Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part I:
Organizational

Philipp Kindermann

Summer Semester 2020

Organizational

Lectures: Pre-recorded (as you see here)

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Release date: Tuesday 10:00 (lecture time slot)

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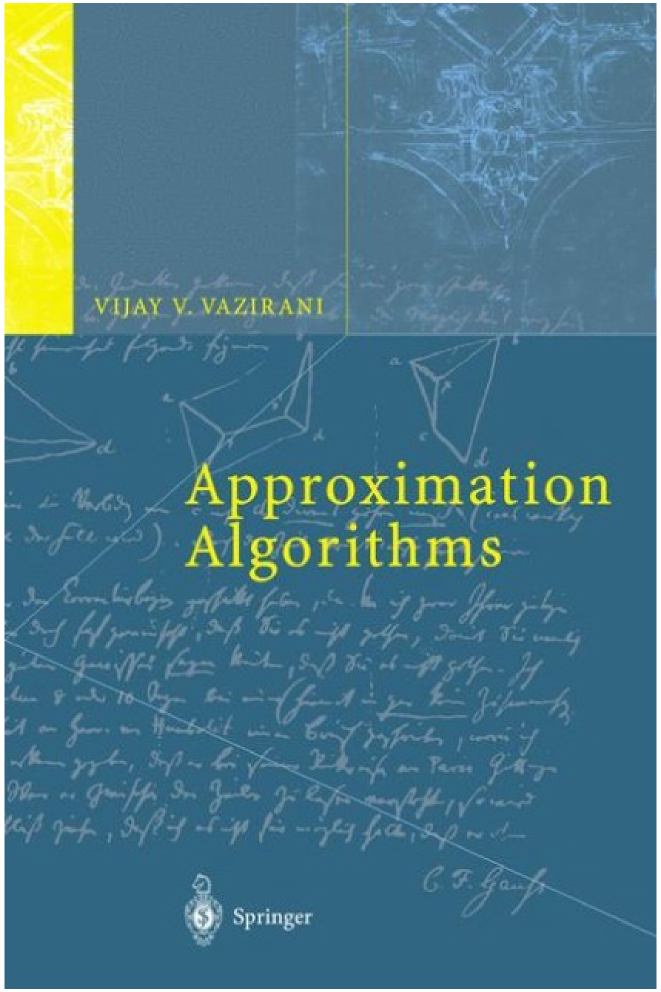
Questions/Tasks during the lecture

Textbooks



Vijay V. Vazirani:
Approximation
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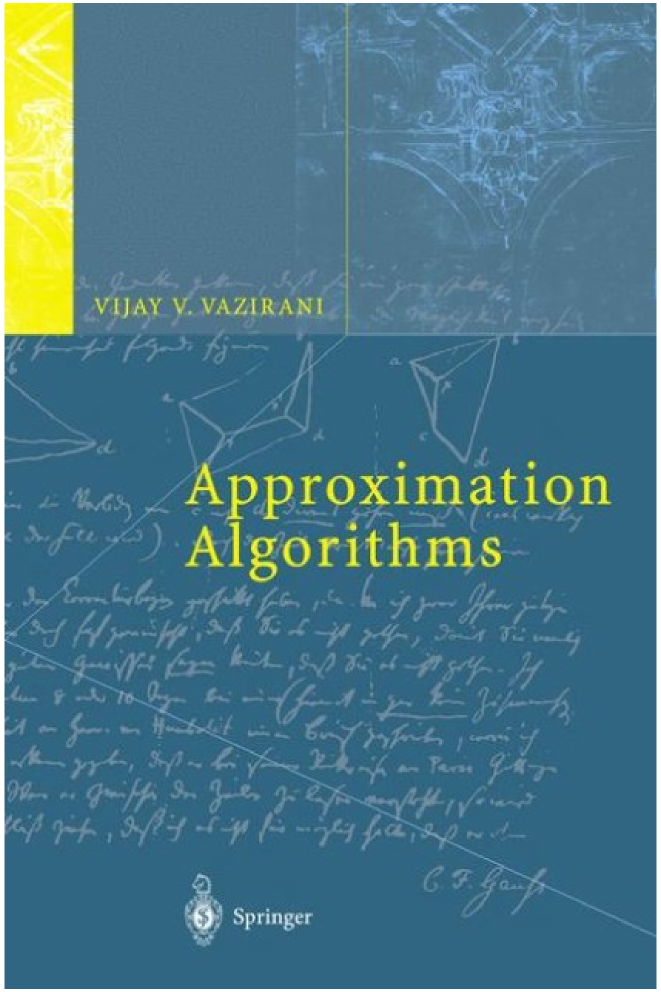
Springer-Verlag, 2003.

Textbooks



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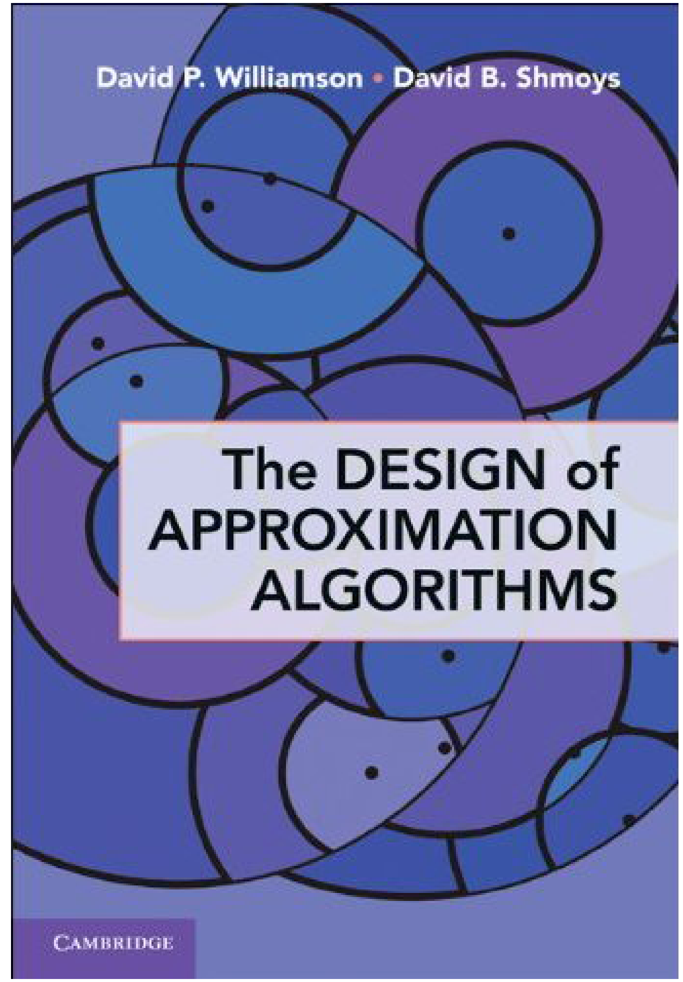
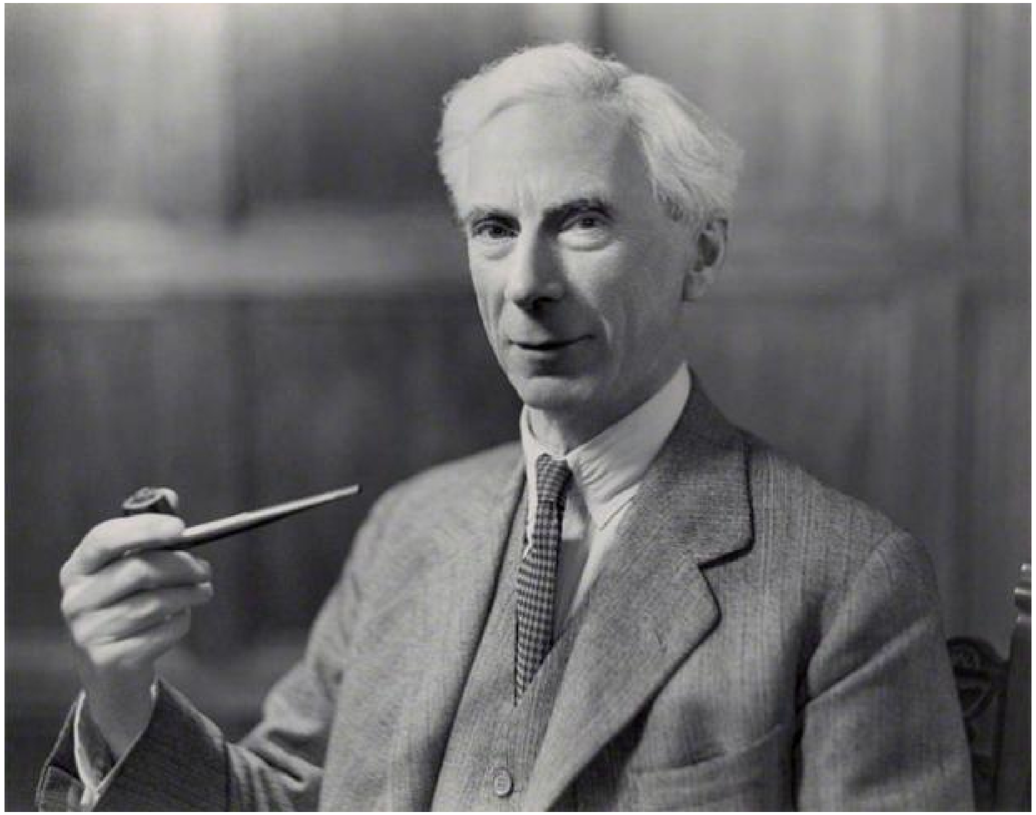


D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.
http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

– Bertrand Russell
(1872 – 1970)



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- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- Techniques for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial Algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- *k*-Center via param. Pruning
- Min-Deg-Spanning-Tree & local search
- Knapsack via DP & Scaling
- Euclidean TSP via Quadtrees

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LP-based Algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal-Dual Schema
- Maximum Satisfiability
- Scheduling und Extrem point solutions
- Steiner Forest via Primal-Dual

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part II:
Vertex Cover (card.)

Philipp Kindermann

Summer Semester 2020

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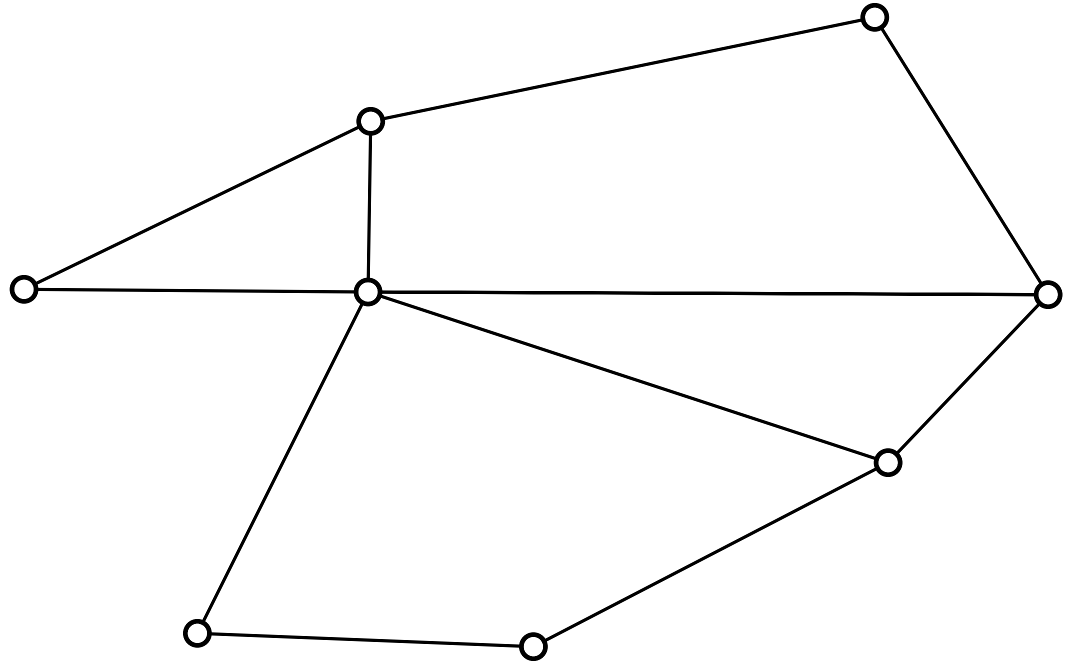

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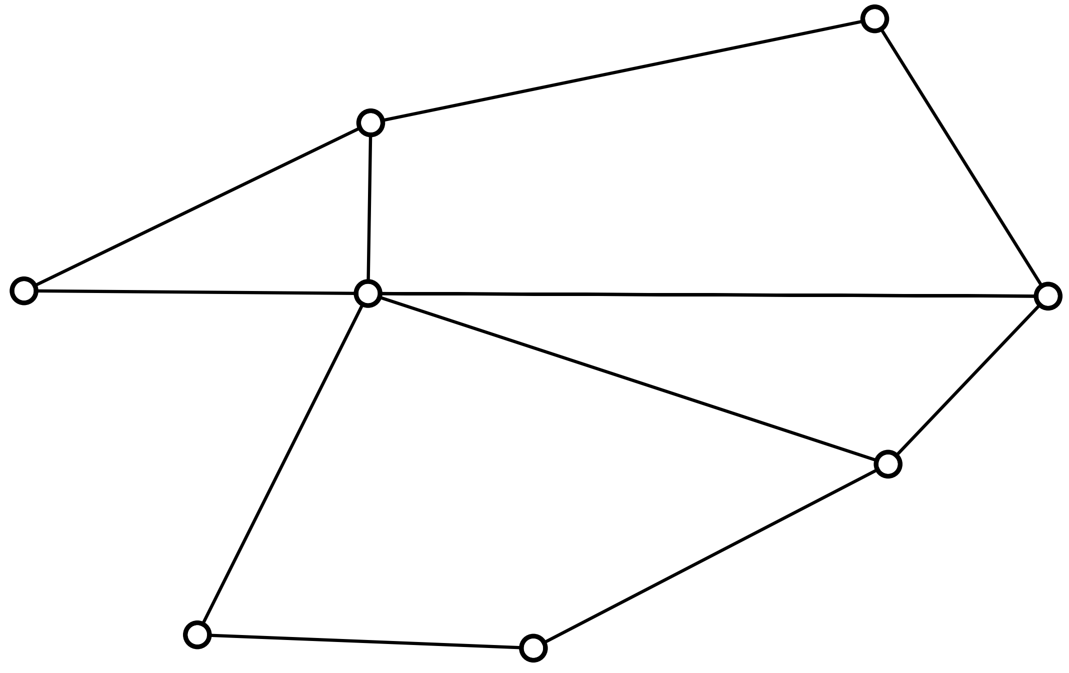

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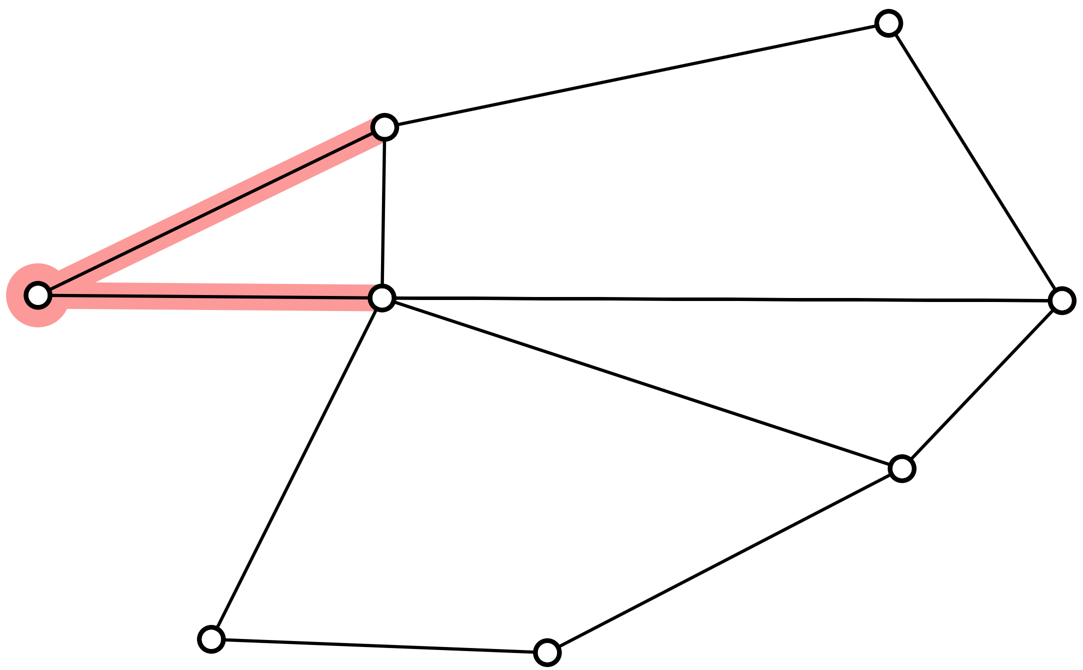

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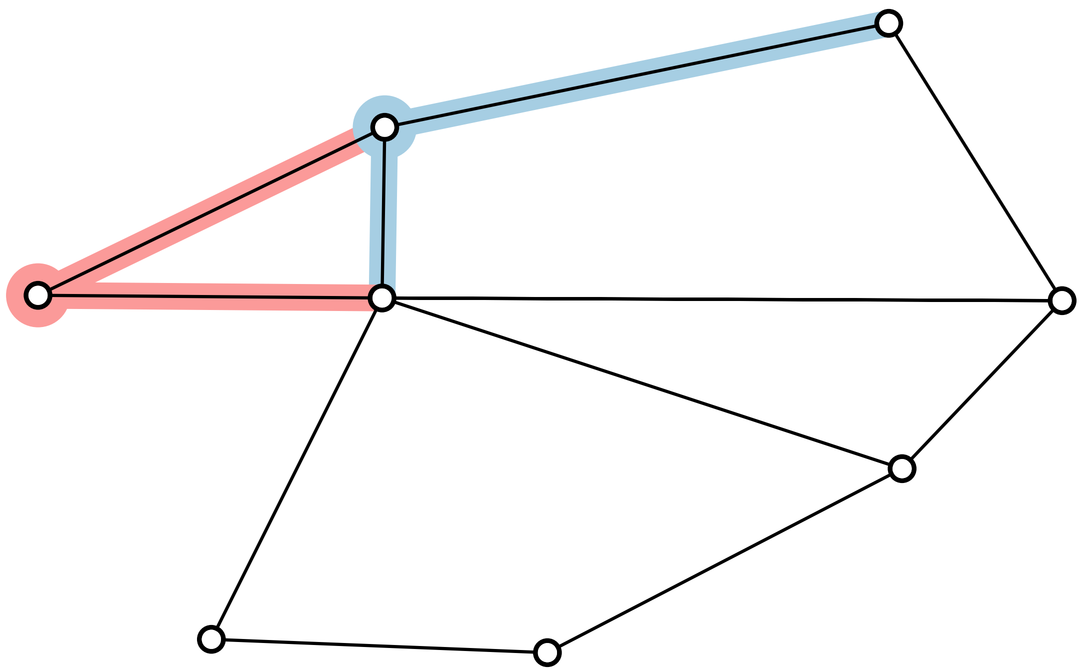

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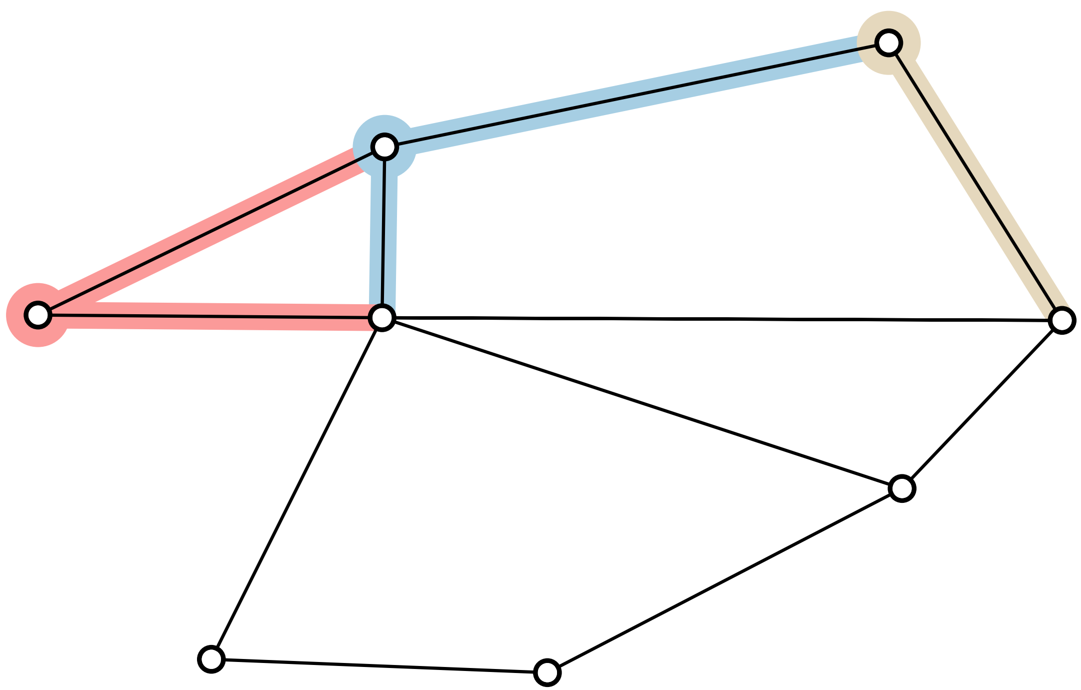

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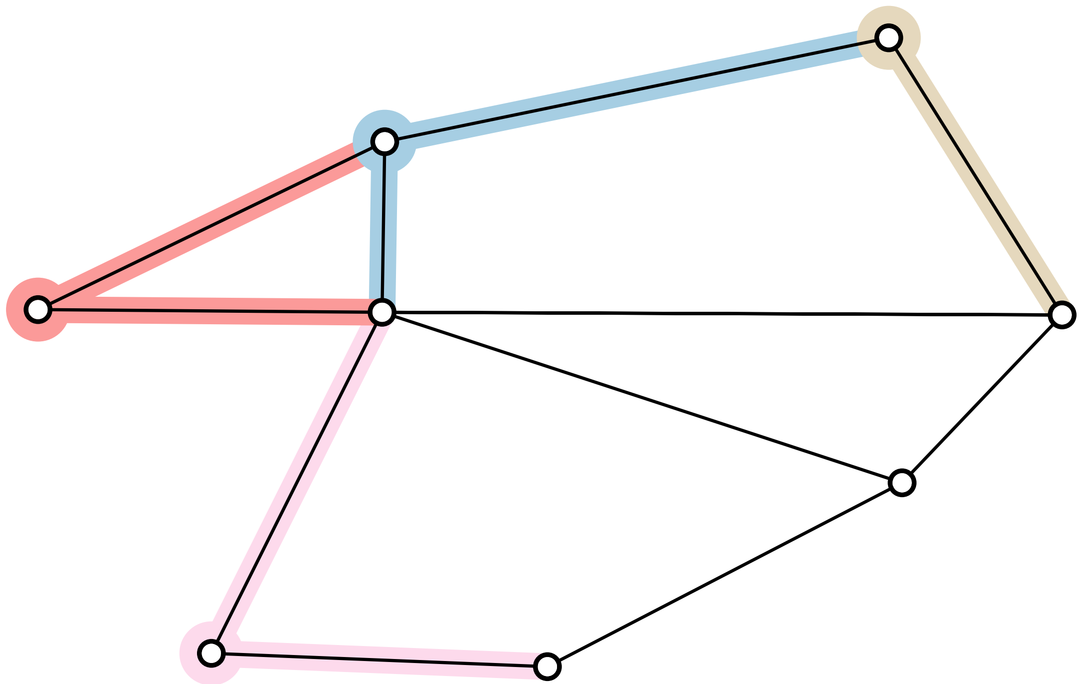

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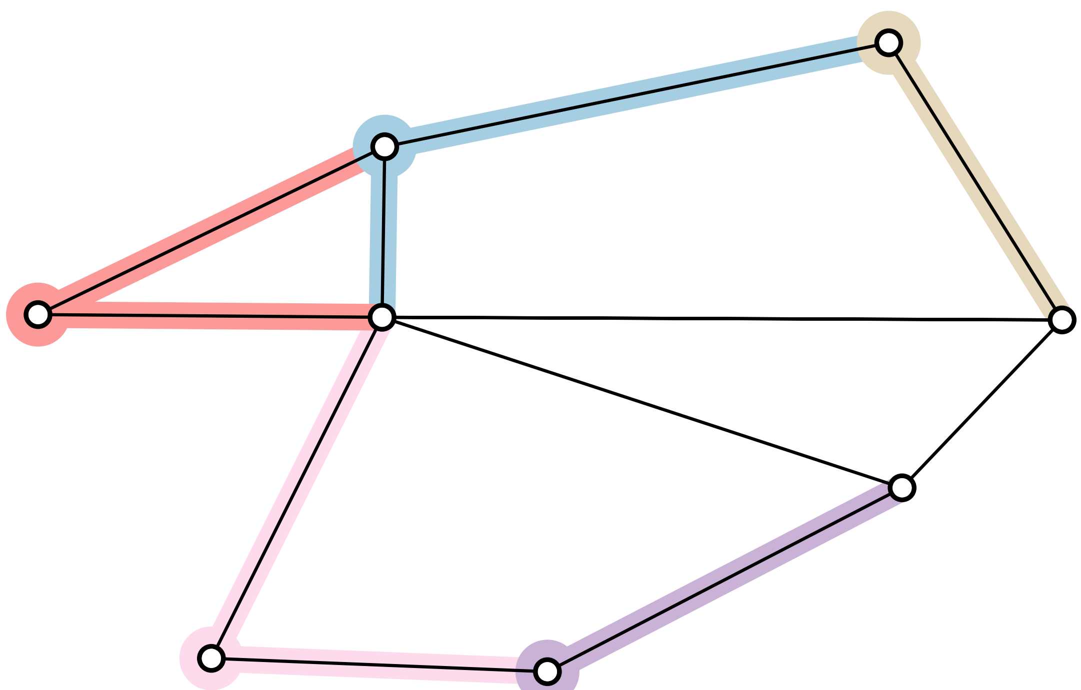

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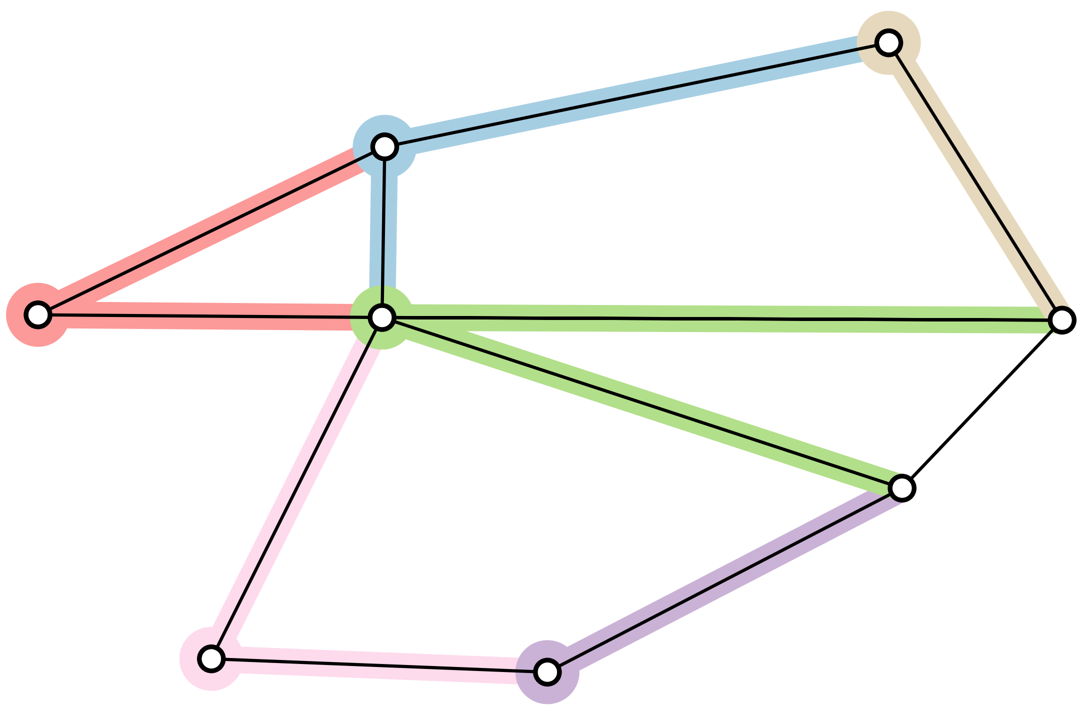

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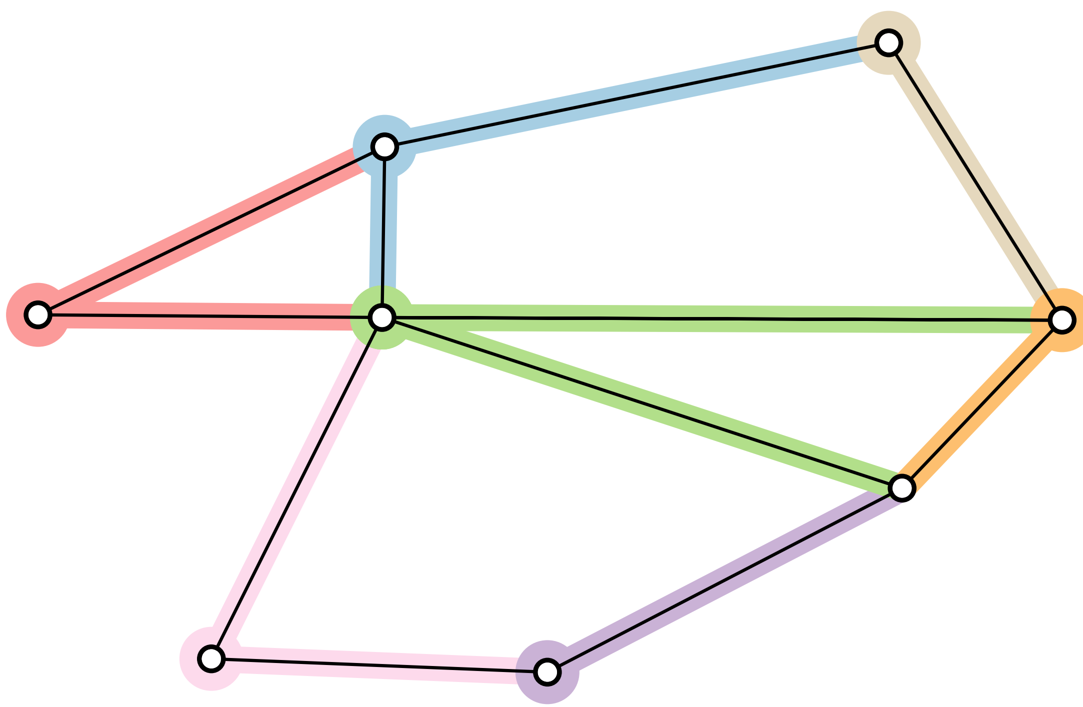

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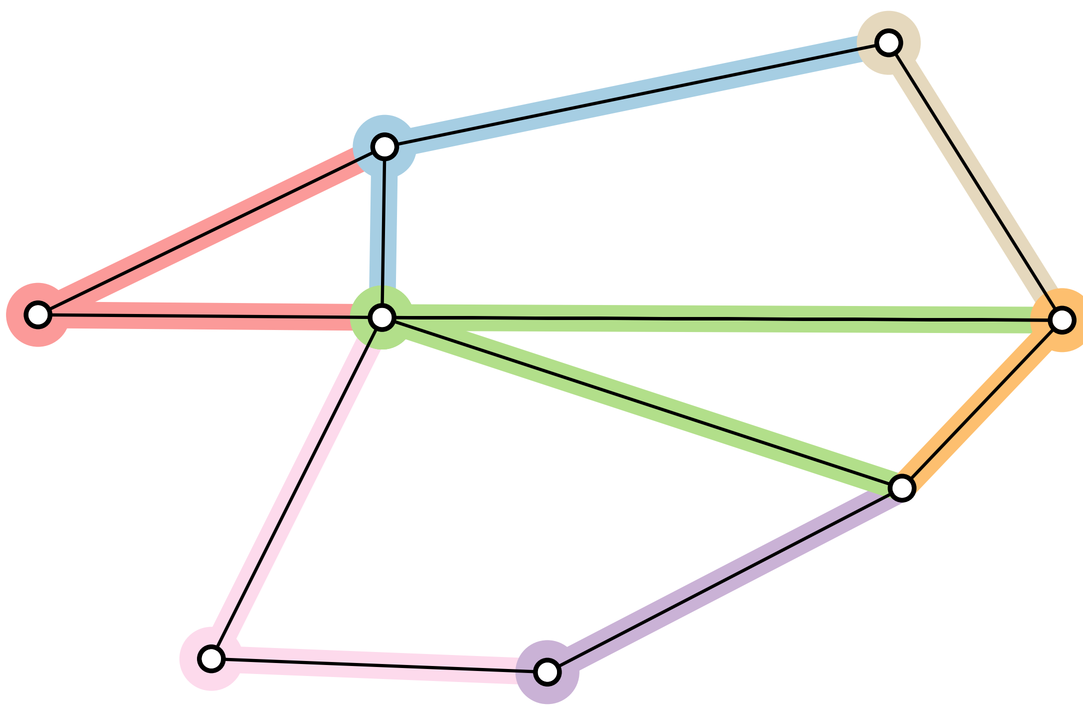

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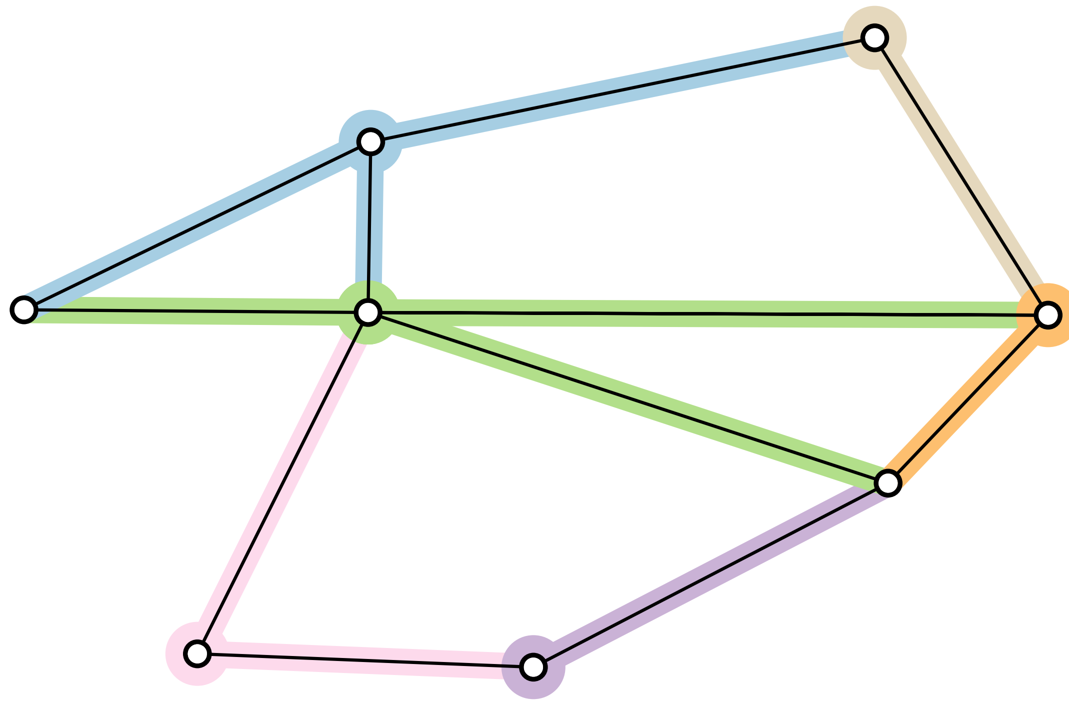

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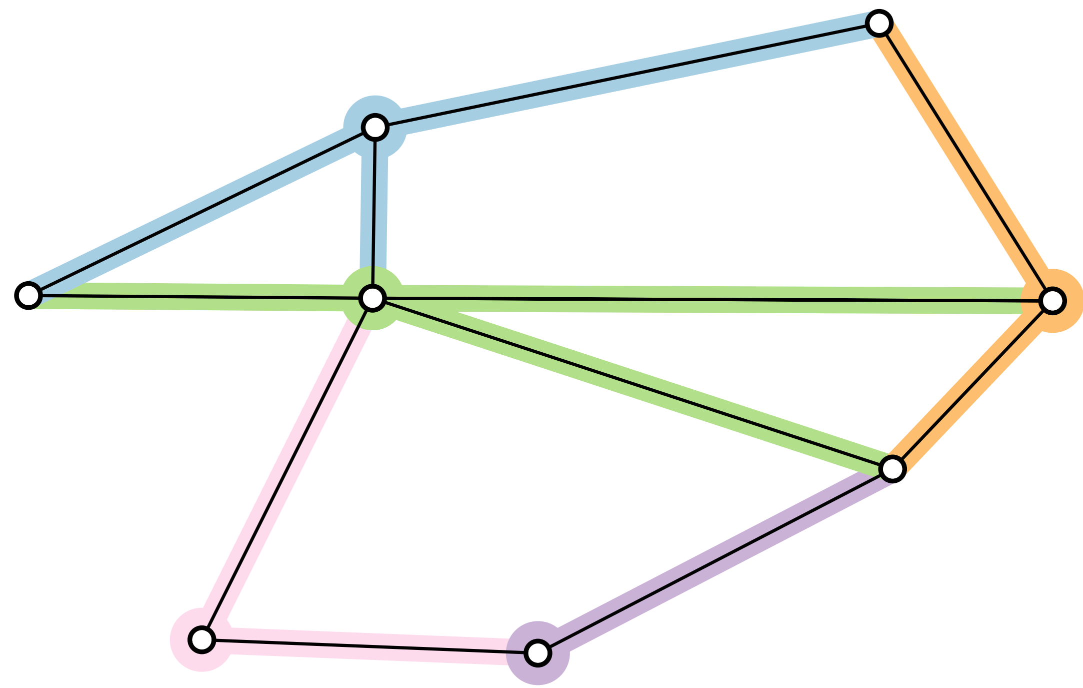

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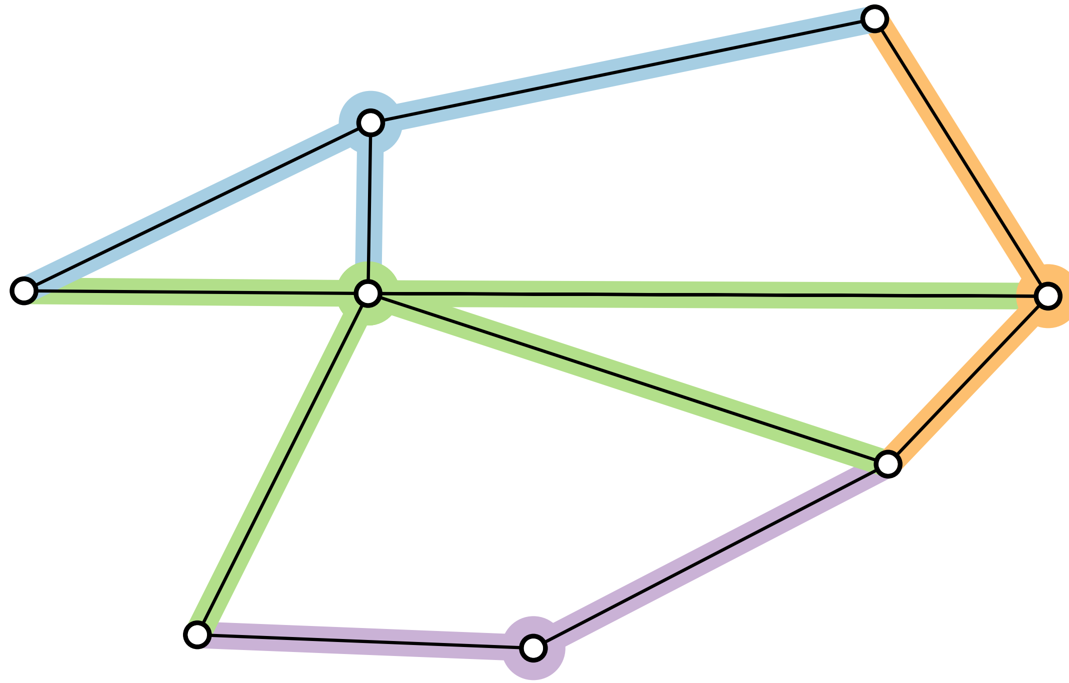

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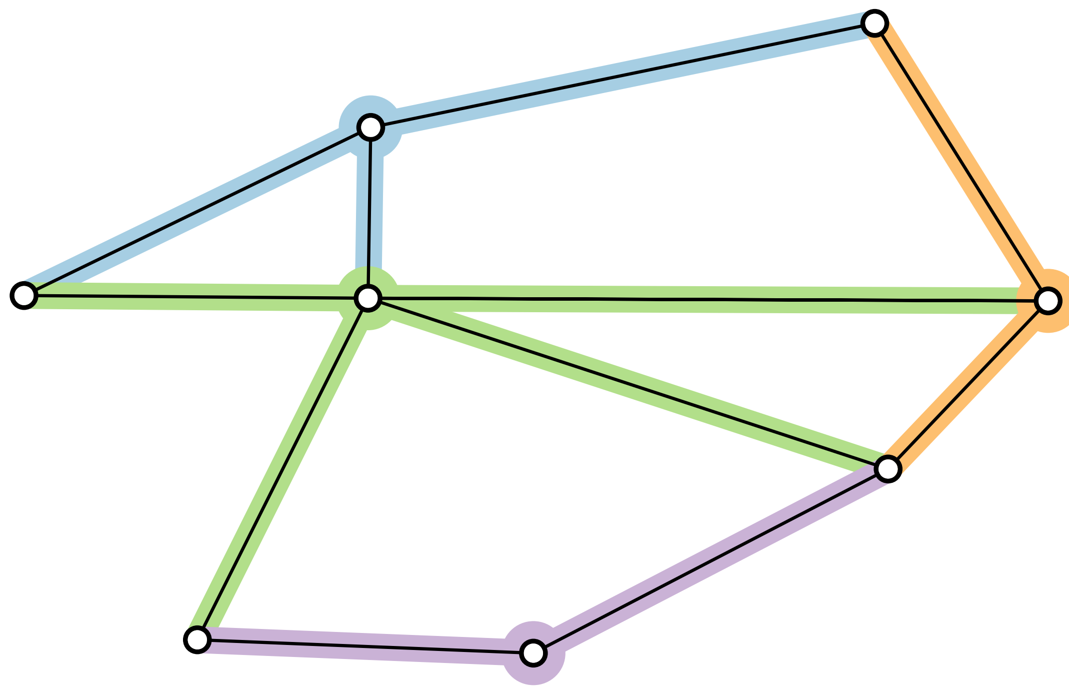

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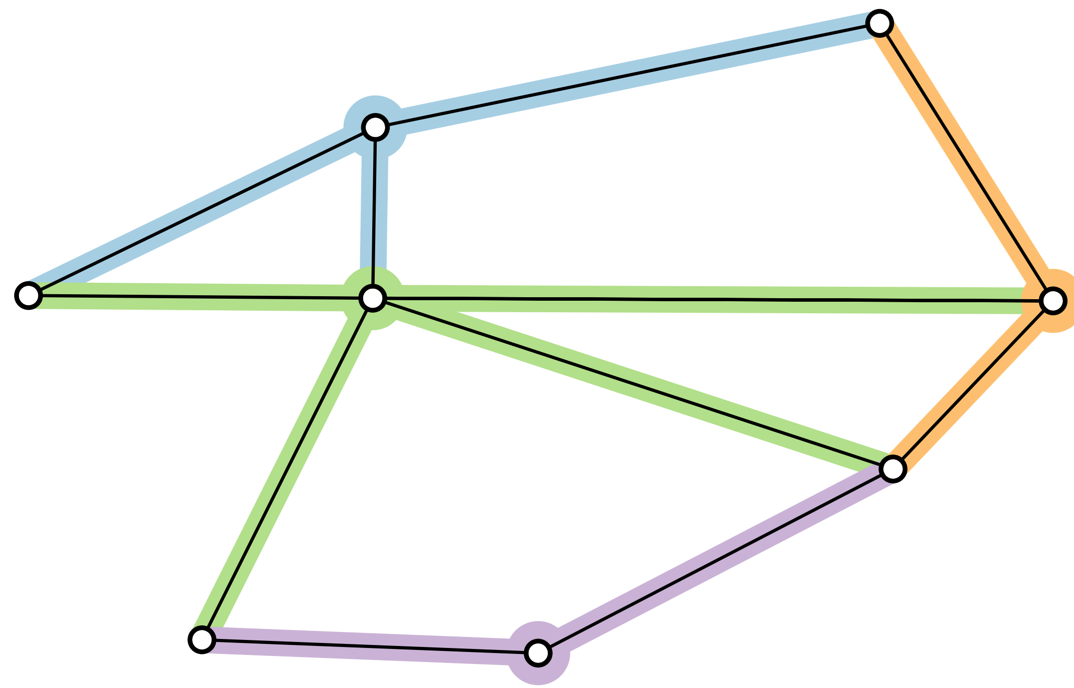

Optimum (OPT $= 4$) – but in general NP-hard to find :-(

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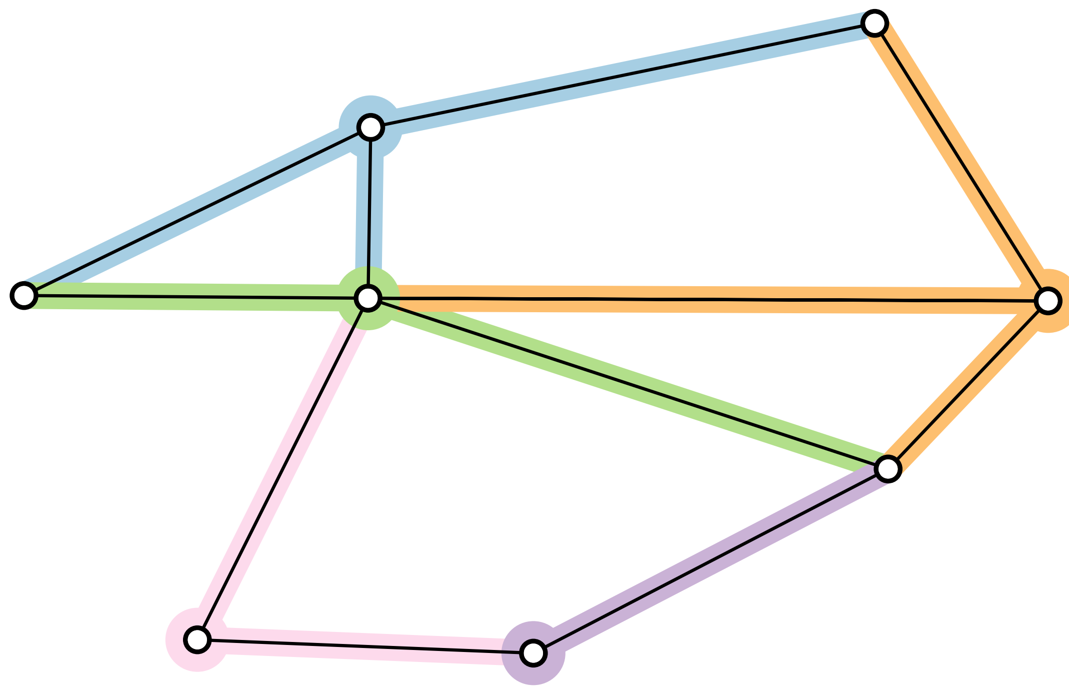

"good" approximate solution (5/4-approximation)

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part III:
NP-Optimization Problem

Philipp Kindermann

Summer Semester 2020

NP-Optimization Problem

An **NP-optimization problem Π** is given by:

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- Π is either a minimization or maximization problem.

VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi = \textsc{Vertex Cover}$.

$D_\Pi =$

For $I \in D_\Pi$: $|I| =$

$S_\Pi(I) =$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

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Π is Minimization problem.

Optimum and optimal objective value

Let Π be a minimization problem and $I \in D_\Pi$ be an instance of Π.

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The optimal value $\text{obj}_\Pi(I, s^*)$ of the objective function is also denoted by $\text{OPT}_\Pi(I)$ or simply OPT in context.

Approximation Algorithms

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Approximation Algorithms

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Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part IV:
Approximation Algorithm for VertexCover

Philipp Kindermann

Summer Semester 2020

Approximation Alg. for VertexCover

Ideas?

Approximation Alg. for VertexCover

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- Edge-Greedy

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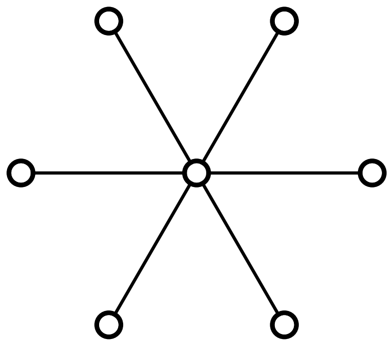

Approximation Alg. for VertexCover

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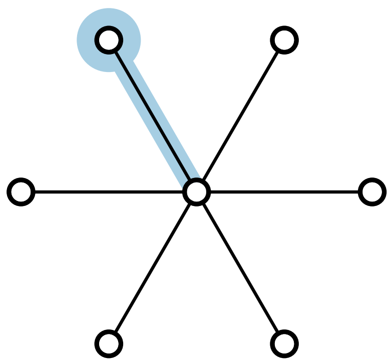

Approximation Alg. for VertexCover

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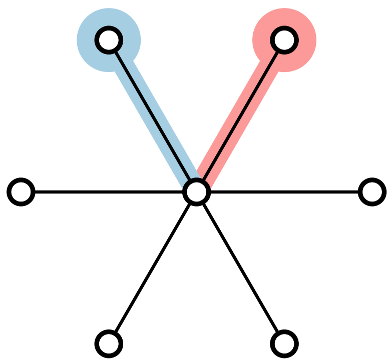

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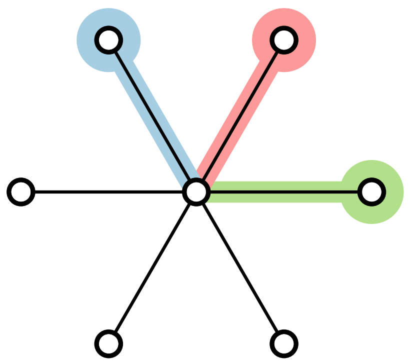

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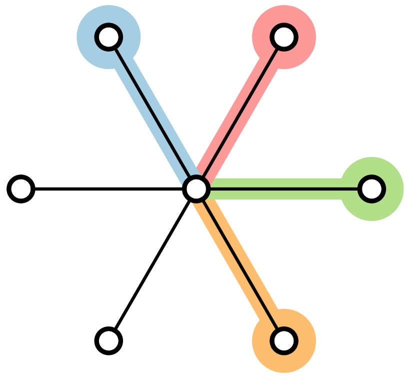

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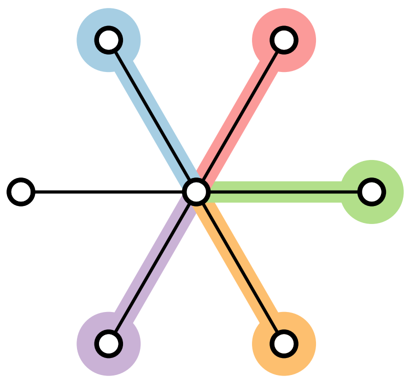

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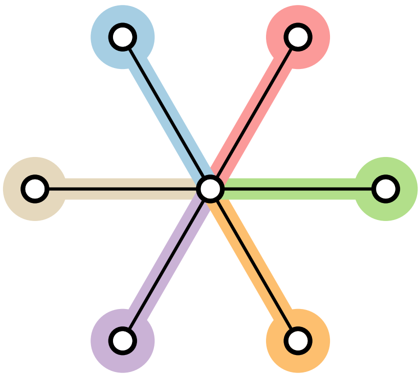

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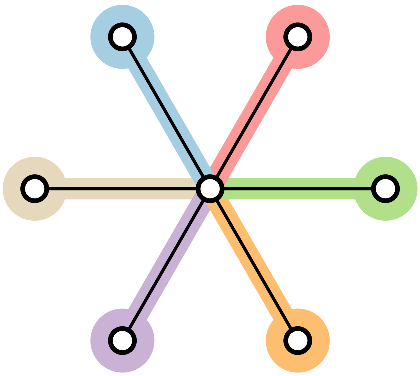

Quality?

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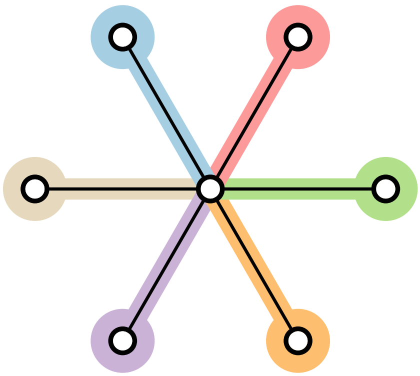

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Problem: How can we estimate $\mathrm{obj}_\Pi(I, s) / \mathrm{OPT}$, when it is hard to calculate OPT?

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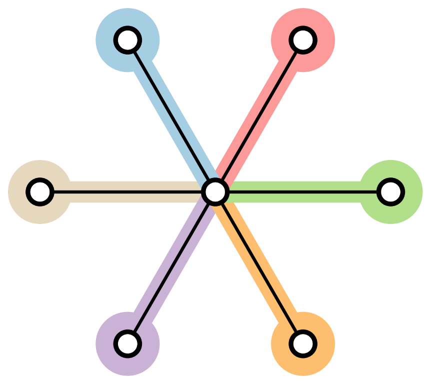

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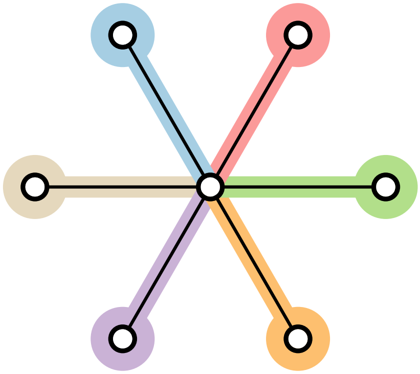

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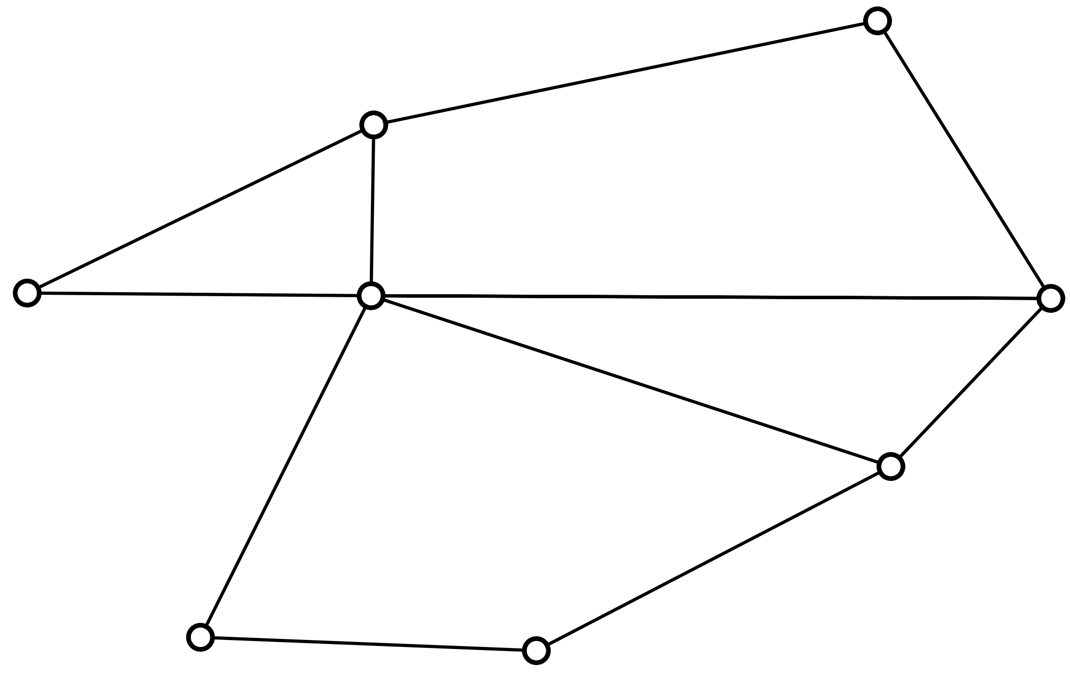
$$\frac{\mathrm{obj}_\Pi(I,s)}{\mathrm{OPT}} \leq \frac{\mathrm{obj}_\Pi(I,s)}{U}$$

Lower Bound by Matchings



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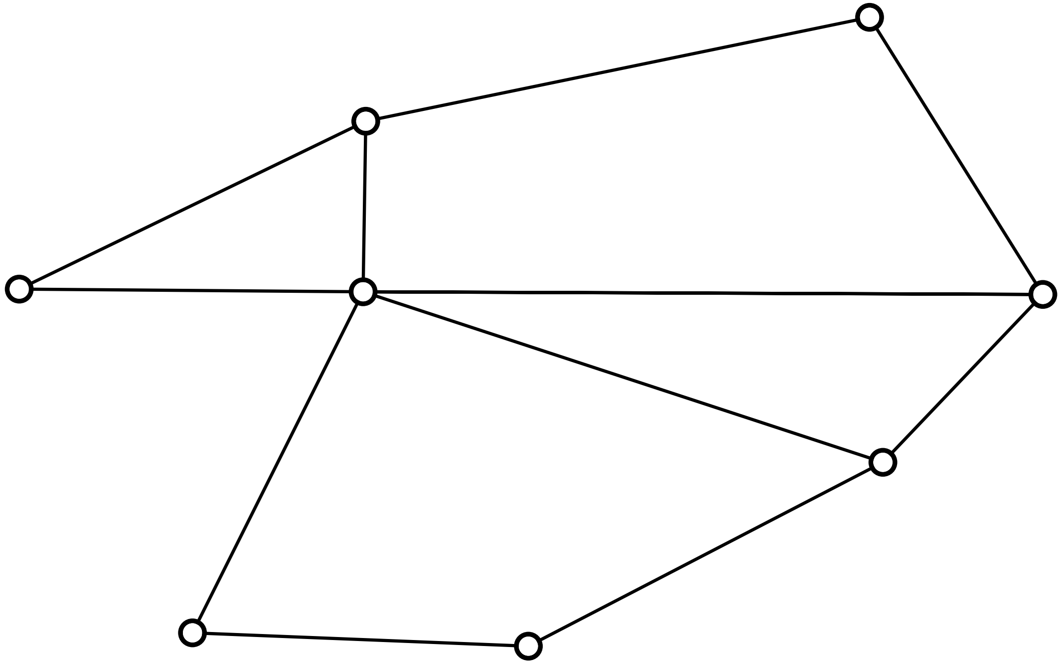
An edge set $M \subseteq E$ of a graph $G = (V, E)$ is a **matching** if no two edges of M are adjacent (i.e., share an end vertex).



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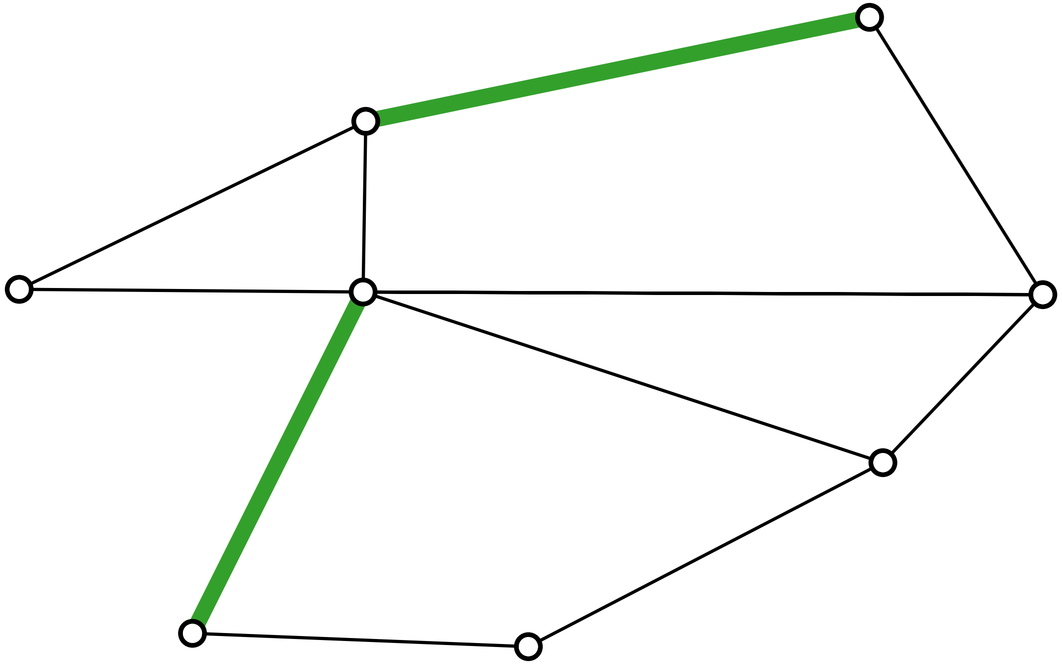

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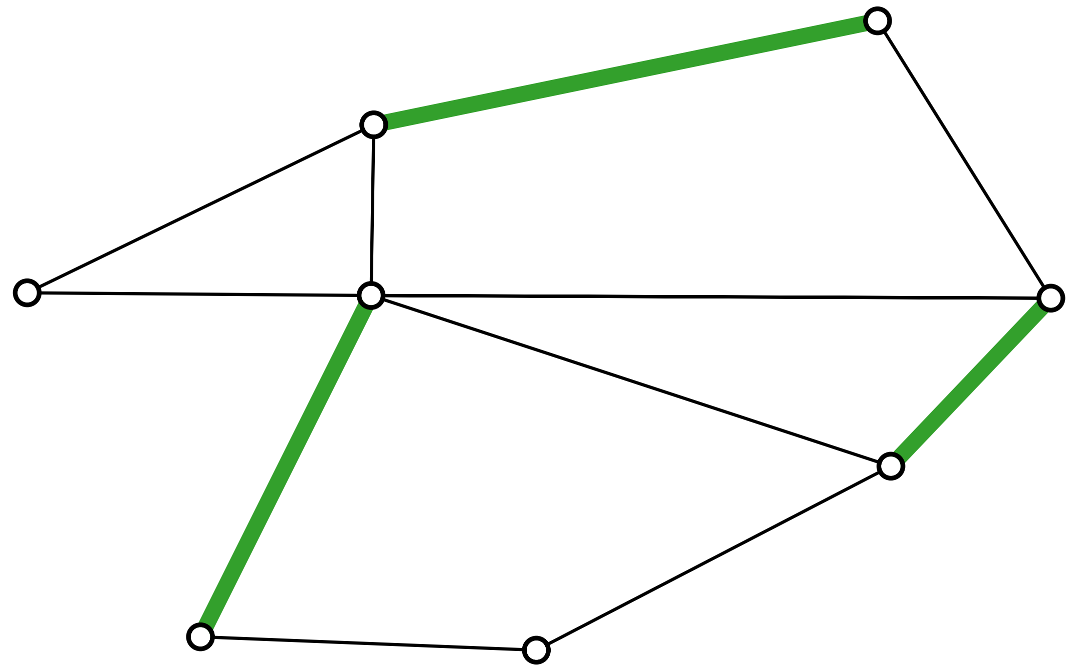

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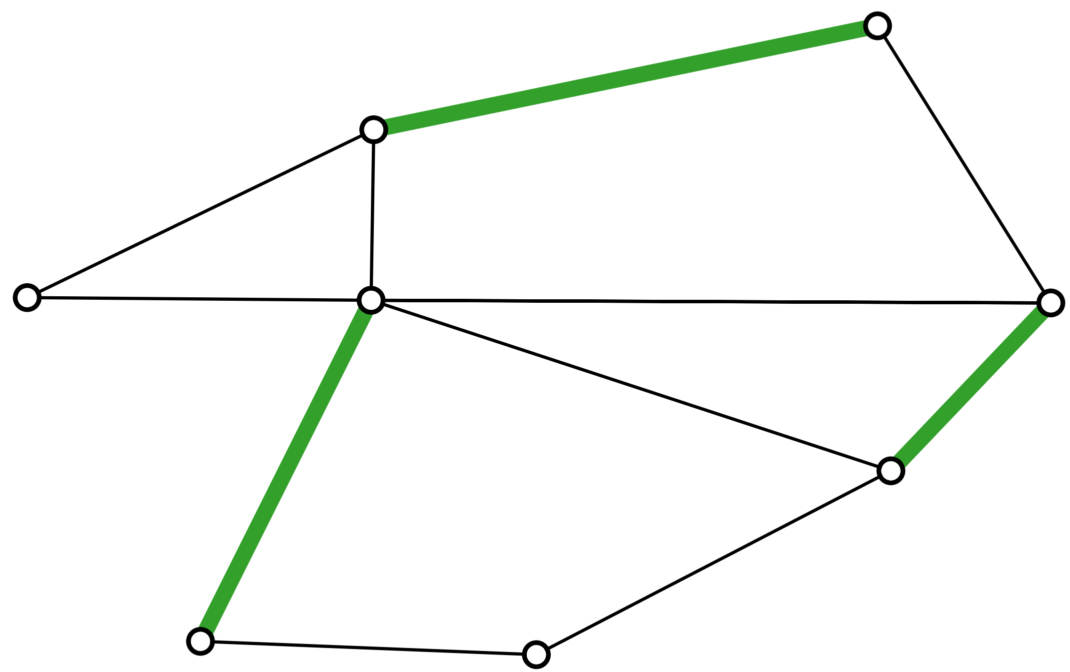

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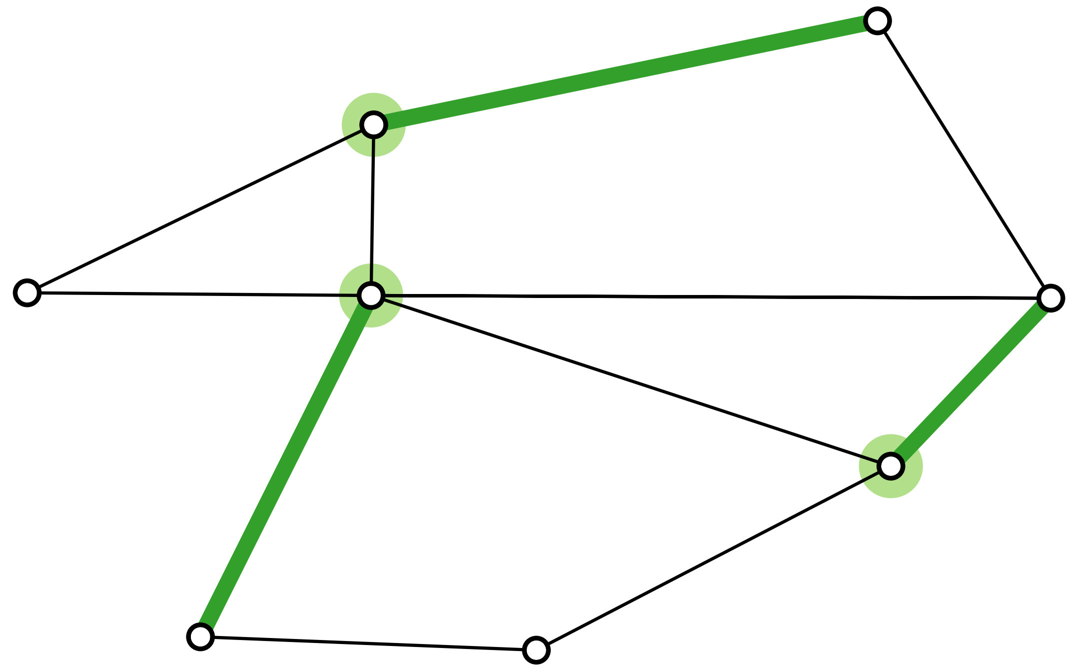
Vertex cover of M



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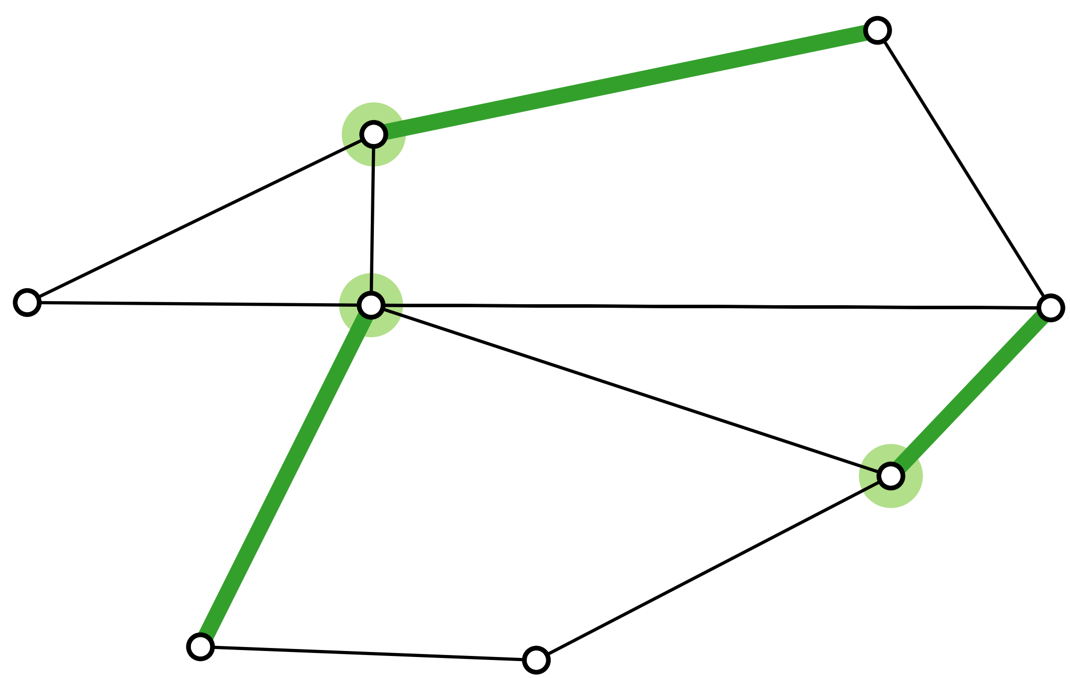

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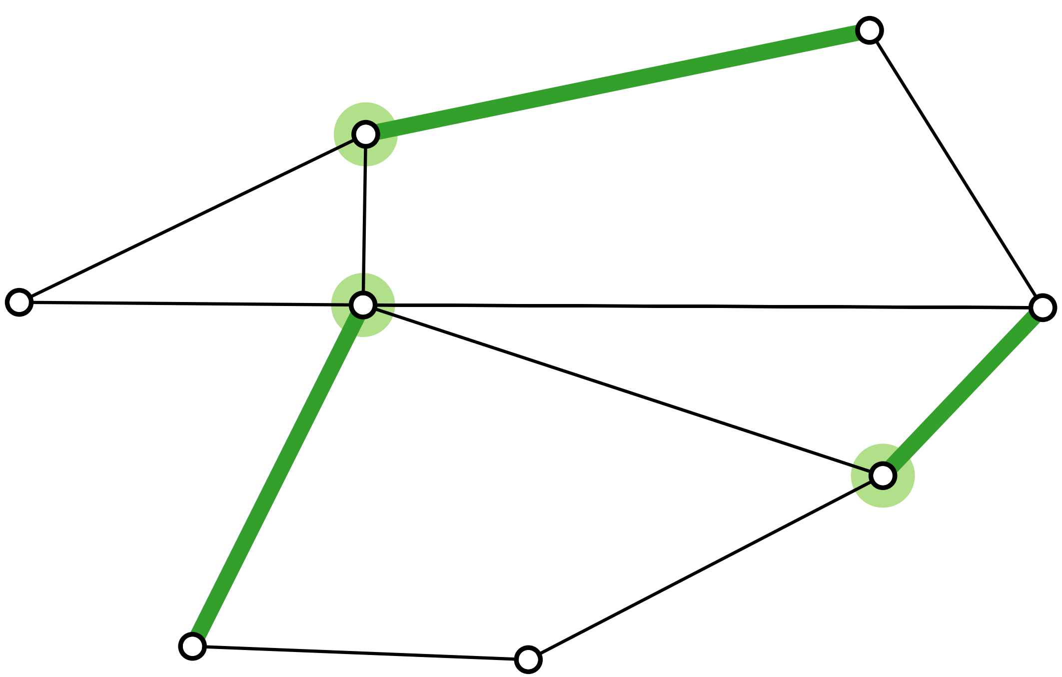

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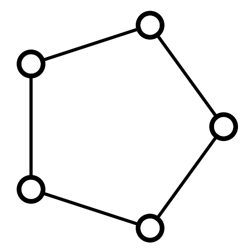

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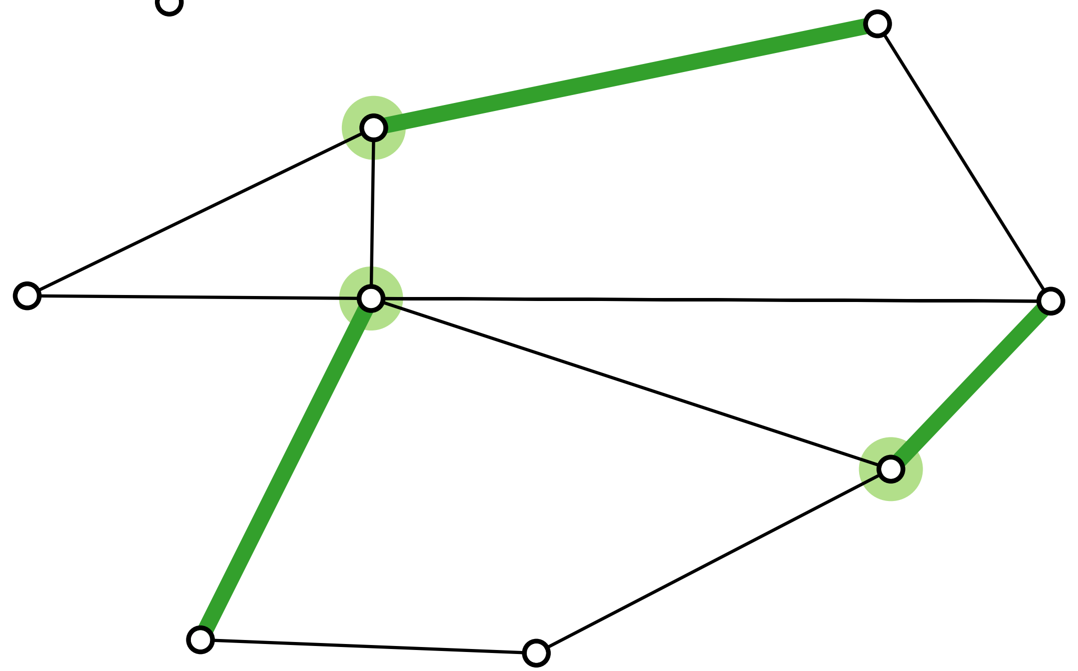

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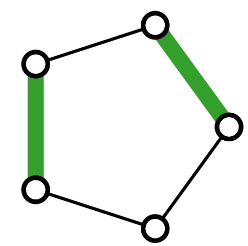

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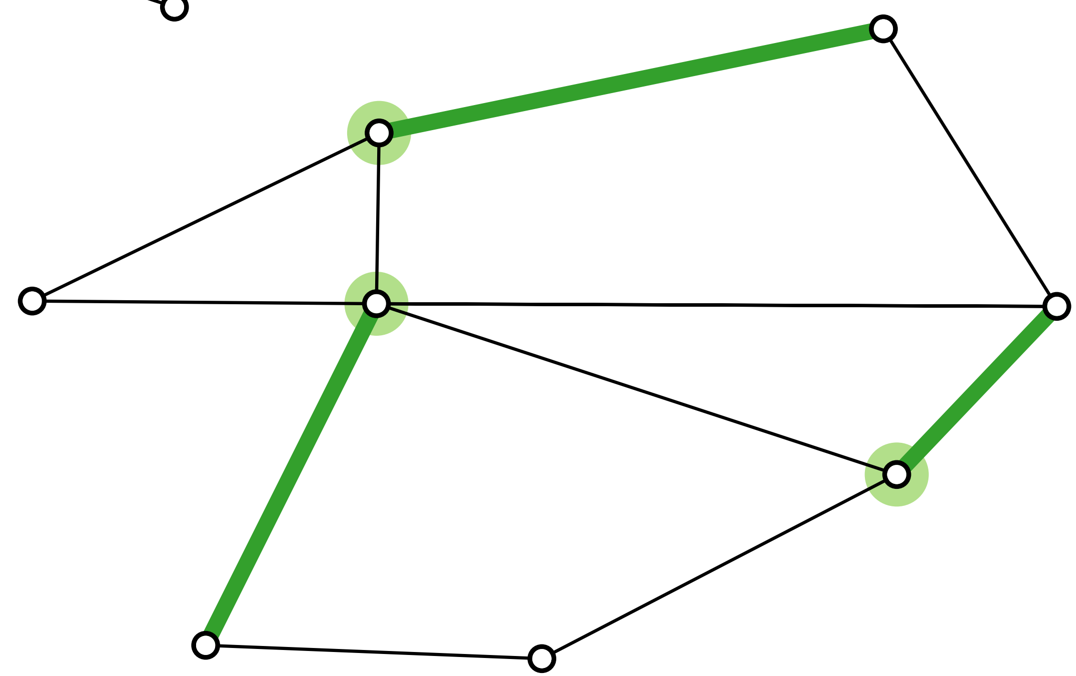

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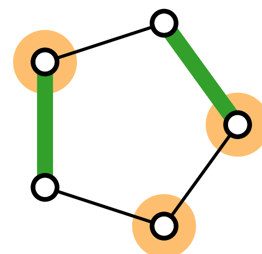

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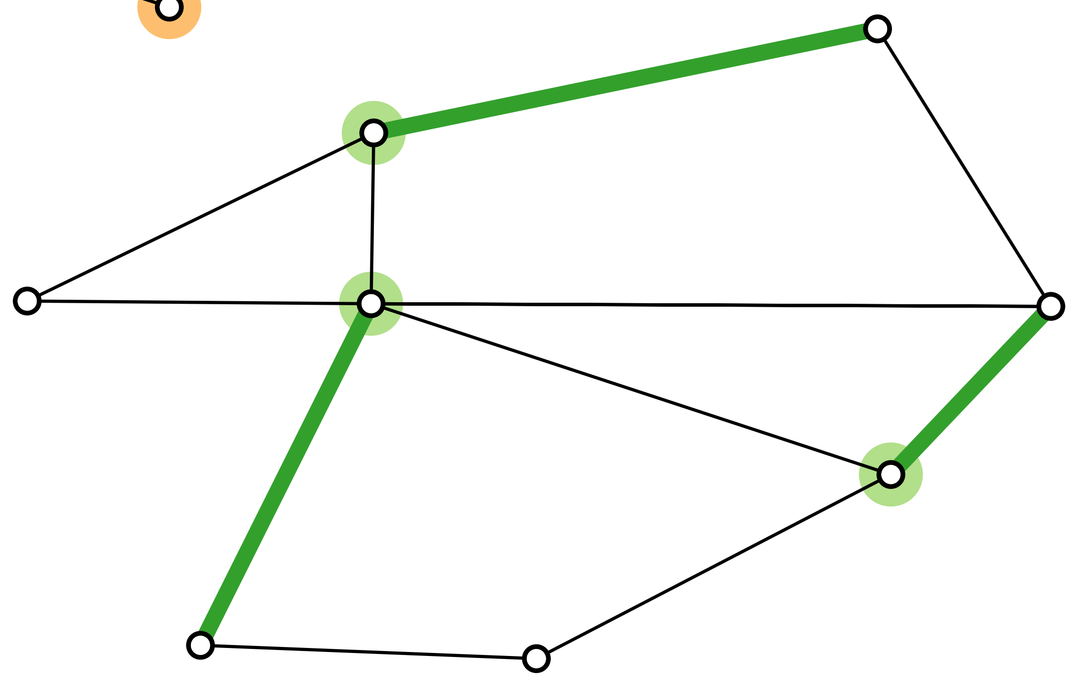

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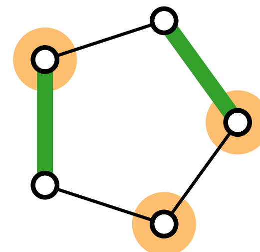

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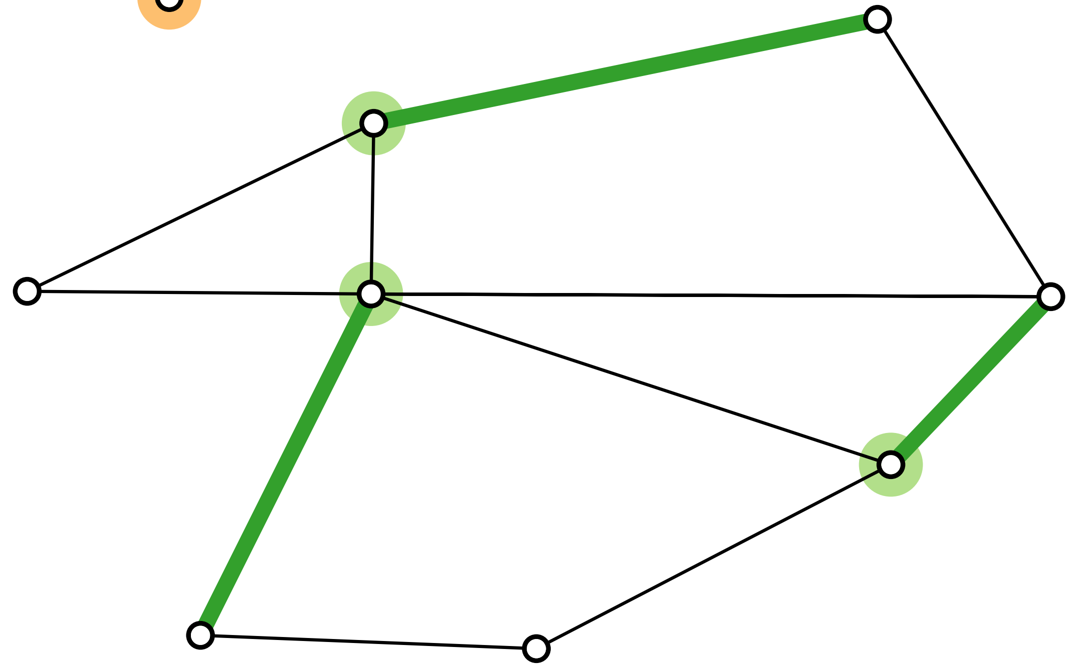

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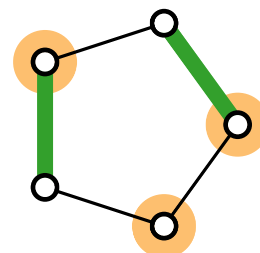

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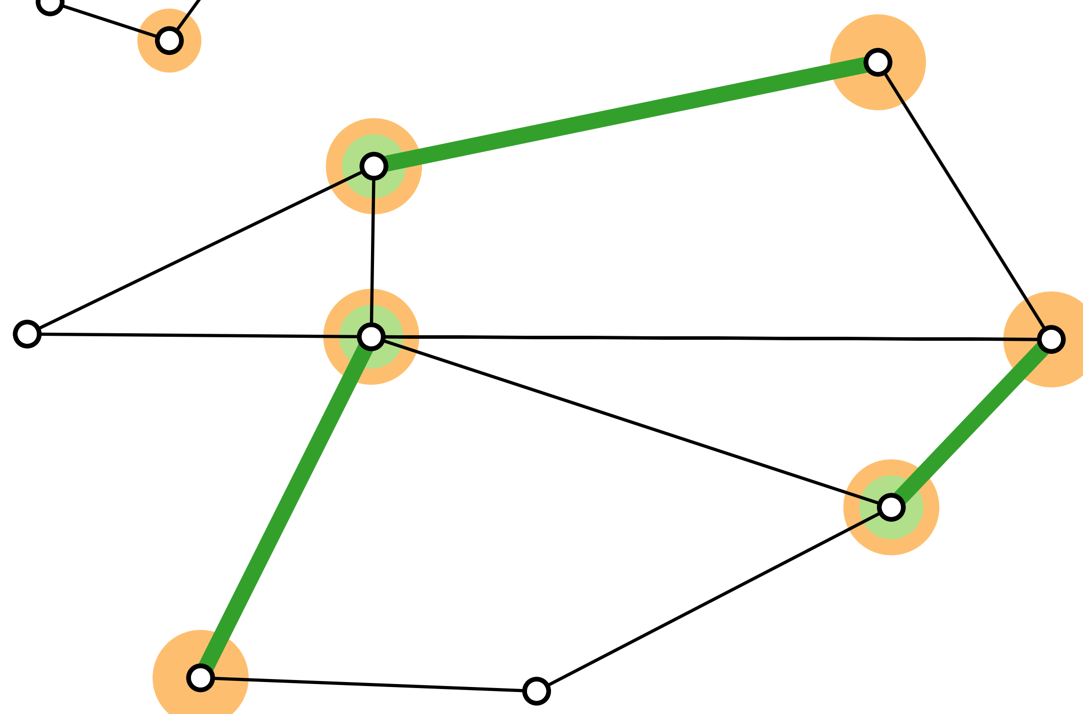
Vertex cover of M

Vertex cover of E



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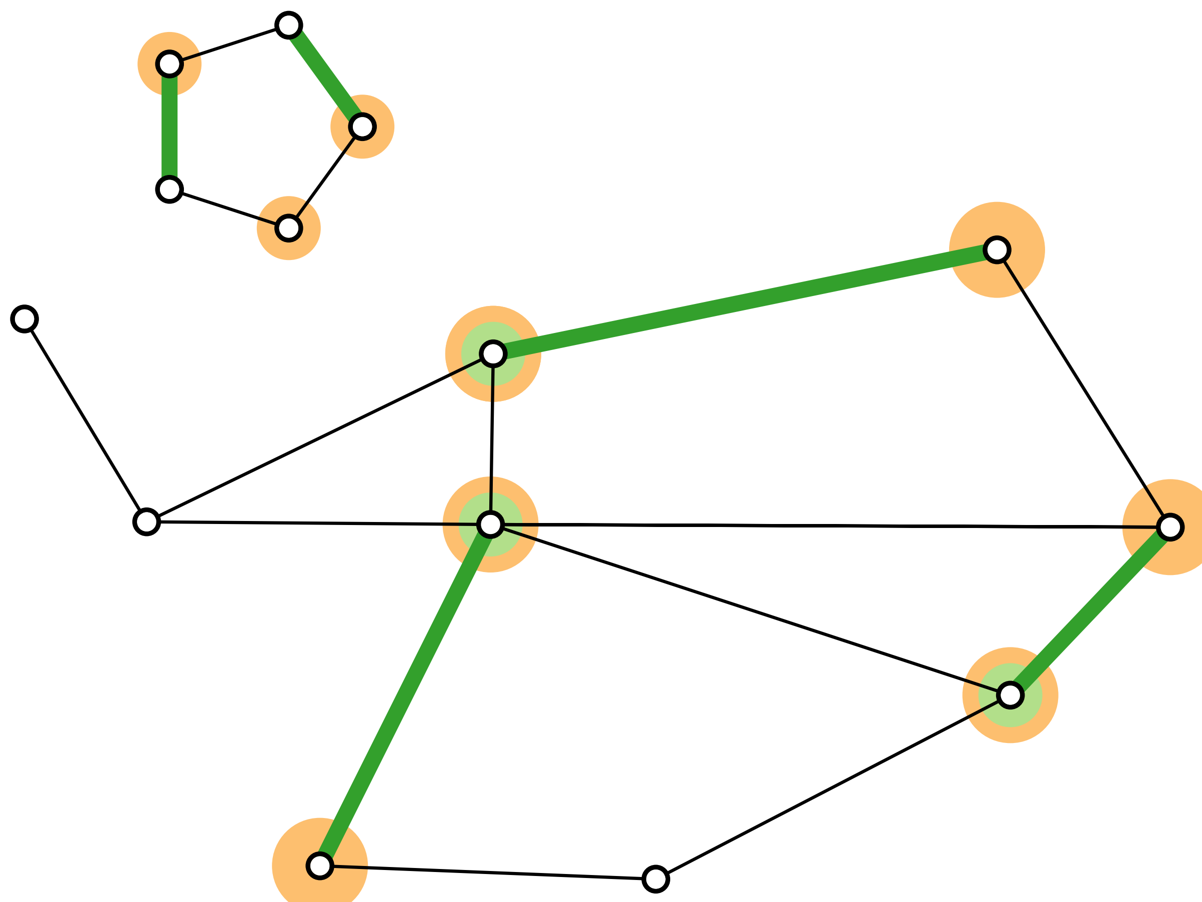
Vertex cover of M

Vertex cover of E



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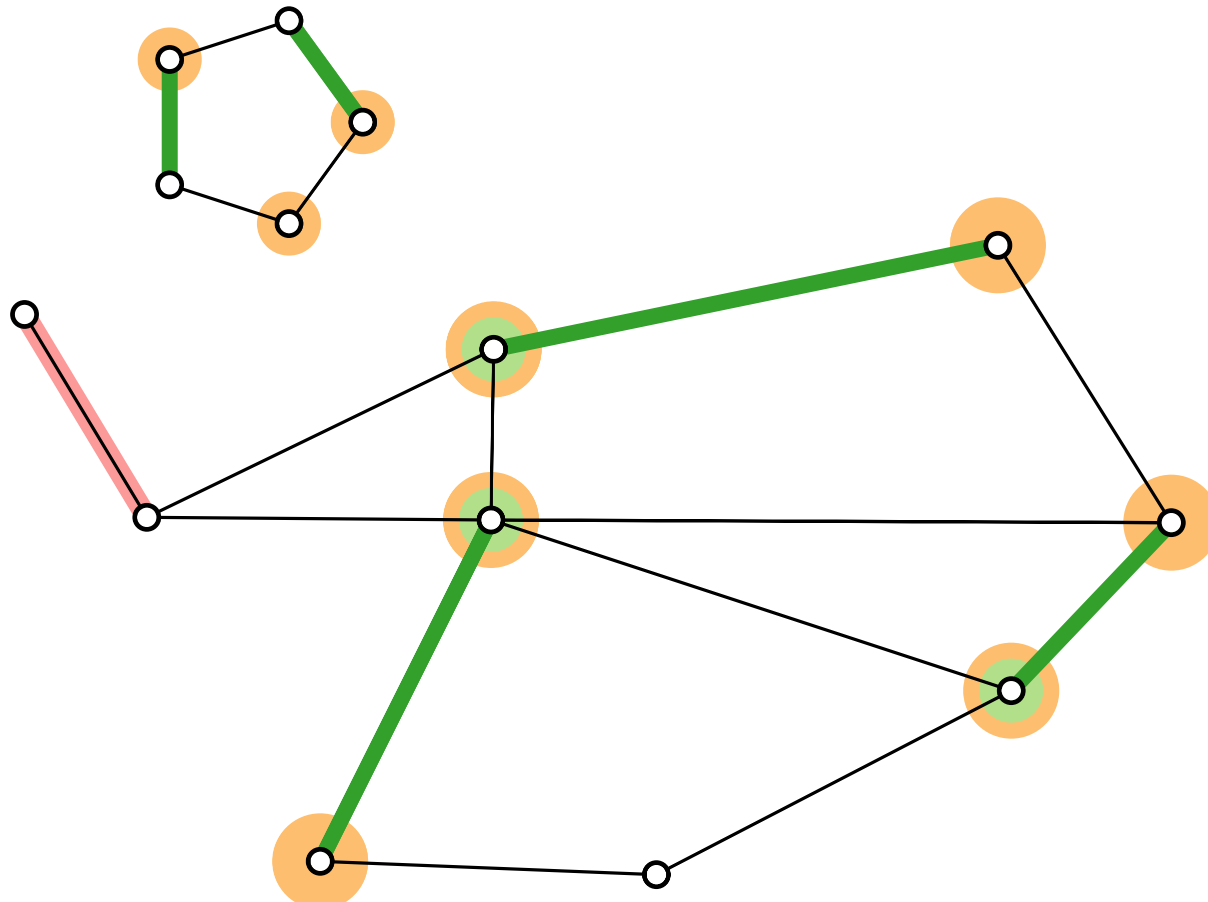
Vertex cover of M

Vertex cover of E



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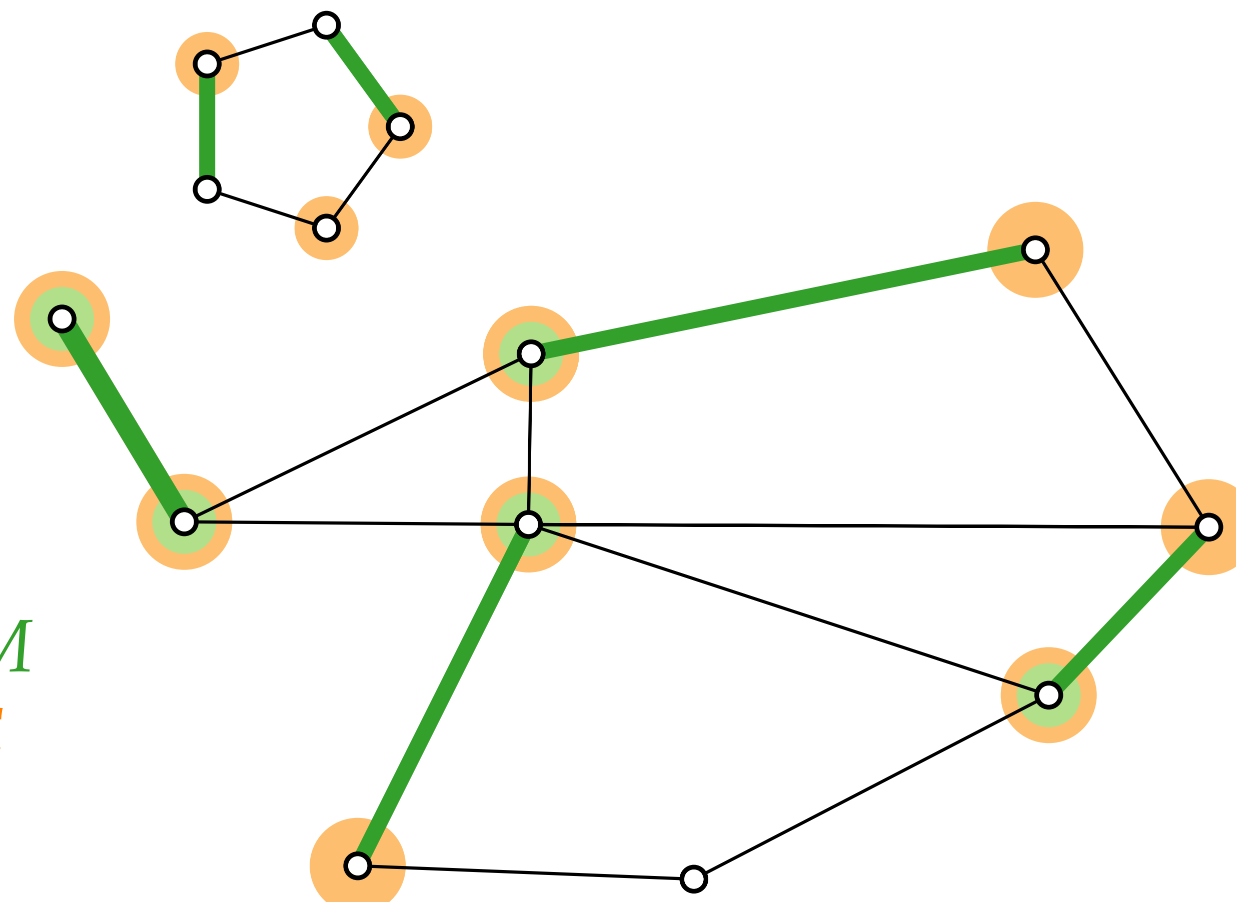

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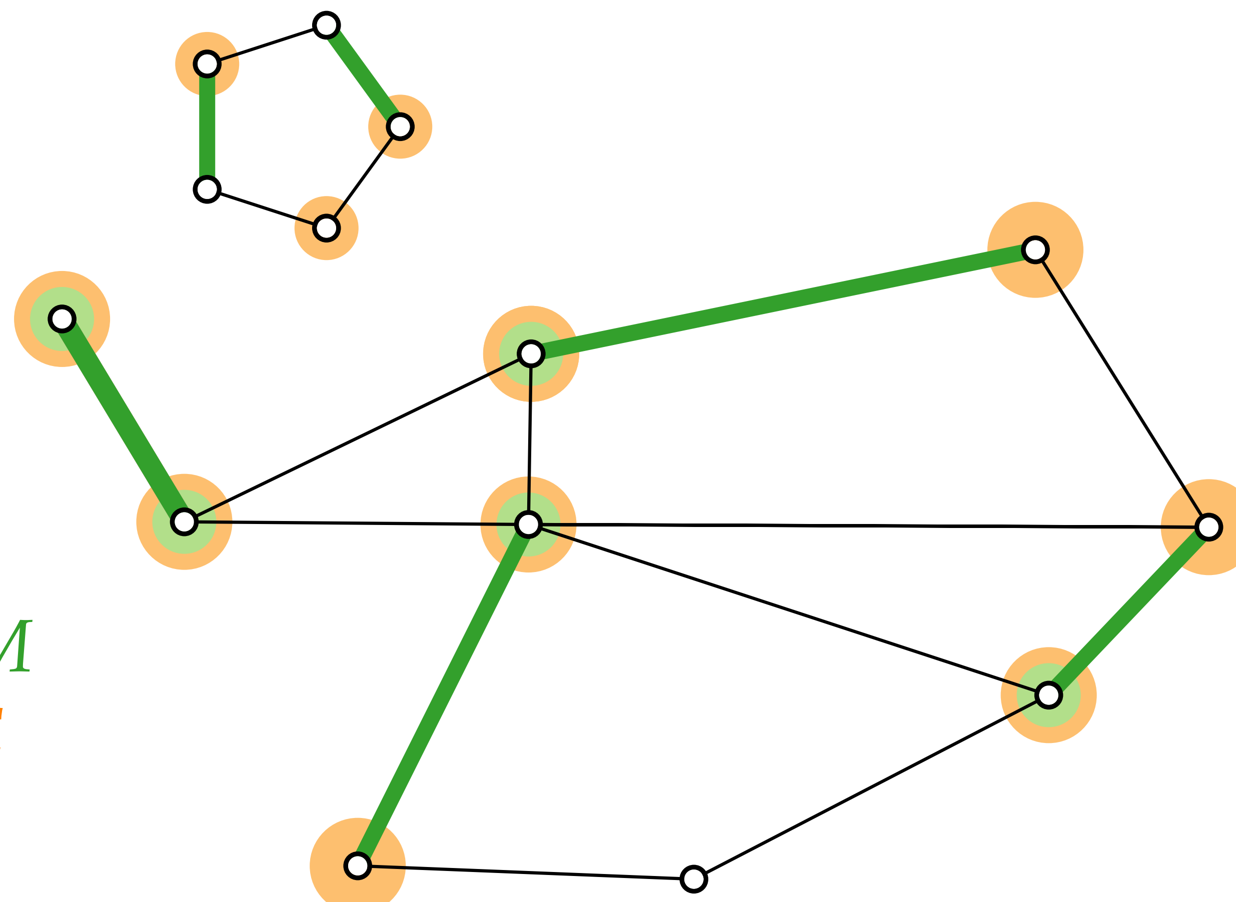

Vertex cover of M

Vertex cover of E

$\mathrm{ALG} = 2 \cdot |M| \leq$

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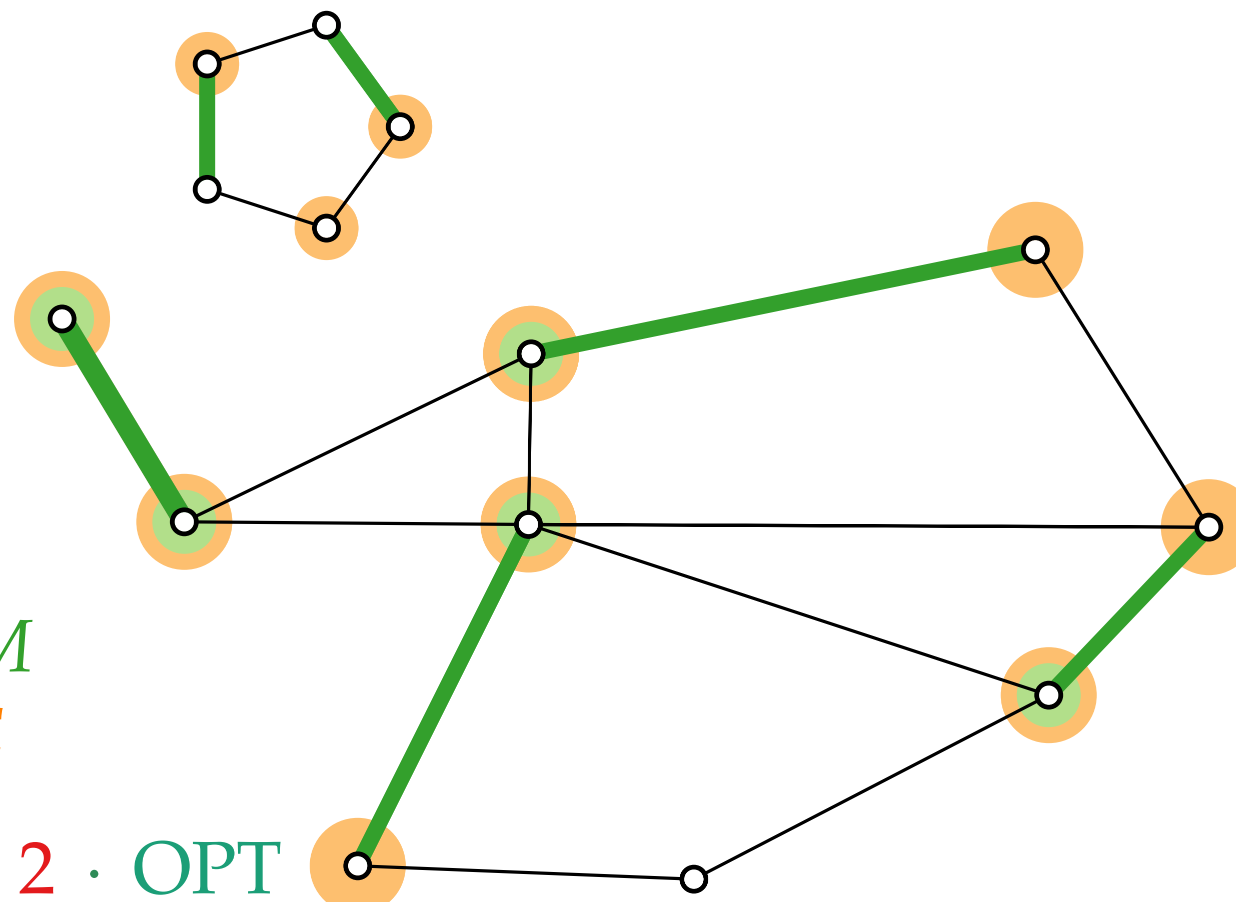

OPT $\geq |M|$

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Vertex cover of M

Vertex cover of E

ALG $= 2 \cdot |M| \leq 2 \cdot$ OPT

Approximation Alg. for VertexCover

Algorithm VertexCover(G)

$M \leftarrow \emptyset$

Approximation Alg. for VertexCover

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foreach $e \in E(G)$ **do**

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Approximation Alg. for VertexCover

Algorithm VertexCover(G)

```
M ← ∅
foreach e ∈ E(G) do
    if e not adjacent to edge in M then
        M ← M ∪ {e}
return { u, v | uv ∈ M }
```

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VertexCover cannot be approximated within factor 1.3606 (unless P=NP)

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VertexCover cannot be approximated within factor $2 - \Theta(1)$, if "Unique Game Conjecture" holds.