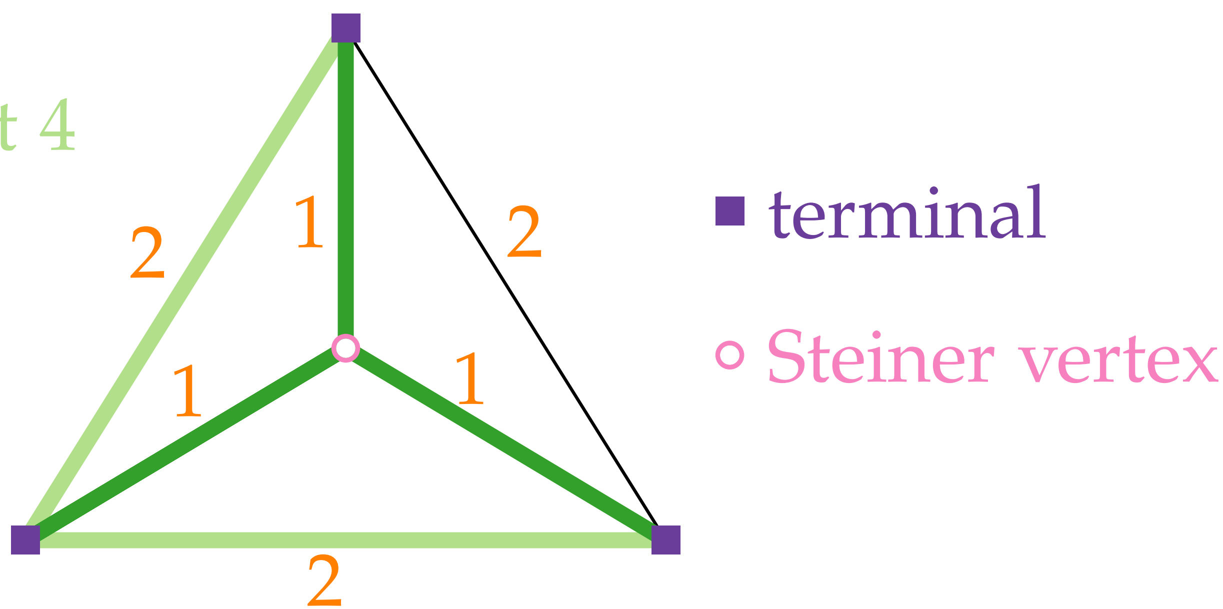

MetricSteinerTree

Restriction of SteinerTree where the graph G is complete and the cost function is **metric**

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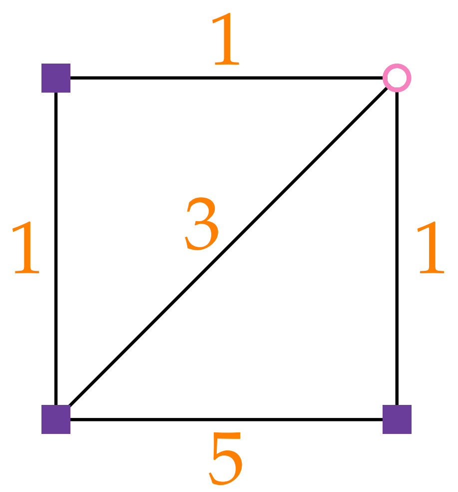

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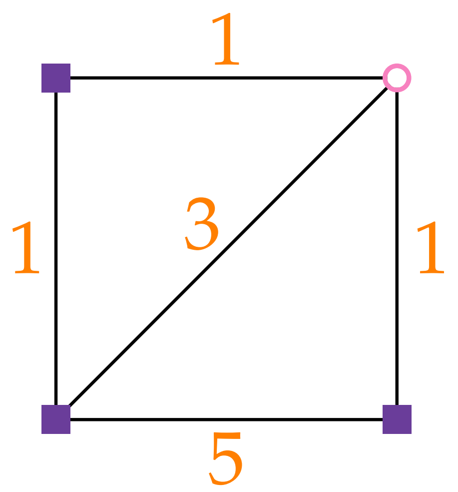

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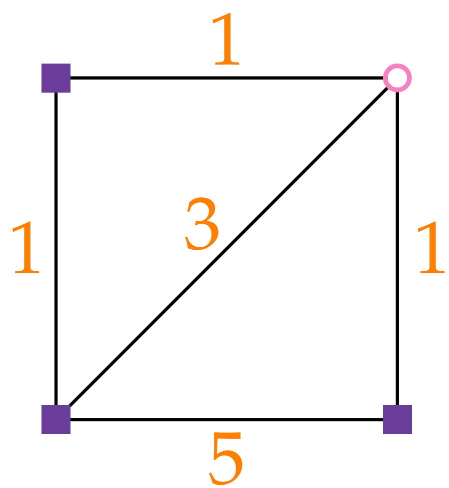

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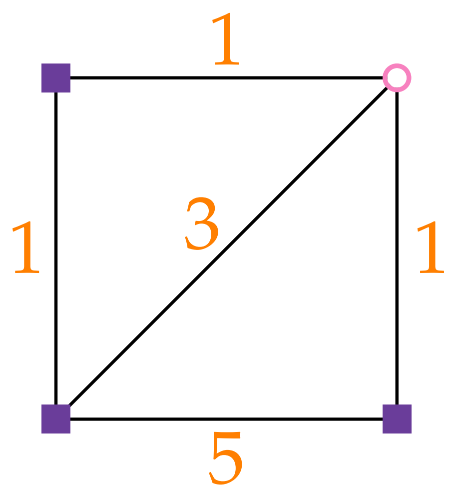

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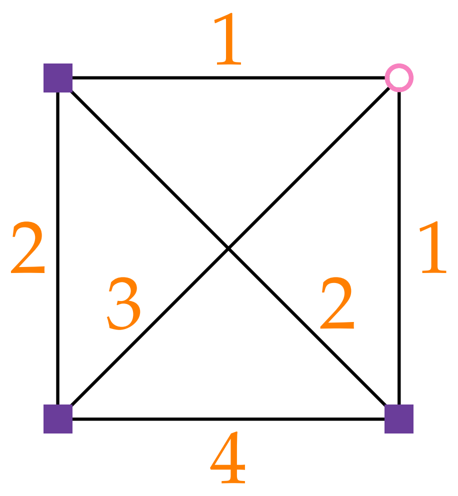

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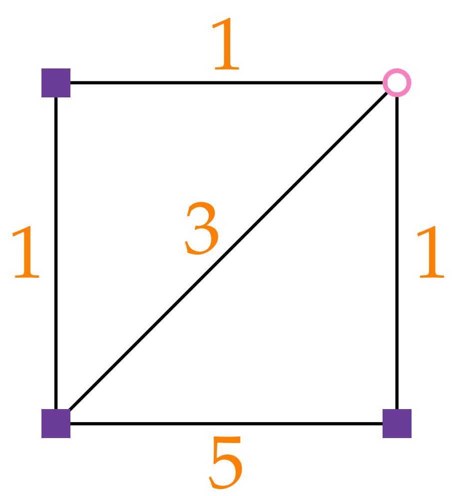

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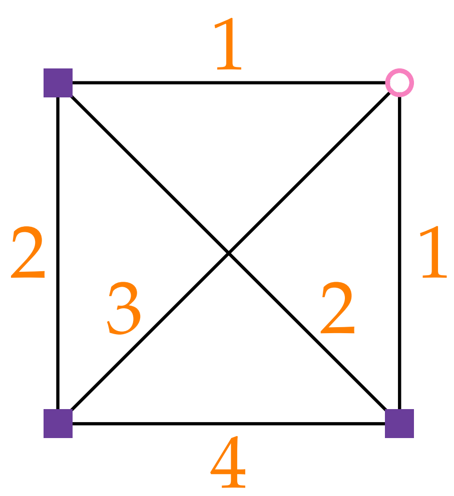

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part II:
Approximation Preserving Reduction

Philipp Kindermann

Summer Semester 2020

Approximation Preserving Reduction

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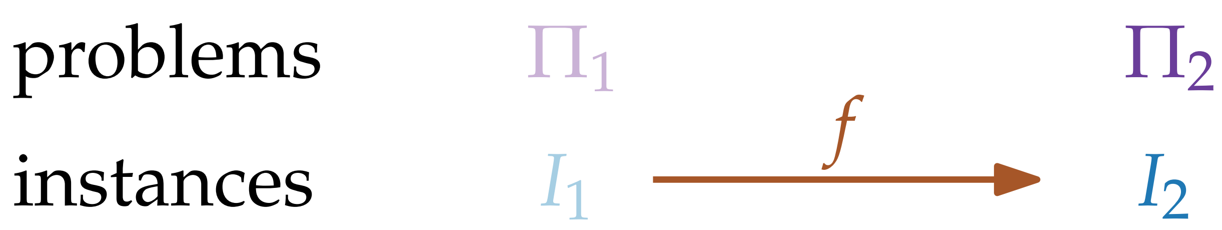

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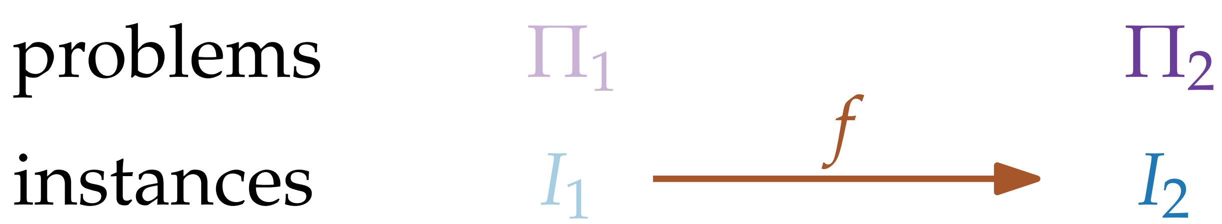

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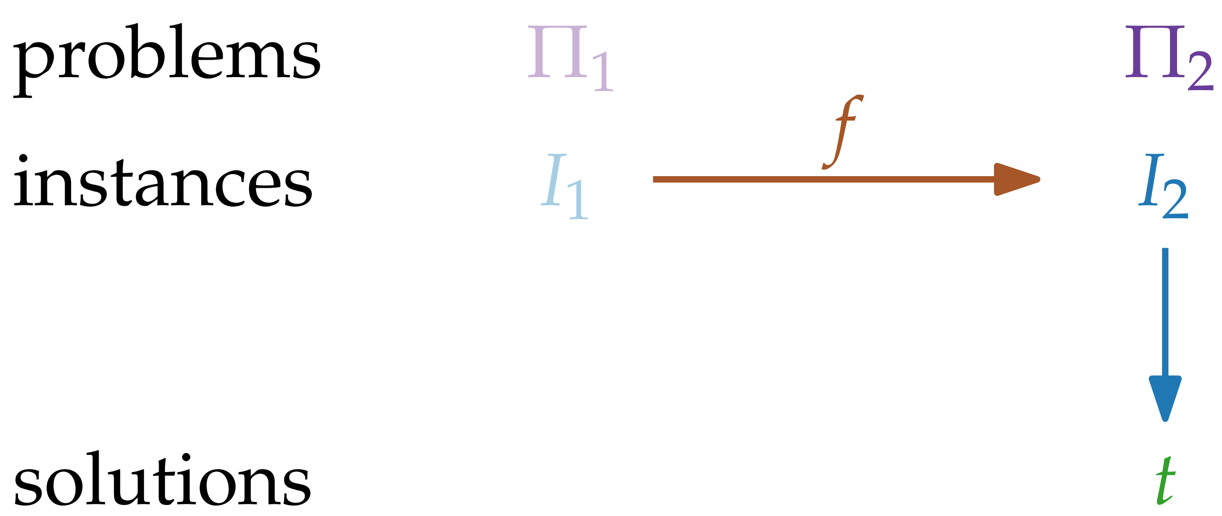

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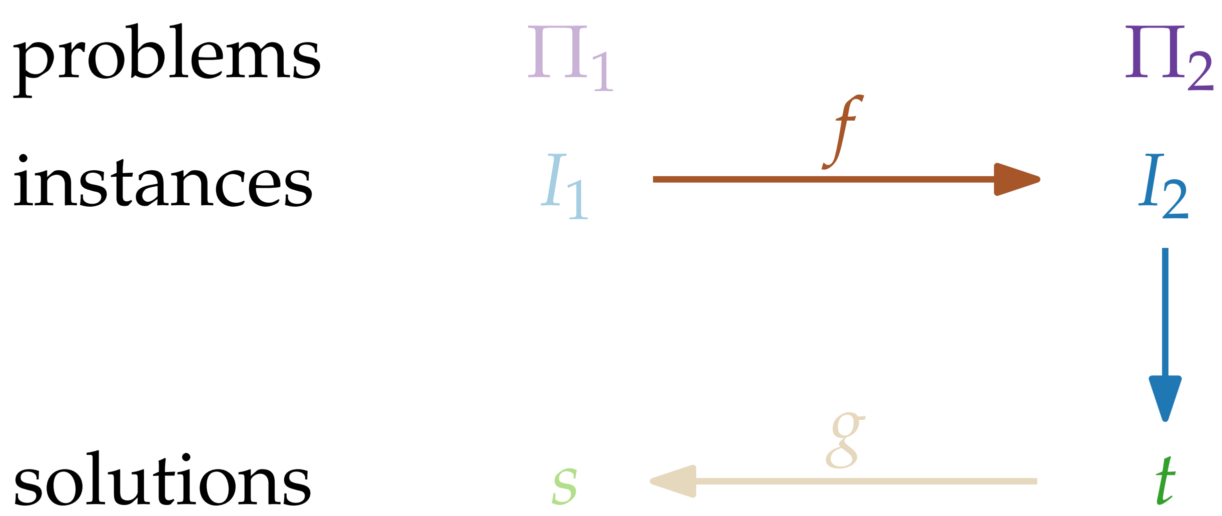

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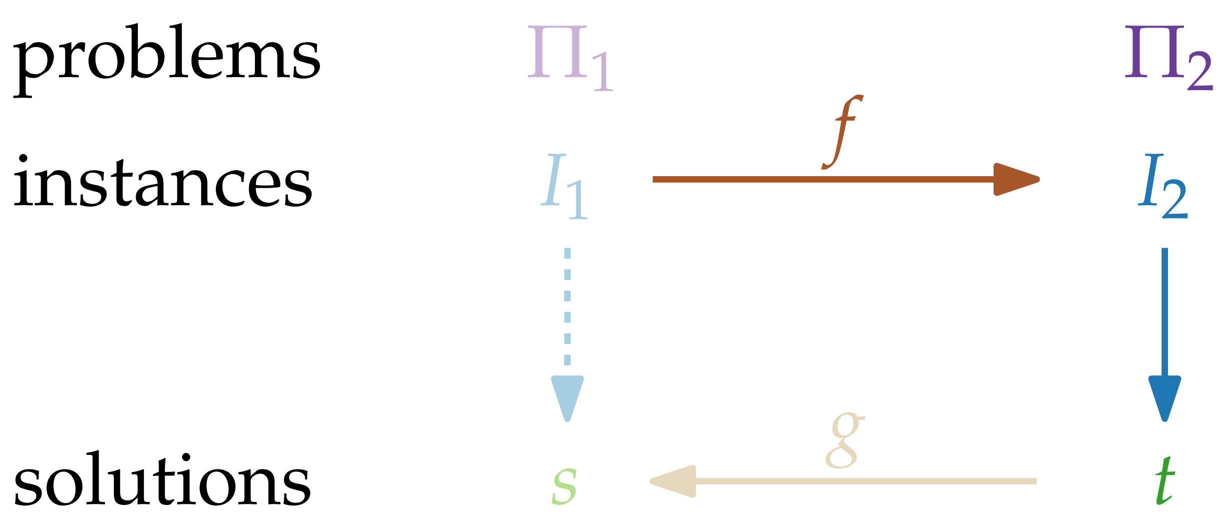

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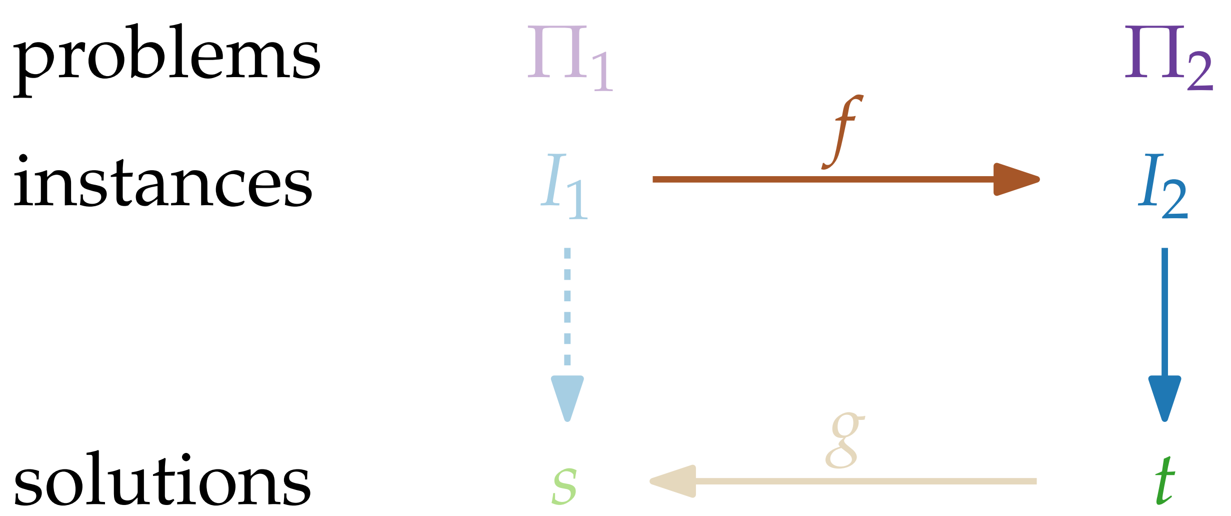

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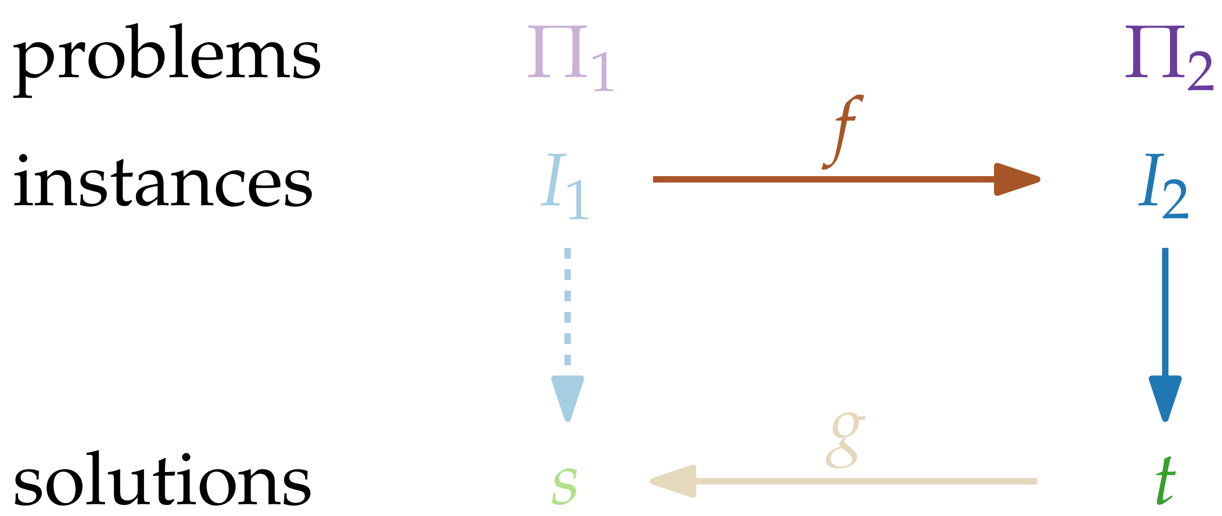

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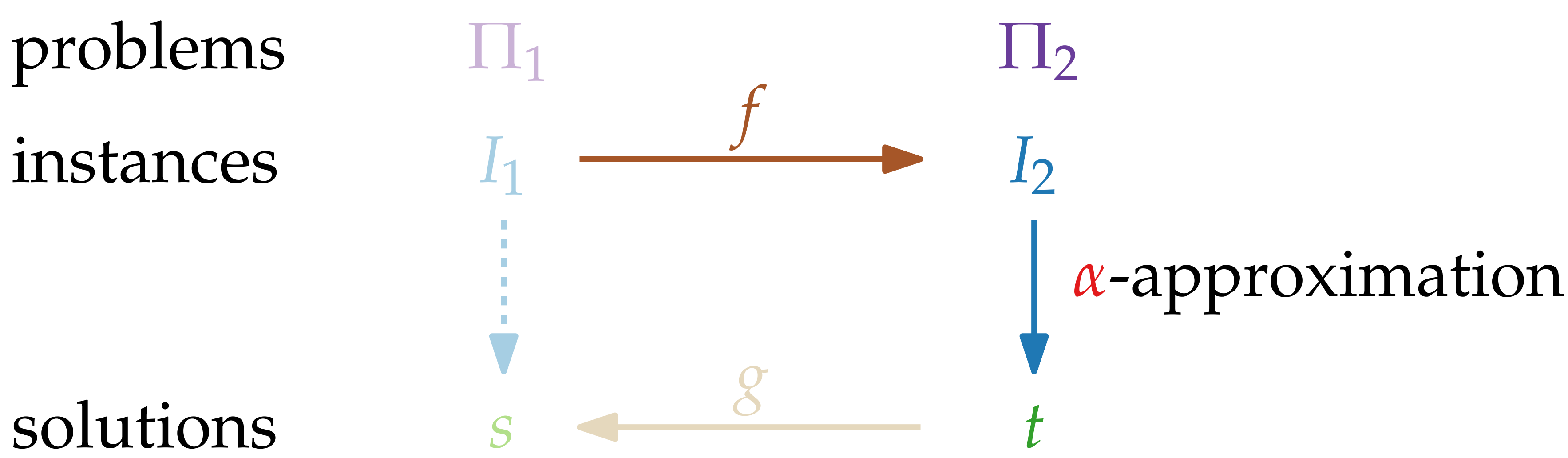

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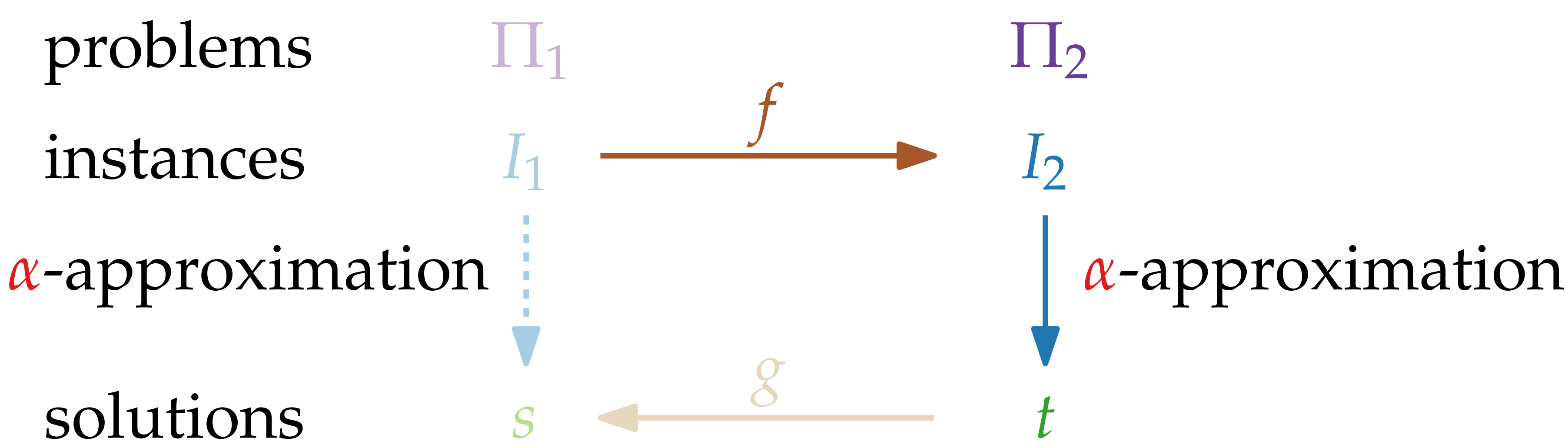

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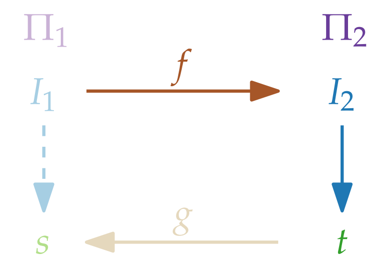

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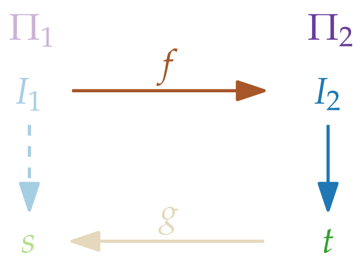

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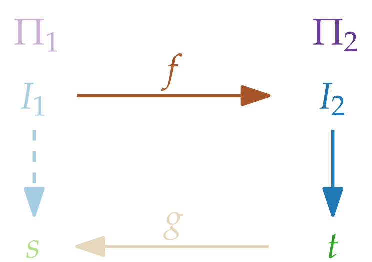

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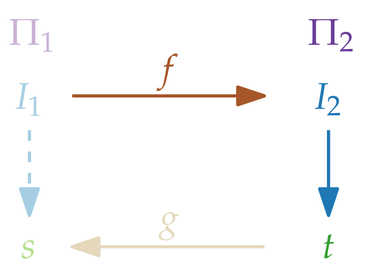

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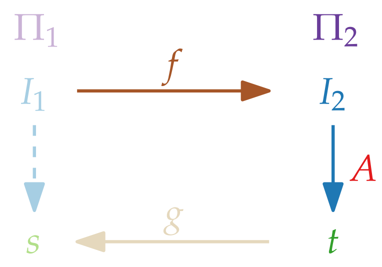

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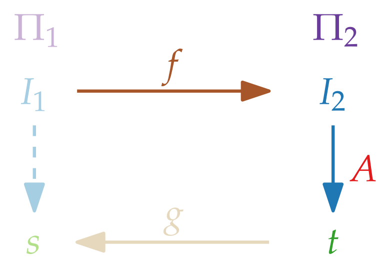

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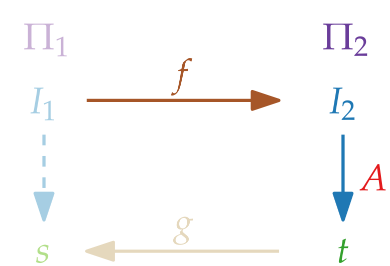

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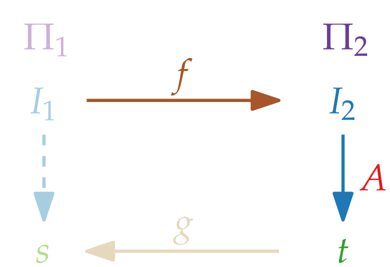

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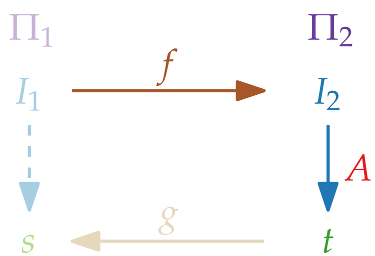

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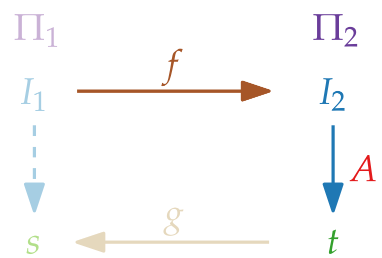

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part III:
Reduction to MetricSteinerTree

Philipp Kindermann Summer Semester 2020

MetricSteinerTree

Theorem. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

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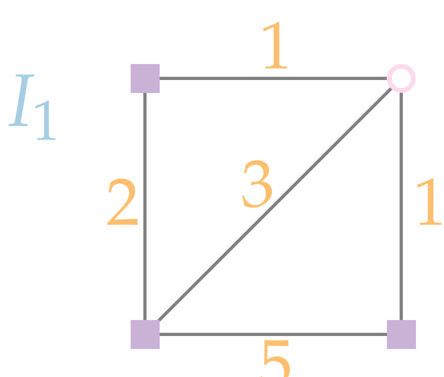

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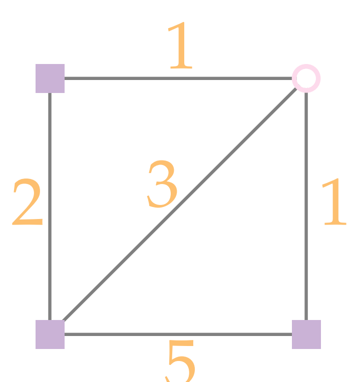

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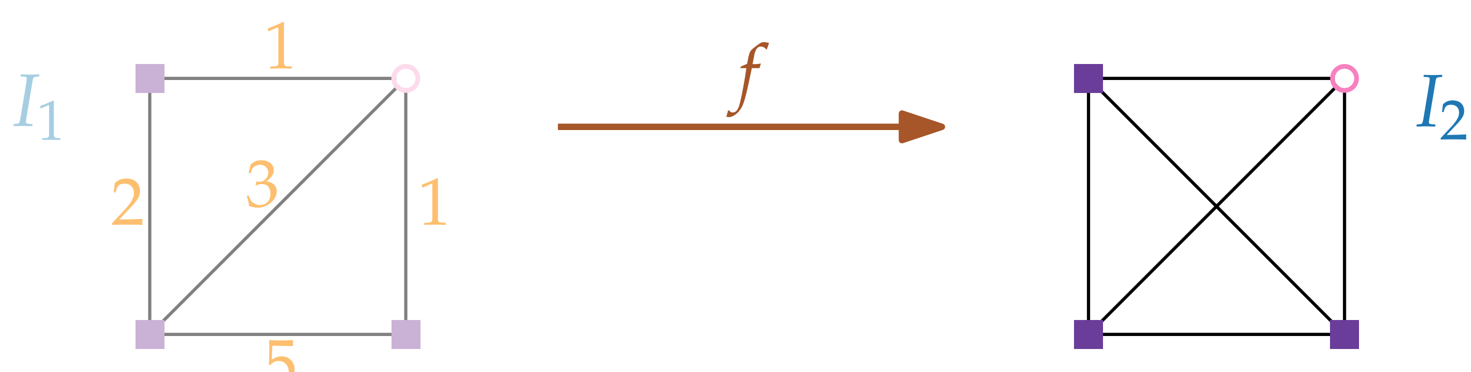

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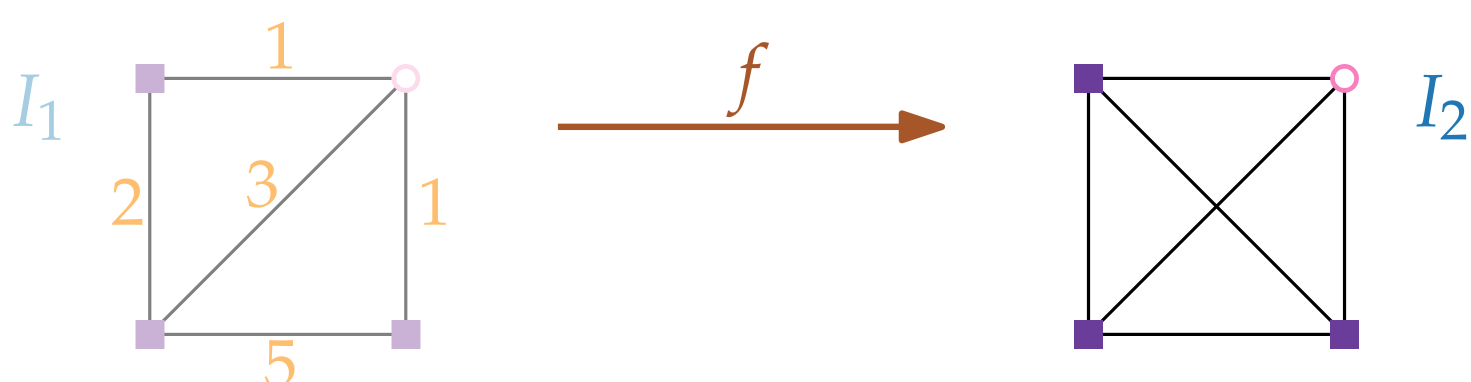

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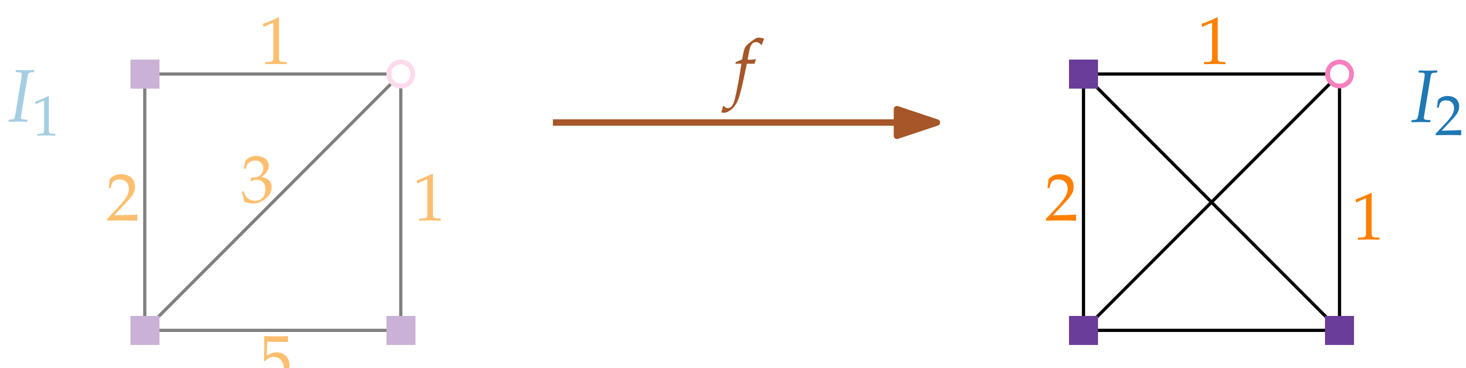

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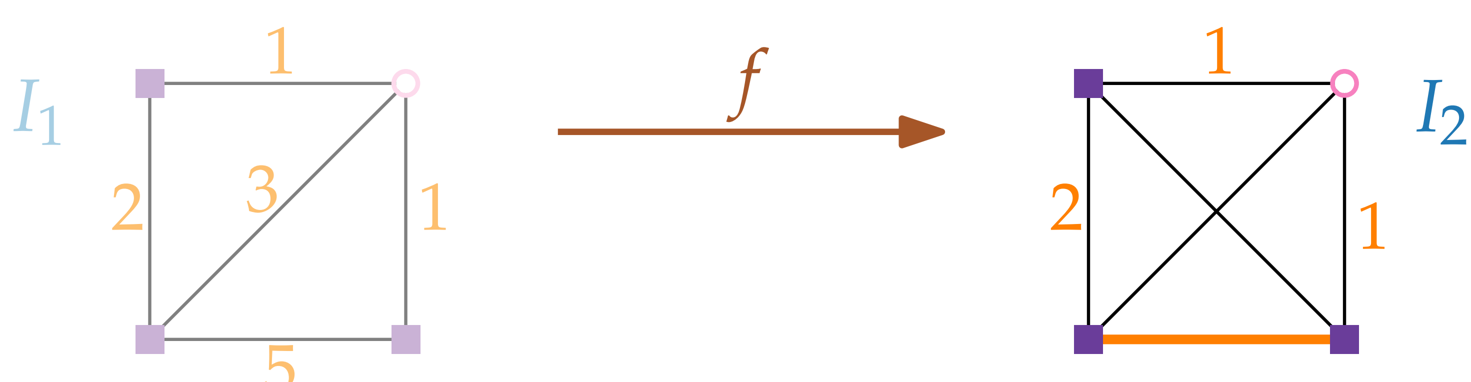

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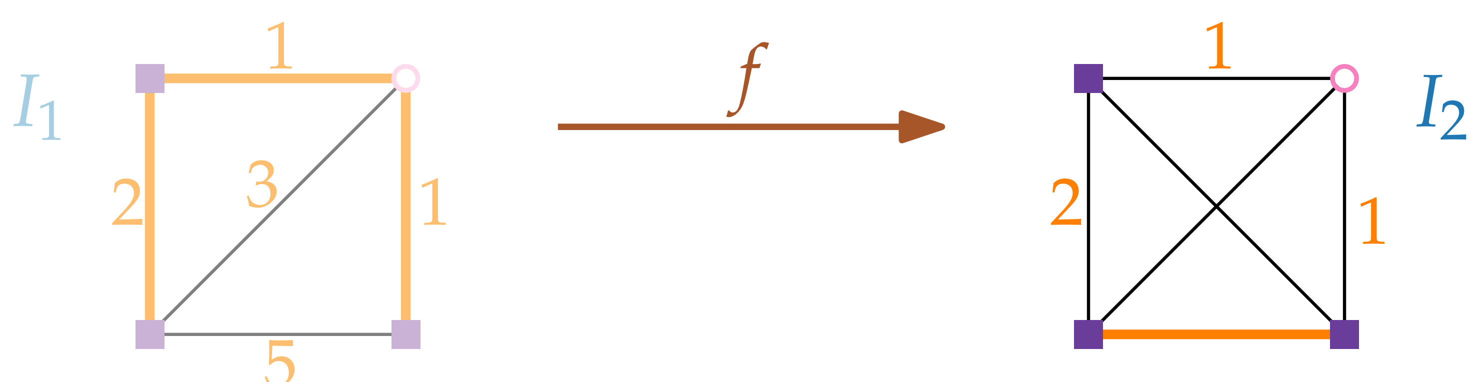

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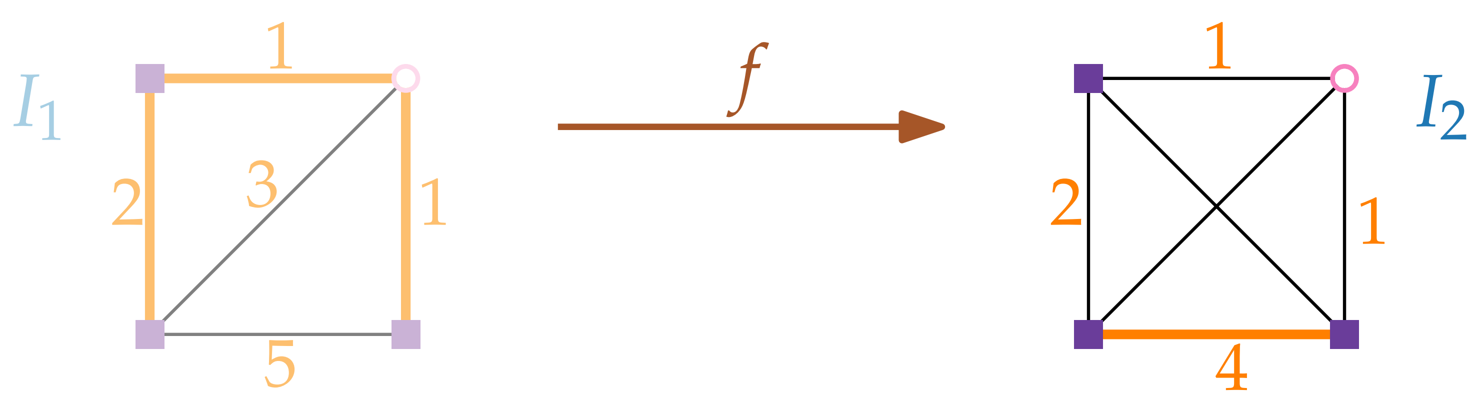

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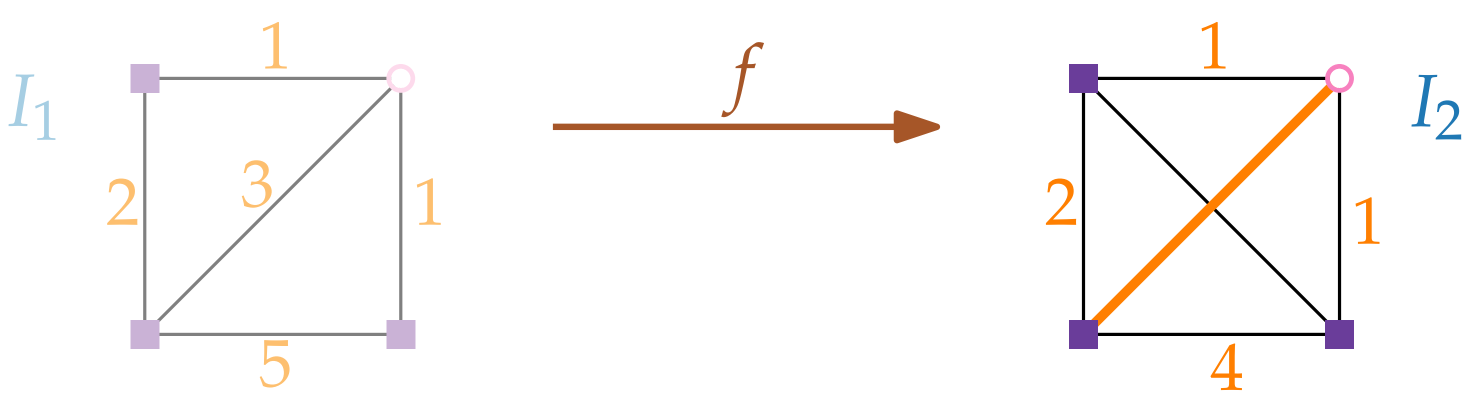

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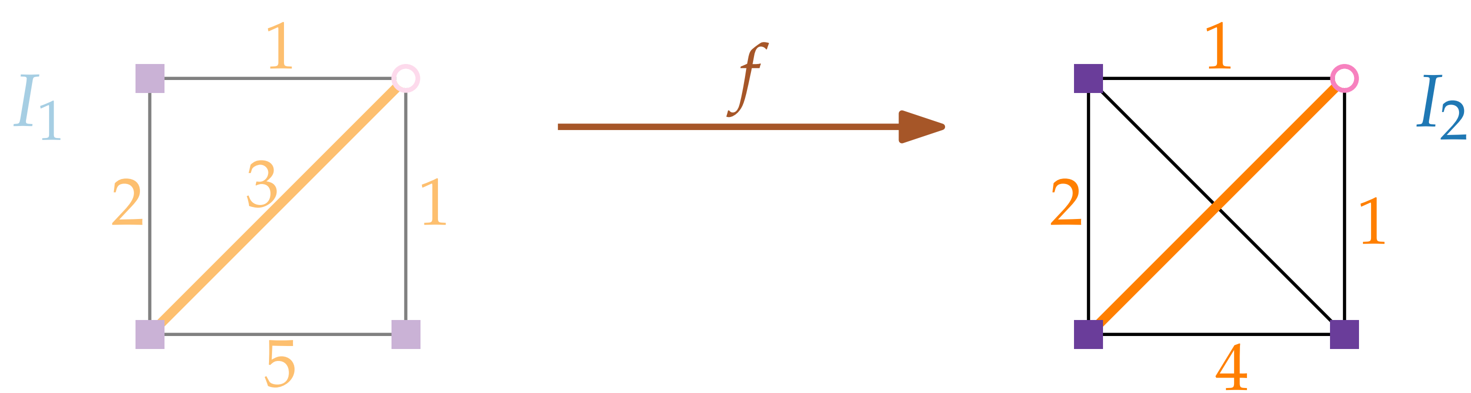

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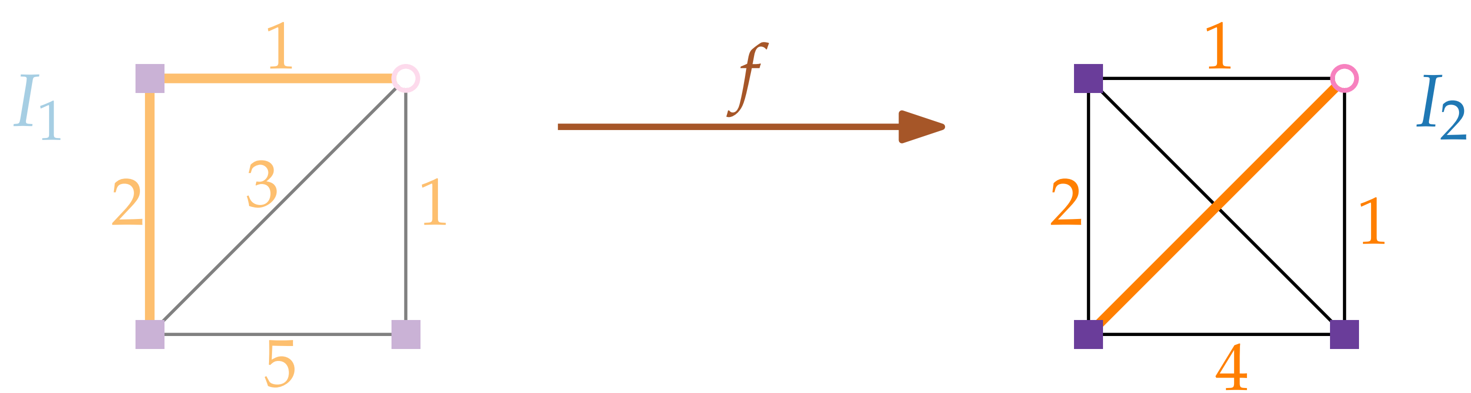

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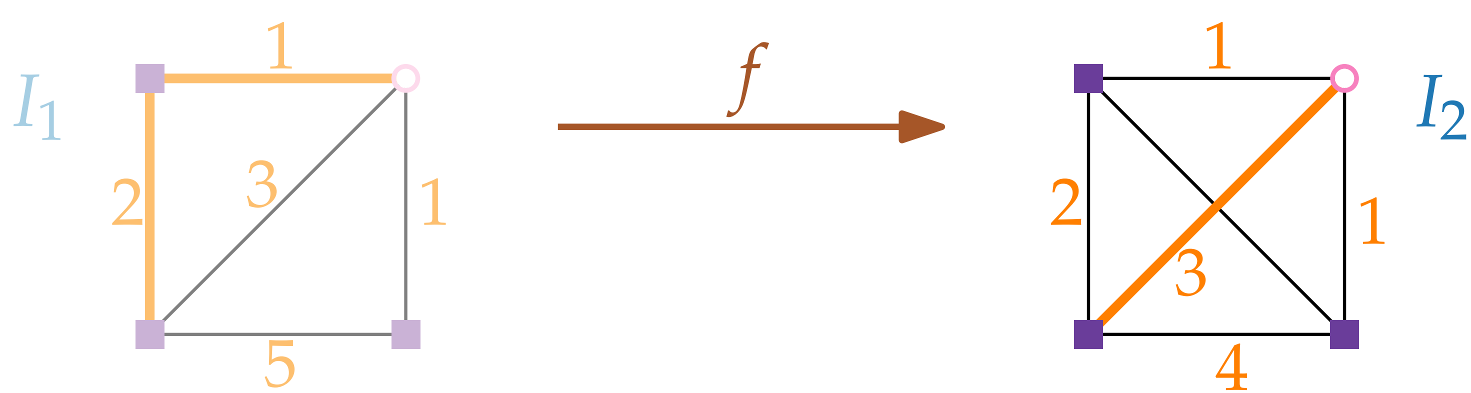

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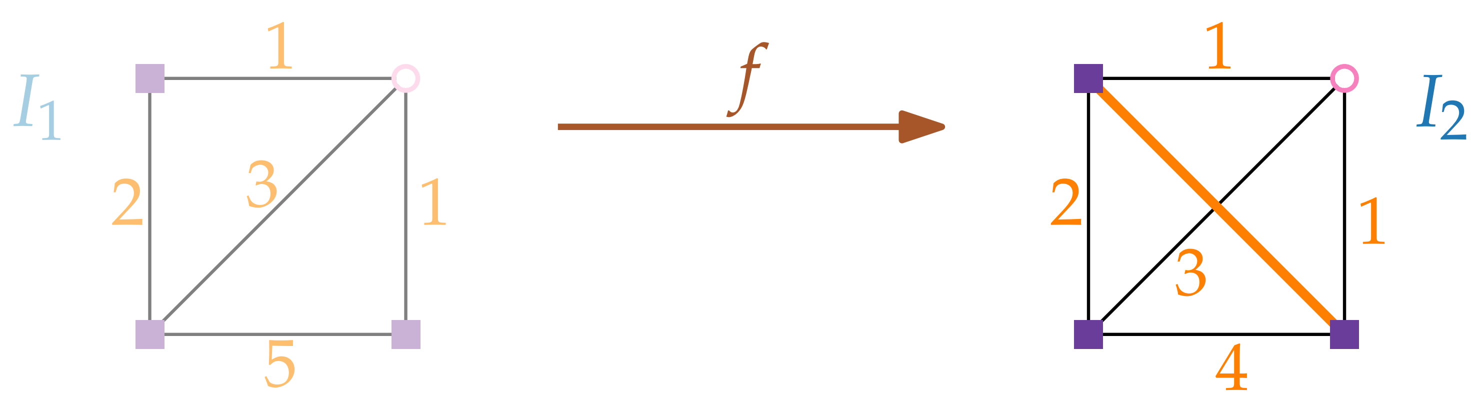

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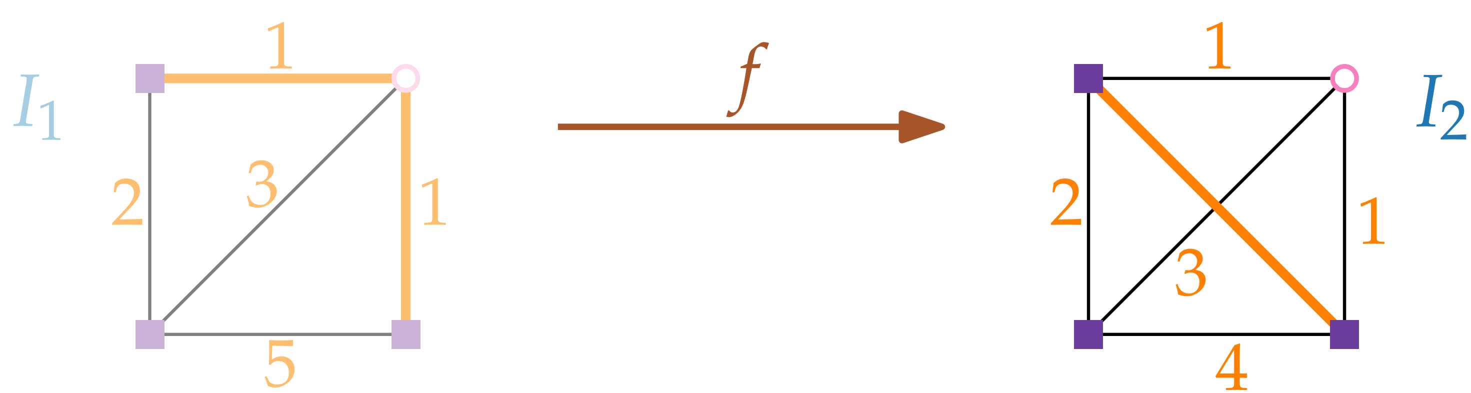

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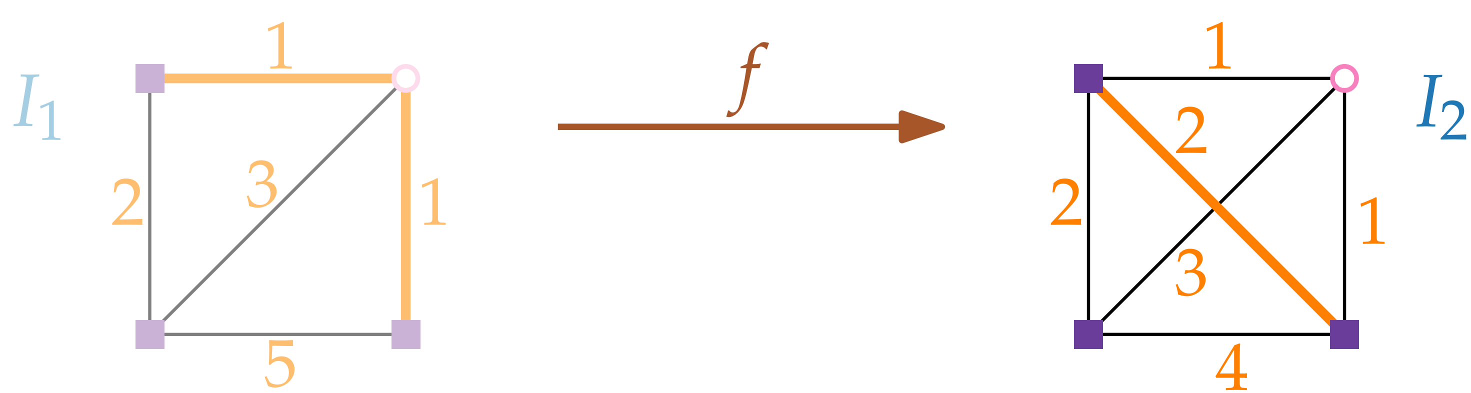

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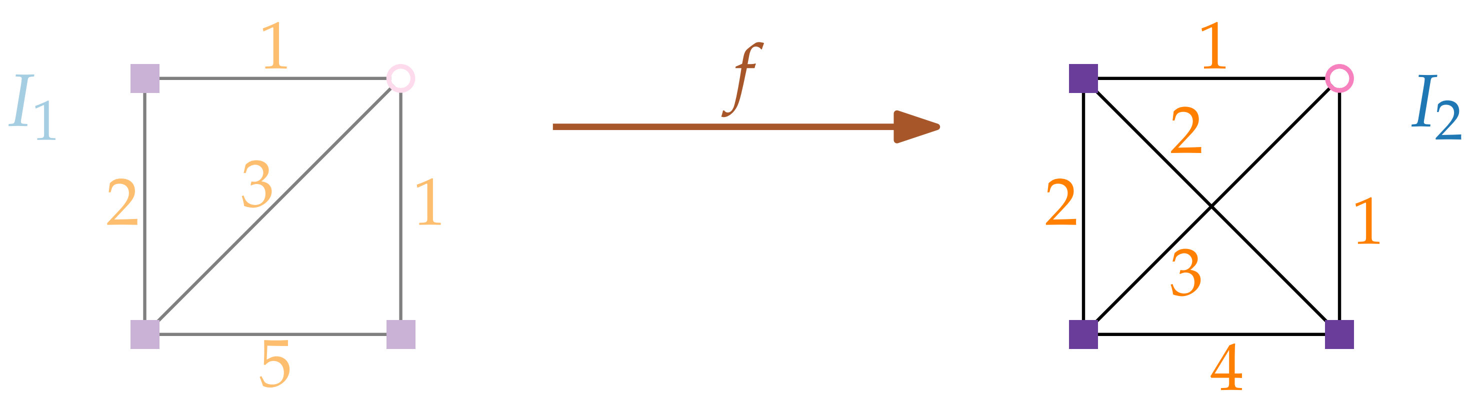

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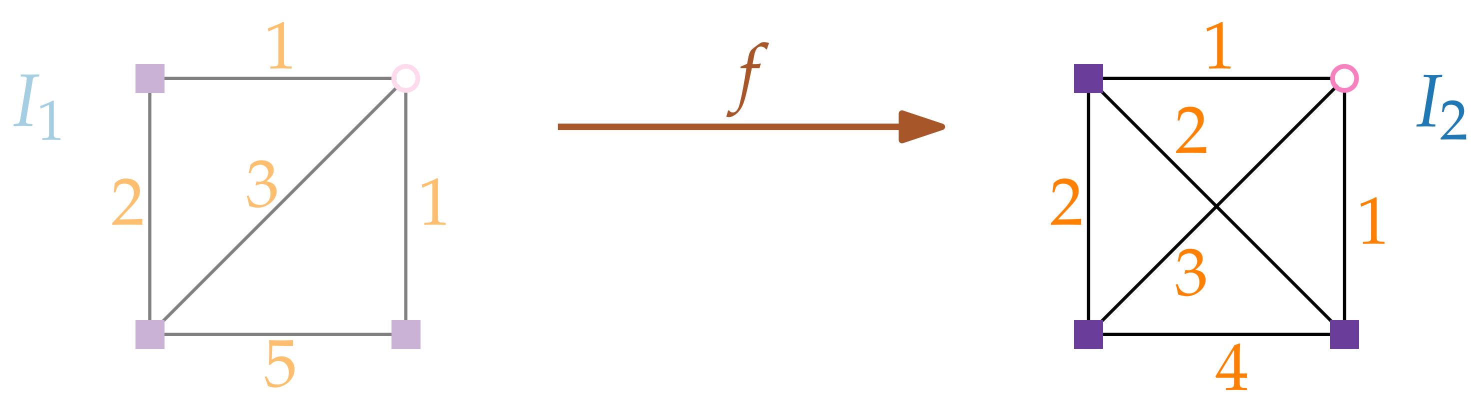
$c_2(u, v) \leq c_1(u, v)$ for all $(u, v) \in E$



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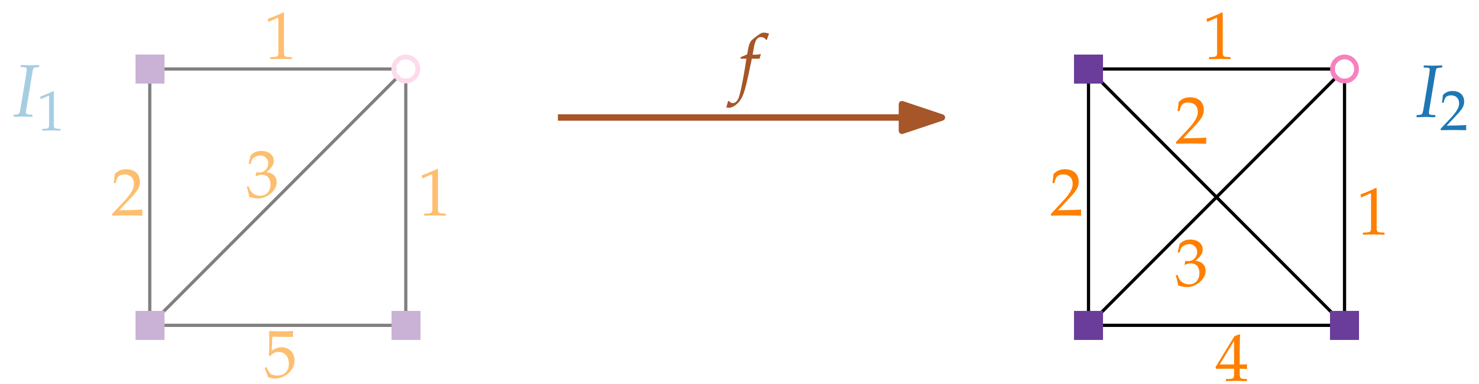

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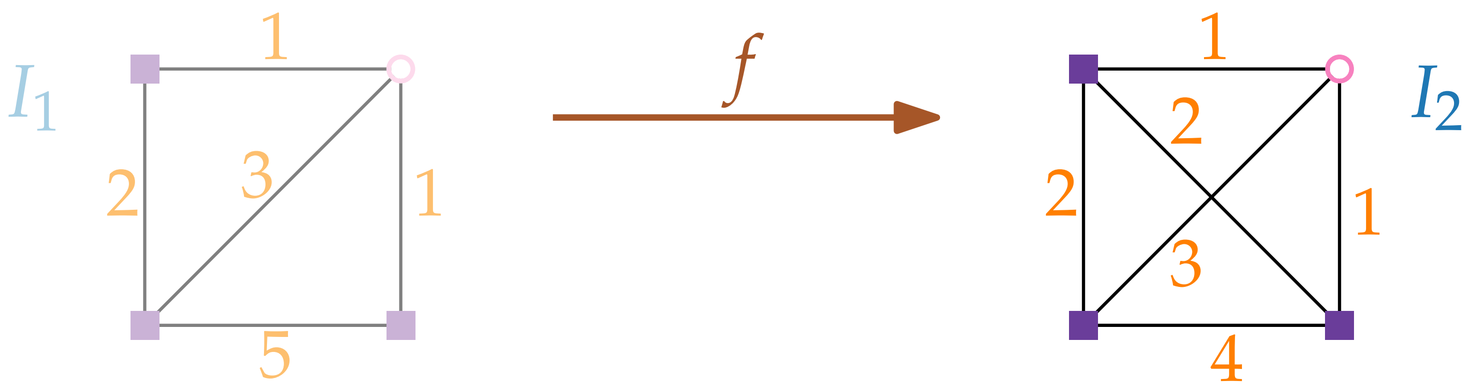

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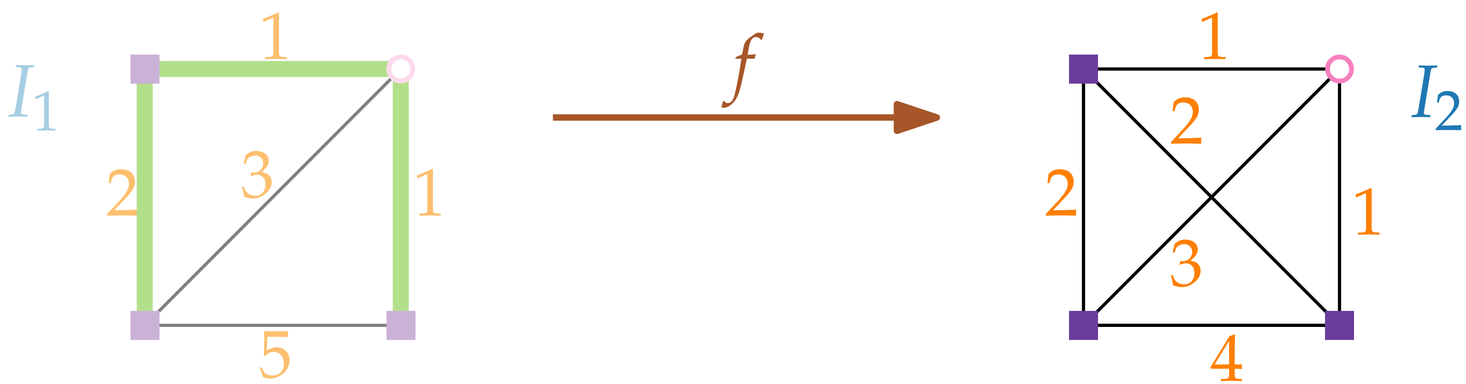

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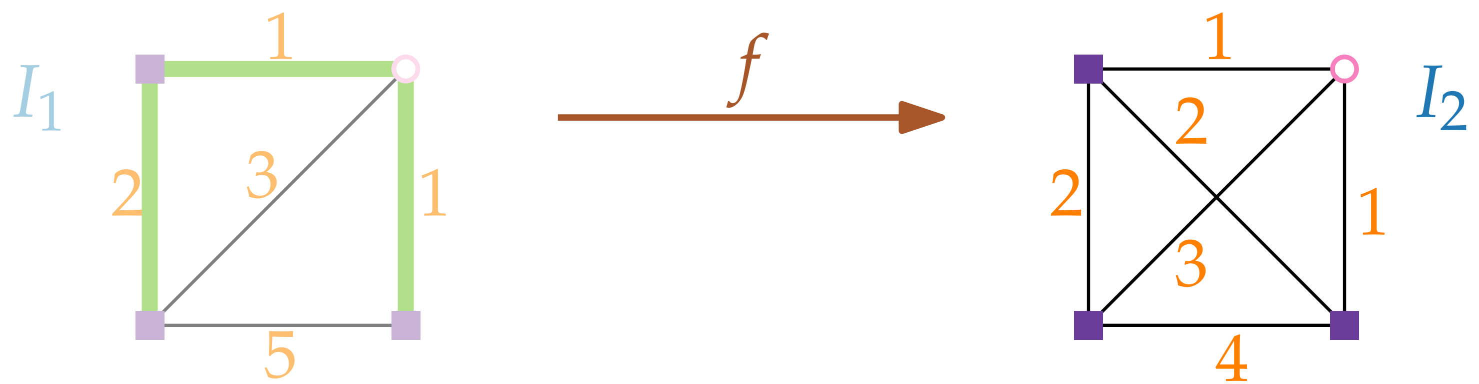

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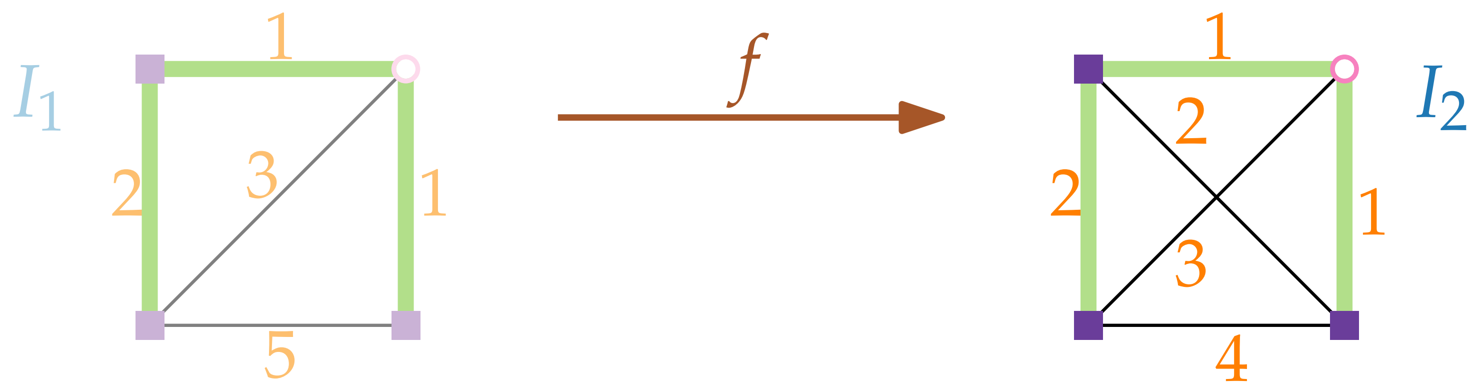

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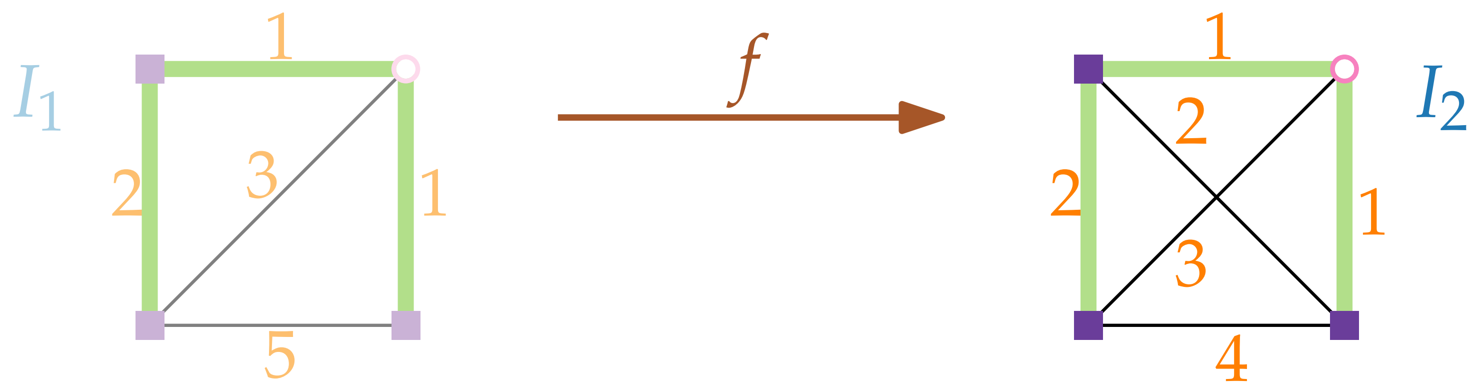

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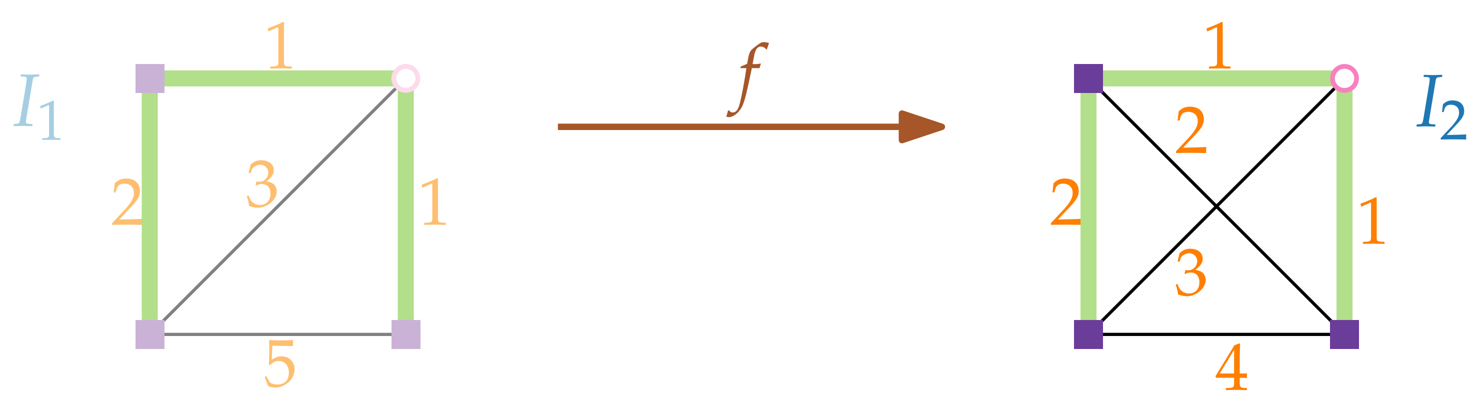

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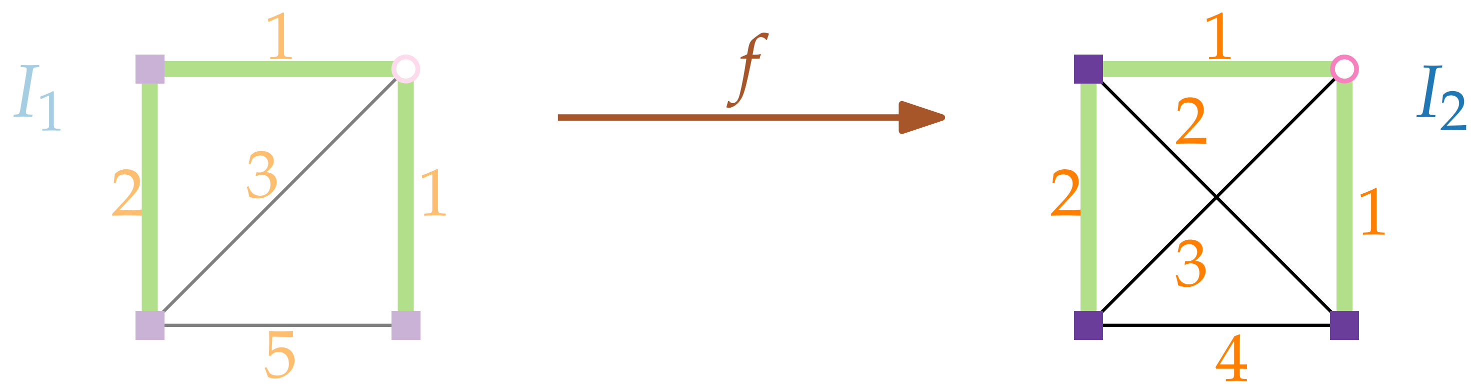

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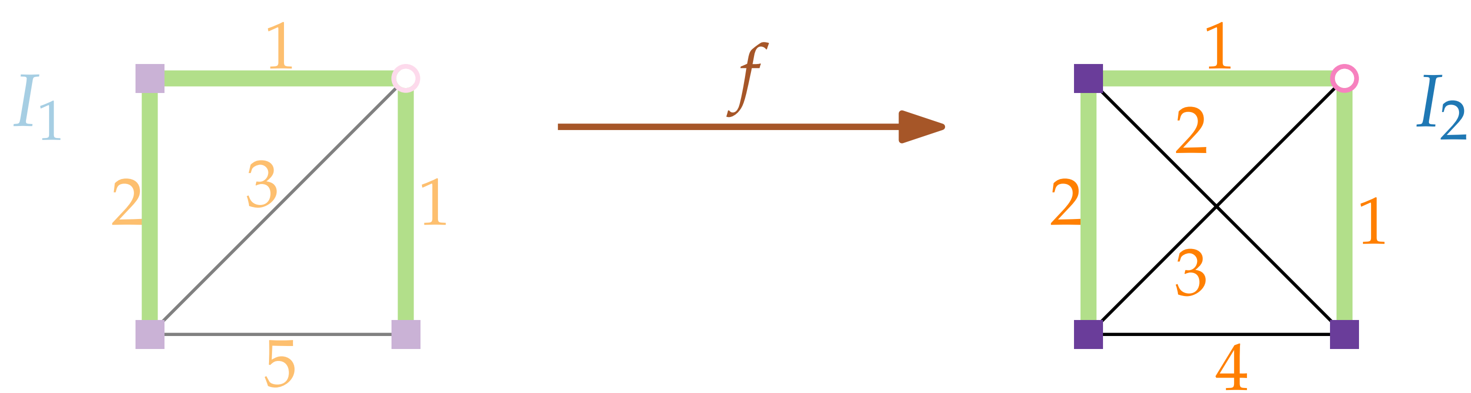

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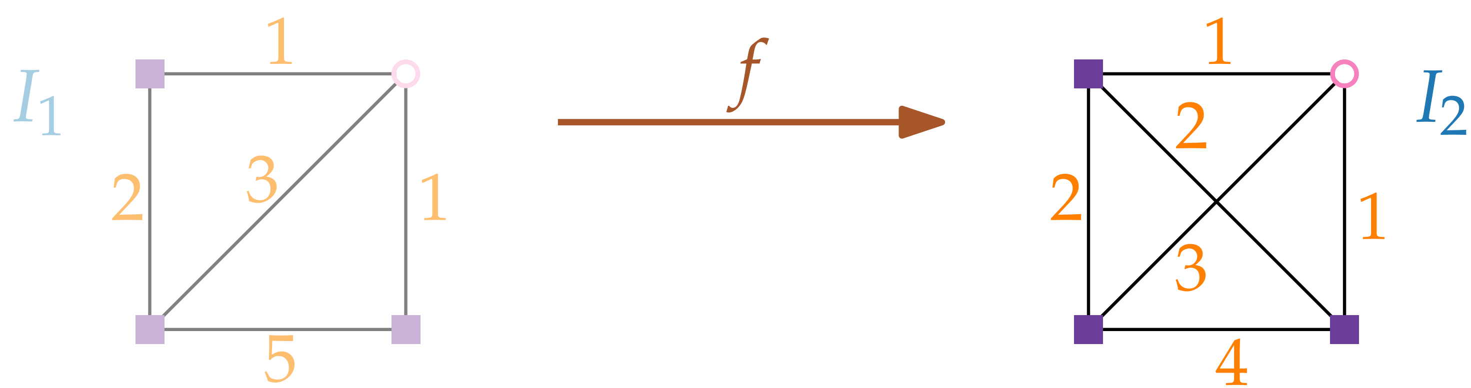

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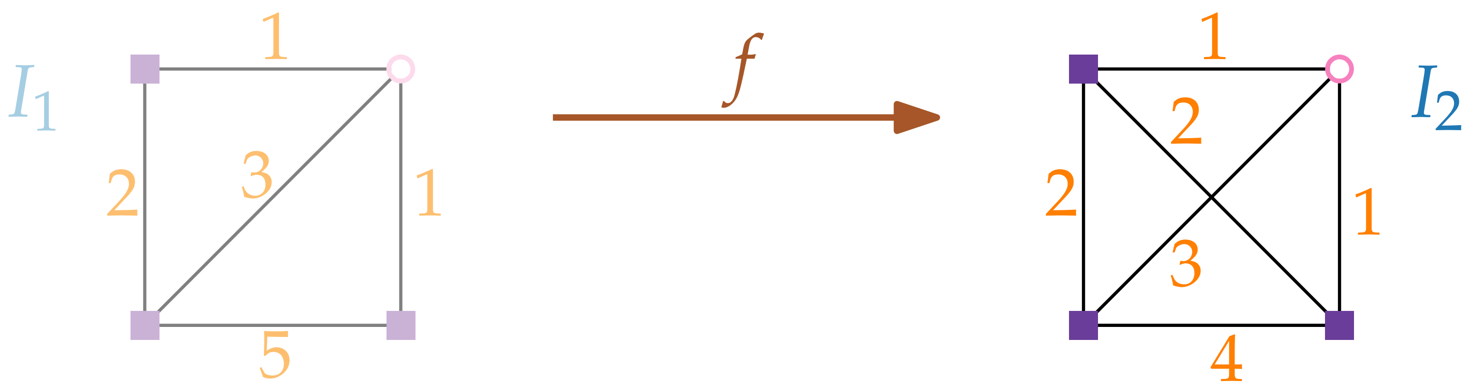

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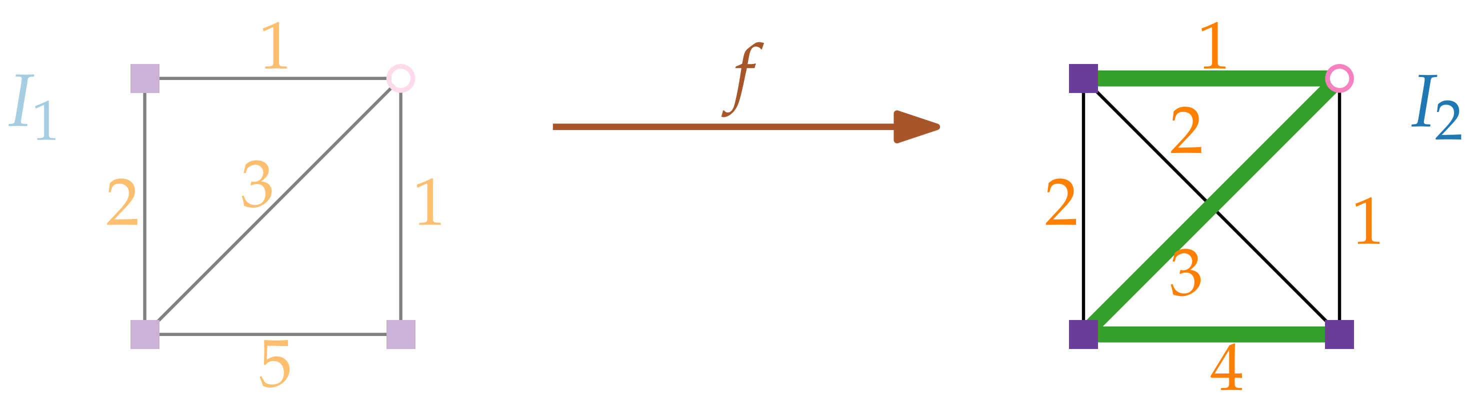

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Proof. (3) Mapping g $s \xleftarrow{g} t$

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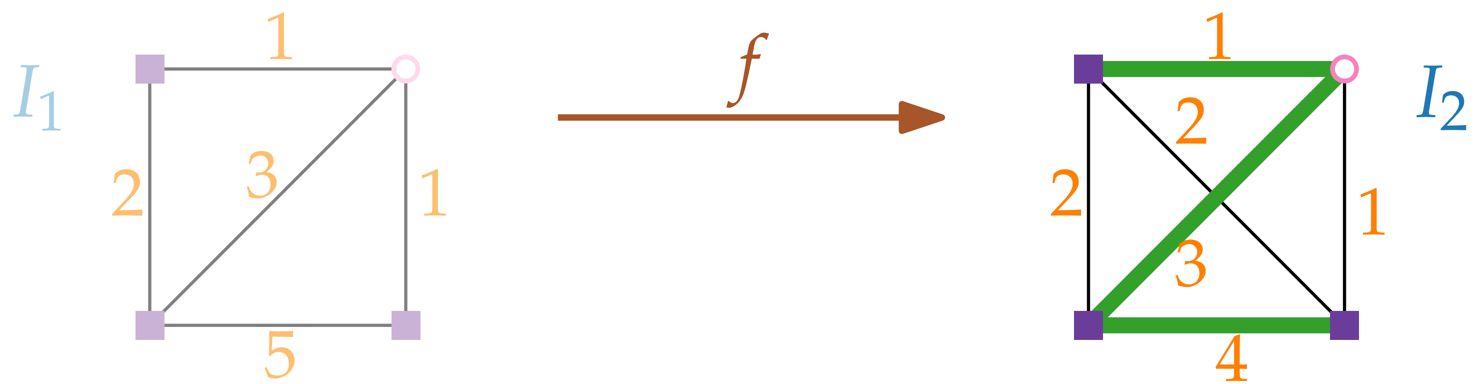
Construct $G_1' \subseteq G_1$ from B_2 by replacing each edge (u, v) of B_2 by a shortest u–v-path in G_1.



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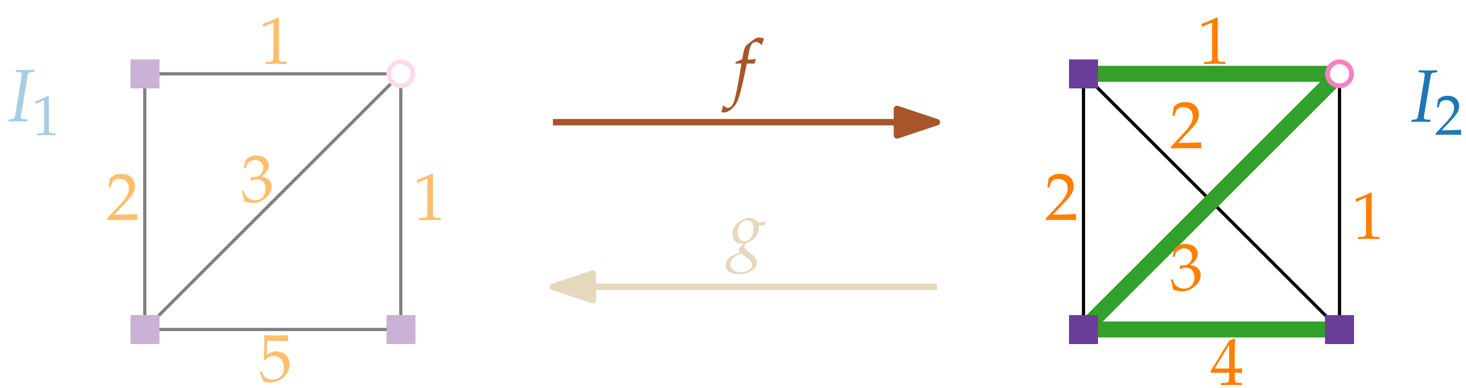

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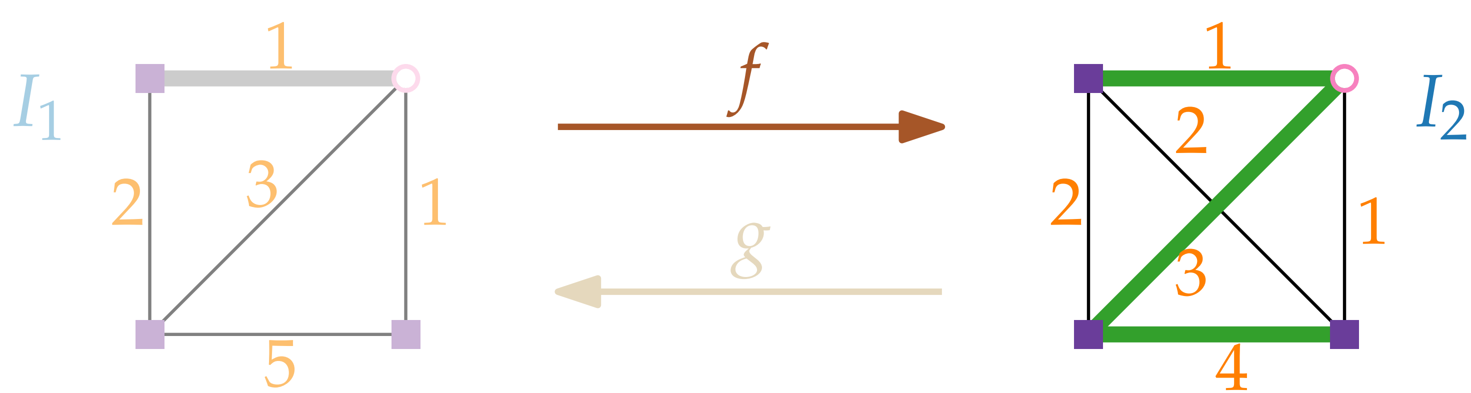

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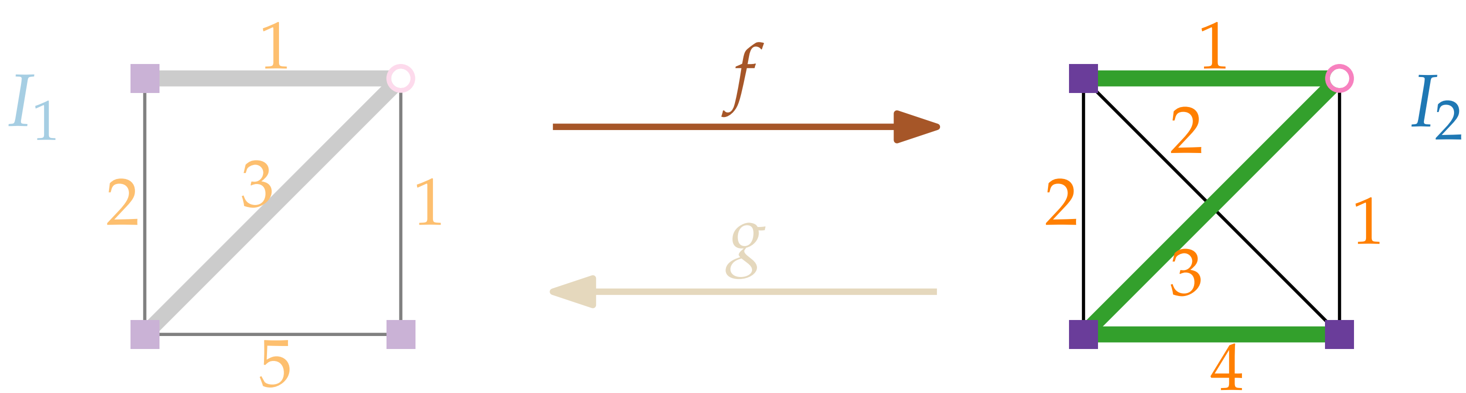

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Proof. (3) Mapping g $s \xleftarrow{g} t$

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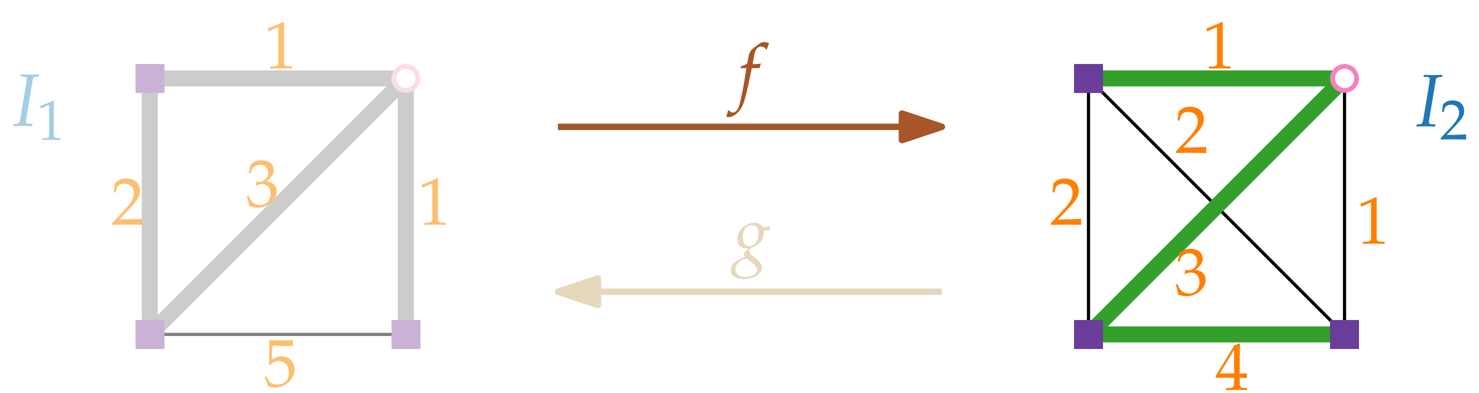

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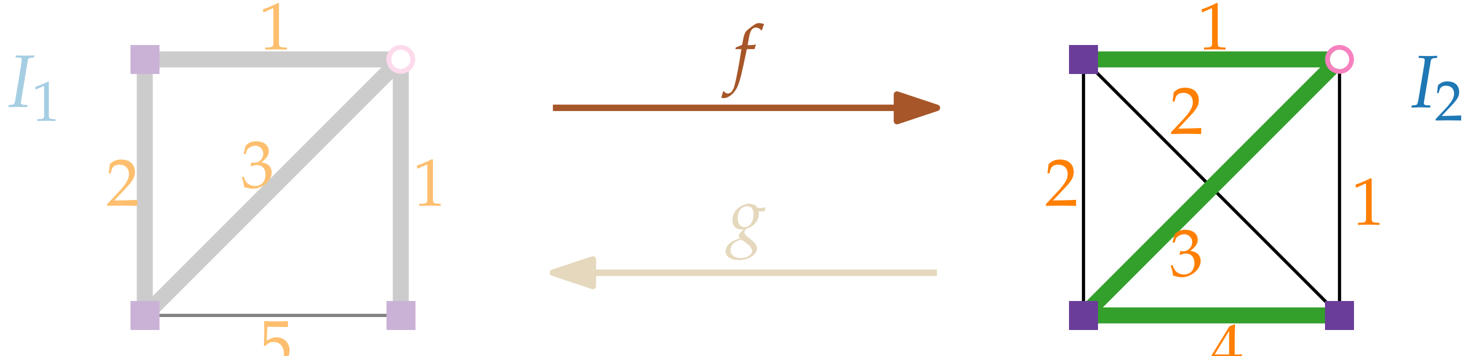
Construct $G_1' \subseteq G_1$ from B_2 by replacing each edge (u, v) of B_2 by a shortest u–v-path in G_1.

$c_1(G_1') \leq c_2(B_2)$



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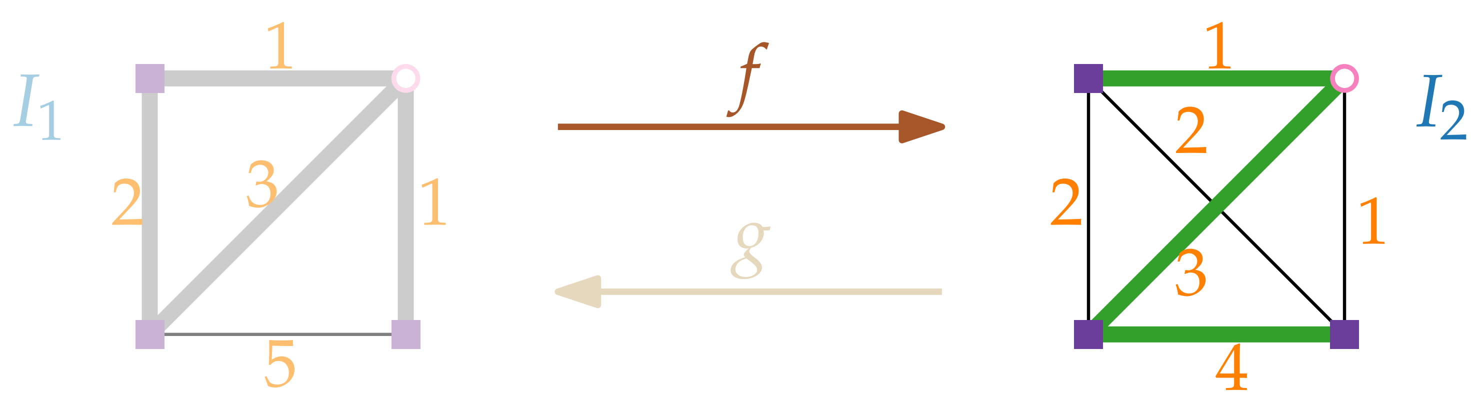
$c_1(G_1') \leq c_2(B_2)$; G_1' connects all terminals



MetricSteinerTree

Theorem. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (3) Mapping g $s \xleftarrow{g} t$

Let B_2 be Steiner tree of G_2

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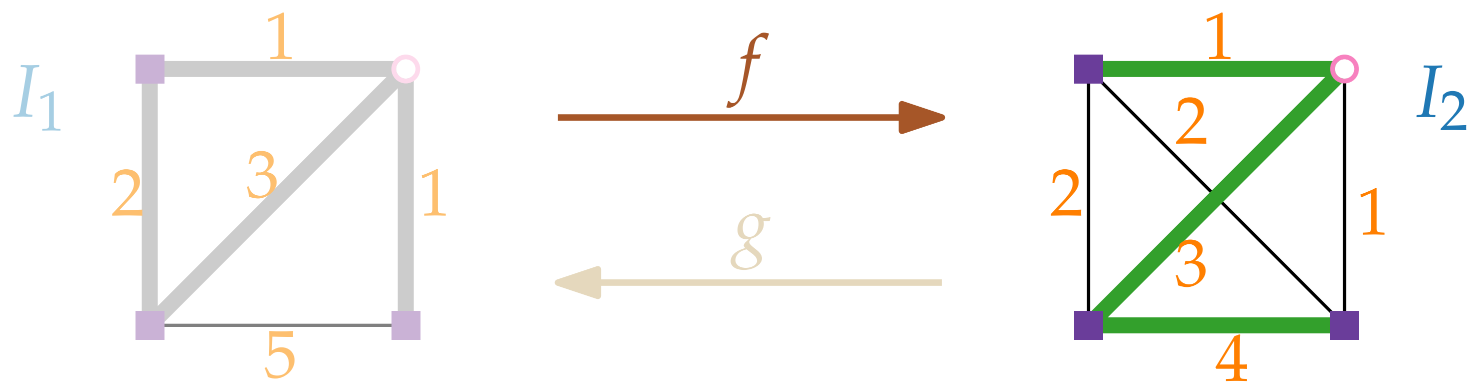

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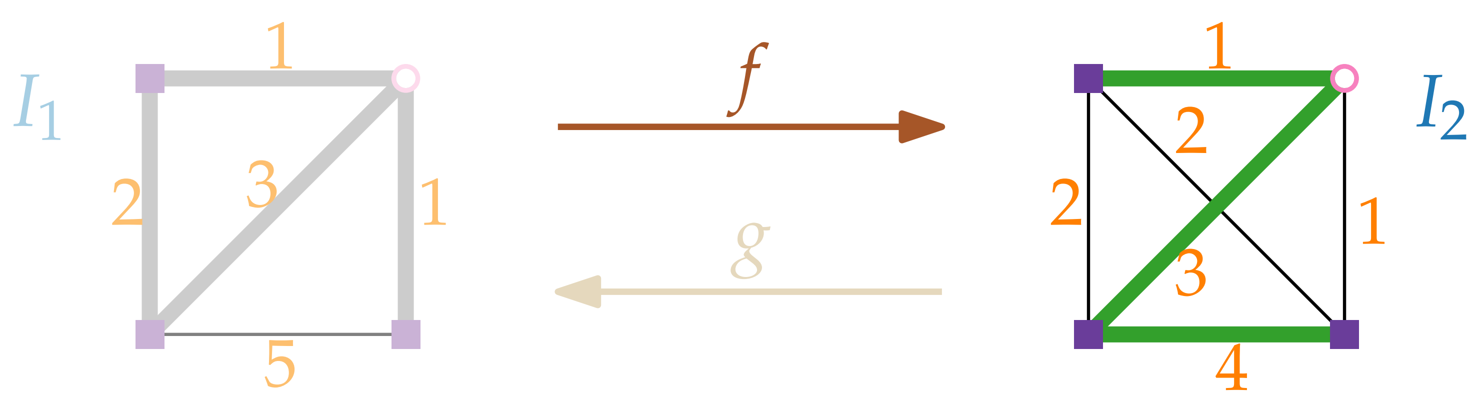

METRICSTEINERTREE

Theorem. There is an approximation preserving reduction from STEINERTREE to METRICSTEINERTREE.

Proof. (3) Mapping g

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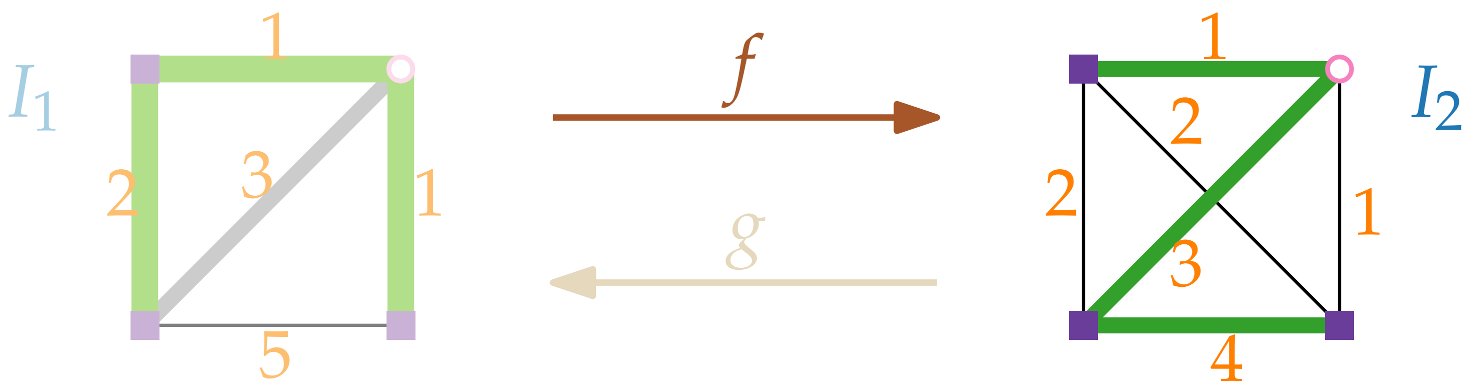

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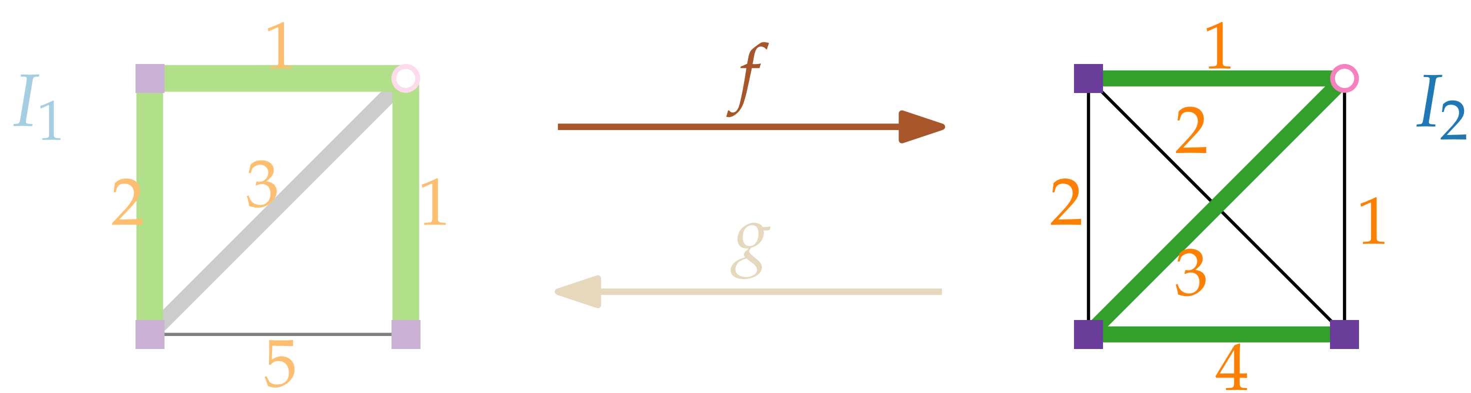
Consider spanning tree B_1 of G'_1 $\rightsquigarrow$ Steiner tree B_1 of G_1



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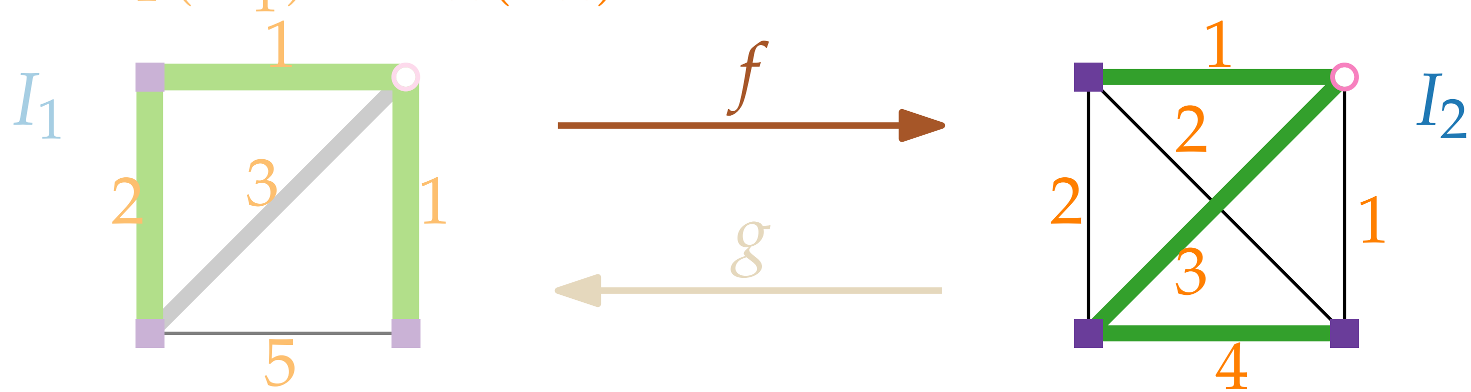

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part IV:
2-Approximation for SteinerTree

Philipp Kindermann

Summer Semester 2020

2-Approximation for SteinerTree

2-Approximation for SteinerTree

Theorem. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T]$ induced by the terminal set T. Then $c(B) \leq 2 \cdot \mathrm{OPT}$.

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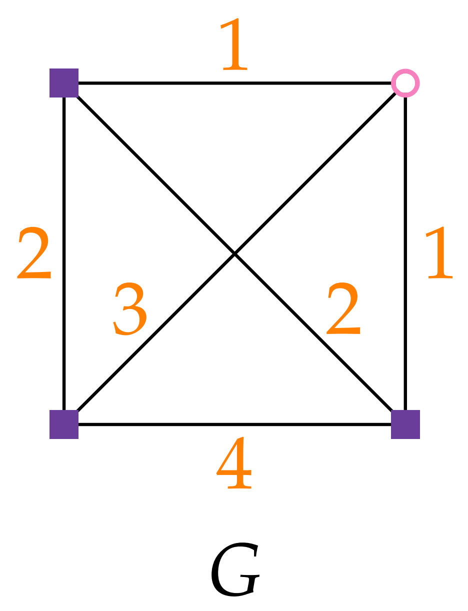

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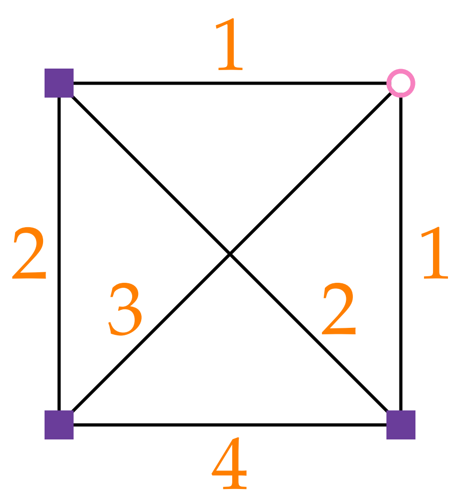
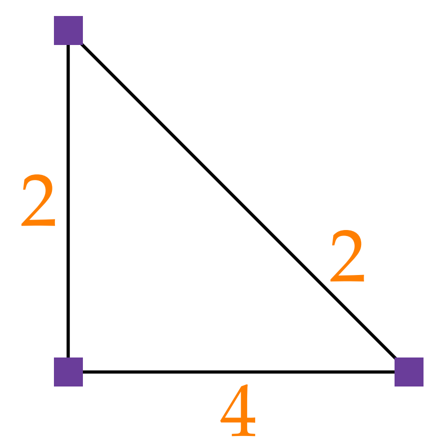


G



$G[T]$

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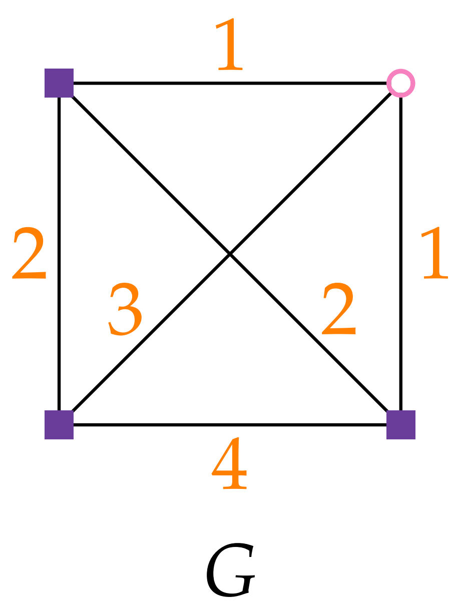

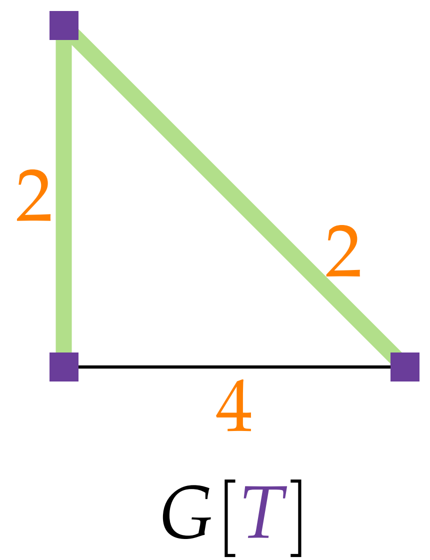

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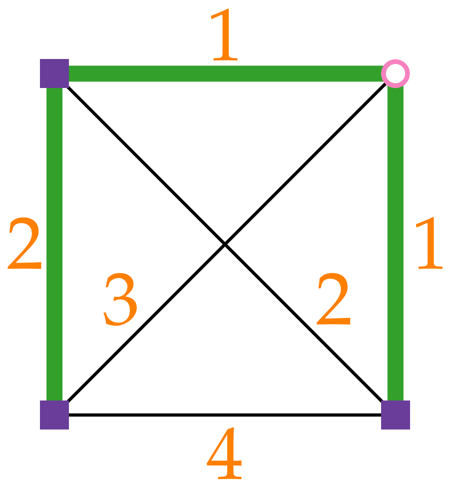

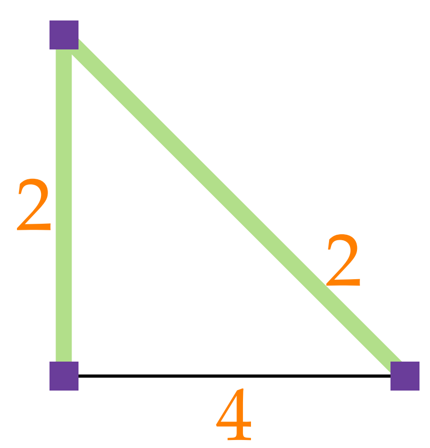

Proof of Approximation Factor

Consider optimal Steiner tree B^*

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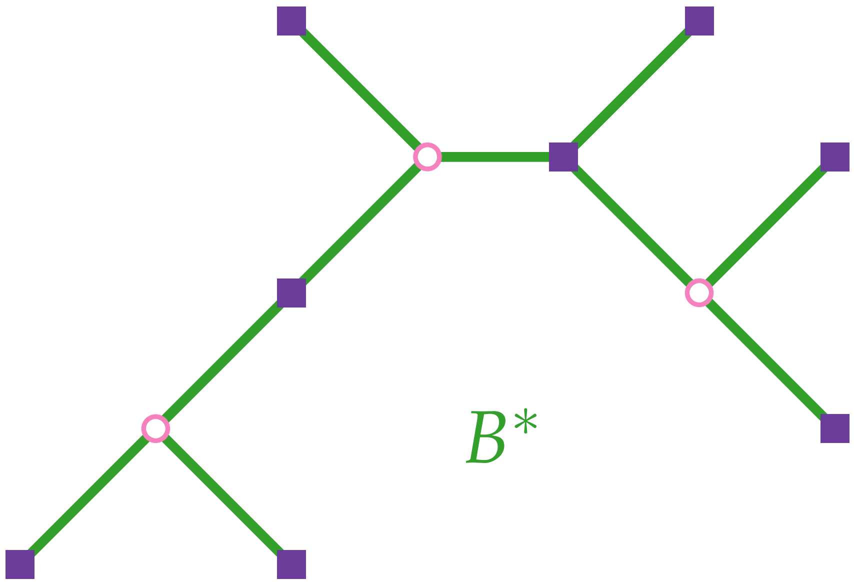

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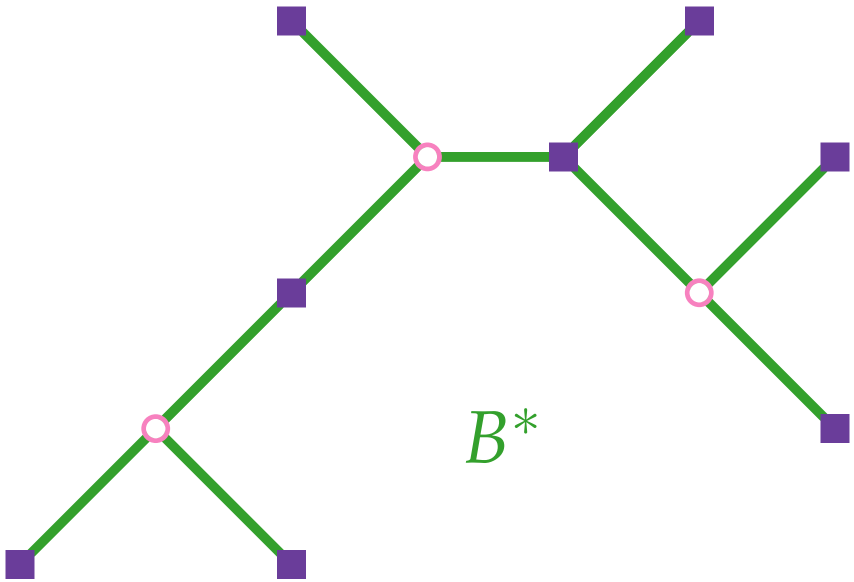
Duplicate all edges in B^* $\rightsquigarrow$ Eulerian (multi-)graph B' with cost $c(B') = 2 \cdot \mathrm{OPT}$



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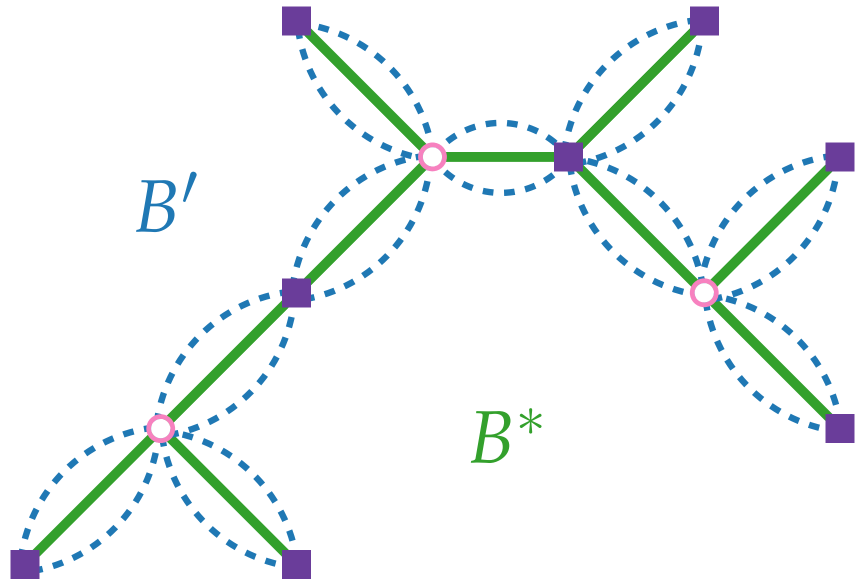

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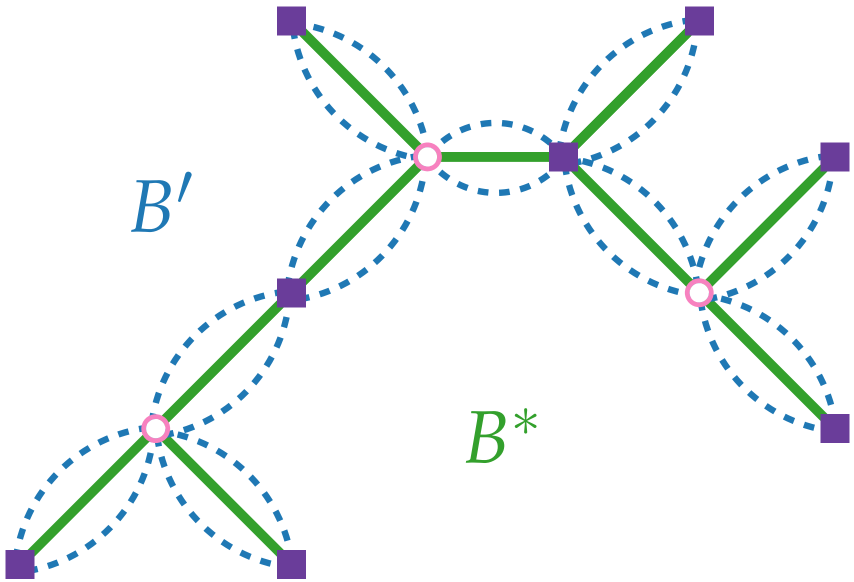

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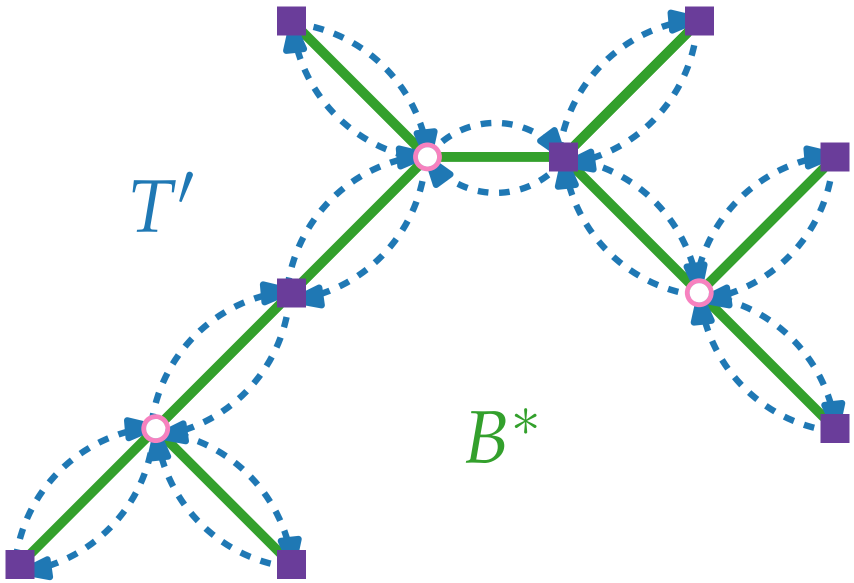

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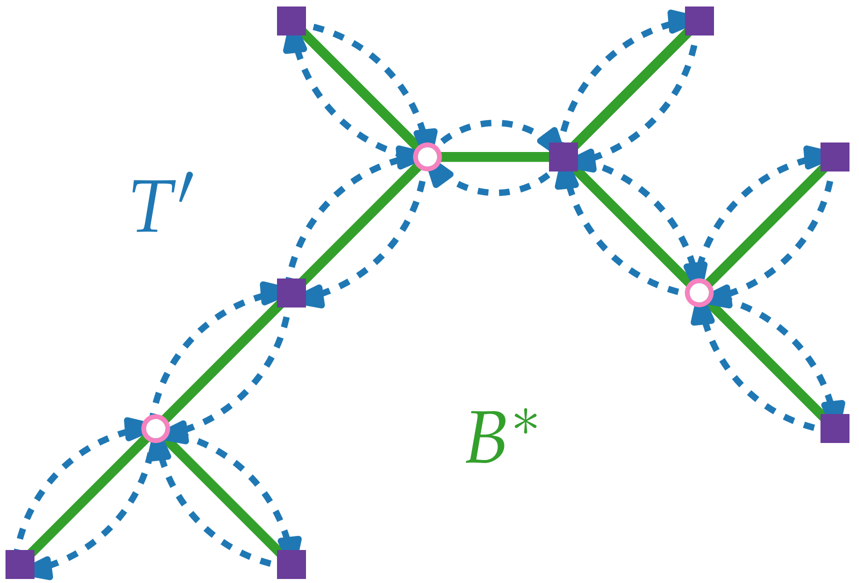

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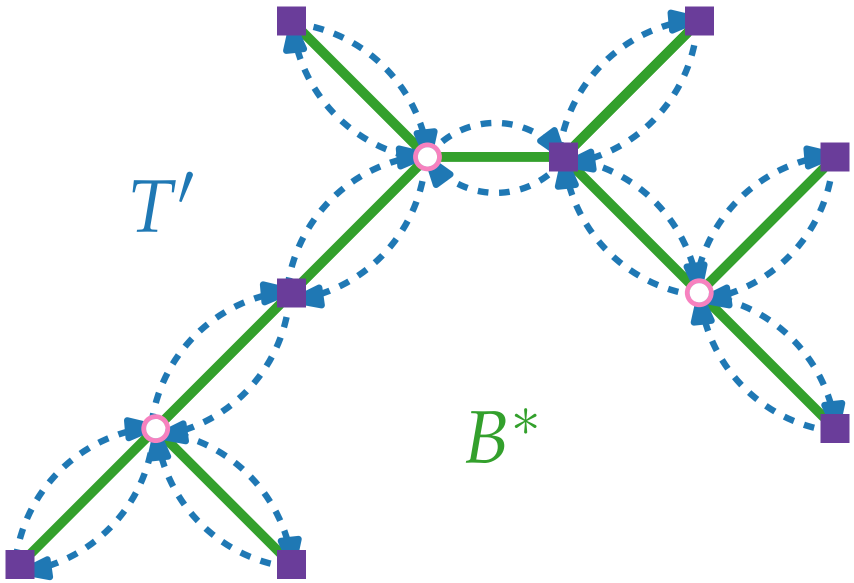

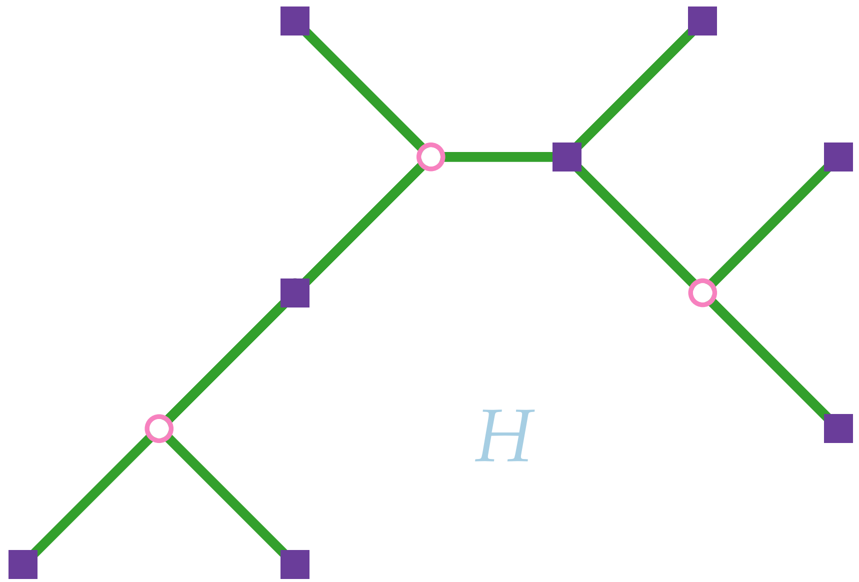

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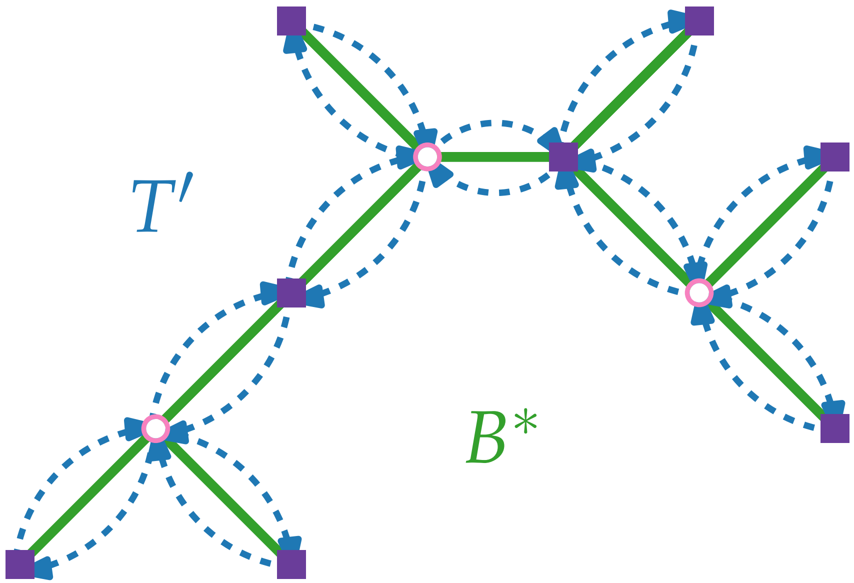

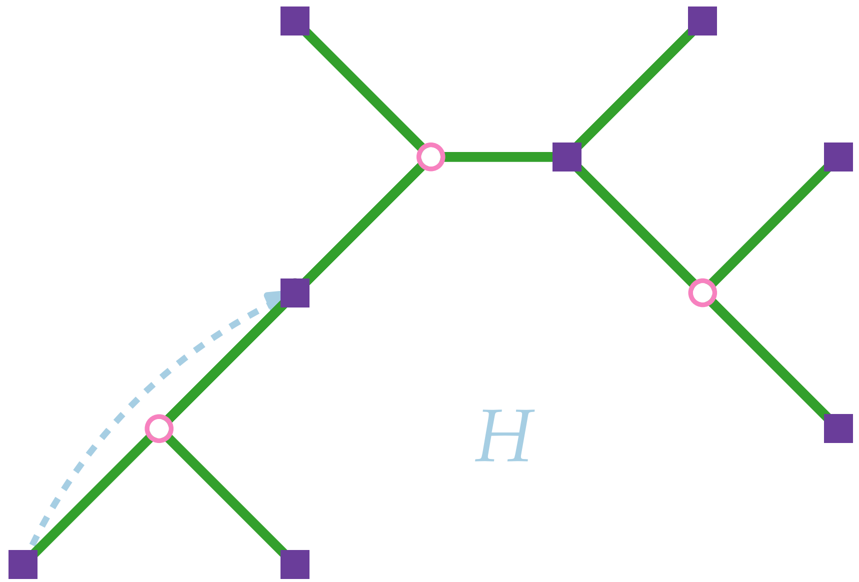

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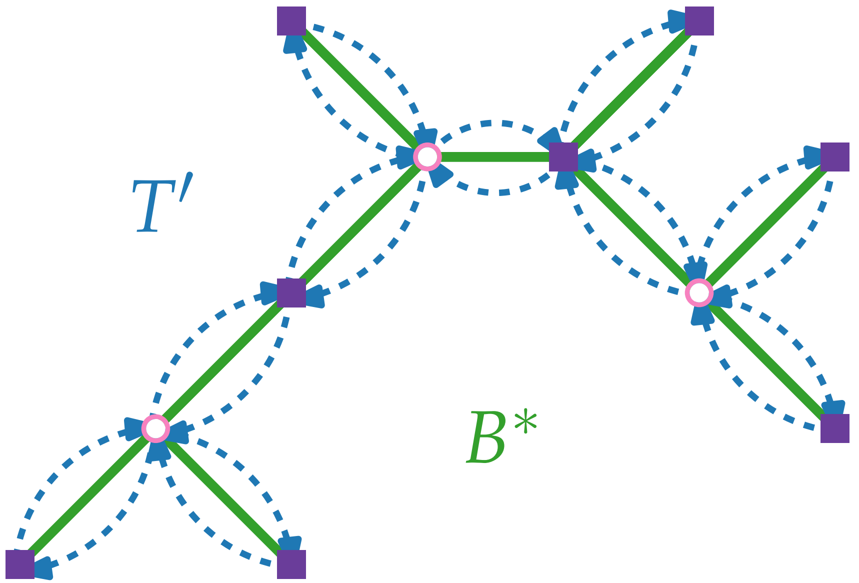

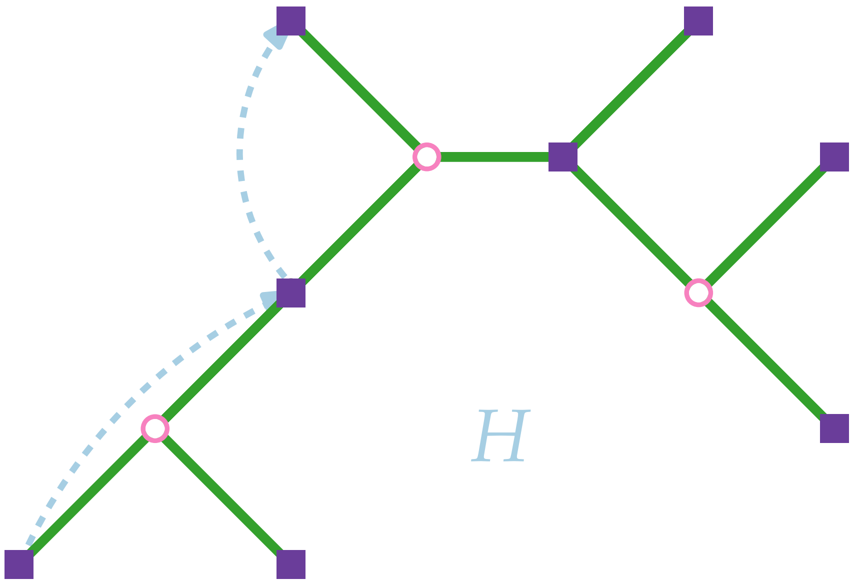

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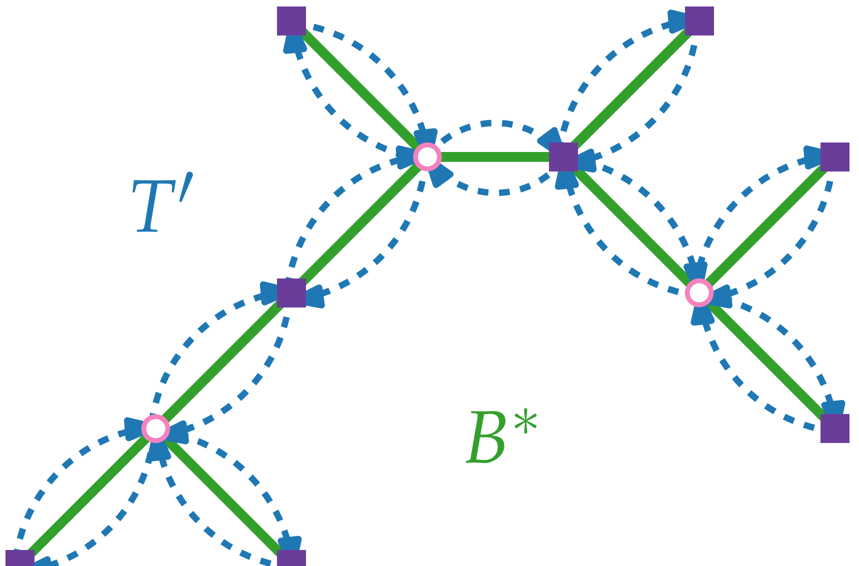

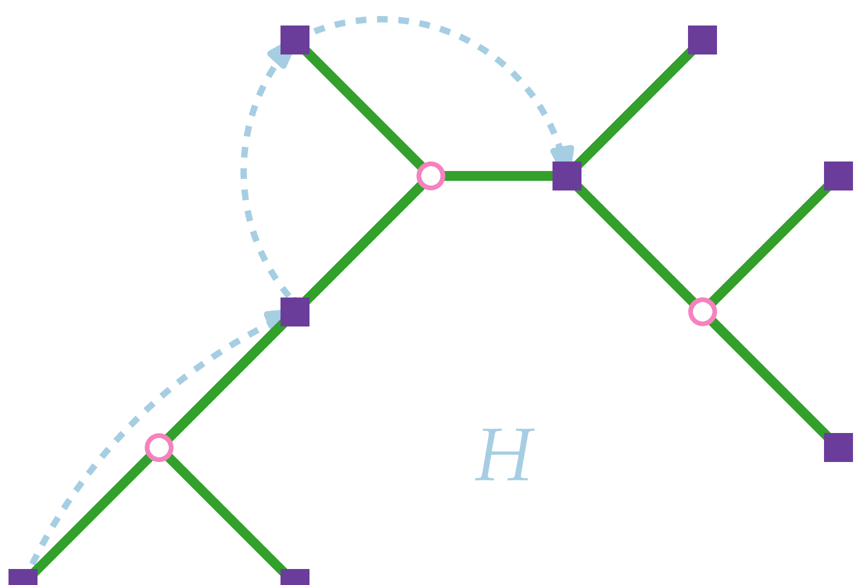

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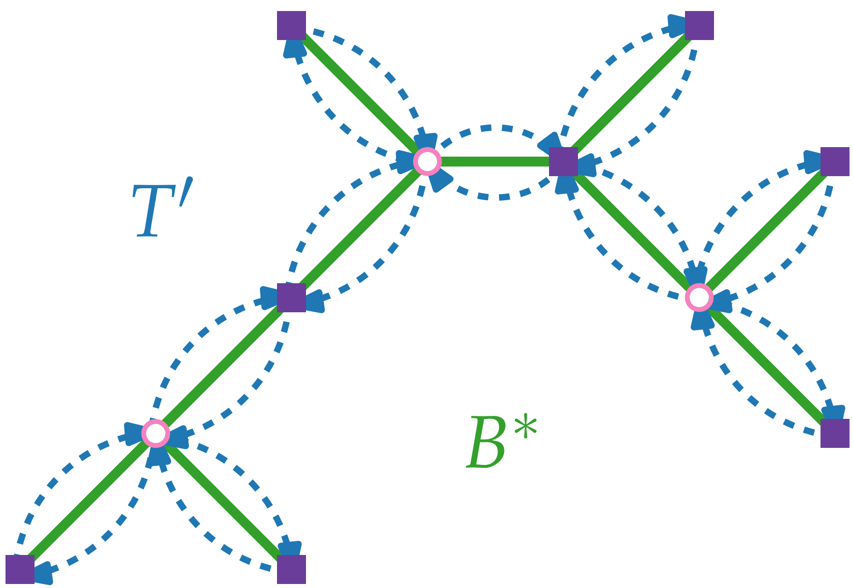

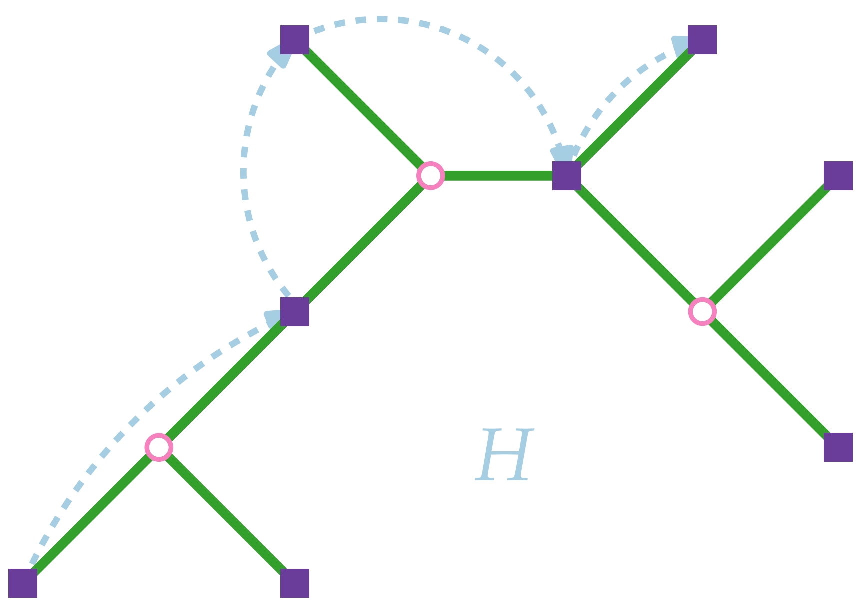

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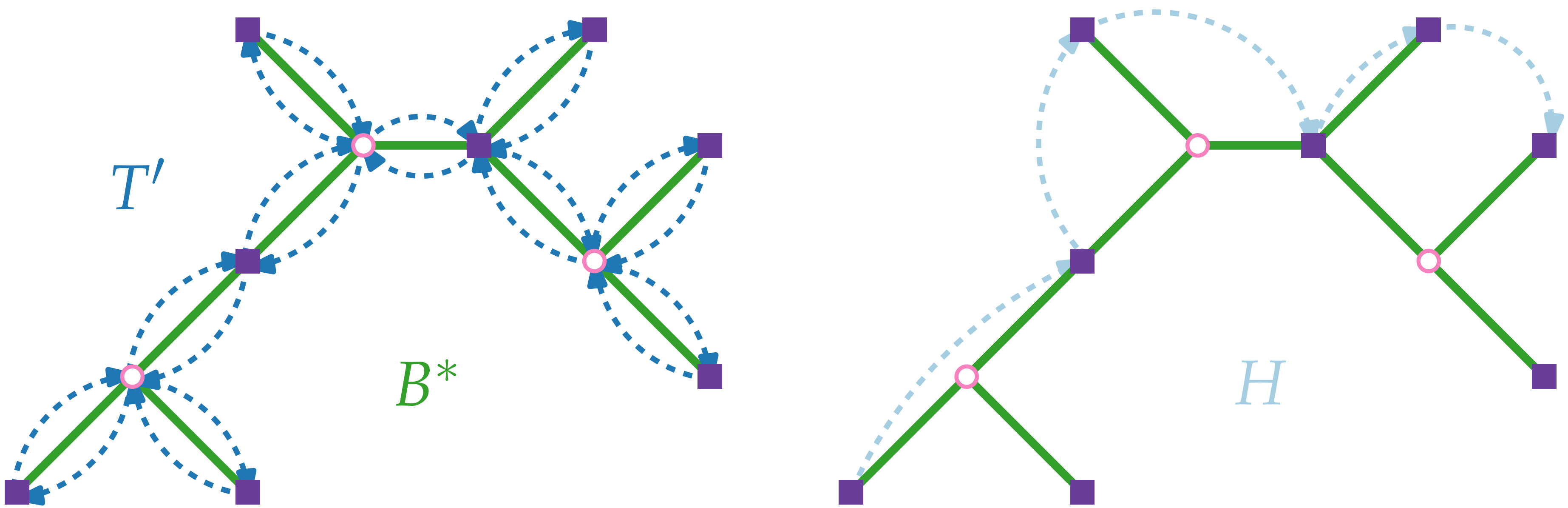

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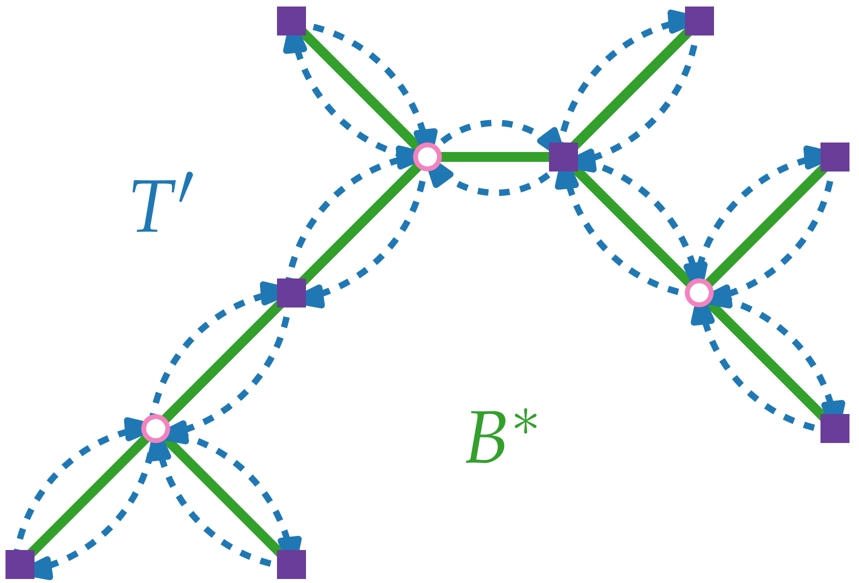

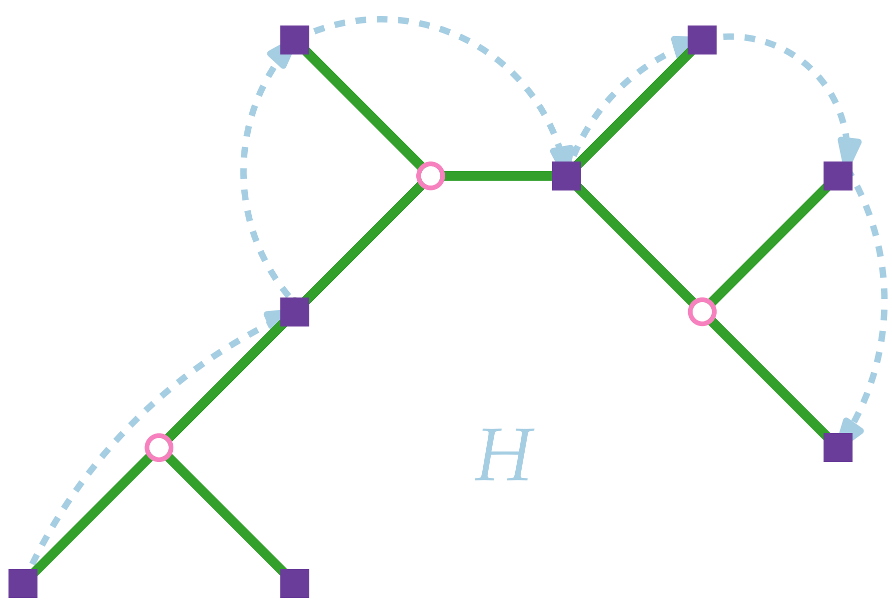

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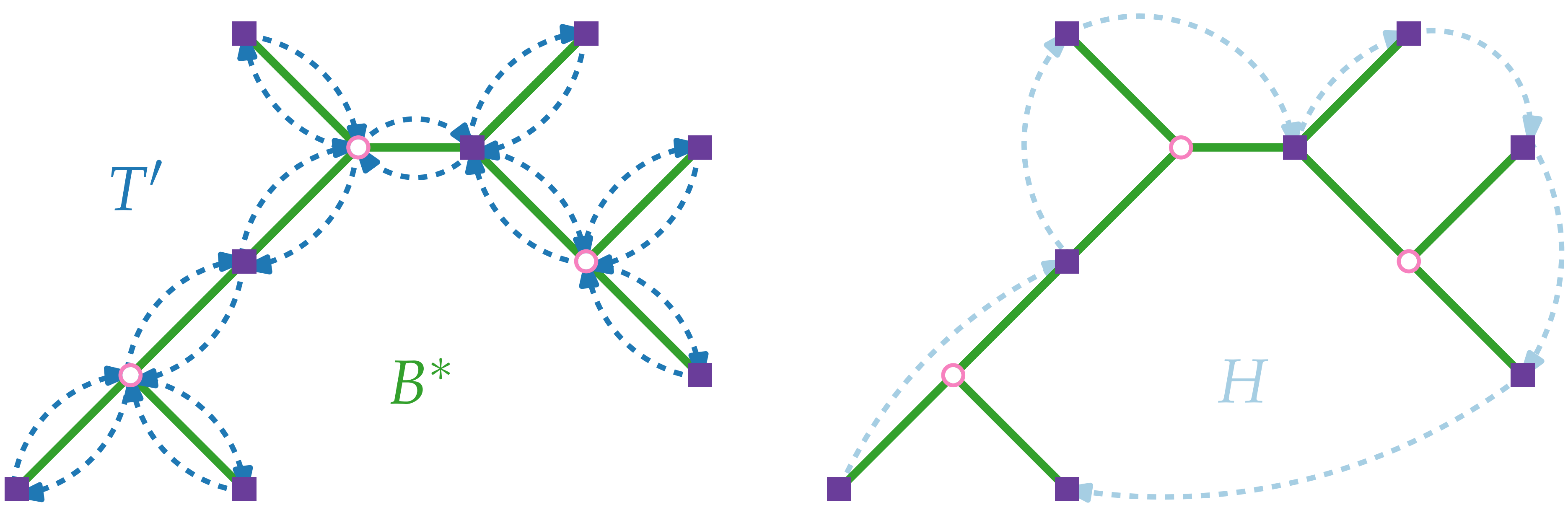

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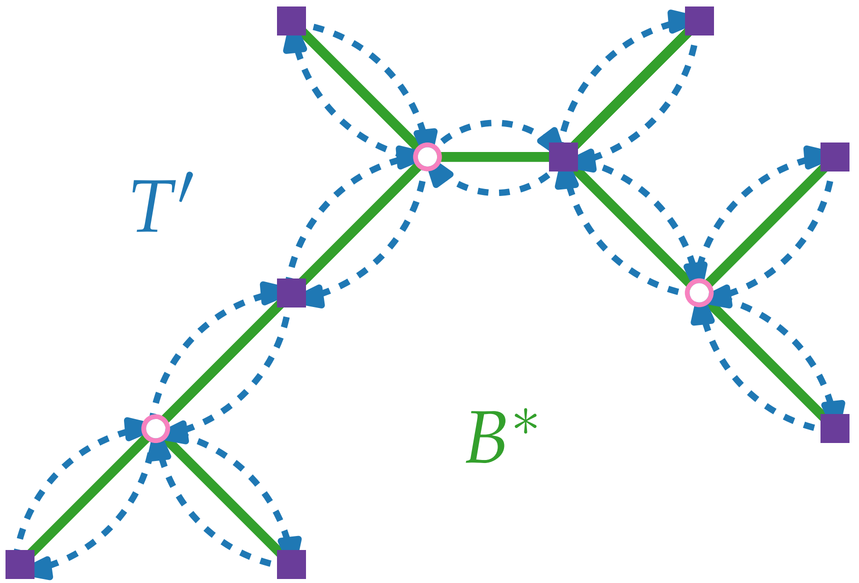

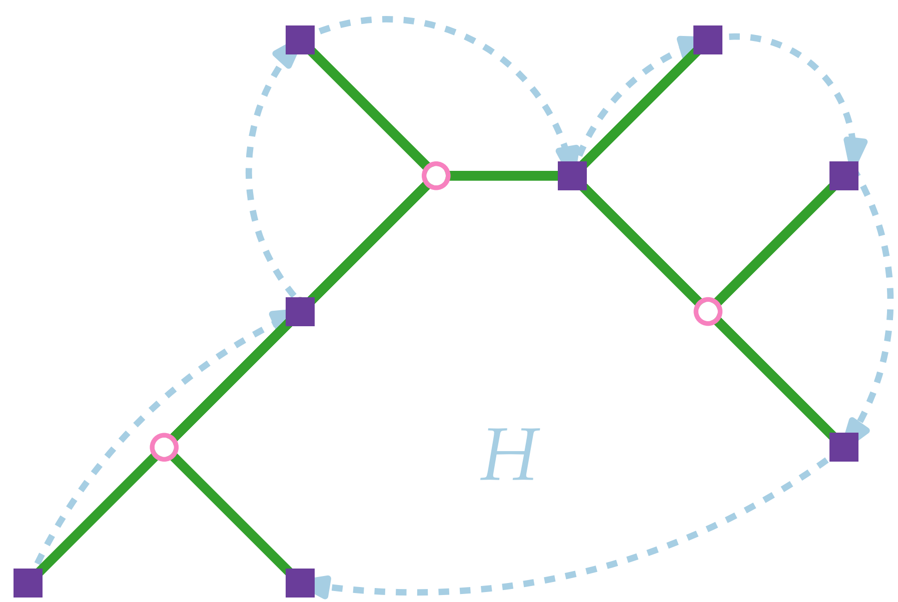

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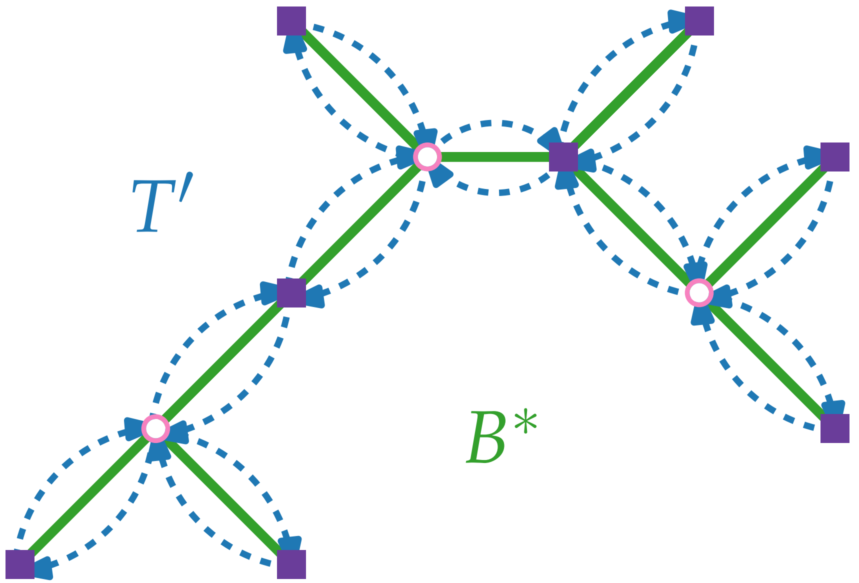

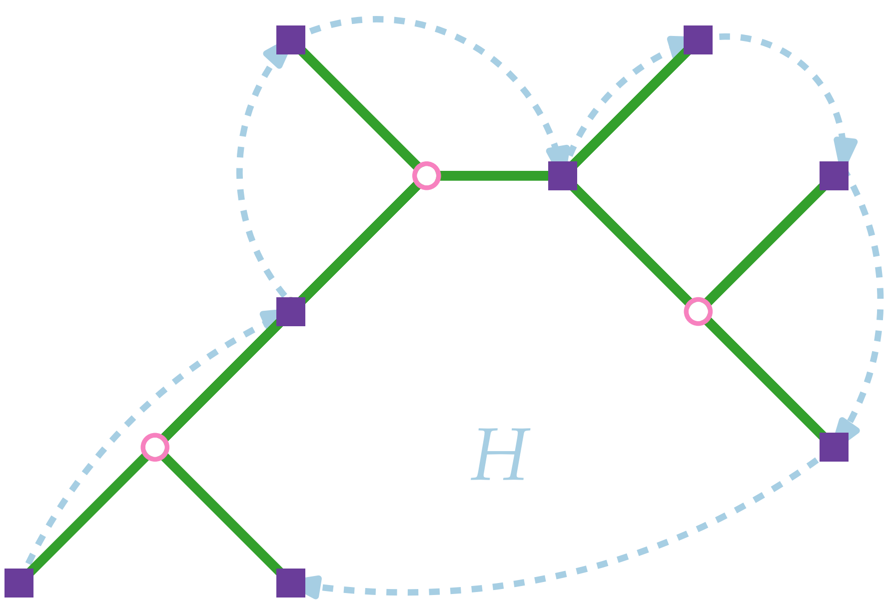

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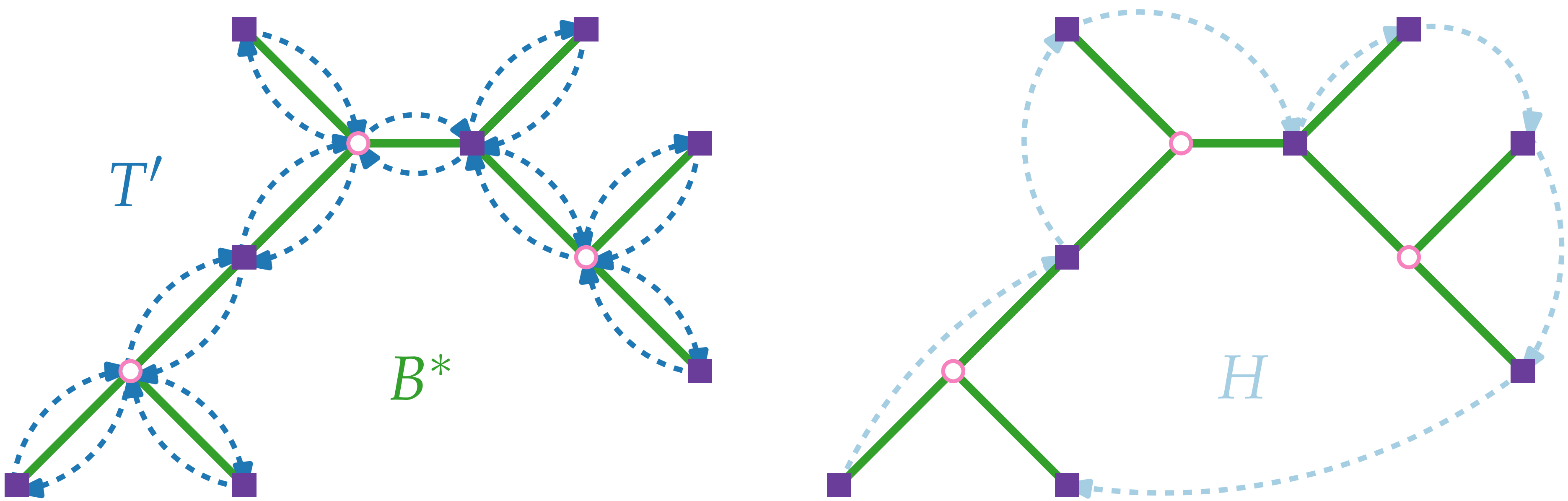

Analysis Sharp?

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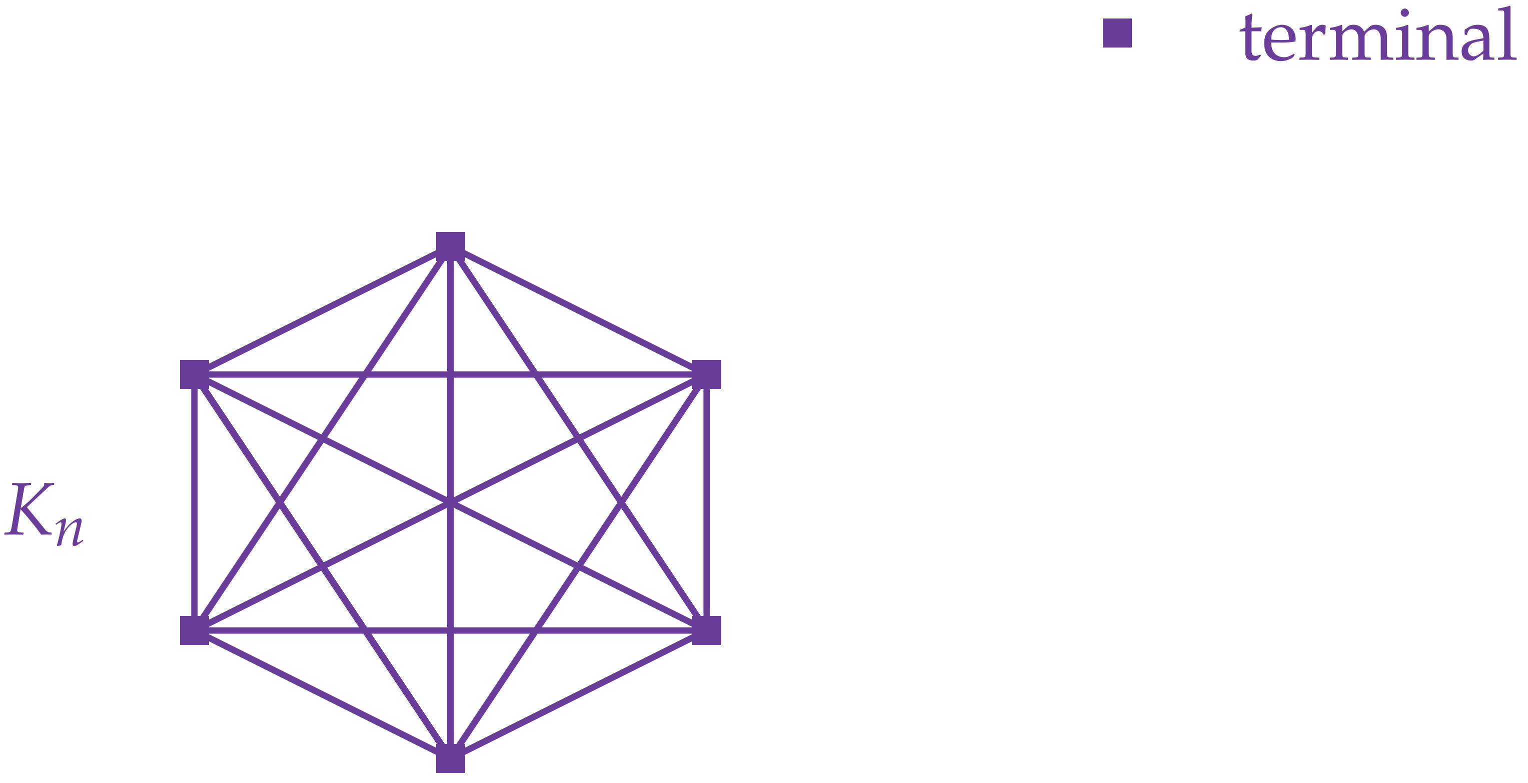

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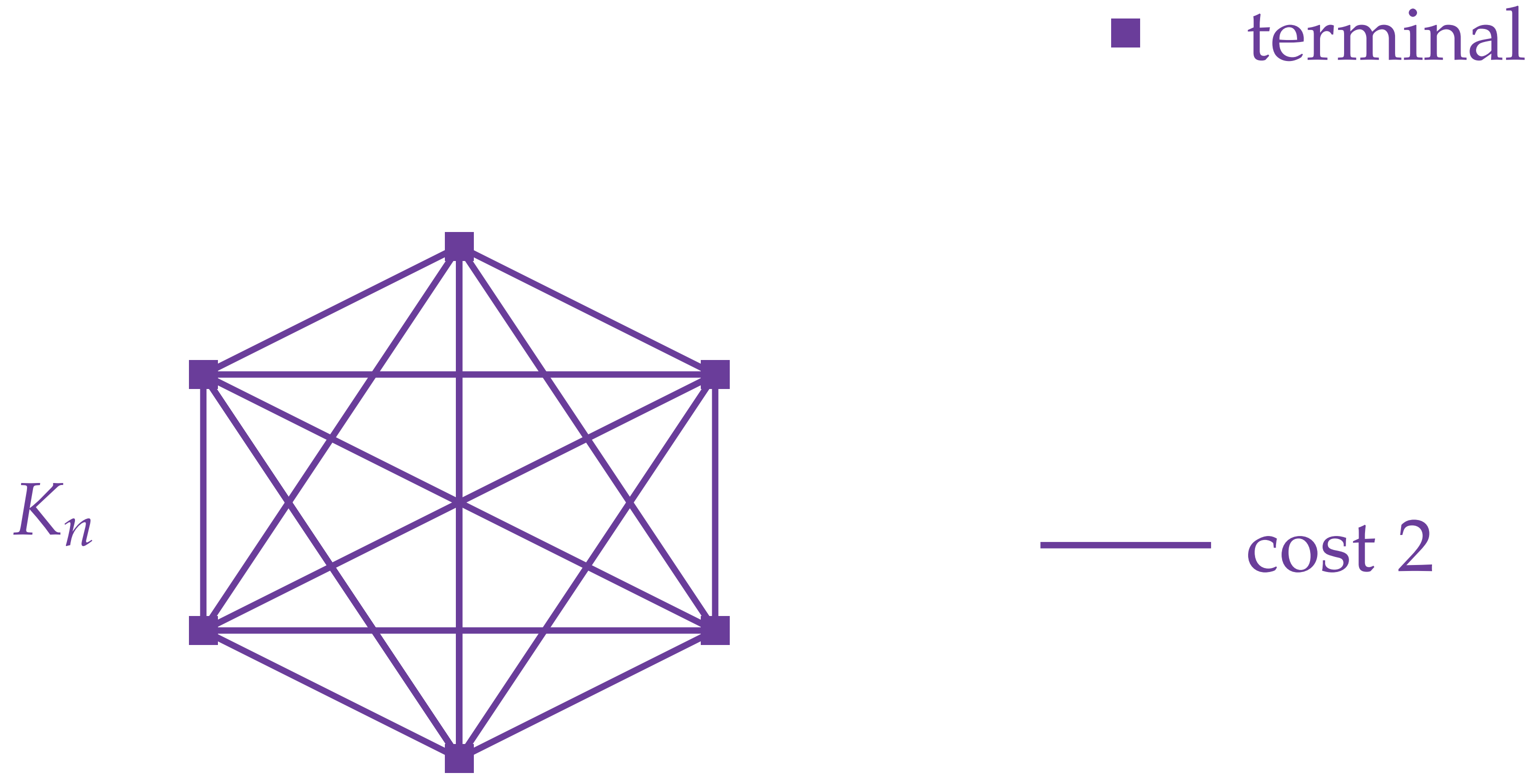

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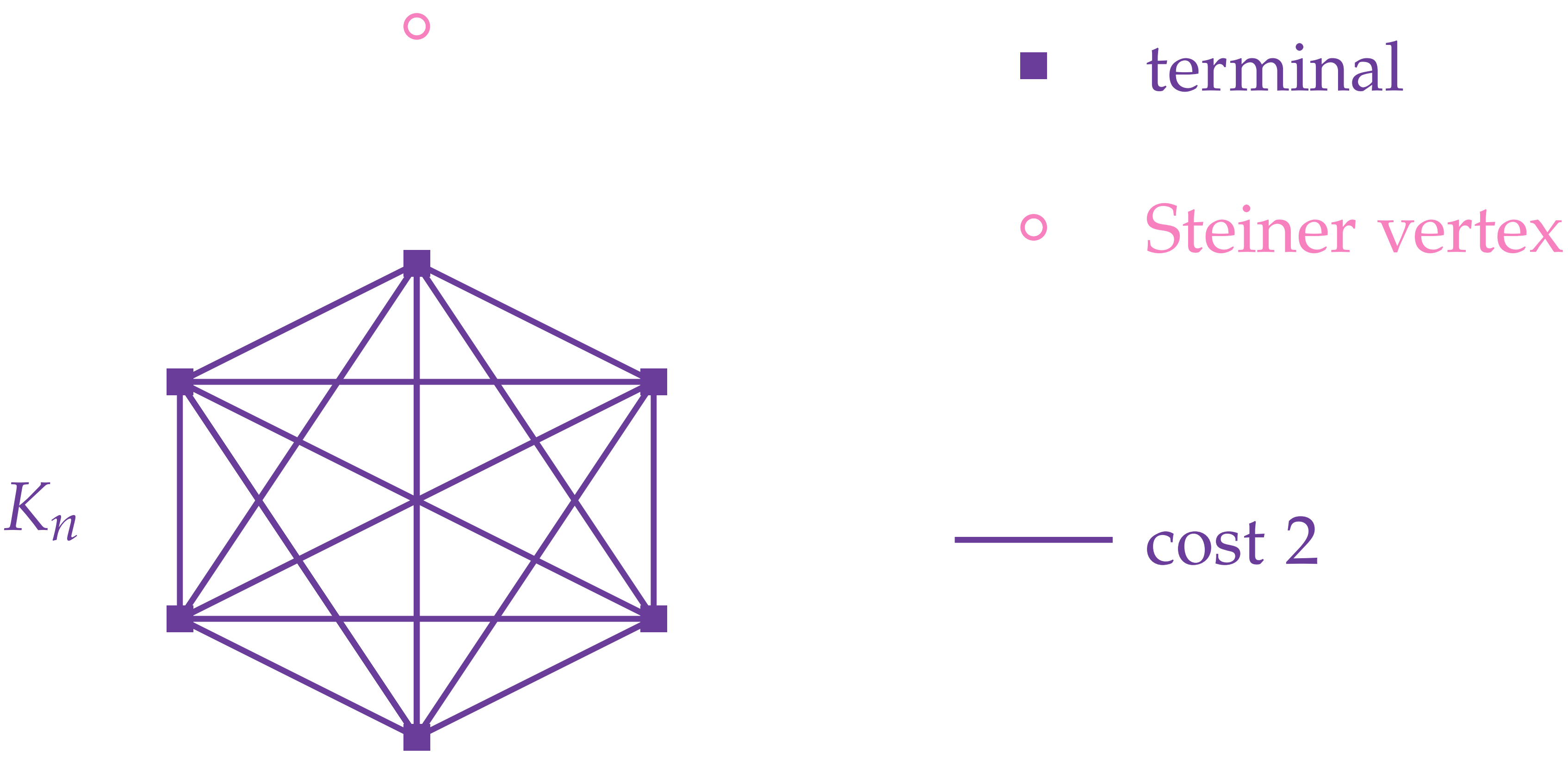

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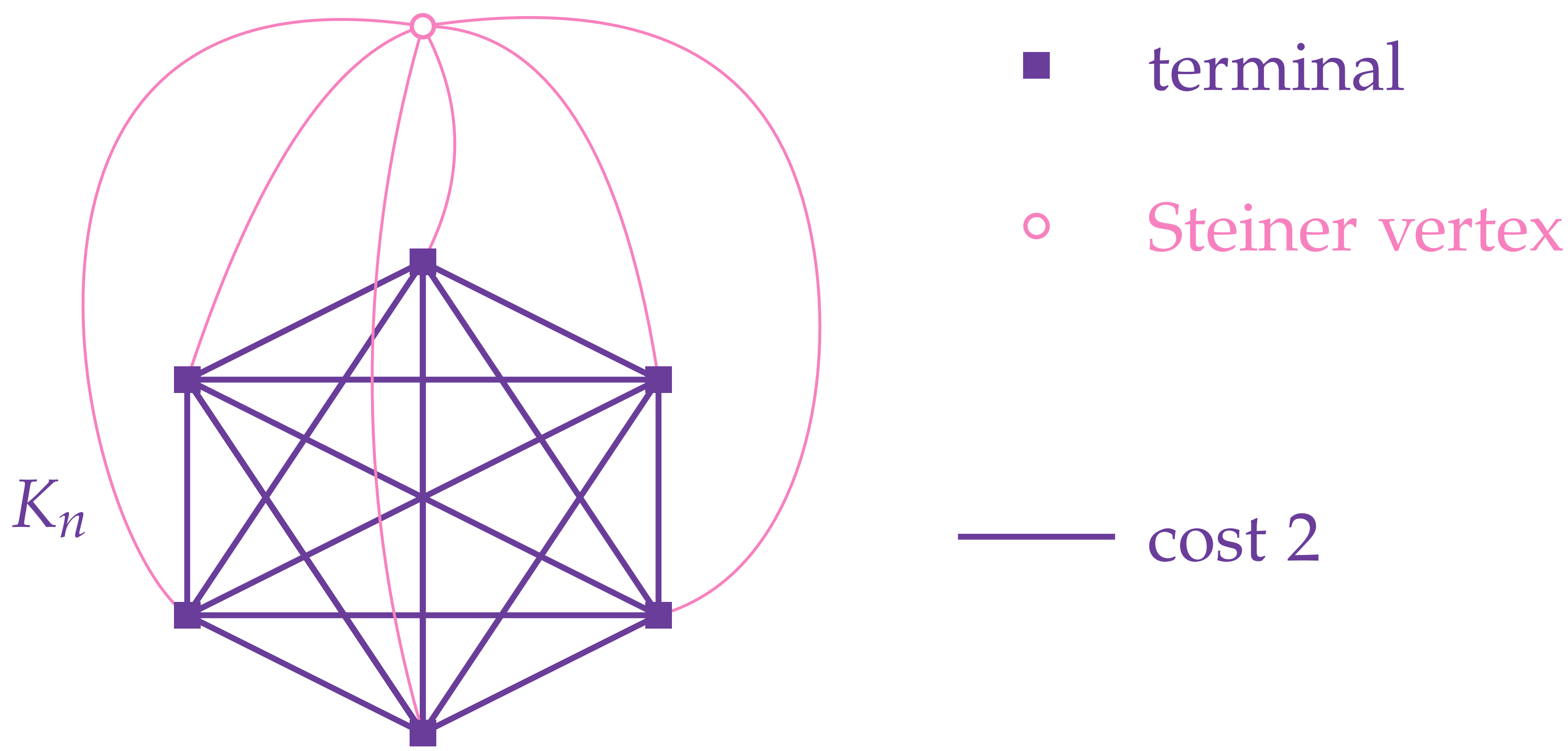

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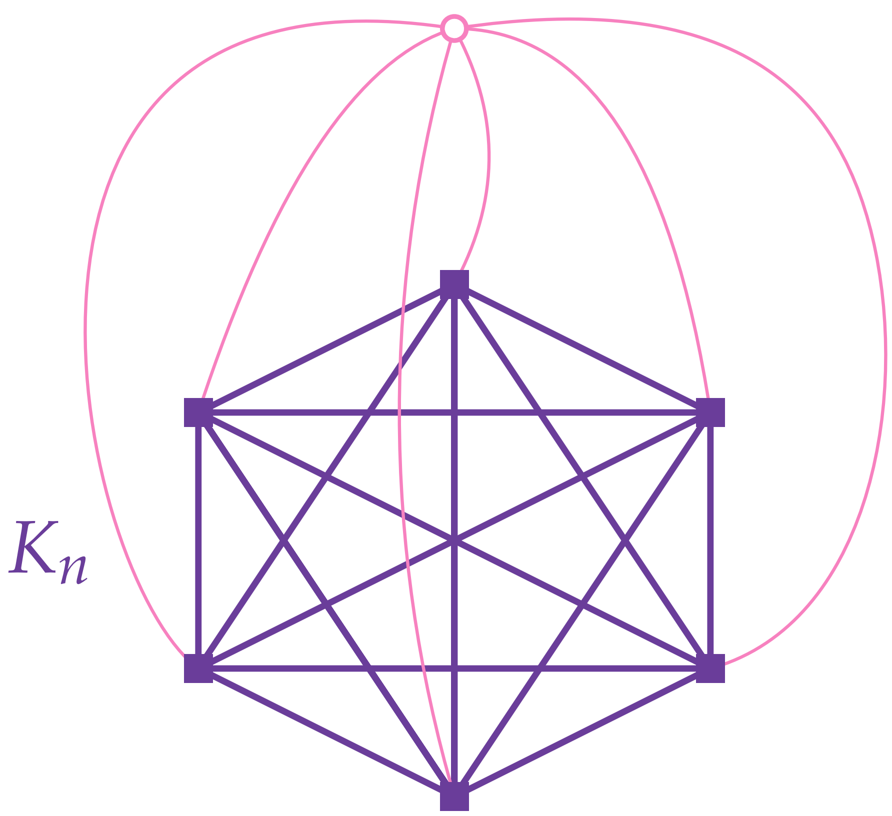
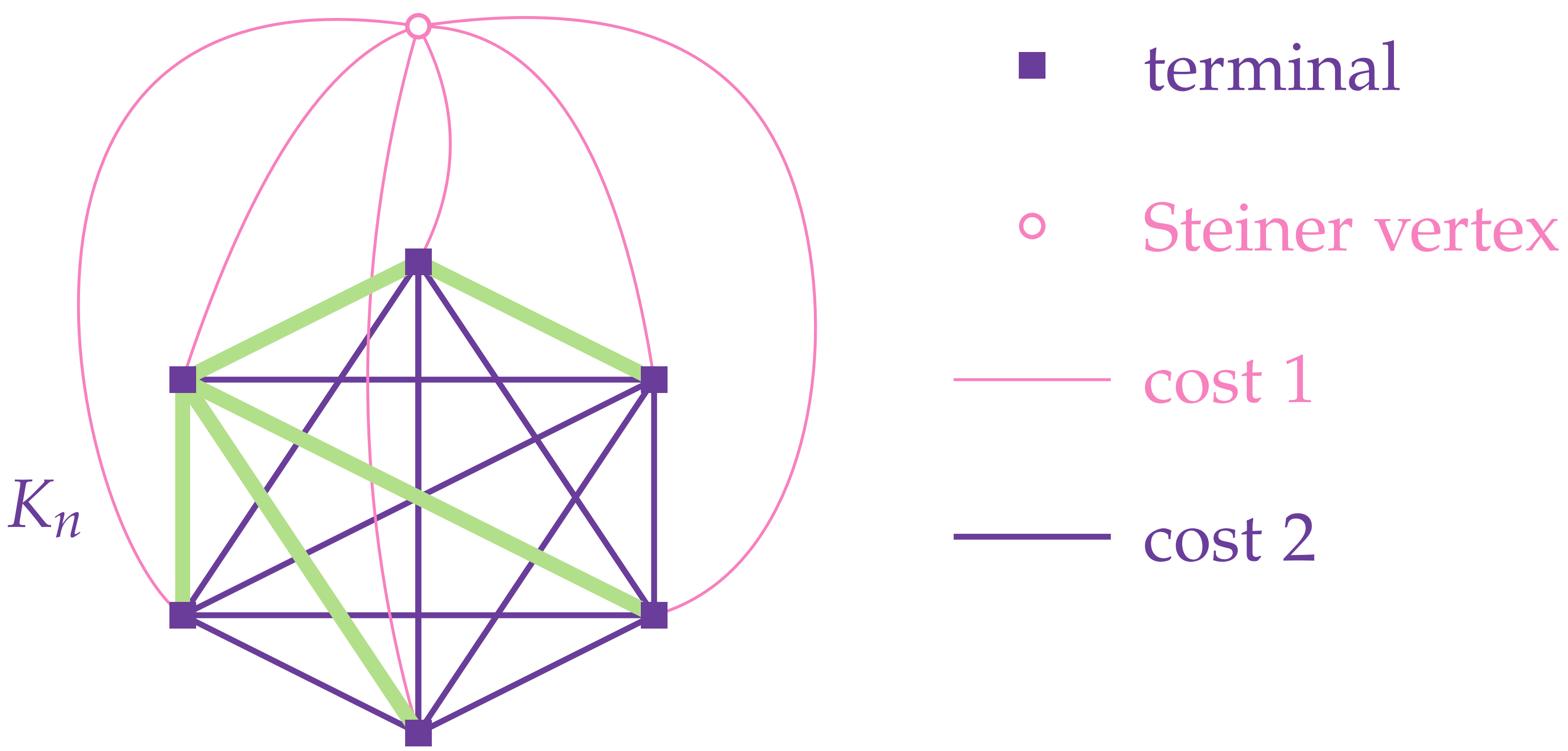


- ■ terminal
- ○ Steiner vertex
- cost 1
- cost 2

Analysis Sharp?

MST of $G[T]$ with cost $2(n-1)$



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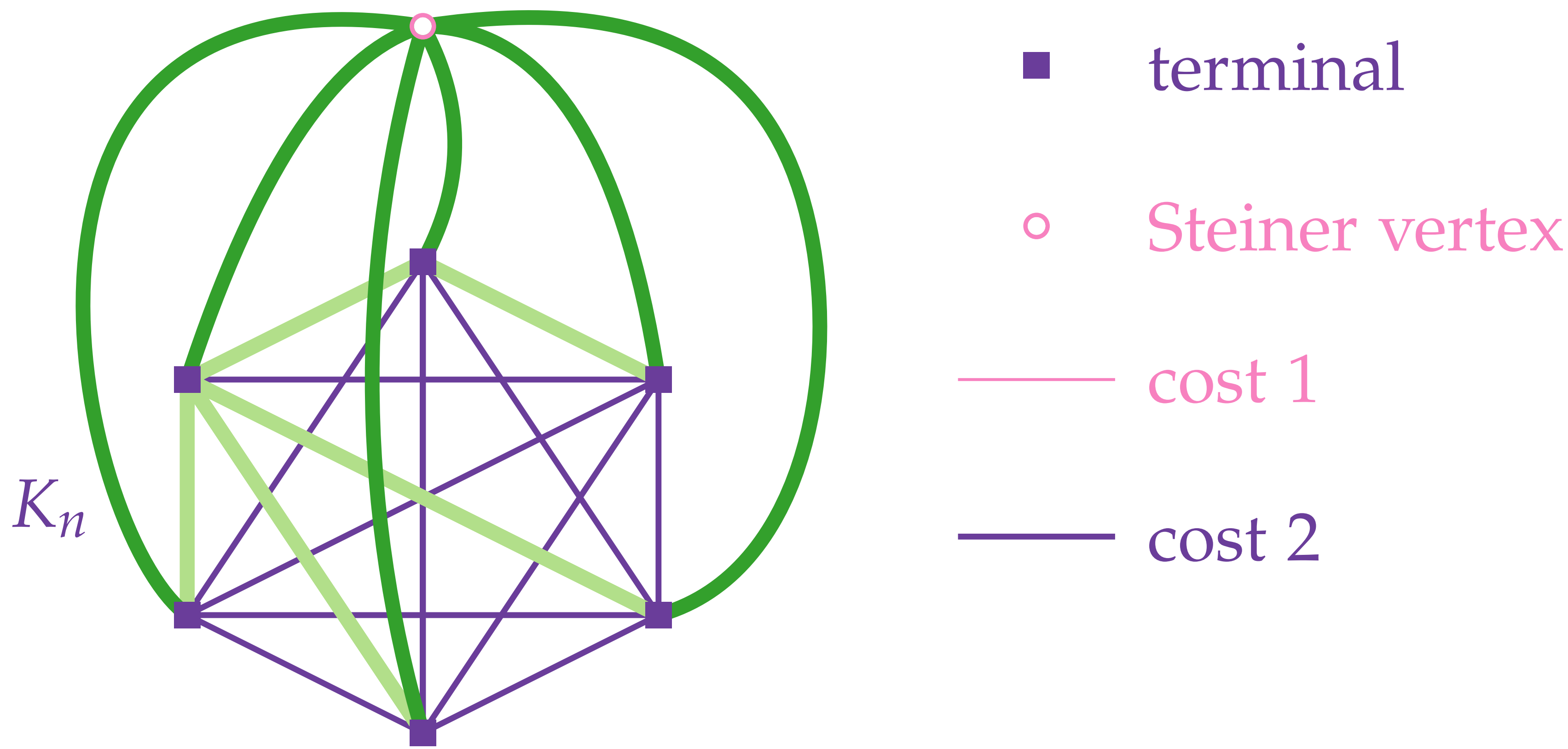
MST of $G[T]$ with cost $2(n-1)$
Optimal solution with cost n



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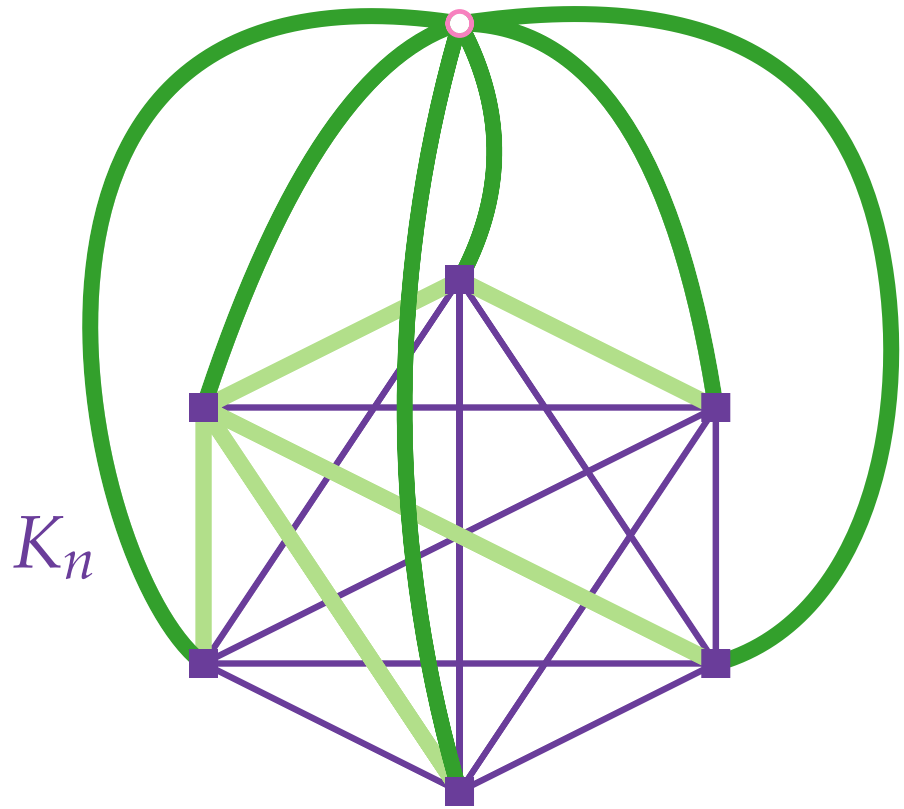
$$\frac{2(n-1)}{n} \to 2$$



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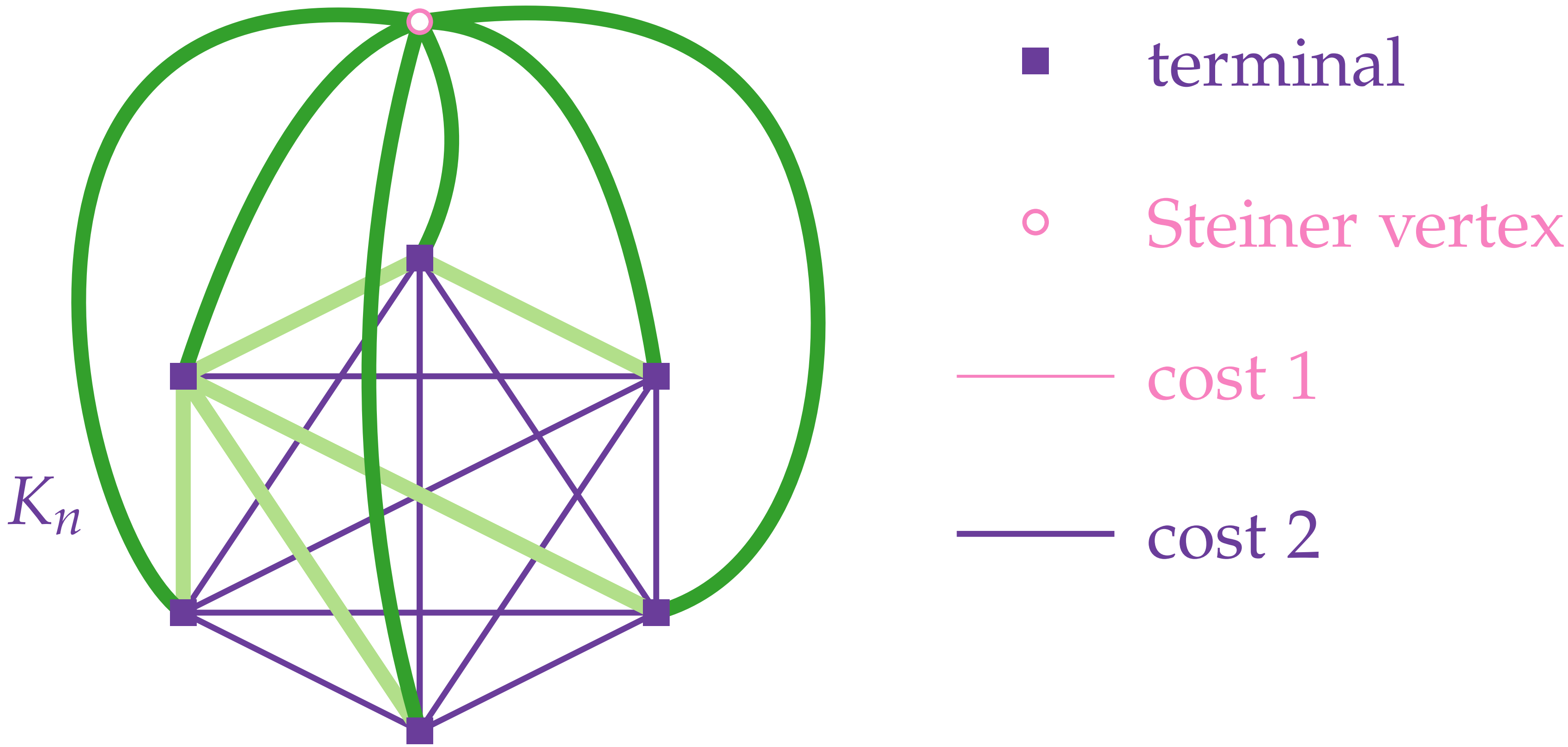
Analysis Sharp?

MST of $G[T]$ with cost $2(n-1)$
Optimal solution with cost n

$$\frac{2(n-1)}{n} \to 2$$



better?

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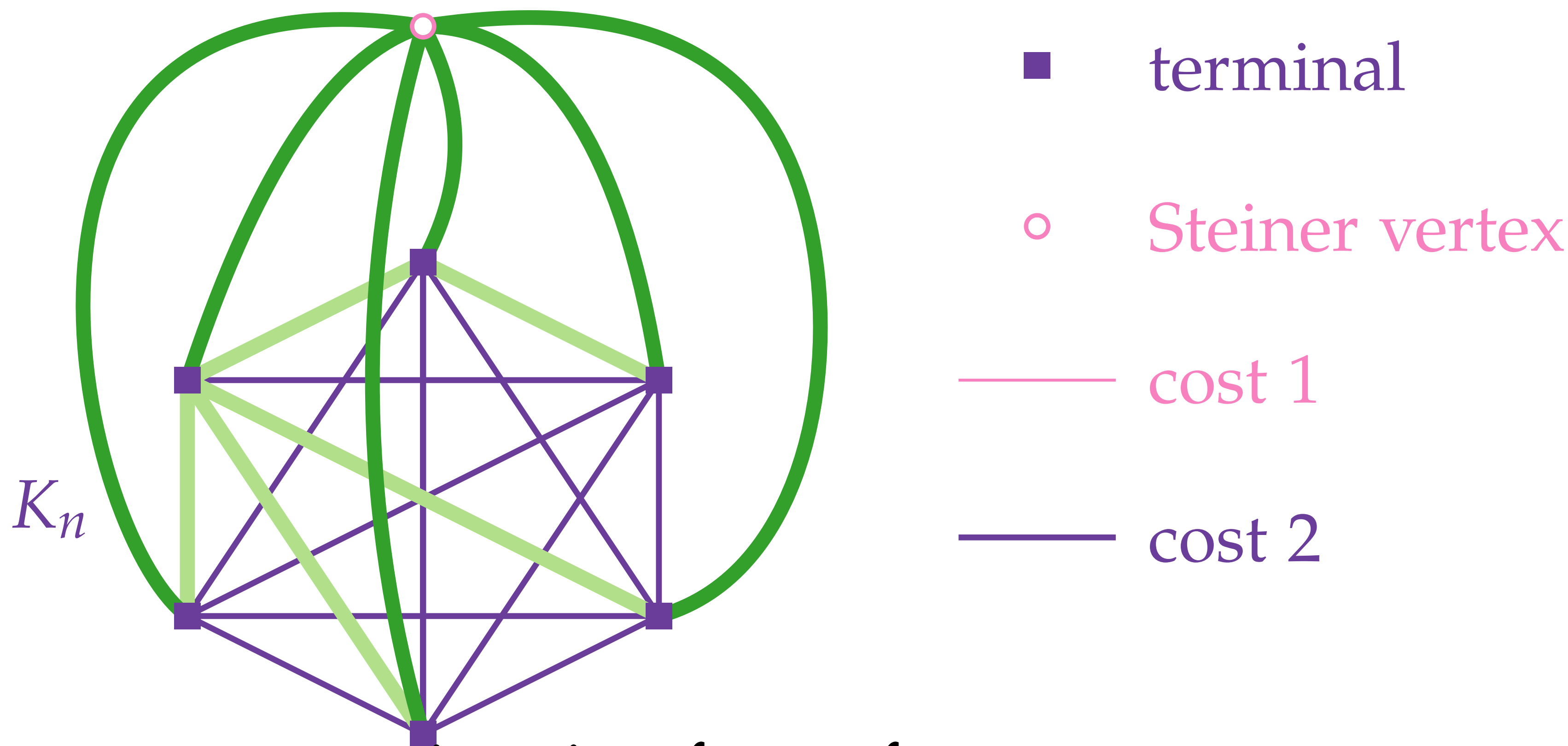

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The best-known approximation factor for SteinerTree is $\ln(4) + \varepsilon \approx 1.39$

[Byrka, Grandoni, Rothvoß & Sanita '10]

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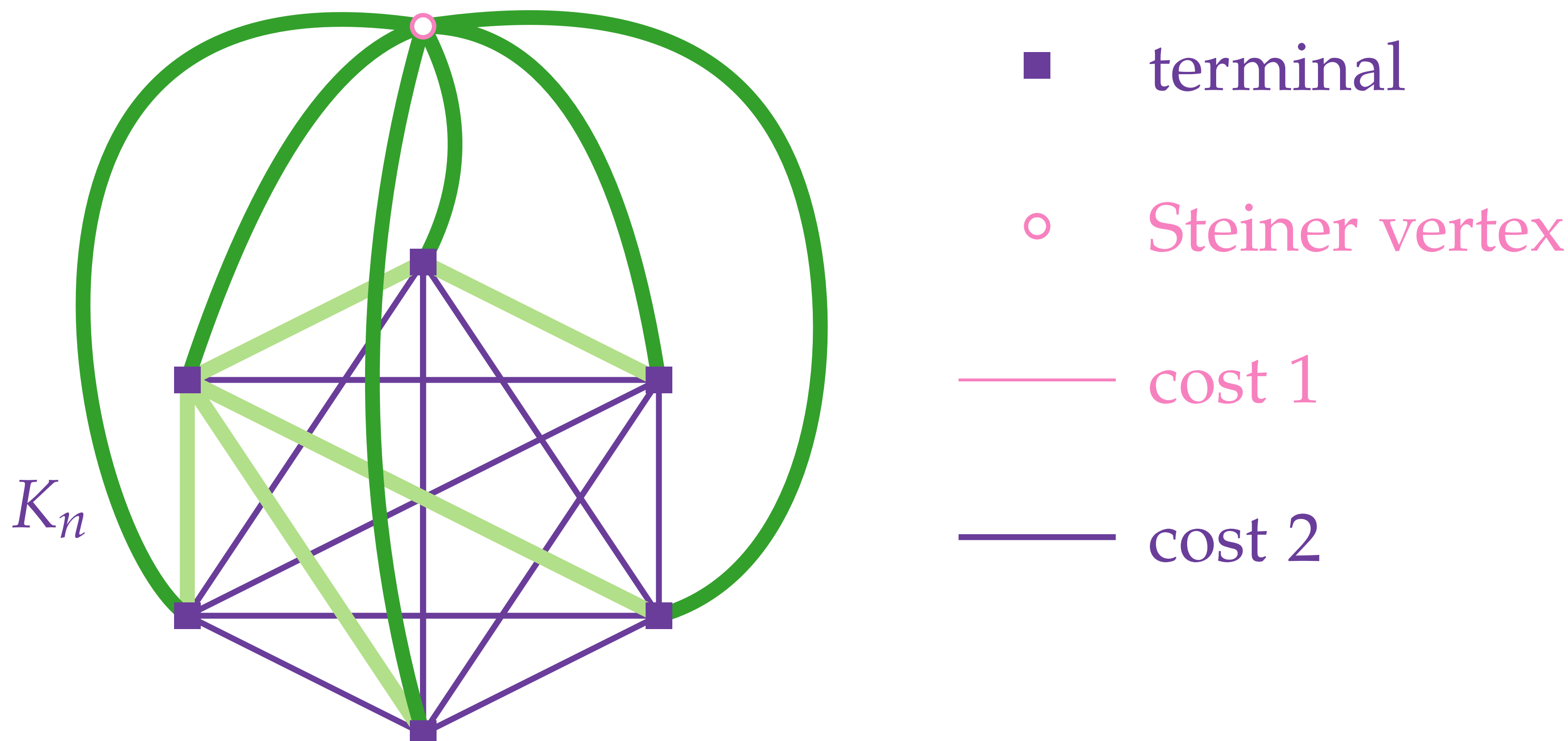

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SteinerTree cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P=NP) [Chlebik & Chlebikova '08]

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part V:
MultiwayCut

Philipp Kindermann

Summer Semester 2020

MultiwayCut

Given: A connected graph $G = (V, E)$

MultiwayCut

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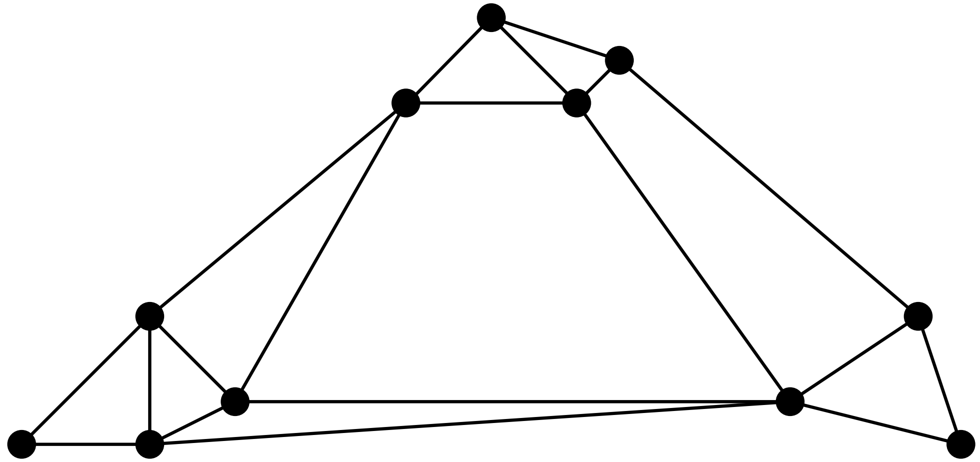

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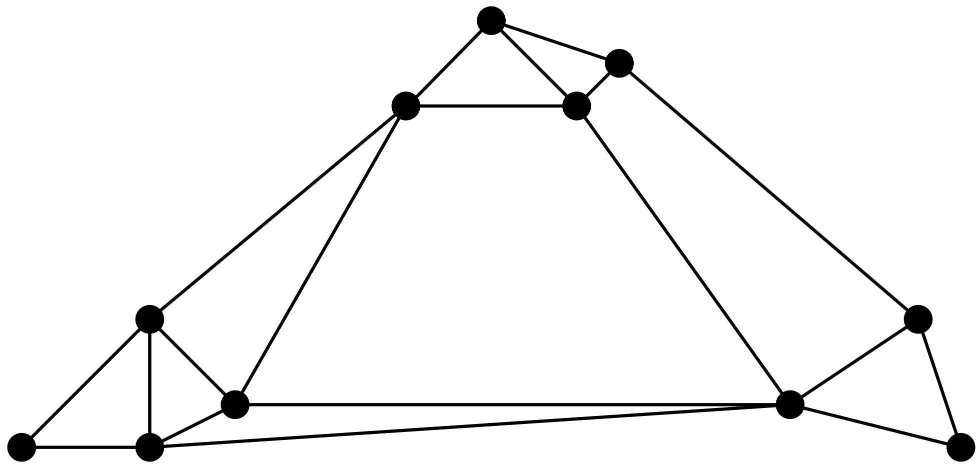

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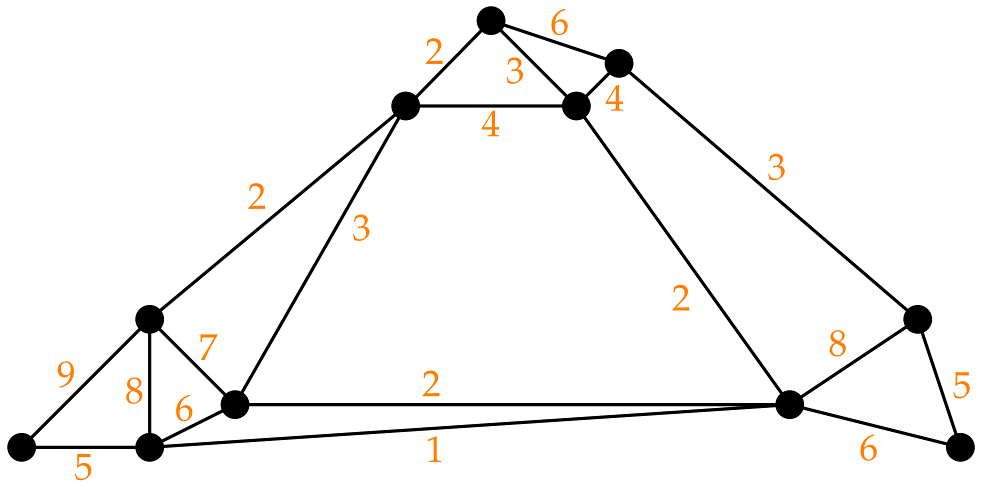

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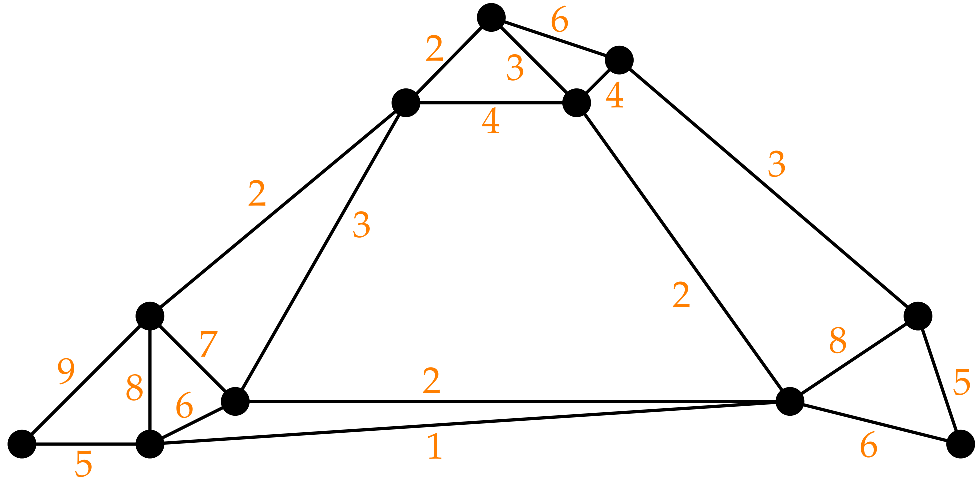
Given: A connected graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{Q}^+$ and a set $T = \{t_1, \ldots, t_k\} \subseteq V$ of **terminals**.



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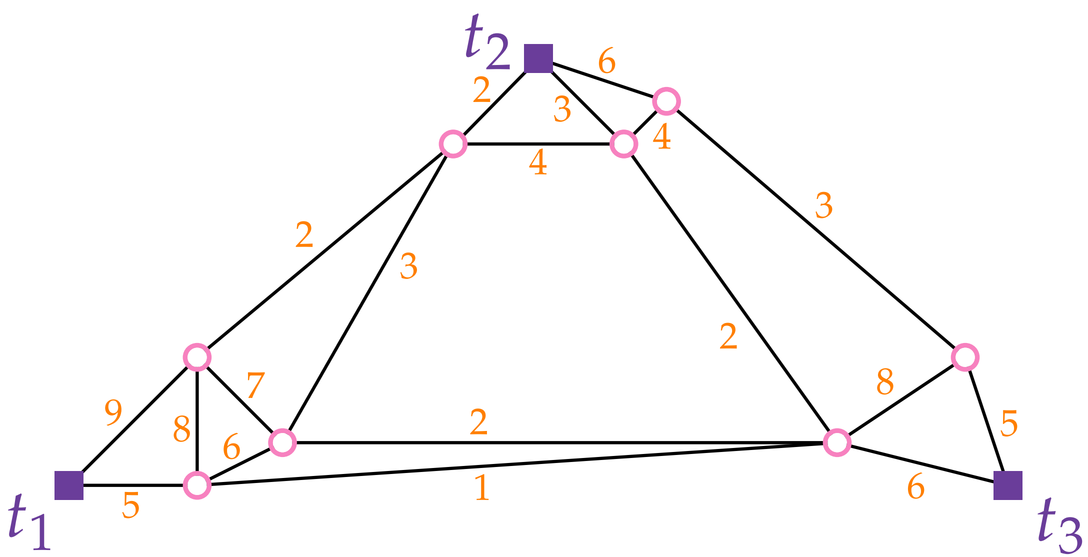

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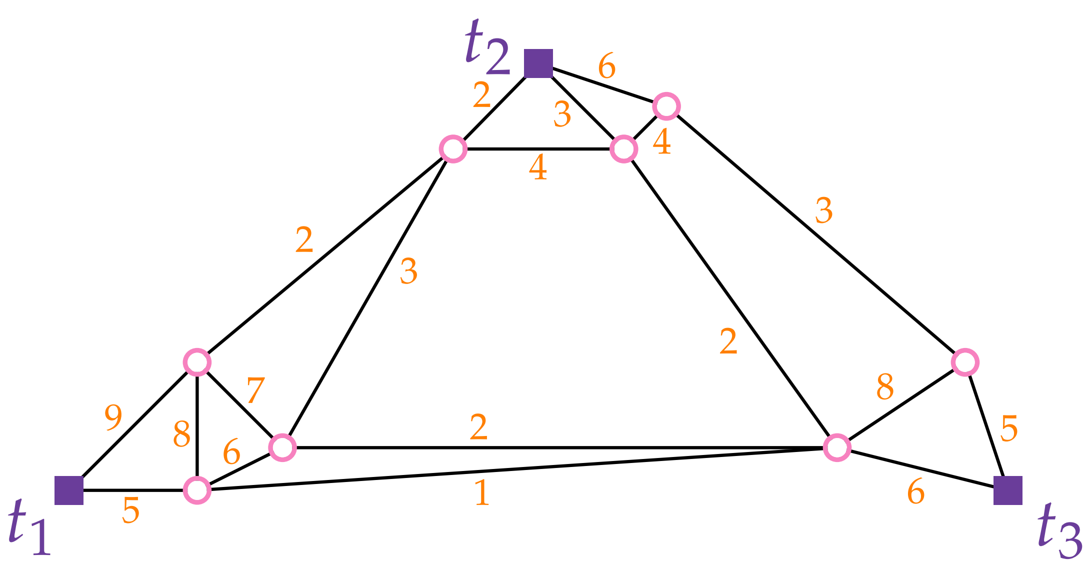

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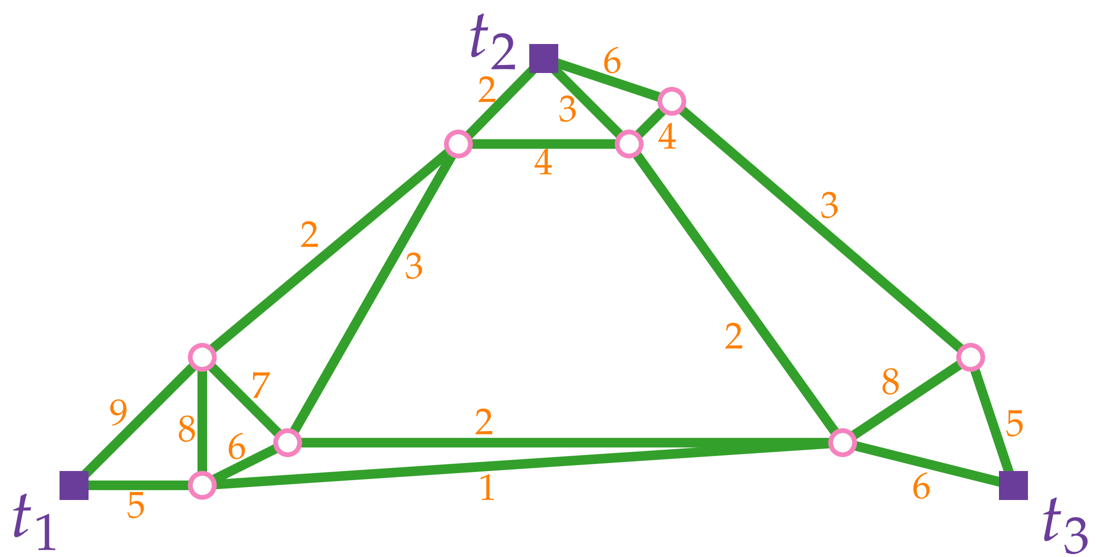

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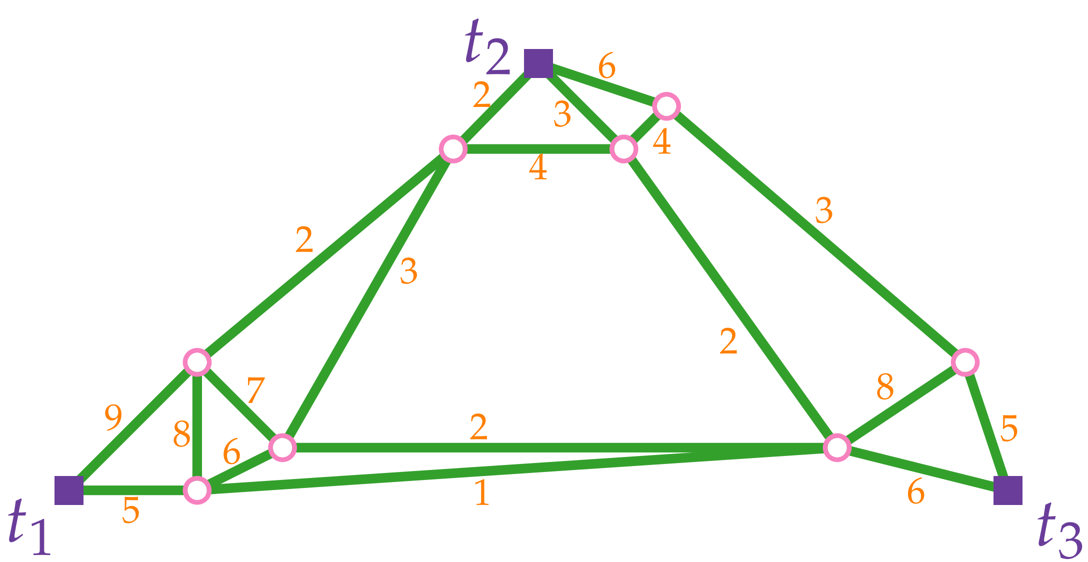

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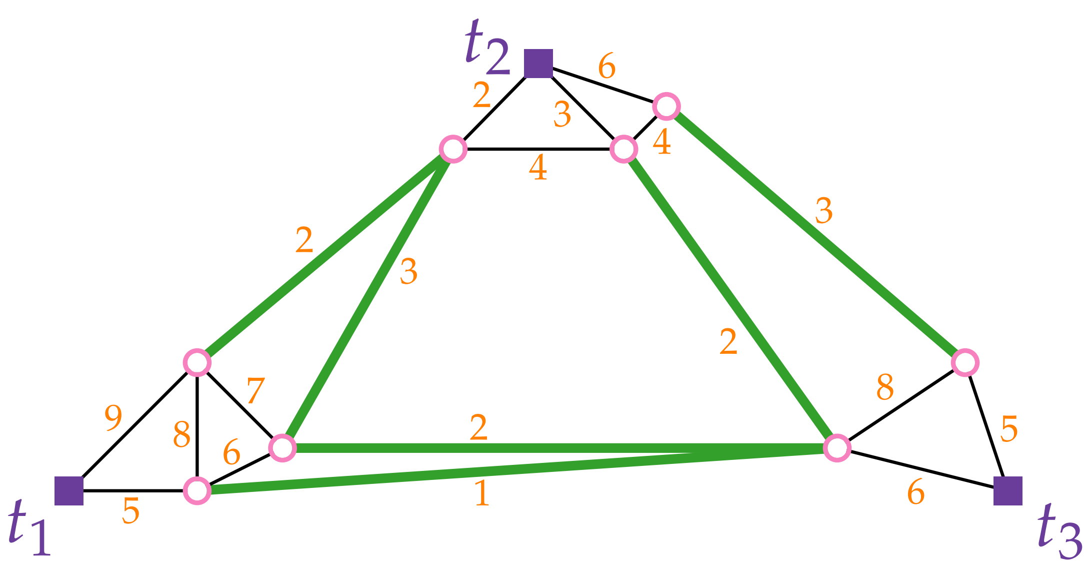

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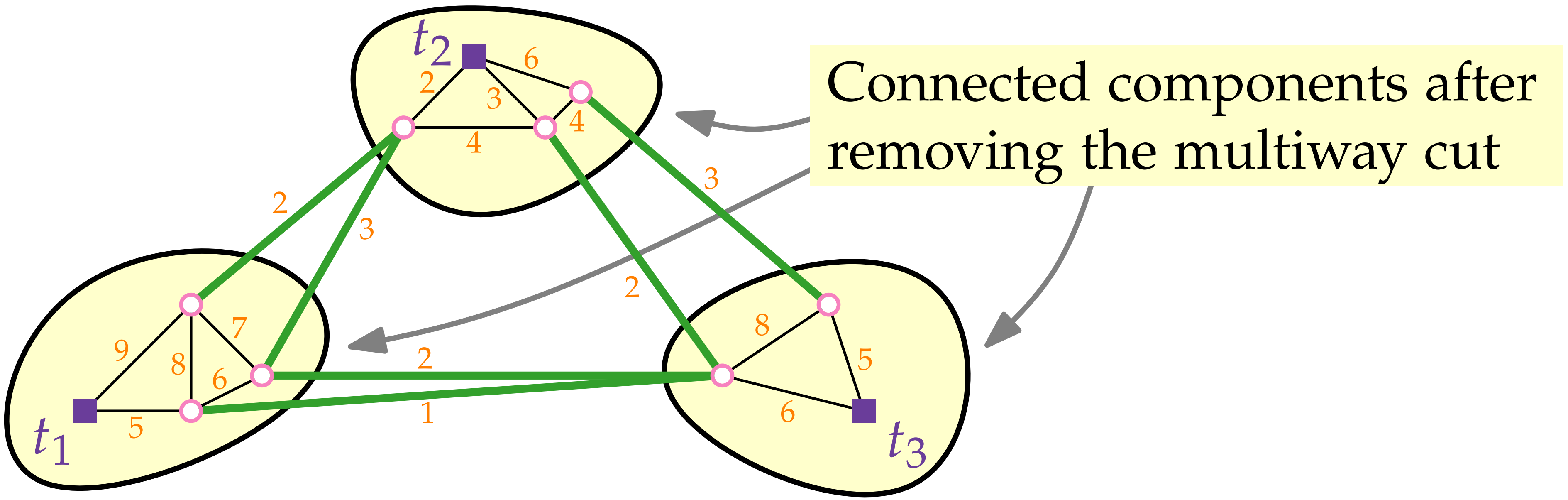

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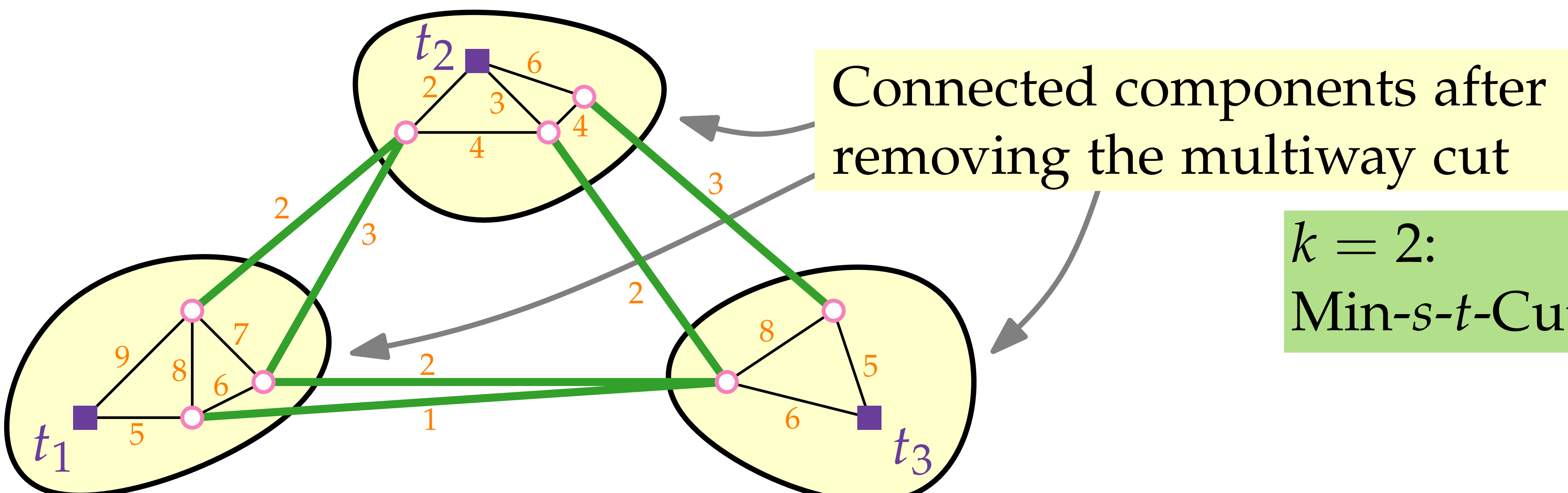

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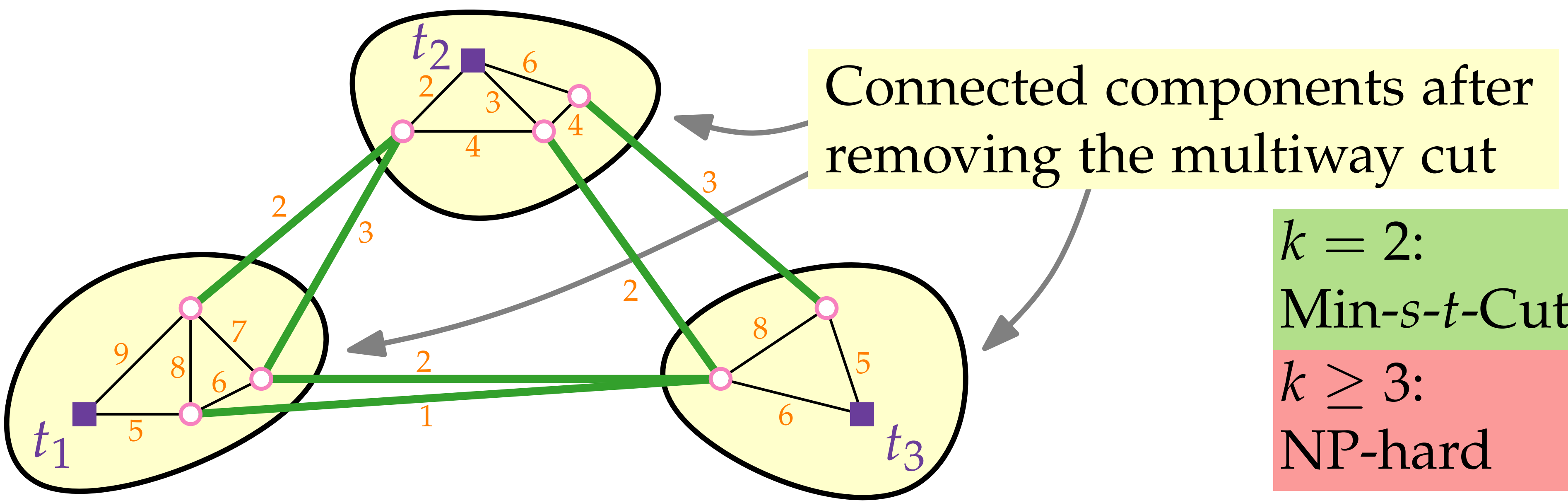

$k = 2$: Min-s-t-Cut

$k \geq 3$: NP-hard

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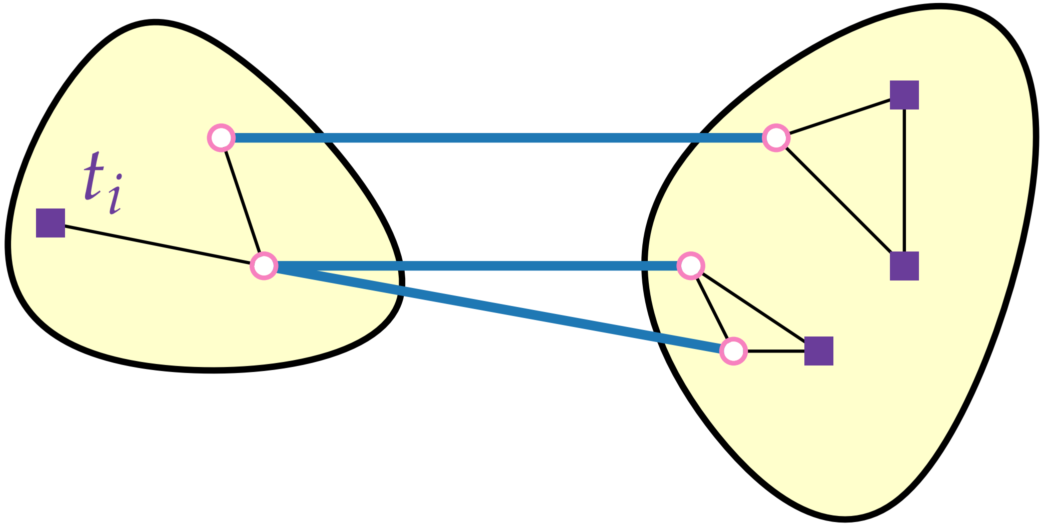

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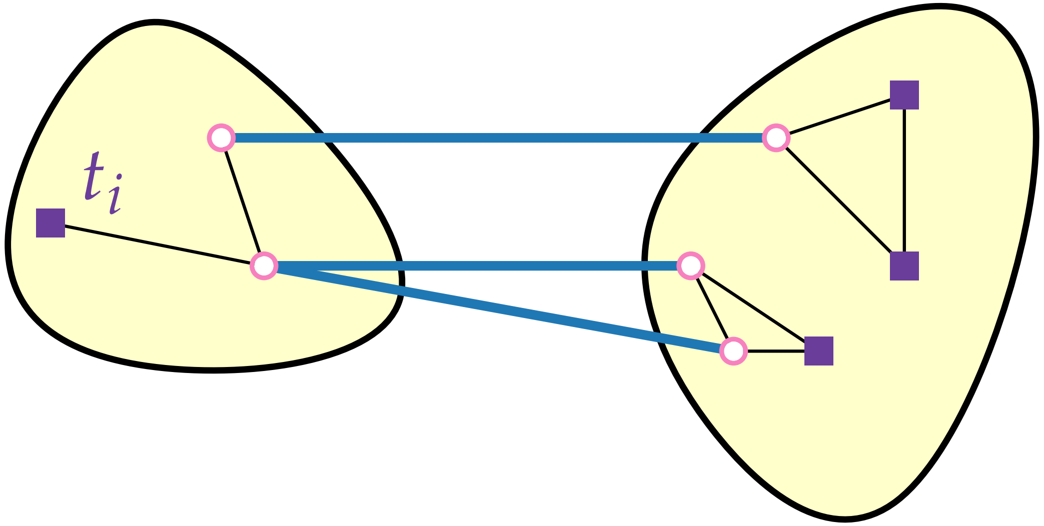

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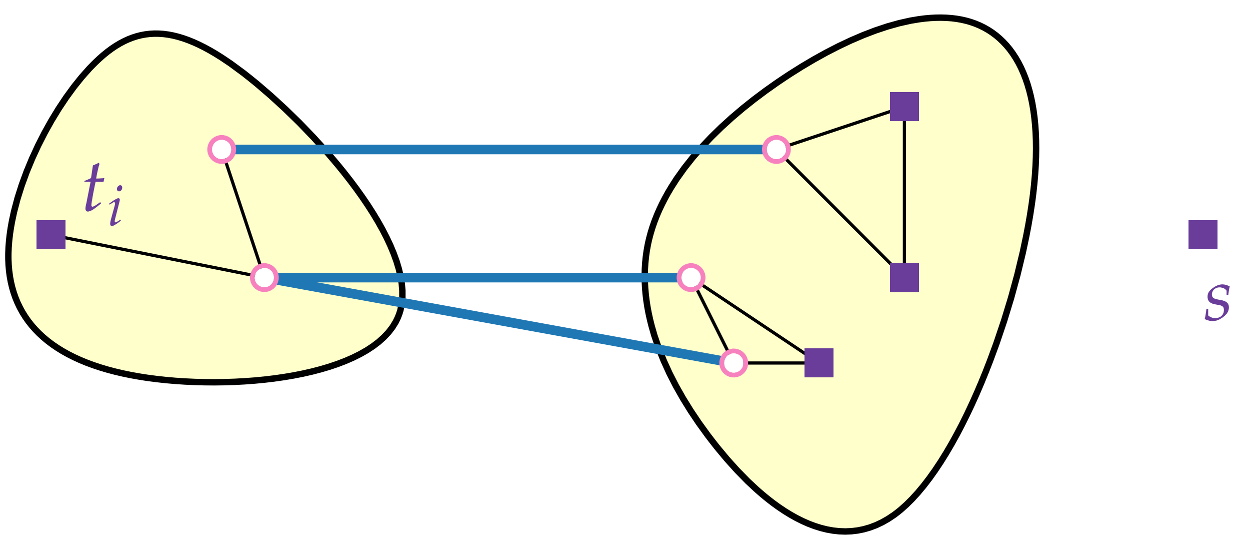

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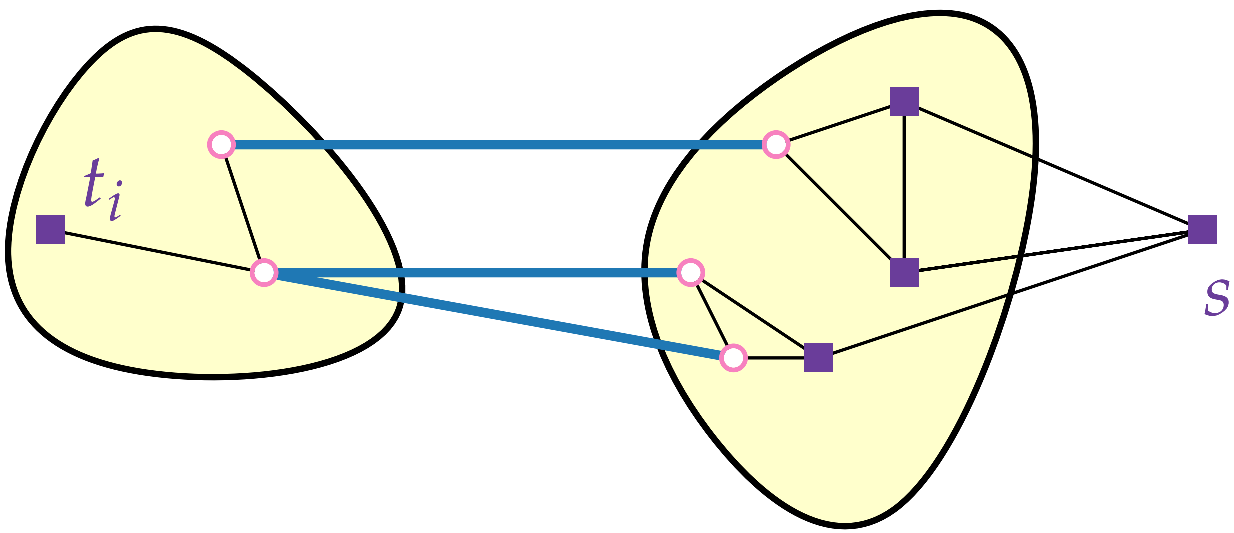

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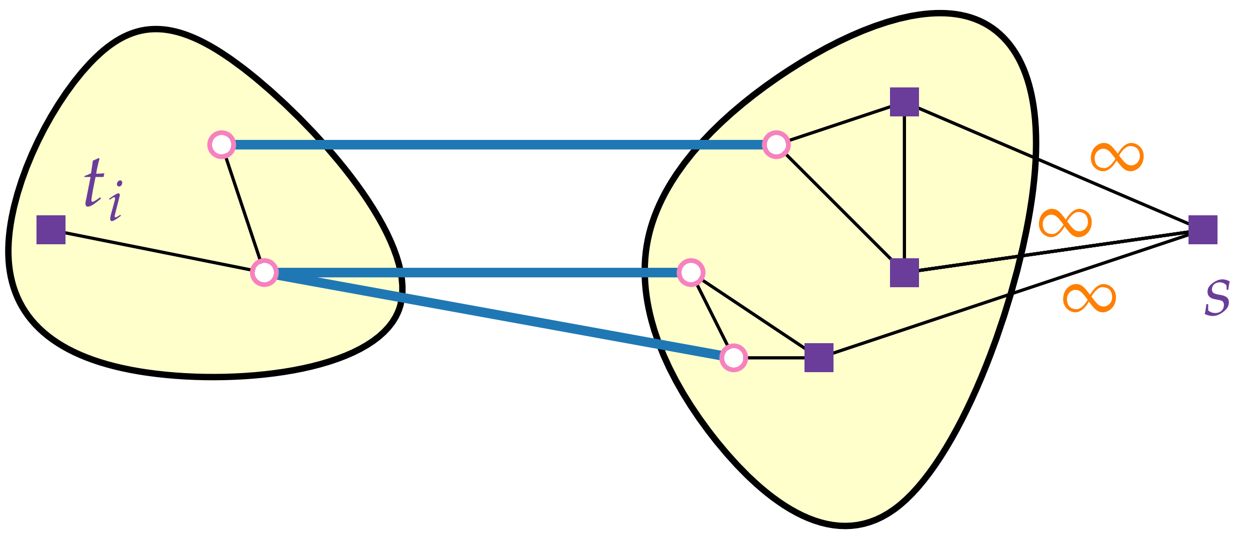

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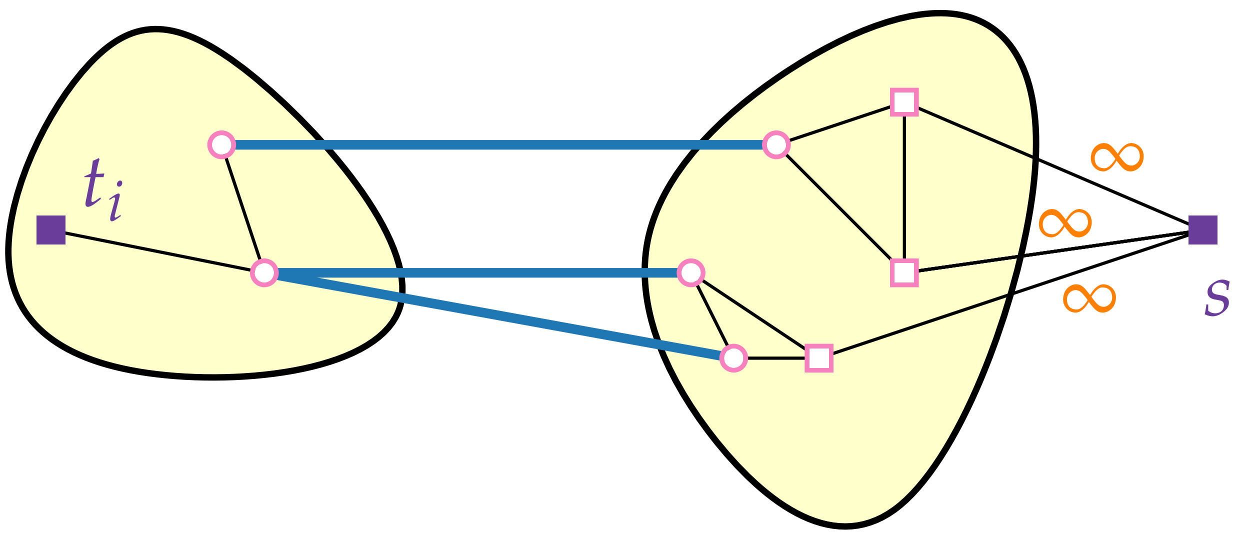

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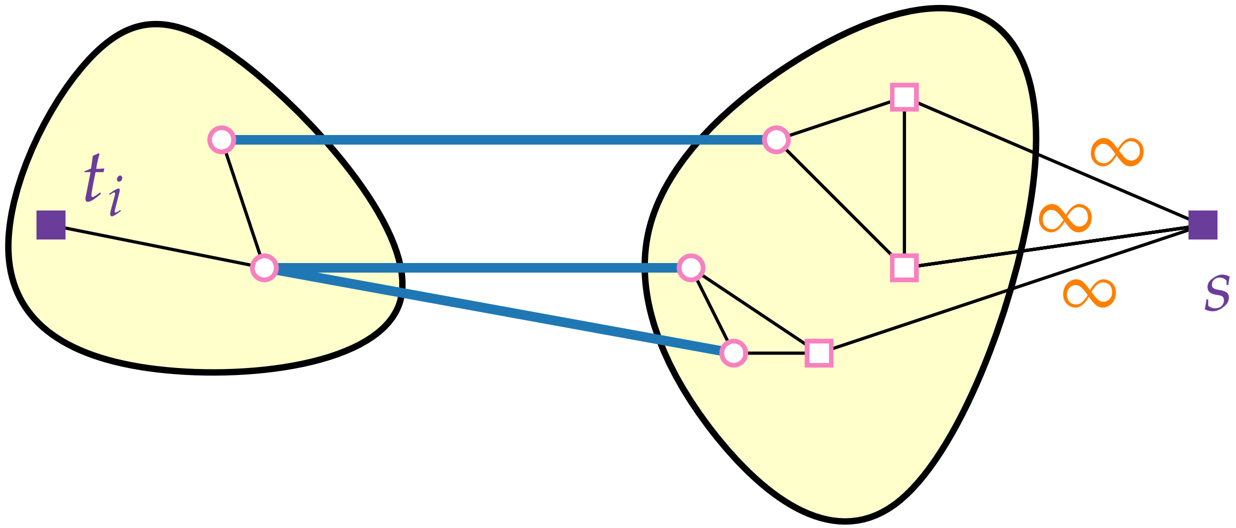

Add dummy terminal s and find minimum cost s-t_i-cut.

Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

Part VI: Algorithm for MultiwayCut

Philipp Kindermann

Summer Semester 2020

Algorithm MultiwayCut

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Compute a minimum cost isolating cut C_i for t_i.

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Approximation Factor

Theorem. This algorithm is a factor-( )-approximation algorithm for MultiwayCut.

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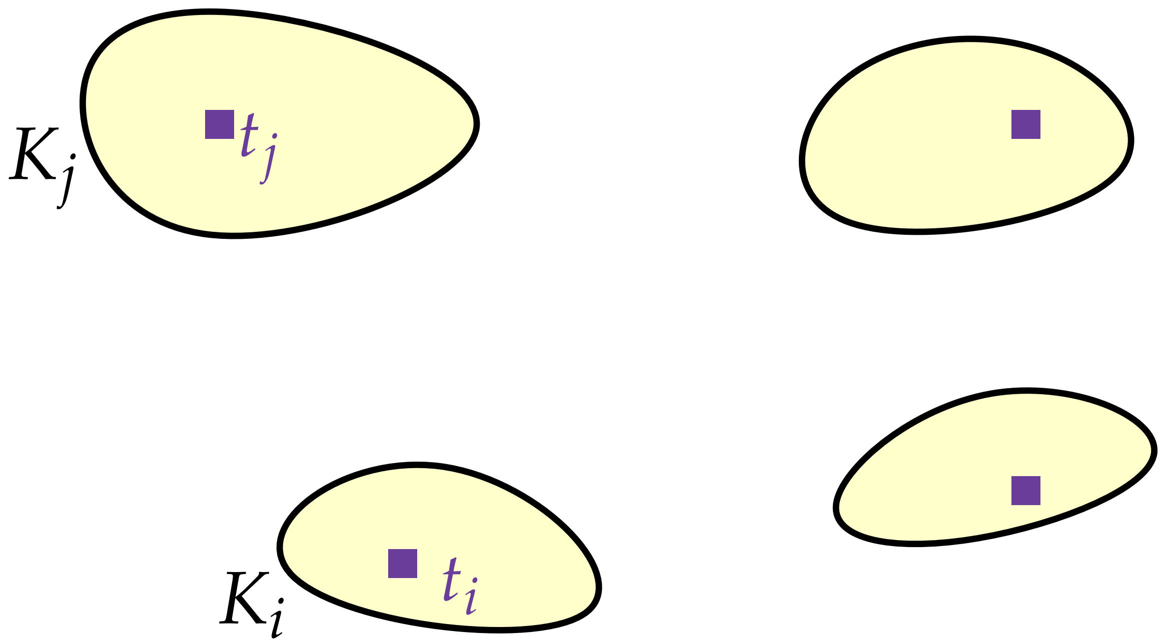

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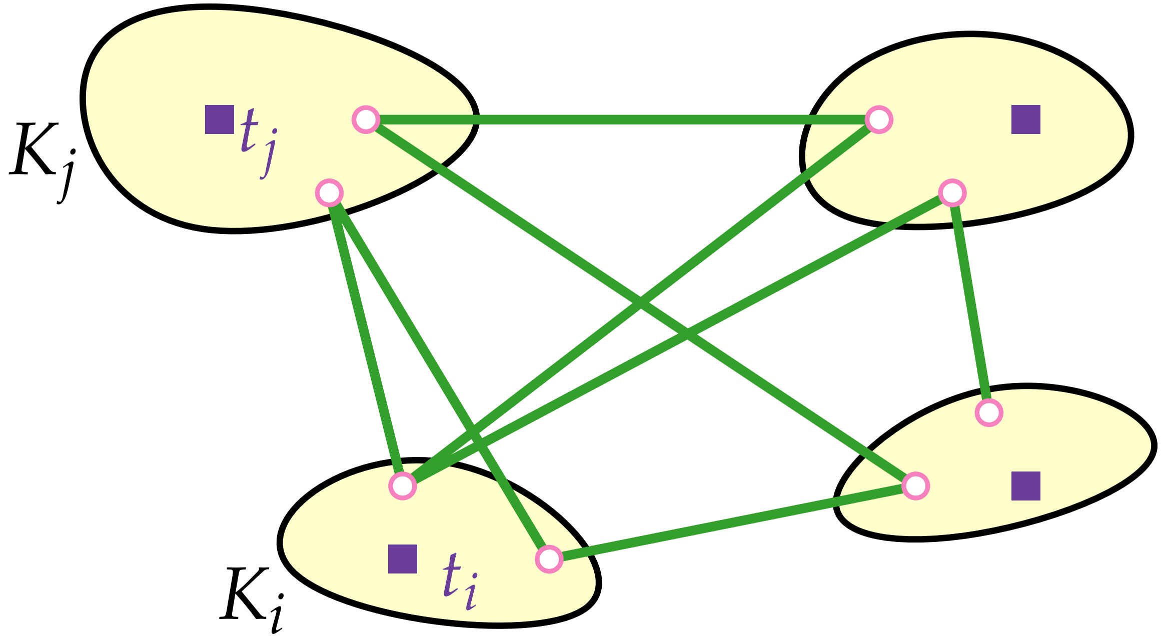

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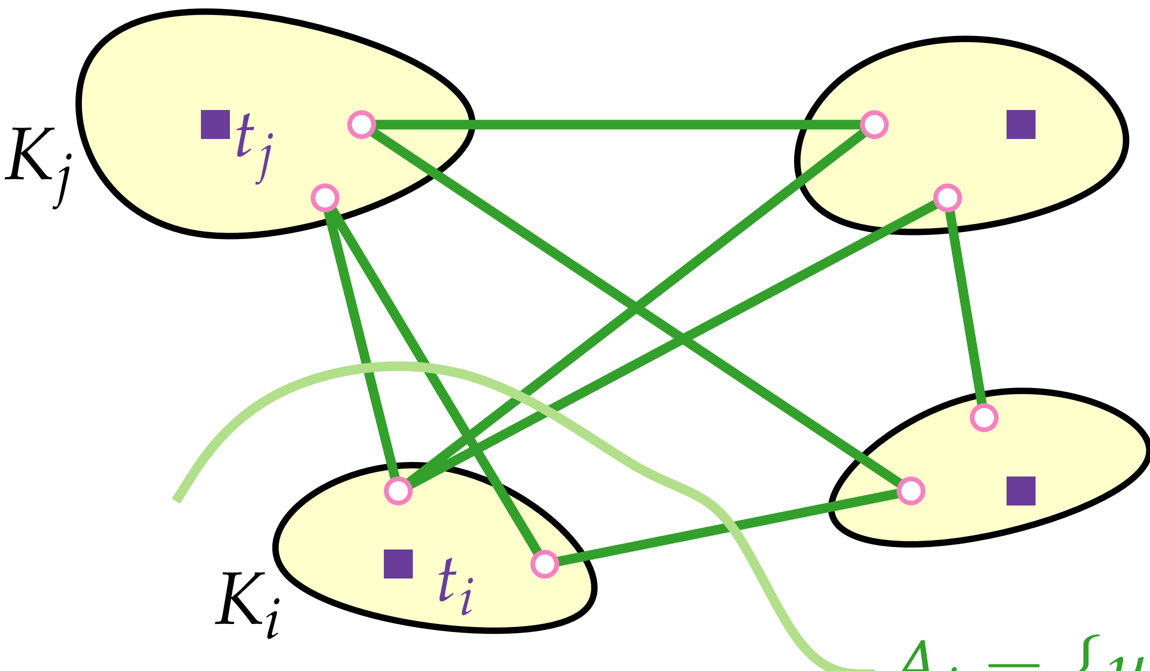

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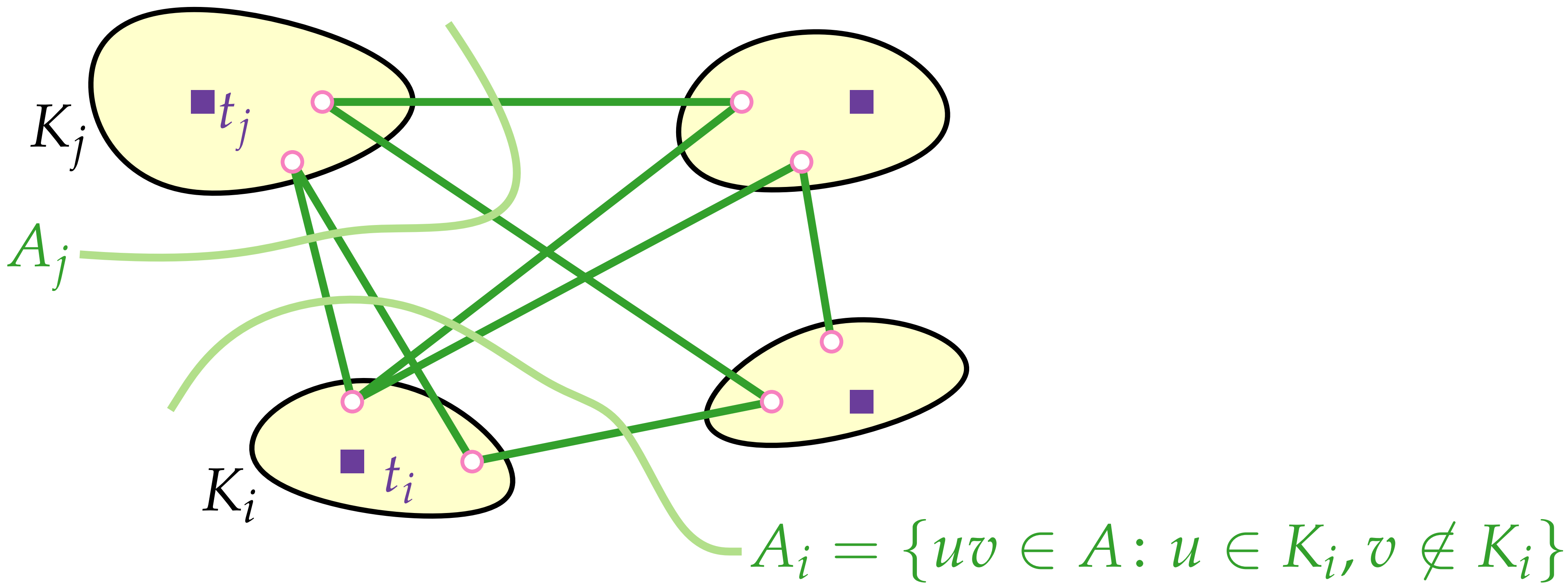

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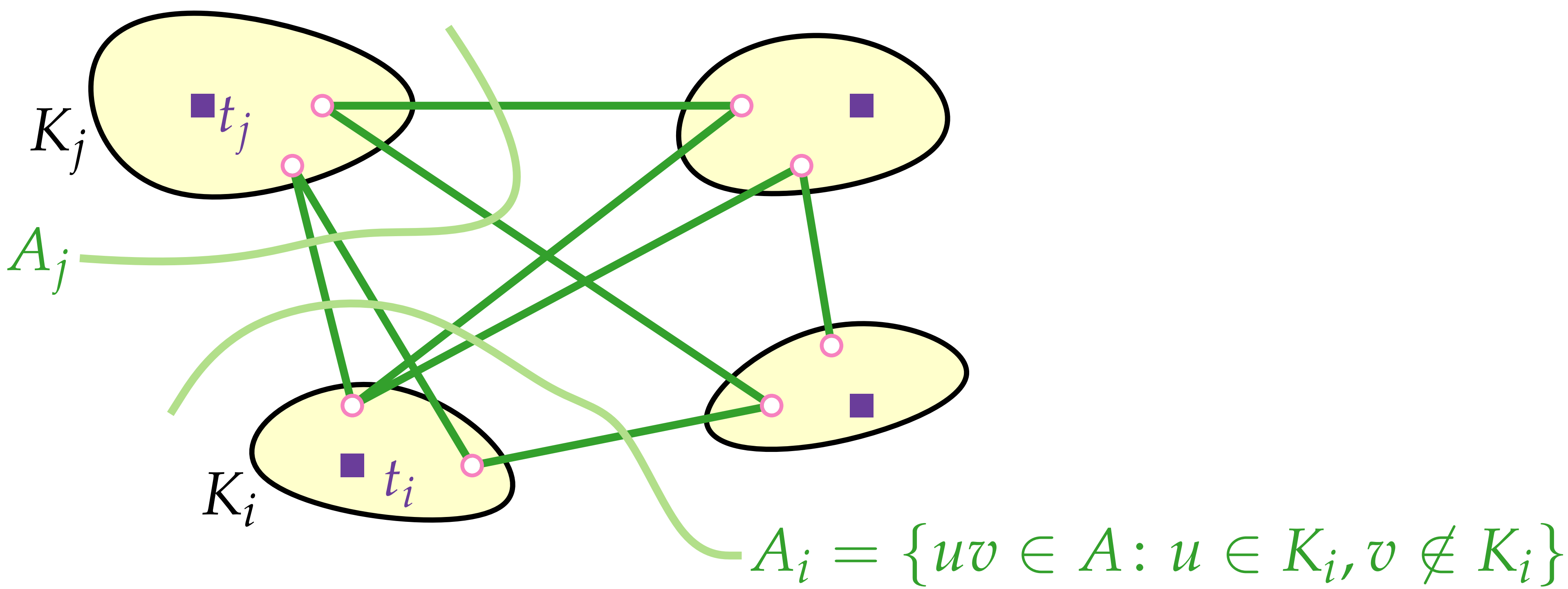

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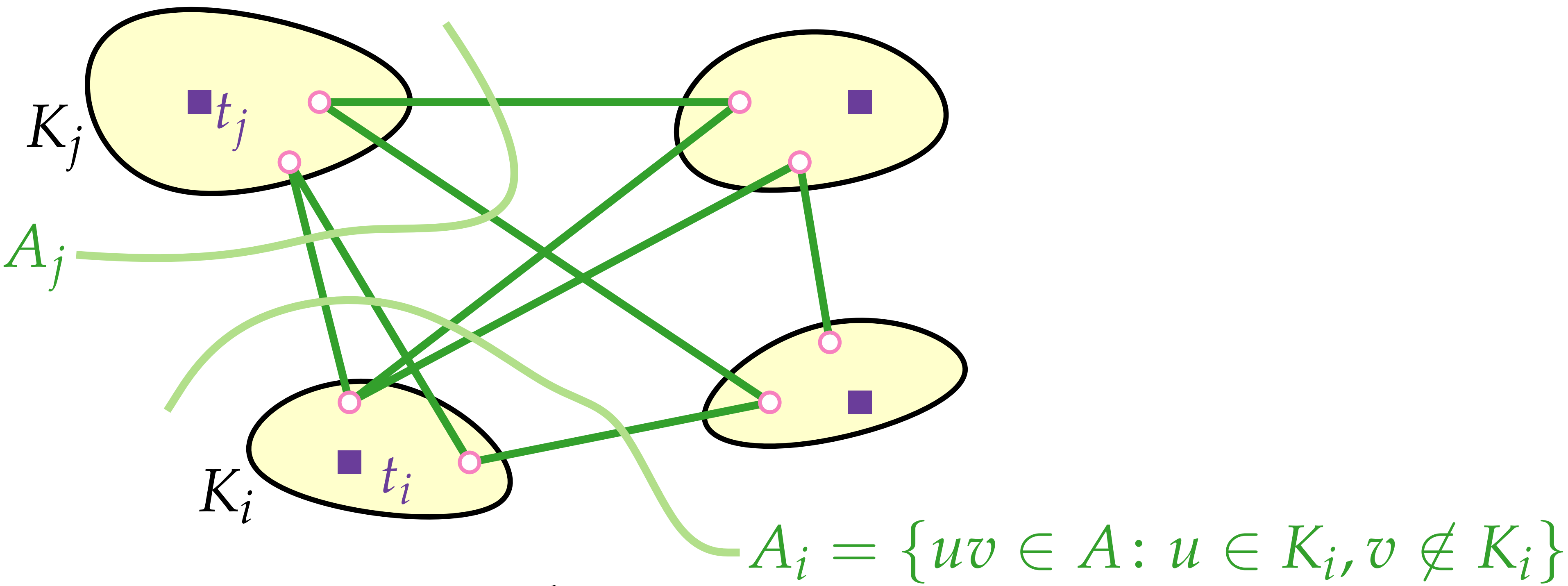

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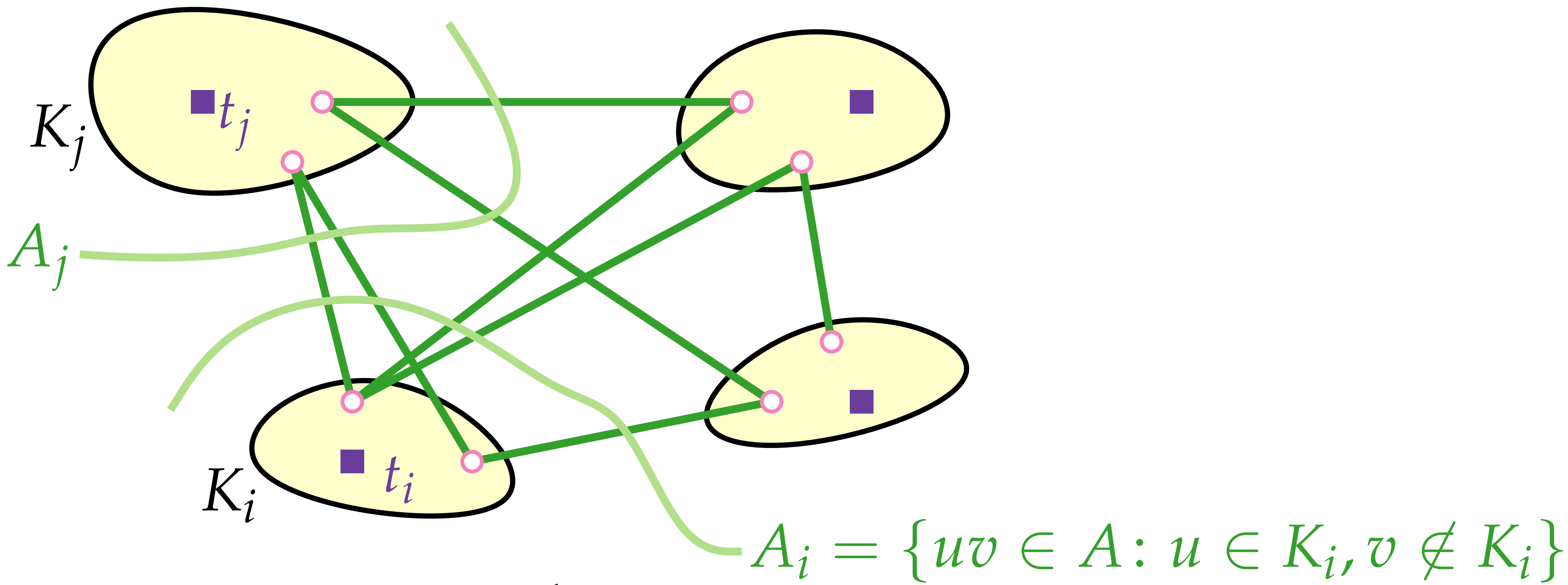

Observation. $A = \bigcup_{i=1}^{k} A_i$ and $c(A) =$

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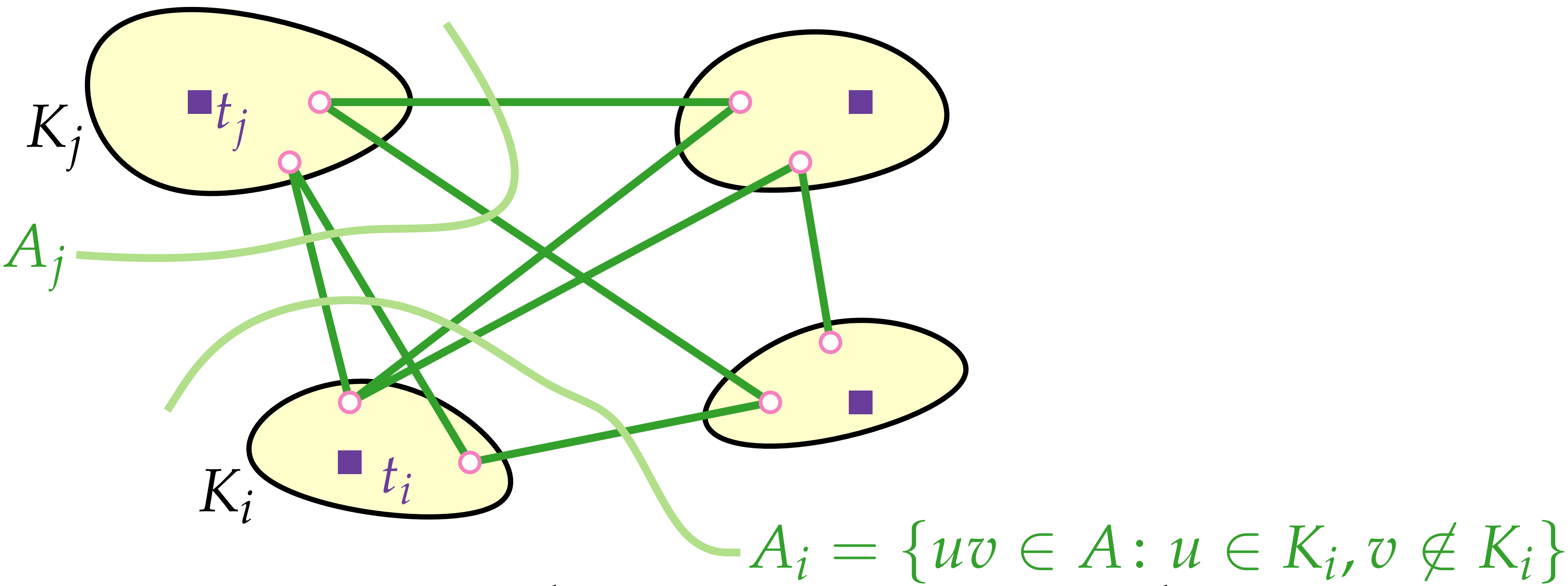
Theorem. This algorithm is a factor-($2 - 2/k$)-approximation algorithm for MULTIWAYCUT.

Proof. Consider optimal multiway cut A:



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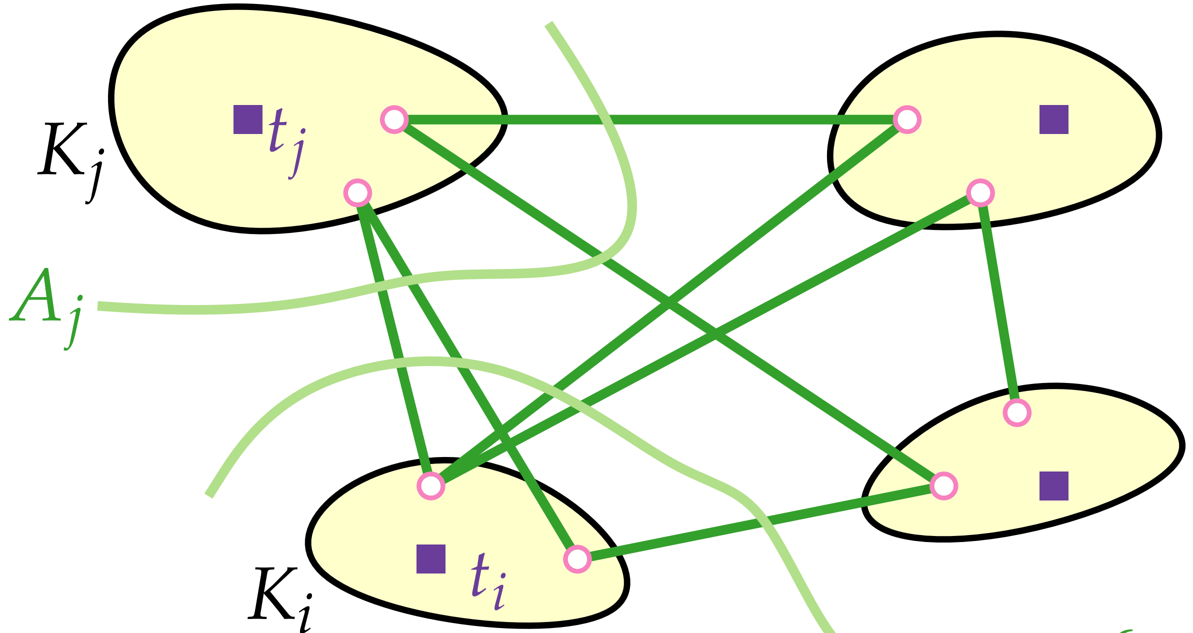

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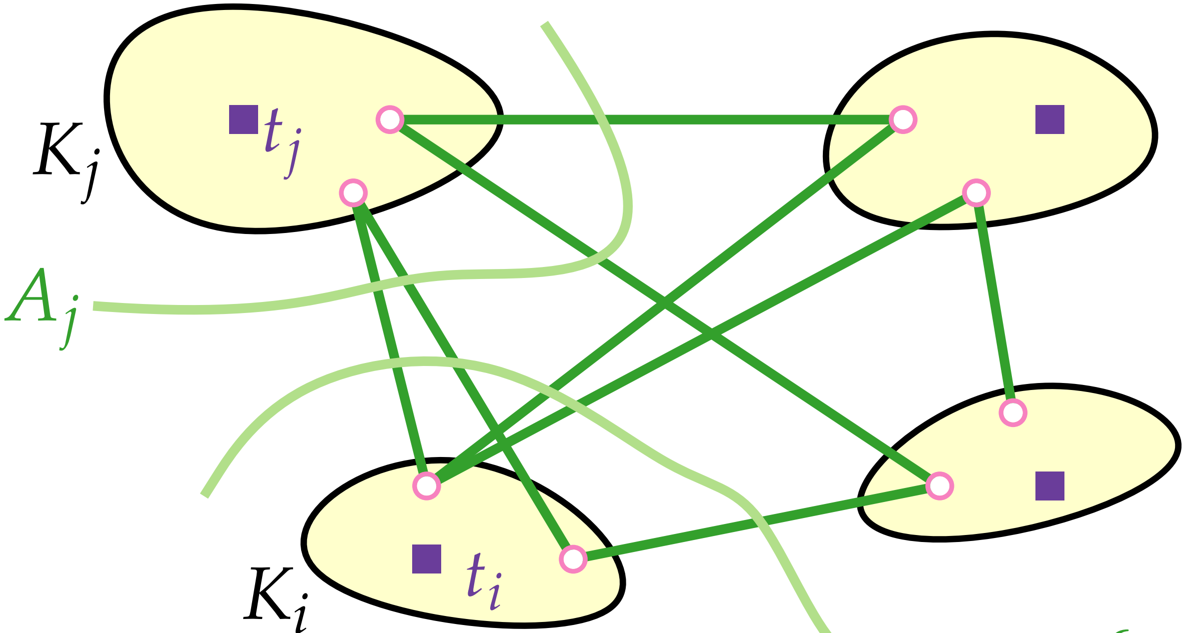

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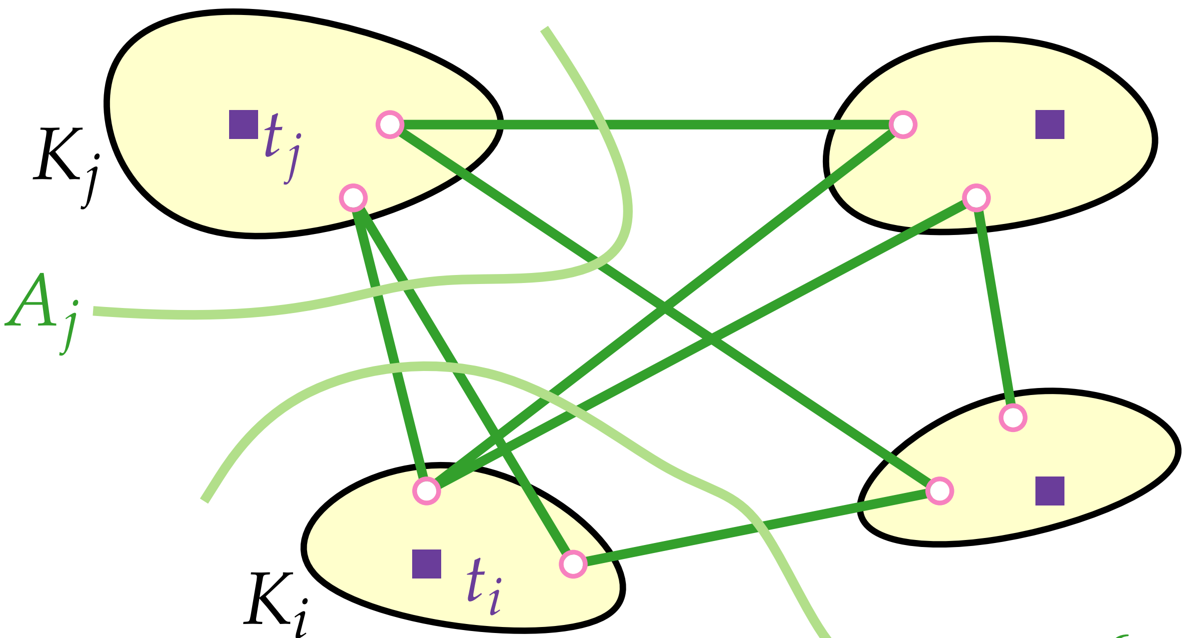

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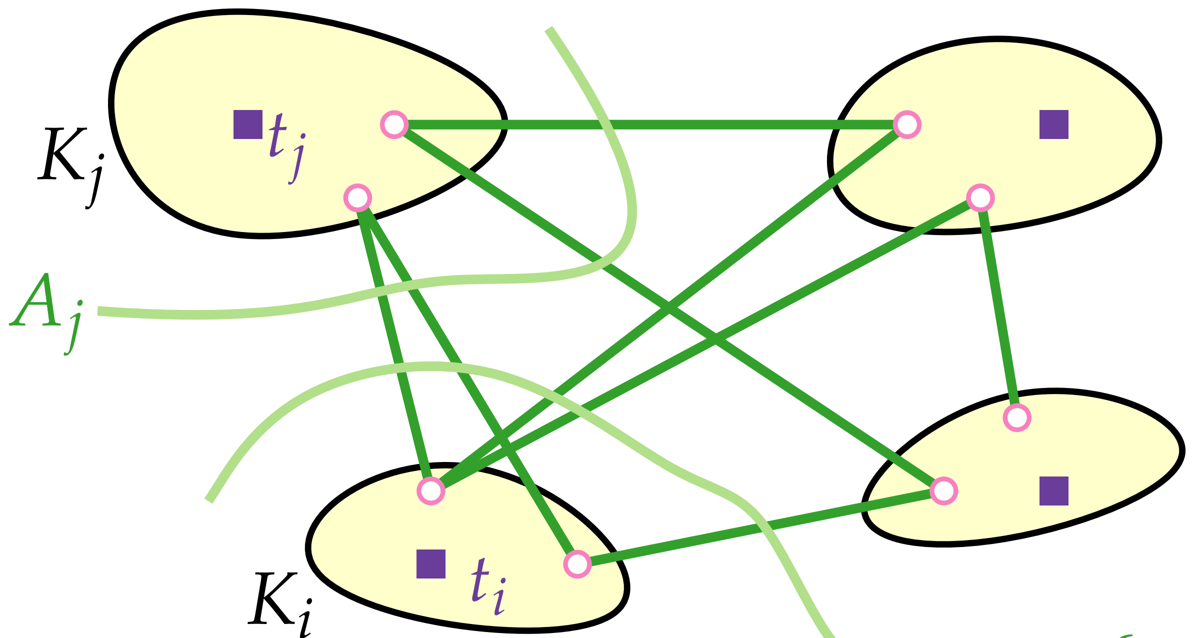

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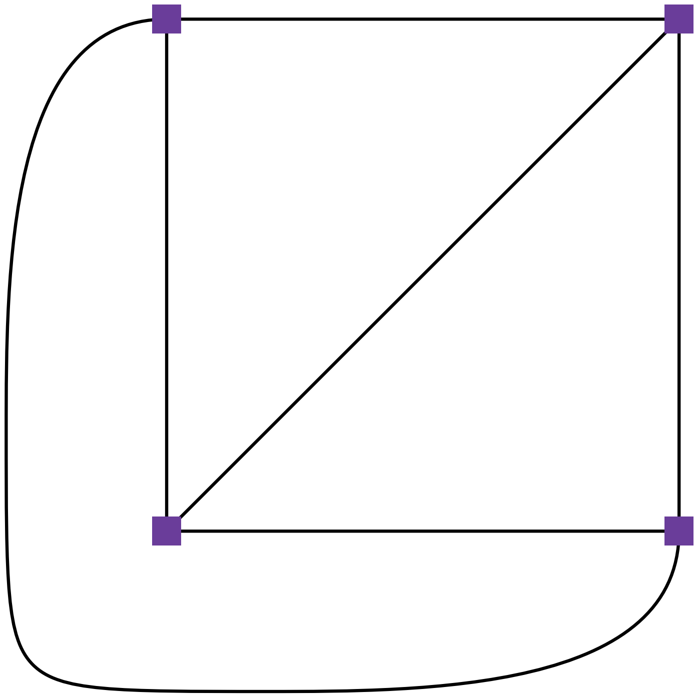

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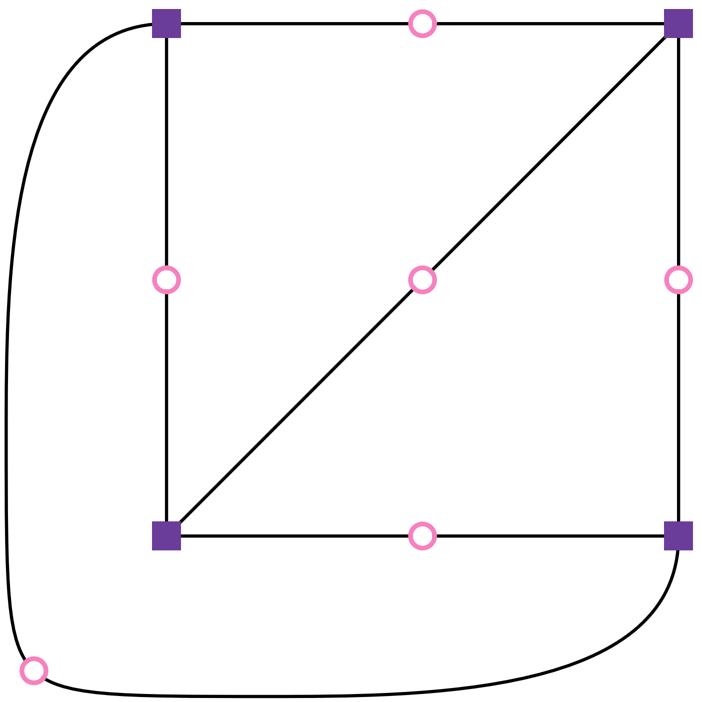

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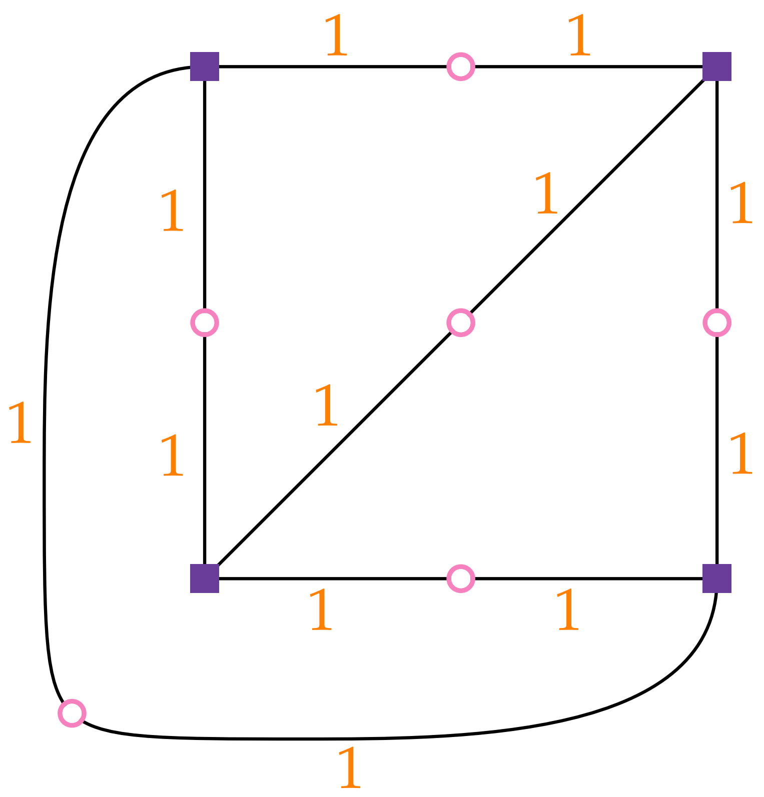

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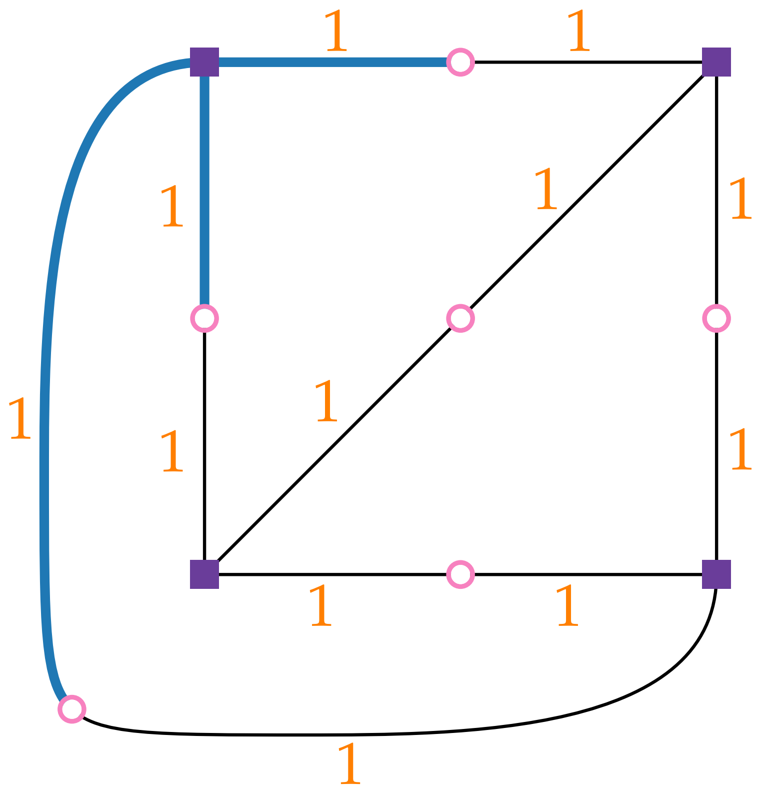
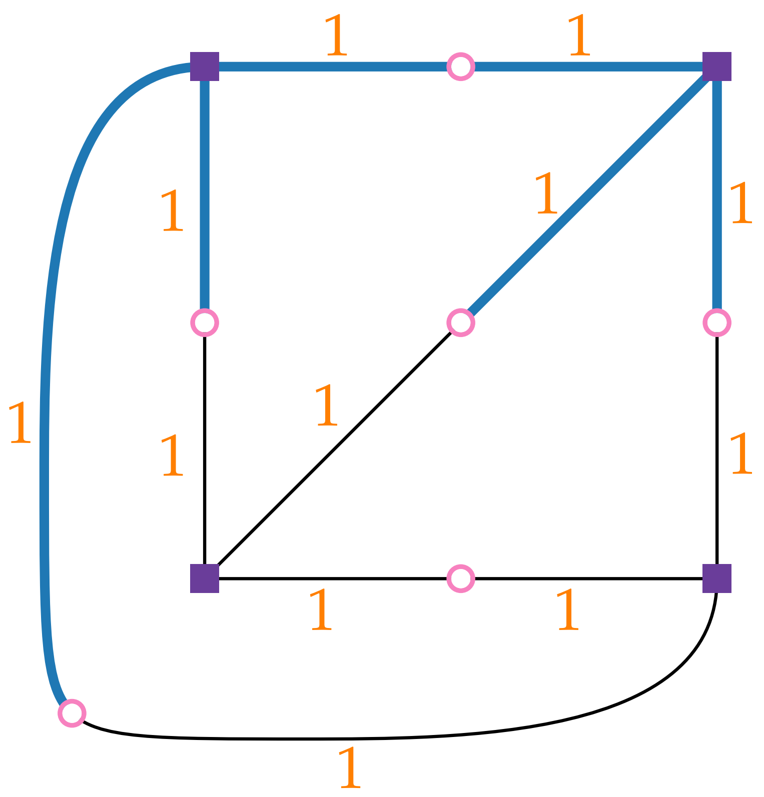


Analysis Sharp?

K_k



Analysis Sharp?

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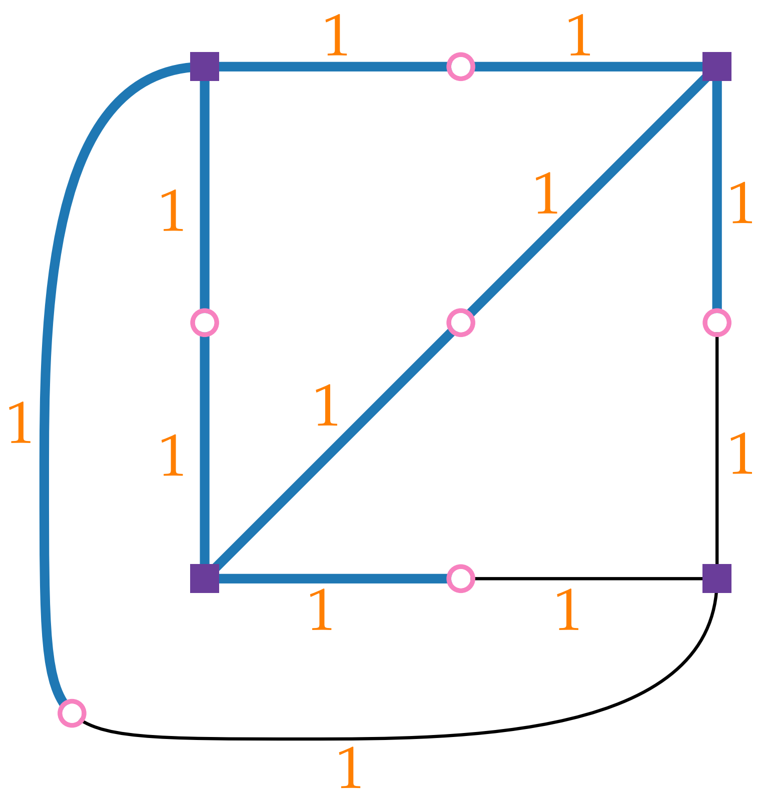

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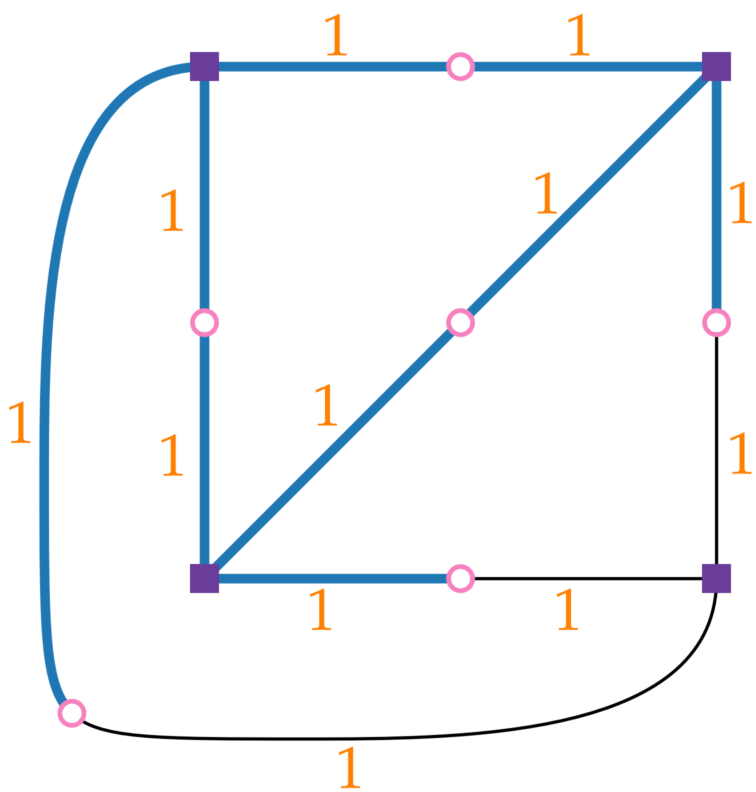
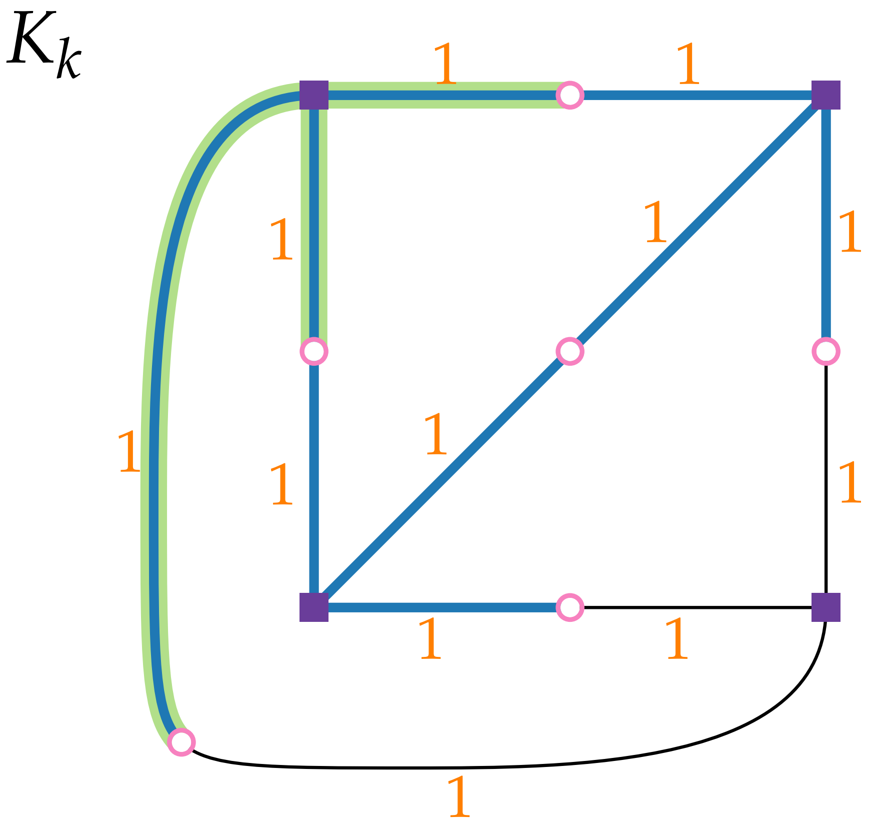


$\text{ALG} = (k-1)(k-1)$

Analysis Sharp?



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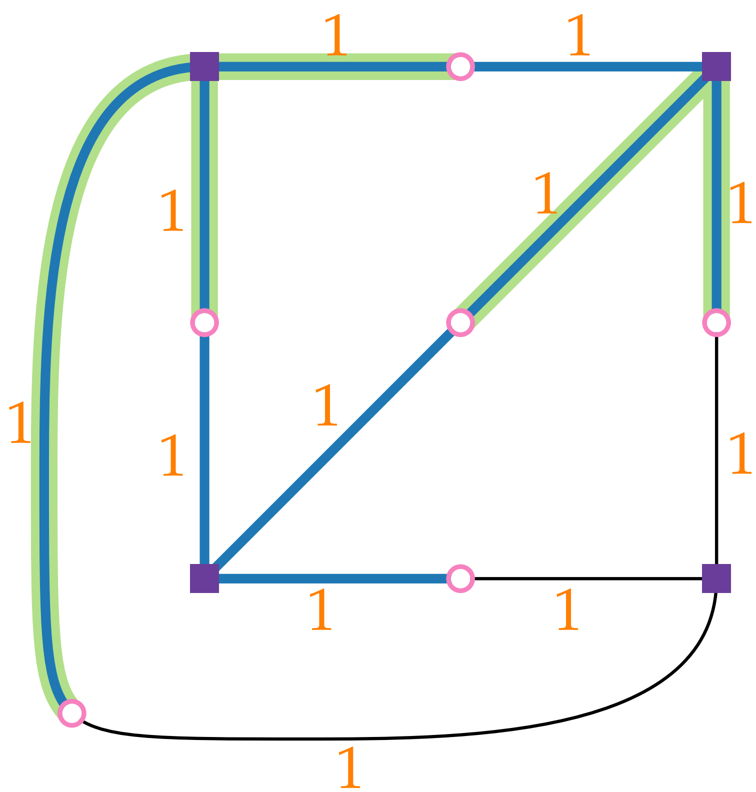

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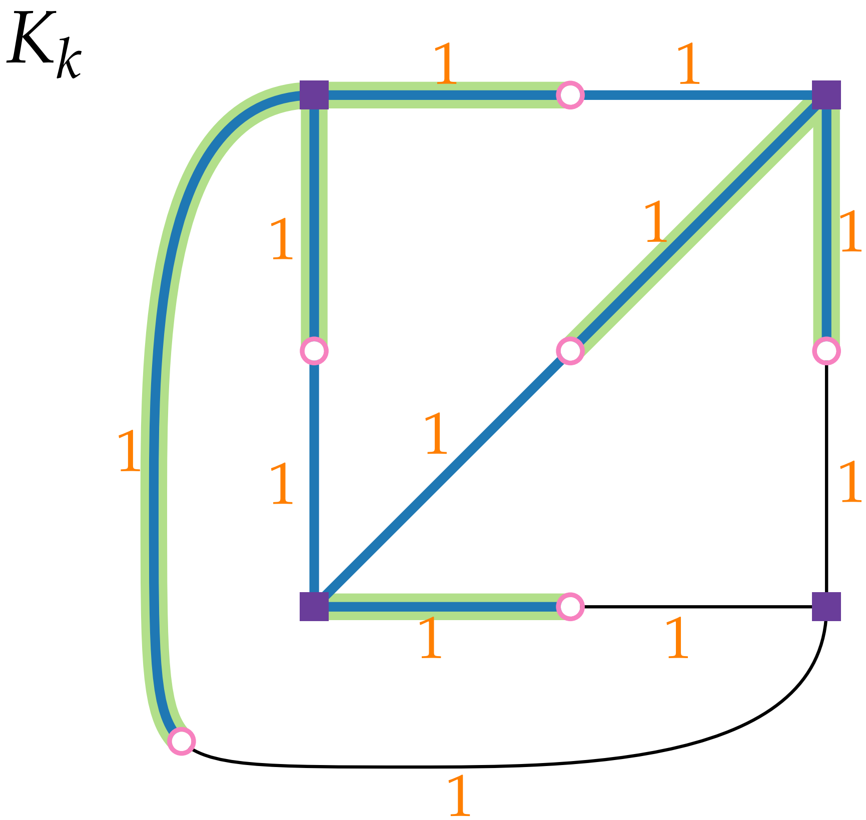

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Analysis Sharp?

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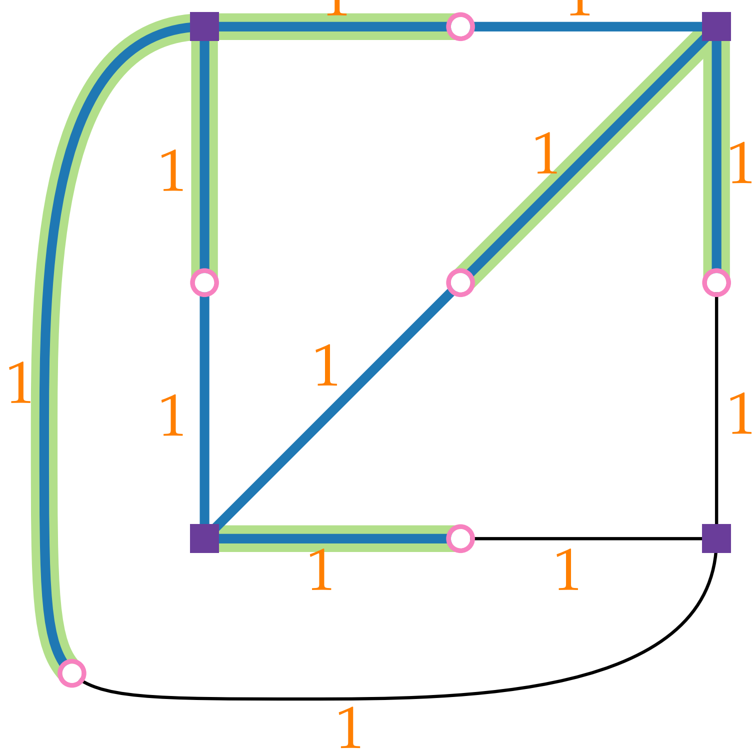
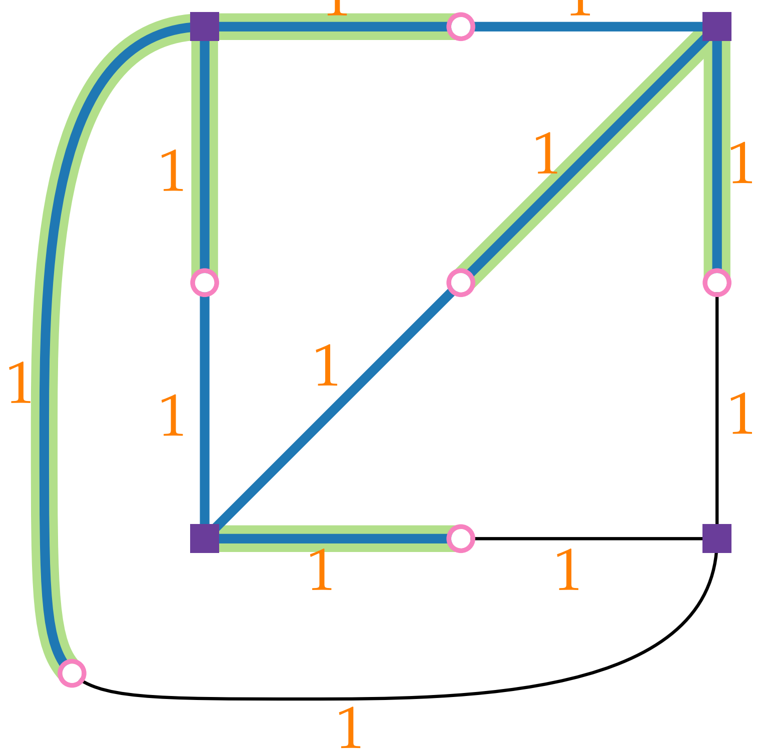


$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i =$

Analysis Sharp?

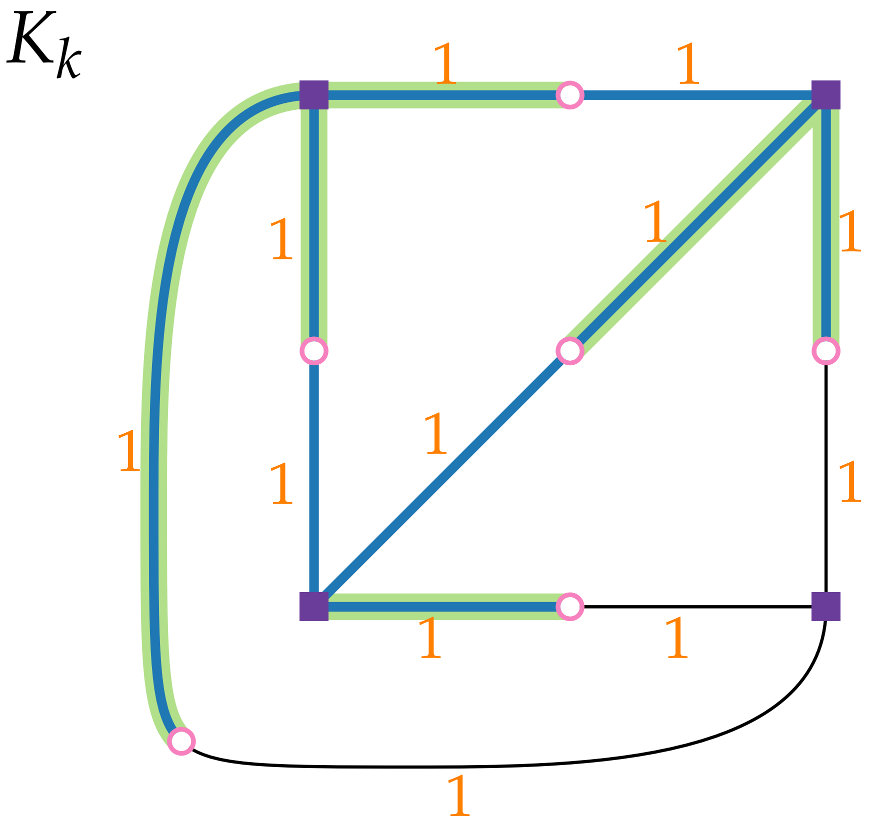
K_k



$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k\cdot(k-1)}{2}$

Analysis Sharp?



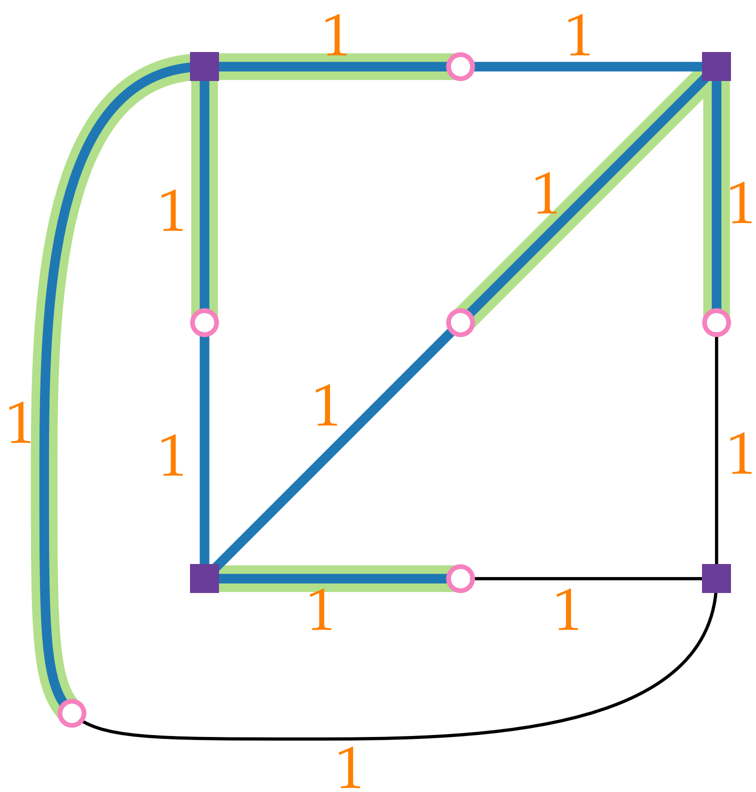
$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k\cdot(k-1)}{2}$

$\text{ALG}/\text{OPT} =$

Analysis Sharp?

K_k



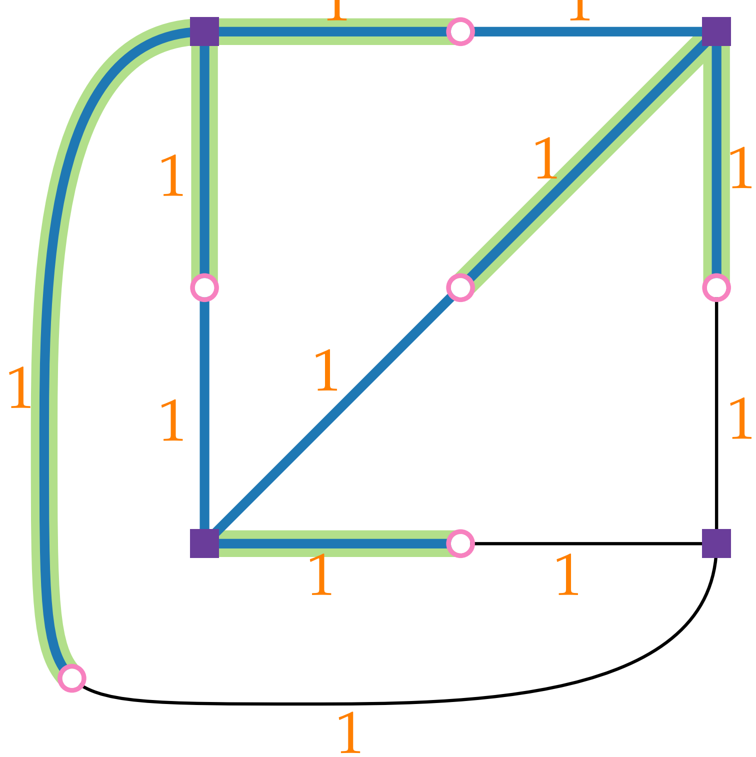
$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k \cdot (k-1)}{2}$

$\text{ALG}/\text{OPT} = \frac{2k-2}{k} =$

Analysis Sharp?

K_k



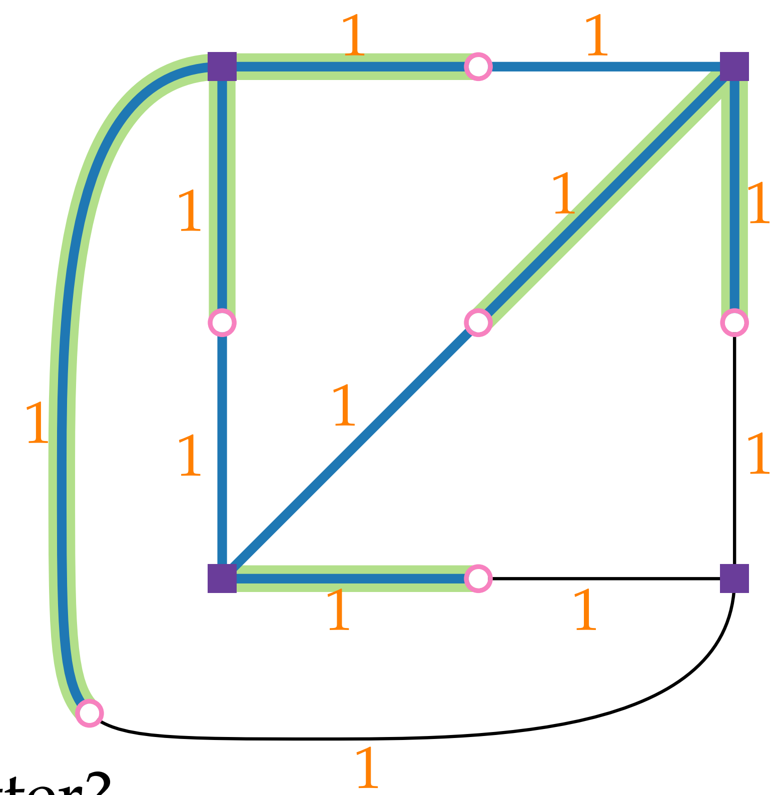
$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k \cdot (k-1)}{2}$

$\text{ALG}/\text{OPT} = \frac{2k-2}{k} = 2 - \frac{2}{k}$

Analysis Sharp?

K_k



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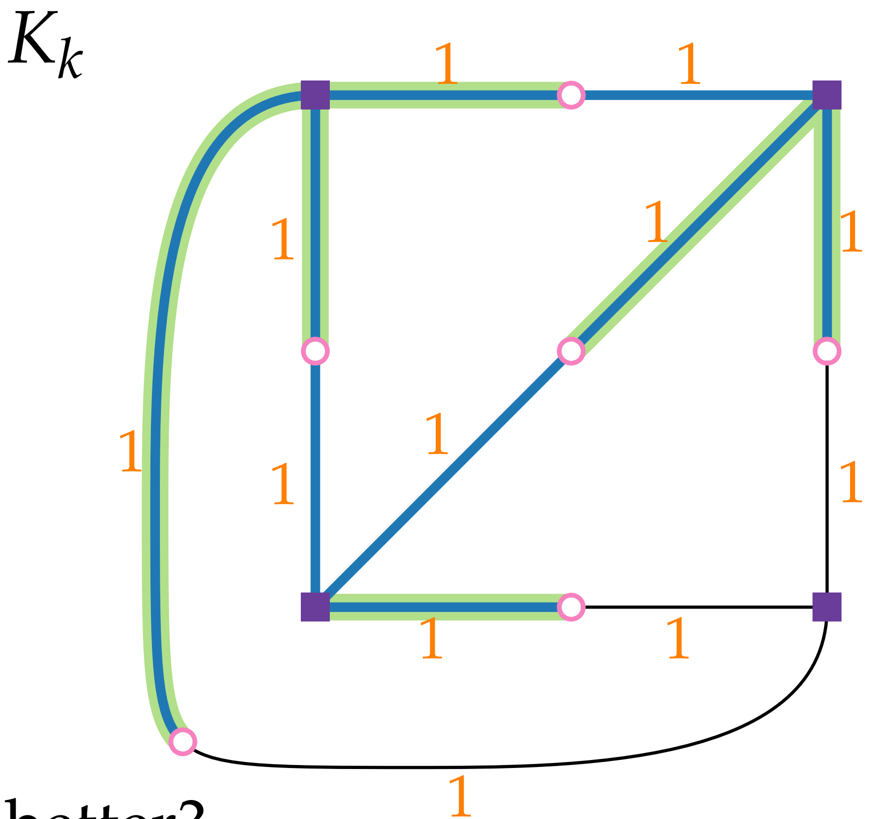
better?

Analysis Sharp?



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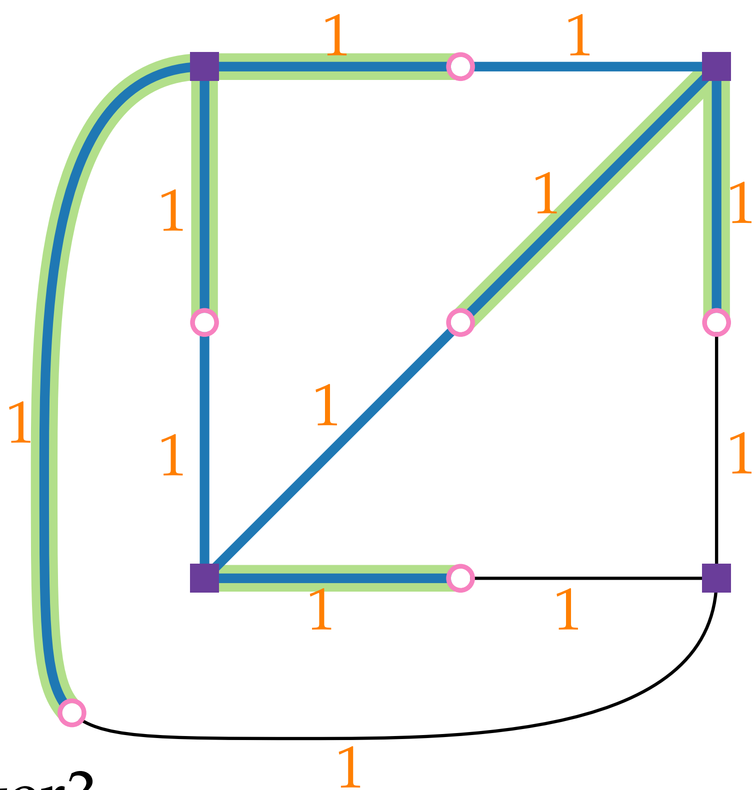
better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$. [Sharma & Vondrák '14]

Analysis Sharp?

K_k



$$\text{ALG} = (k-1)(k-1)$$

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$$\text{ALG}/\text{OPT} = \frac{2k-2}{k} = 2 - \frac{2}{k}$$

better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$. [Sharma & Vondrák '14]

MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless P=NP).

[Bérczi, Chandrasekaran, Király & Madan '18]