Julius-Maximilians-
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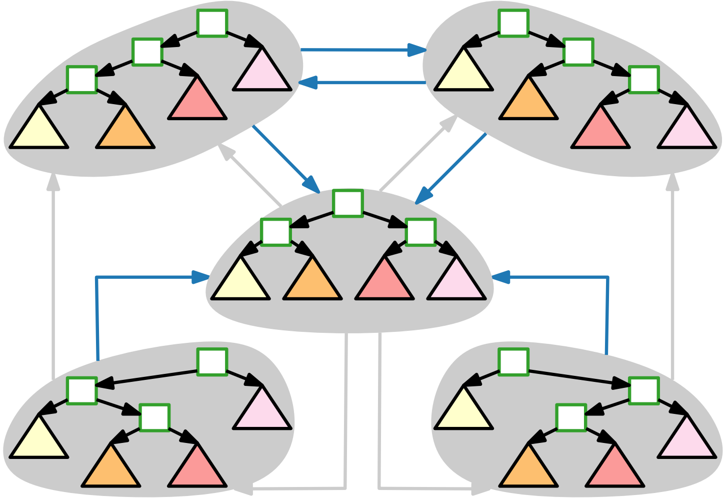
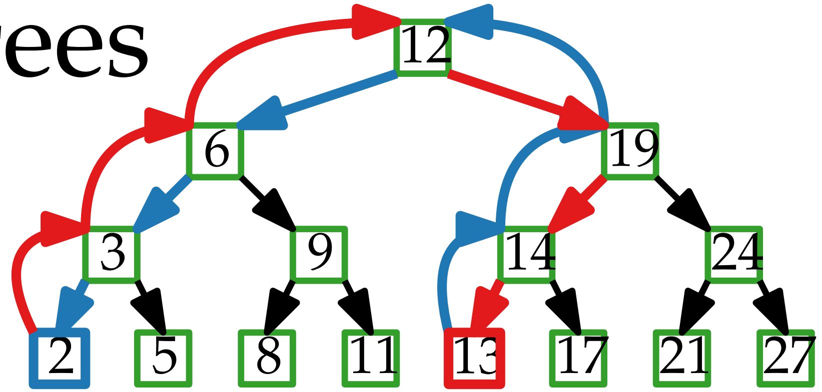
Advanced Algorithms

Optimal Binary Search Trees

Splay Trees





Philipp Kindermann · WS20

Part I:
How Good is a Binary Search Tree?

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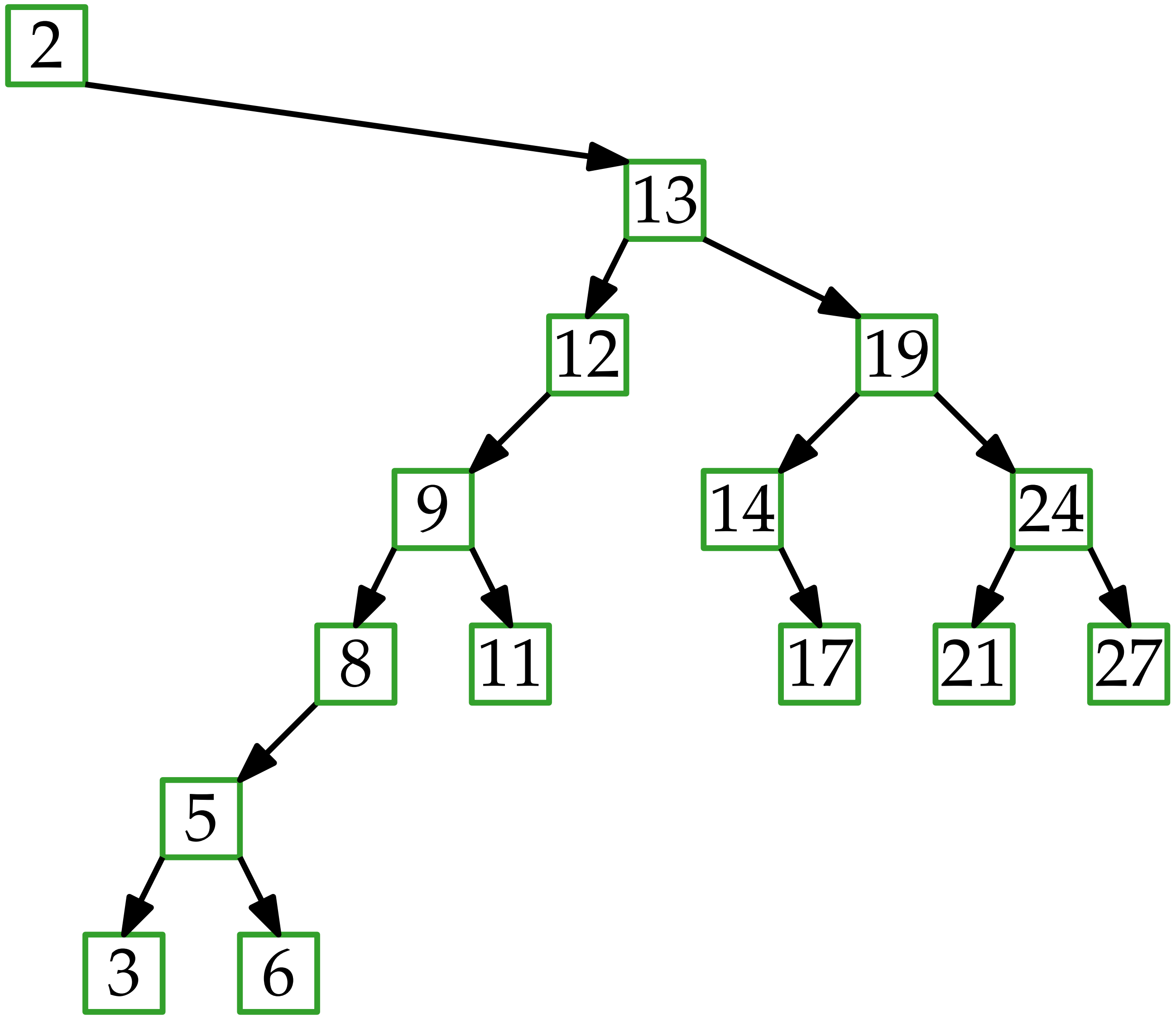
Binary search tree:



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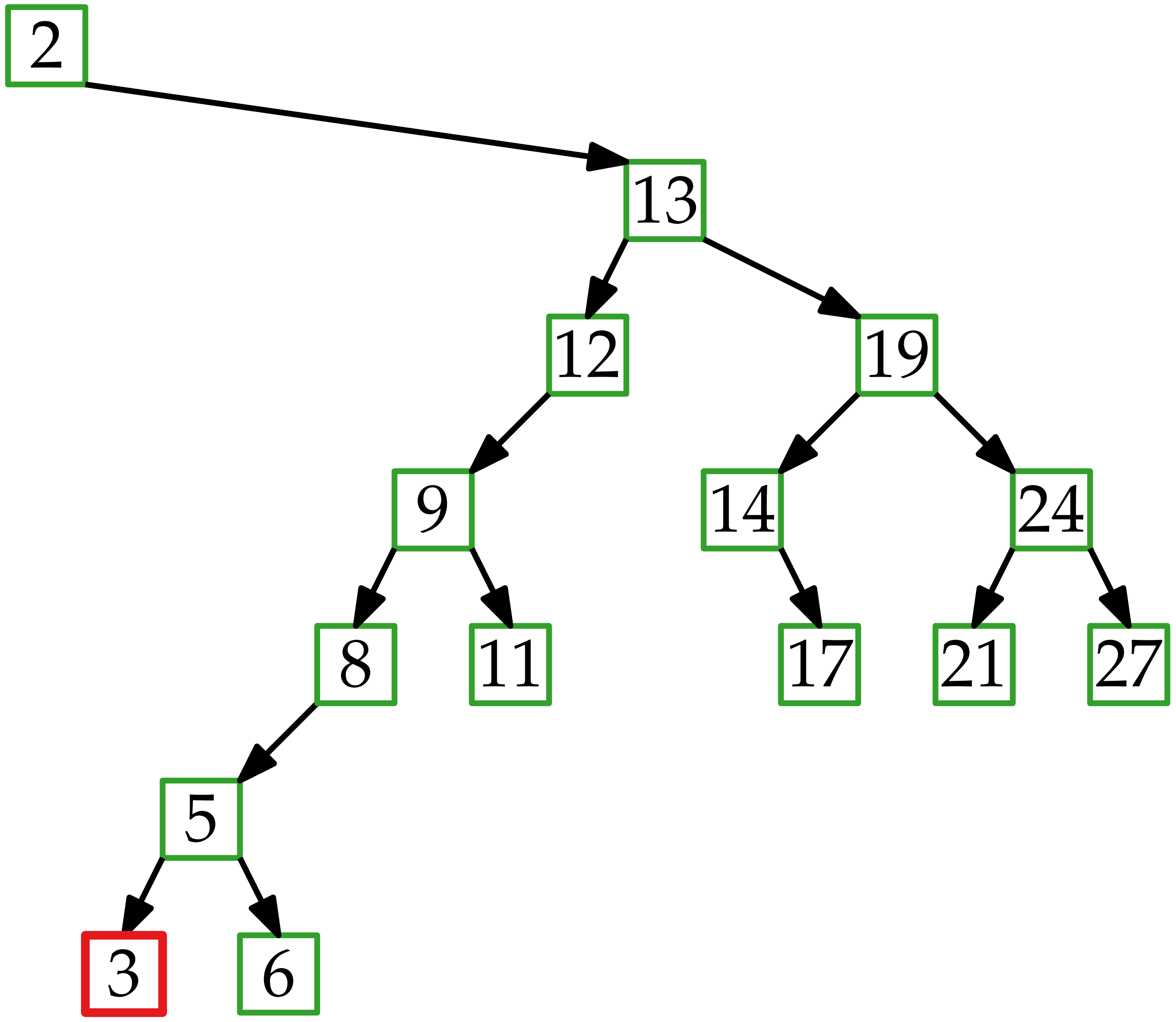

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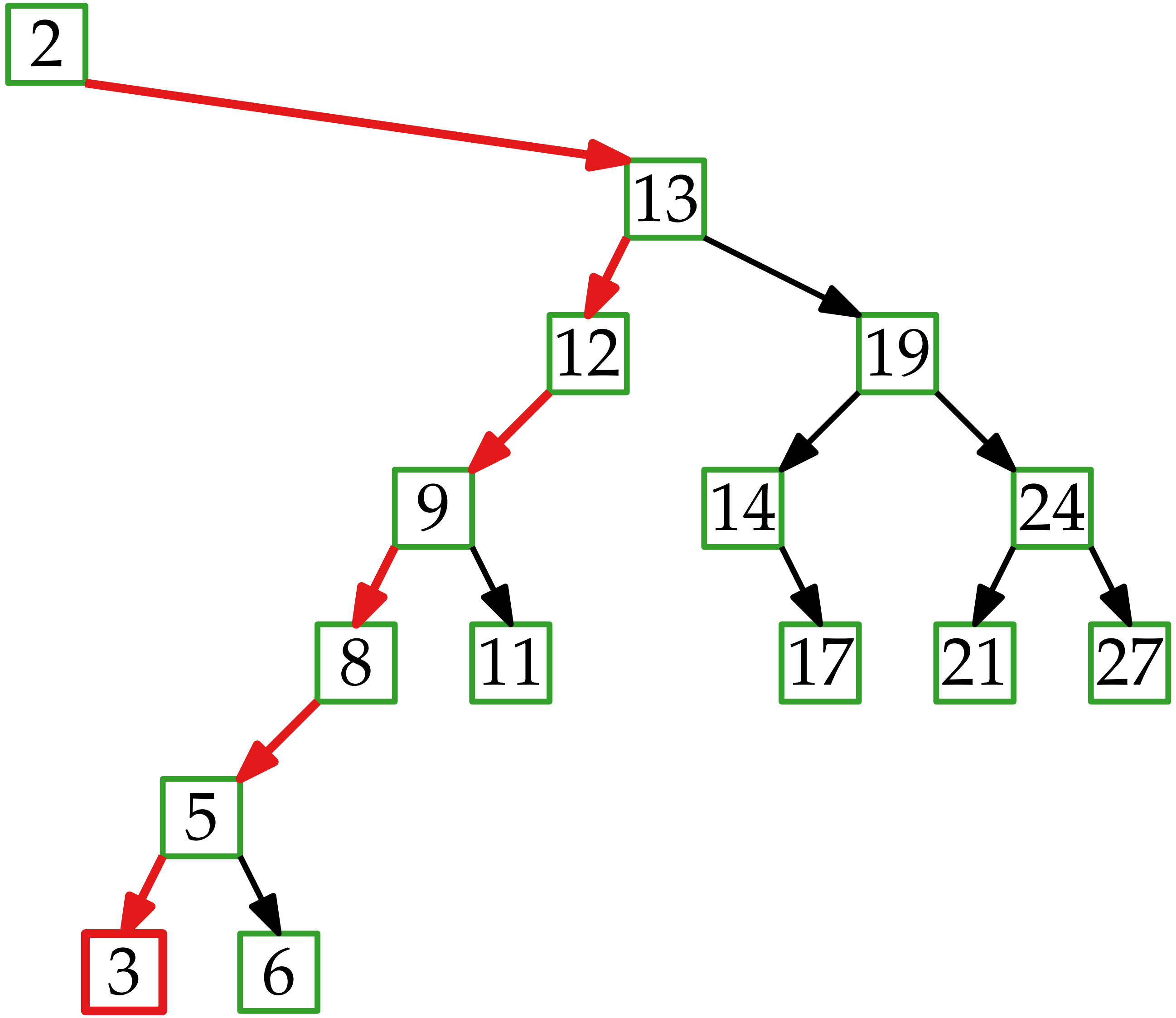

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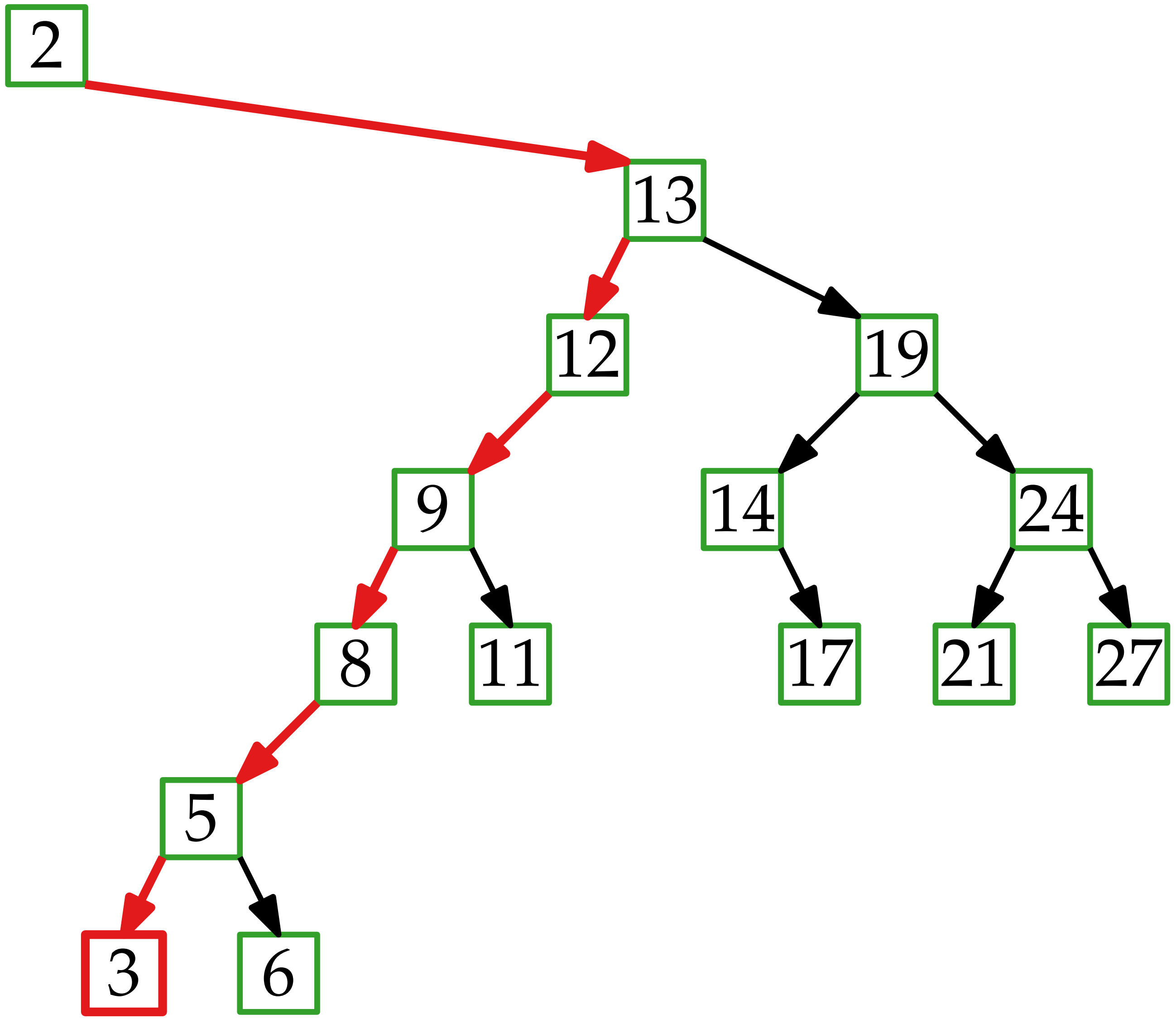
Binary search tree: w.c. query time $\Theta(n)$



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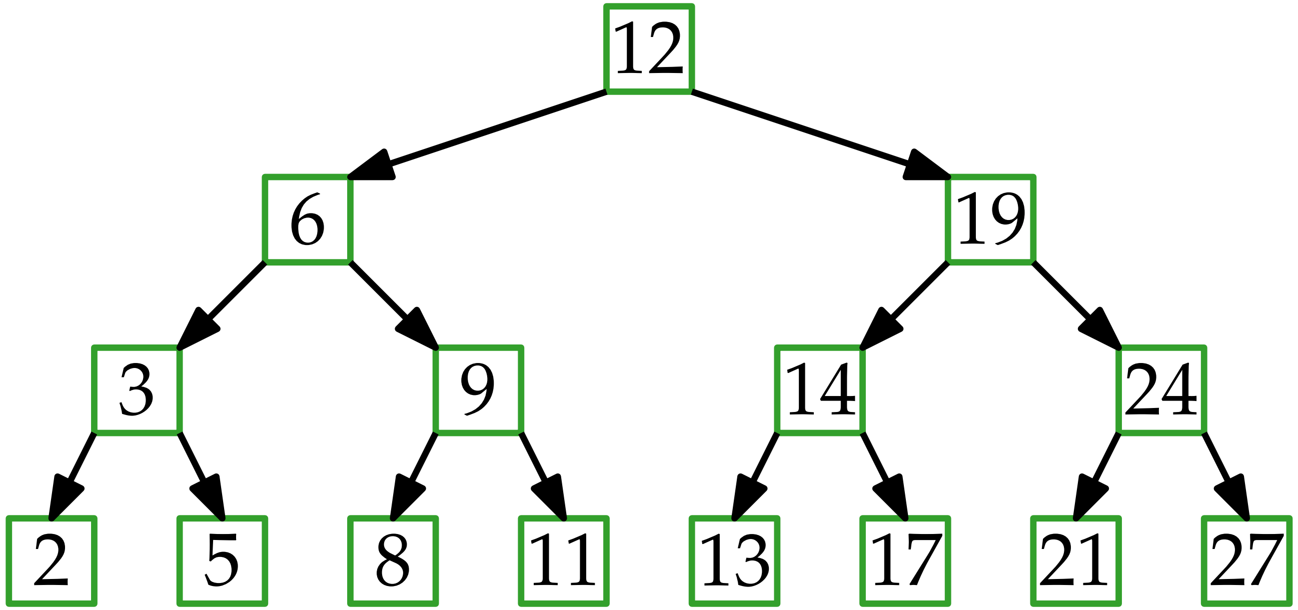

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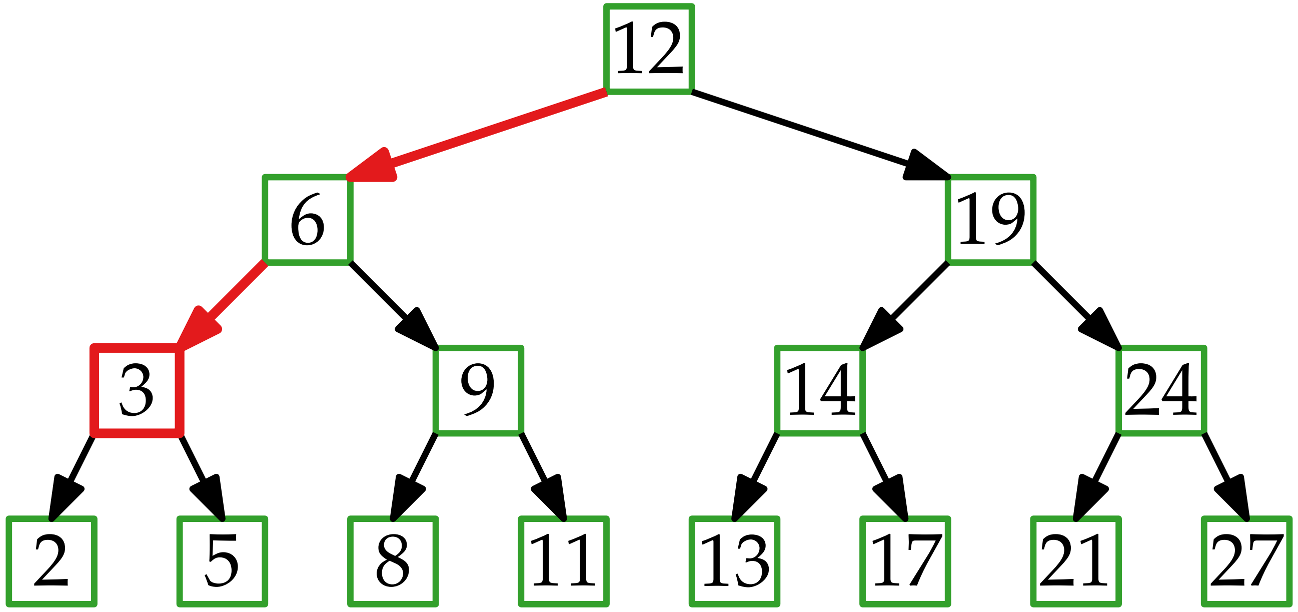

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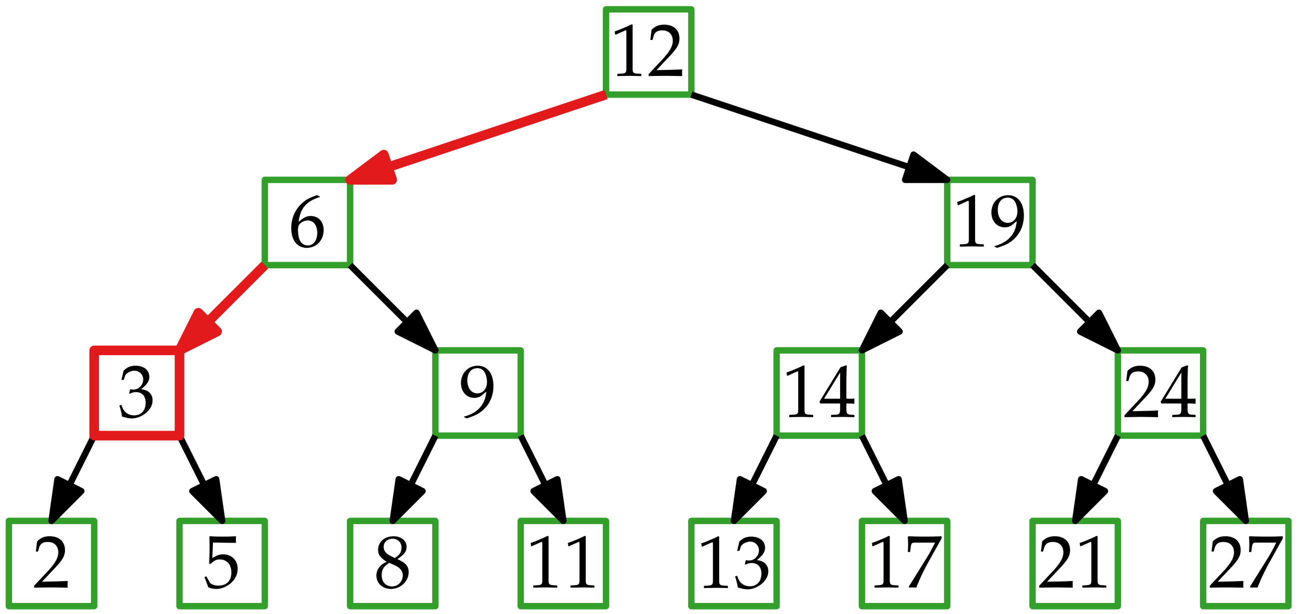

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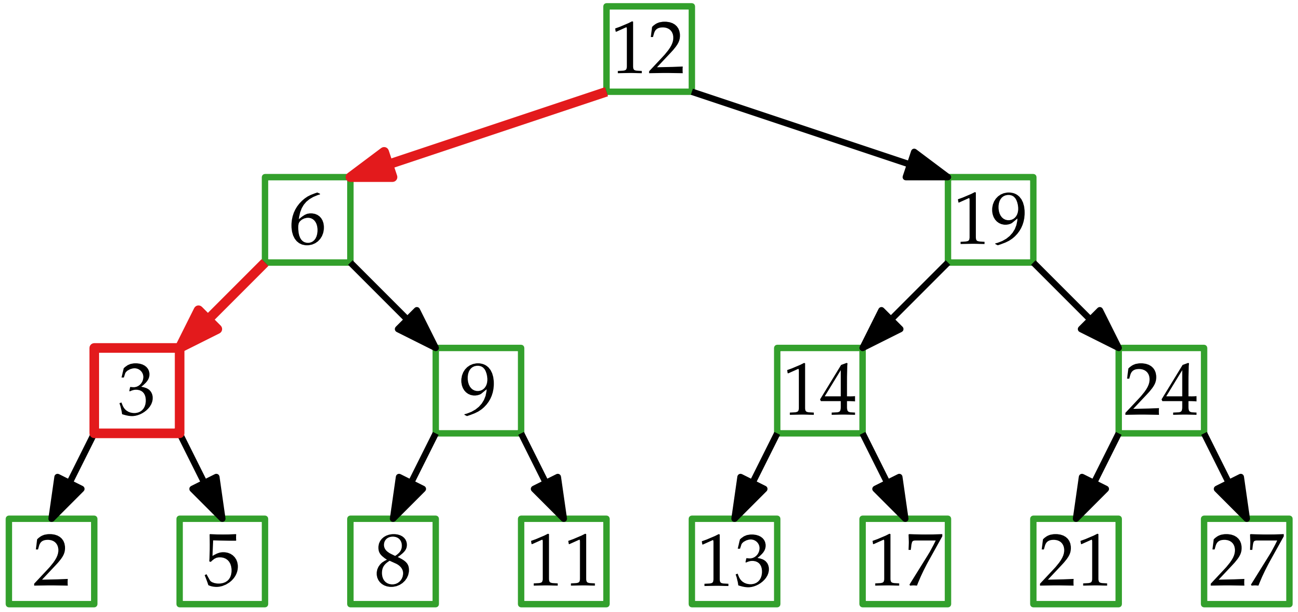

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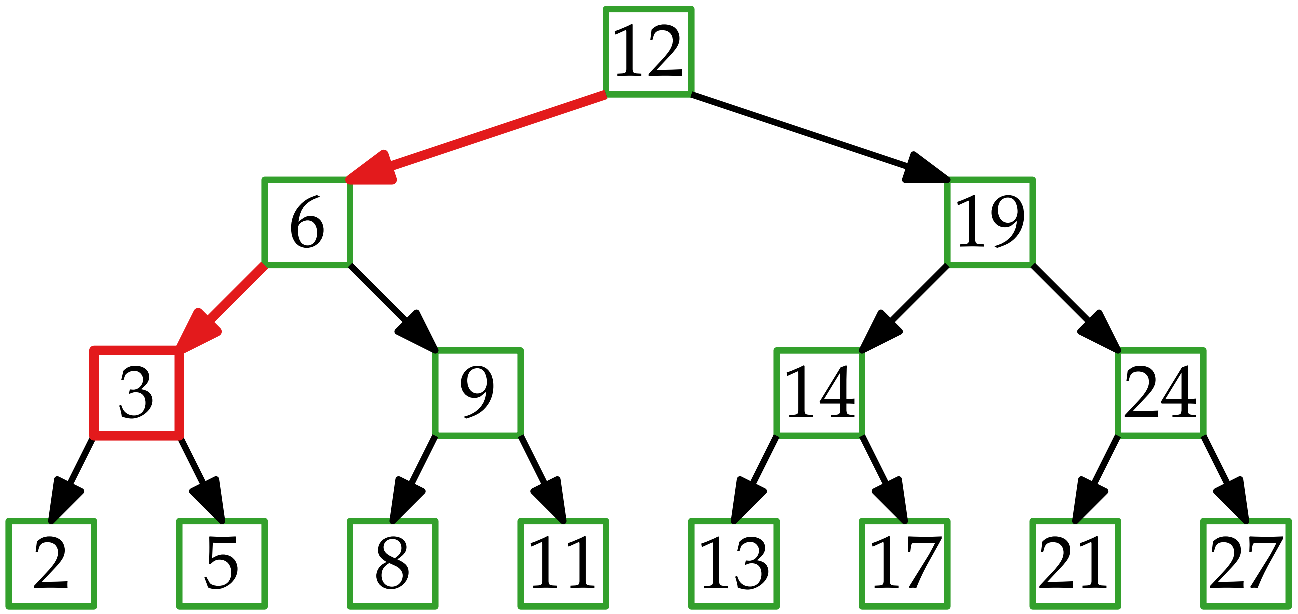

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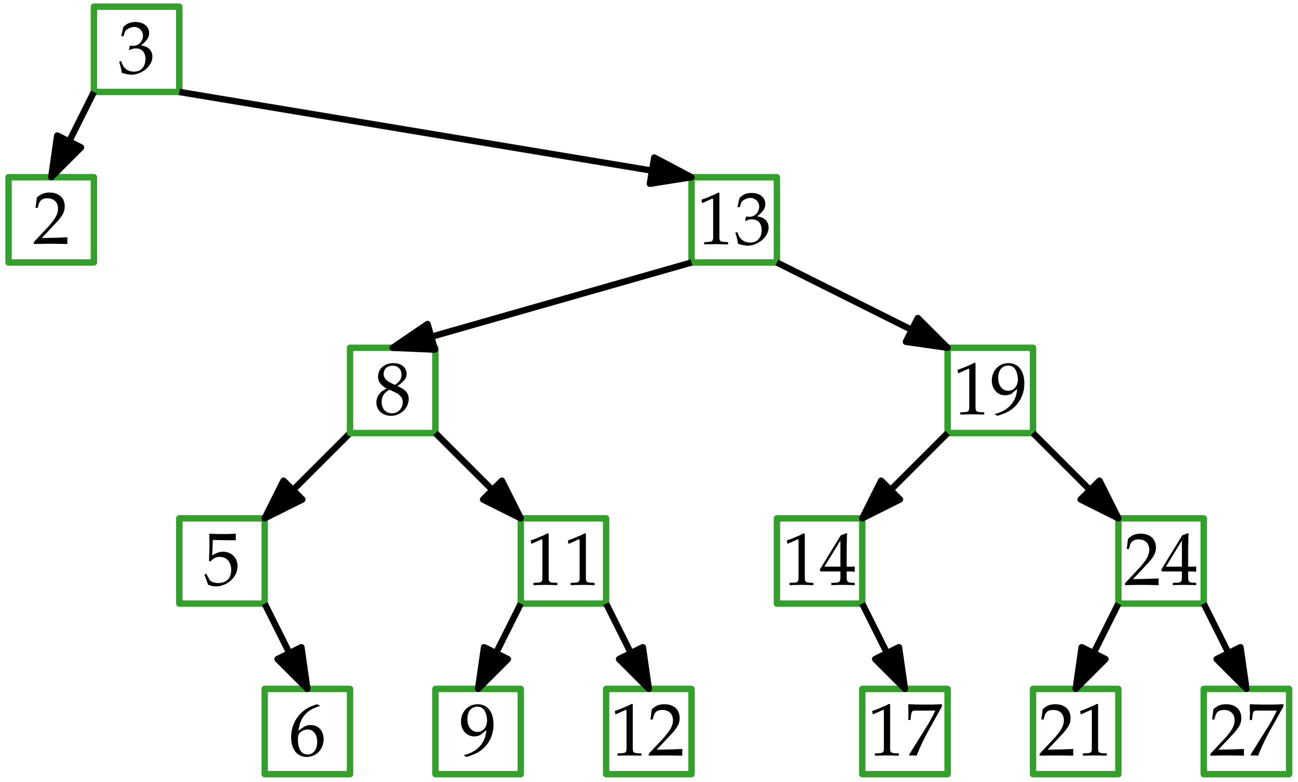

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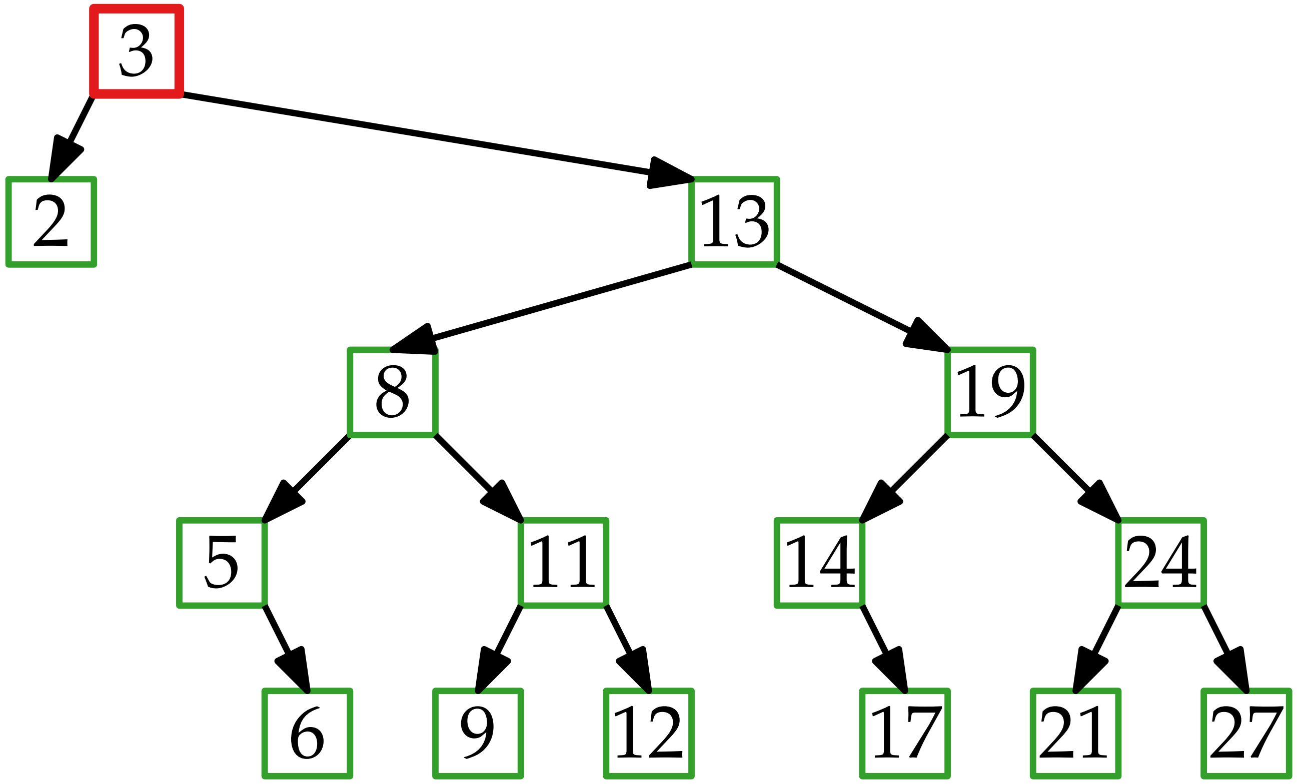

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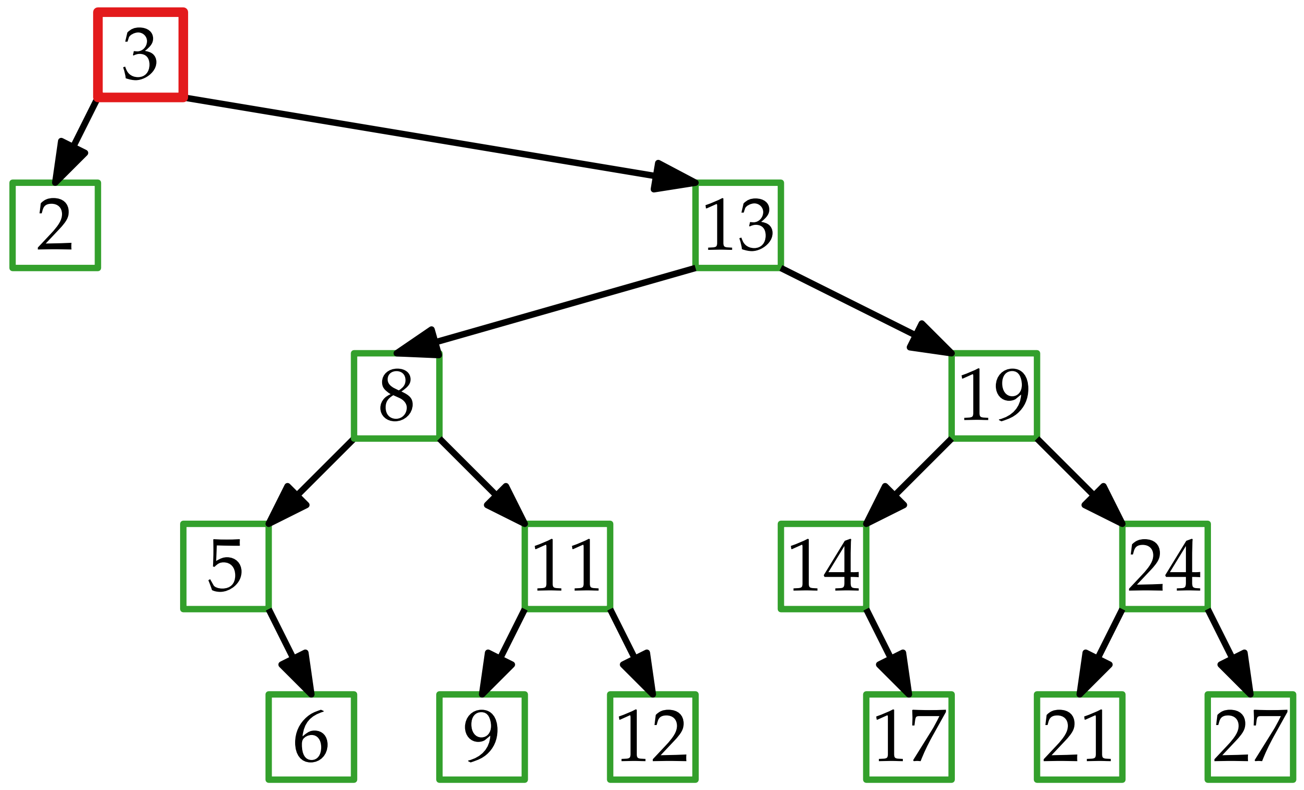

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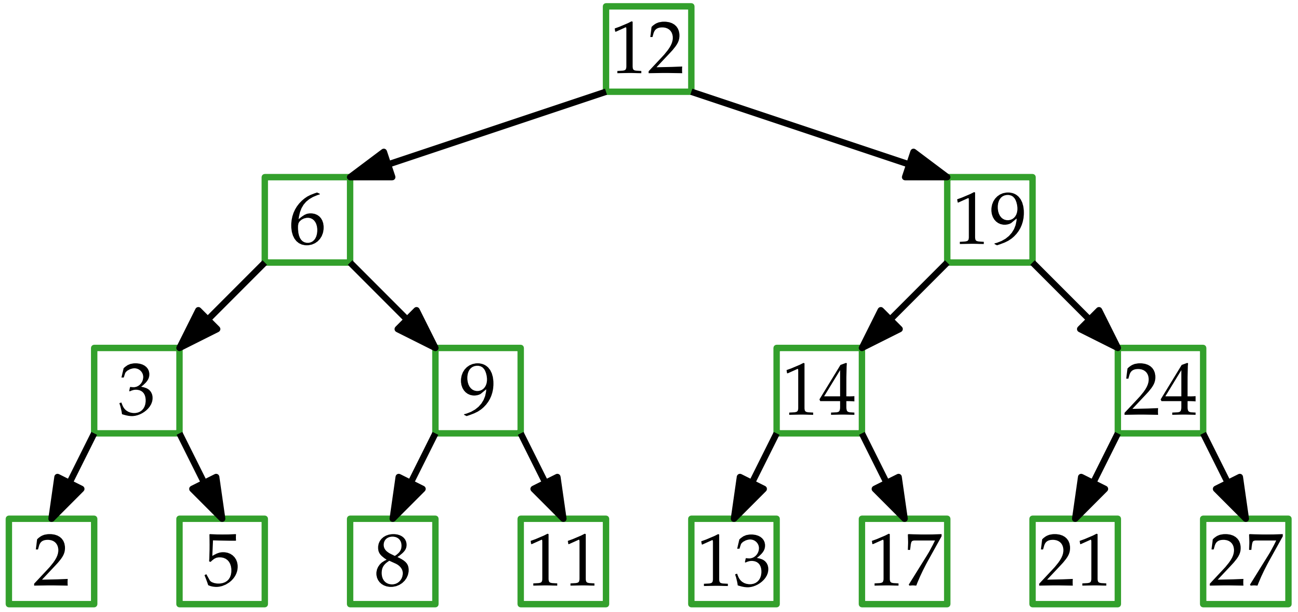

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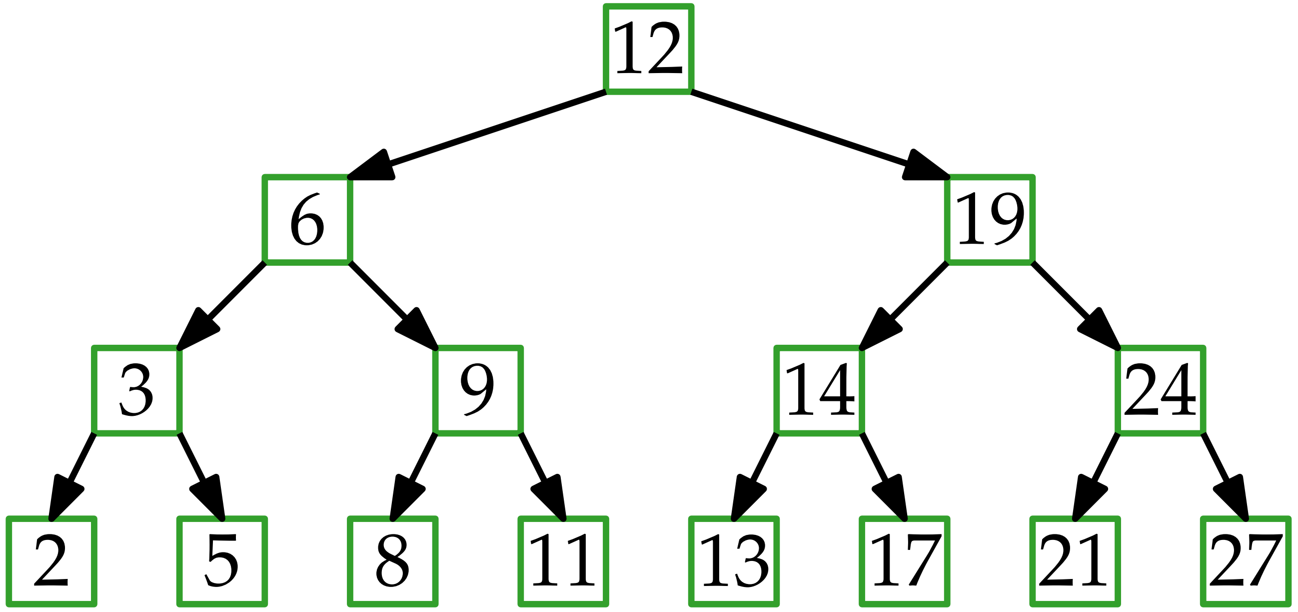
e.g. 2—13—5



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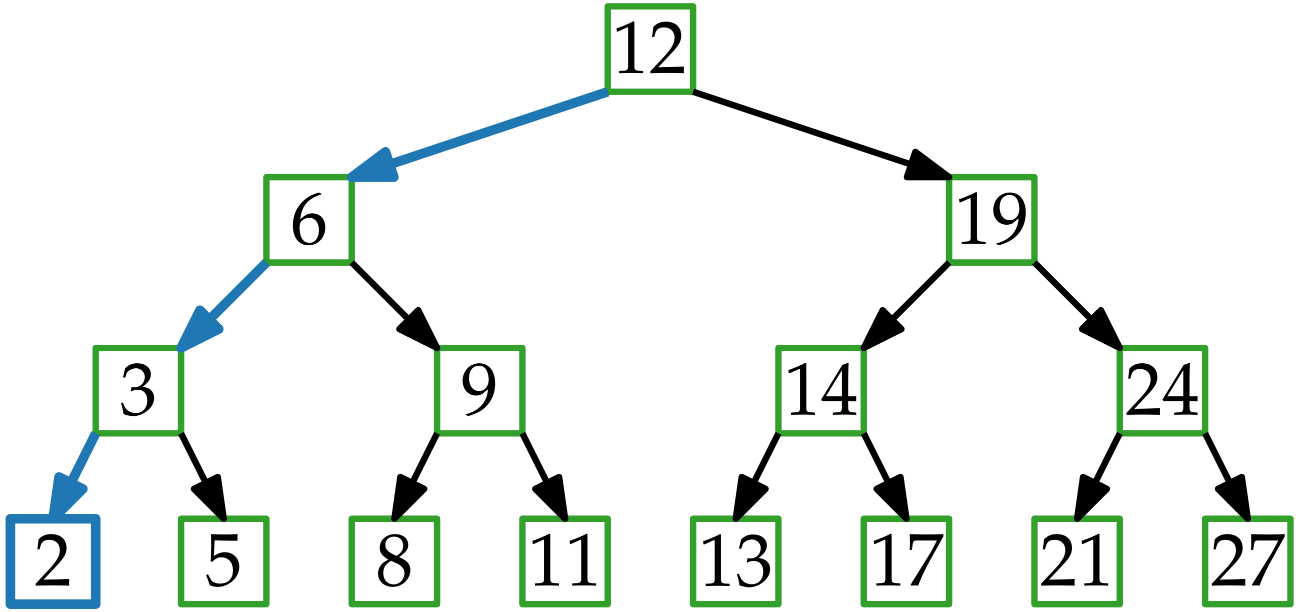

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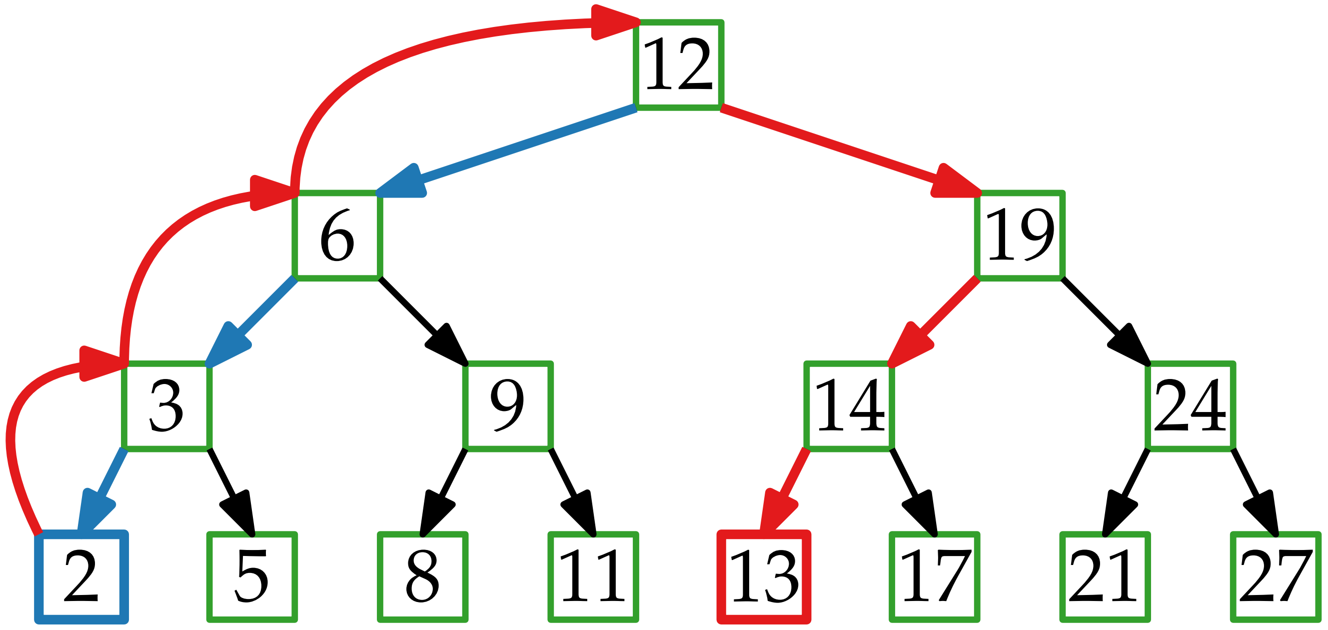

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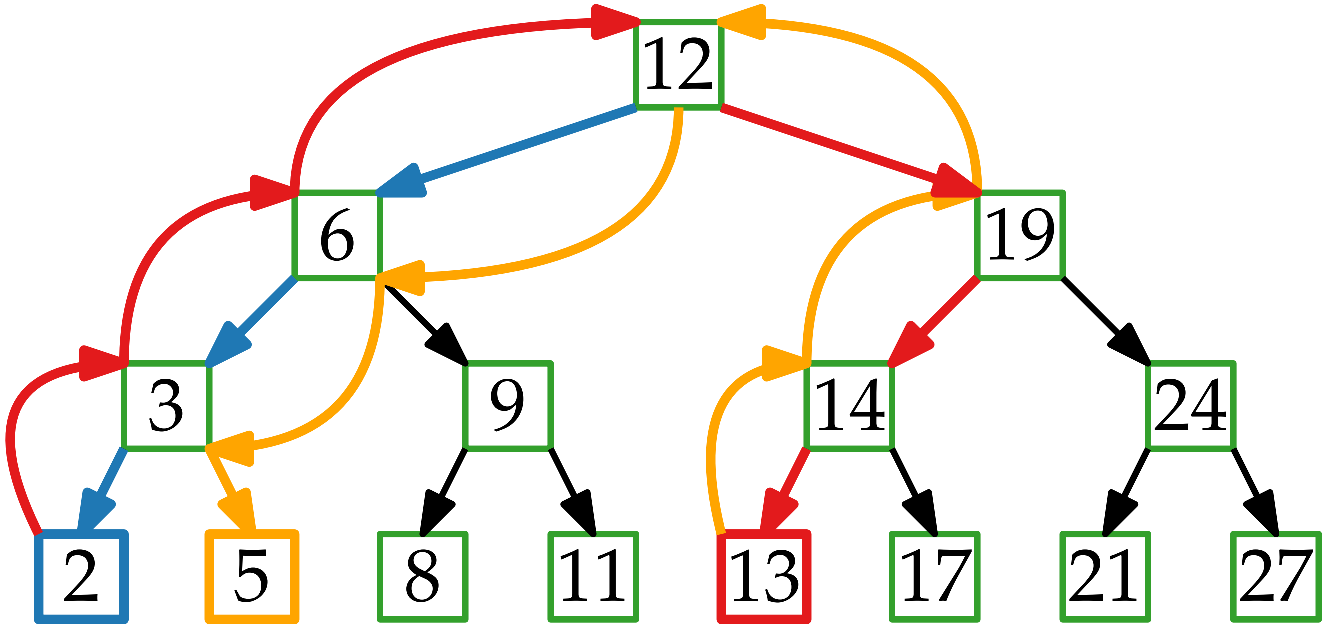

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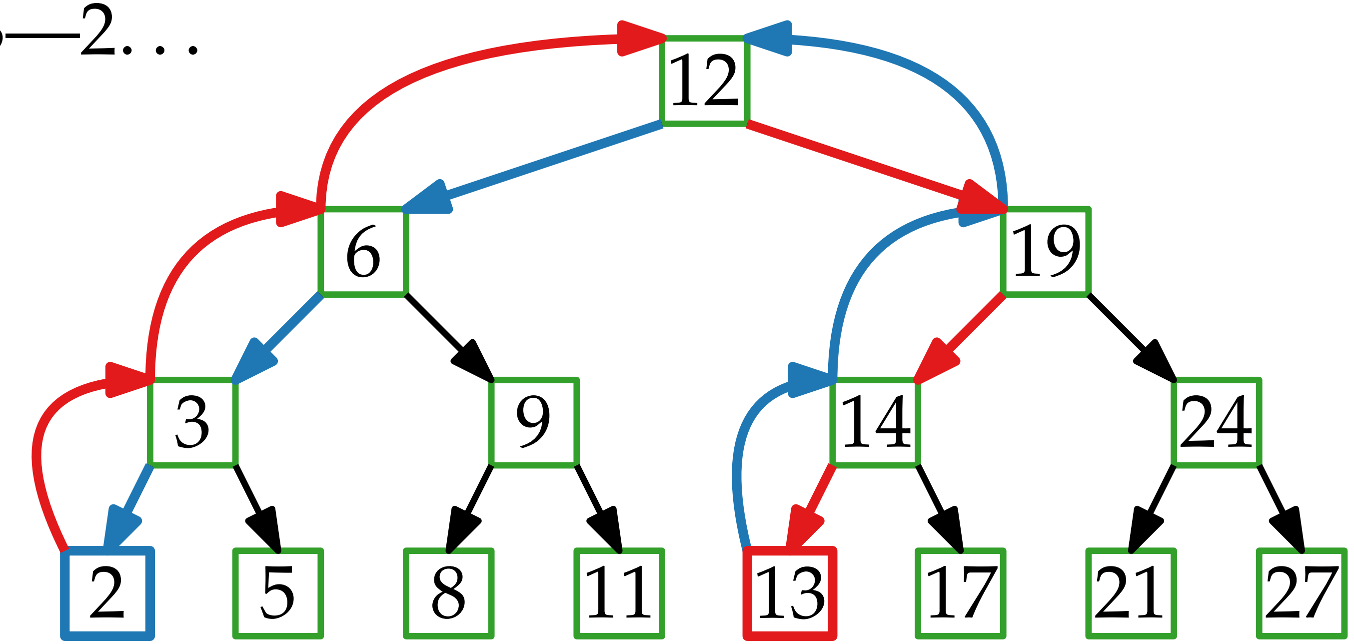

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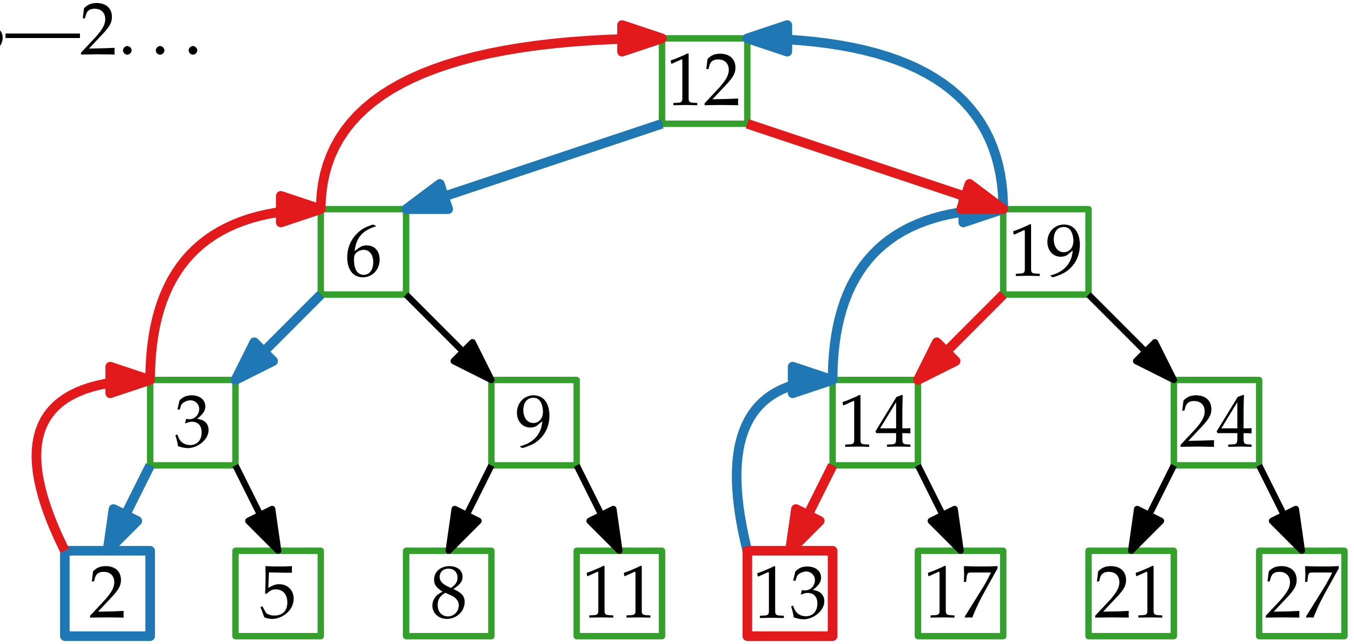

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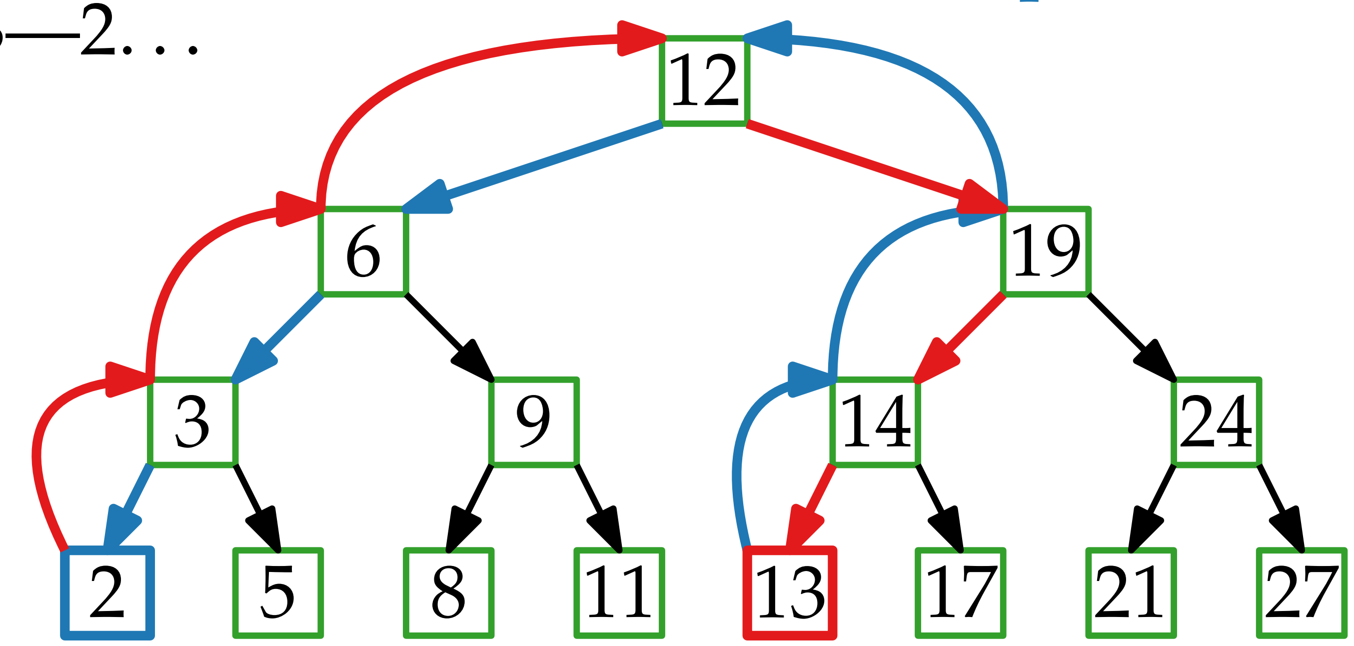

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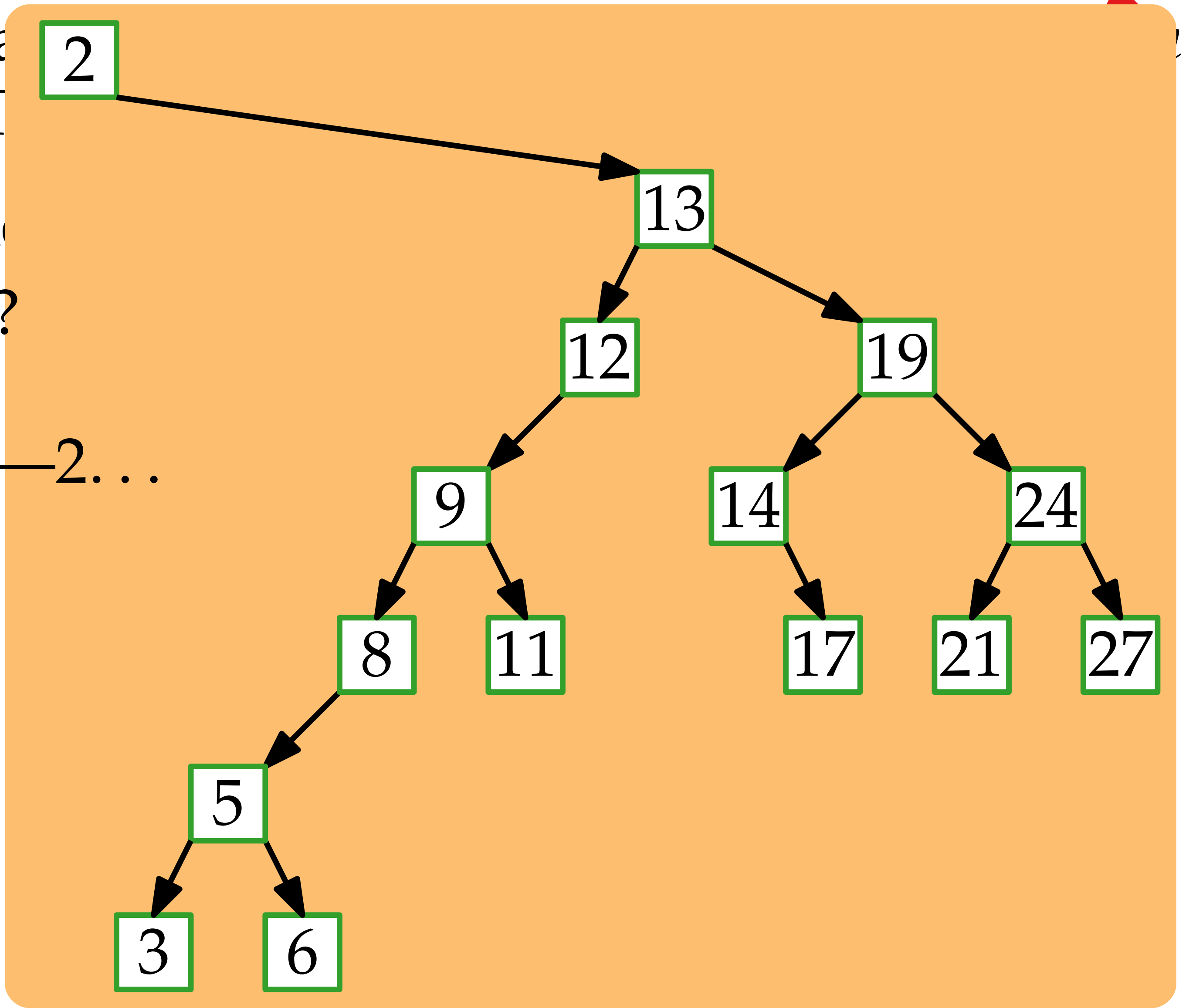

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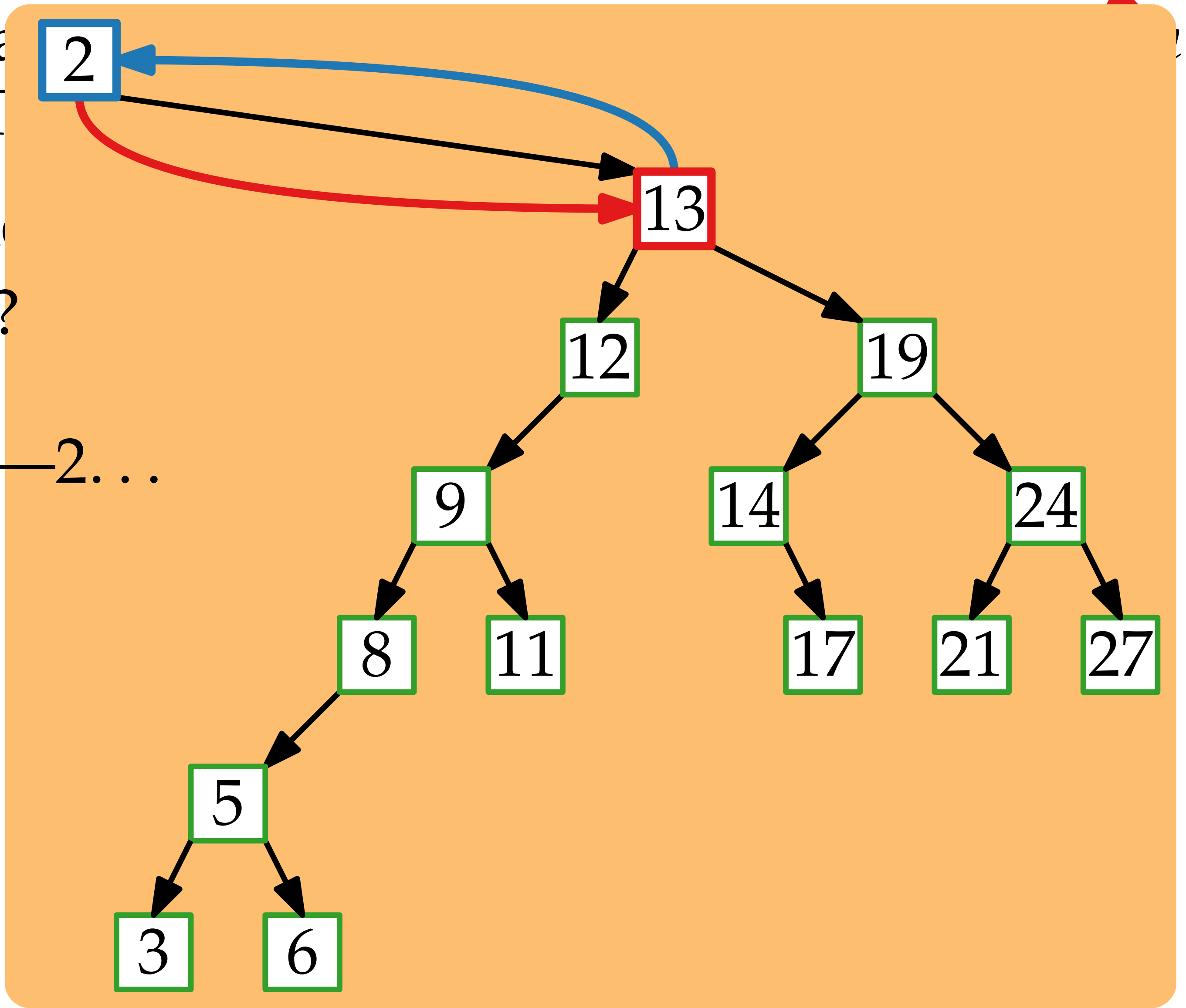

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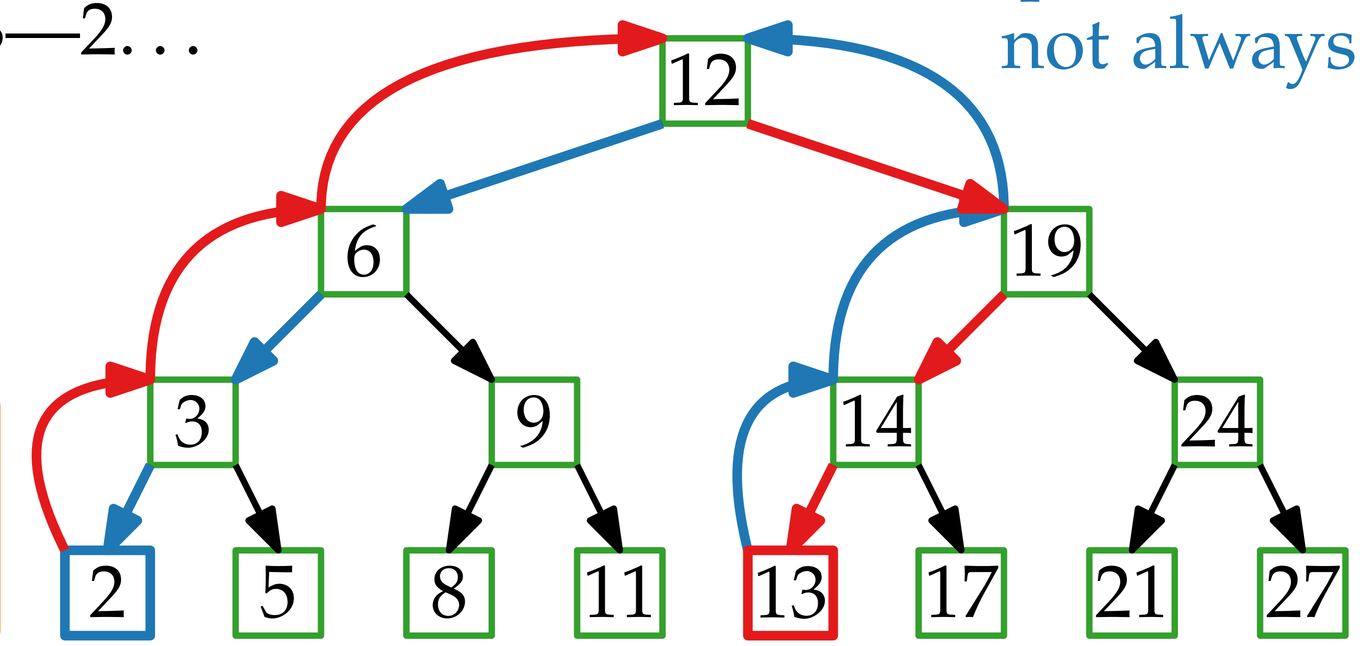
not always!



The performance of a BST depends on the model!

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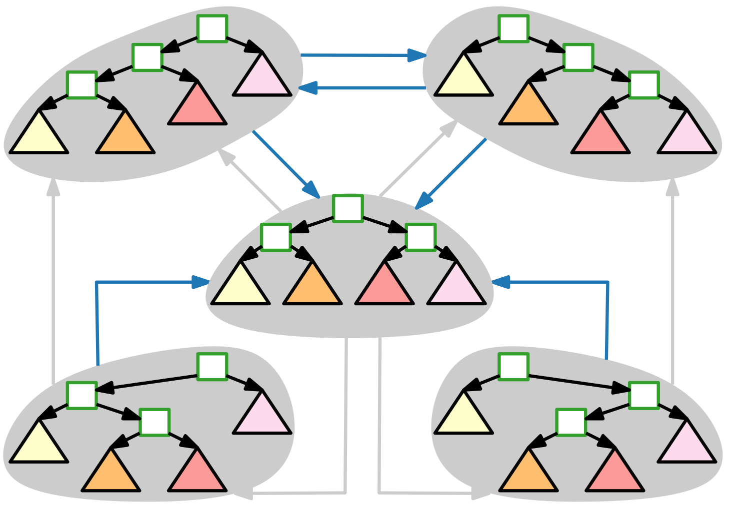

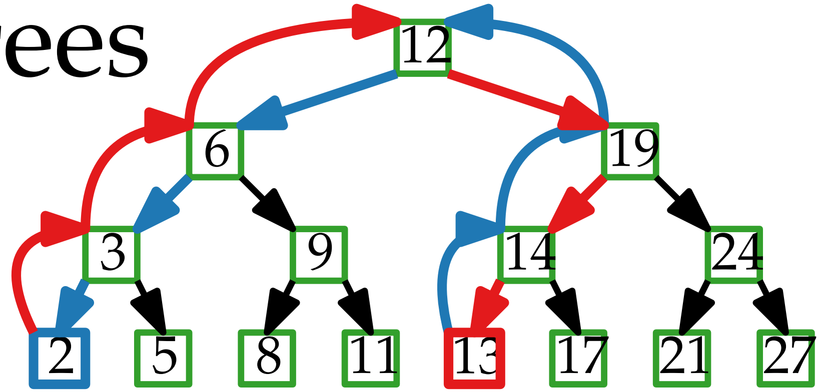

Philipp Kindermann · WS20

Part II:
Models of Optimality

Model 1: Malicious Queries

Given a BST, what is the worst sequence of queries?

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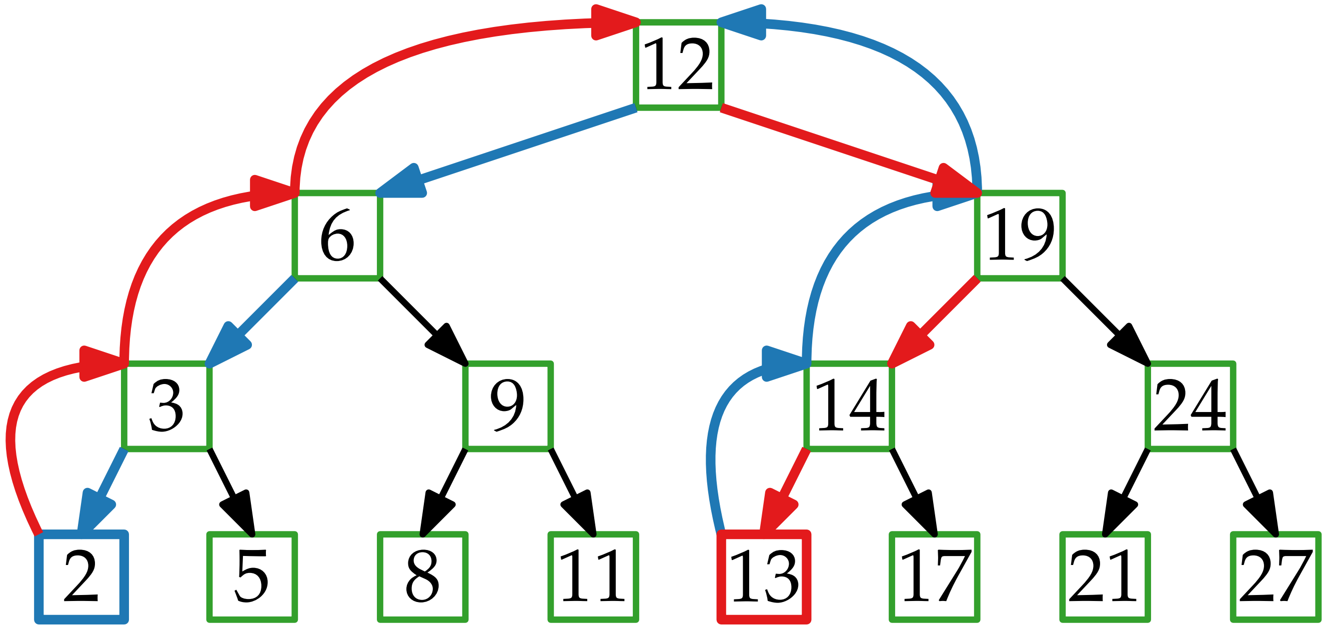

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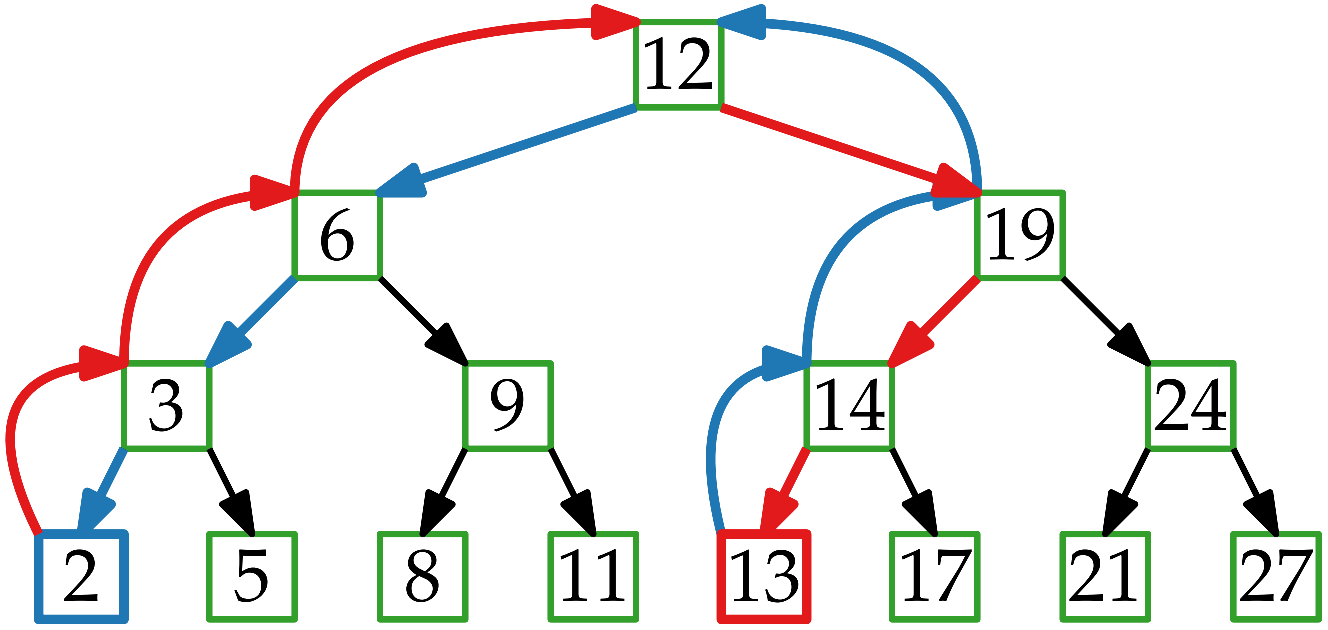
Given a BST, what is the worst sequence of queries?

Lemma. The worst-case malicious query cost in any BST with n nodes is at least $\Omega(\log n)$ per query.



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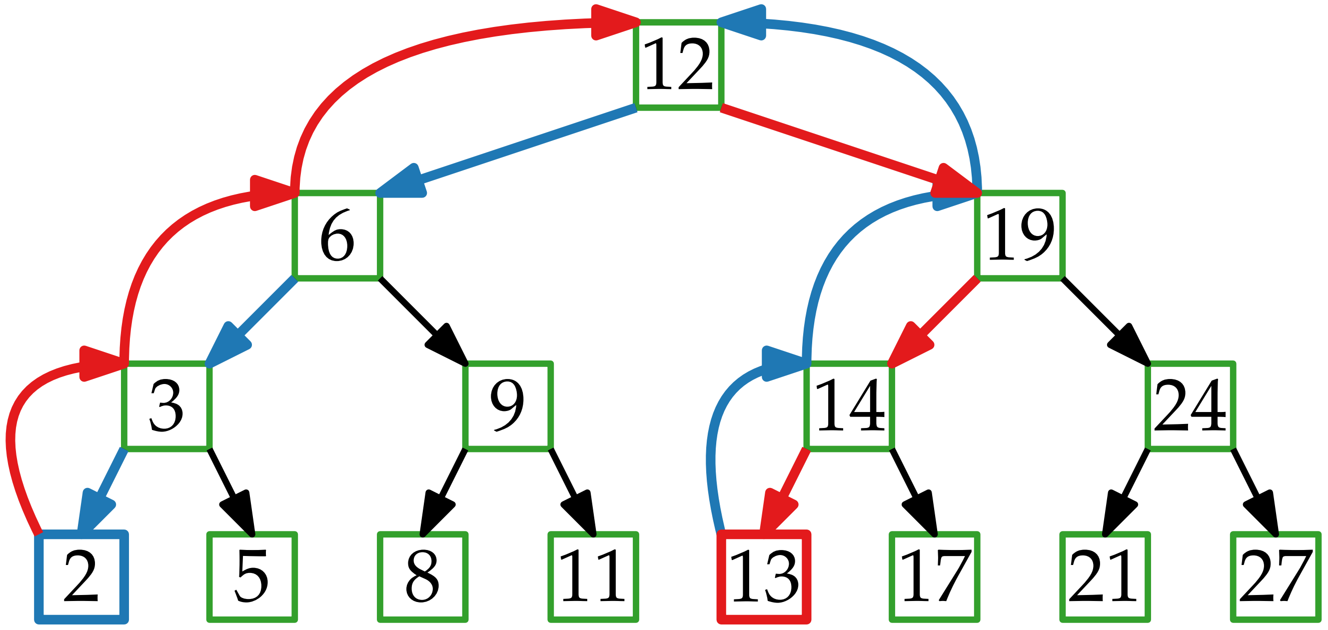

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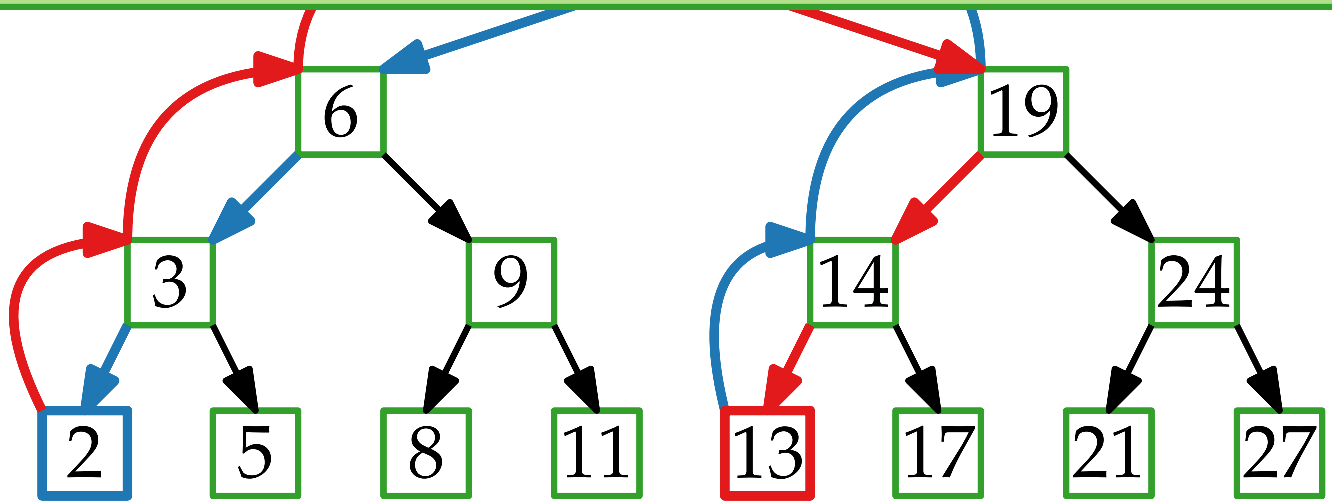
$\Rightarrow$ the (amortized) cost of each query is $O(\log n)$ (for at least n queries)



Model 2: Known Probability Distribution

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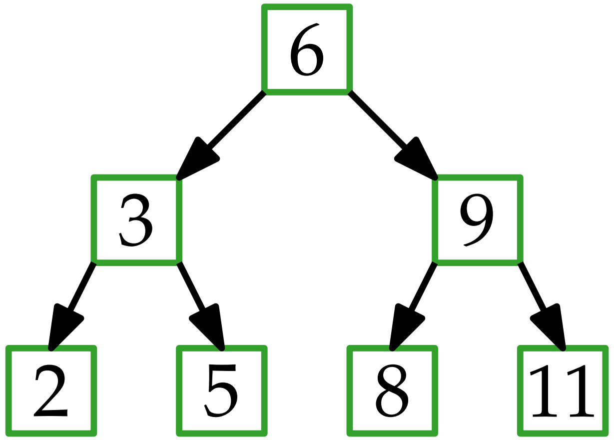
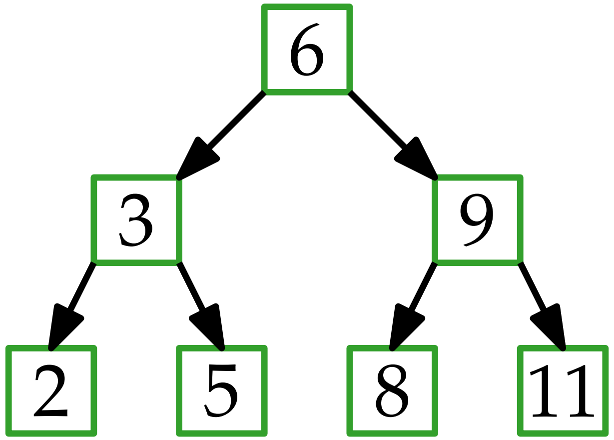


Model 2: Known Probability Distribution

Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%



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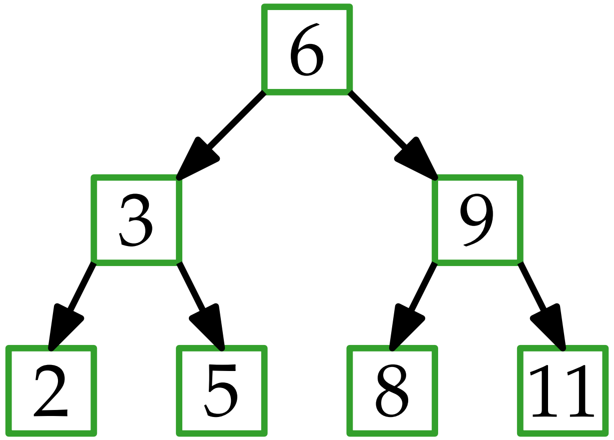

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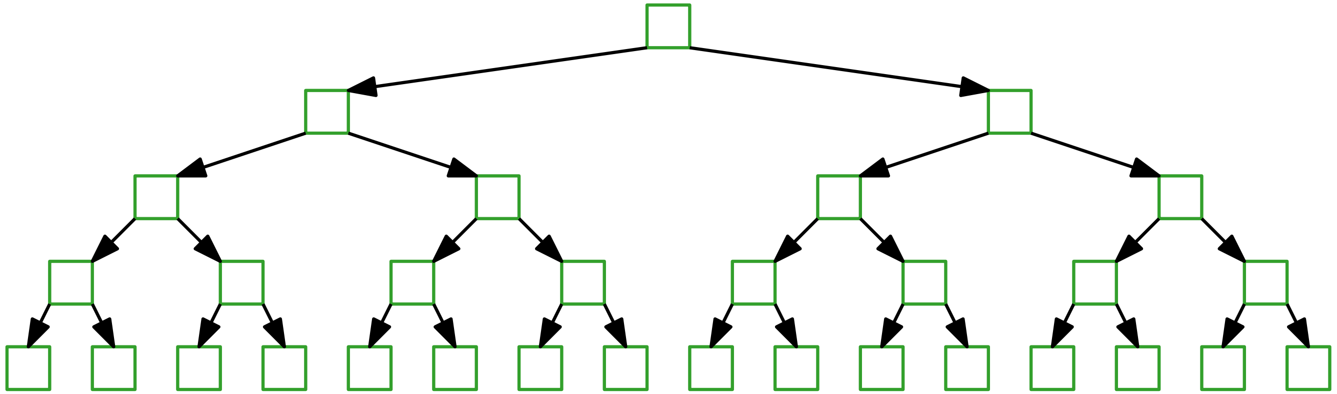

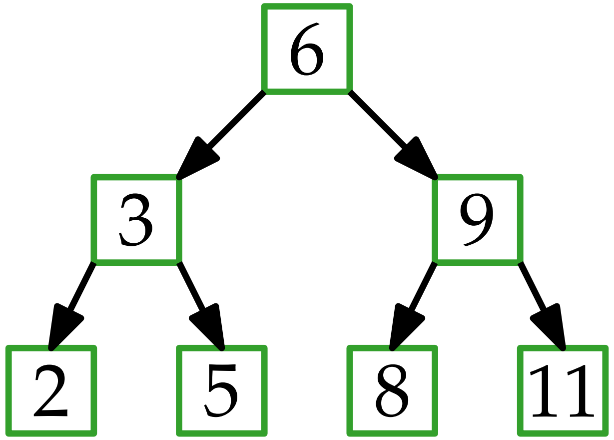

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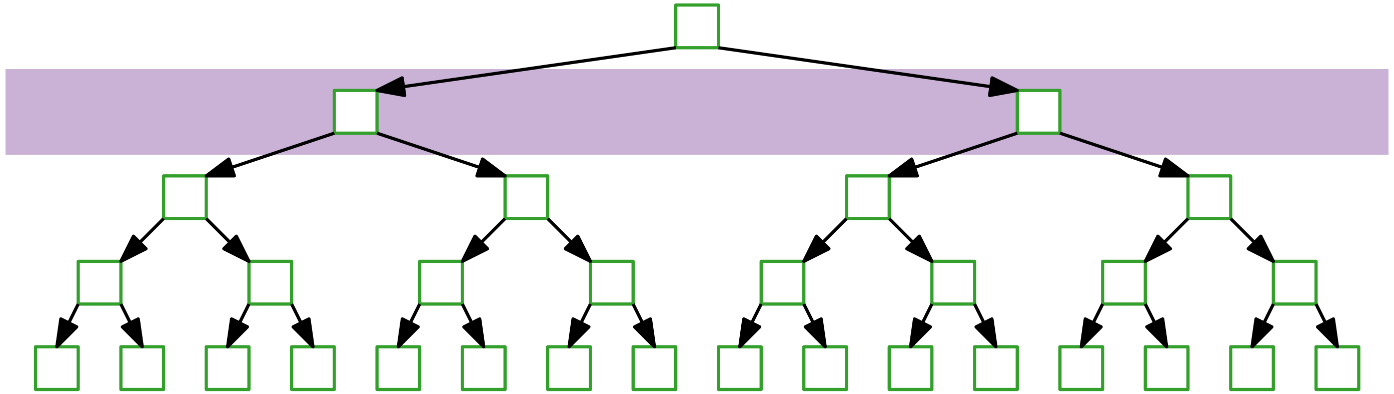
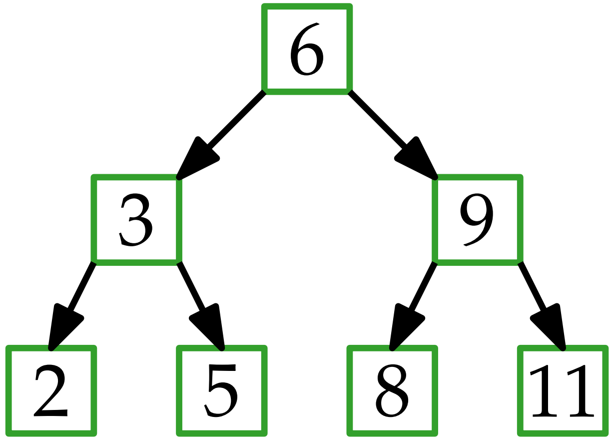


prob. $\leq 1/2$



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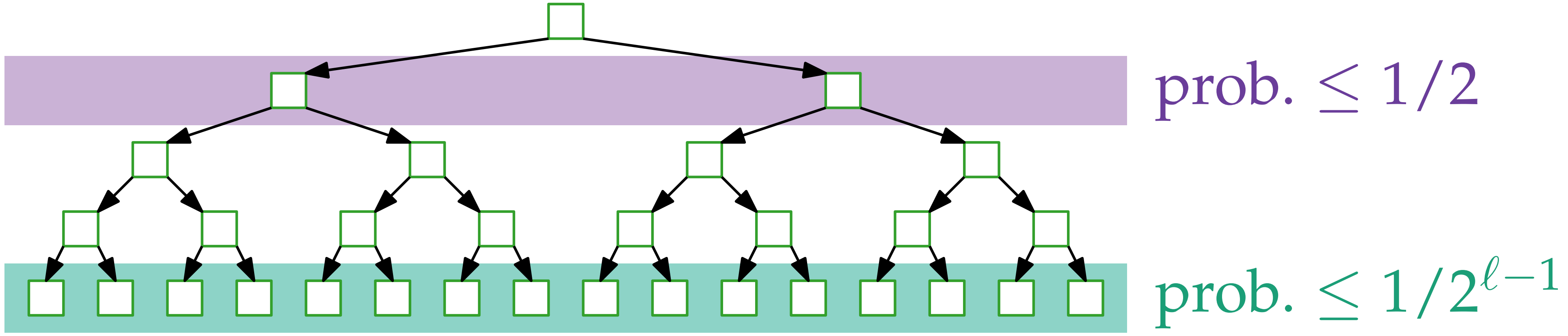

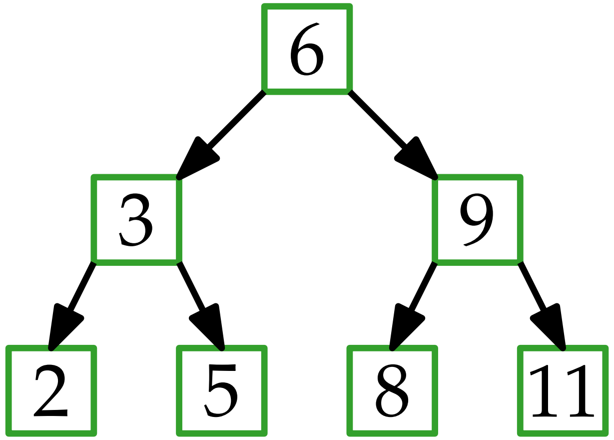

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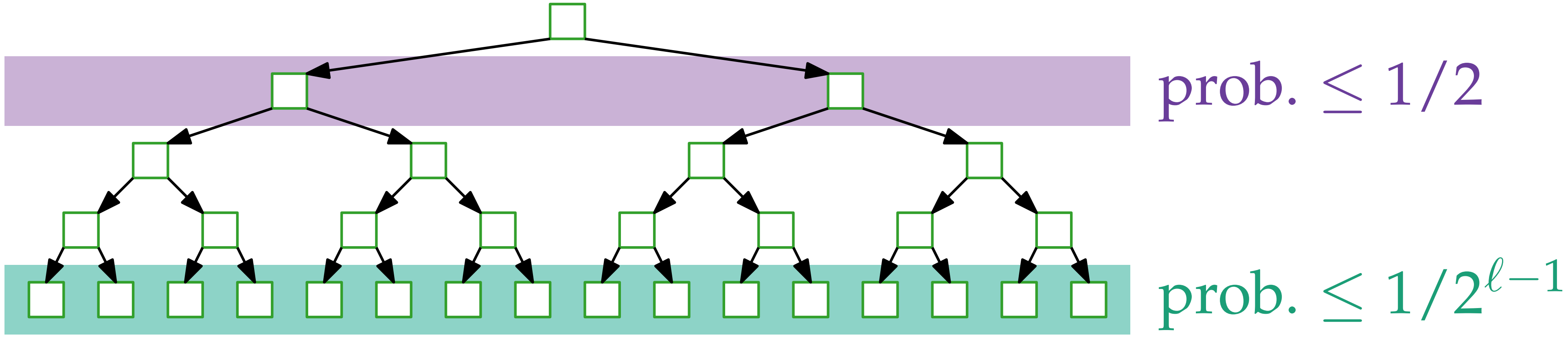

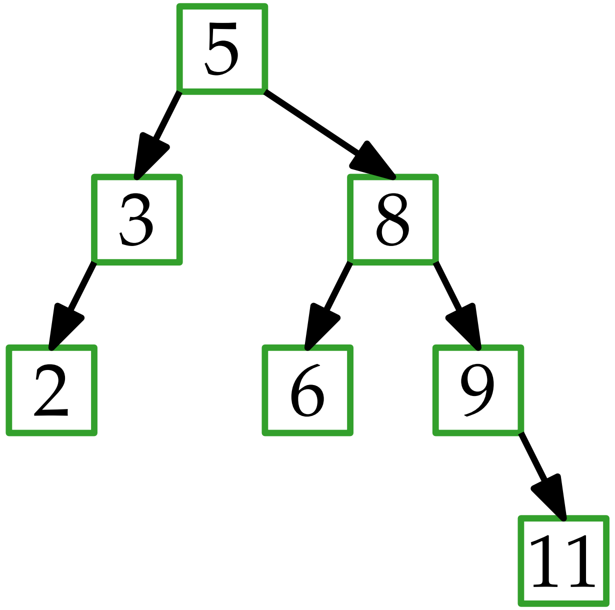

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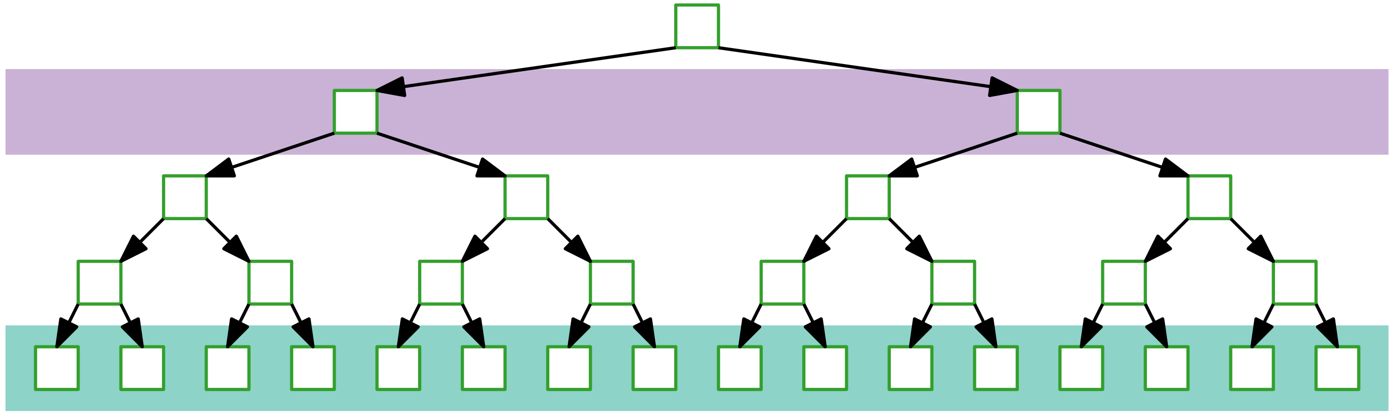

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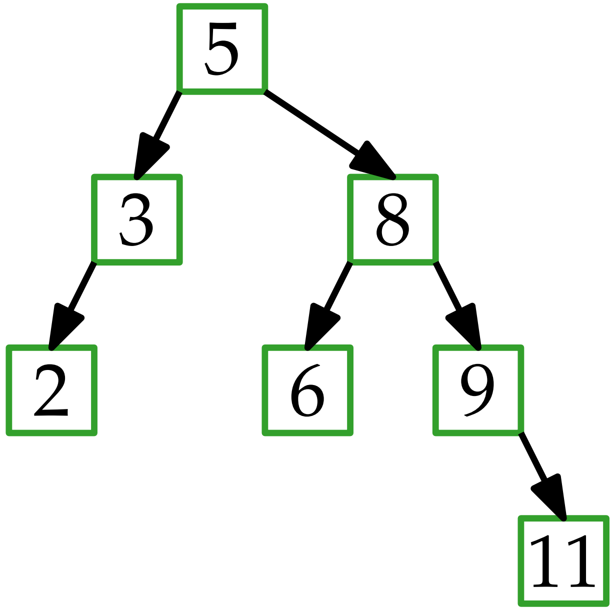

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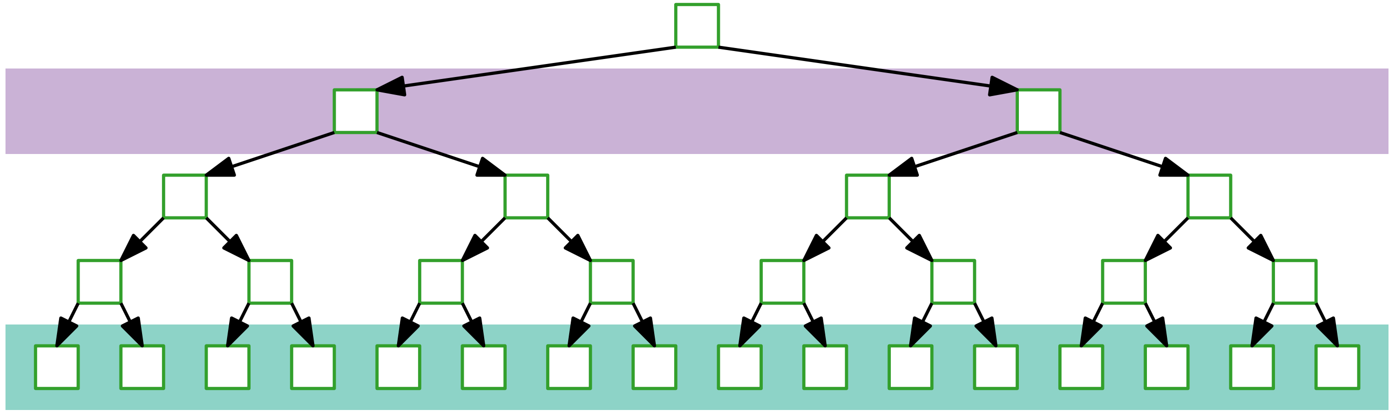

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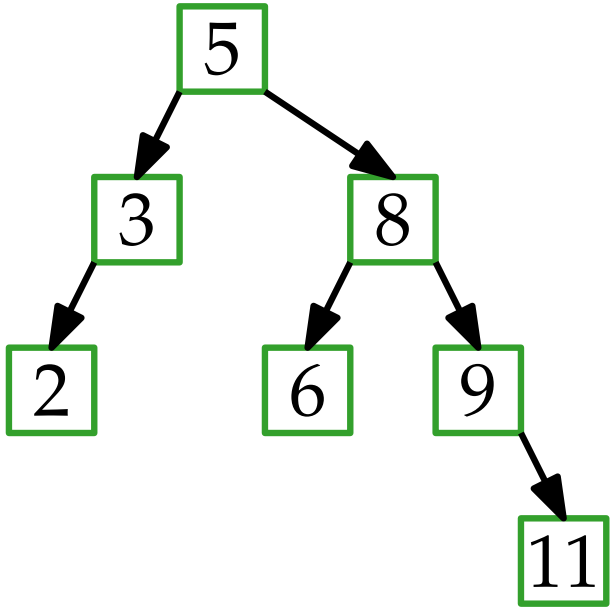

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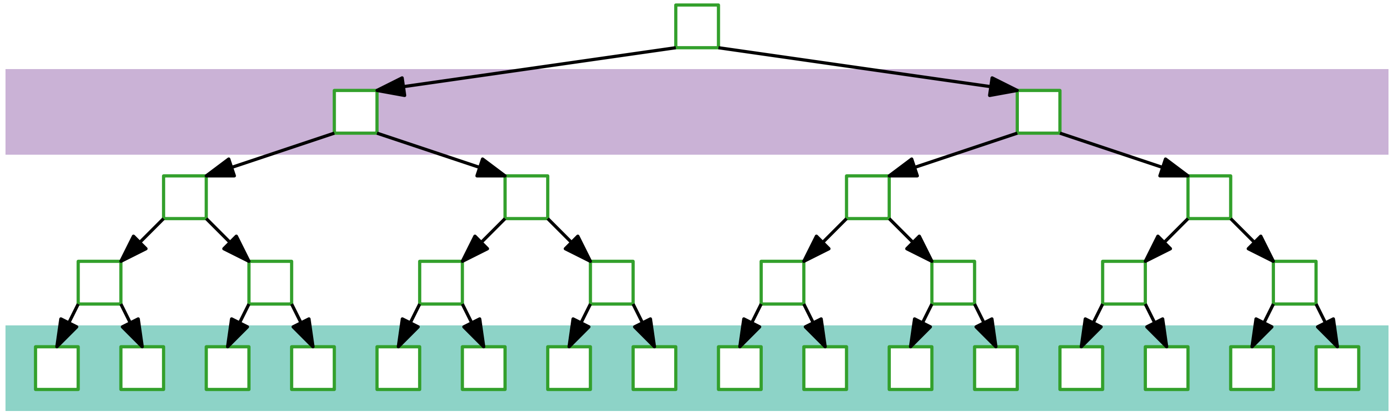

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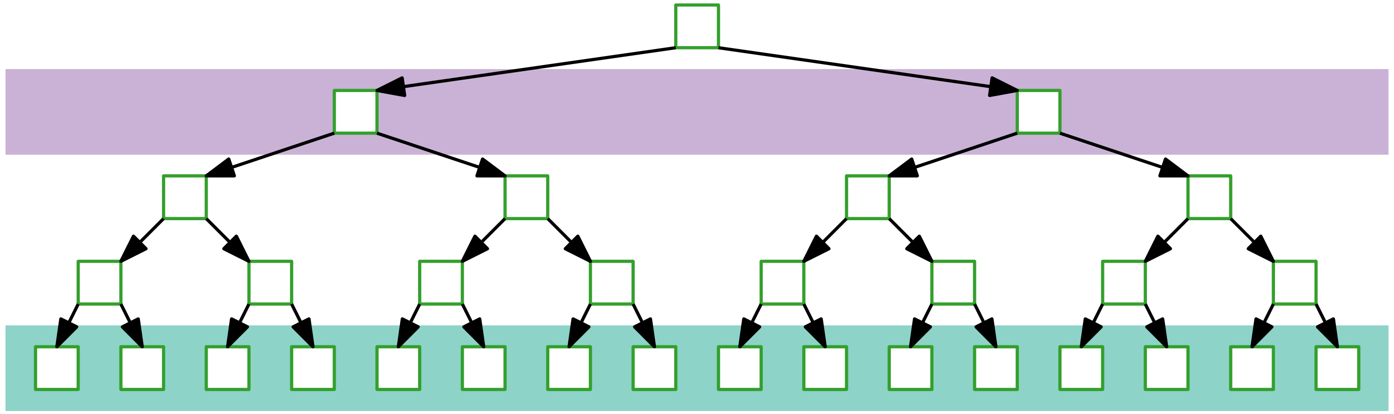

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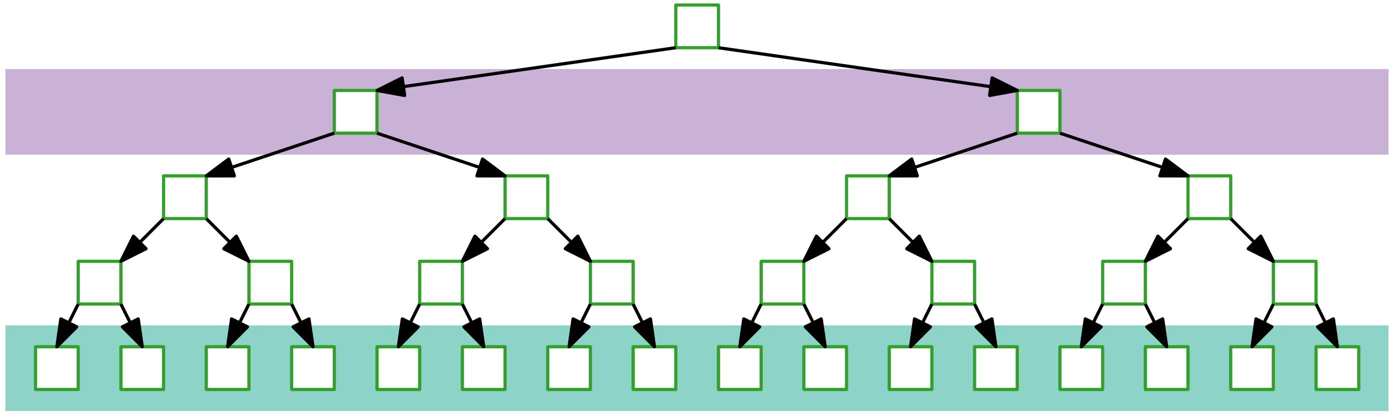
Model 2: Known Probability Distribution

Access Probabilities:

2	3	5	6	8	9	11
2%	20%	30%	8%	20%	15%	5%

Idea: Place nodes with higher probability higher in the tree.



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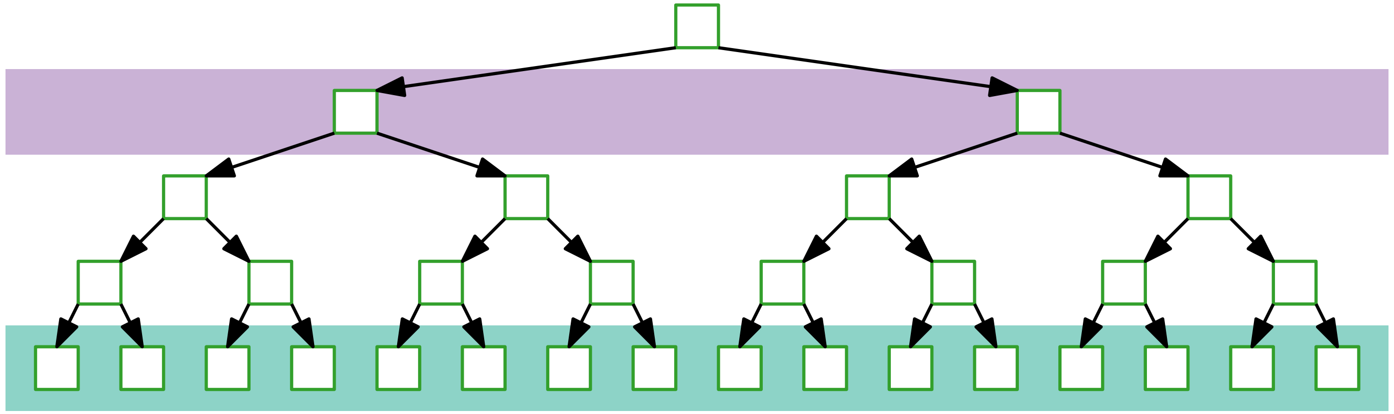

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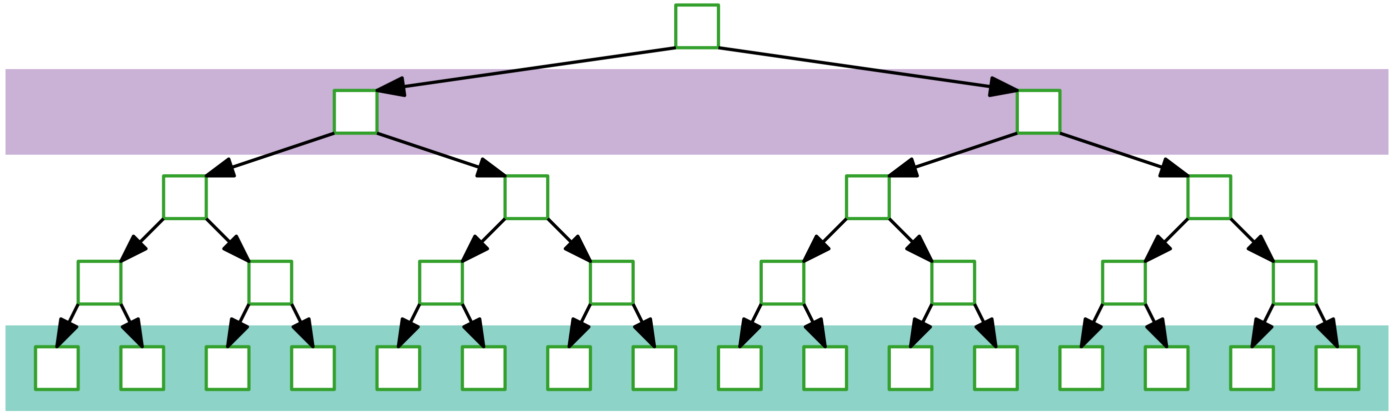

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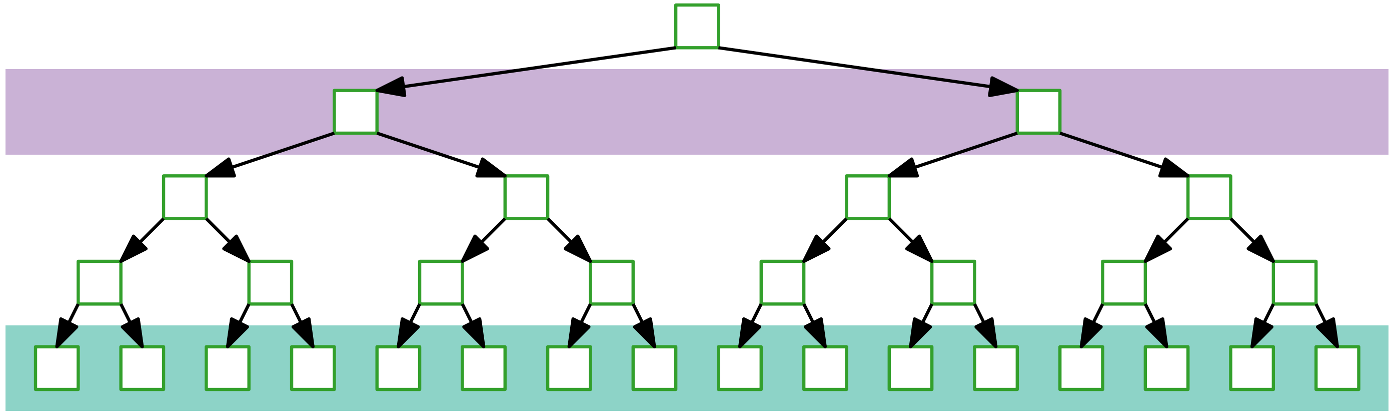

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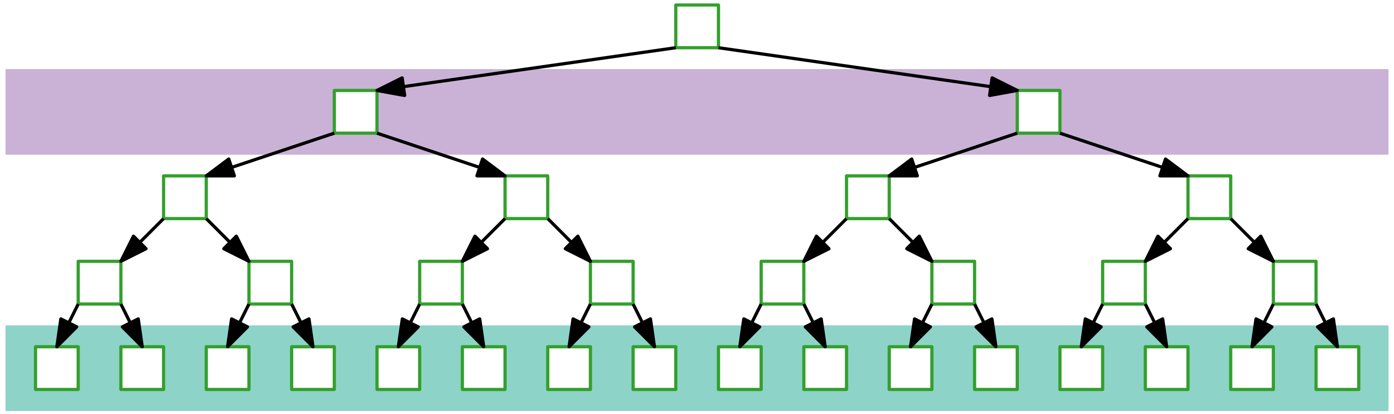

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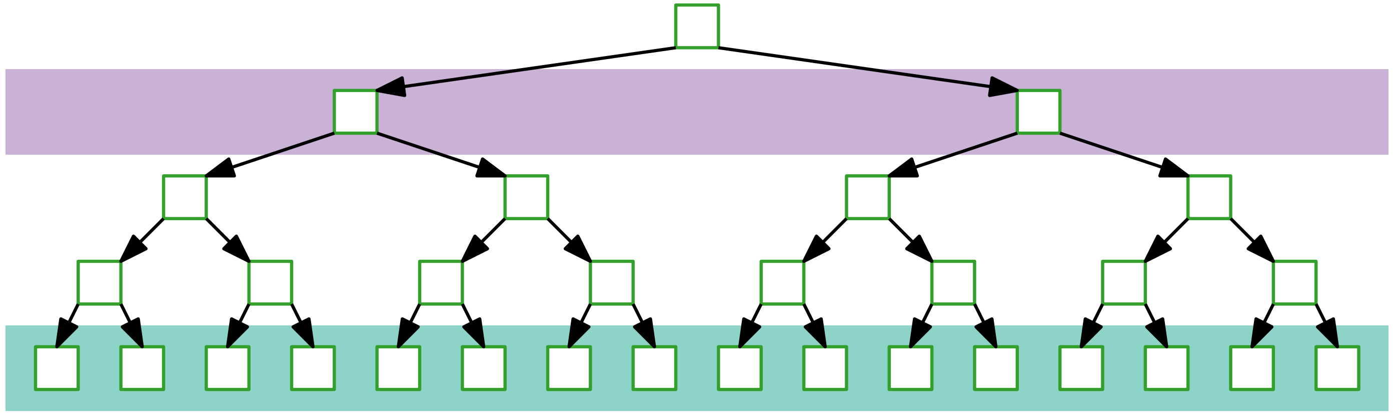

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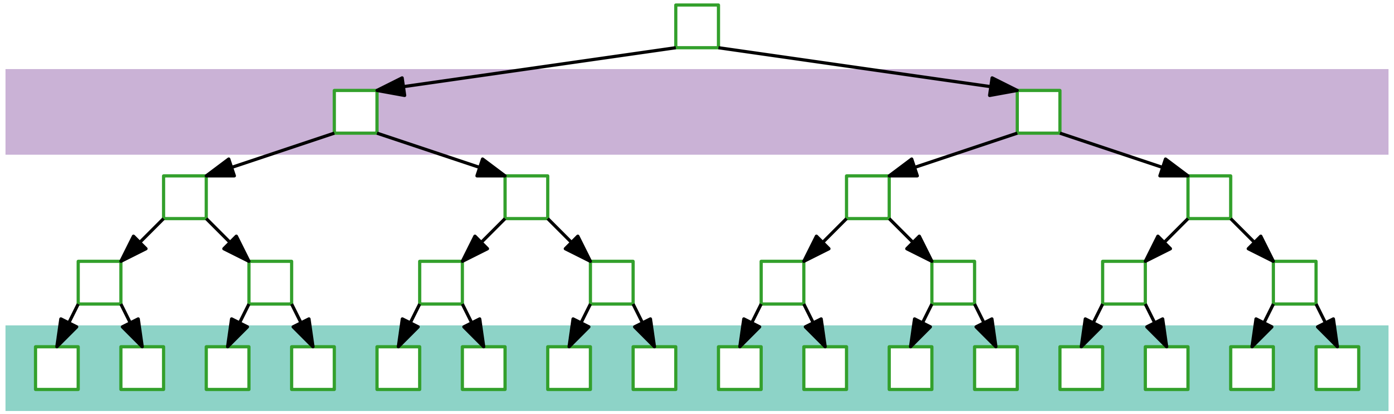

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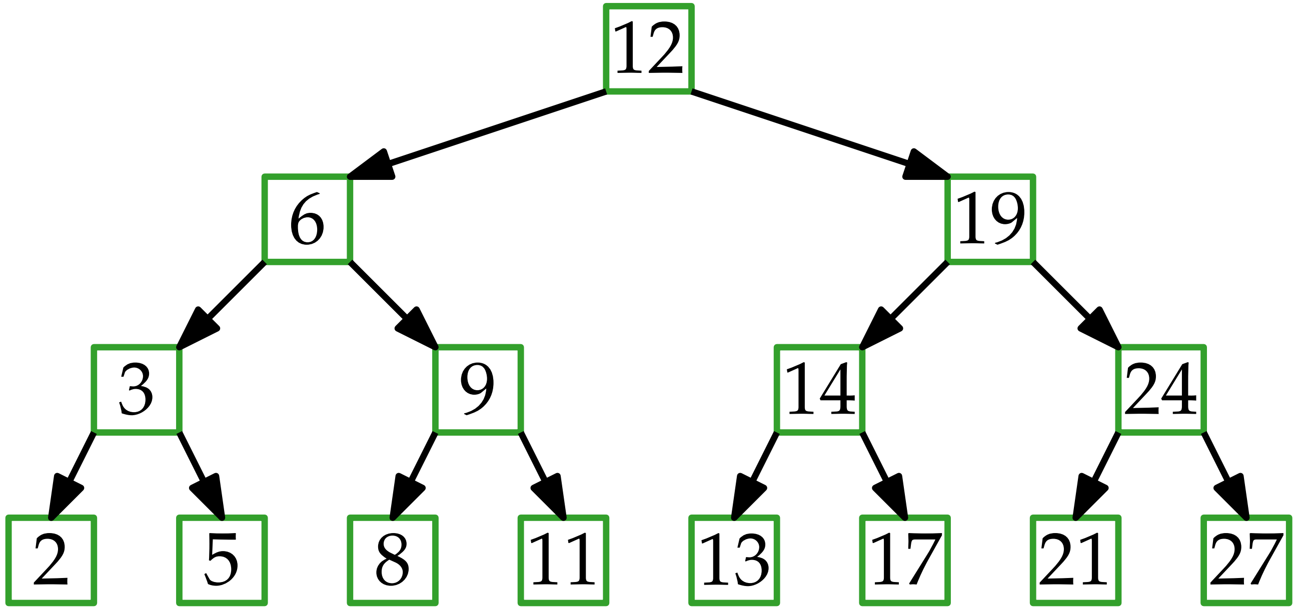

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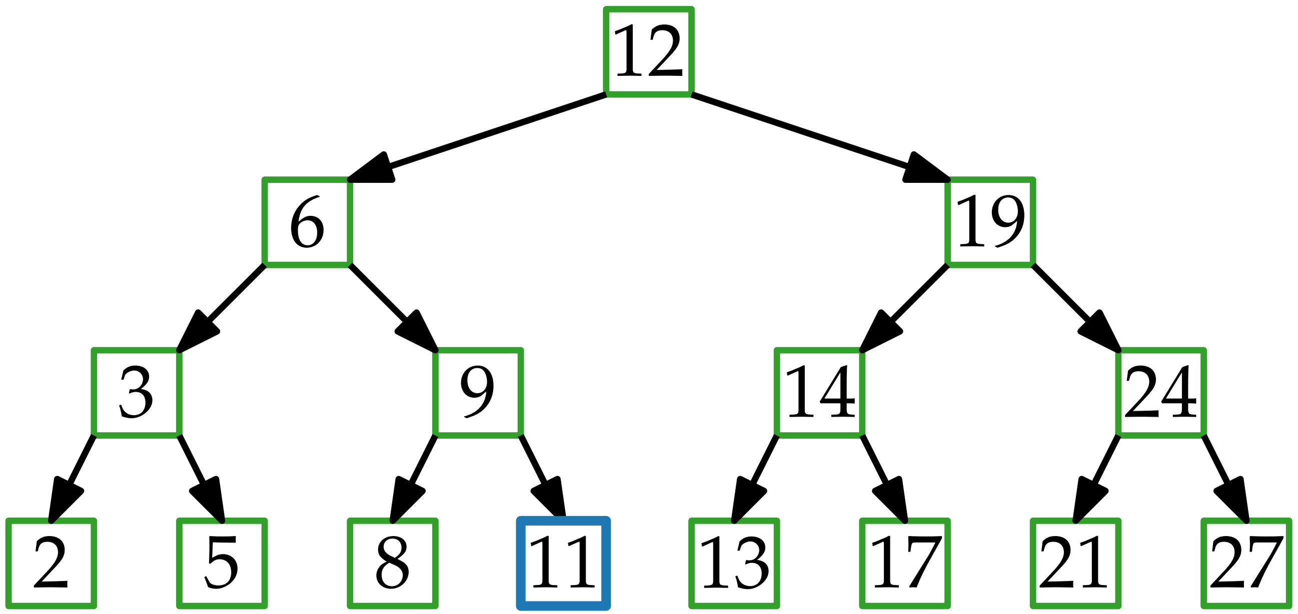

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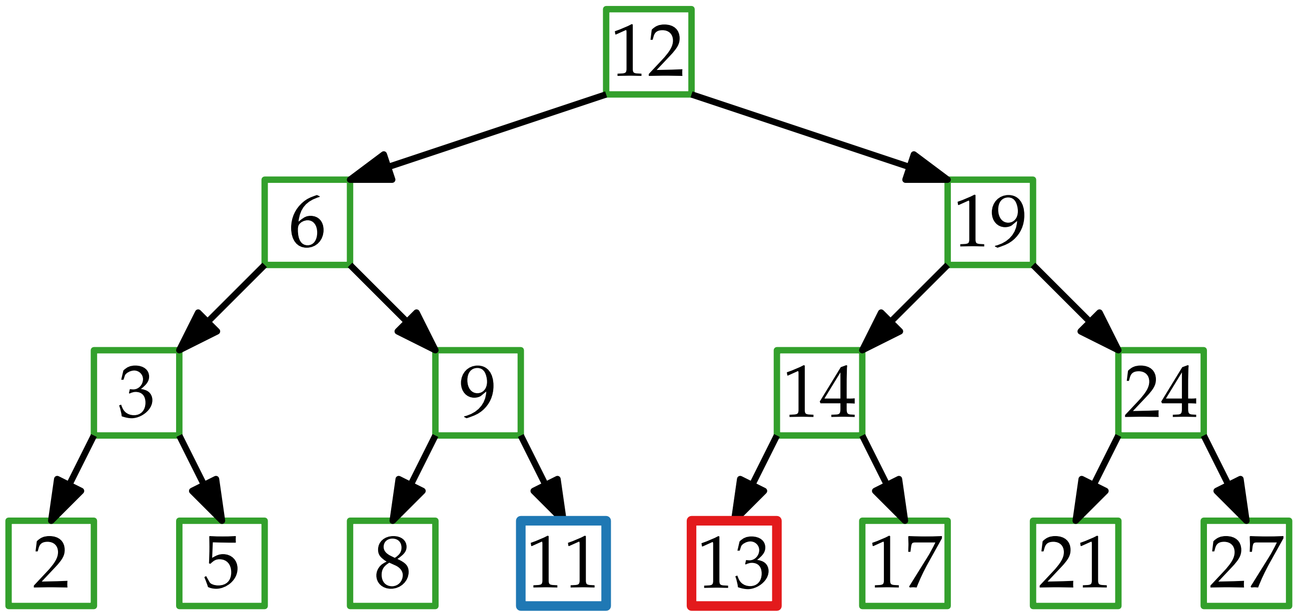

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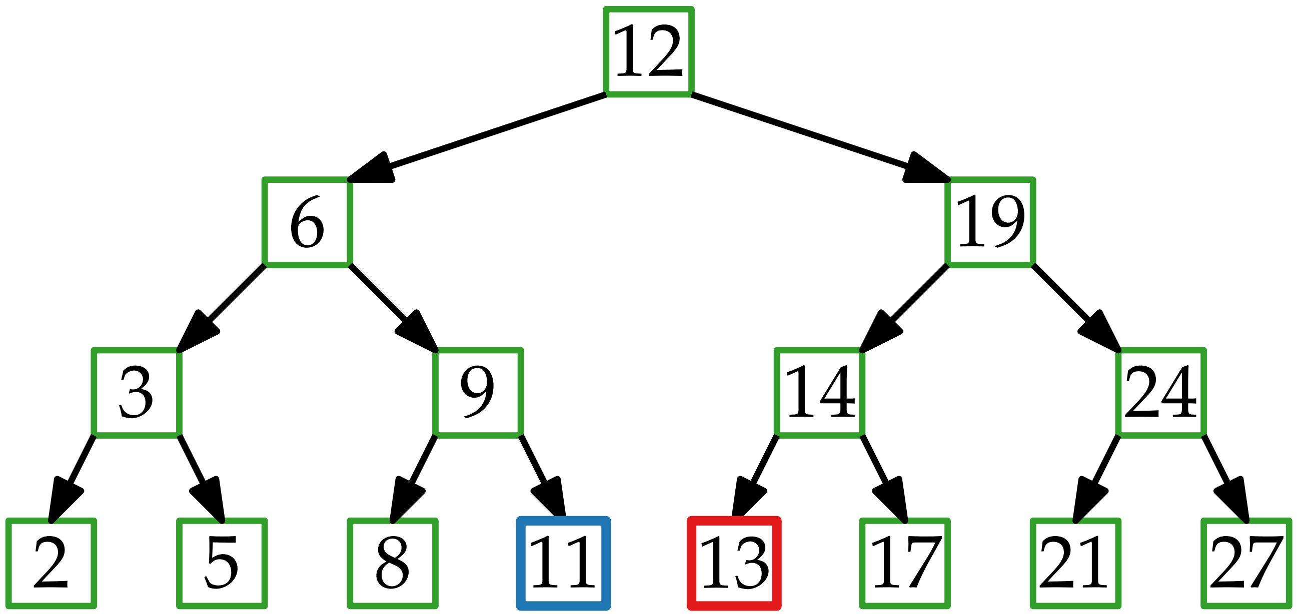

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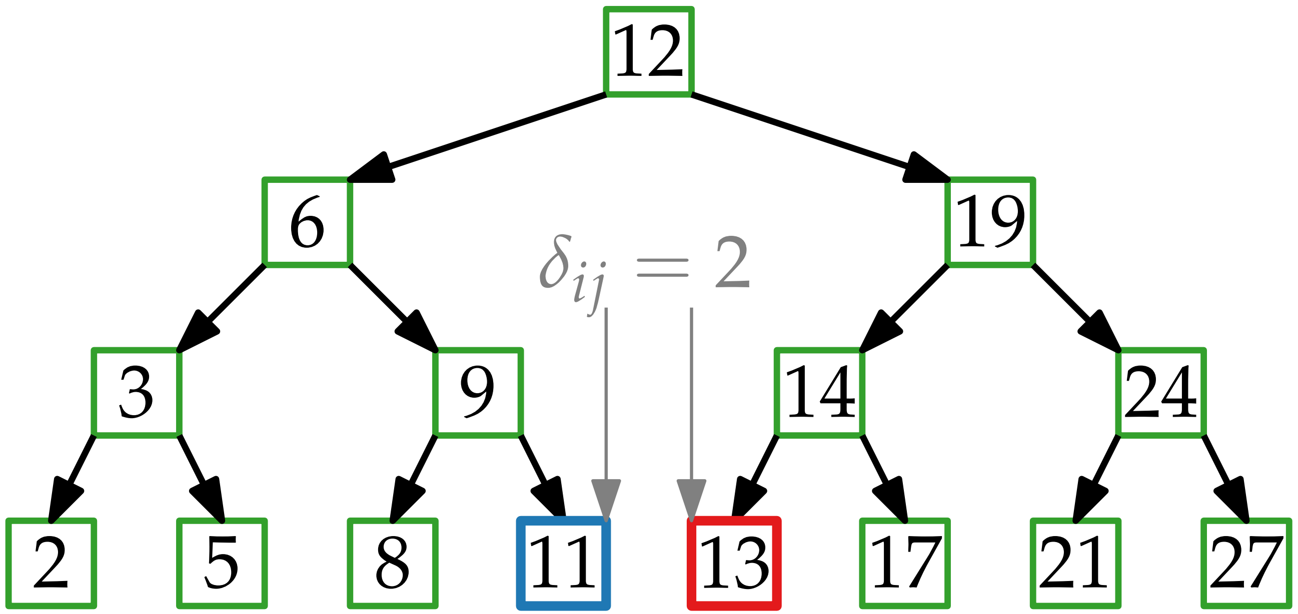

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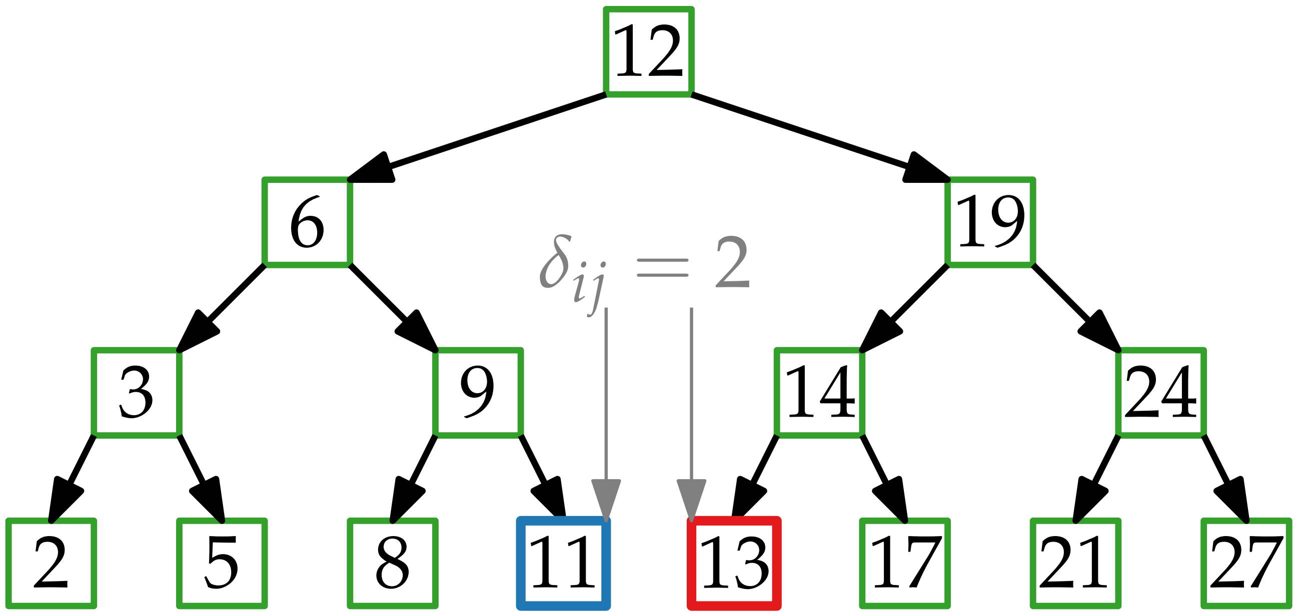

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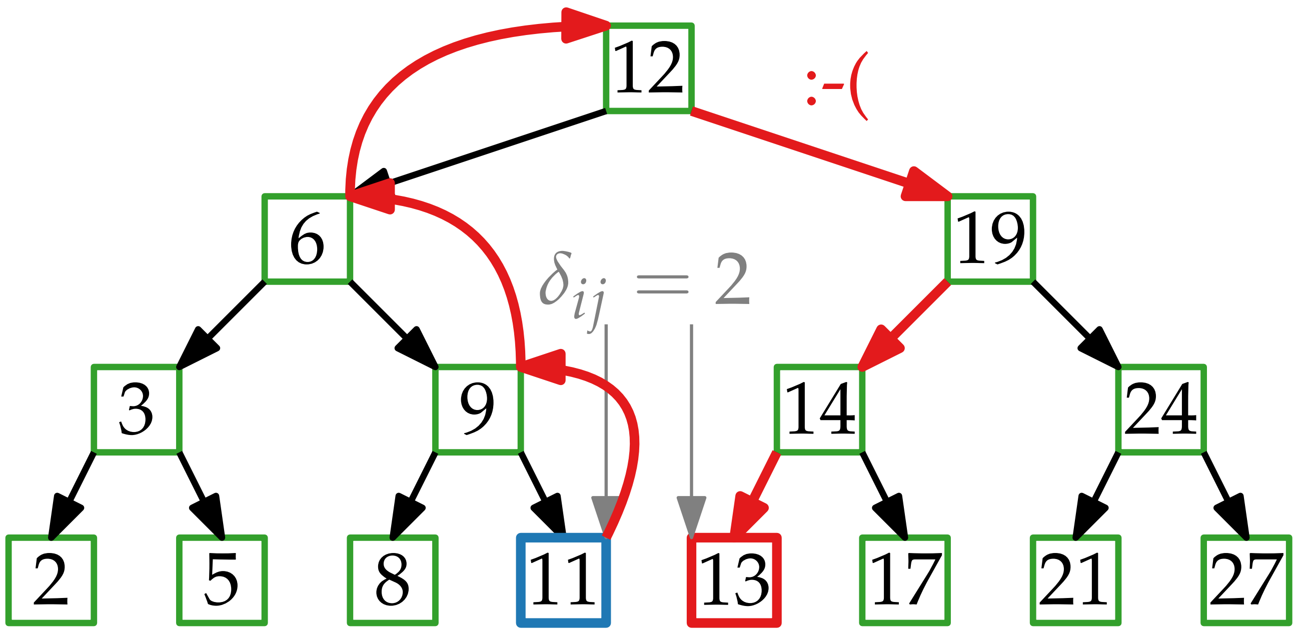

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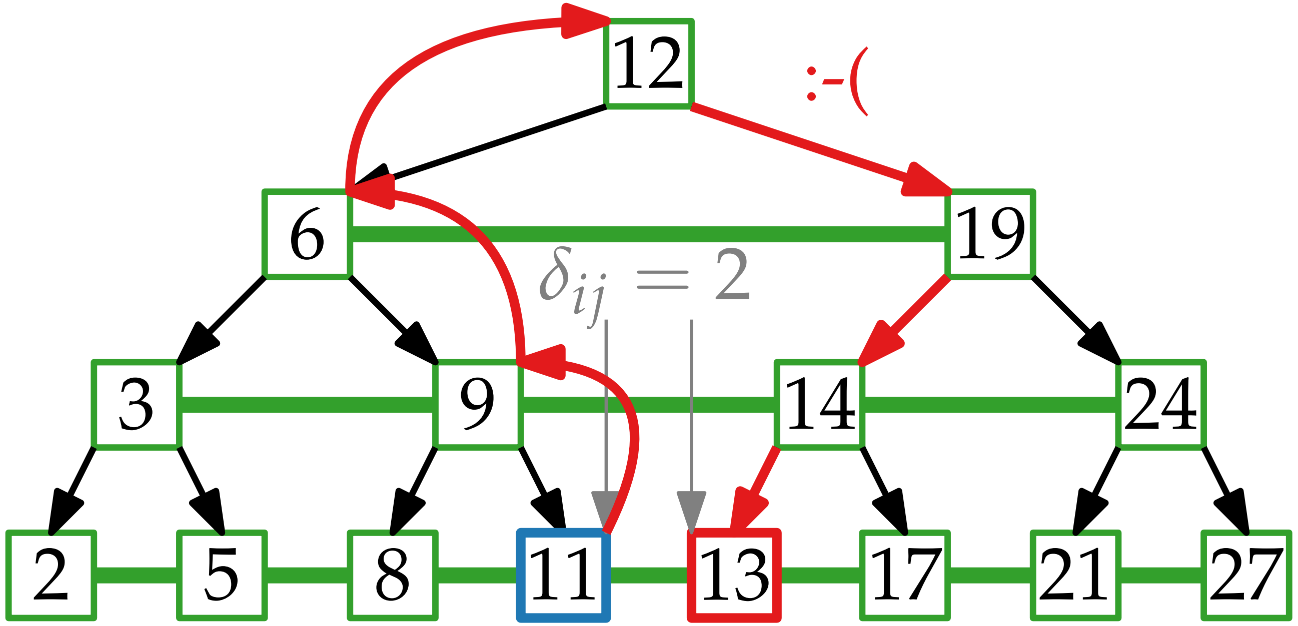

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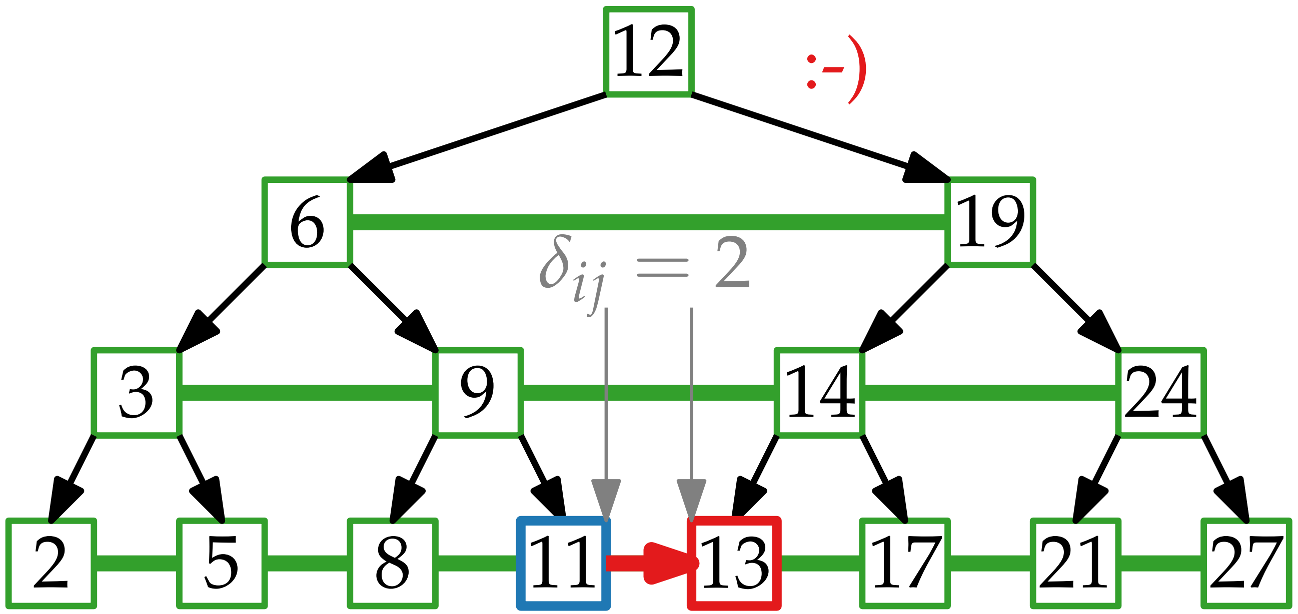

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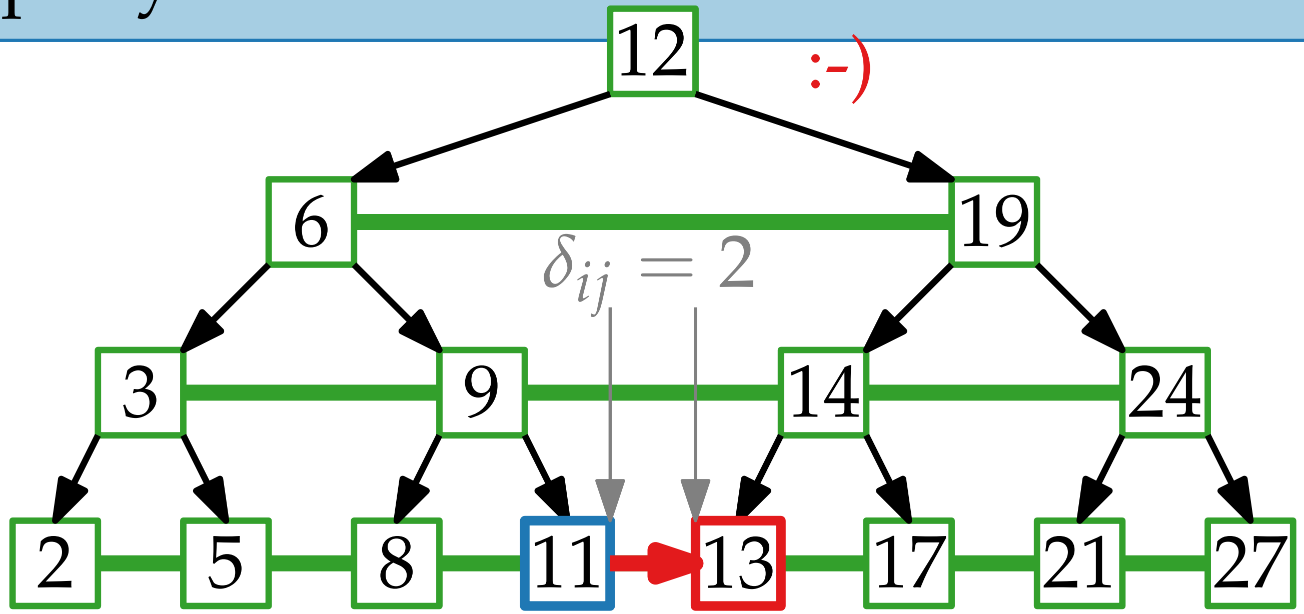
Lemma. A level-linked Red-Black-Tree has the dynamic finger property.



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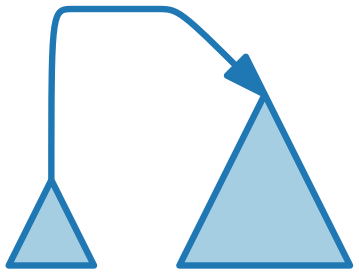

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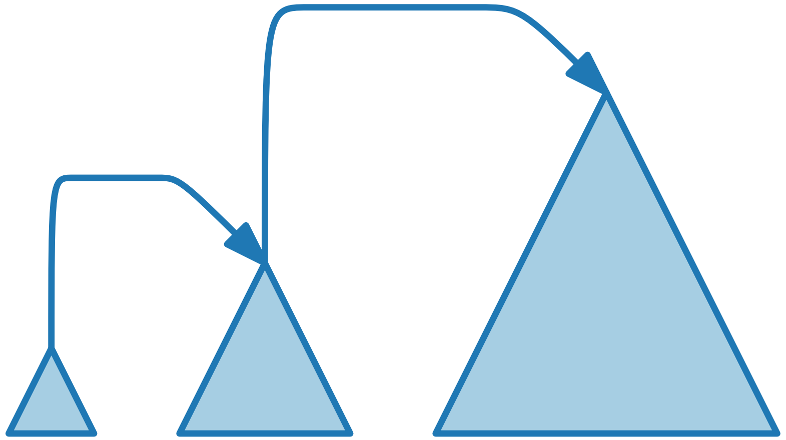

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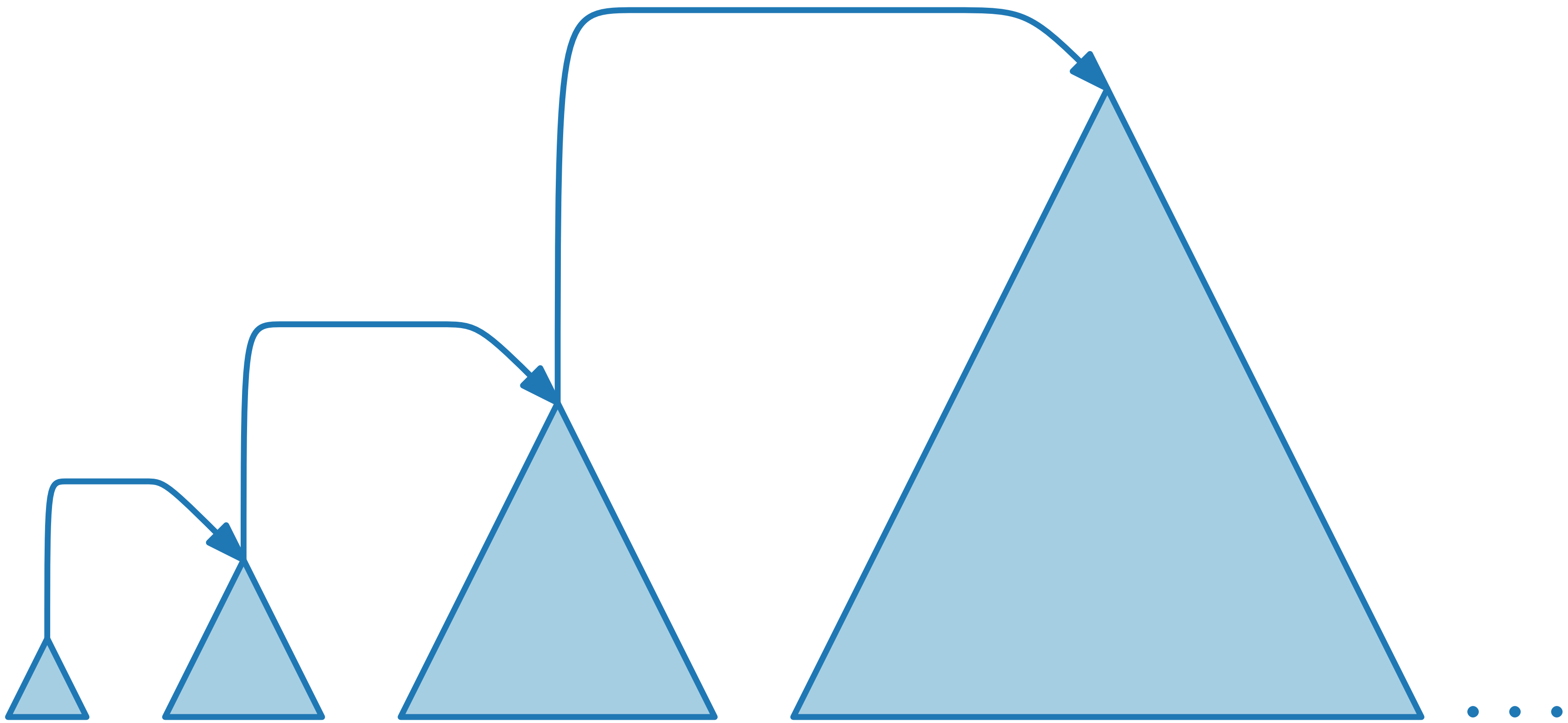

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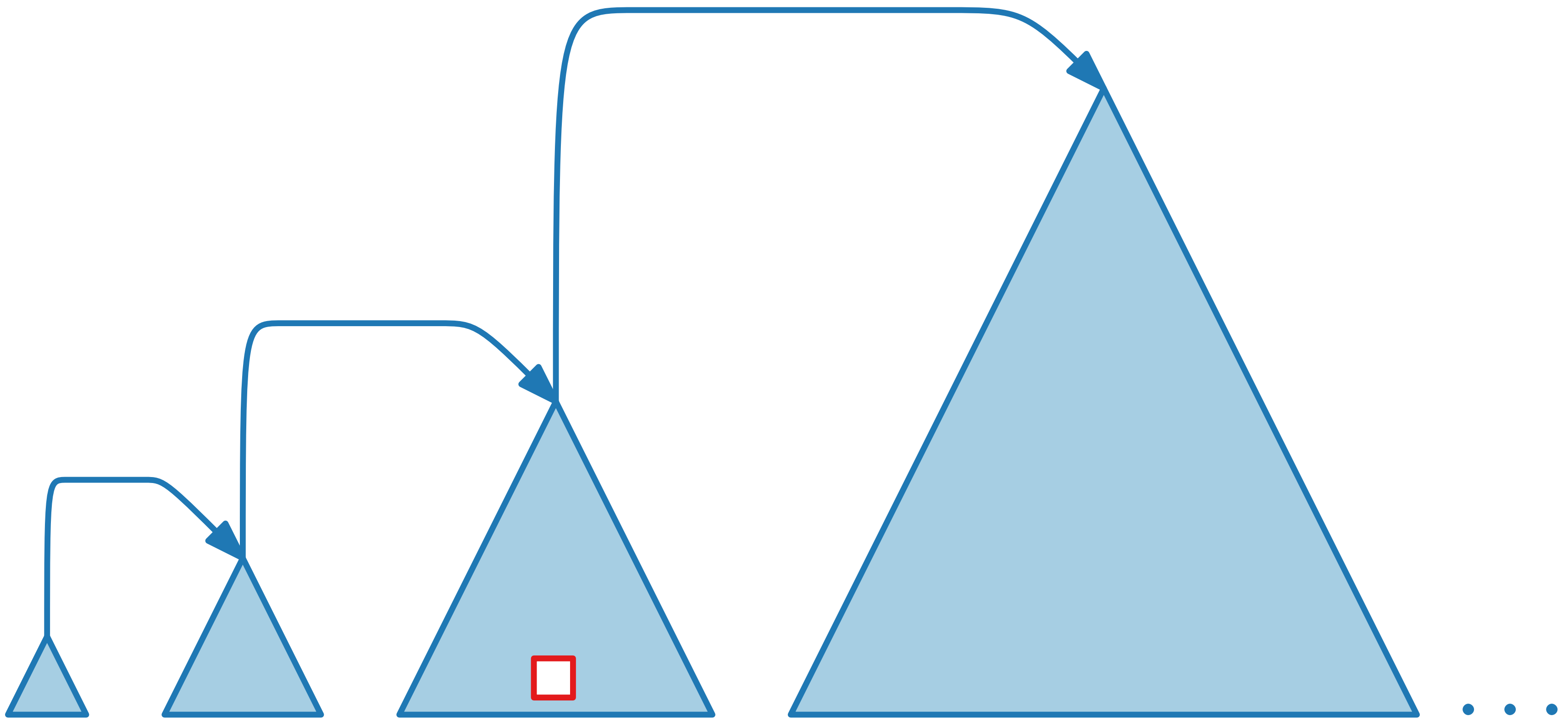

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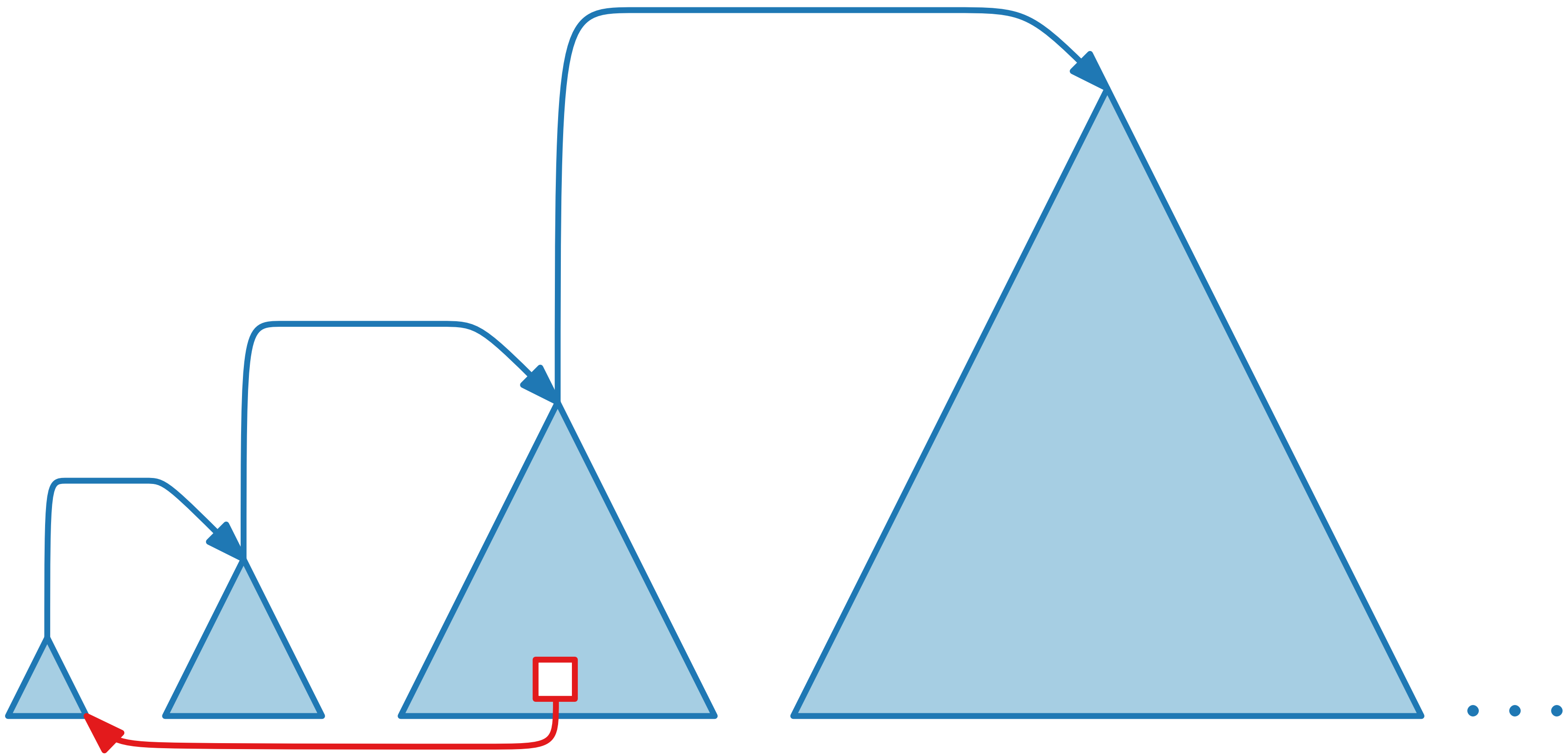

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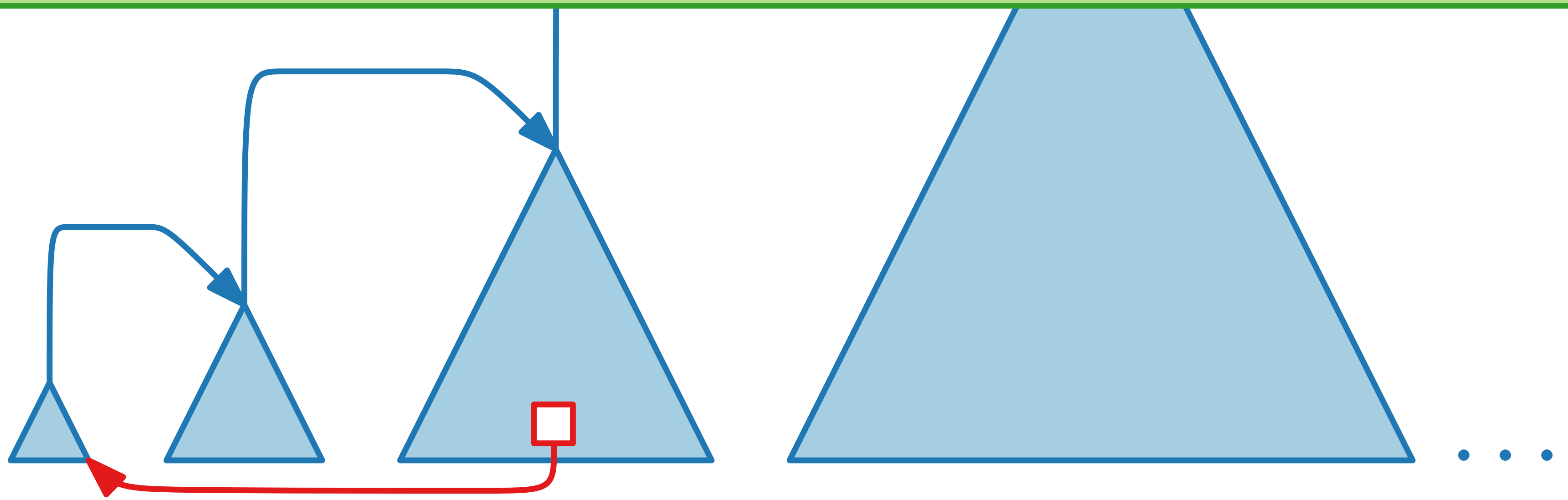
Definition. A BST has the **working set property** if the (amortized) cost of a query for key x is $O(\log t)$, where t is the number of keys queried more recently than x.



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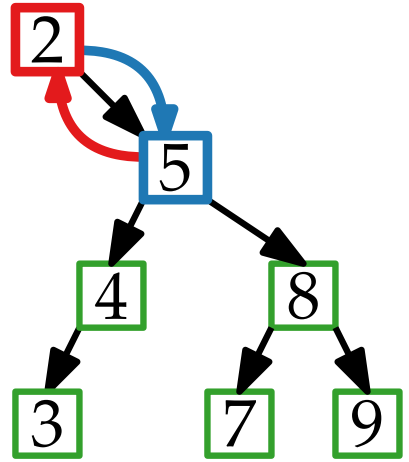

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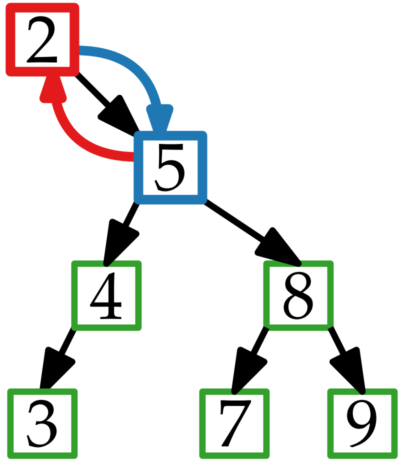

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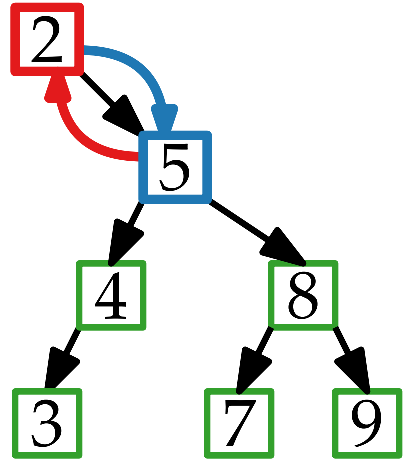

Definition. A BST is **statically optimal** if queries take (amort.) $O(\mathrm{OPT}_S)$ time for every S.

All These Properties...

Balanced:	Queries take (amort.) $O(\log n)$ time
Entropy:	Queries take expected $O(1 + H)$ time
Dynamic Finger:	Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)
Working Set:	Queries take $O(\log t)$ time (t: recency)
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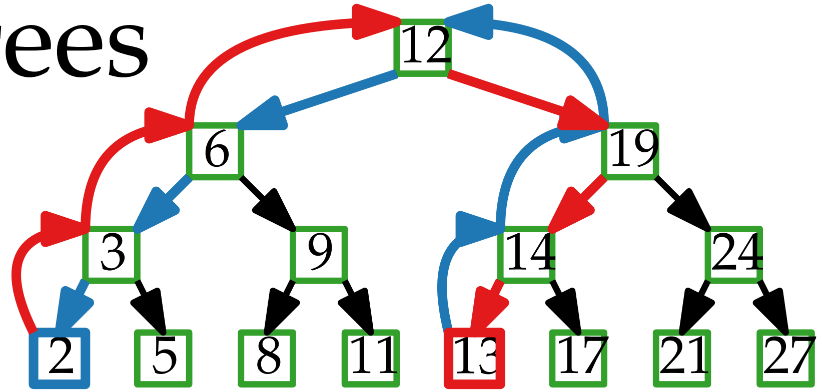
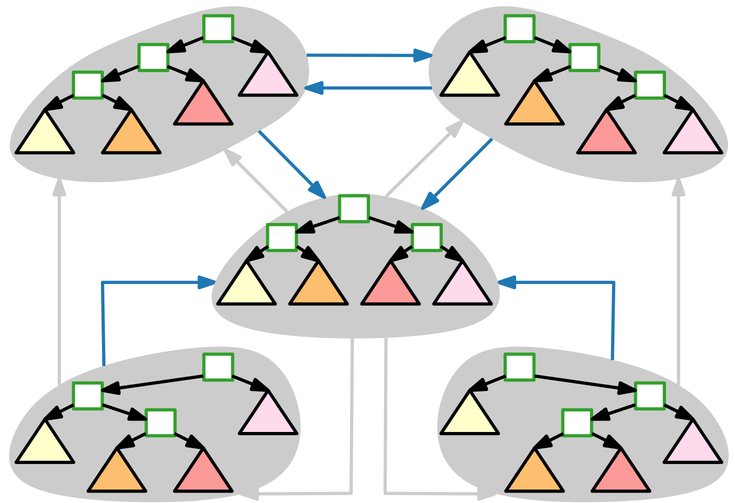
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Optimal Binary Search Trees



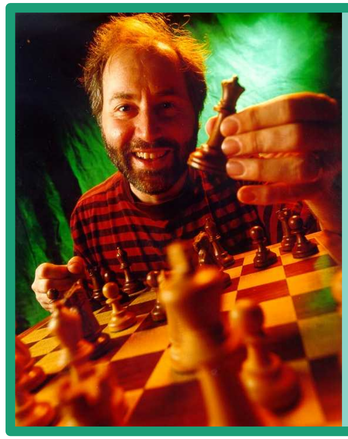
Splay Trees



Philipp Kindermann · WS20

Part III:
Splay Trees

Splay Trees



Daniel D. Sleator

Robert E. Tarjan

J. ACM 1985

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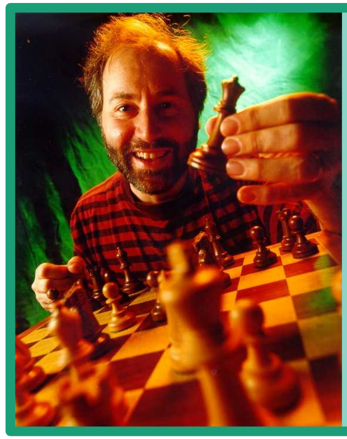

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Idea: Whenever we query a key, rotate it to the root.

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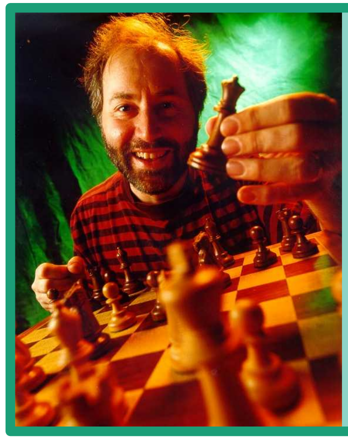

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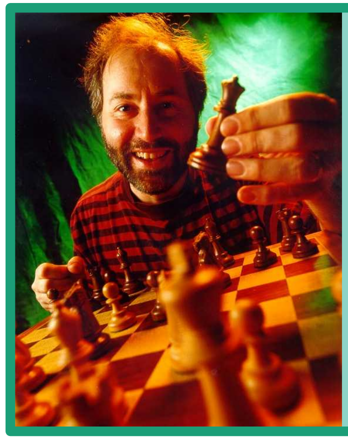

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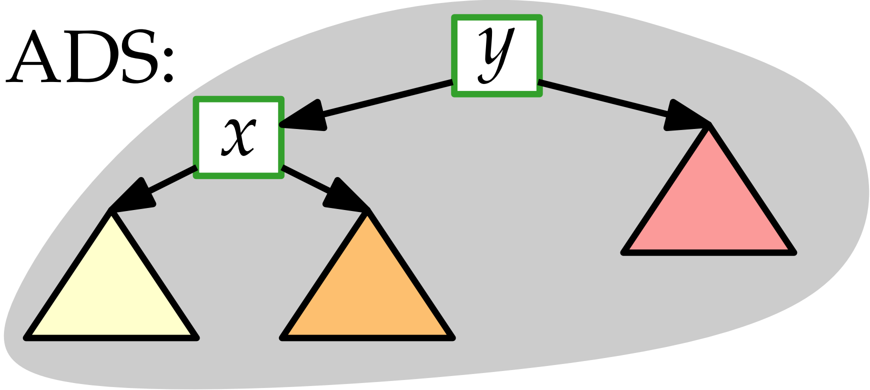

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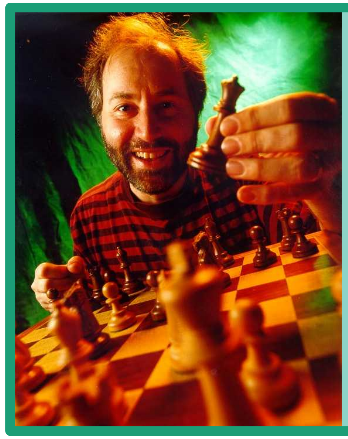

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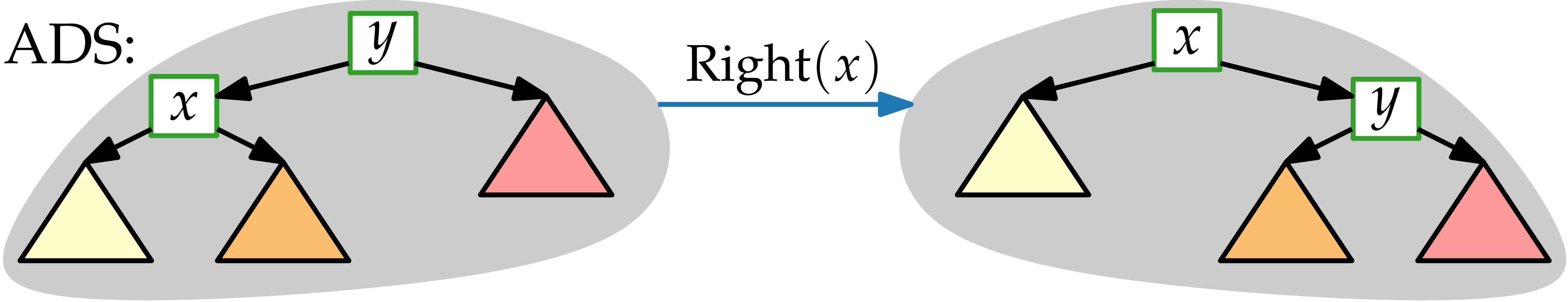

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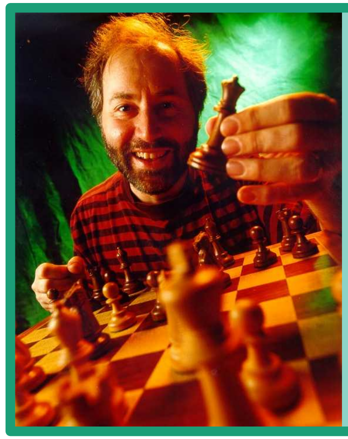

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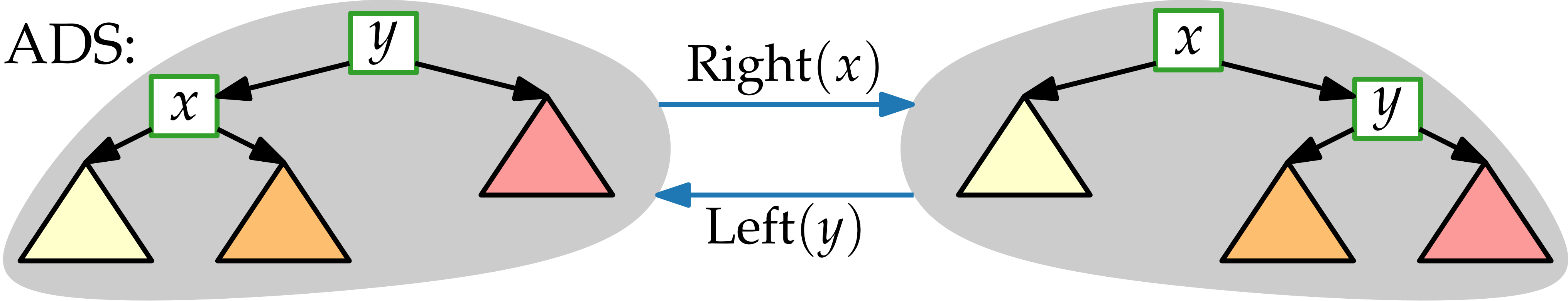
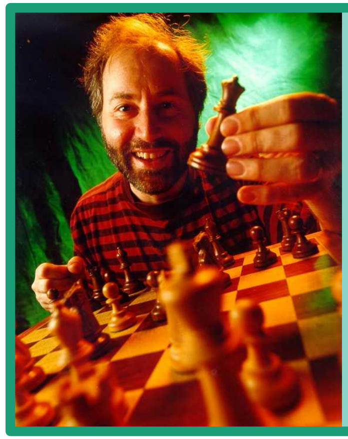


Splay Trees



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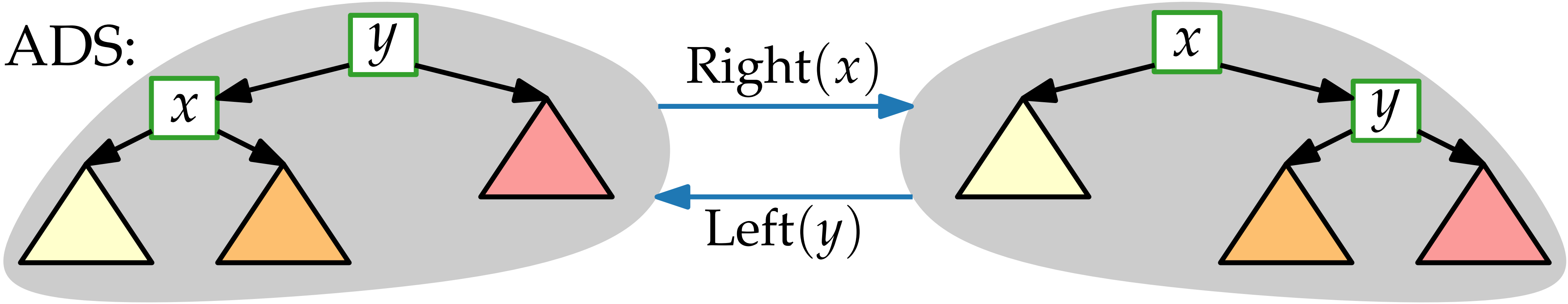

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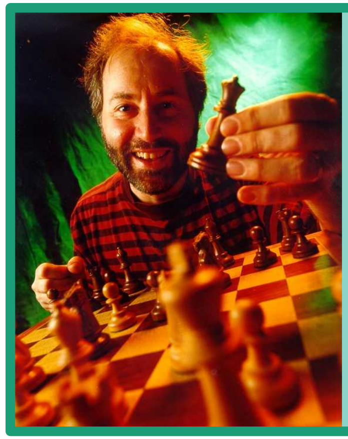

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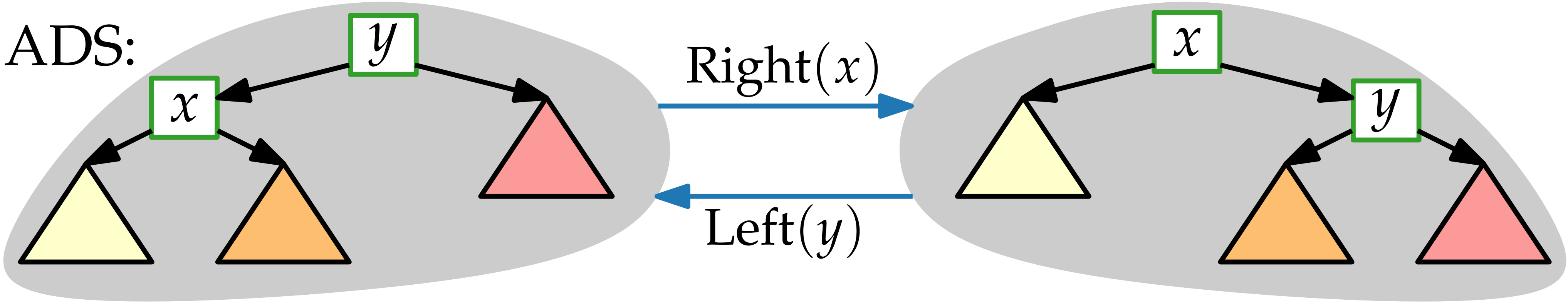

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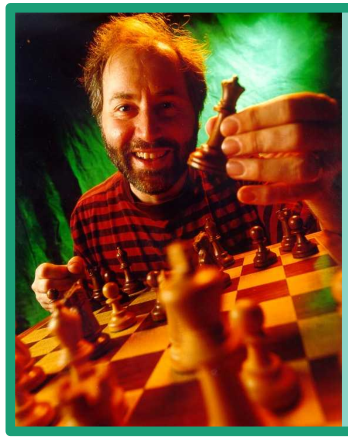

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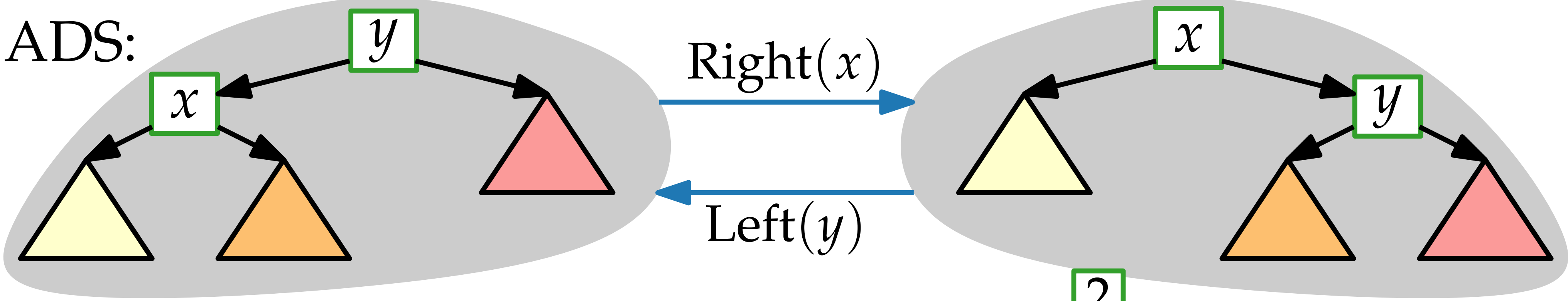

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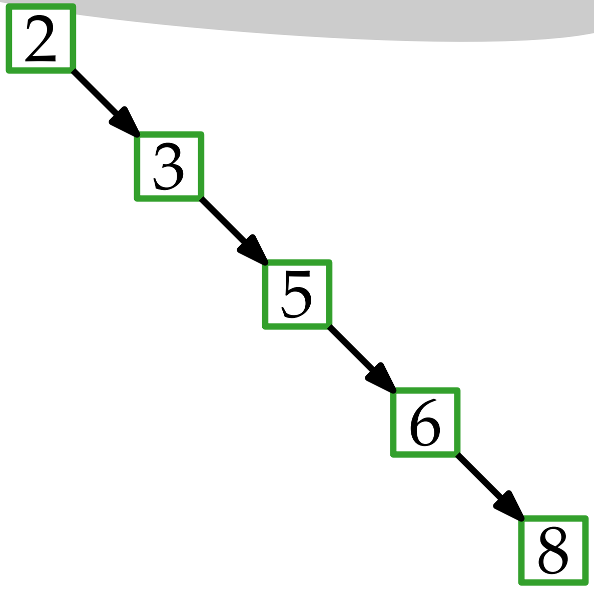

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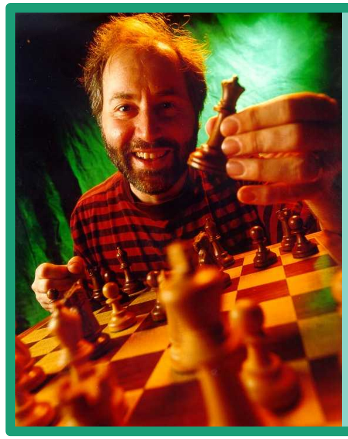

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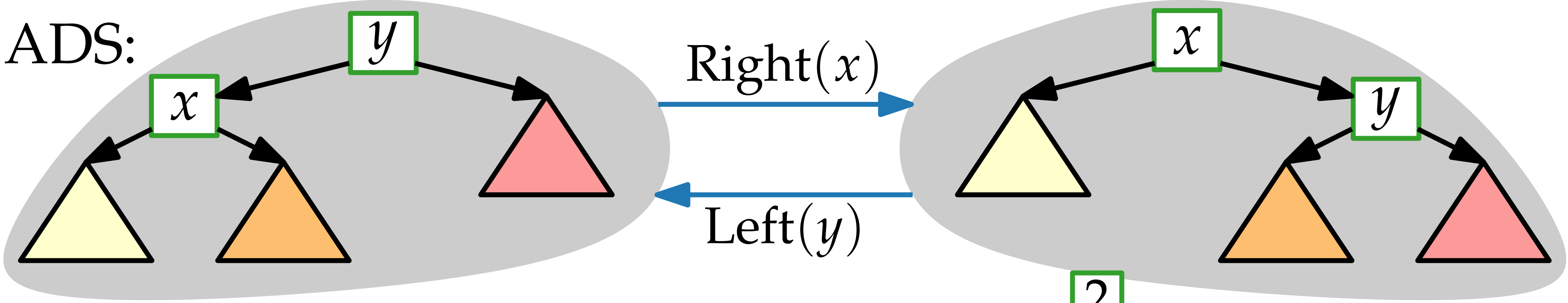

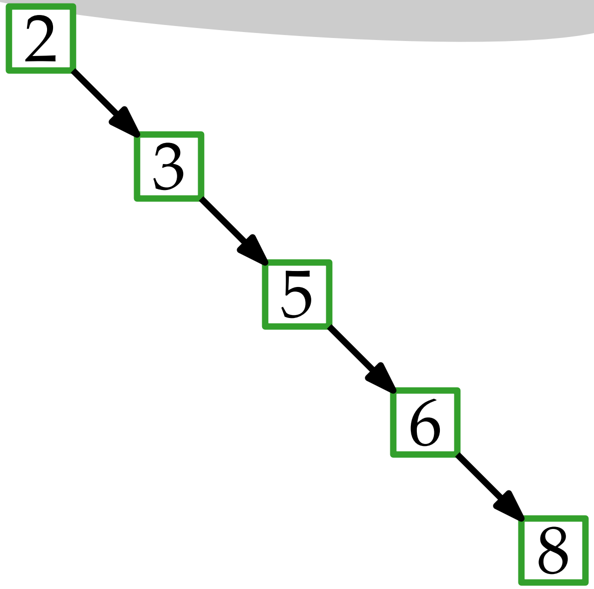

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Query(8)

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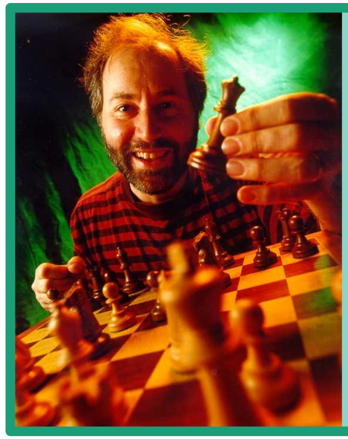

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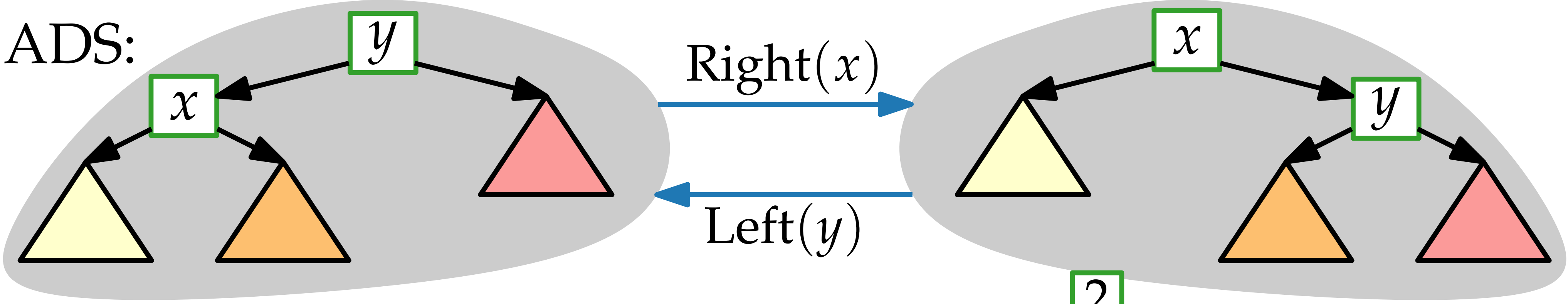

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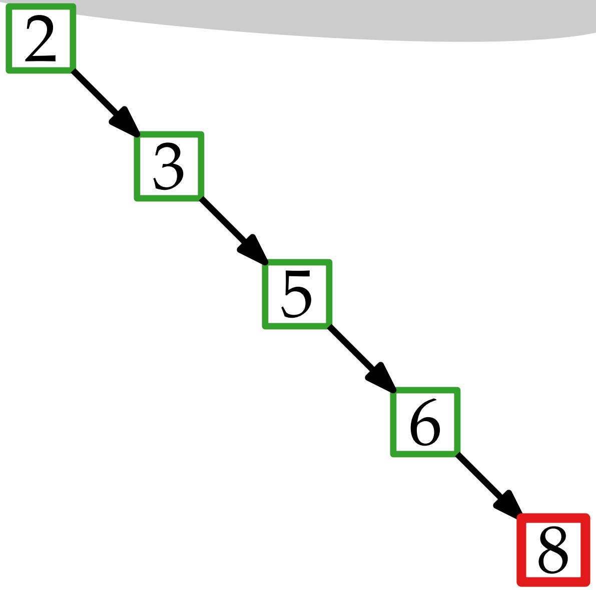

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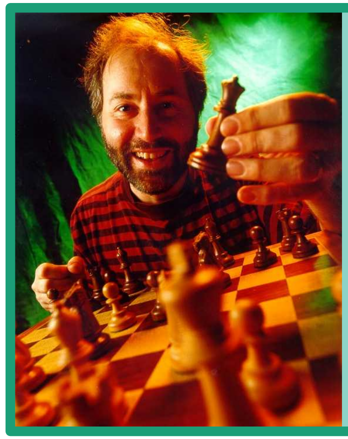

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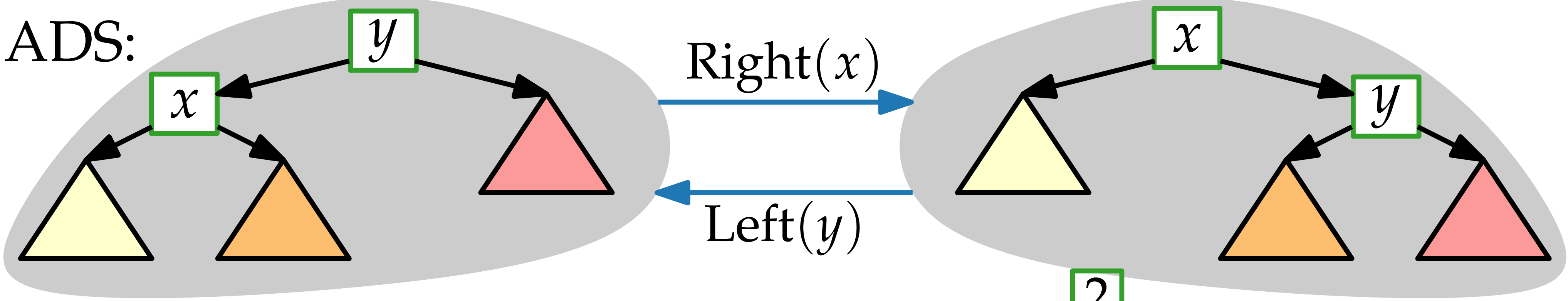

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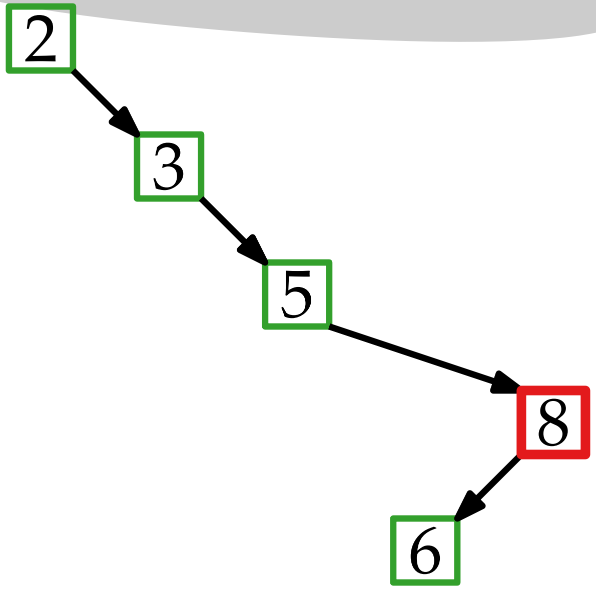

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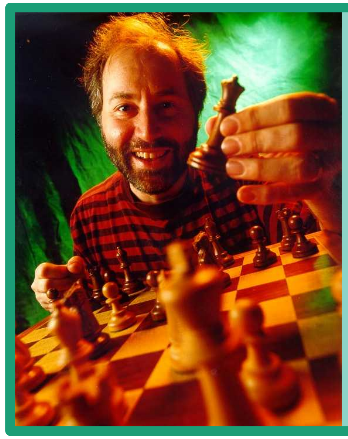

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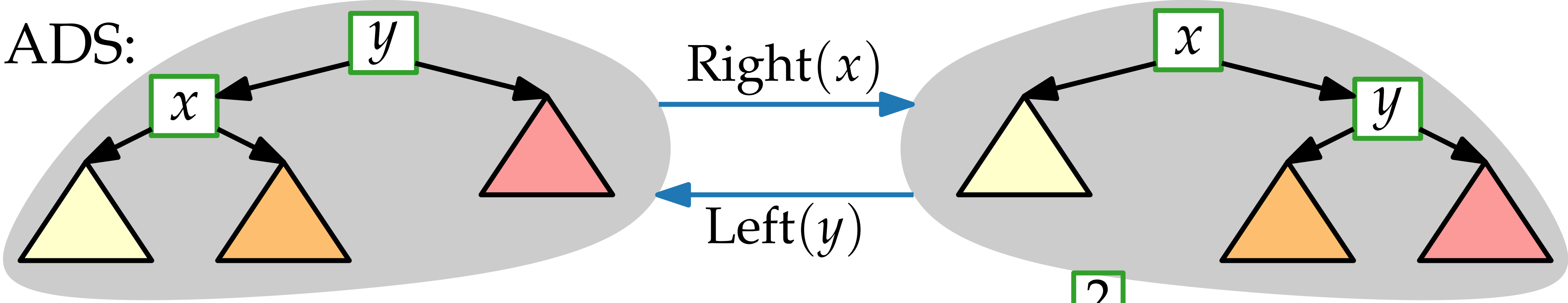

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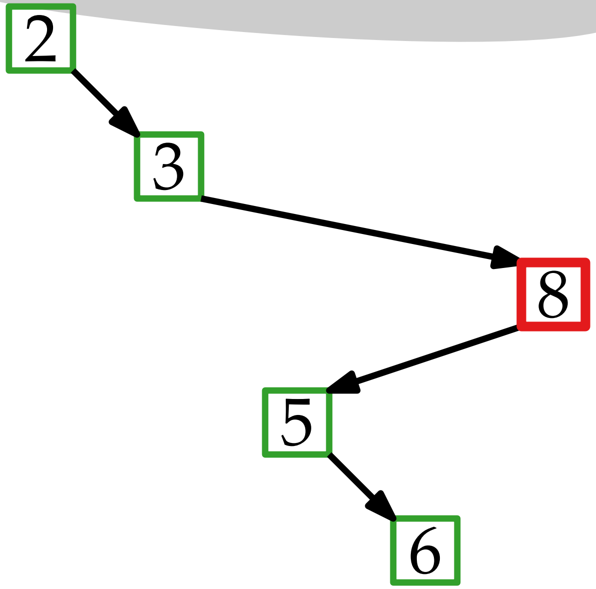

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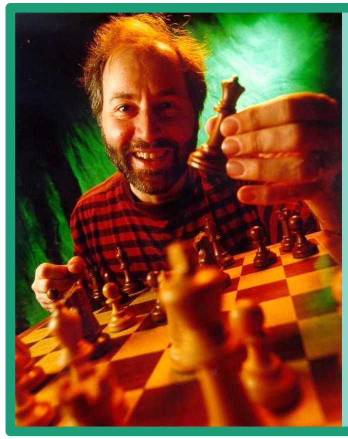

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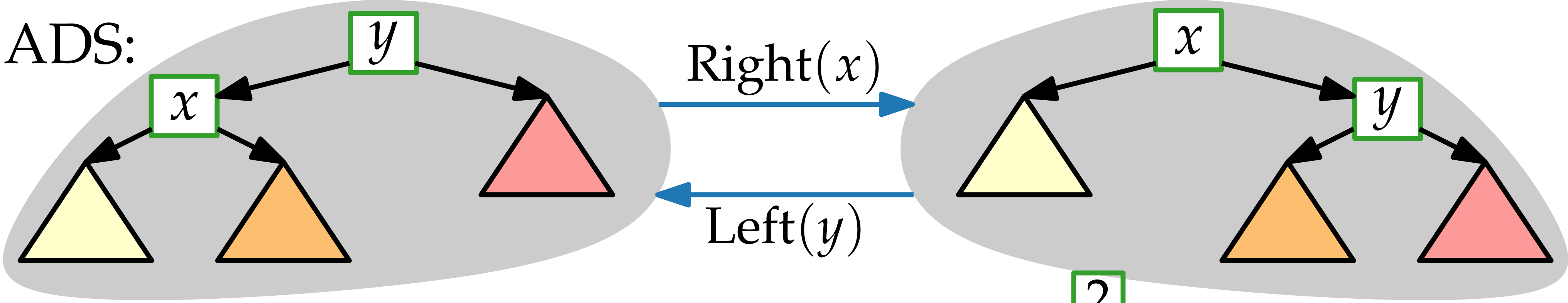

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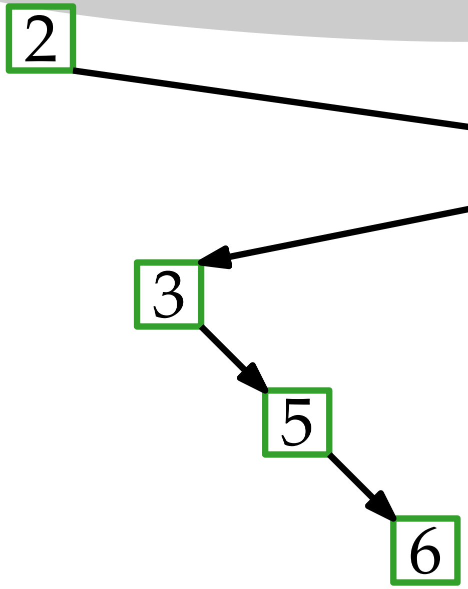

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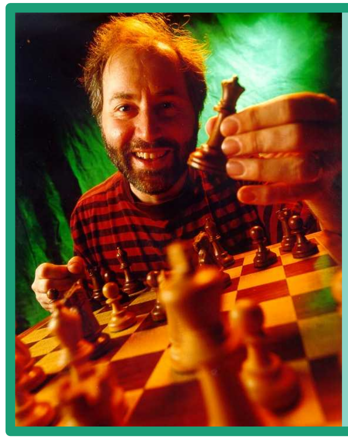

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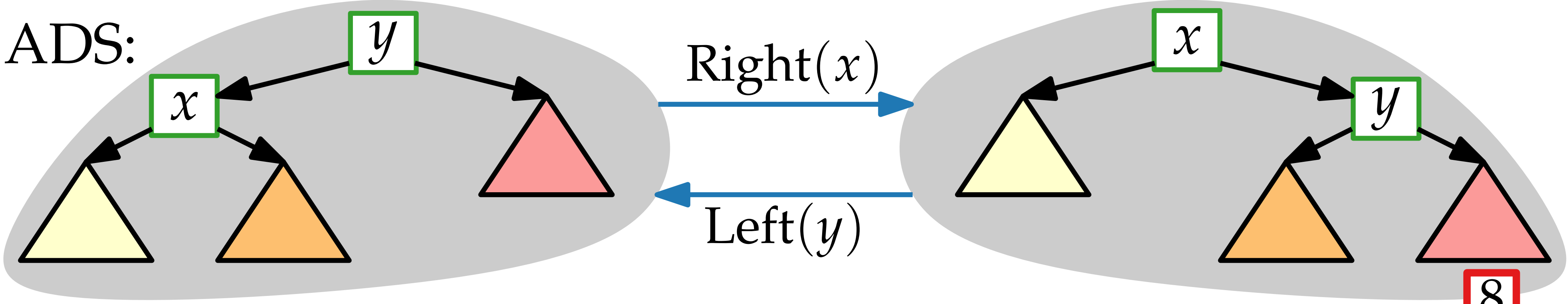

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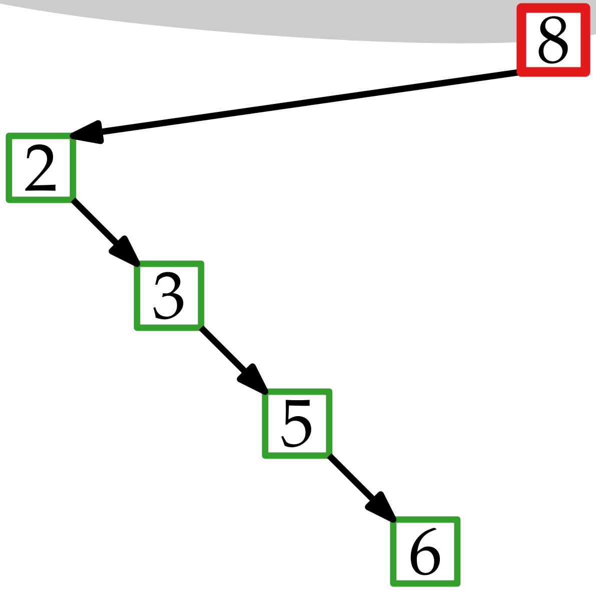

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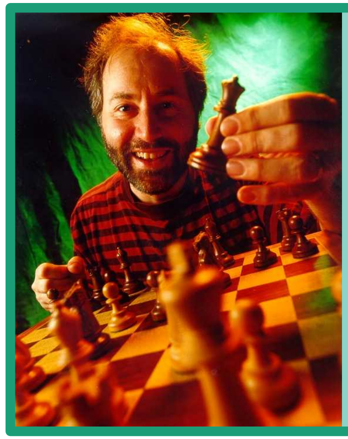

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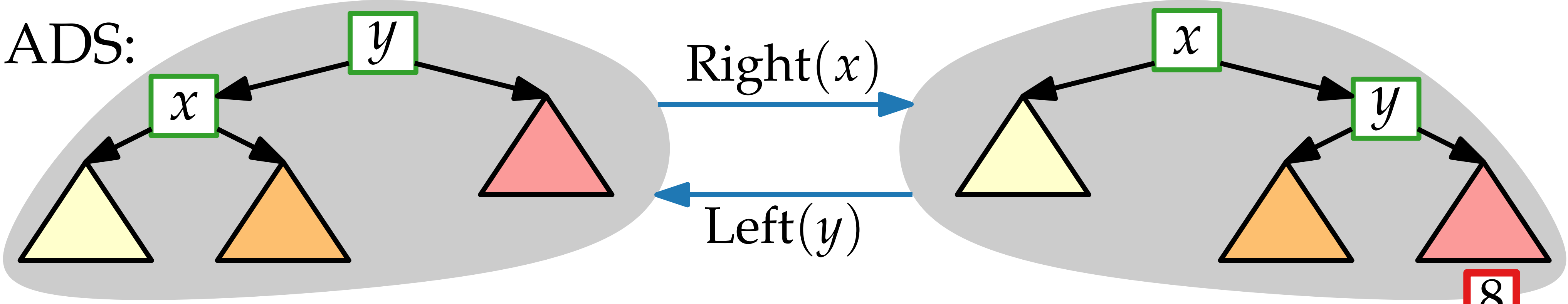

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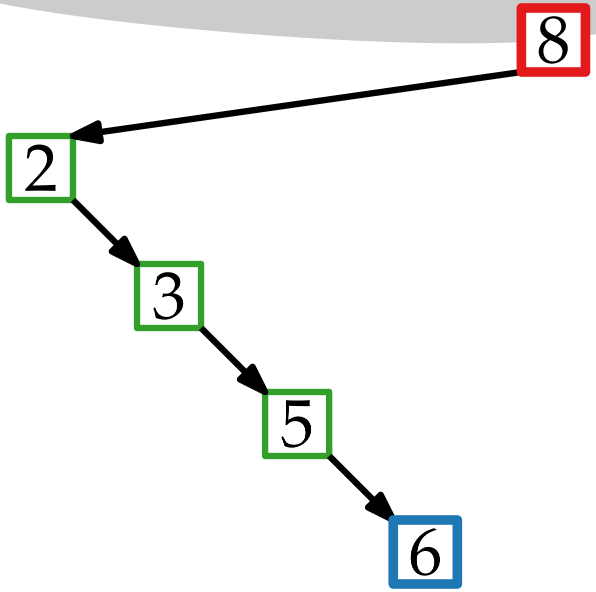
Query(8) Query(6)



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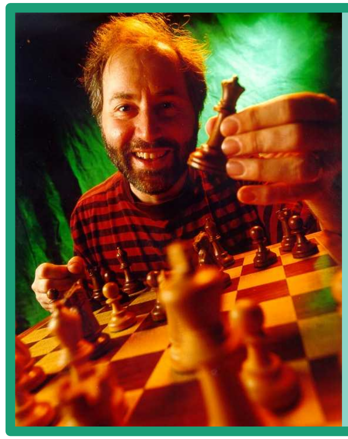

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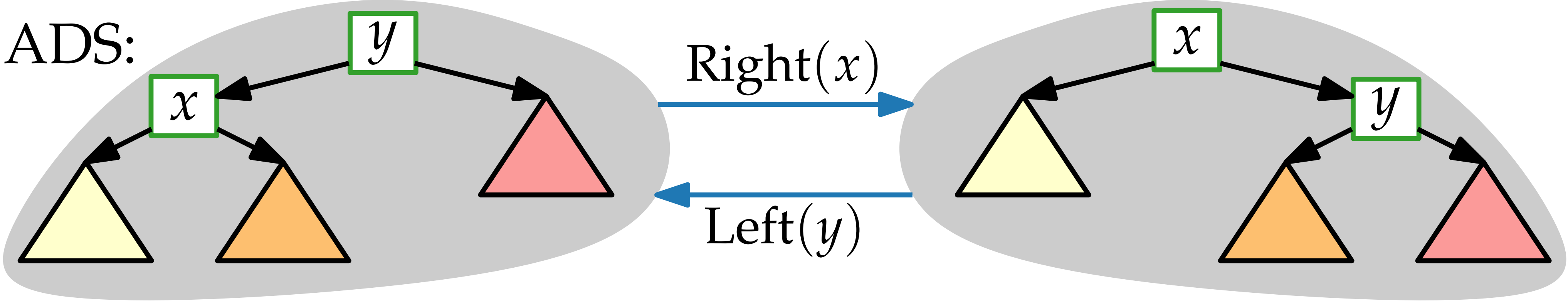

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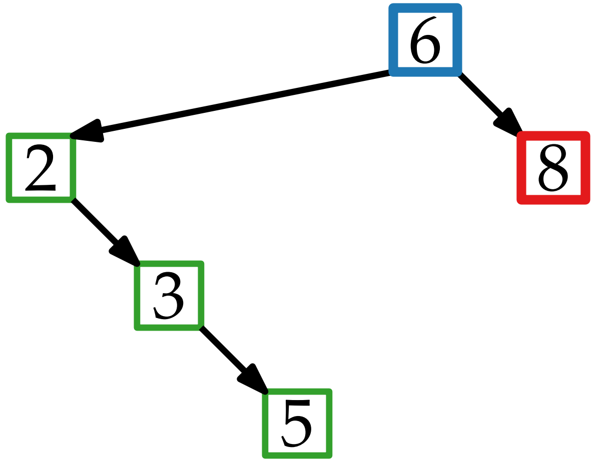

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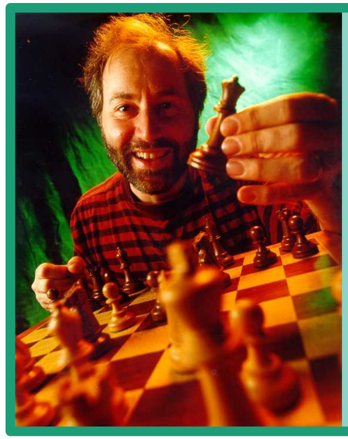

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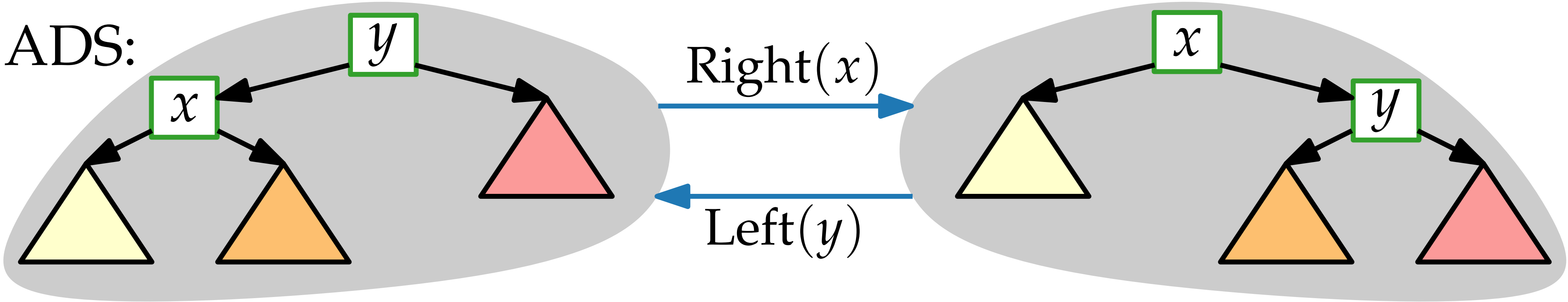

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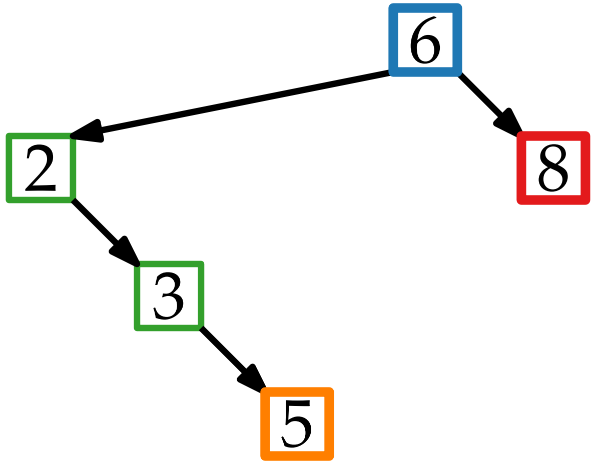

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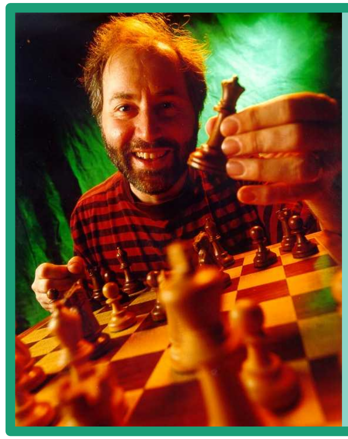

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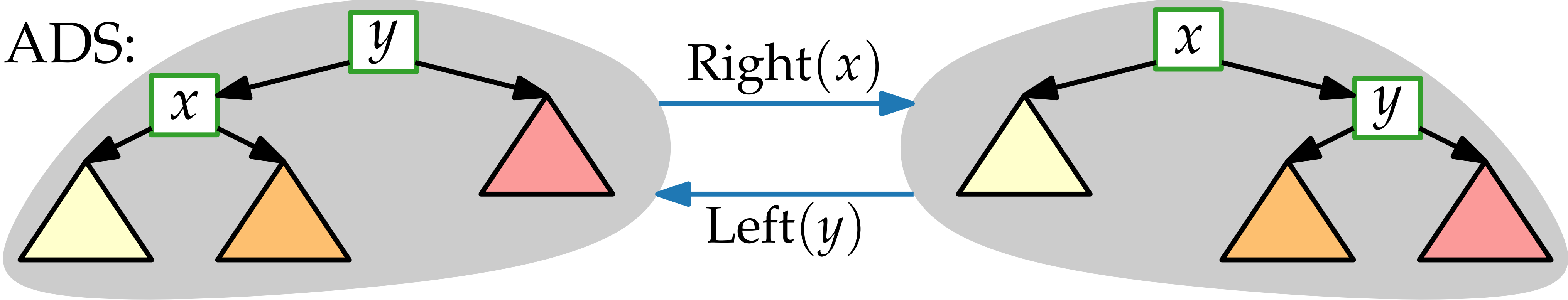

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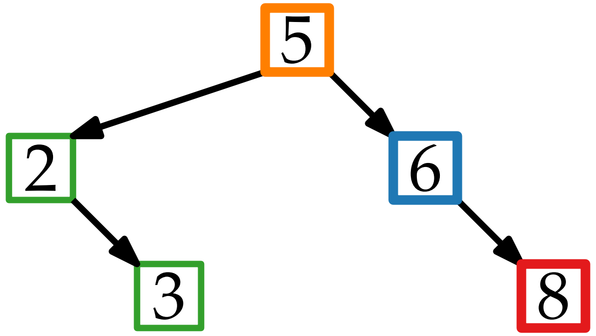

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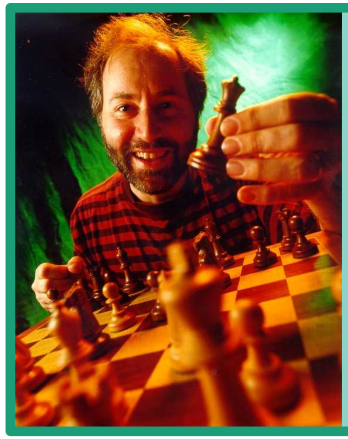

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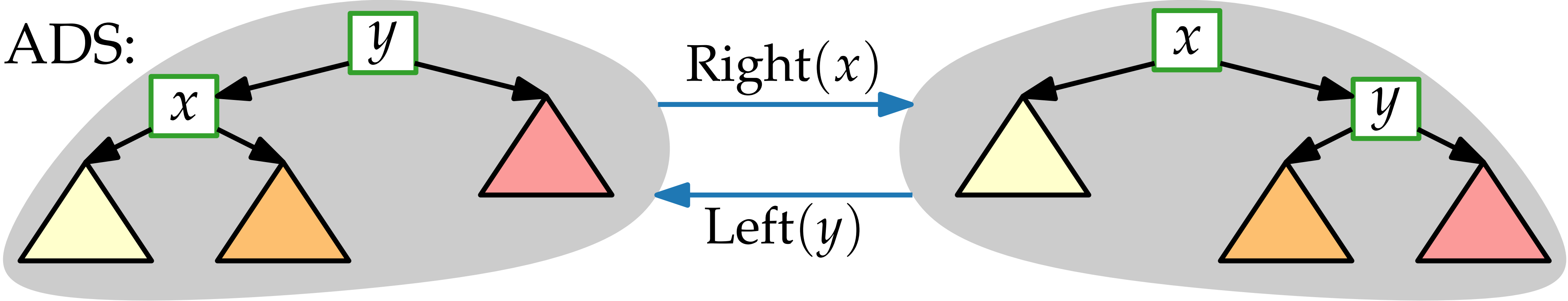

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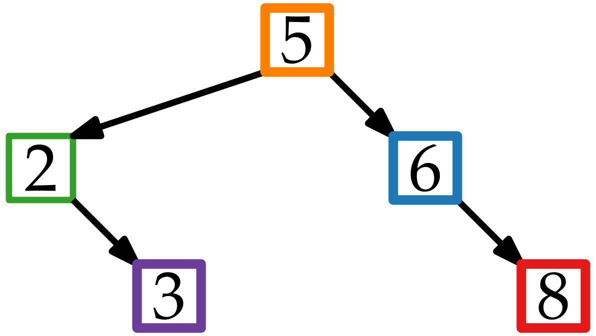
Query(8) Query(6) Query(5)
Query(3)



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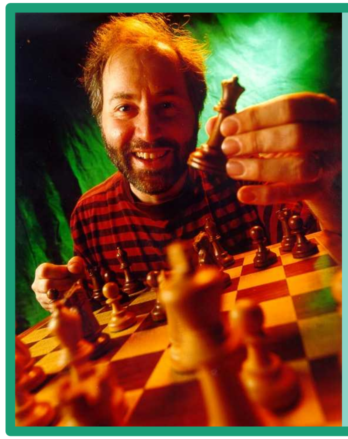

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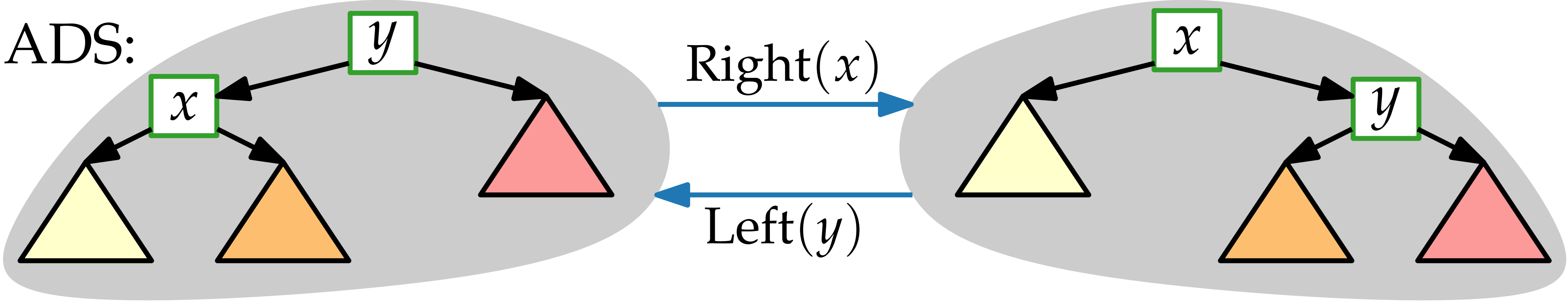

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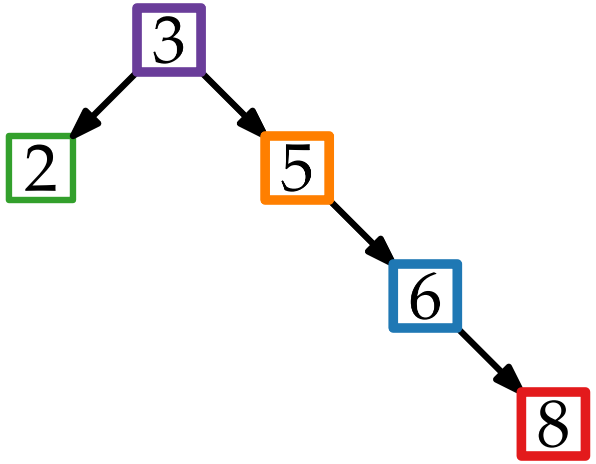

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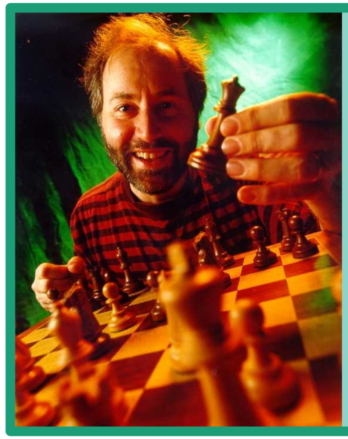

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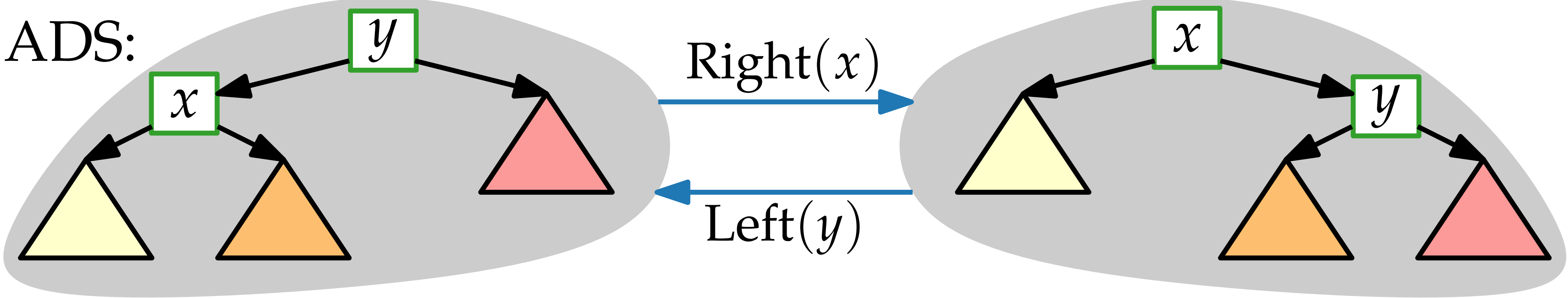

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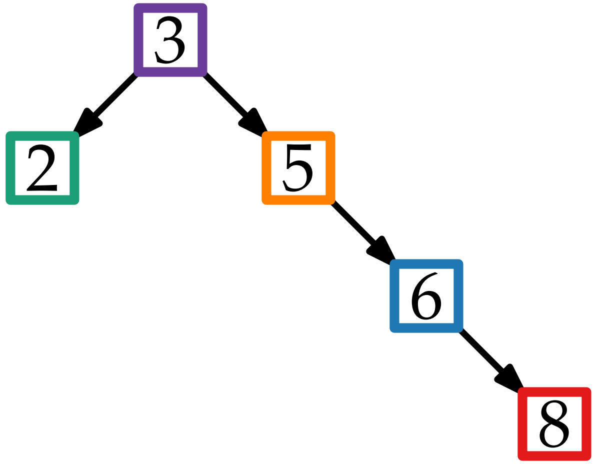

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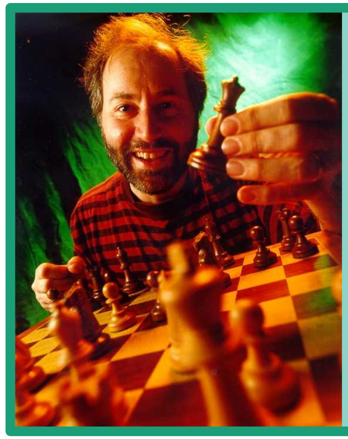

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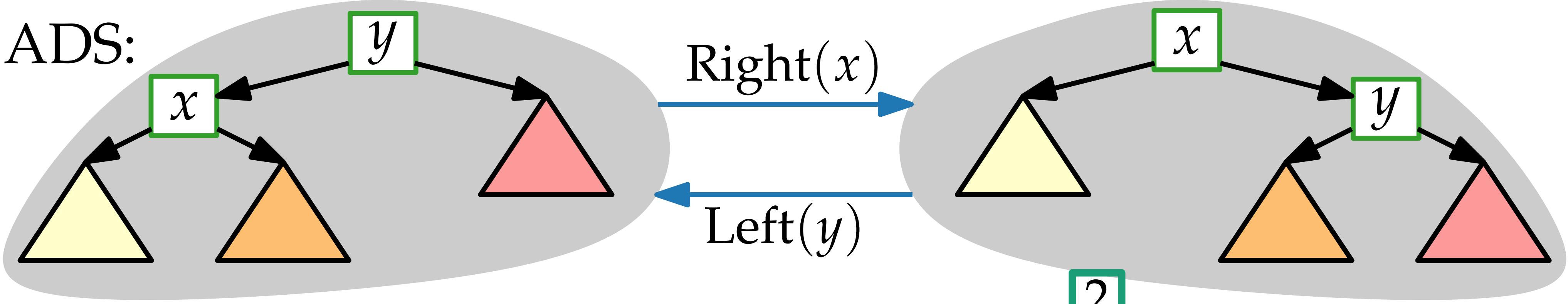

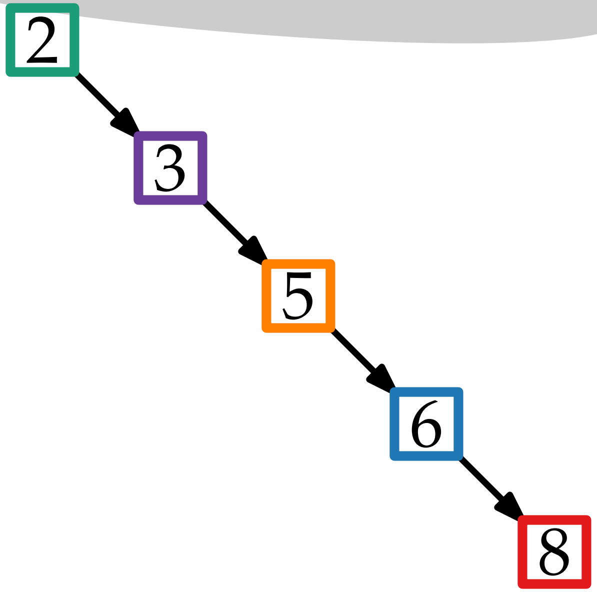

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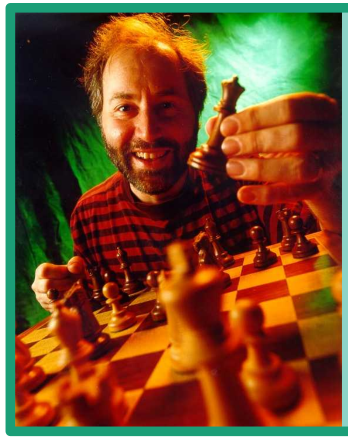

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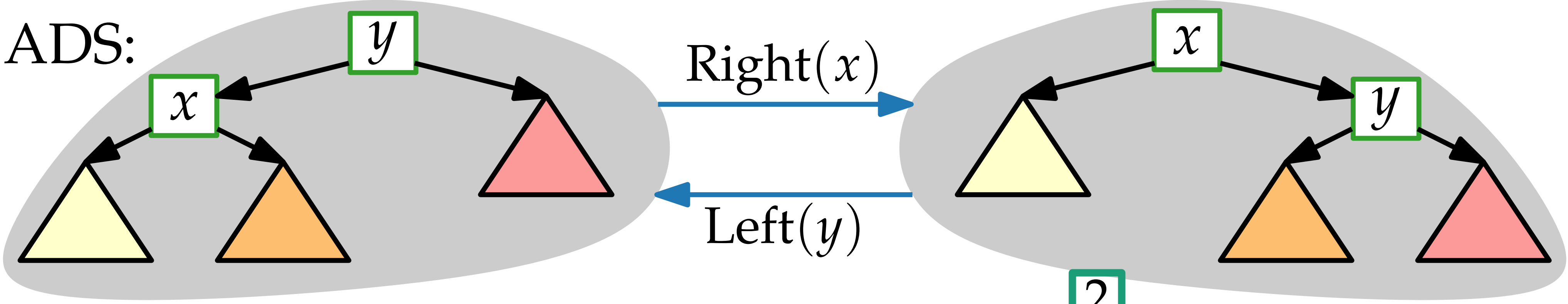

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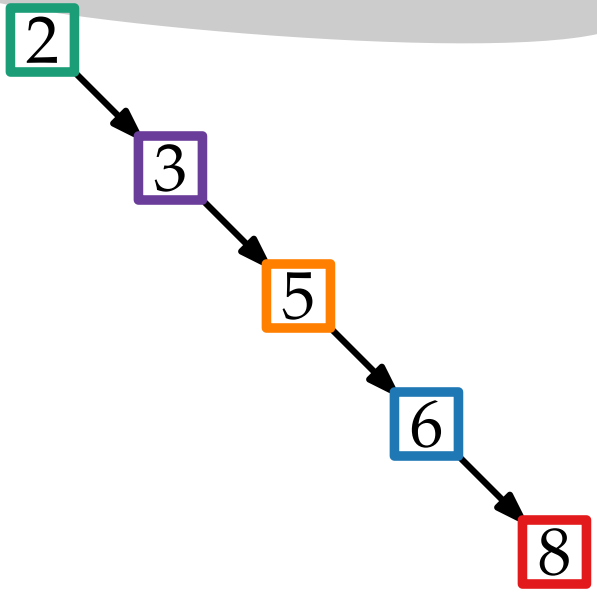
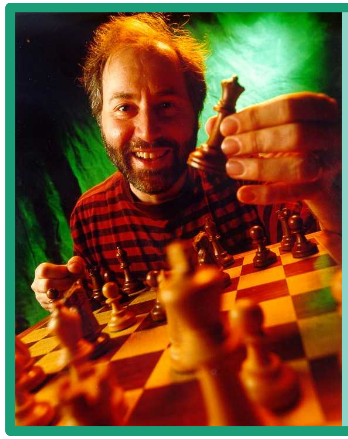


Query(8) Query(6) Query(5)
Query(3)
Query(2)

We're back at the start...

Splay Trees



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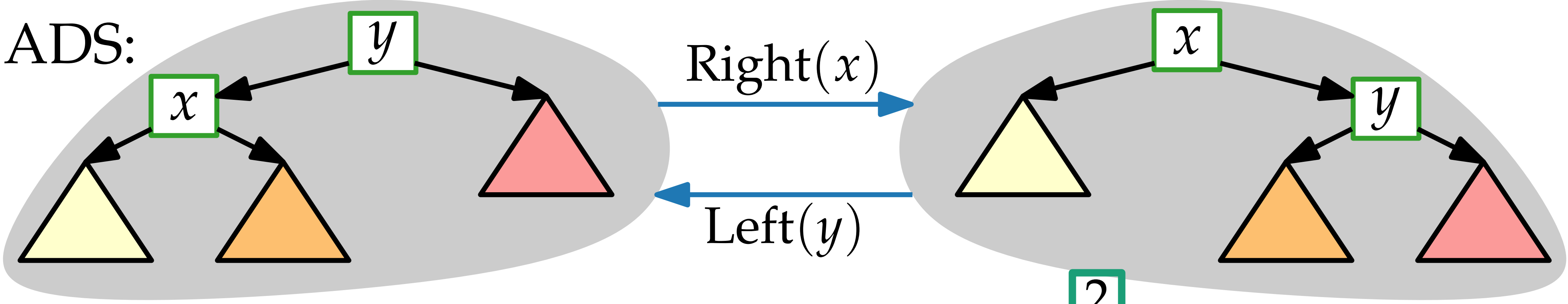

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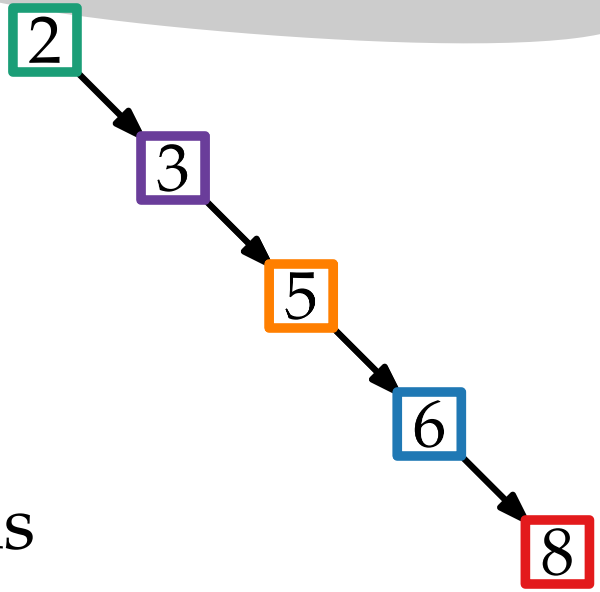
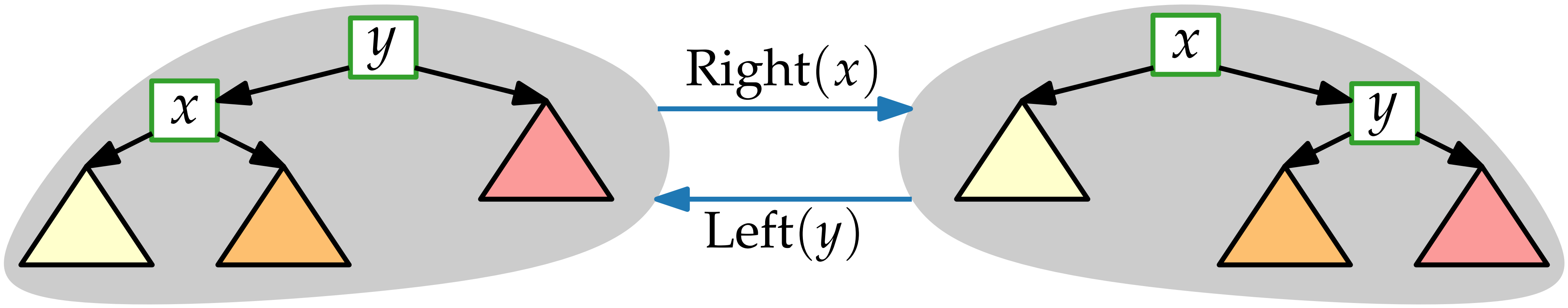
Query(8) Query(6) Query(5)
Query(3)
Query(2)

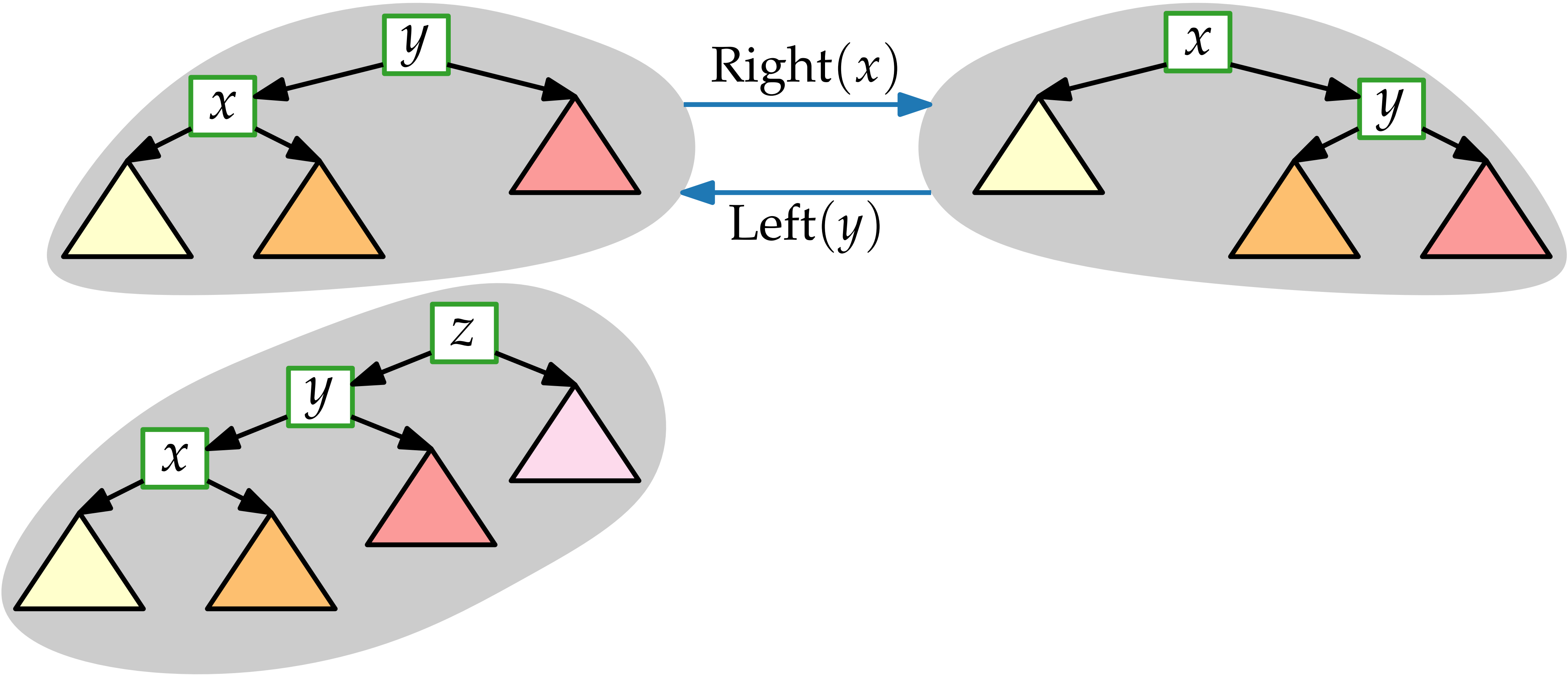
We're back at the start...
and we did $\Theta(n^2)$ rotations



Rotations II



Rotations II



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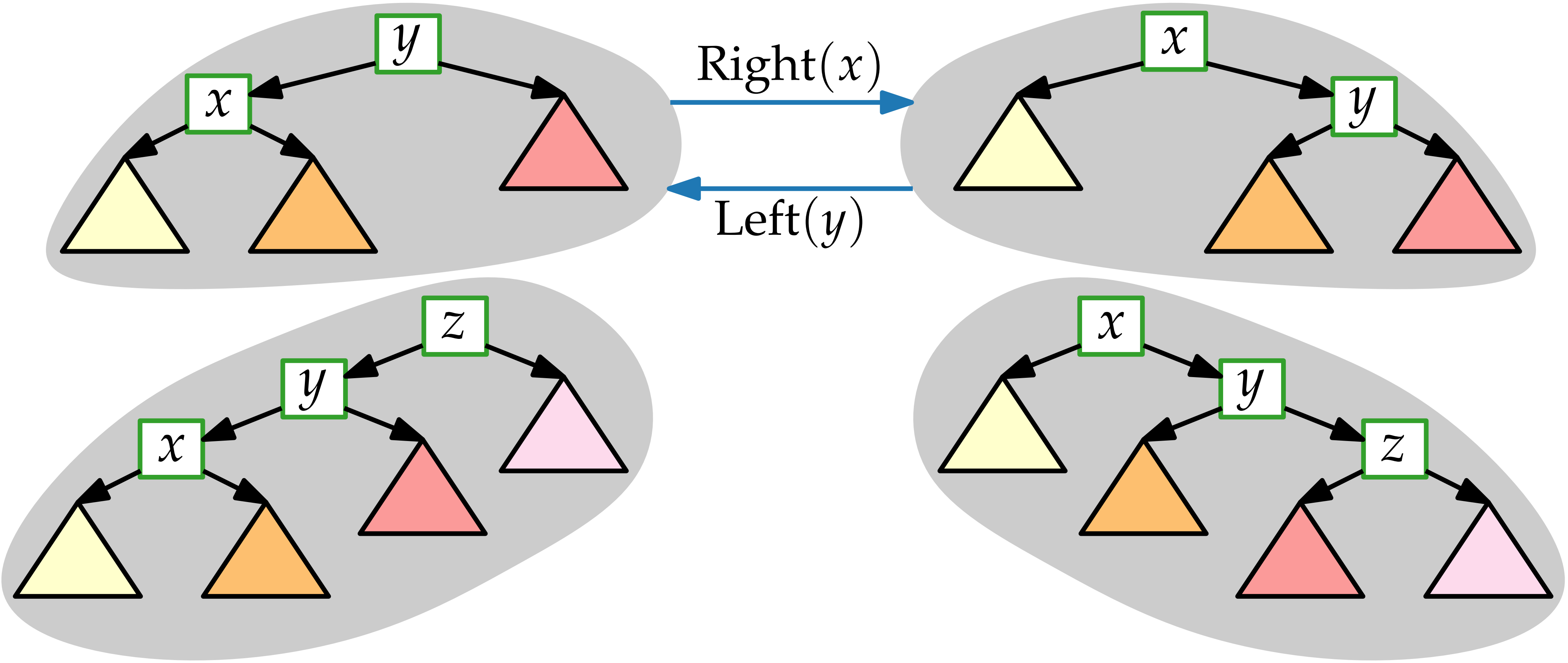

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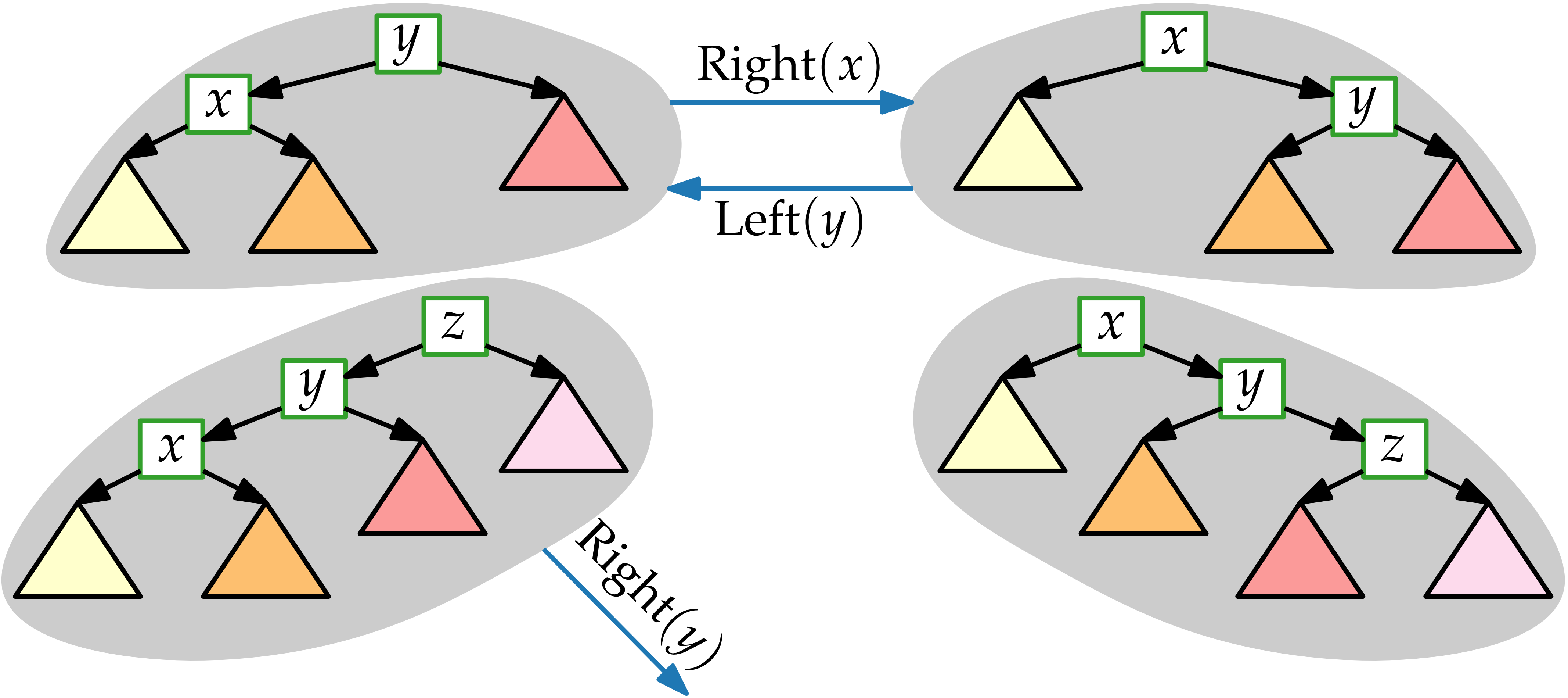

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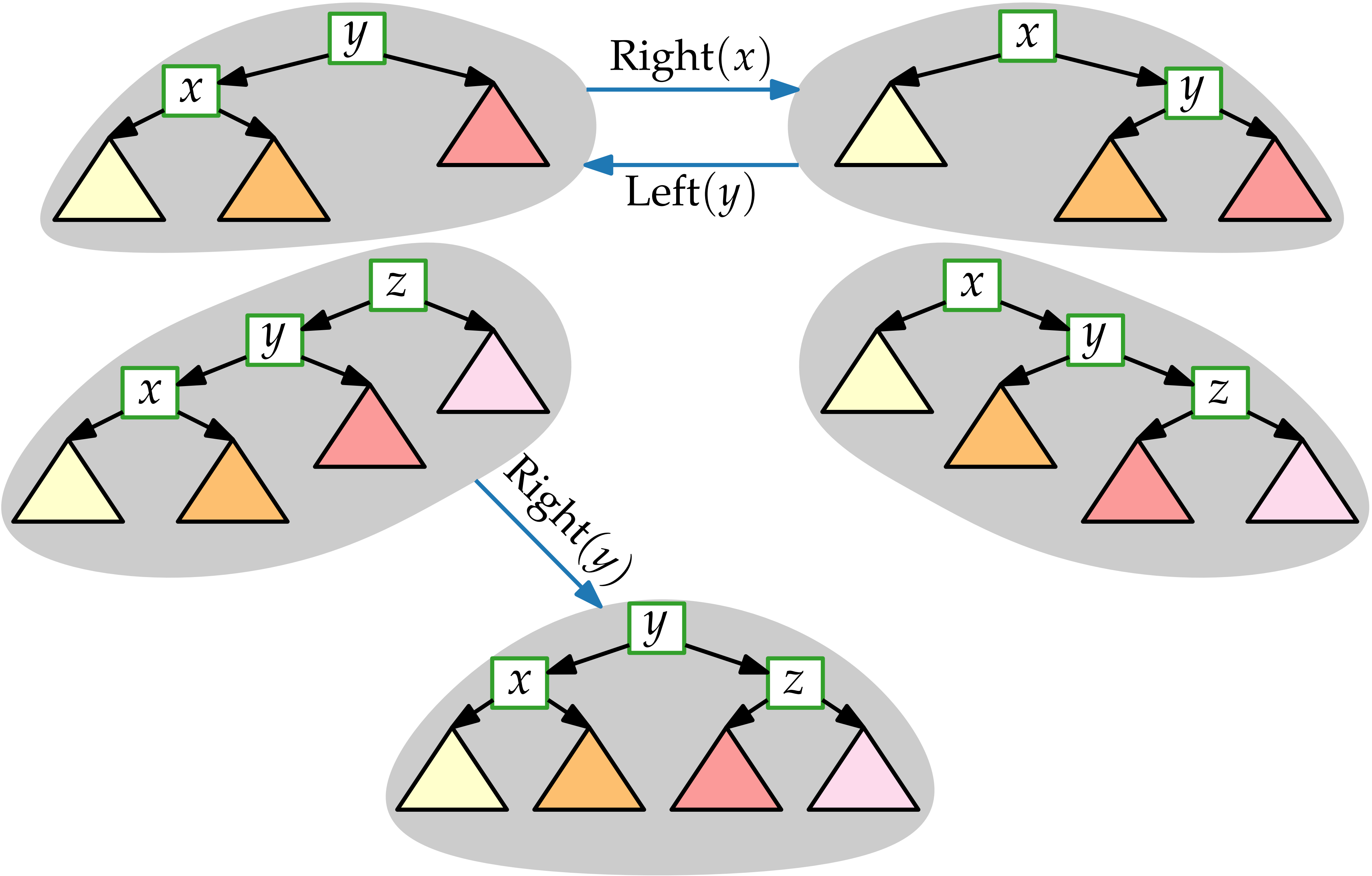

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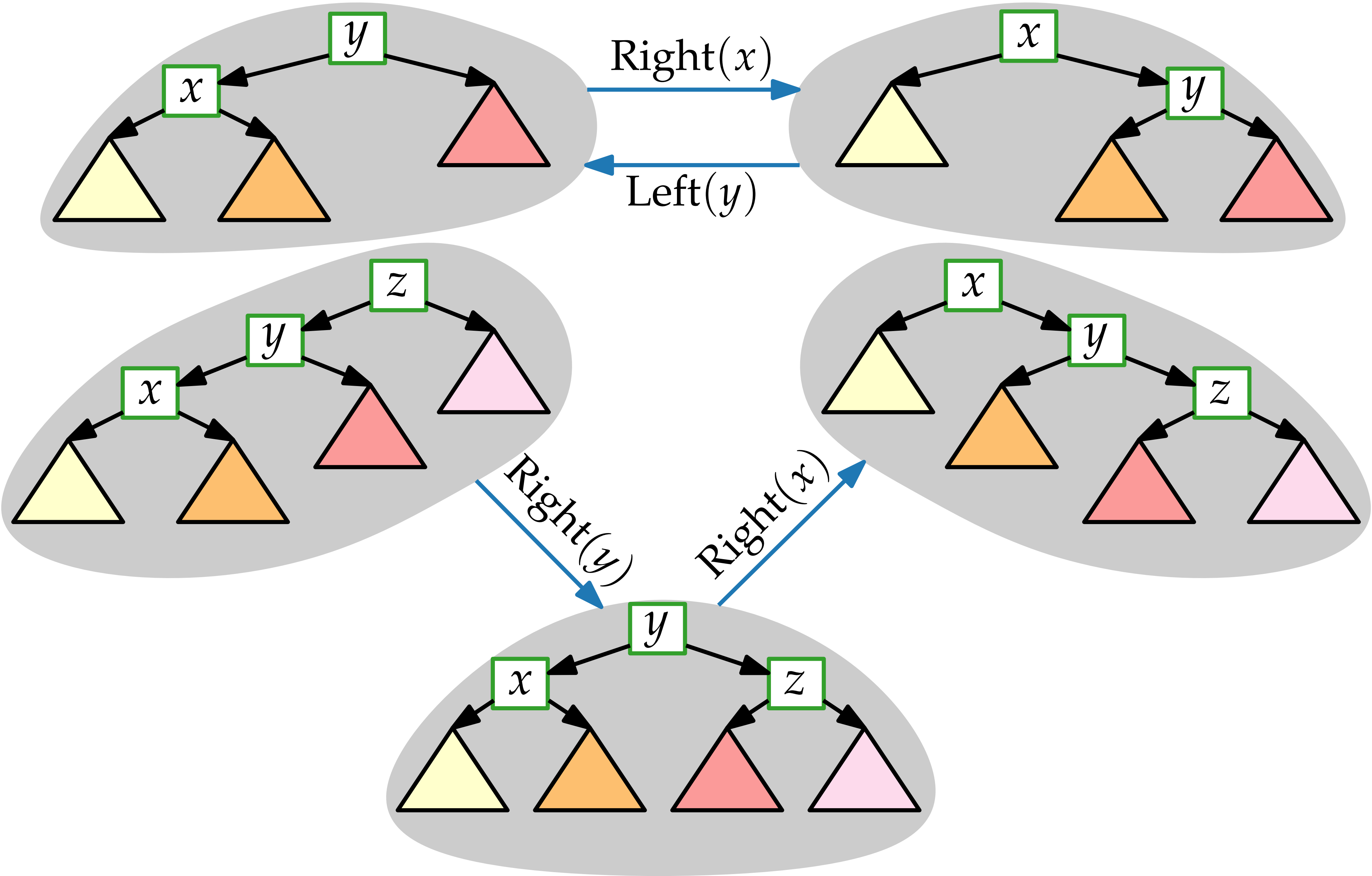
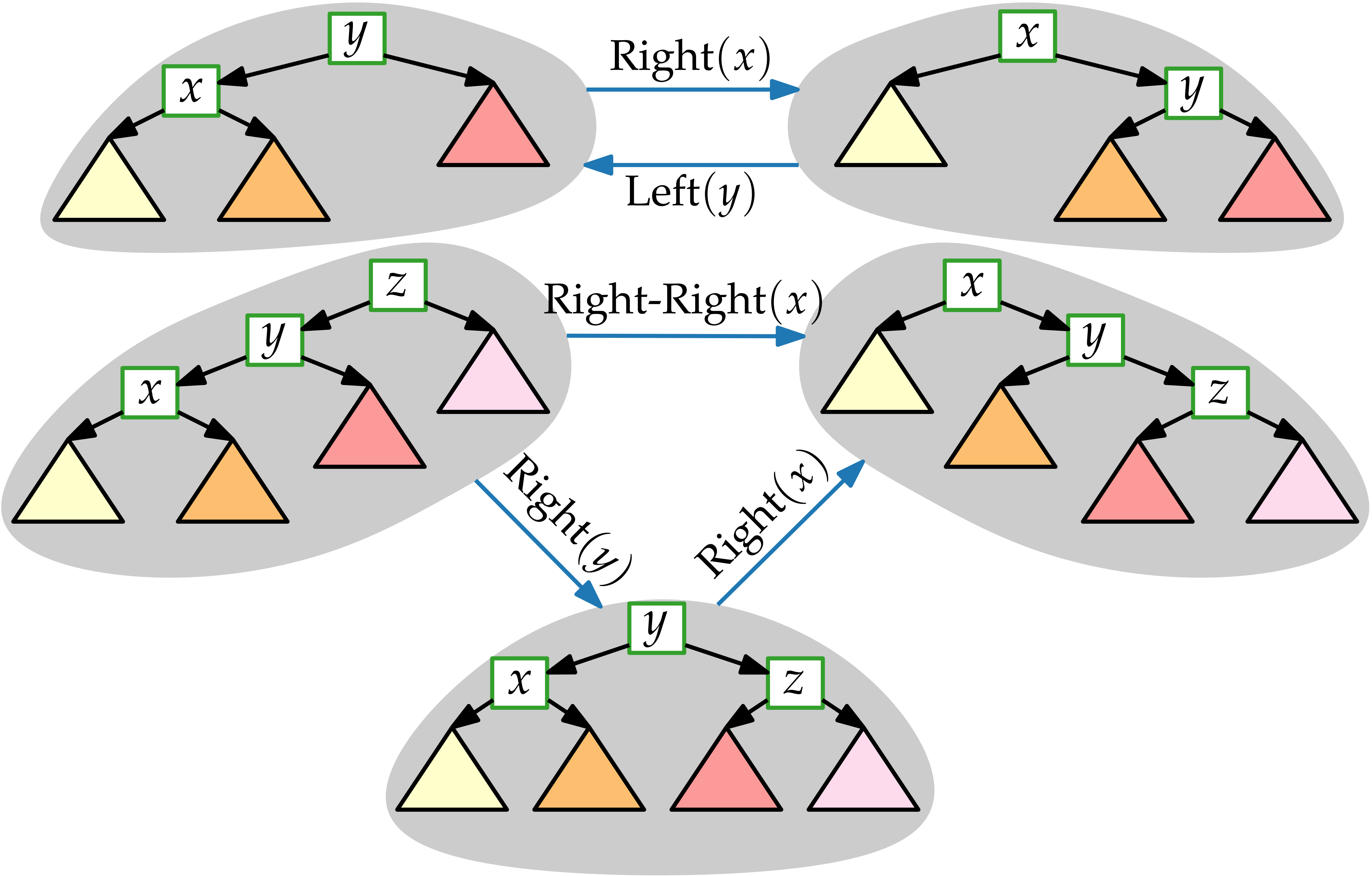
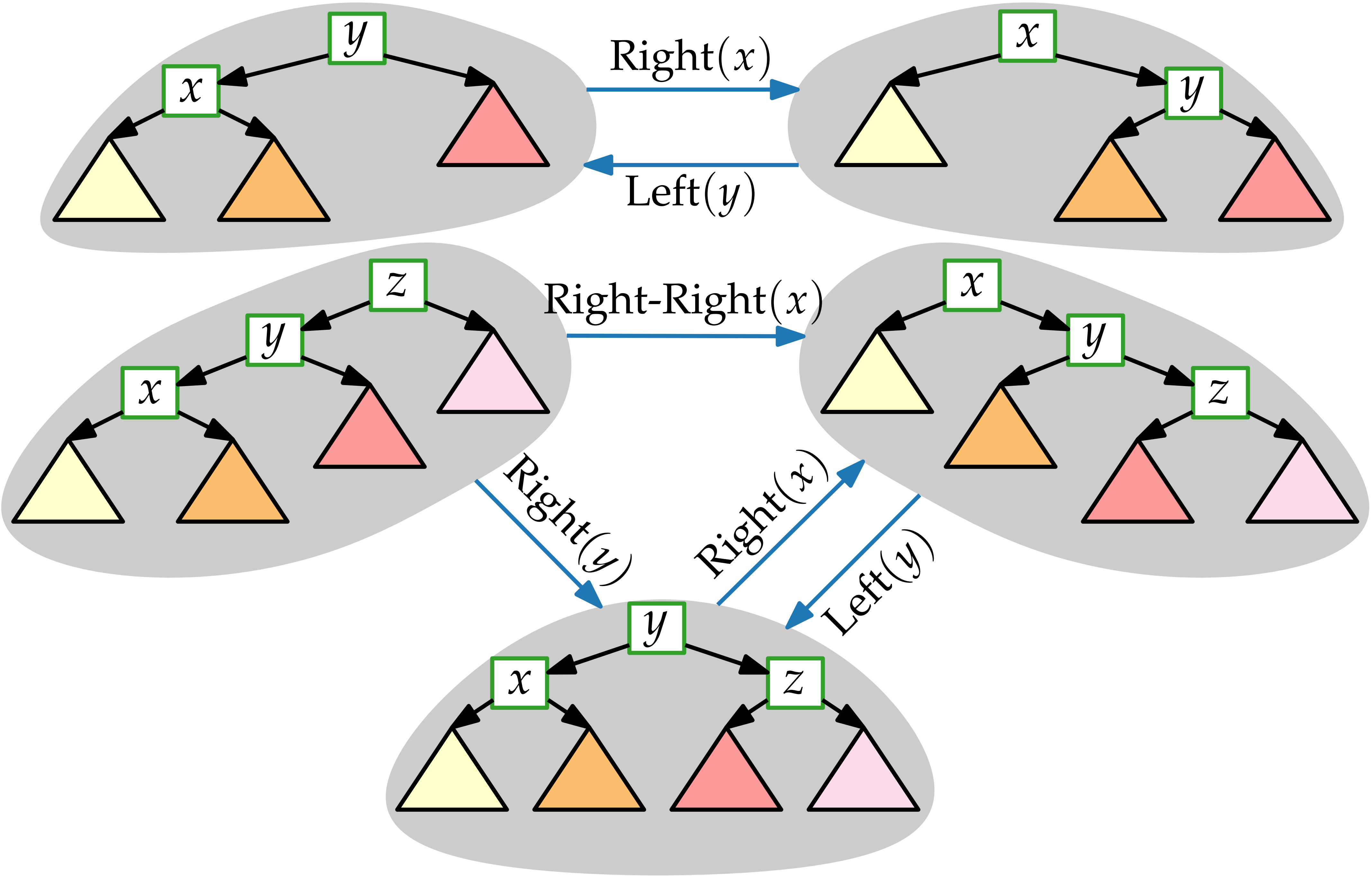


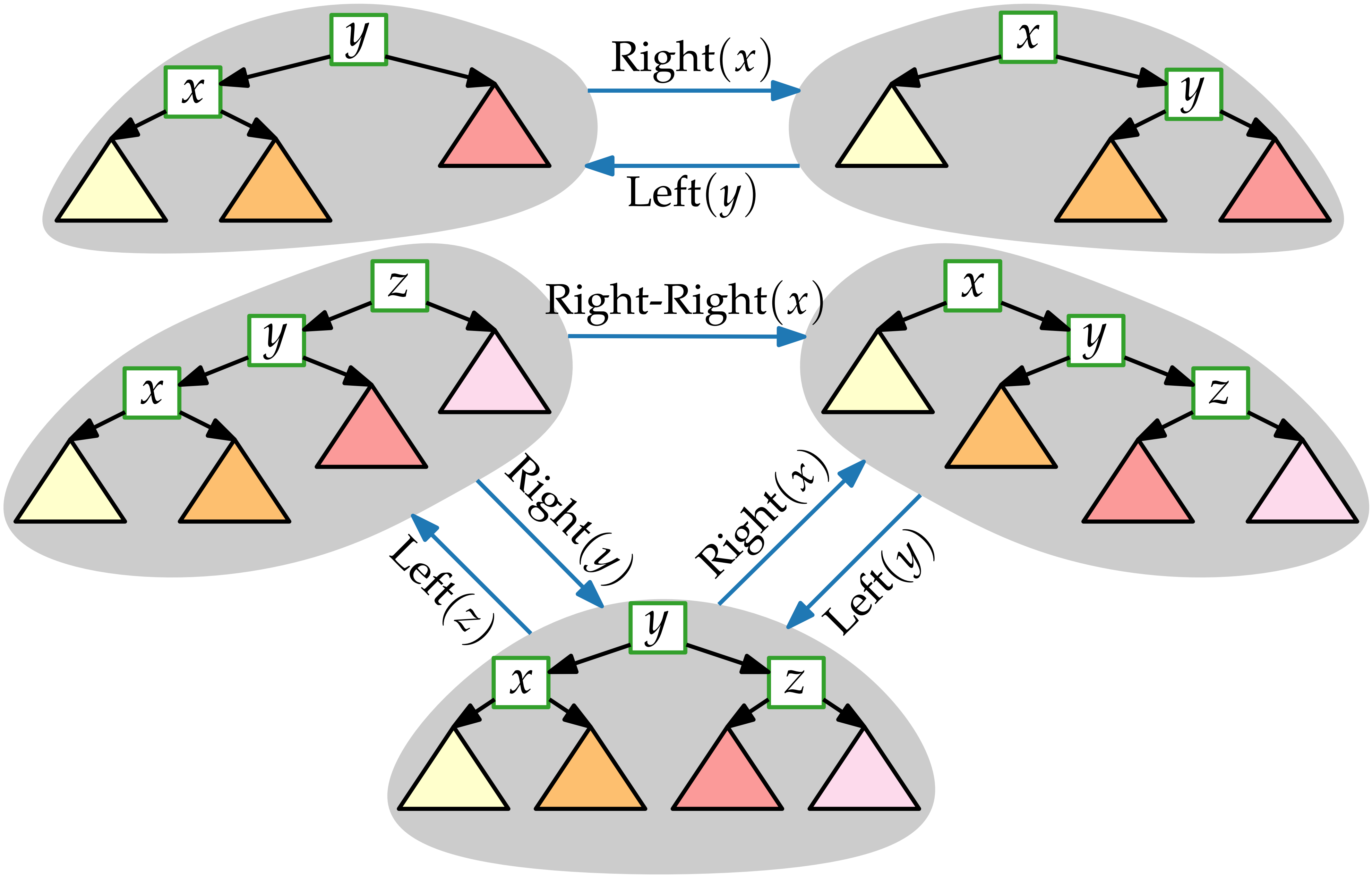
Rotations II



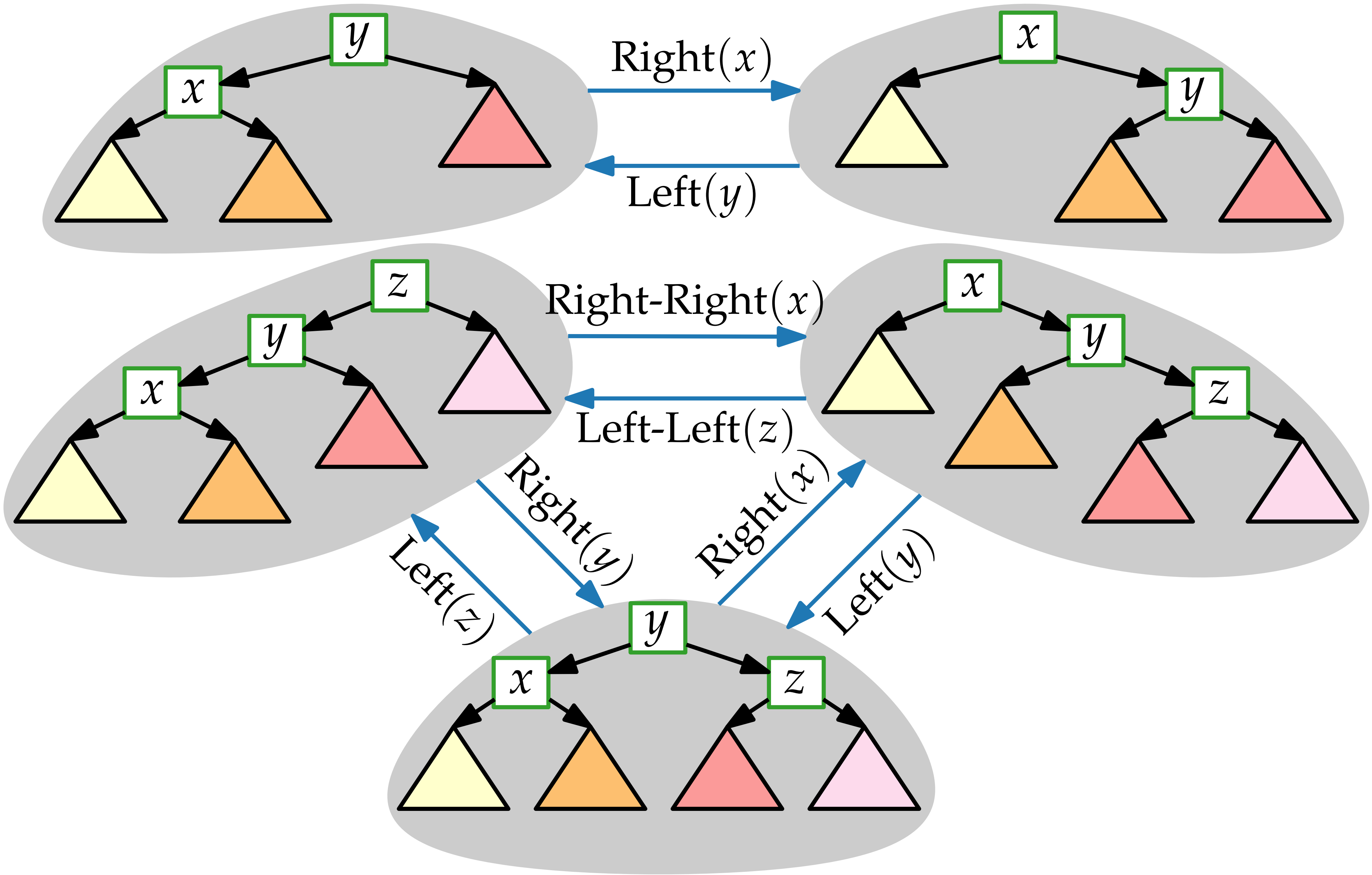
Rotations II



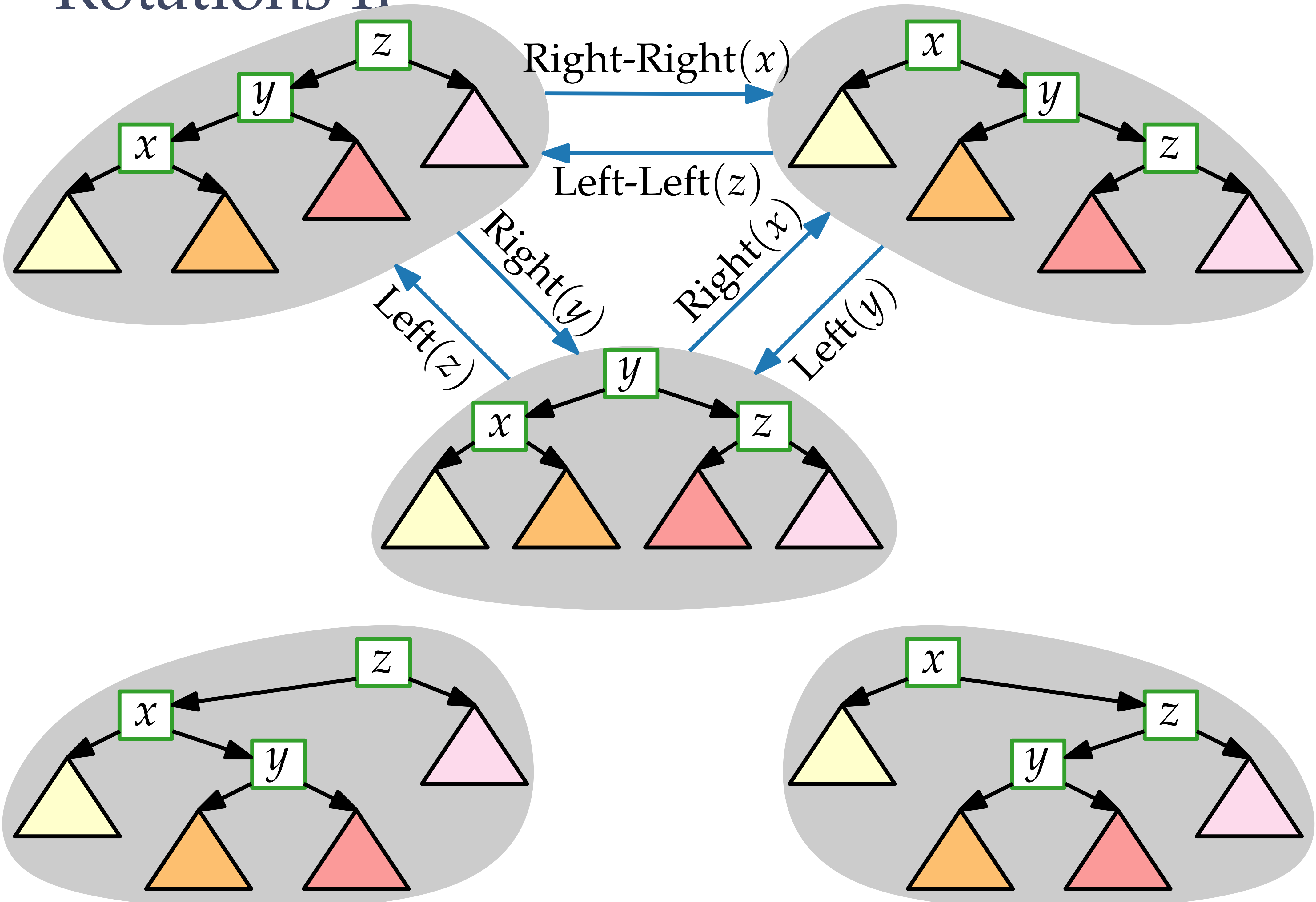
Rotations II



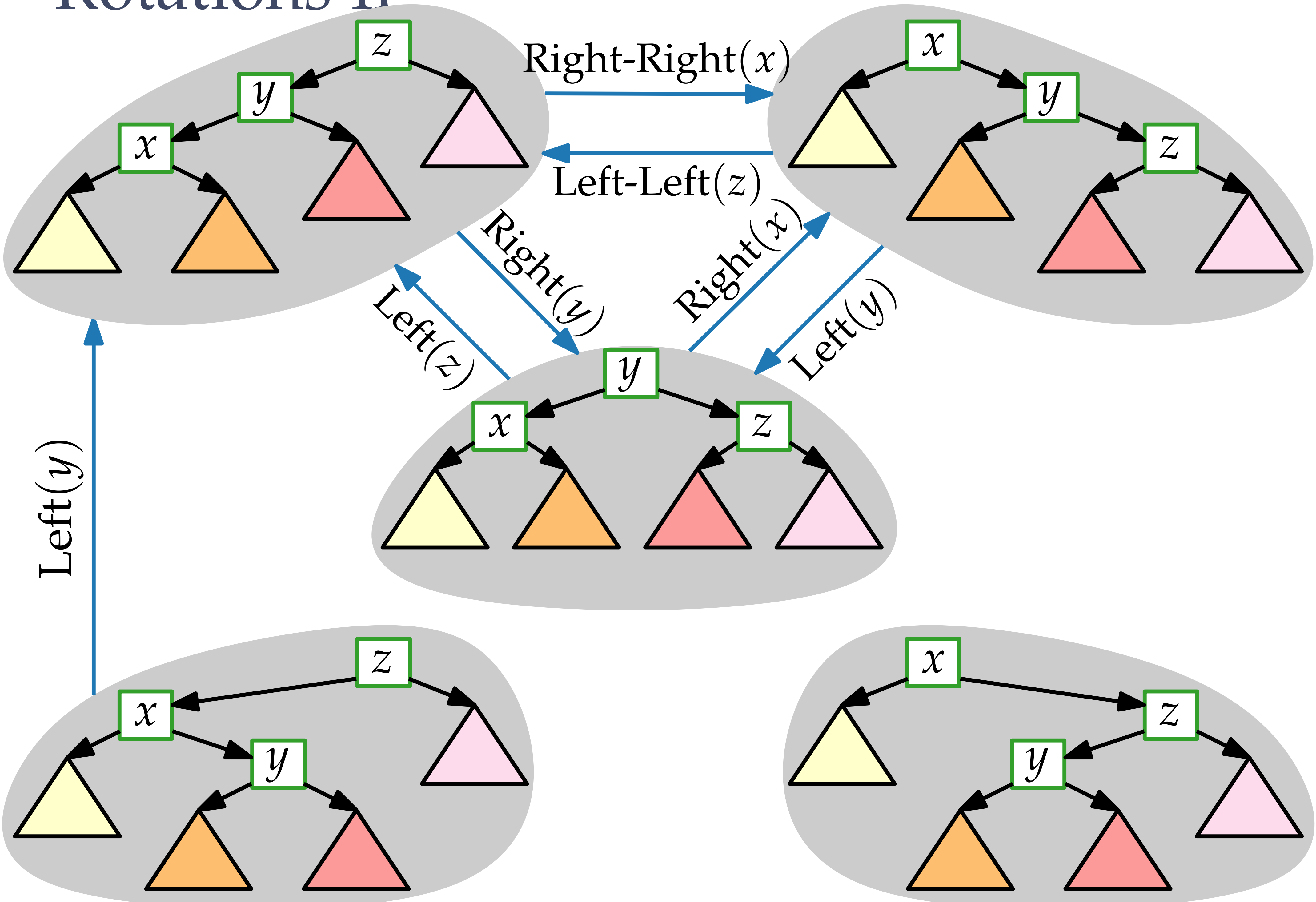
Rotations II



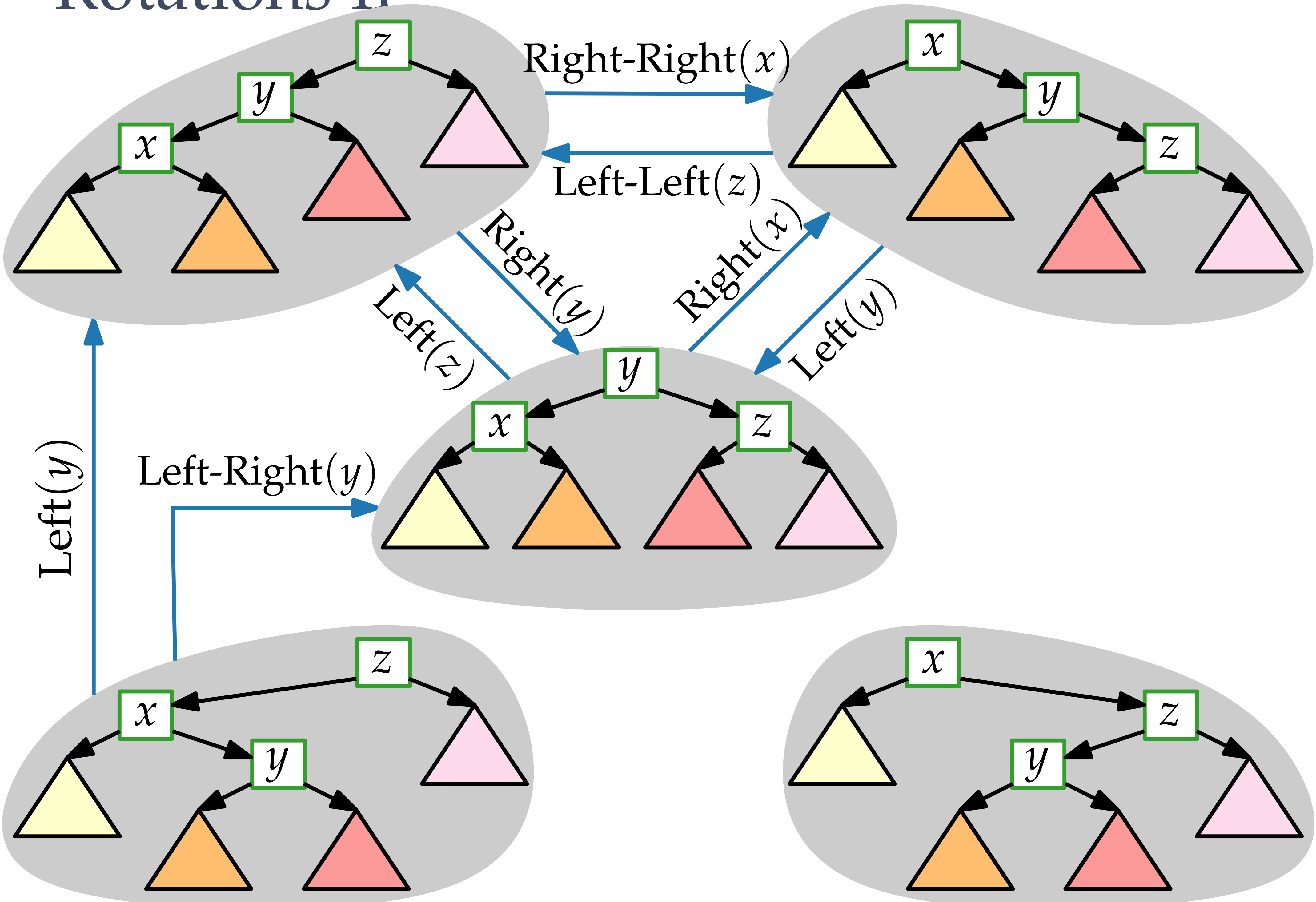
Rotations II



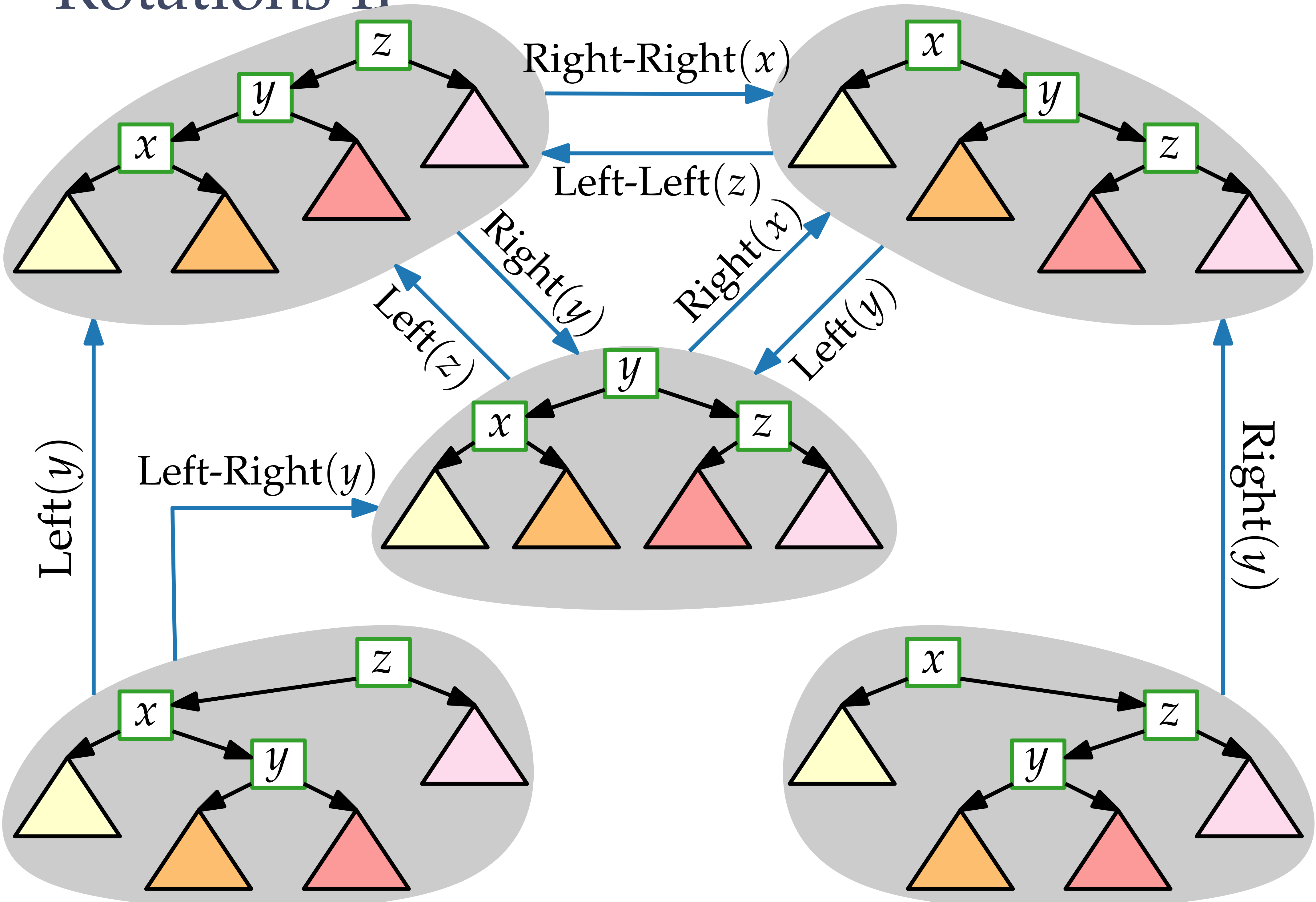
Rotations II



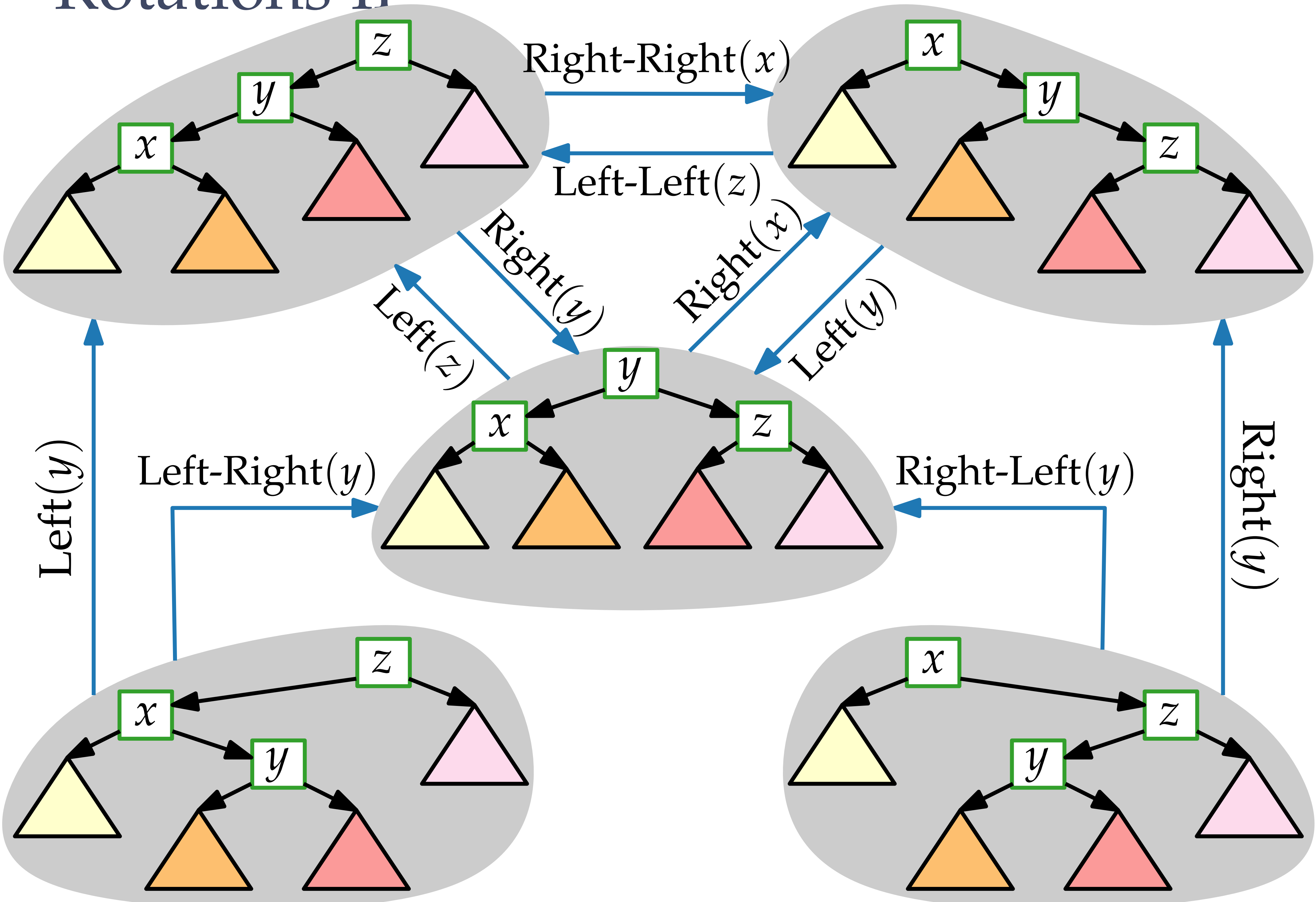
Rotations II



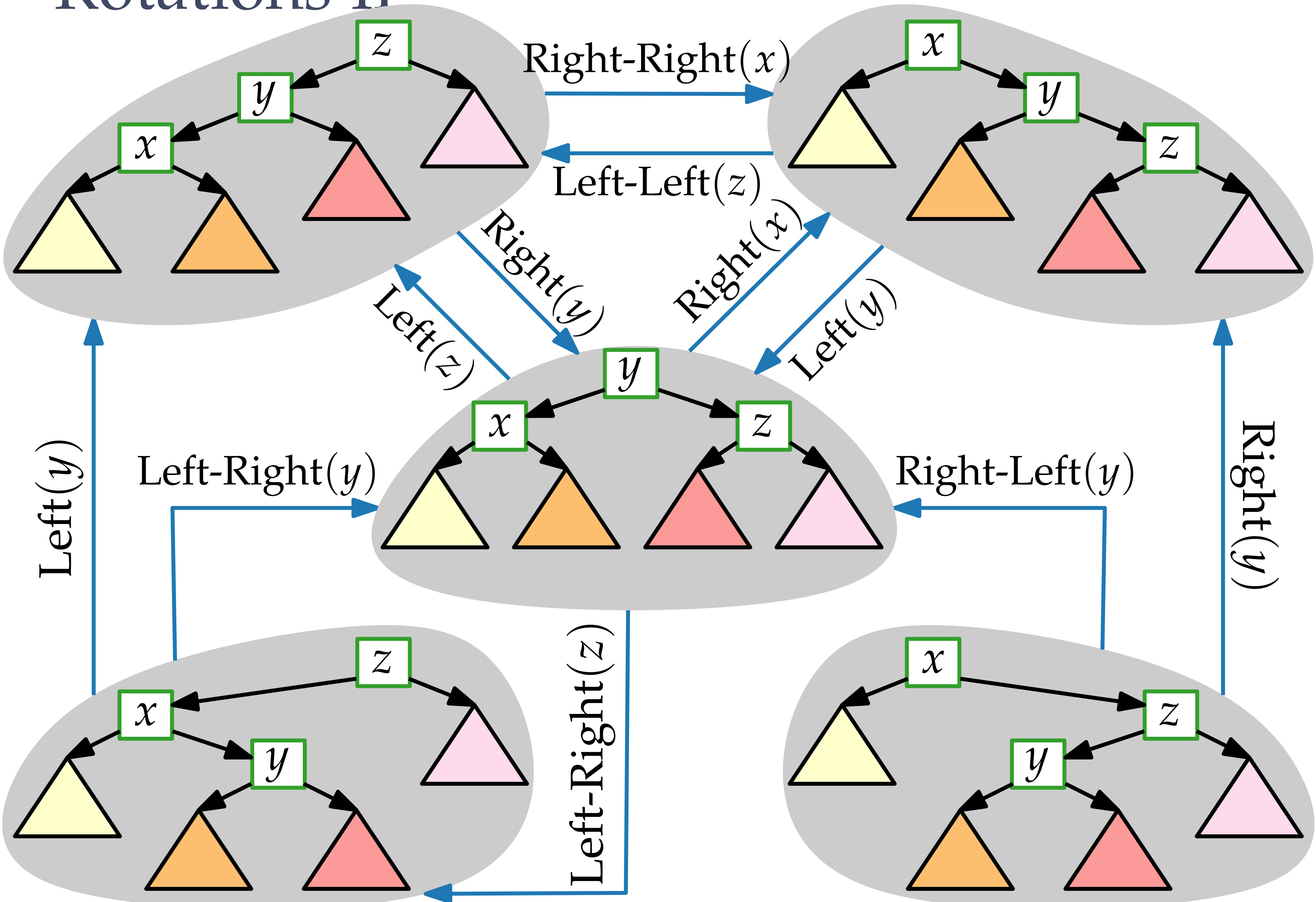
Rotations II



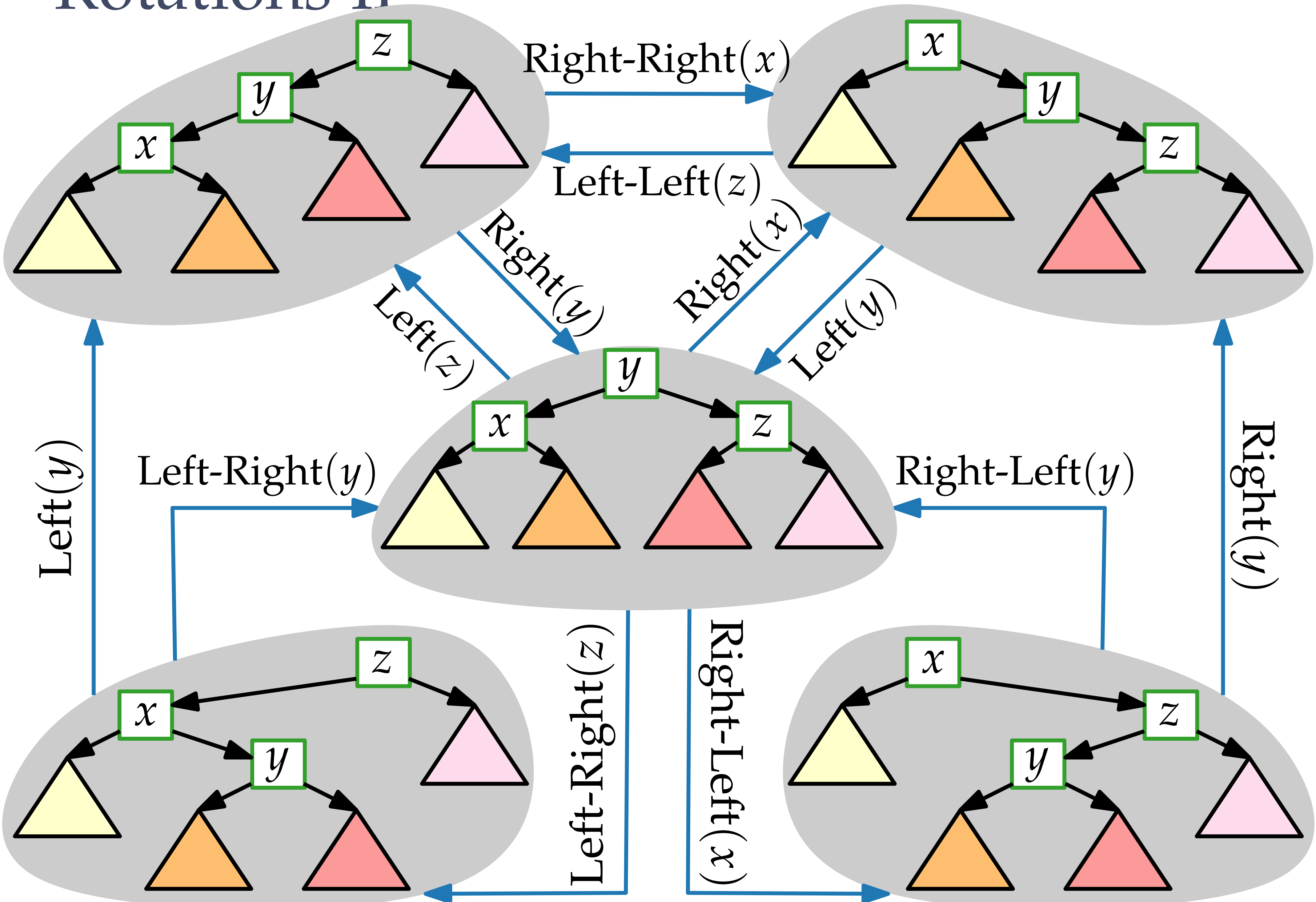
Rotations II



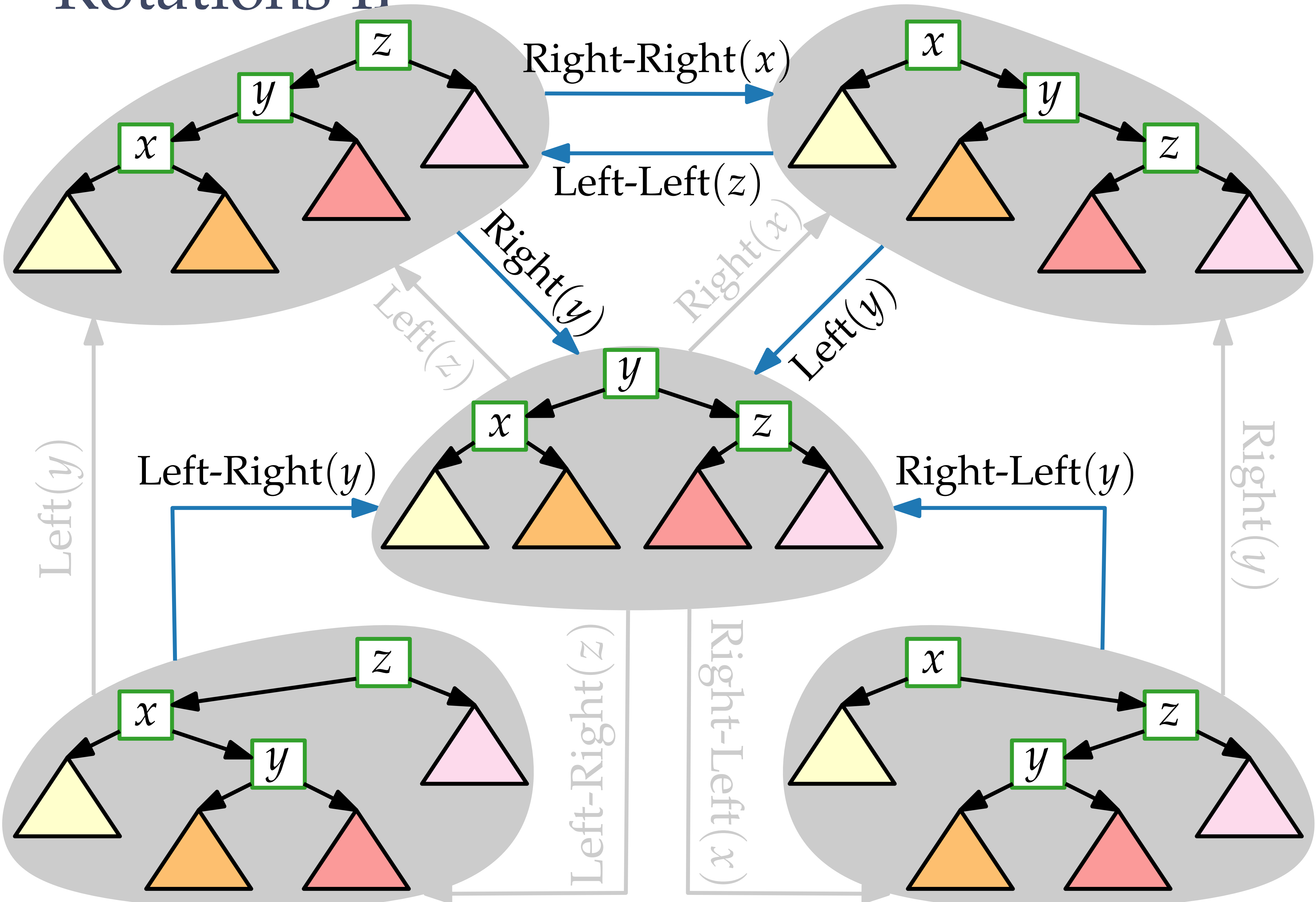
Rotations II



Rotations II



Rotations II



Splay

Algorithm: Splay(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

  y = parent of x

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

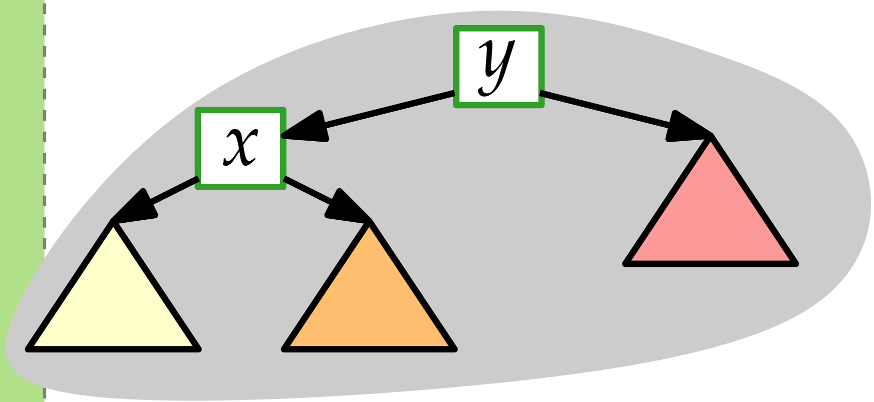
  y = parent of x

  if $y = root$ **then**

Splay

Algorithm: Splay(x)

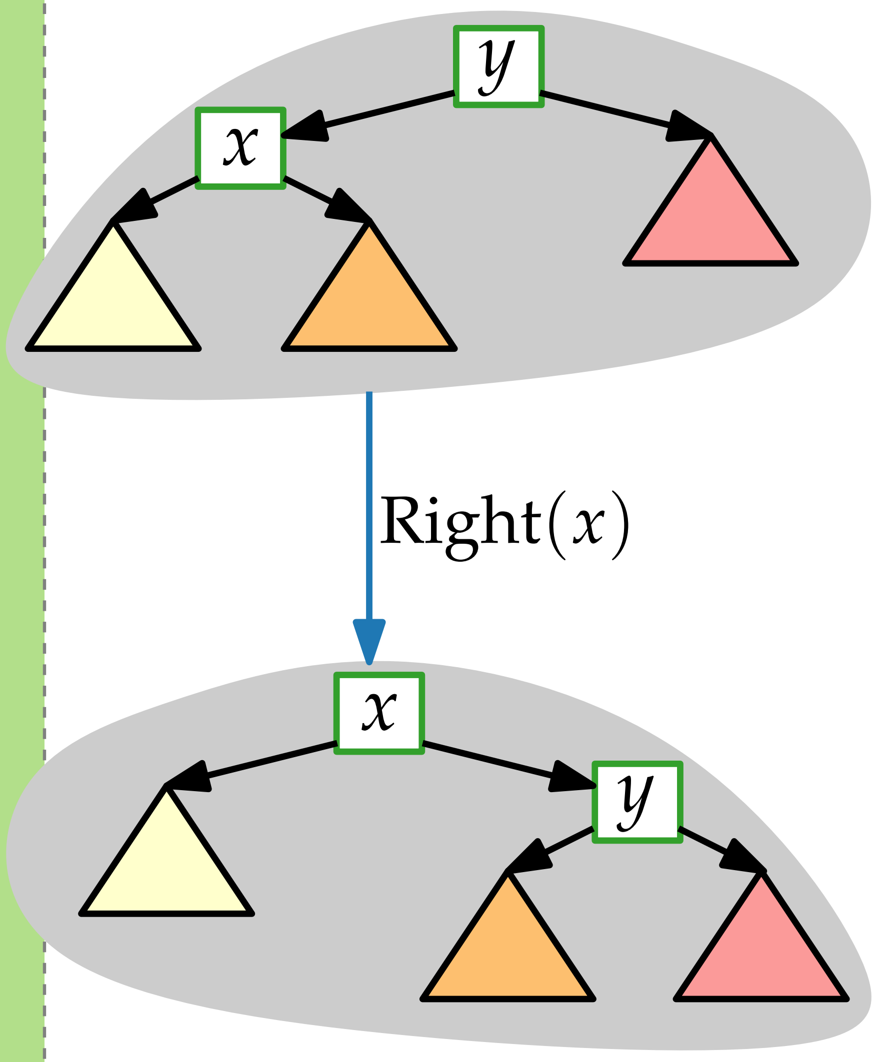
if $x \neq root$ **then**
  $y =$ parent of x
  if $y = root$ **then**
    if $x < y$ **then**



Splay

Algorithm: Splay(x)

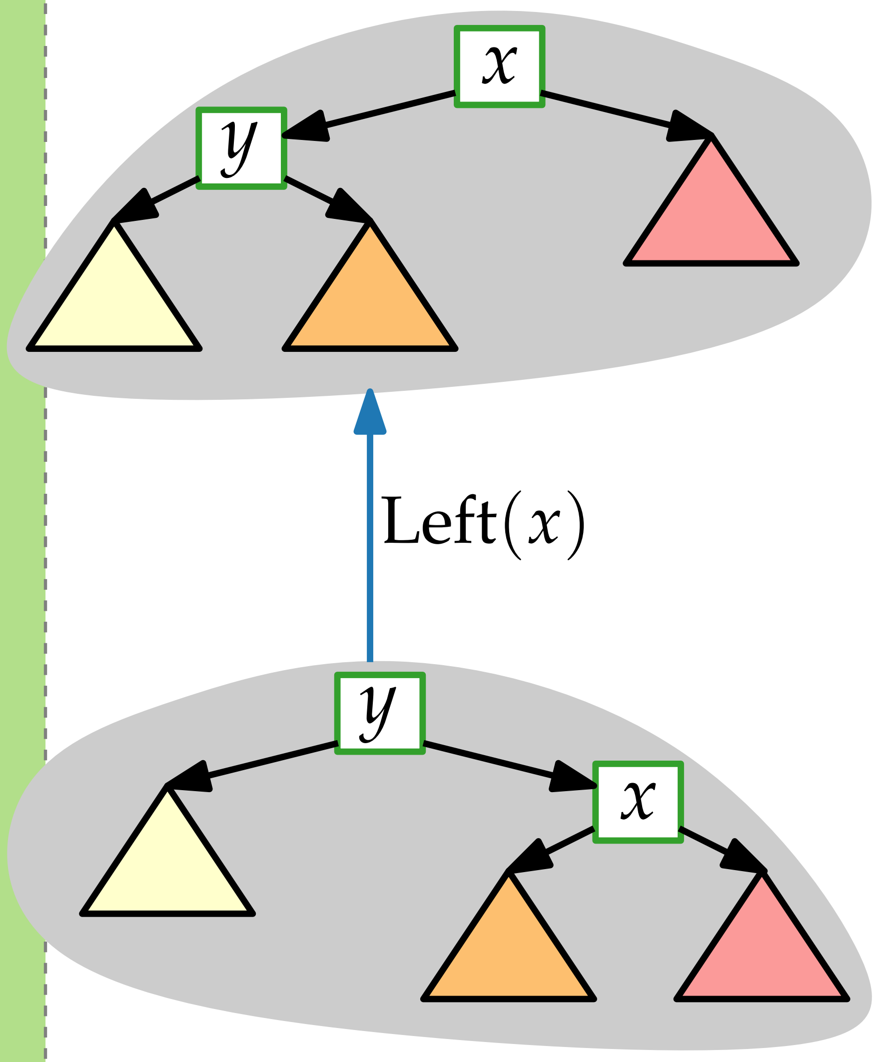
if $x \neq \mathit{root}$ **then**
  y = parent of x
  if $y = \mathit{root}$ **then**
    if $x < y$ **then** Right(x)



Splay

Algorithm: Splay(x)

if $x \neq \textit{root}$ **then**
- $y =$ parent of x
- **if** $y = \textit{root}$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)



Splay

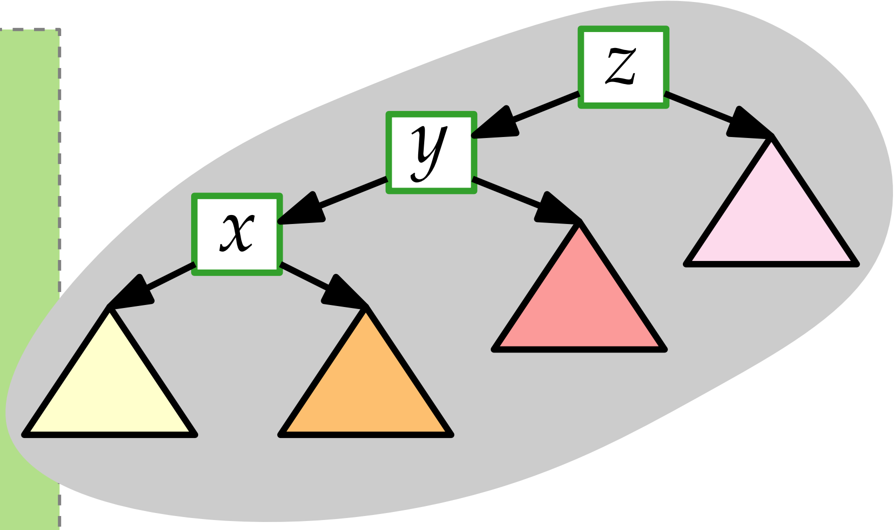
Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
    y = parent of x
    if $y = \mathit{root}$ **then**
        if $x < y$ **then** Right(x)
        if $y < x$ **then** Left(x)
    else
        z = parent of y

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
    $y =$ parent of x
    if $y = root$ **then**
        if $x < y$ **then** Right(x)
        if $y < x$ **then** Left(x)
    else
        $z =$ parent of y
        if $x < y < z$ **then**



Splay

Algorithm: Splay(x)

if $x \neq$ *root* **then**

  $y =$ parent of x

  if $y =$ *root* **then**

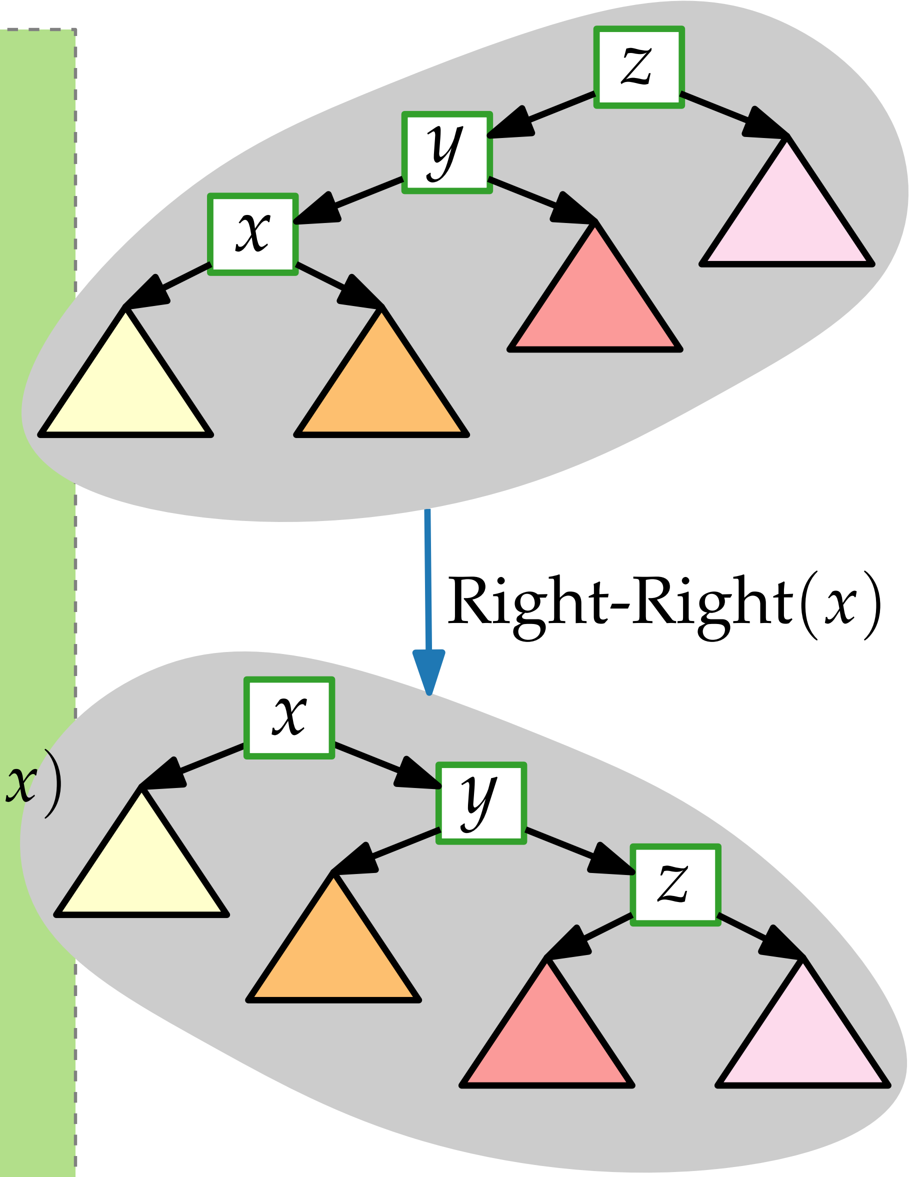
    if $x < y$ **then** Right(x)

    if $y < x$ **then** Left(x)

  else

    $z =$ parent of y

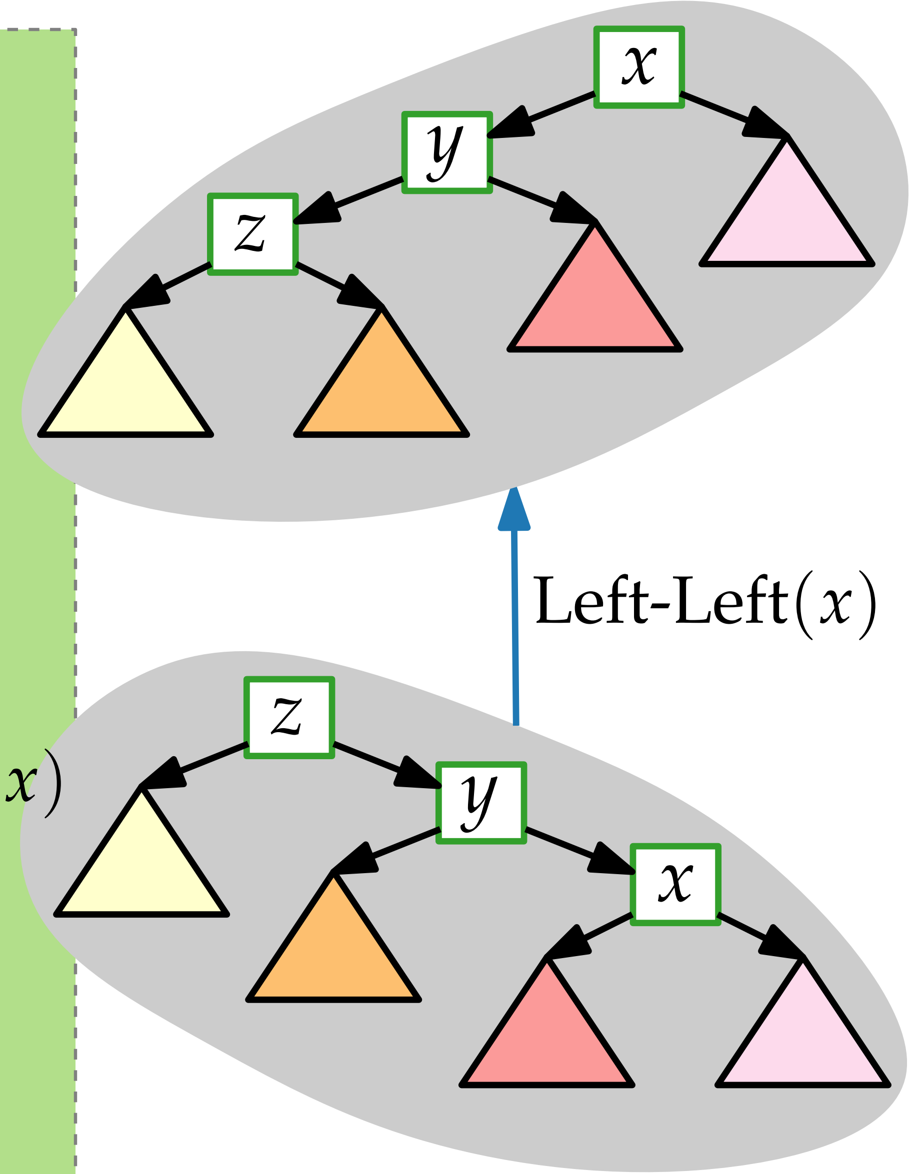
    if $x < y < z$ **then** Right-Right(x)



Splay

Algorithm: Splay(x)

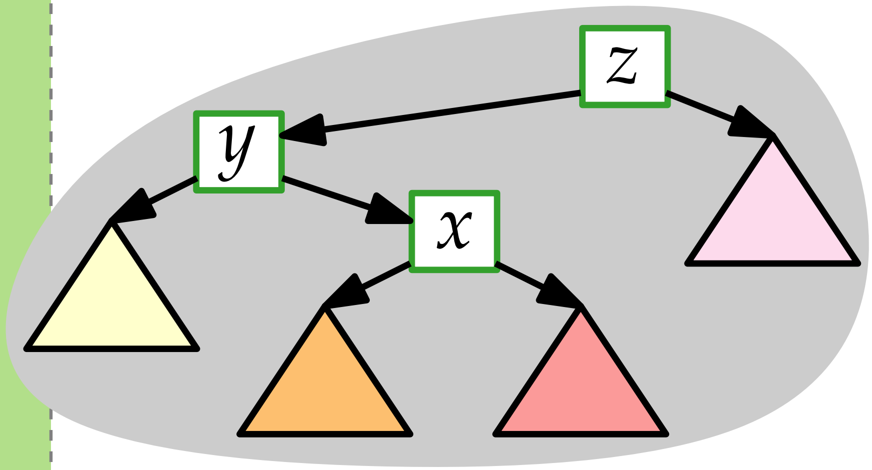
if $x \neq root$ **then**
  $y =$ parent of x
  if $y = root$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
  else
    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)



Splay

Algorithm: Splay(x)

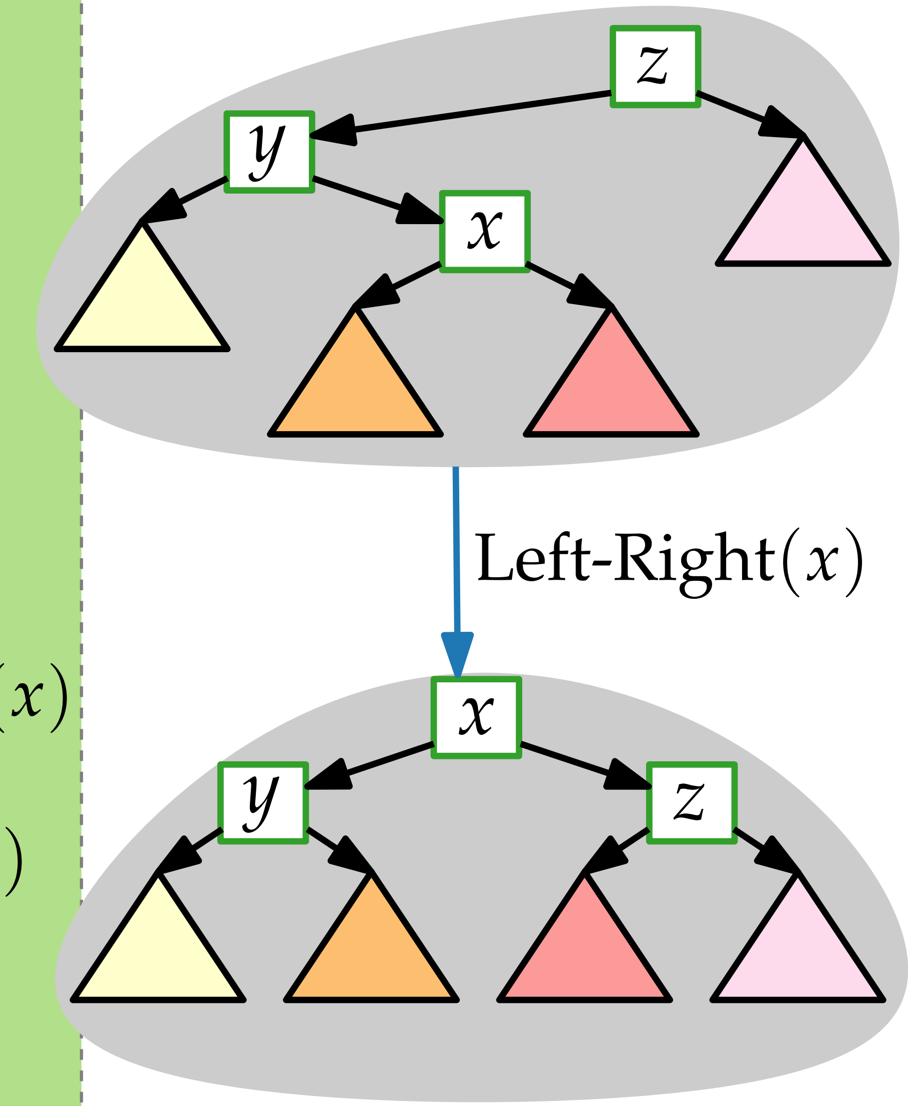
if $x \neq \mathit{root}$ **then**
  y = parent of x
  if $y = \mathit{root}$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
  else
    z = parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)
    if $y < x < z$ **then**



Splay

Algorithm: Splay(x)

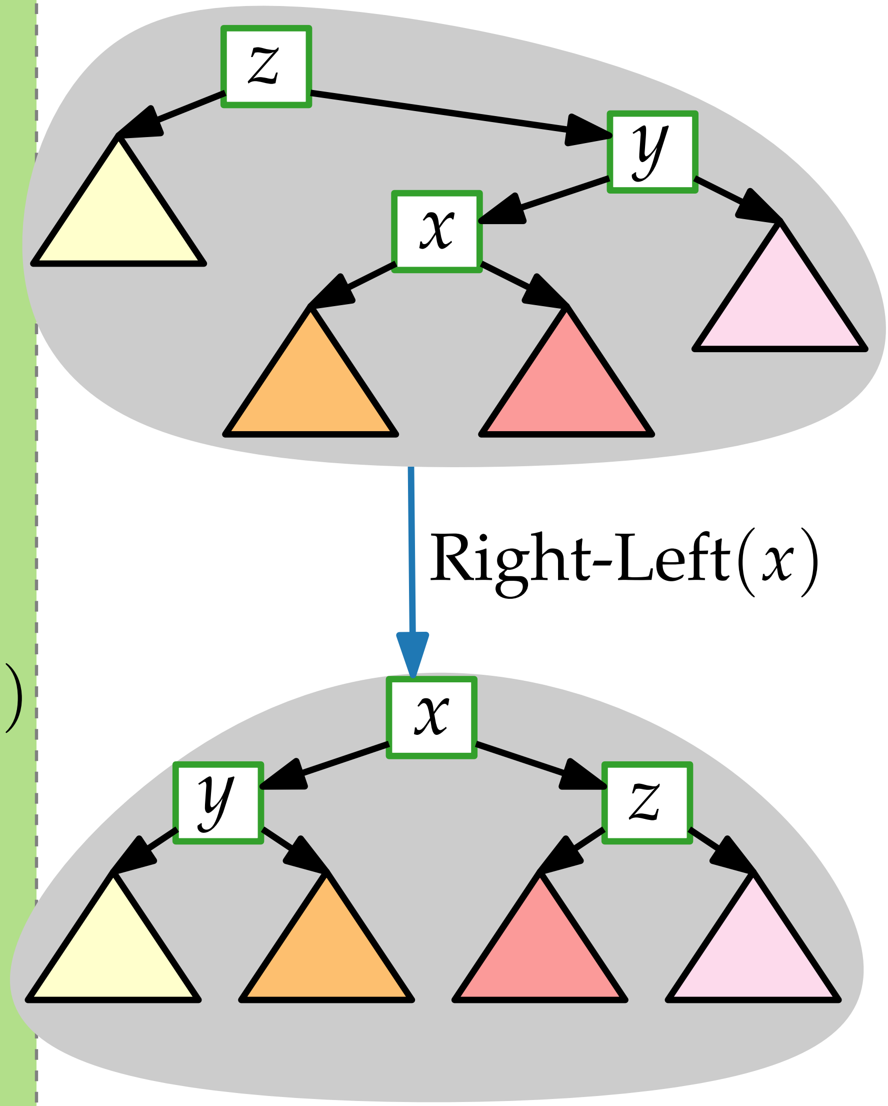
if $x \neq \mathit{root}$ **then**
  $y =$ parent of x
  if $y = \mathit{root}$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
  else
    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)
    if $y < x < z$ **then** Left-Right(x)



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- y = parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - z = parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)



Splay

Algorithm: Splay(x)

if $x \neq \mathit{root}$ **then**
- $y =$ parent of x
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  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

    $y =$ parent of x

    if $y = root$ **then**

        if $x < y$ **then** Right(x)

        if $y < x$ **then** Left(x)

    else

        $z =$ parent of y

        if $x < y < z$ **then** Right-Right(x)

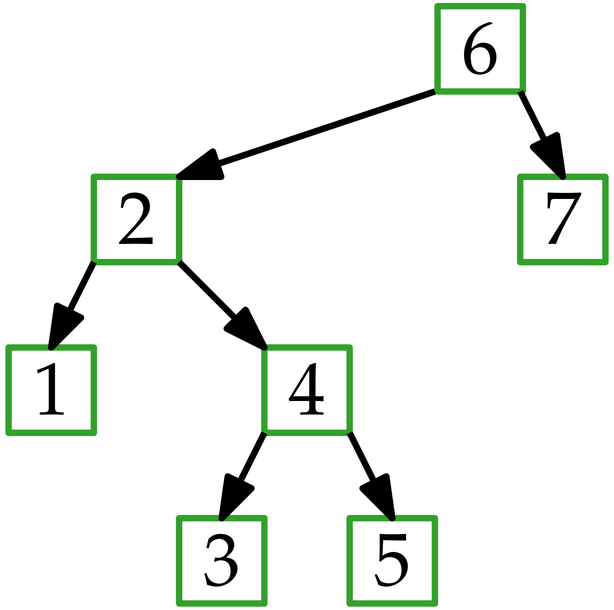
        if $z < y < x$ **then** Left-Left(x)

        if $y < x < z$ **then** Left-Right(x)

        if $z < x < y$ **then** Right-Left(x)

    Splay(x)

Splay(3):

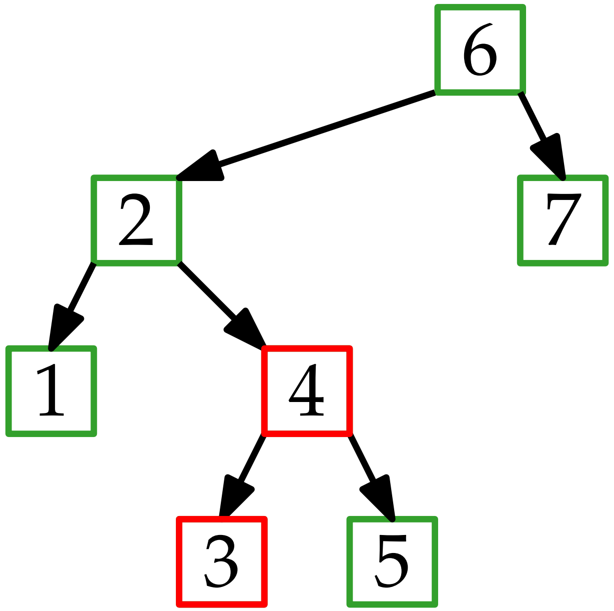


Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- y = parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - z = parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
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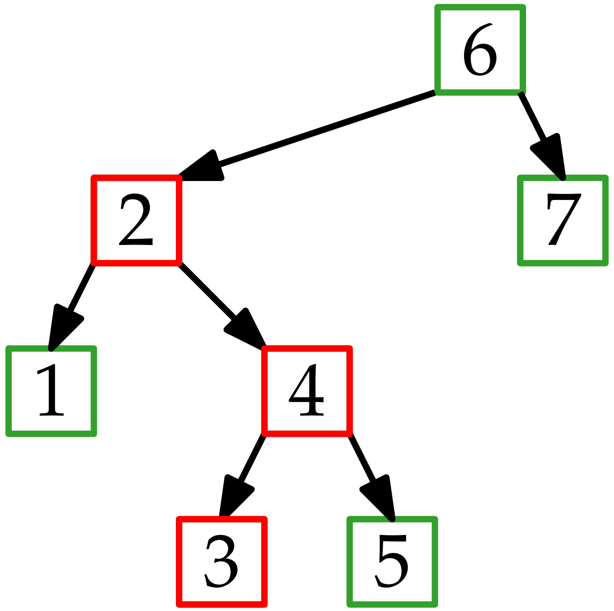
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq \textit{root}$ **then**
- $y =$ parent of x
- **if** $y = \textit{root}$ **then**
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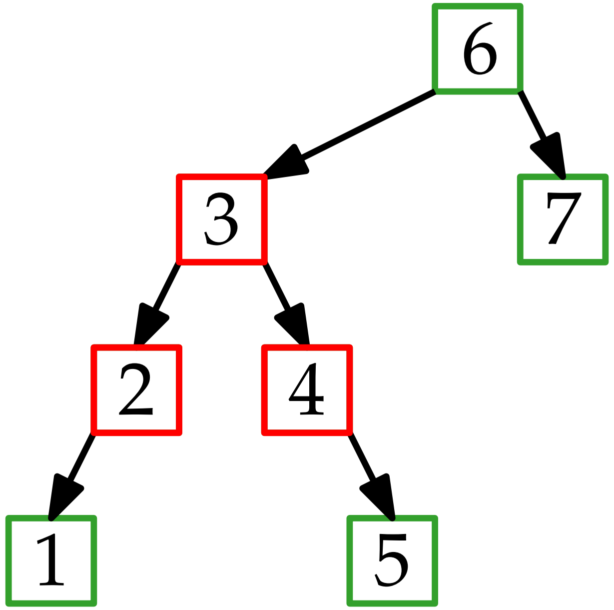
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- $y =$ parent of x
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  - **if** $y < x < z$ **then** Left-Right(x)
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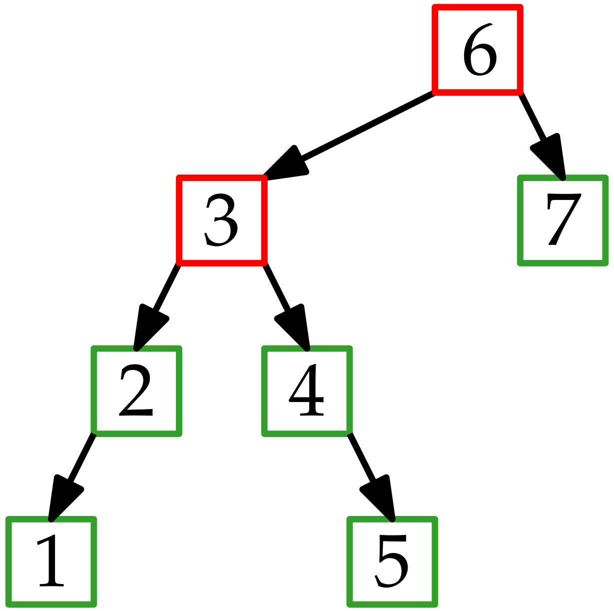
- Splay(x)

Splay(3):



Splay

Algorithm: Splay(x)

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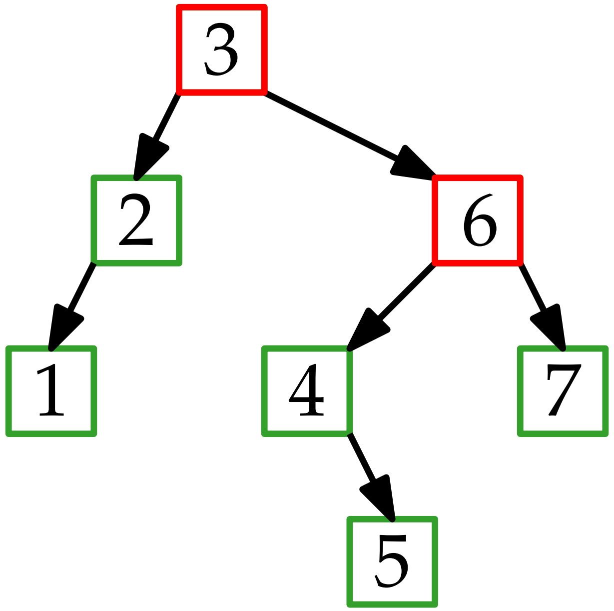
Splay(3):



Splay

Algorithm: Splay(x)

if $x \neq root$ **then**

  $y =$ parent of x

  if $y = root$ **then**

    if $x < y$ **then** Right(x)

    if $y < x$ **then** Left(x)

  else

    $z =$ parent of y

    if $x < y < z$ **then** Right-Right(x)

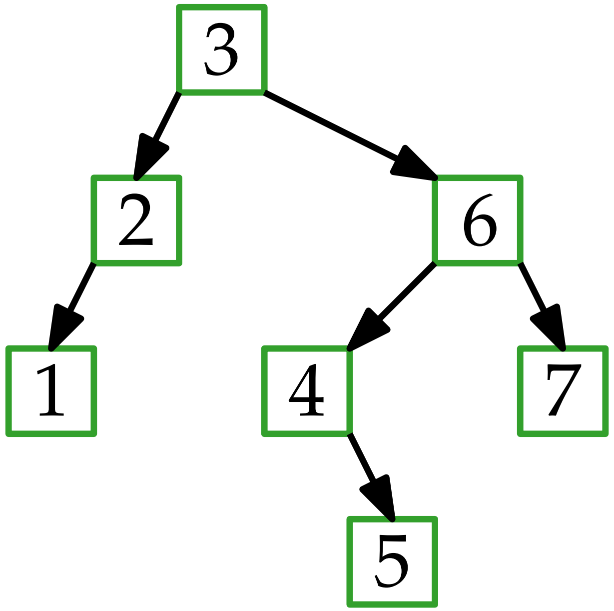
    if $z < y < x$ **then** Left-Left(x)

    if $y < x < z$ **then** Left-Right(x)

    if $z < x < y$ **then** Right-Left(x)

  Splay(x)

Splay(3):



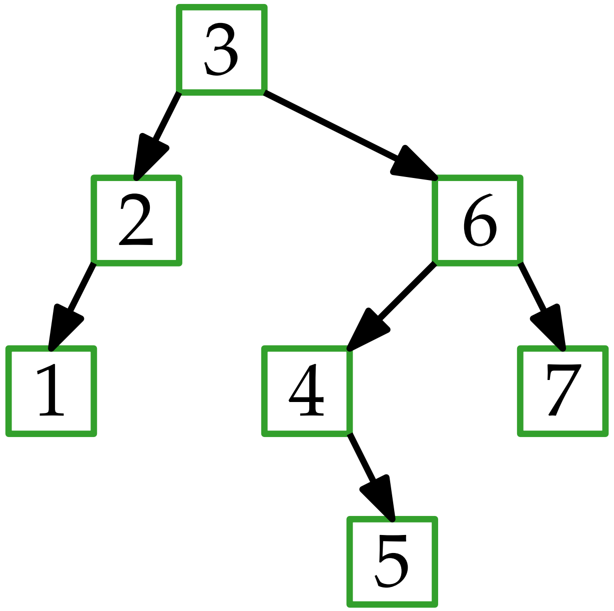
Call Splay(x):

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- y = parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - z = parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



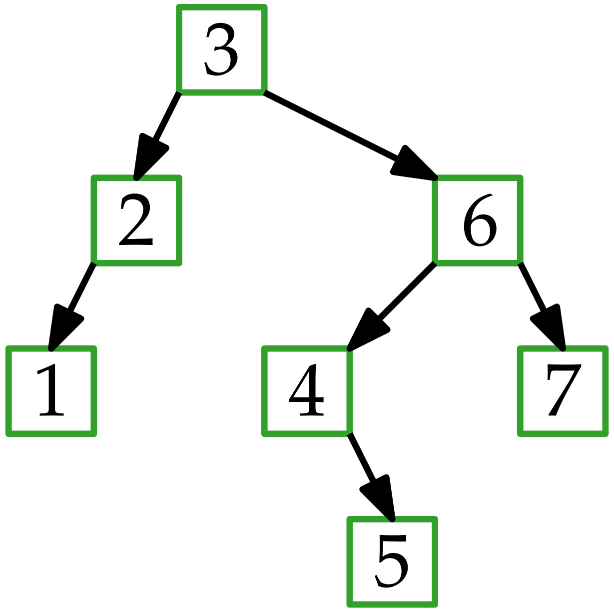
Call Splay(x):
- after Search(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
- $y =$ parent of x
- **if** $y = root$ **then**
  - **if** $x < y$ **then** Right(x)
  - **if** $y < x$ **then** Left(x)
- **else**
  - $z =$ parent of y
  - **if** $x < y < z$ **then** Right-Right(x)
  - **if** $z < y < x$ **then** Left-Left(x)
  - **if** $y < x < z$ **then** Left-Right(x)
  - **if** $z < x < y$ **then** Right-Left(x)
- Splay(x)

Splay(3):



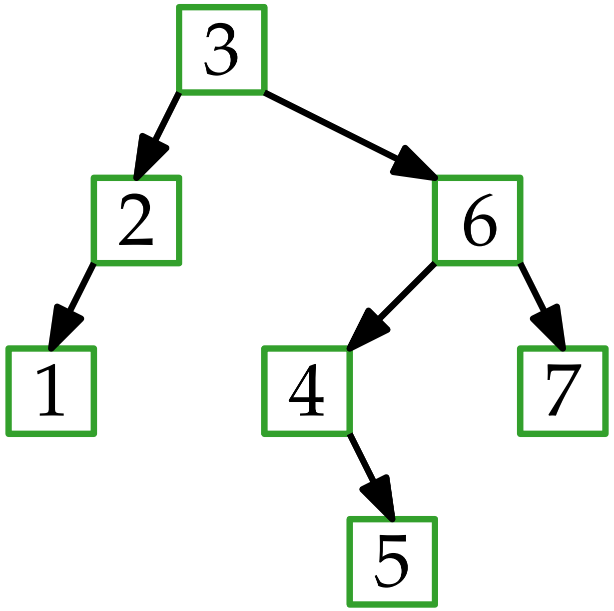
Call Splay(x):
- after Search(x)
- after Insert(x)

Splay

Algorithm: Splay(x)

if $x \neq root$ **then**
  $y =$ parent of x
  if $y = root$ **then**
    if $x < y$ **then** Right(x)
    if $y < x$ **then** Left(x)
  else
    $z =$ parent of y
    if $x < y < z$ **then** Right-Right(x)
    if $z < y < x$ **then** Left-Left(x)
    if $y < x < z$ **then** Left-Right(x)
    if $z < x < y$ **then** Right-Left(x)
  Splay(x)

Splay(3):



Call Splay(x):

- after Search(x)
- after Insert(x)
- before Delete(x)

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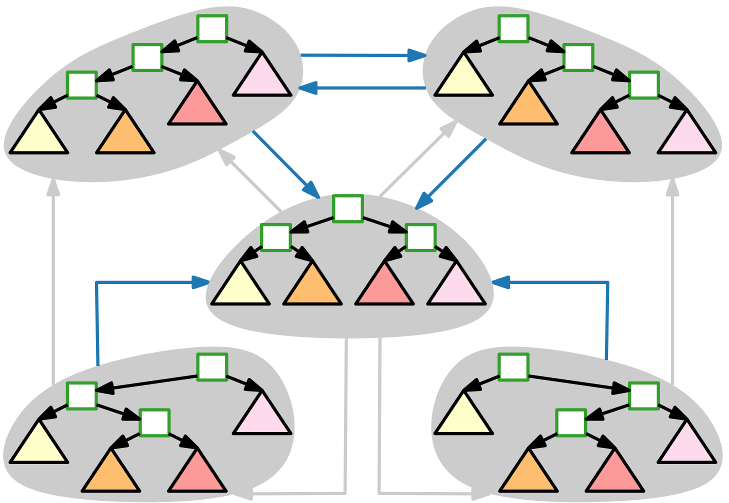
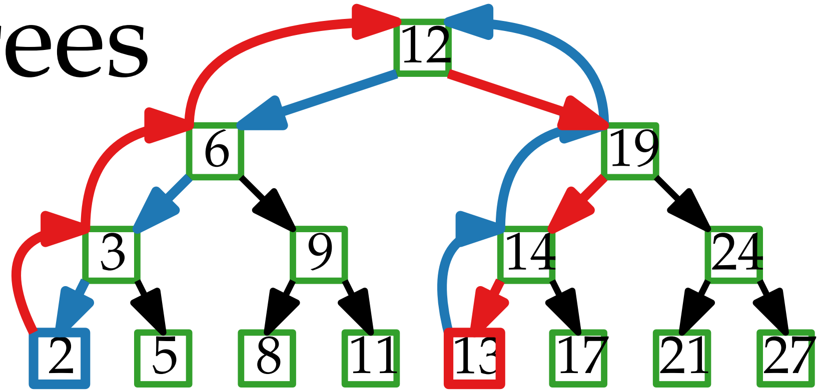
Advanced Algorithms

Optimal Binary Search Trees



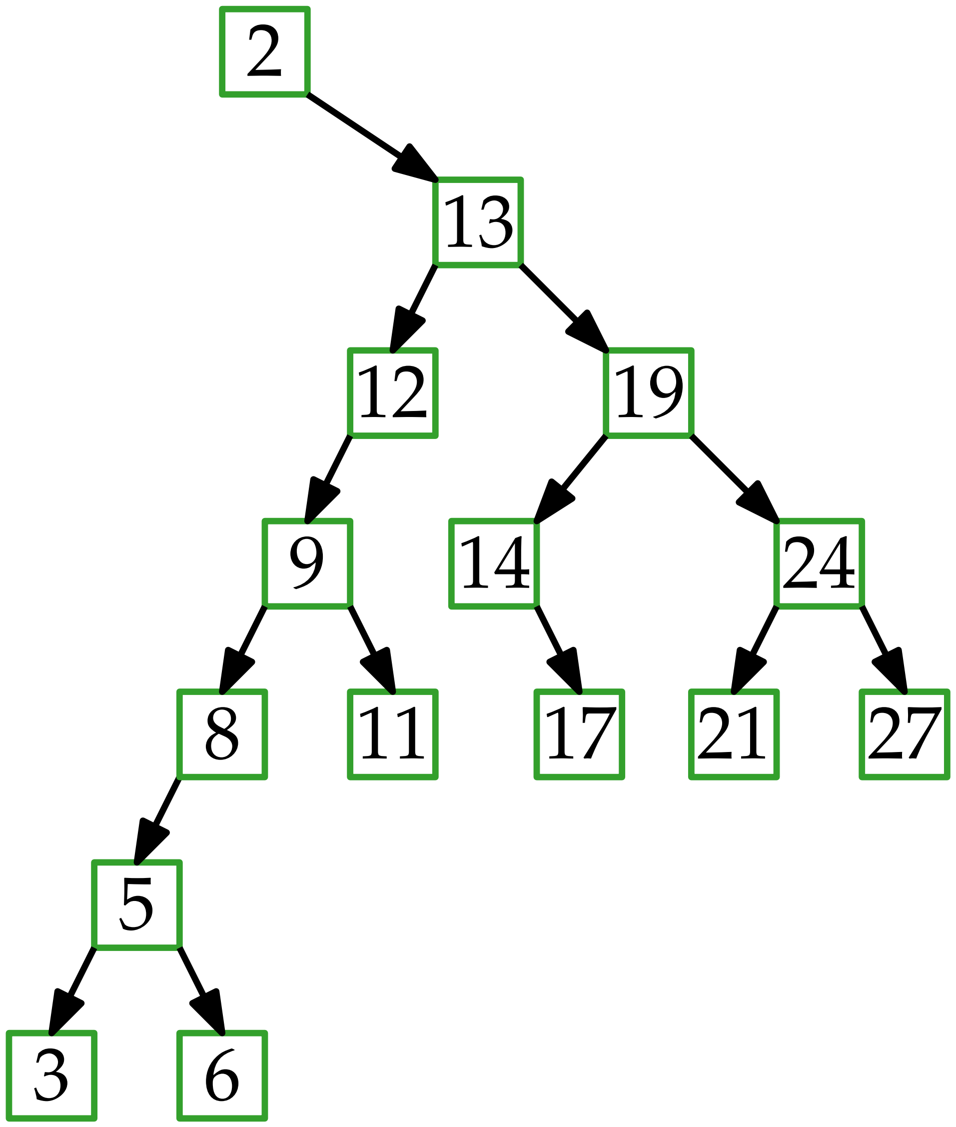


Splay Trees

Philipp Kindermann · WS20

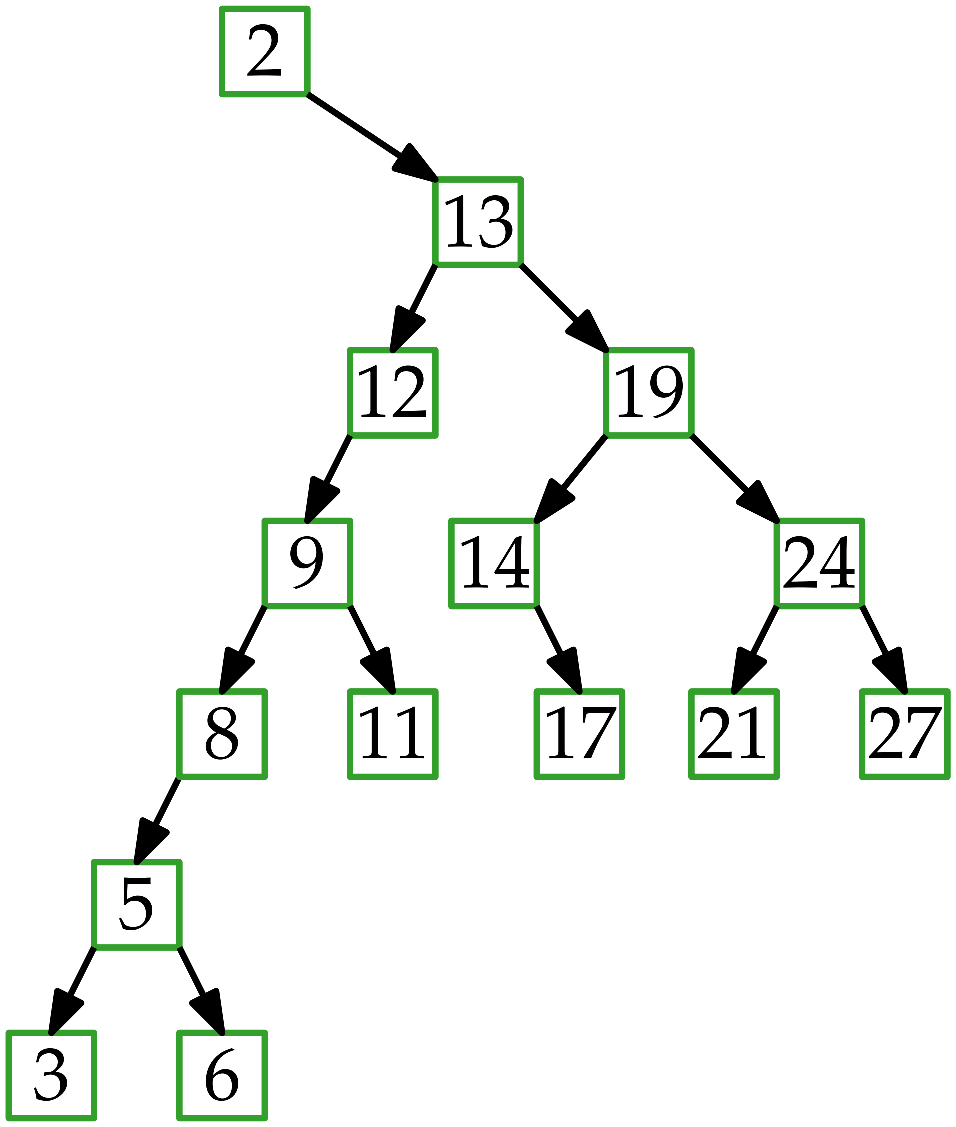
Part IV:
Potential

Why is Splay Fast?



Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)



Why is Splay Fast?

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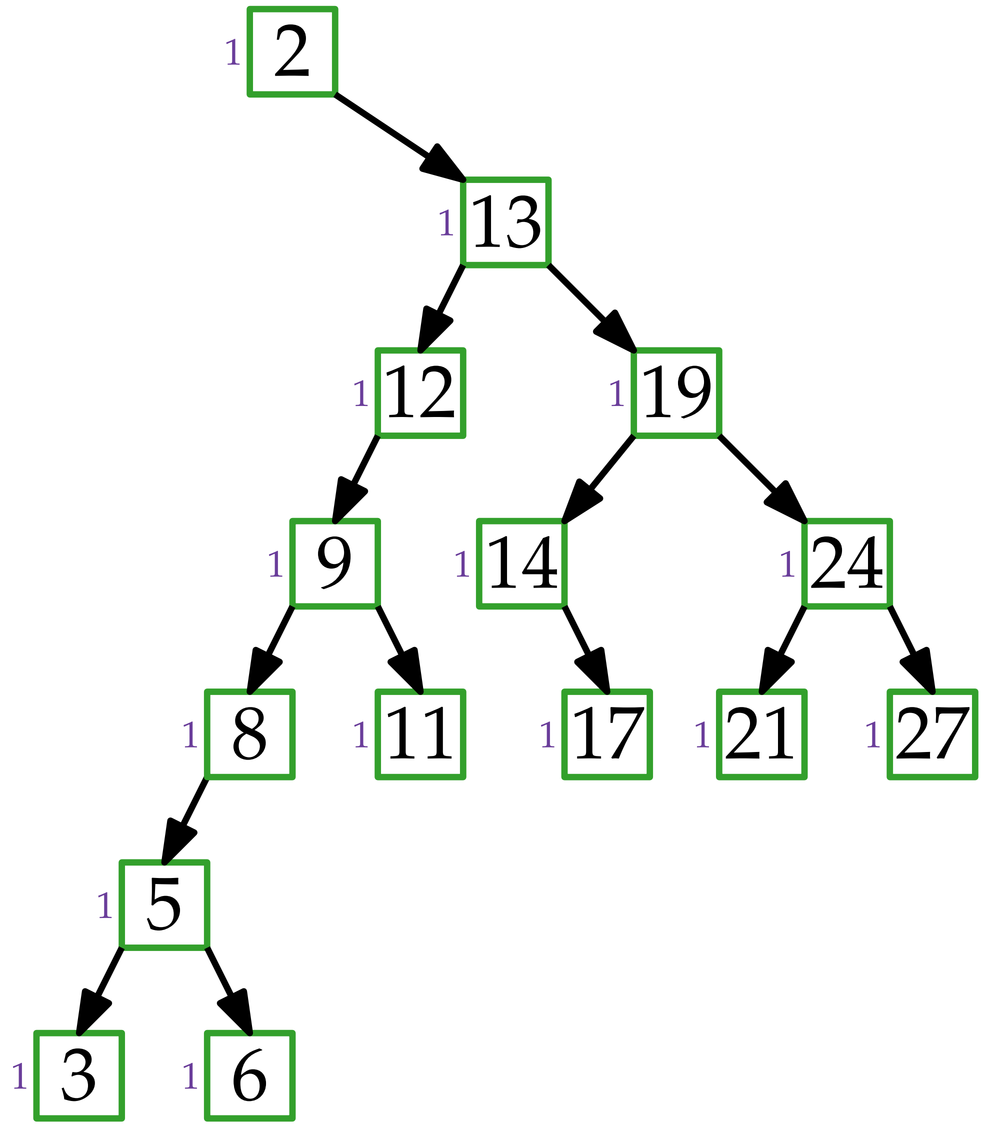
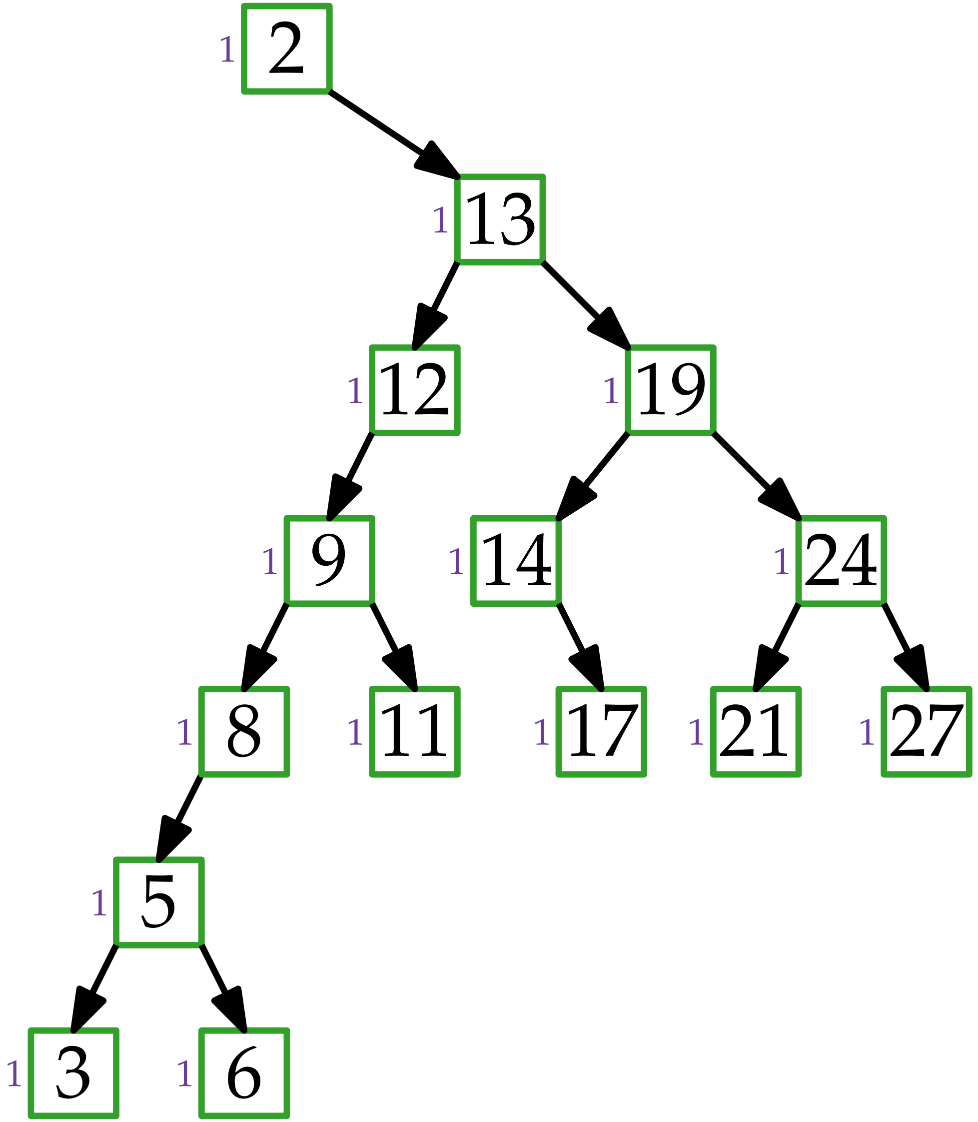


Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)
$s(x)$: sum of all $w(x)$ in subtree of x_i



Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)
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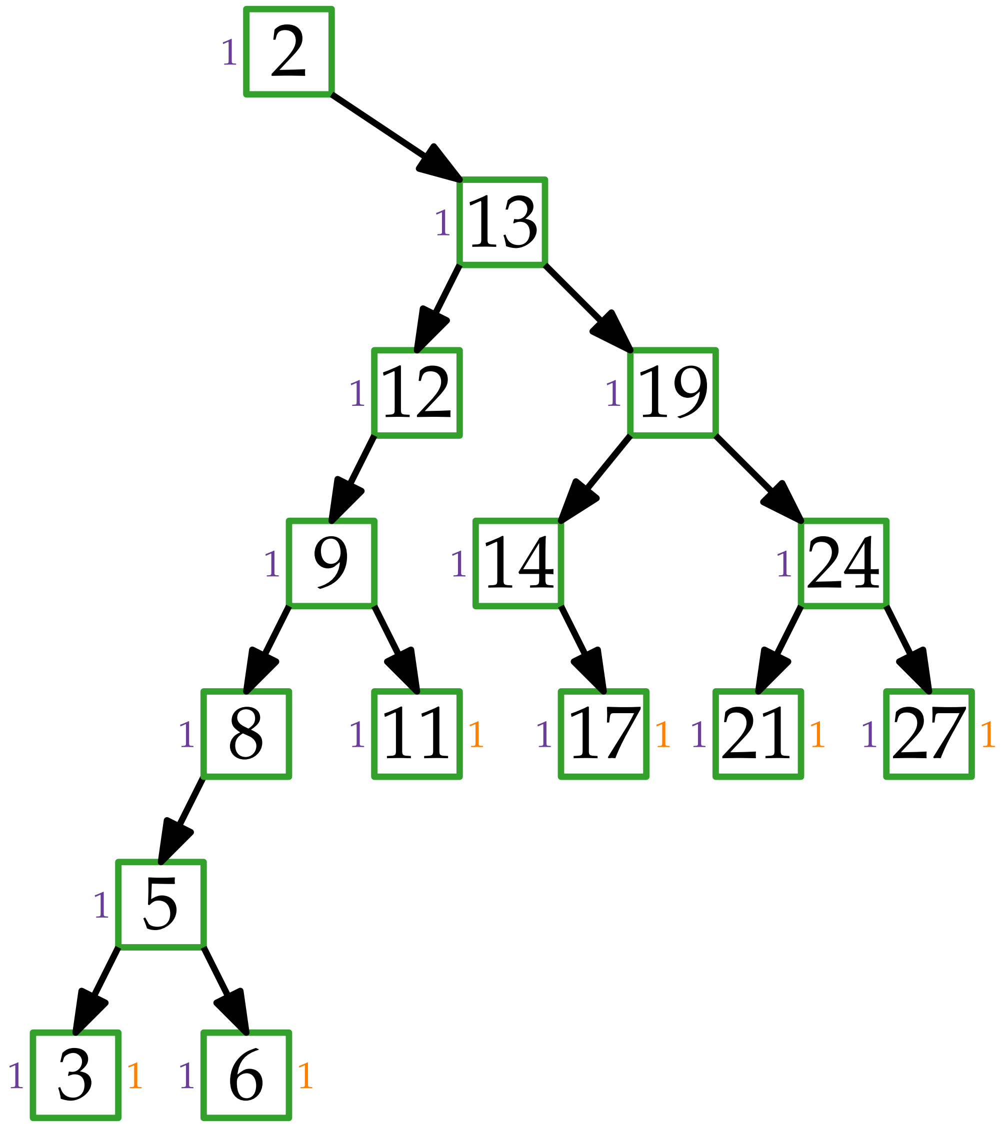

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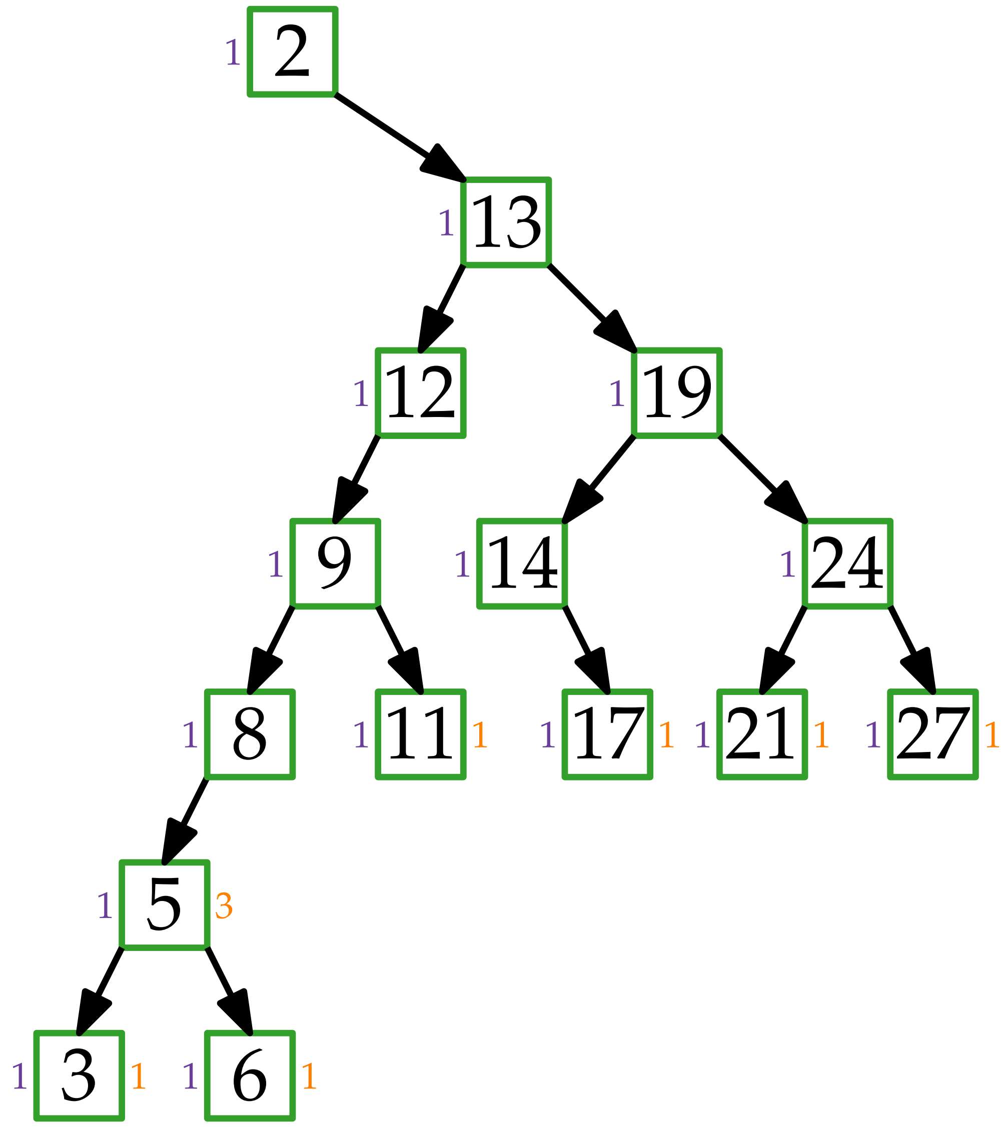
$s(x)$: sum of all $w(x)$ in subtree of x_i



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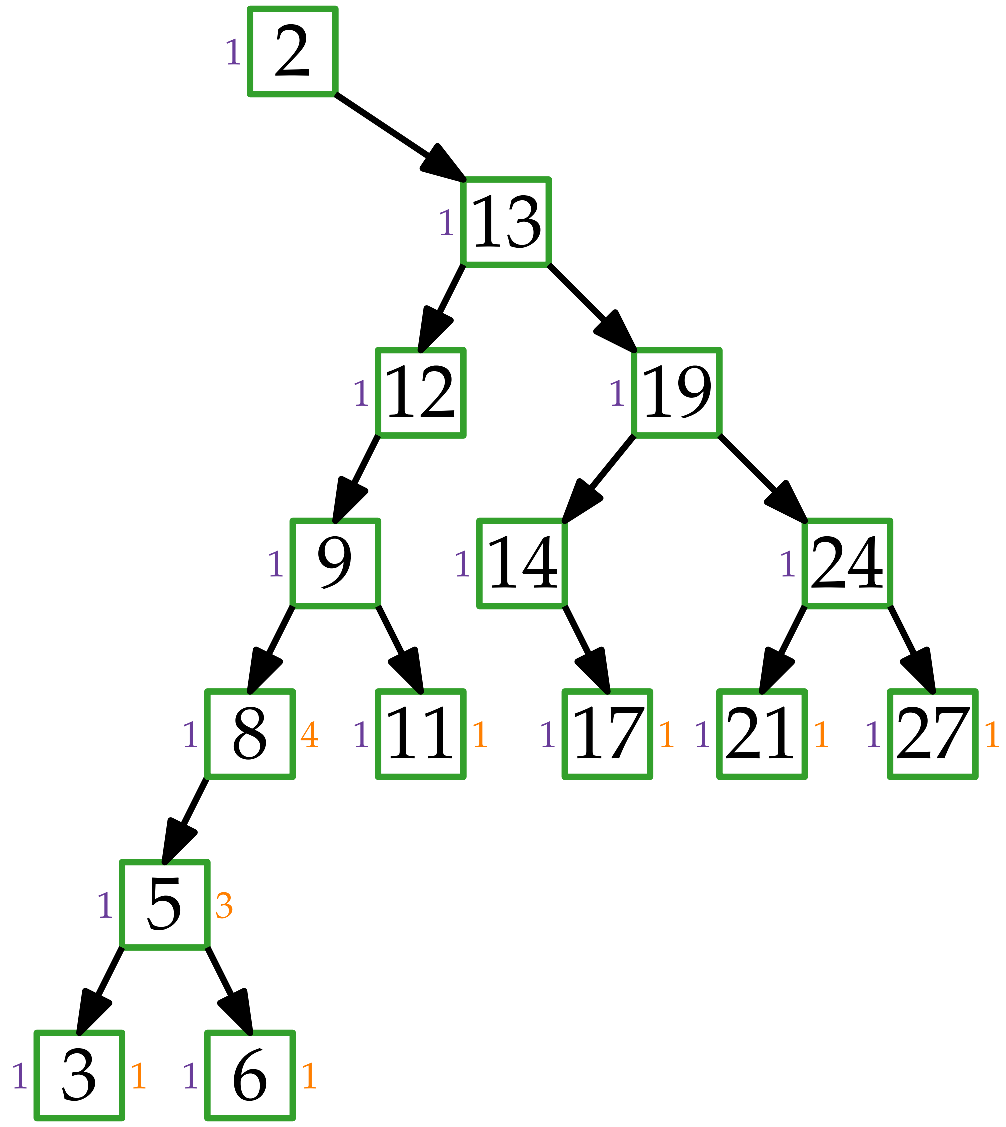

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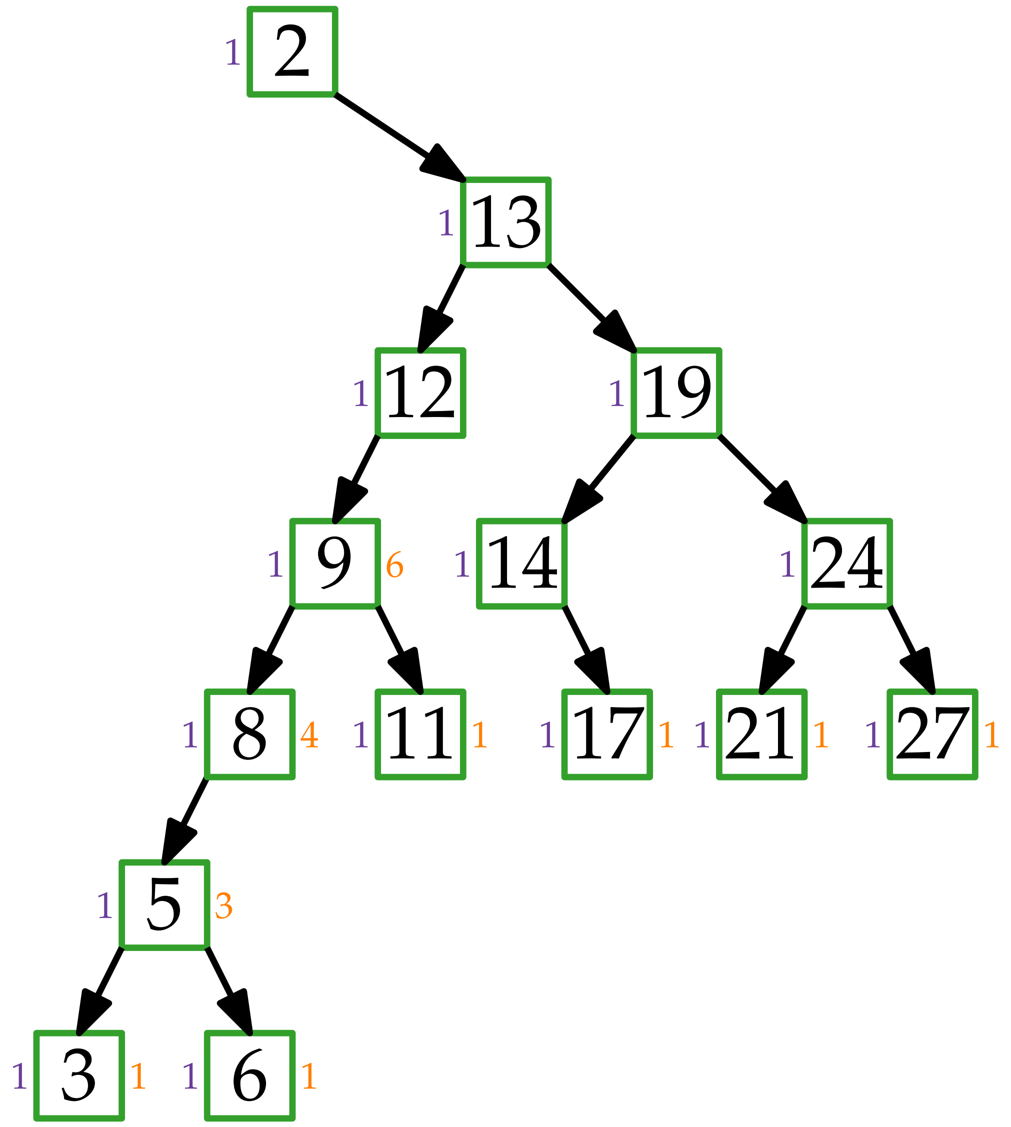

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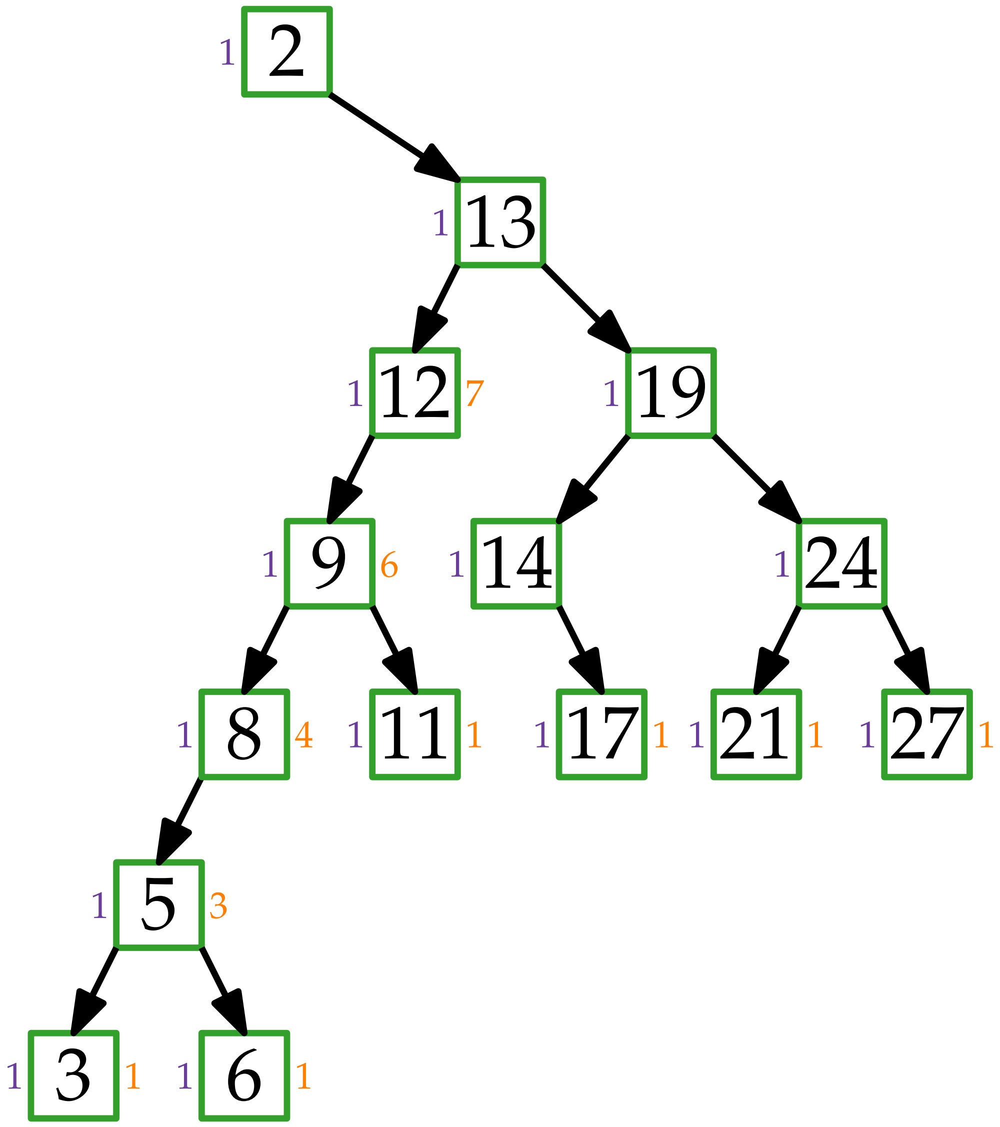

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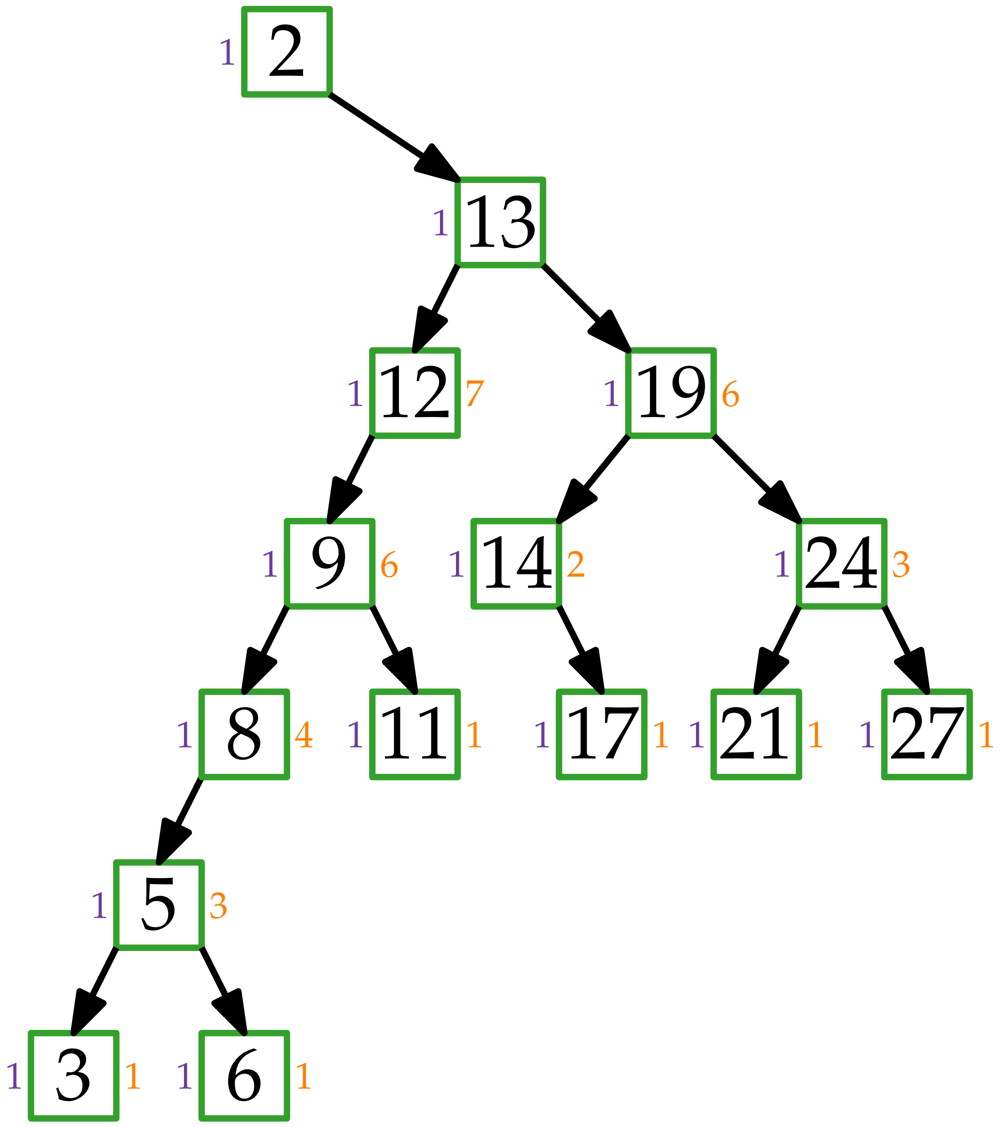

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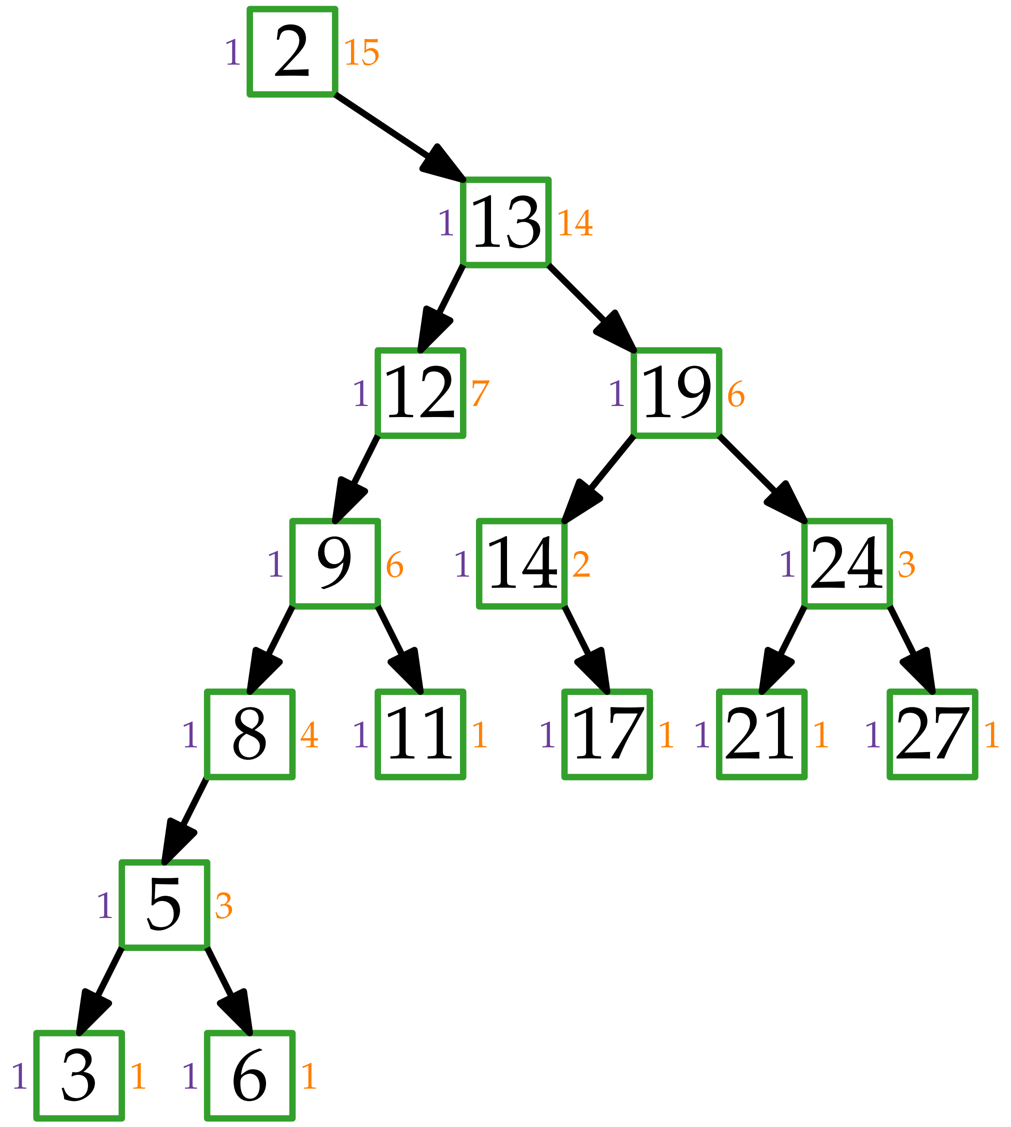

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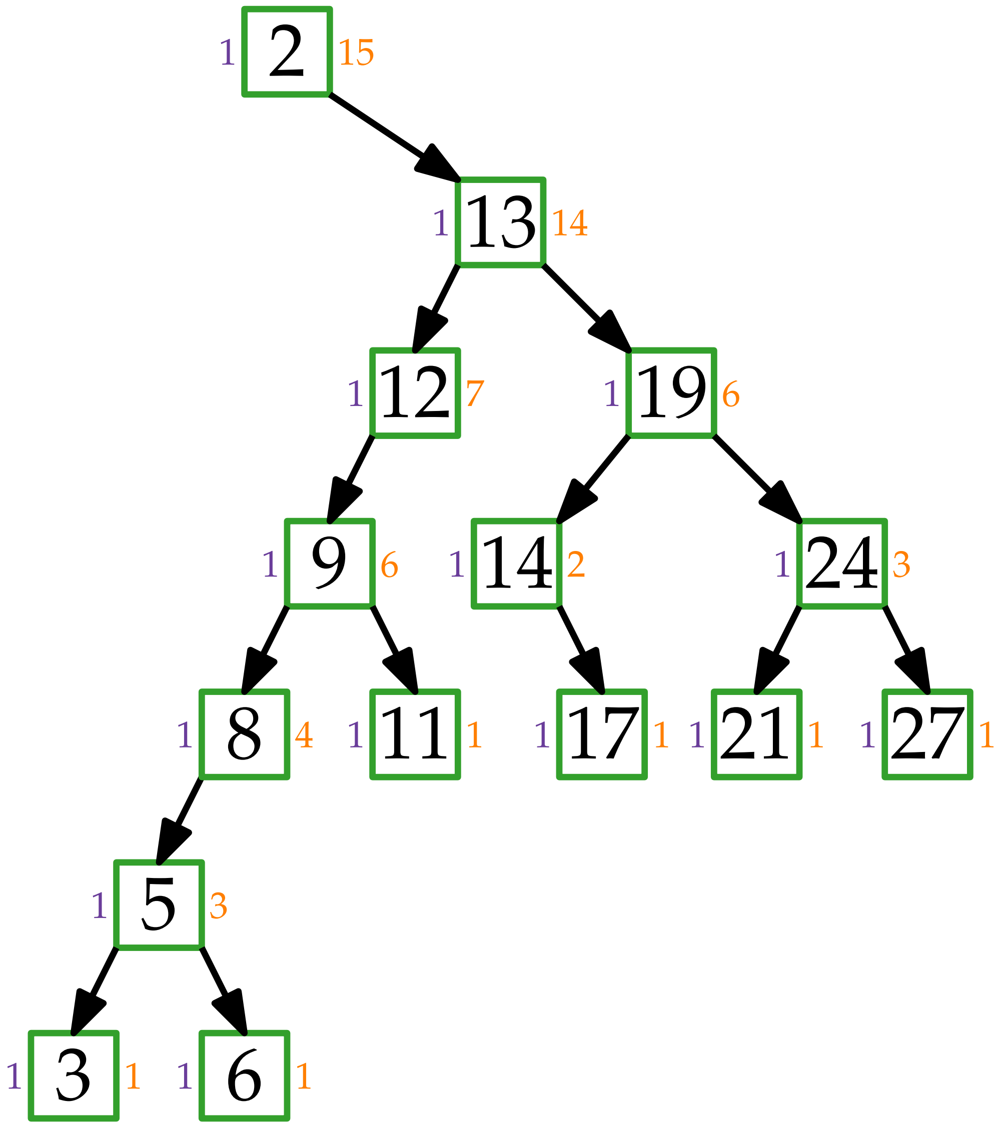

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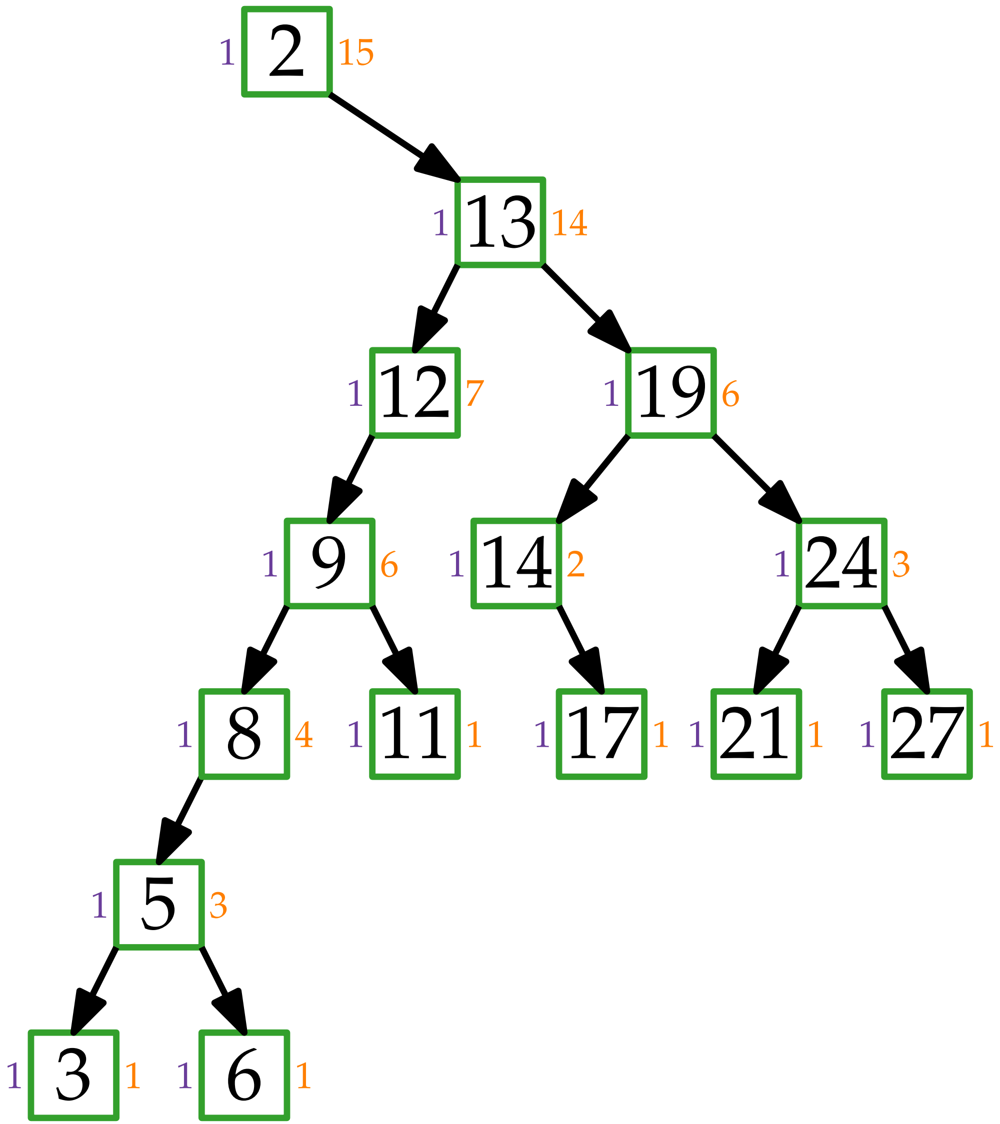

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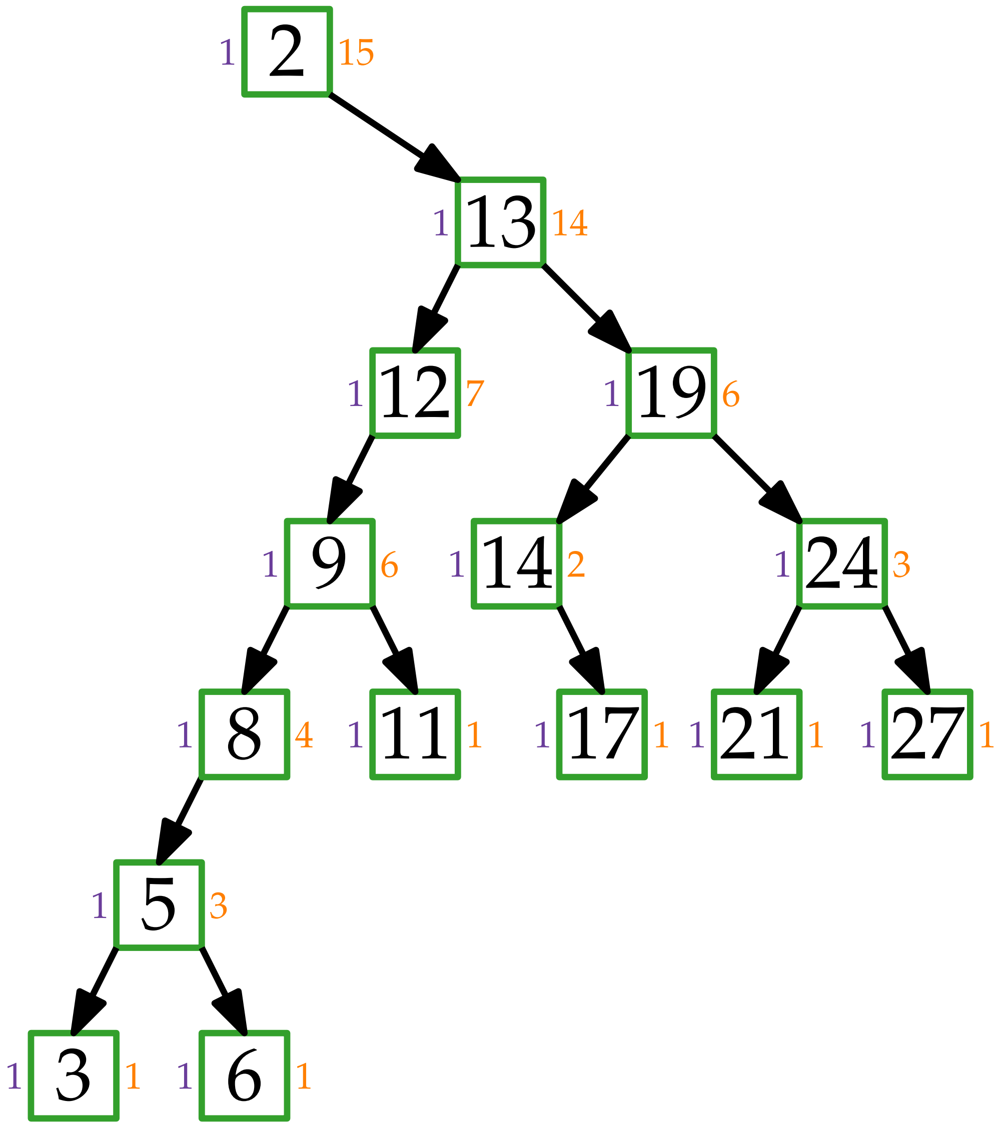

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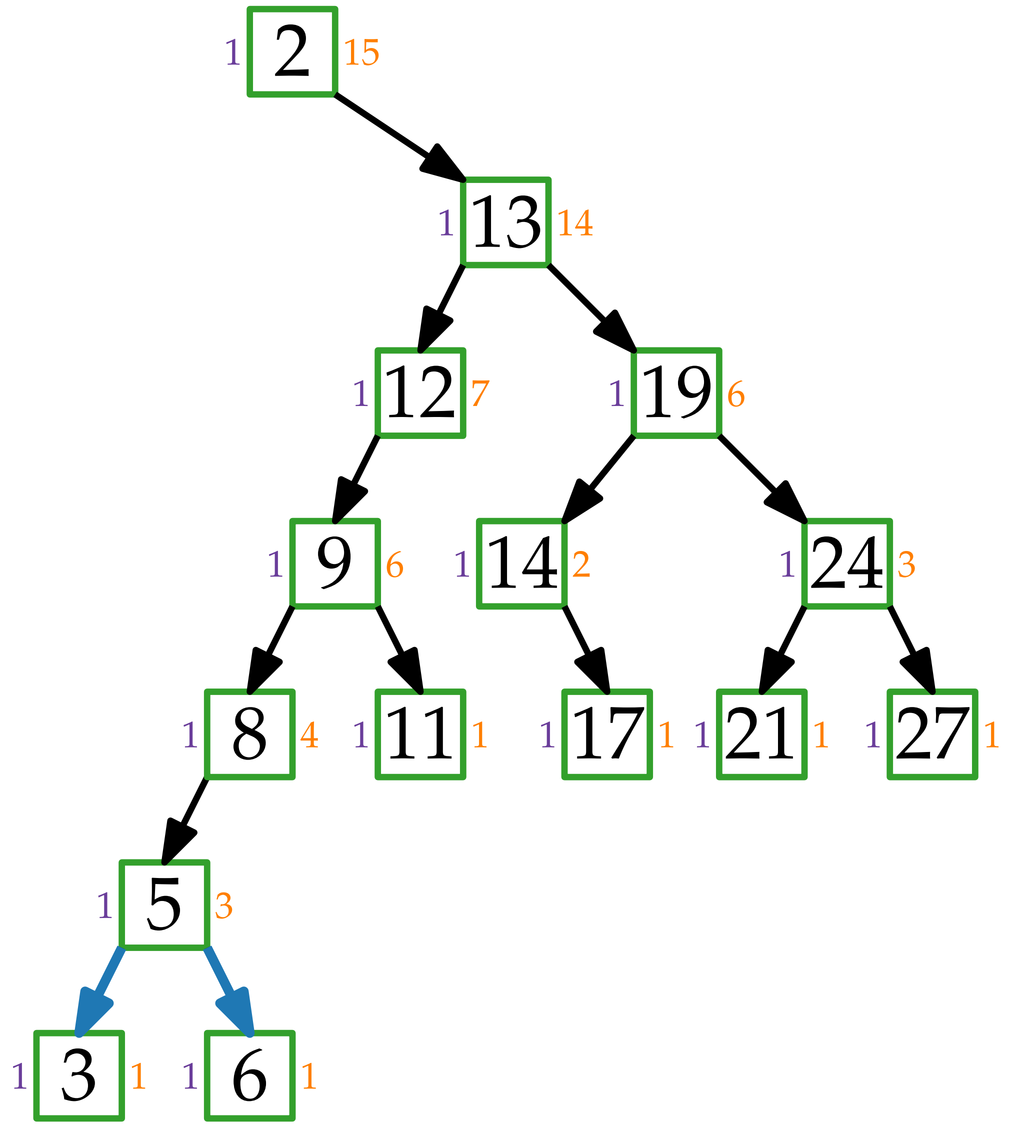

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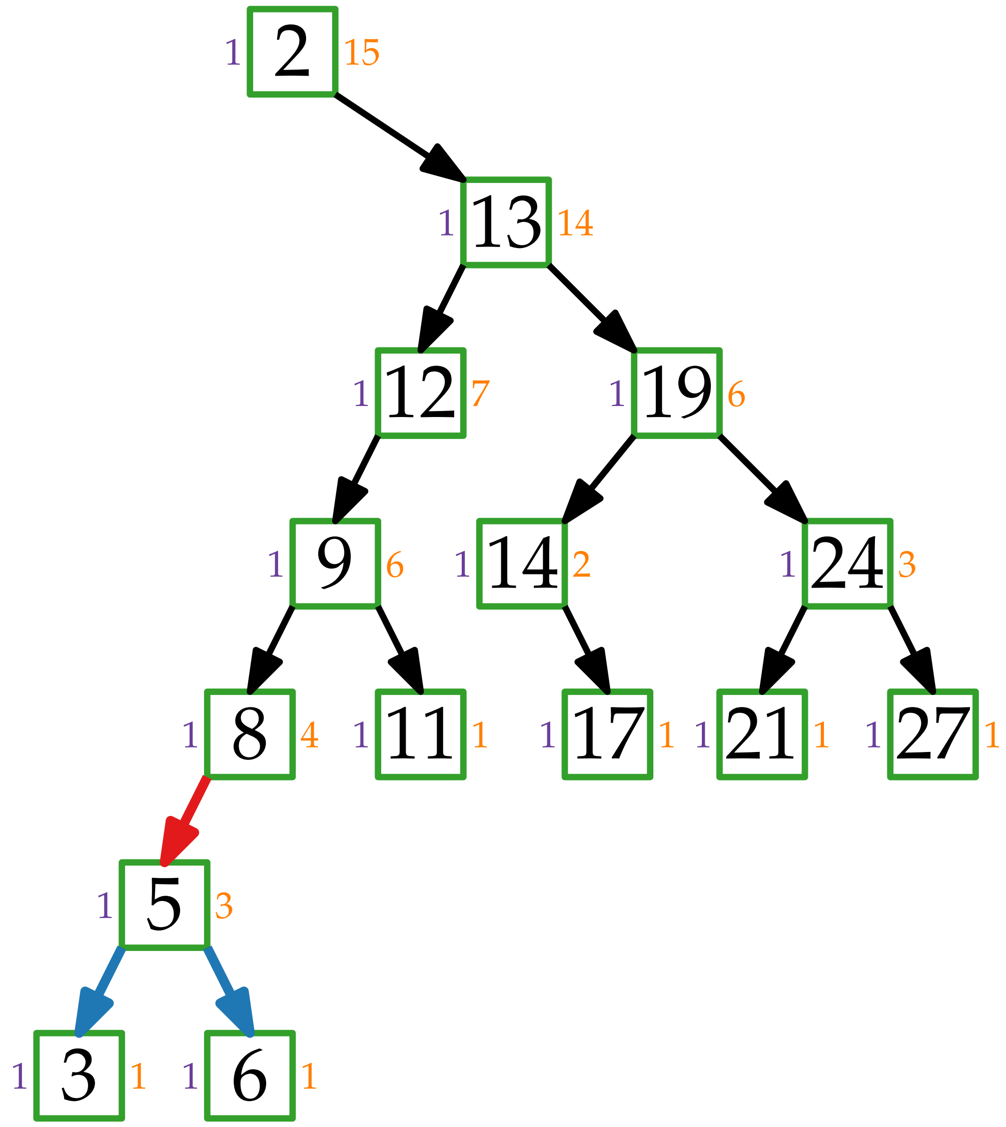

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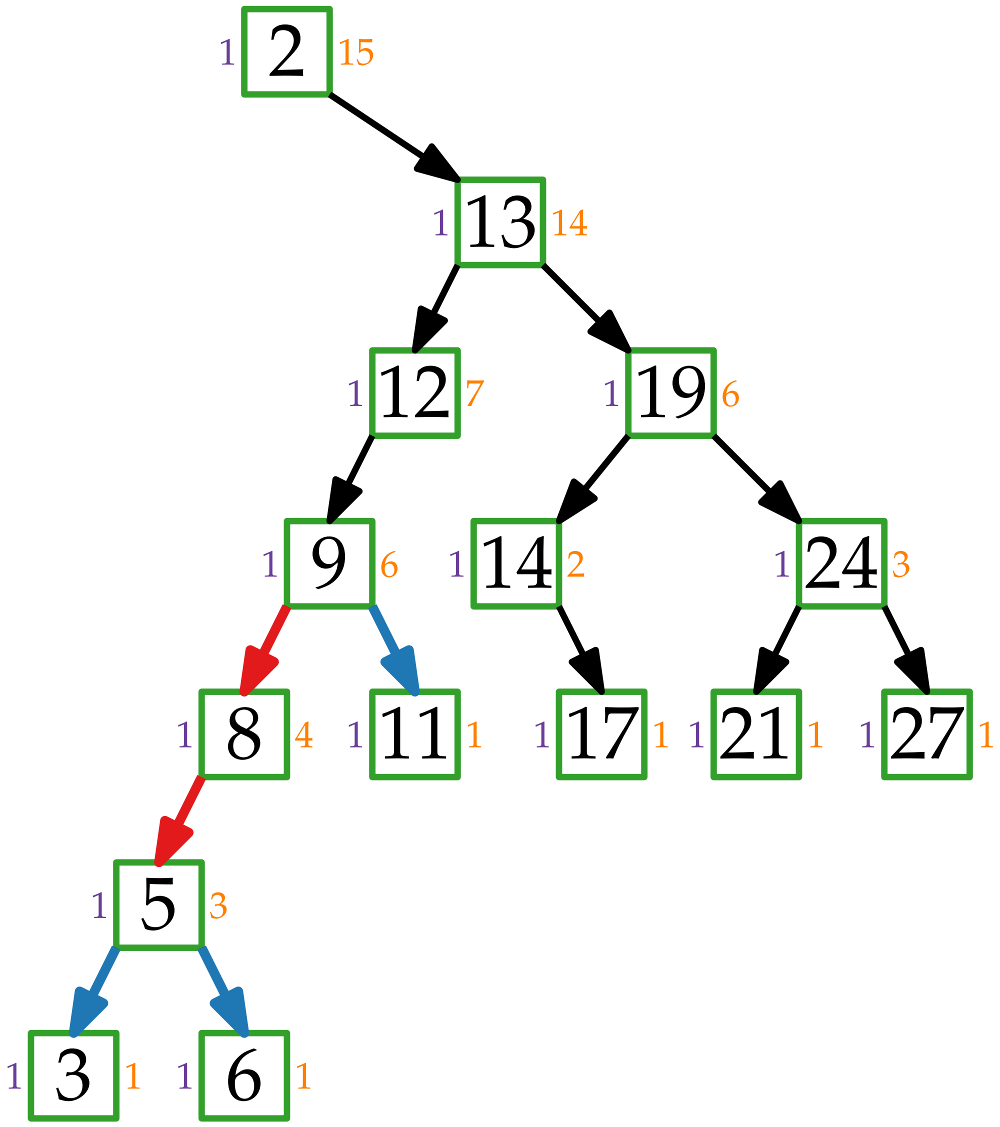

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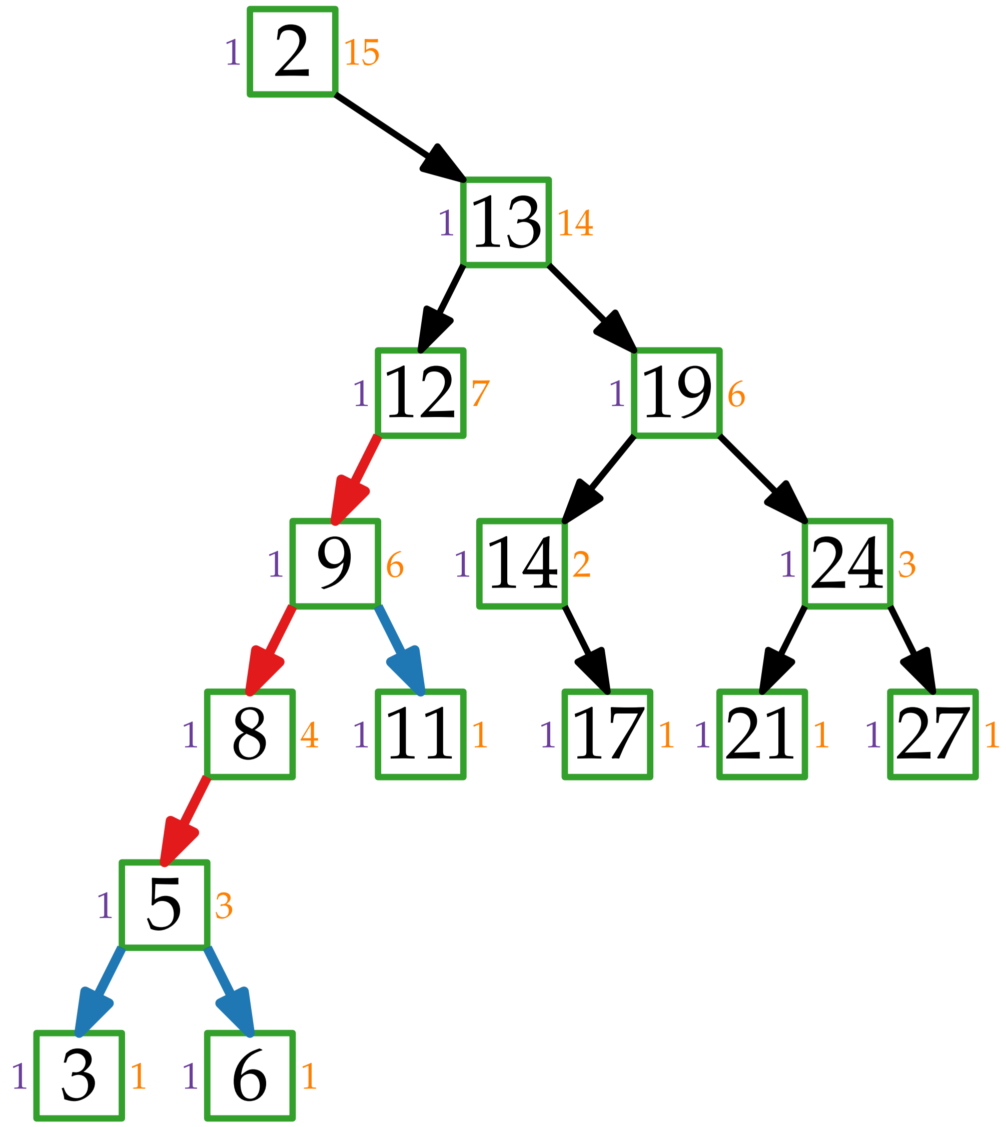

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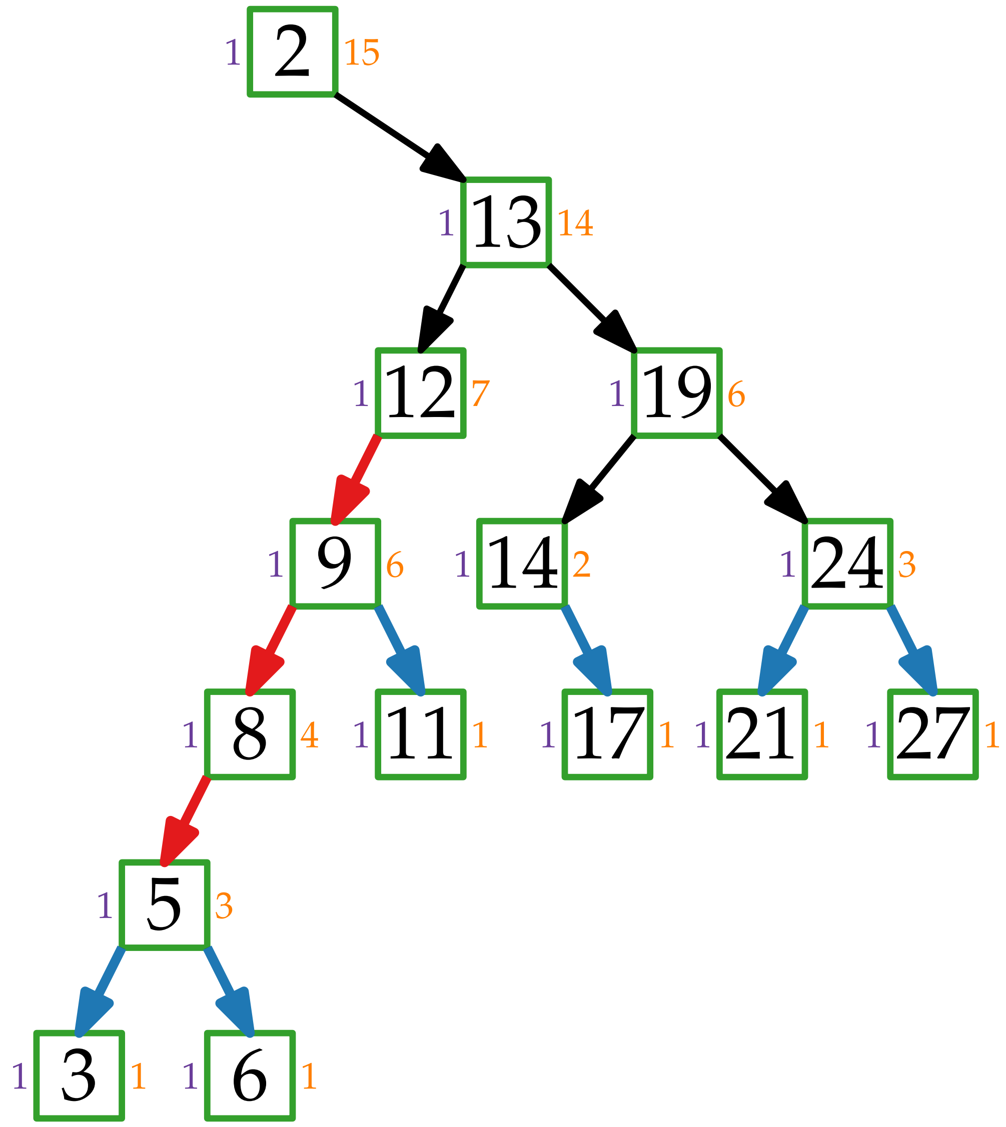

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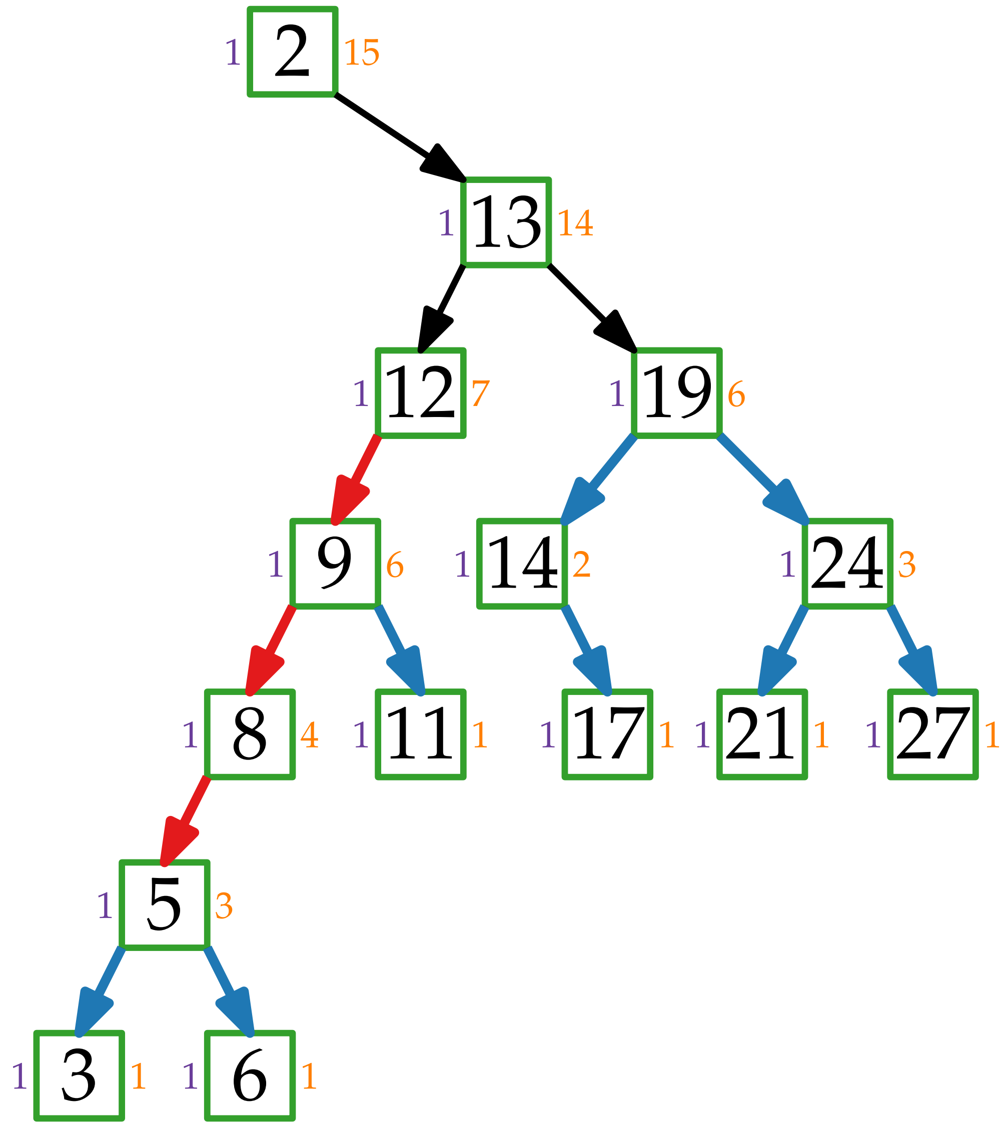

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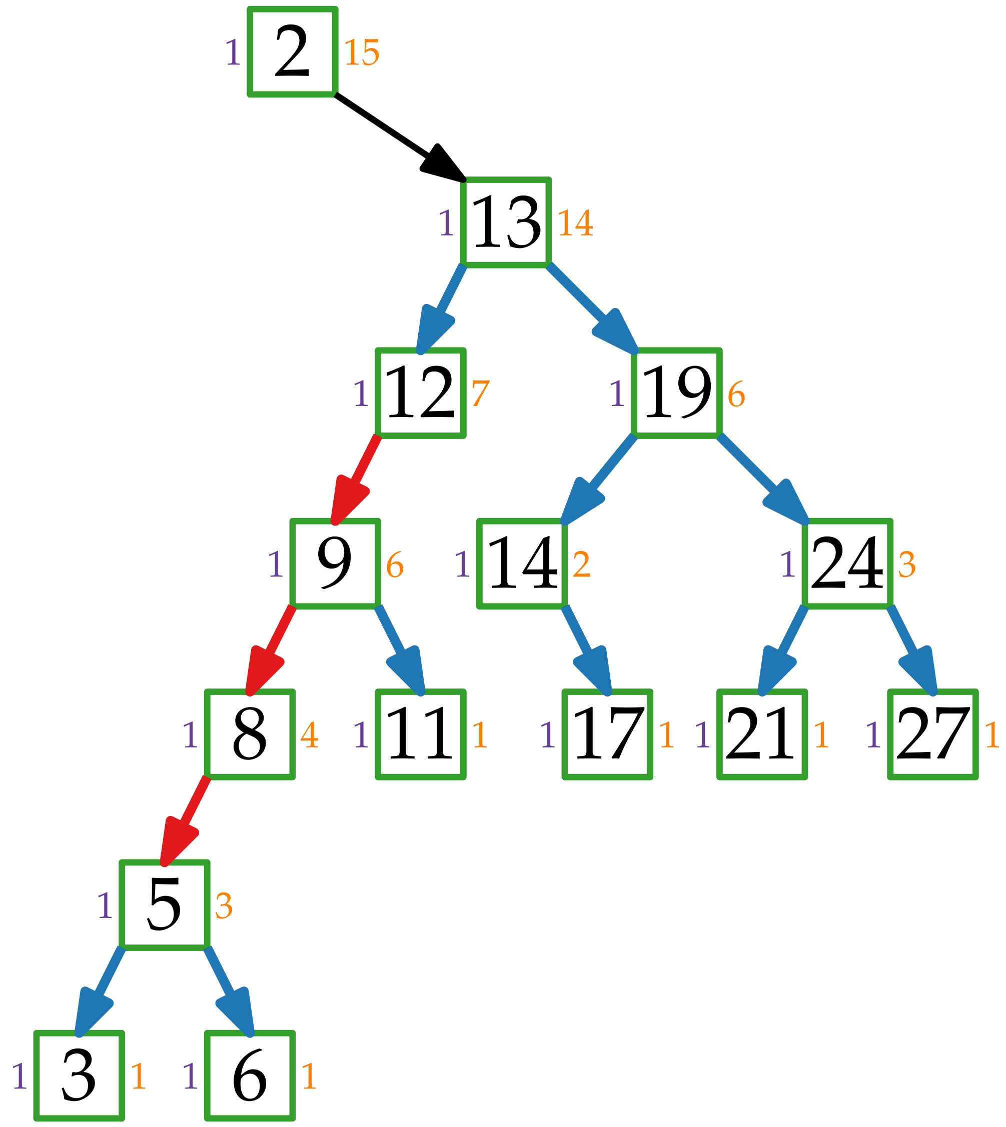

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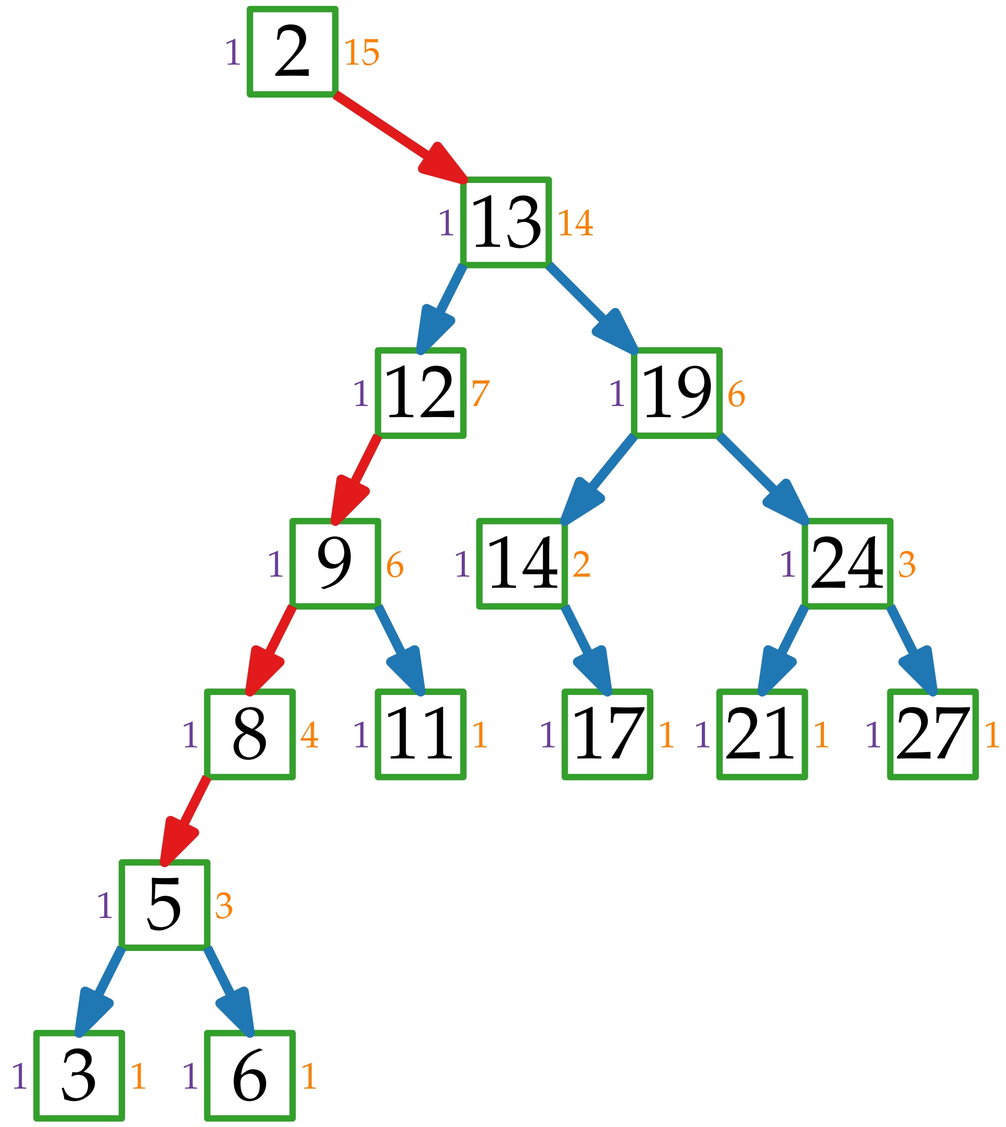

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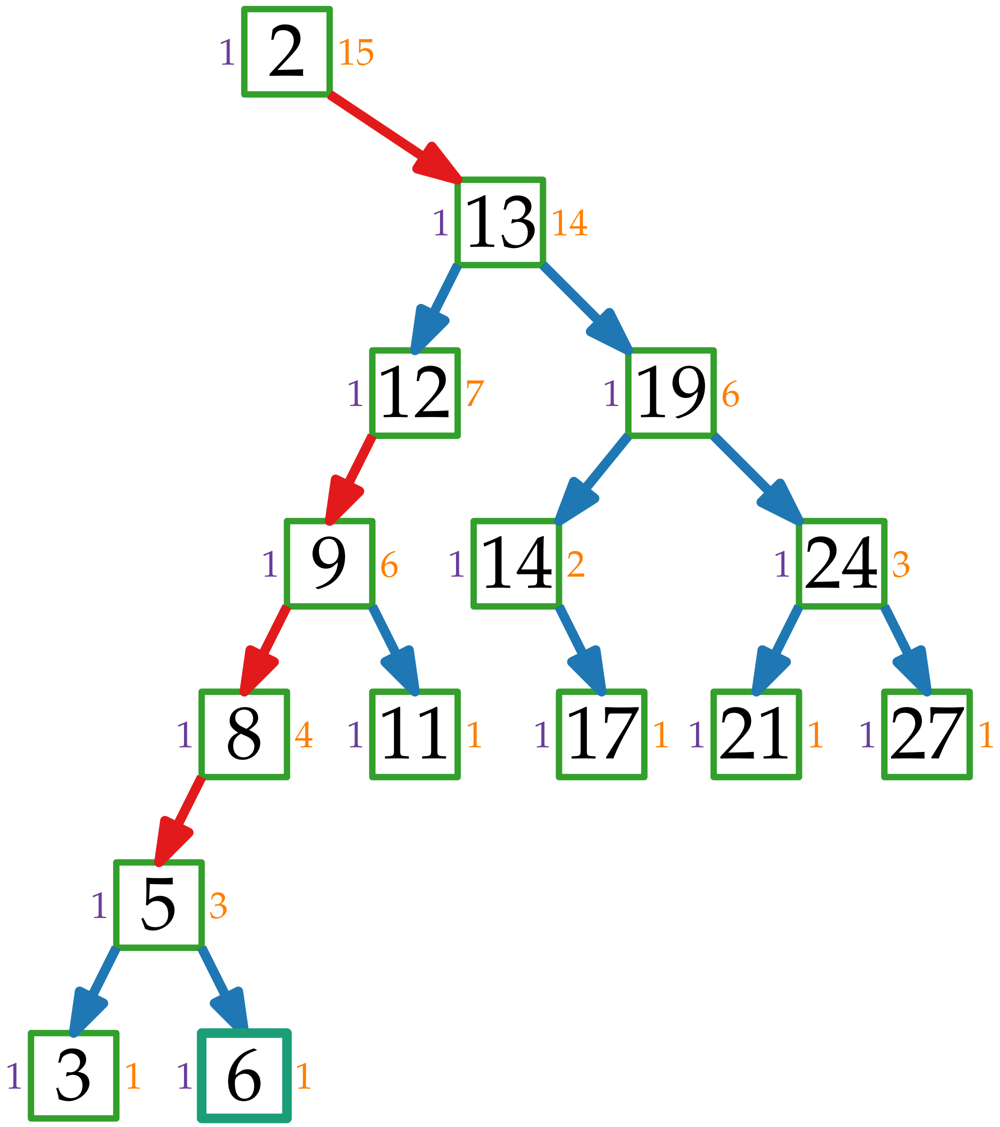

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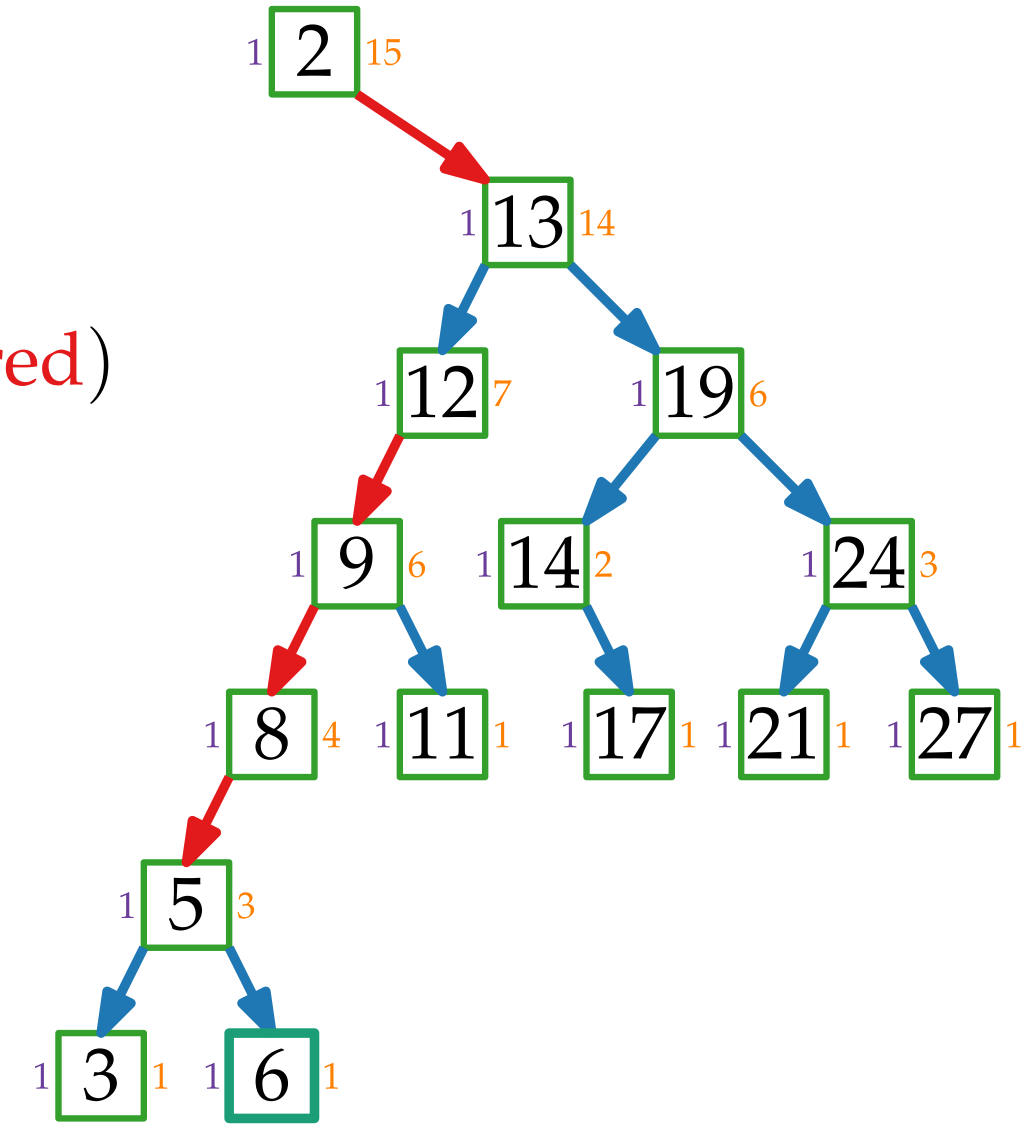

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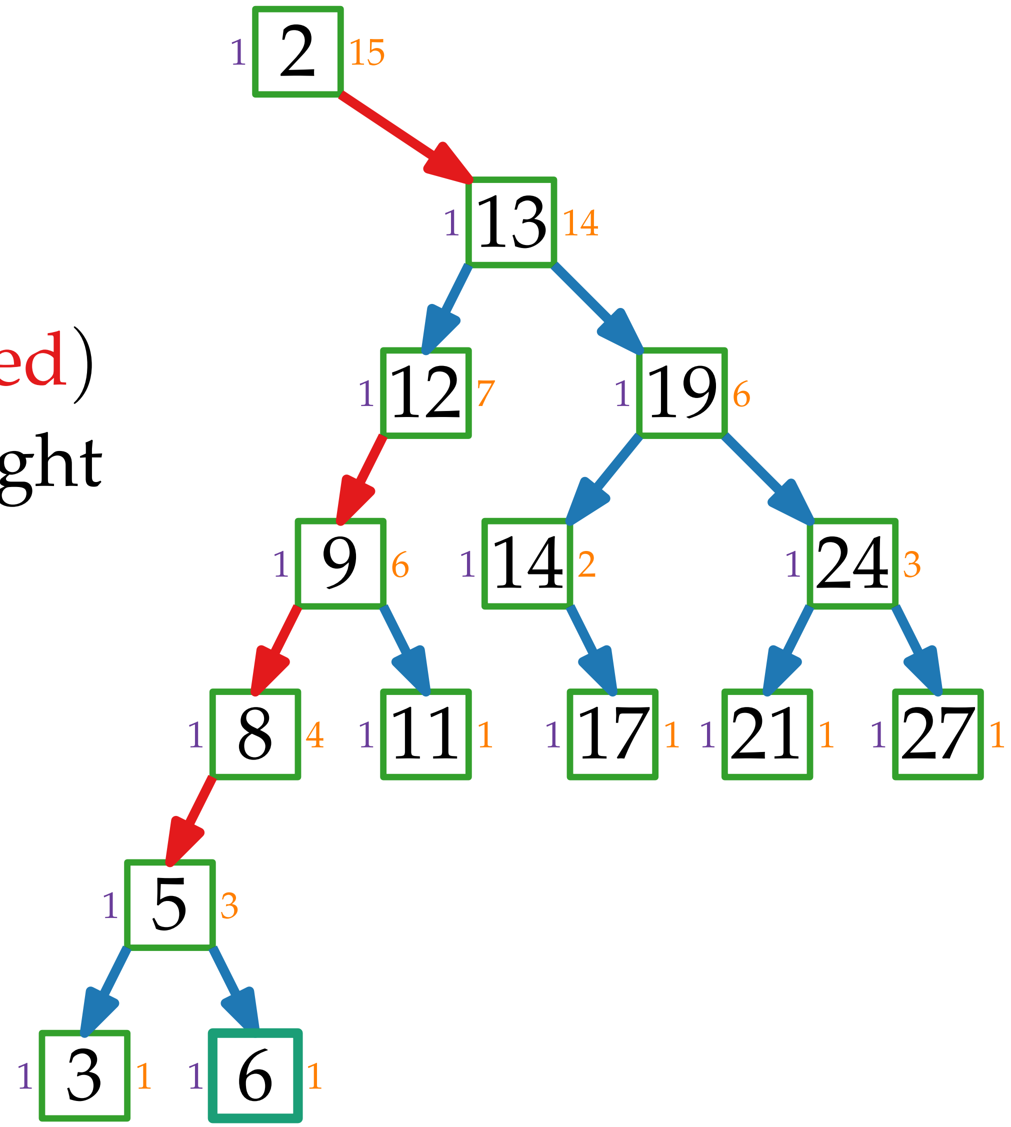

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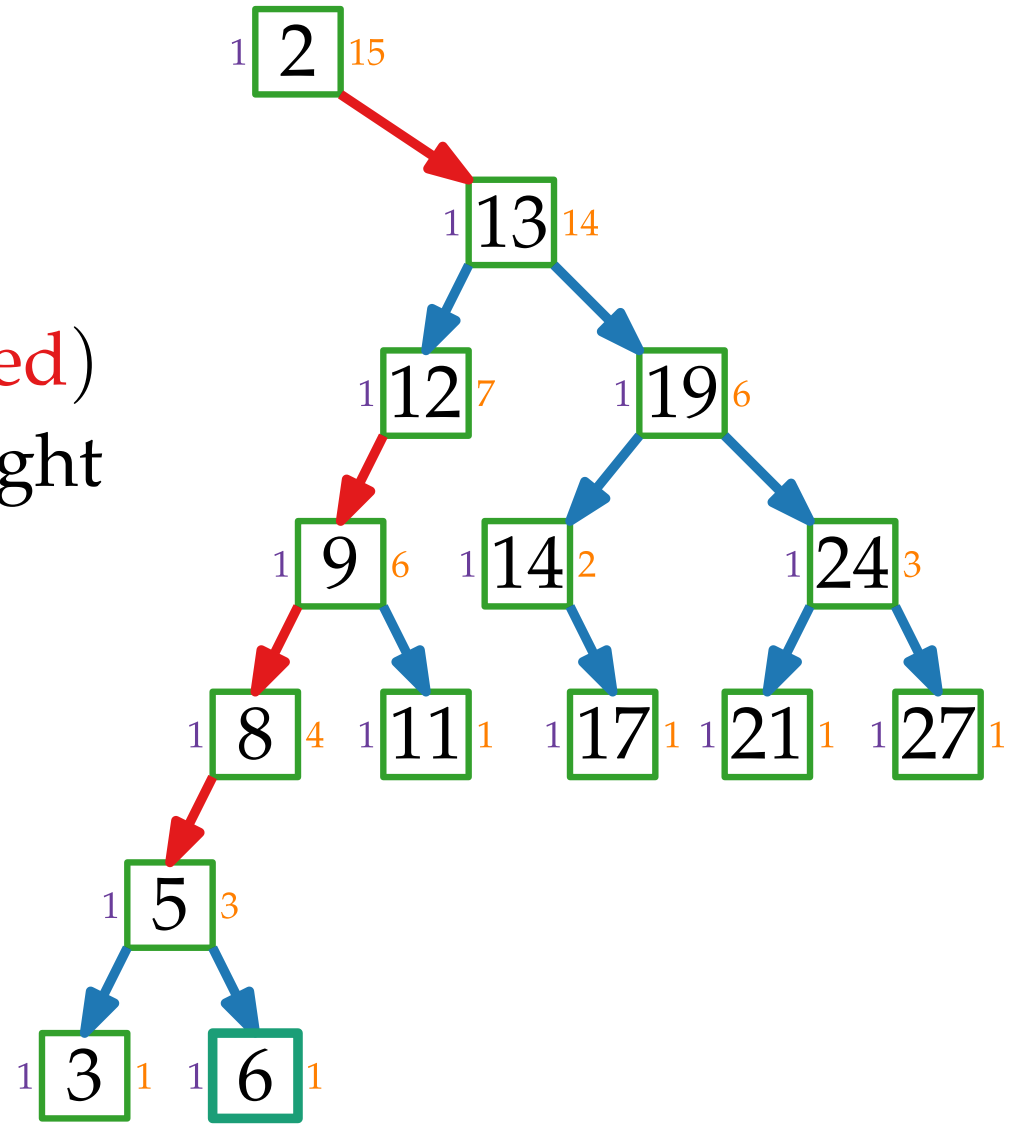

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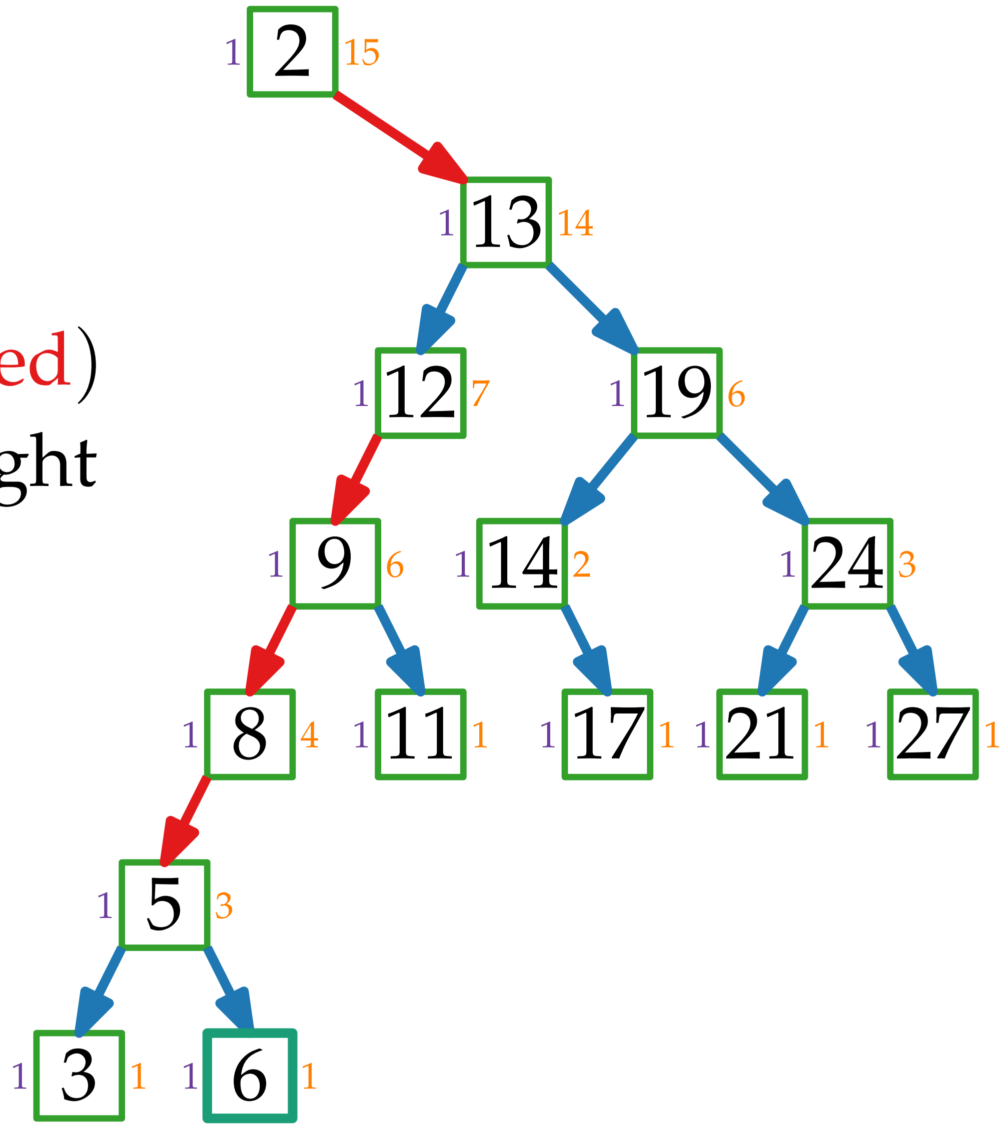

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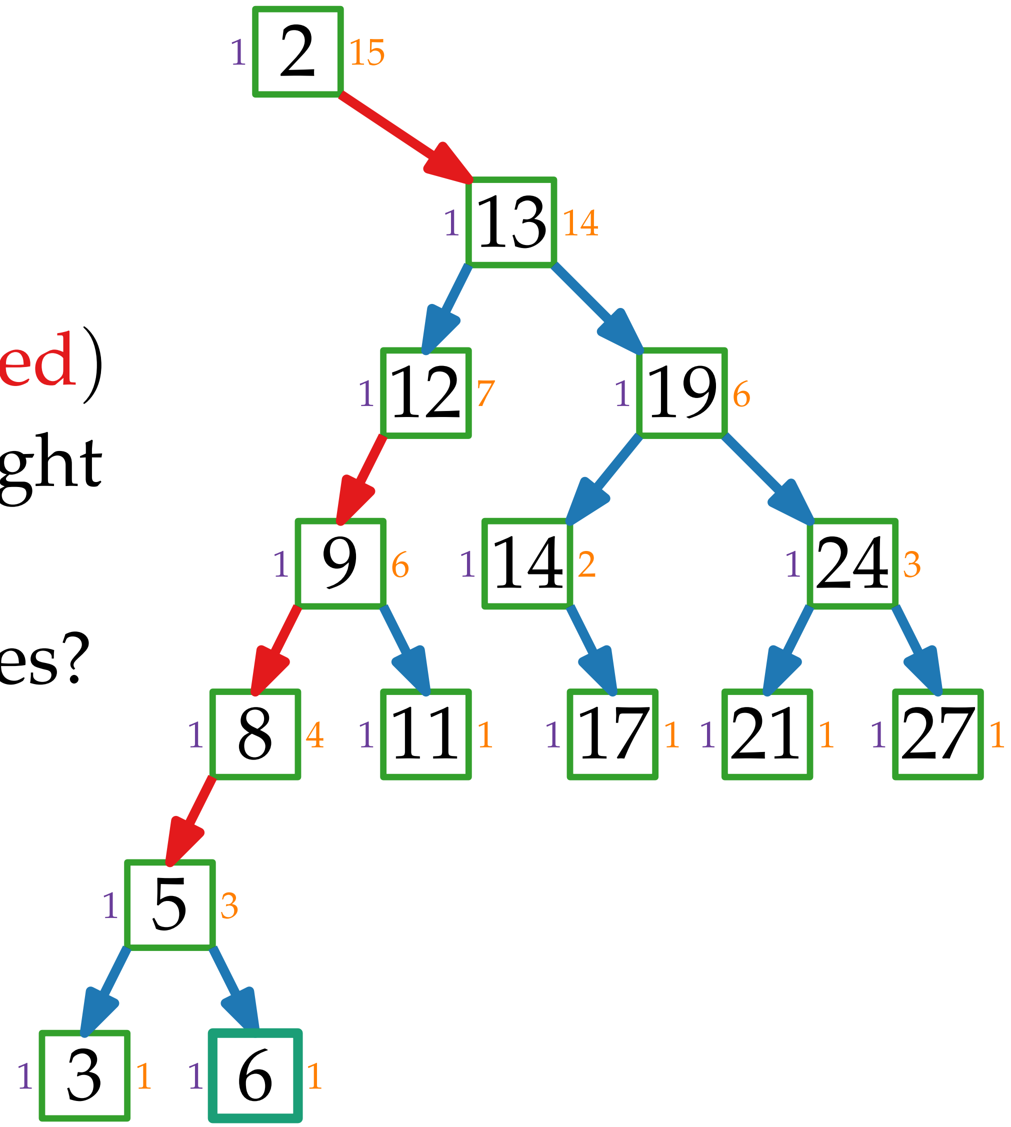

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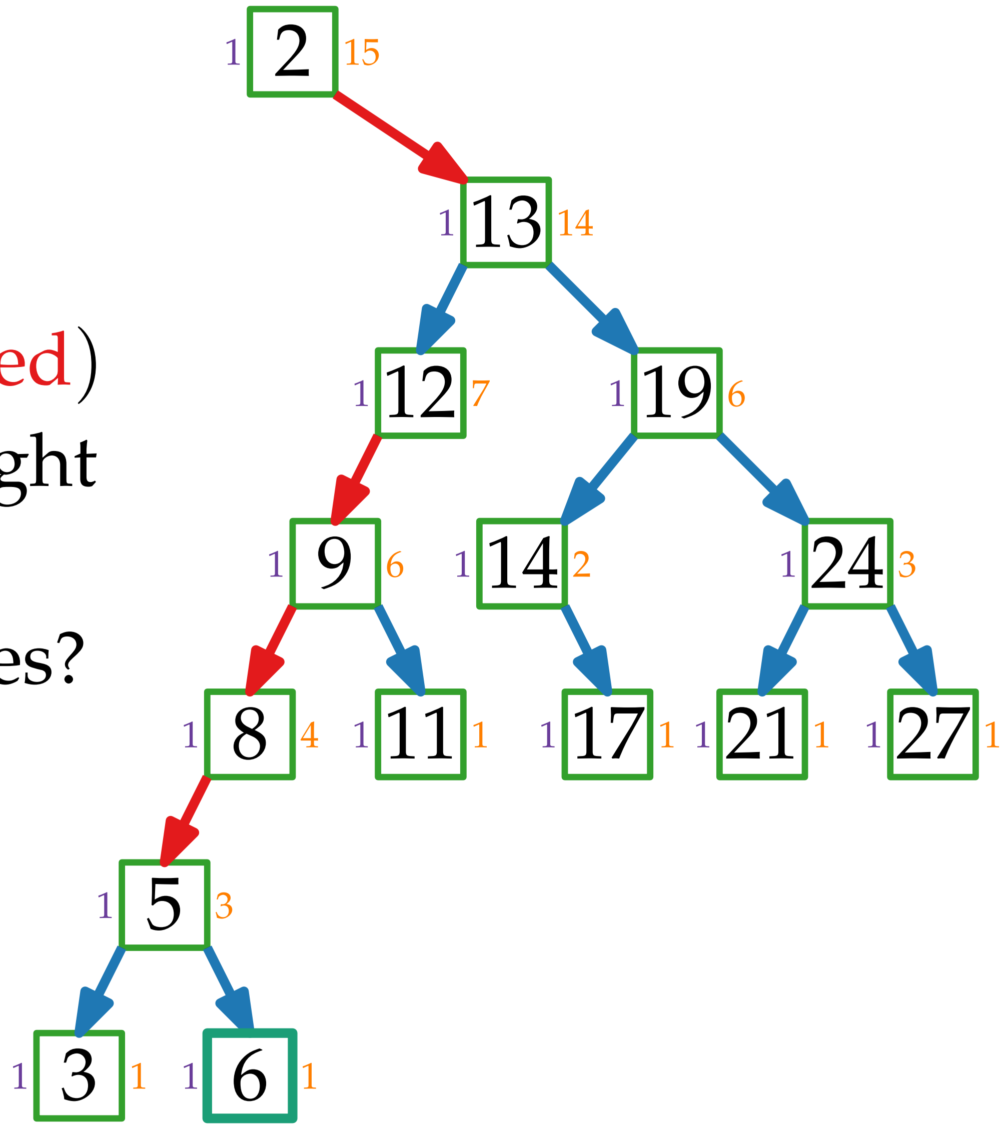
Use sum-of-logs potential

$\Phi = \sum \log s(x)$



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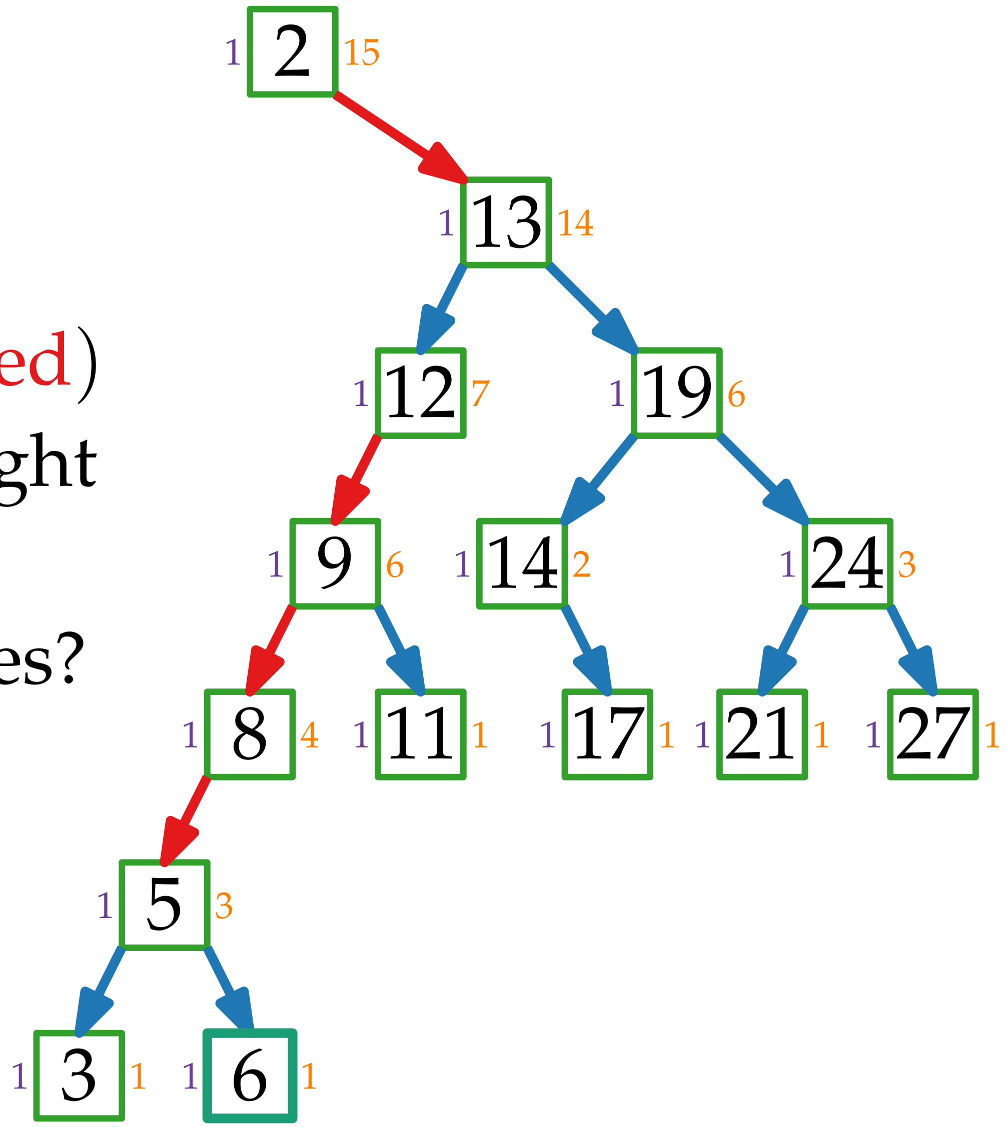
$\Phi = \sum \log s(x)$

Amortized cost:

real cost + $\Phi_+ - \Phi$

(potential after splay)



What is Potential?

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Example (ADS): Stack with multipop

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Φ :=size of the stack

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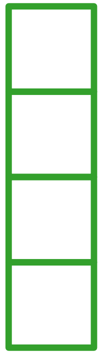

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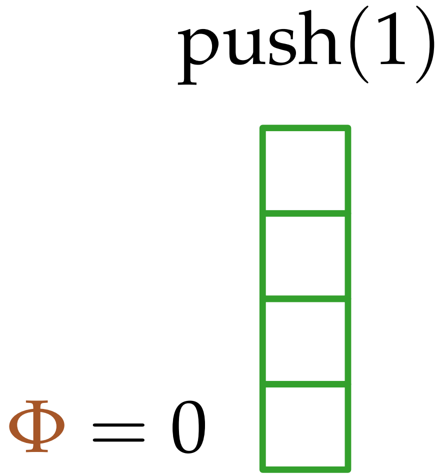

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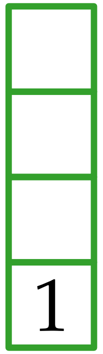
Φ :=size of the stack

push(1)

$\Phi = 1$



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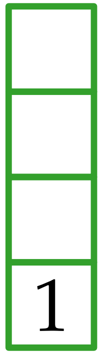
Φ :=size of the stack

push(2)



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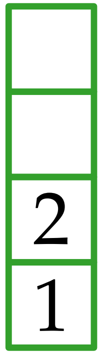

$\Phi = 2$

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push(3)

Example (ADS): Stack with multipop

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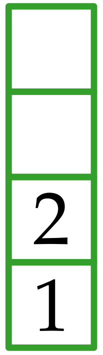

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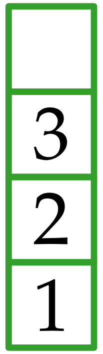
push(3)

Example (ADS): Stack with multipop

Φ :=size of the stack

$\Phi = 3$



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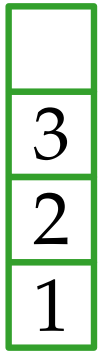
Example (ADS): Stack with multipop

Φ :=size of the stack

pop(2)



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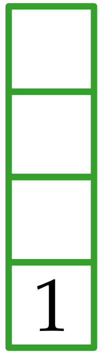

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push:

pop(k):

pop(2)

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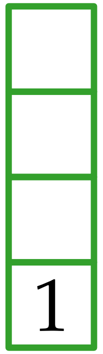

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Example (ADS): Stack with multipop

Φ :=size of the stack

push: $\quad 1 + \Phi_+ - \Phi$

pop(k):

pop(2)

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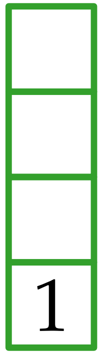

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Φ :=size of the stack

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push: $\quad 1 + \Phi_+ - \Phi \;= 2$

pop(k):

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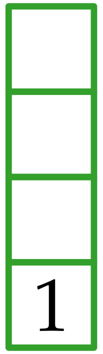

push: $1 + \Phi_+ - \Phi \ = 2$

pop(k): $k + \Phi_+ - \Phi$

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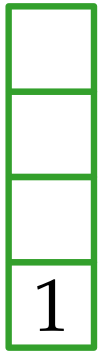
pop(k): $\quad k + \Phi_+ - \Phi \ = 0$

pop(2)

$\Phi = 1$



What is Potential?

Φ represent work that has been "paid for" but not yet performed.

amortized cost per step: real cost $+ \Phi_+ - \Phi$

total cost $= \Phi_0 - \Phi_{\text{end}} + \sum$ amortized cost

Example (ADS): Stack with multipop

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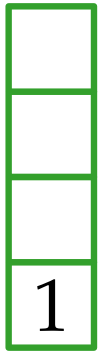

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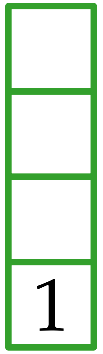

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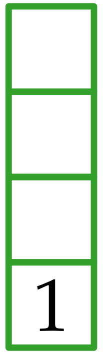

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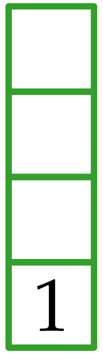

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$$
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\text{total cost} &= \Phi_0 - \Phi_{\text{end}} + \text{amortized cost} \\
&\leq \Phi_0 - \Phi_{\text{end}} + 2n \\
&\leq 2n \;= O(n)
\end{aligned}
$$

Why is Splay Fast?

$w(x)$: weight of x (here 1), $W = \sum w(x)$ (here n)

$s(x)$: sum of all $w(x)$ in subtree of x_i

mark edges:

→ $s(\text{child}) \leq s(\text{parent})/2$

→ $s(\text{child}) > s(\text{parent})/2$

Cost to query x_i: $O(\log W + \#\text{red})$

Idea: blue edges halve the weight

$\Rightarrow \#\text{blue} \in O(\log W)$

How can we amortize red edges?

Use sum-of-logs potential

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Amortized cost:

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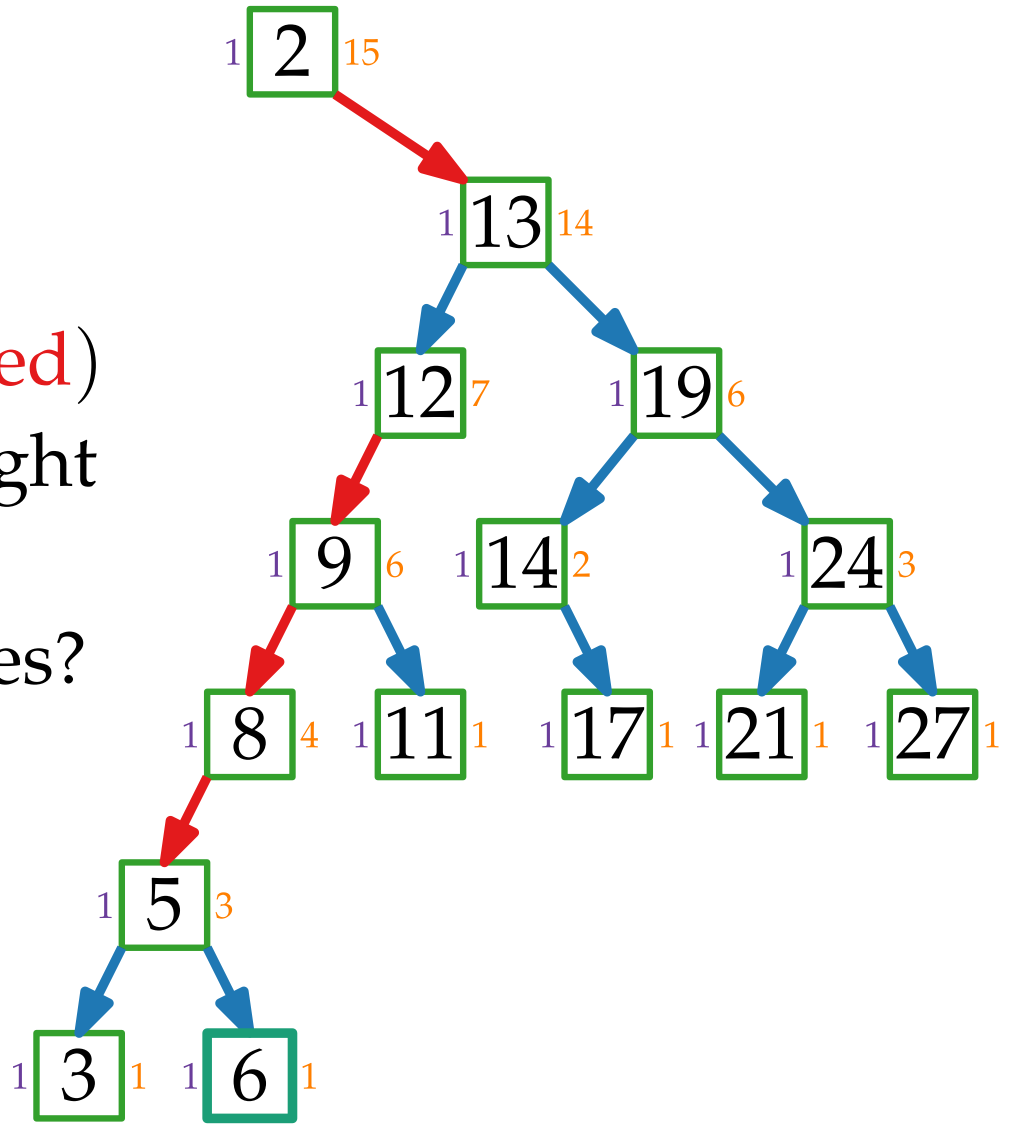
(potential after splay)



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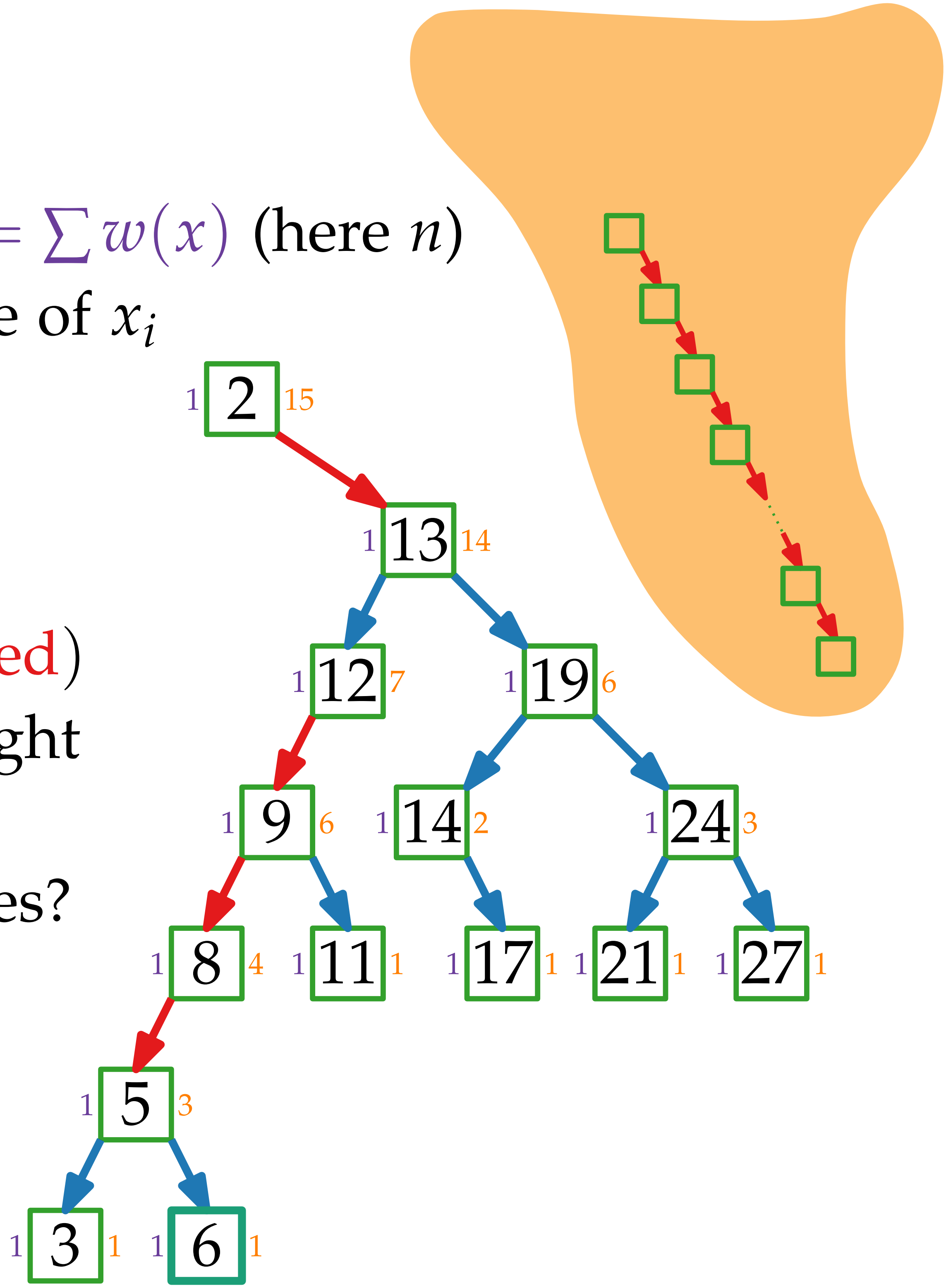

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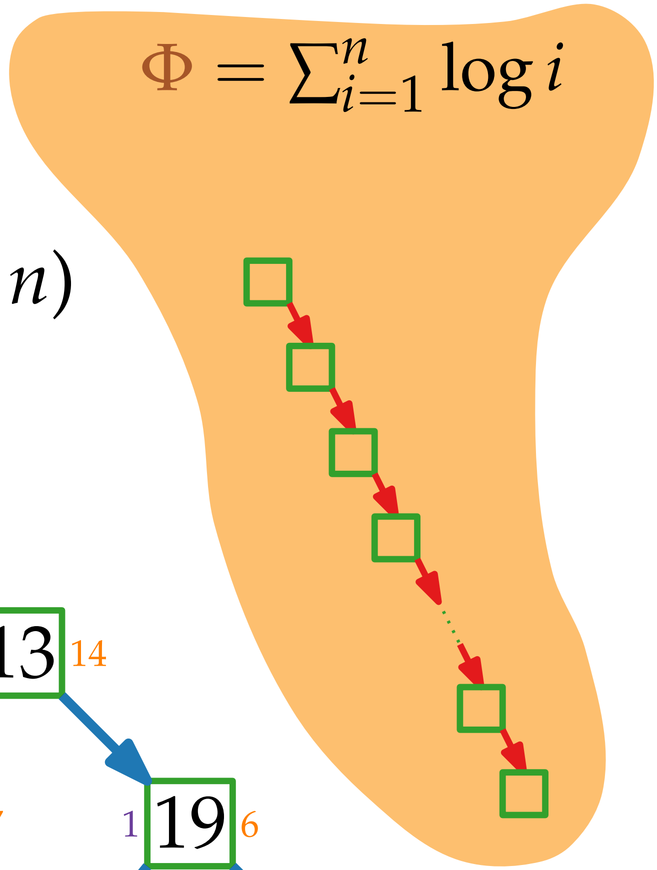

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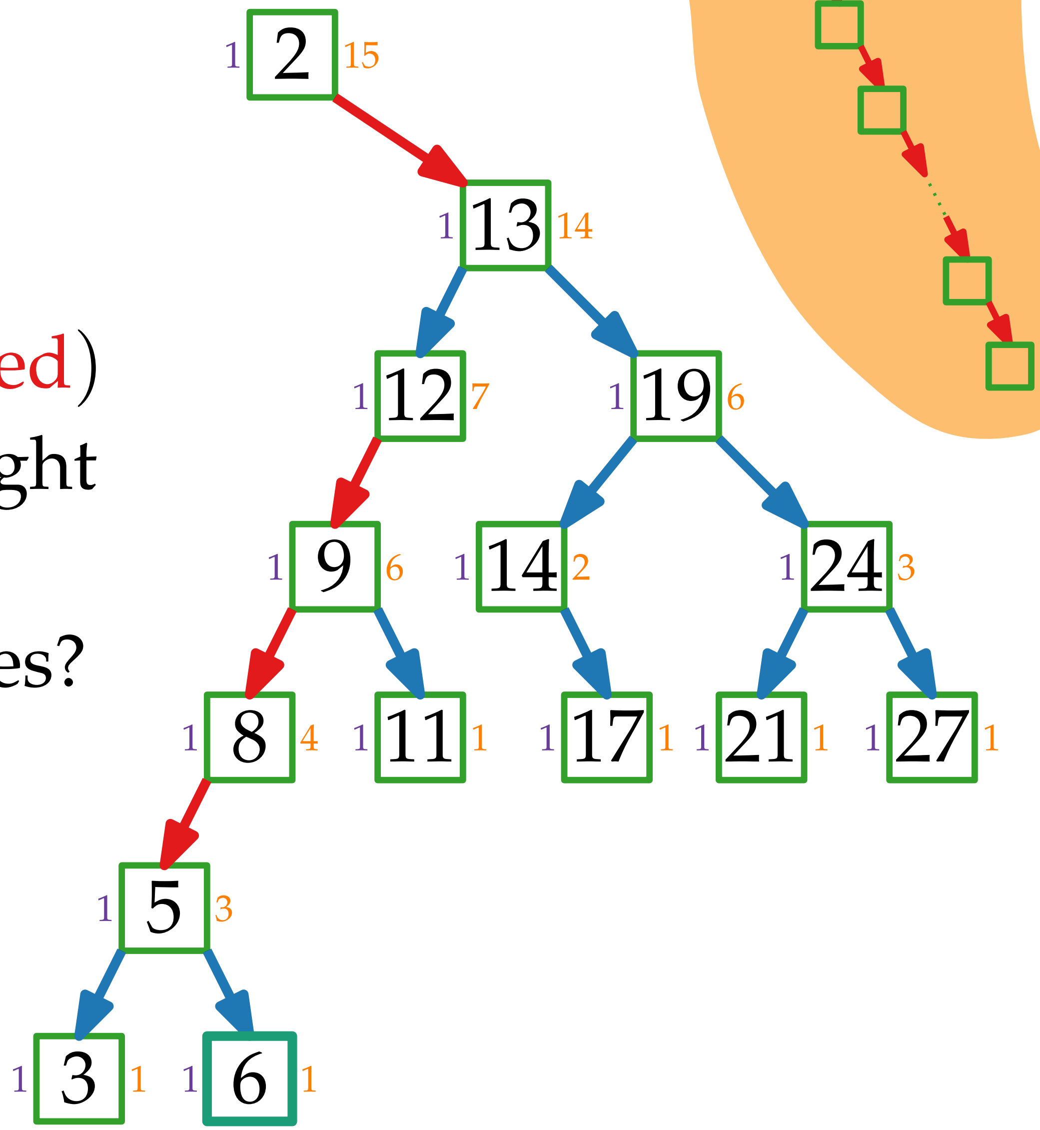

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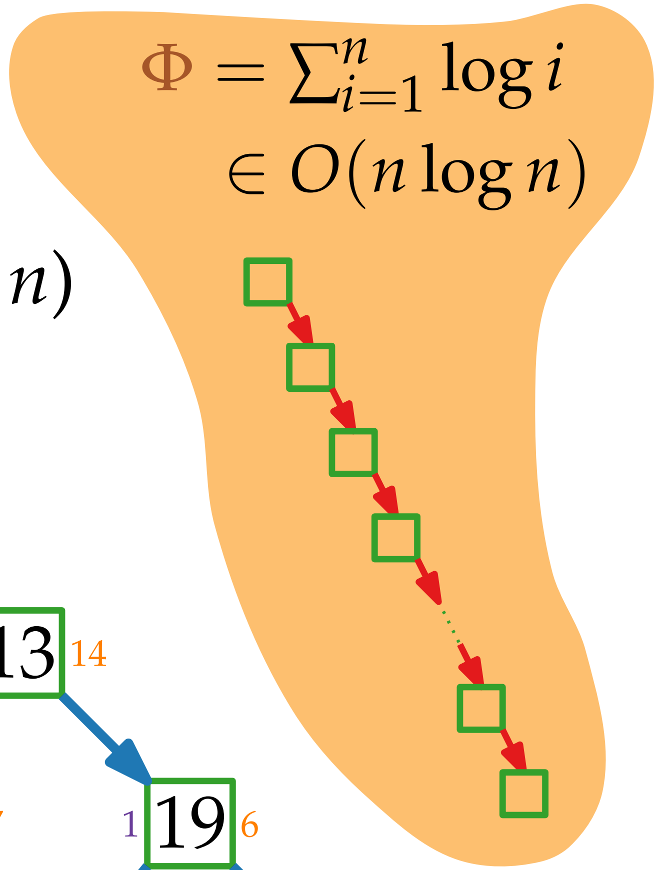

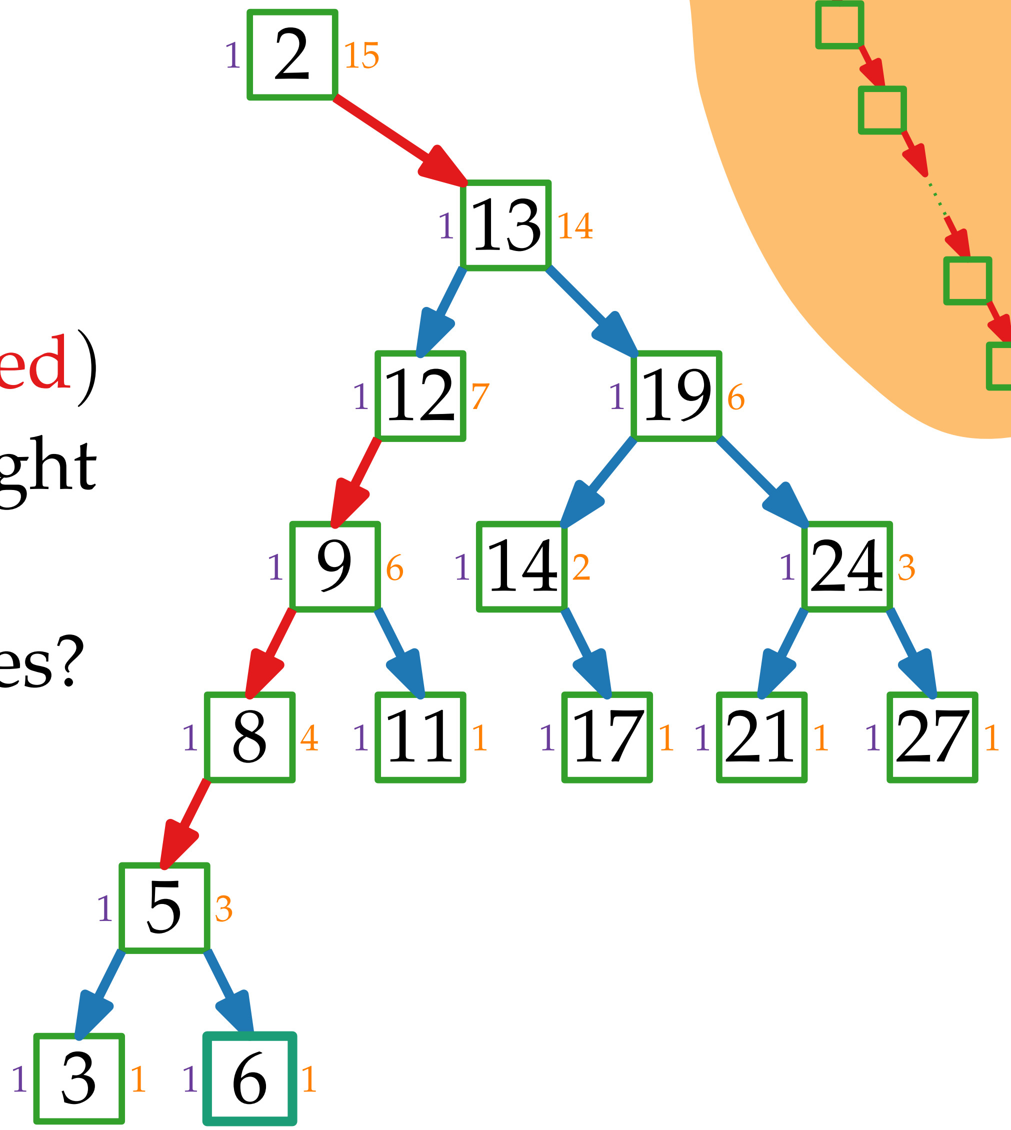

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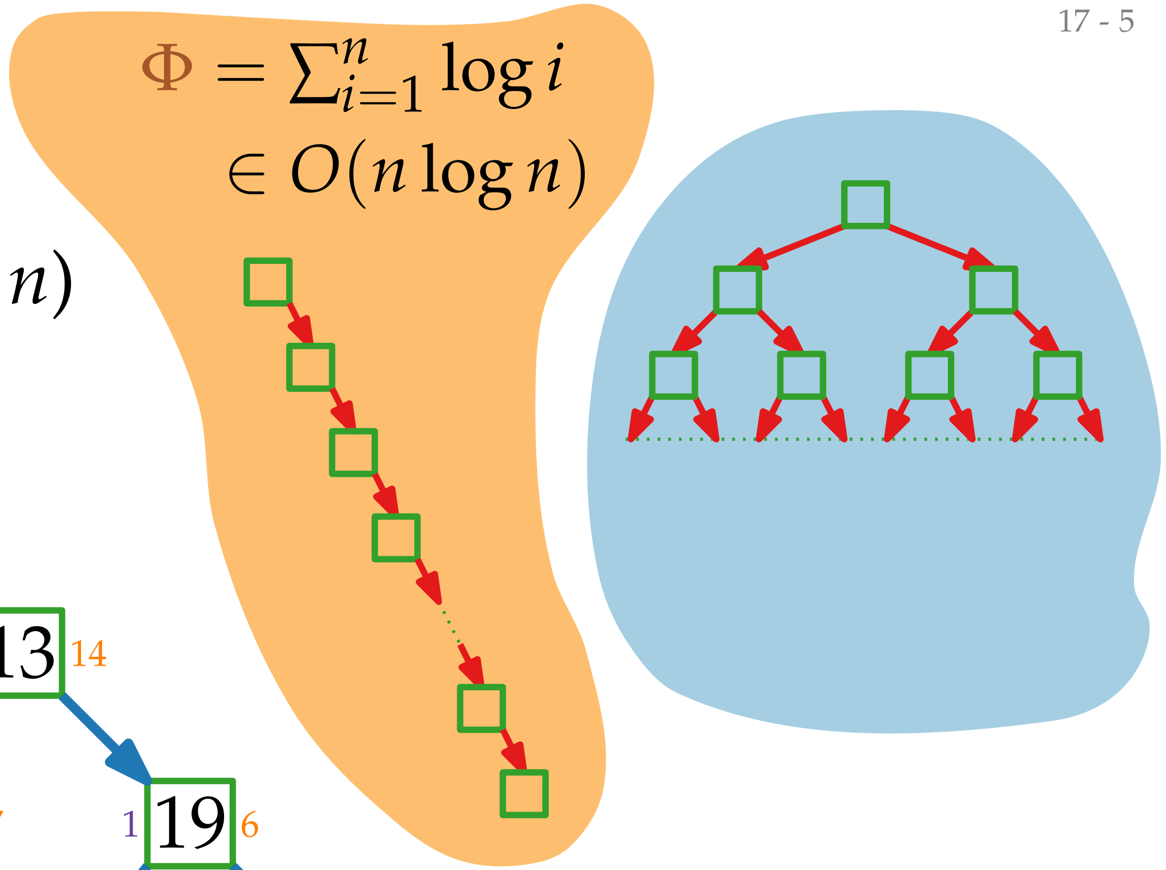

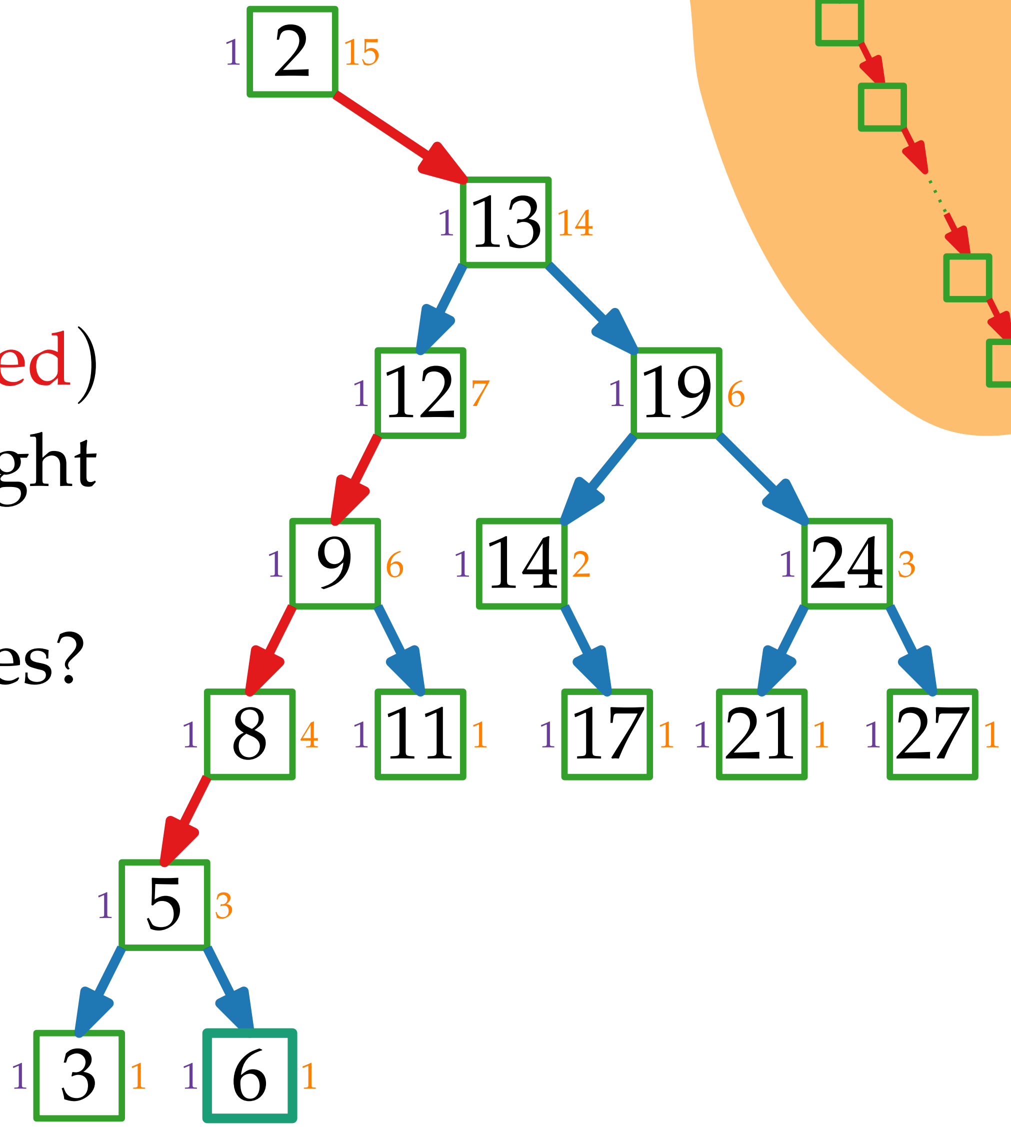

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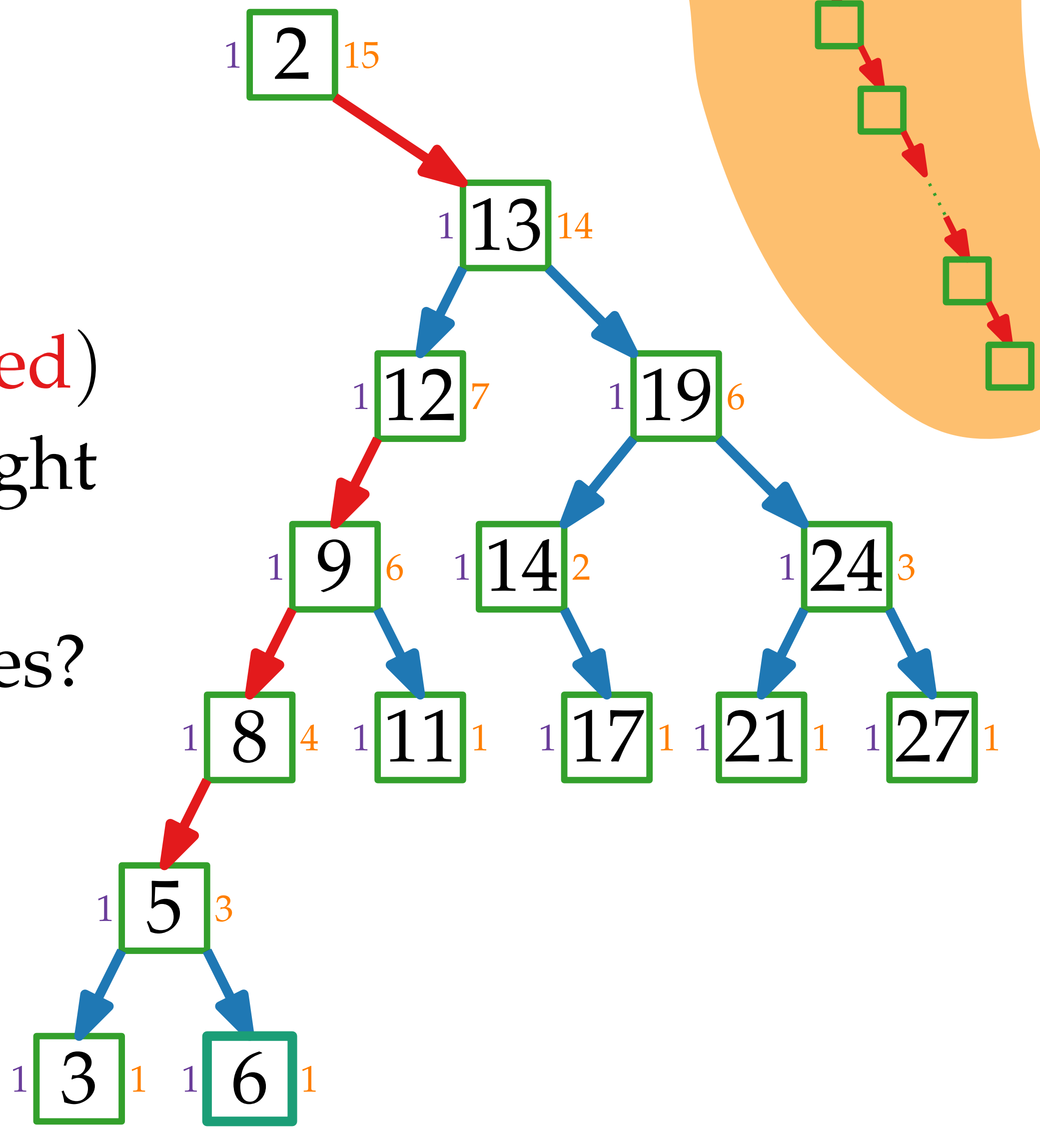

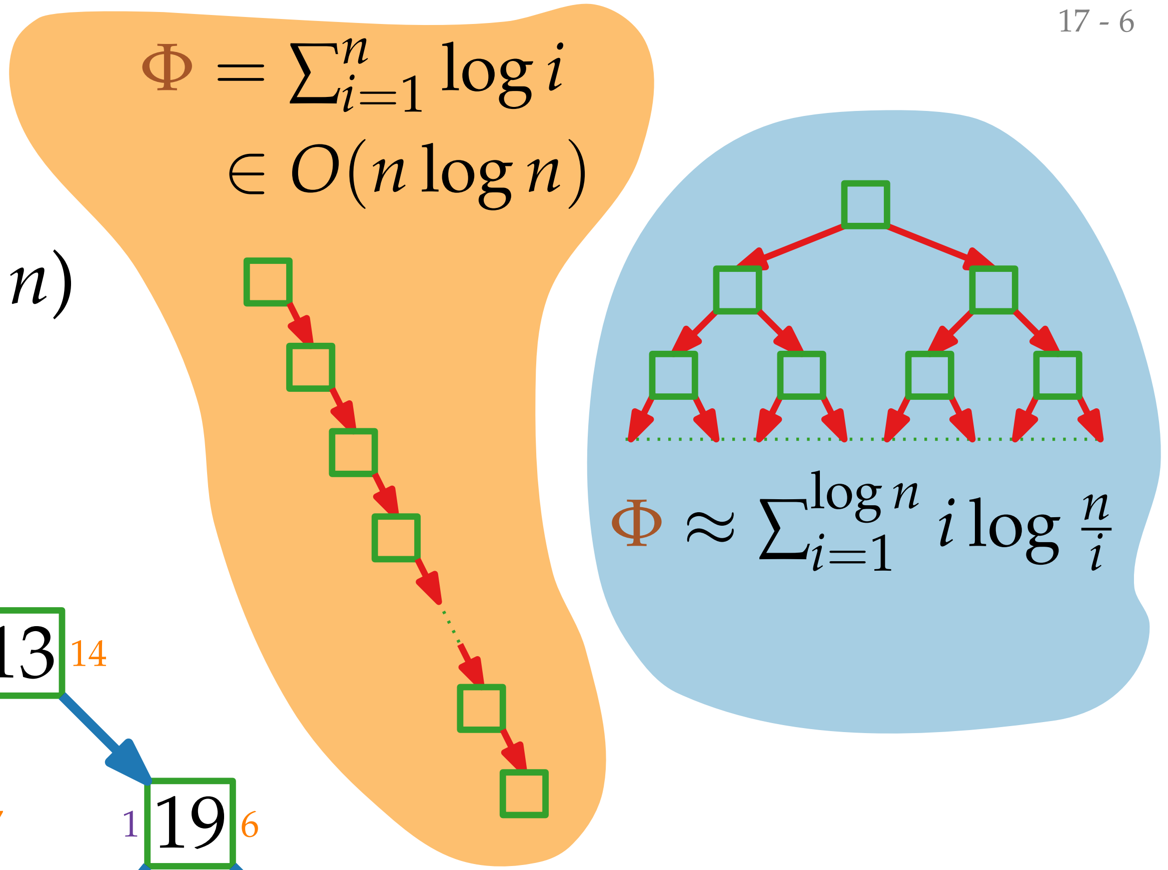

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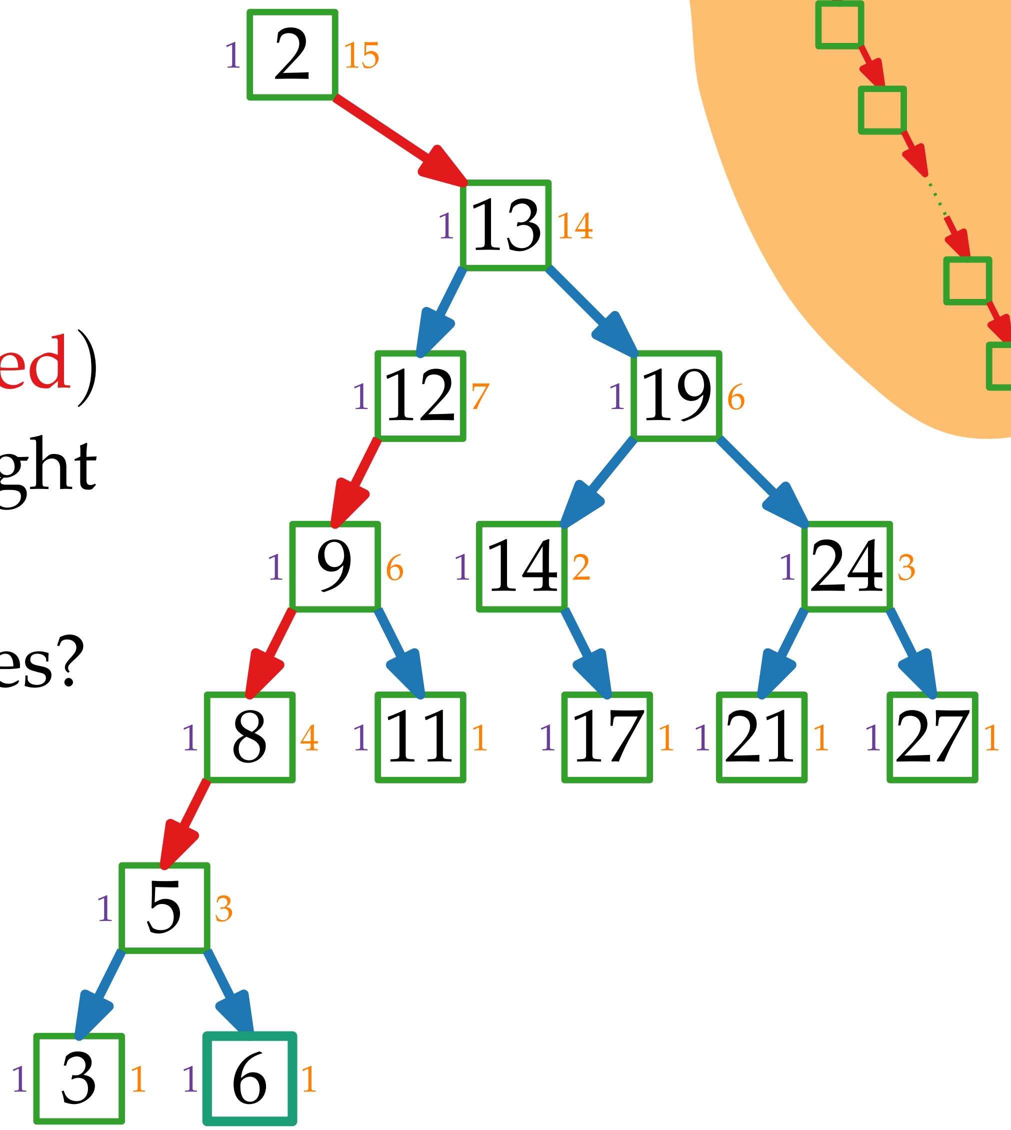
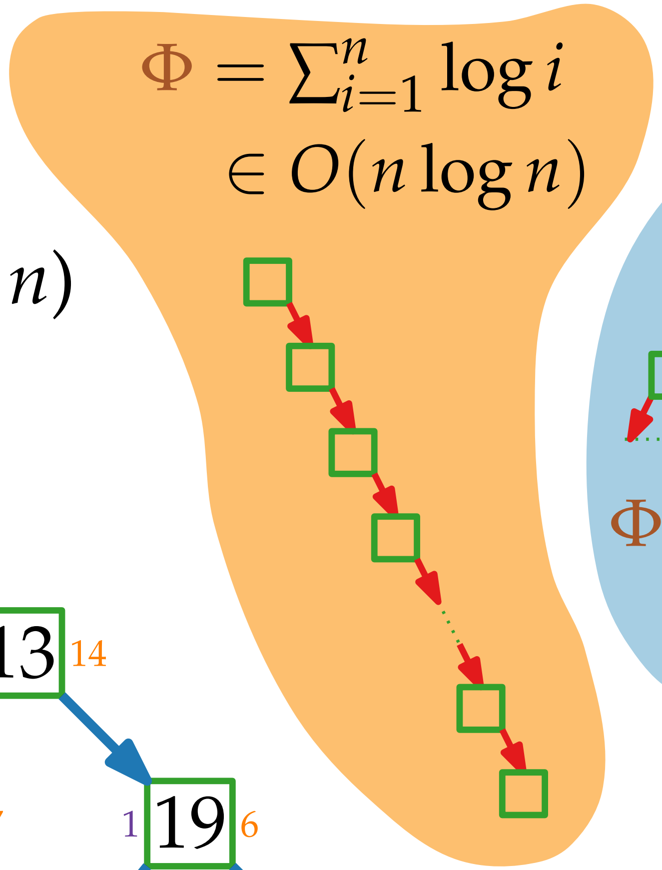
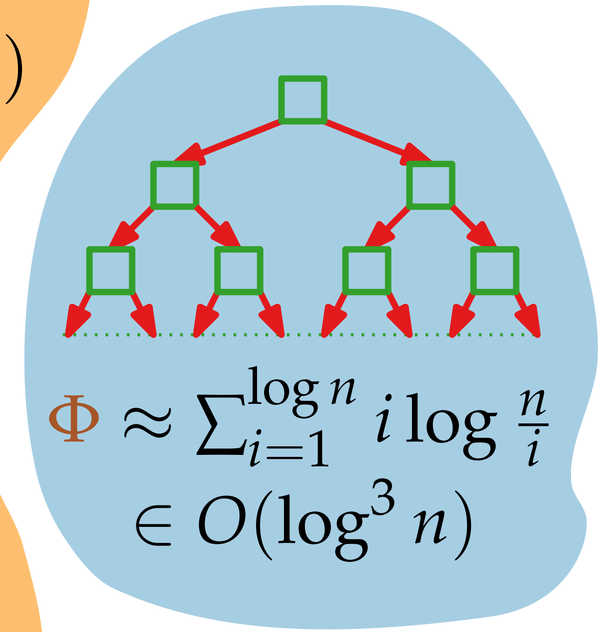




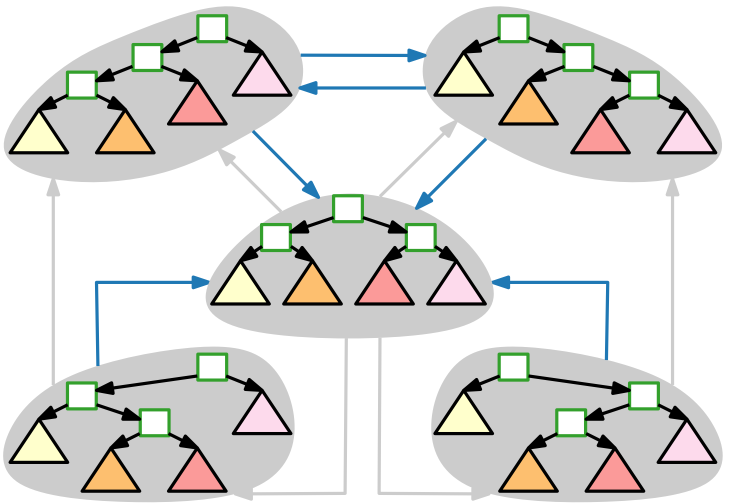


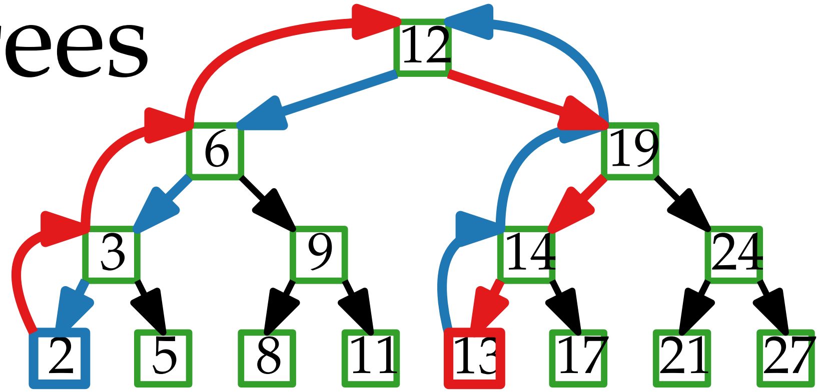
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Optimal Binary Search Trees

Splay Trees





Philipp Kindermann · WS20

Part V:
Access Lemma and Running Time of Splay

Potential after Rotation

Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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Lemma. After a single rotation, the potential increases by $\leq 3\left(\log s_+(x) - \log s(x)\right)$.

Proof. $\mathrm{Right}(x)$

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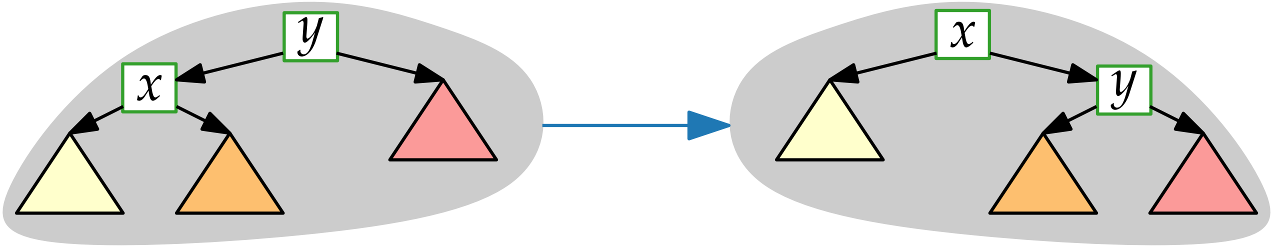
Proof. Right(x)



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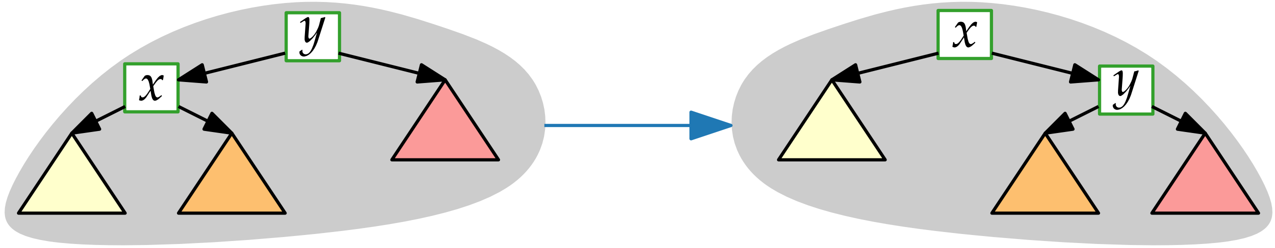

Observe: Only $s(x)$ and $s(y)$ change.

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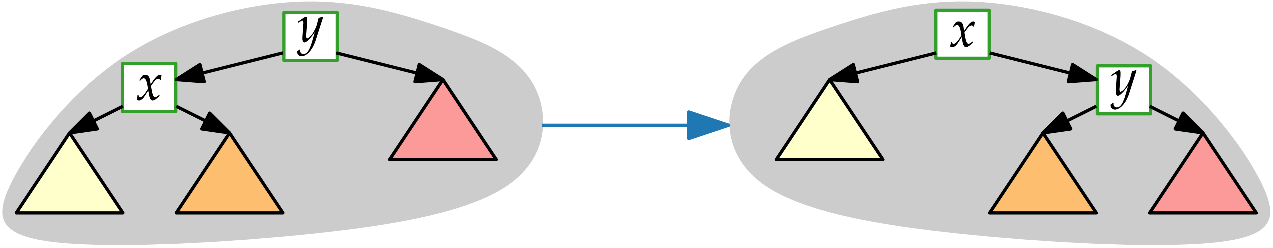

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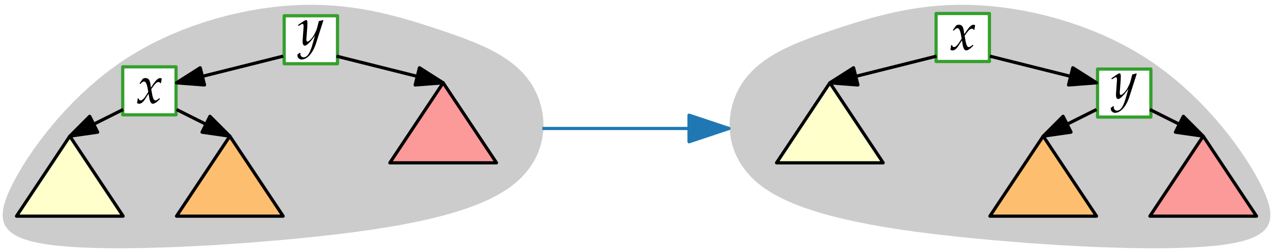

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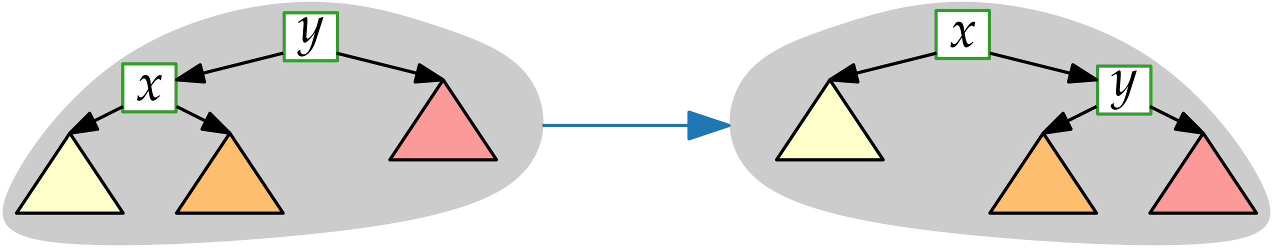

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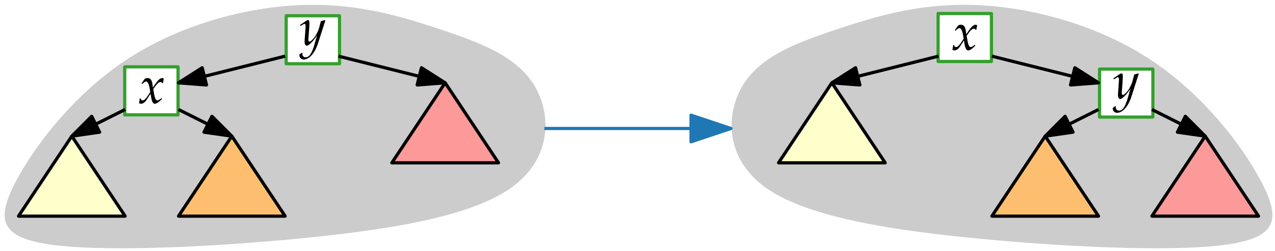

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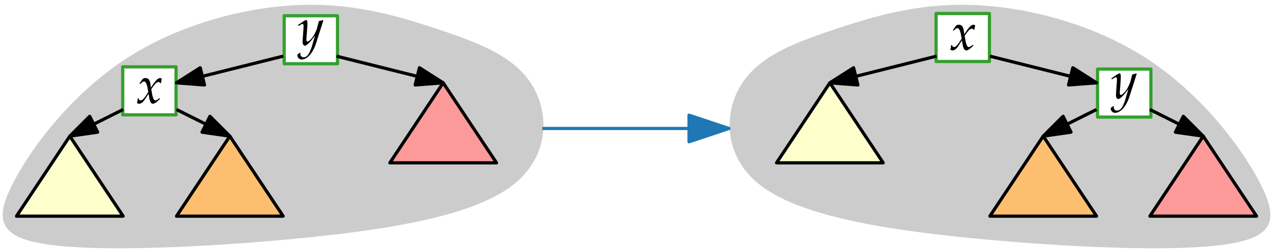

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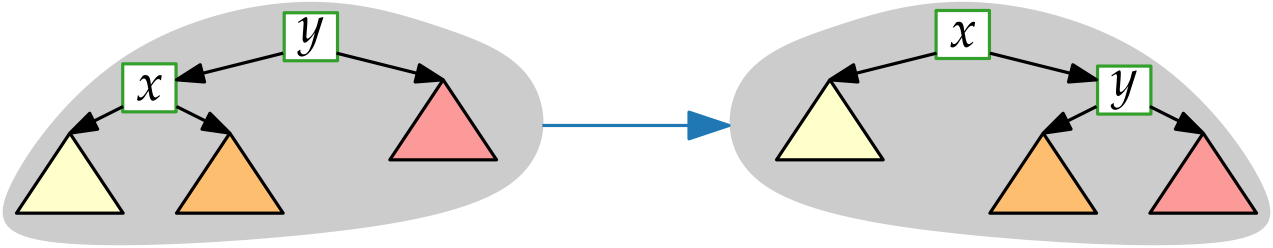

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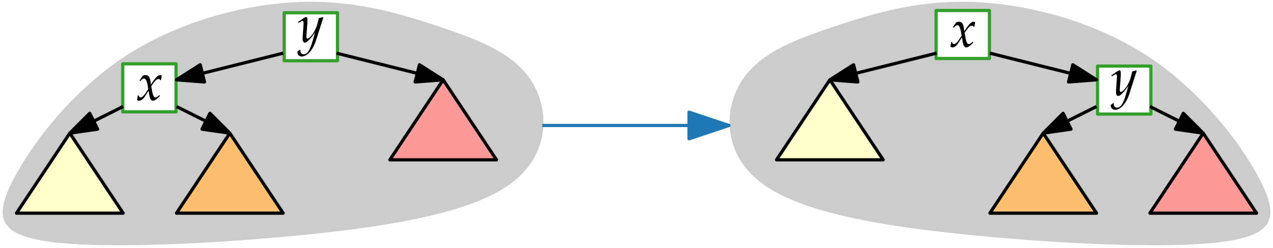

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$\mathrm{Left}(x)$ analogue $\checkmark$

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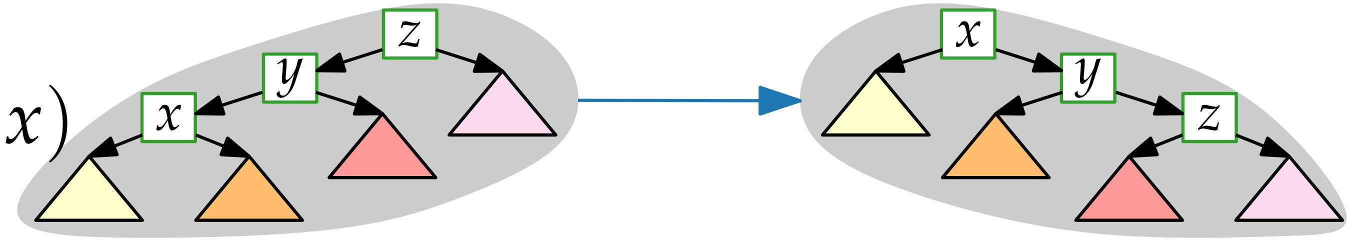

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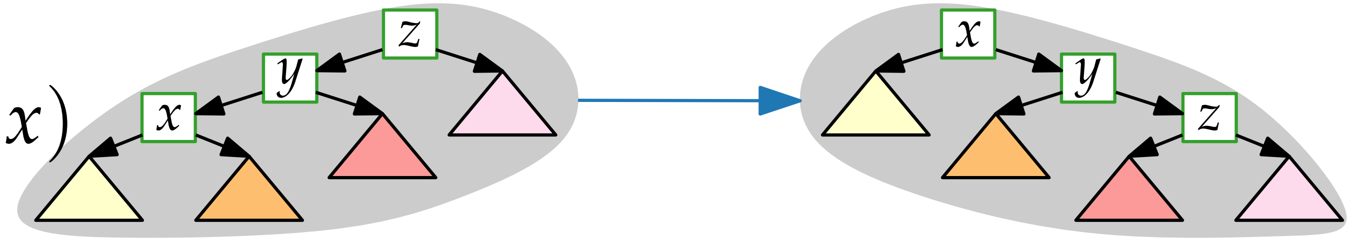

$$\begin{aligned}\text{pot. change} \quad &= \log s_+(x) + \log s_+(y) + \log s_+(z) \\ &\quad - \log s(x) - \log s(y) - \log s(z)\end{aligned}$$

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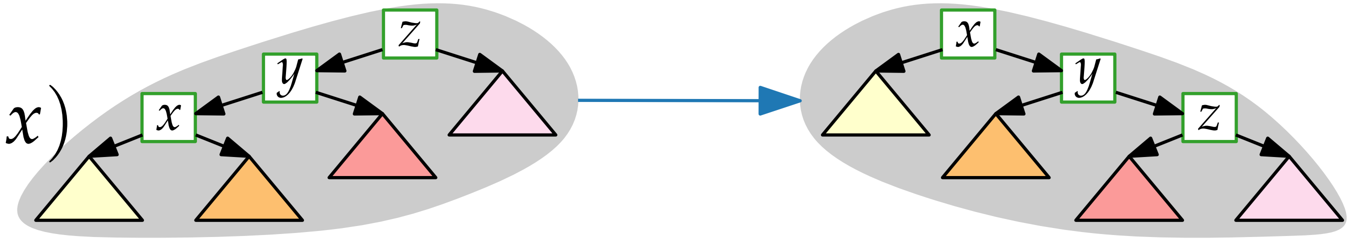

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$- \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$

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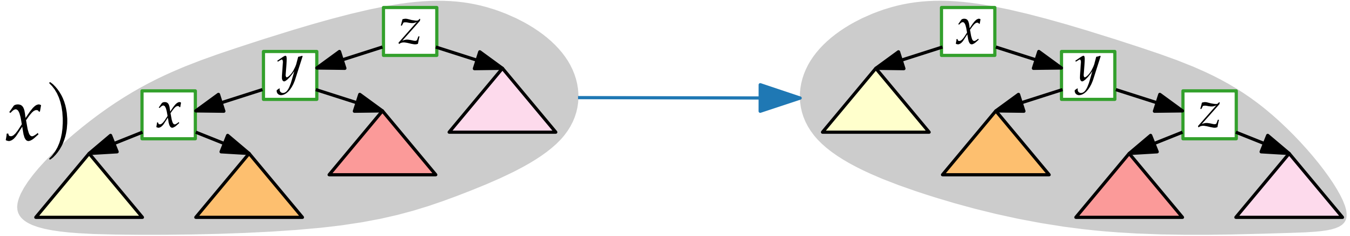

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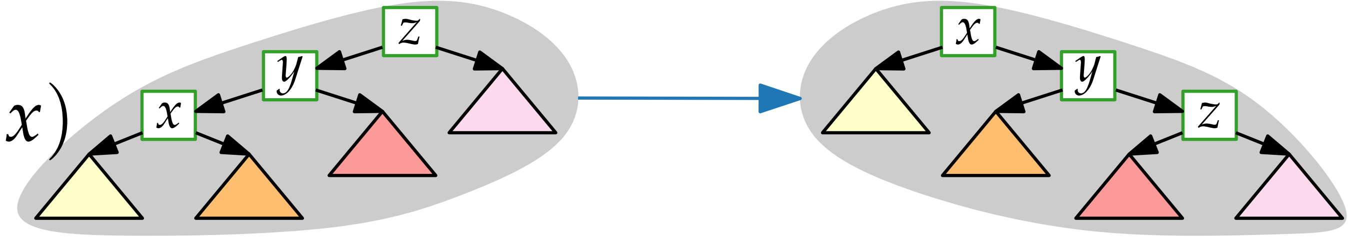
Lemma. After a double rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x)) - 2$.

Proof.

Case 1. Right-Right(x)



$$\text{pot. change} = \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$$

$$(s_+(x) = s(z)) \quad = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$$

$(s(x) \leq s(y))$

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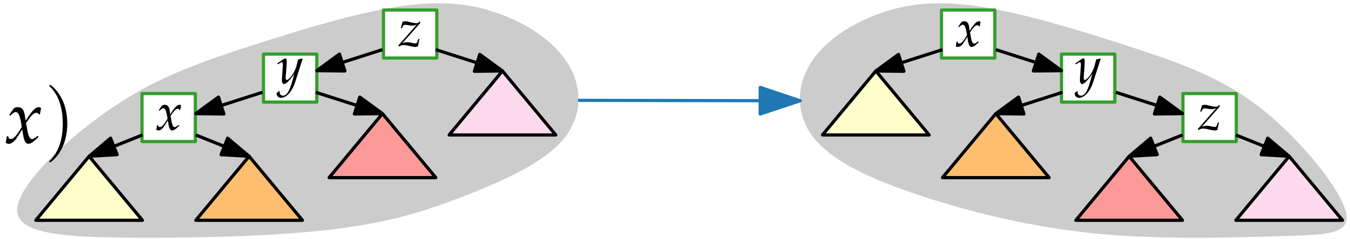

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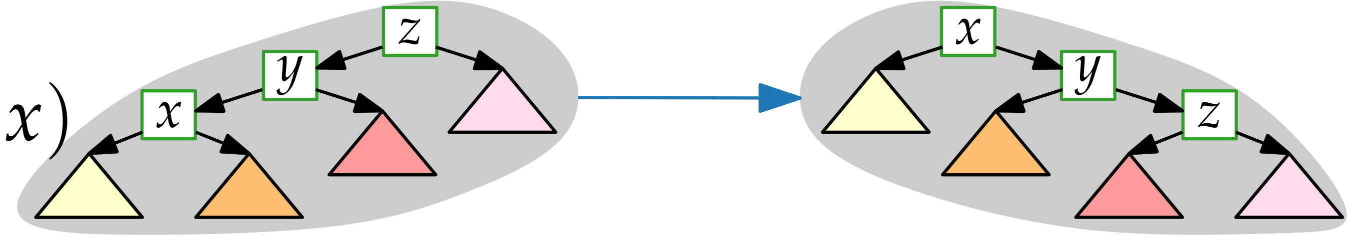

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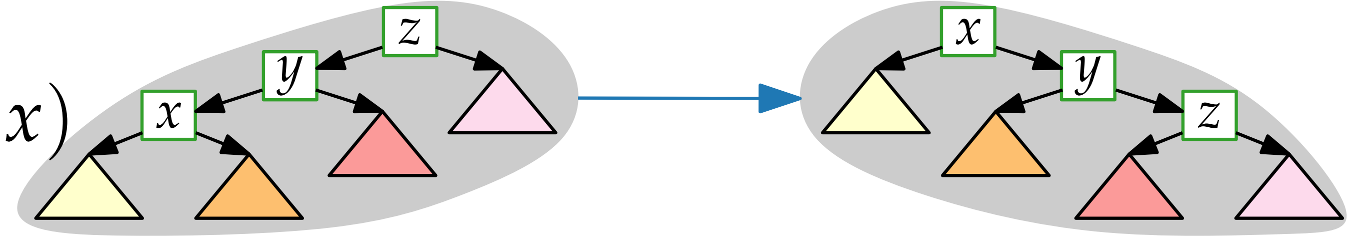

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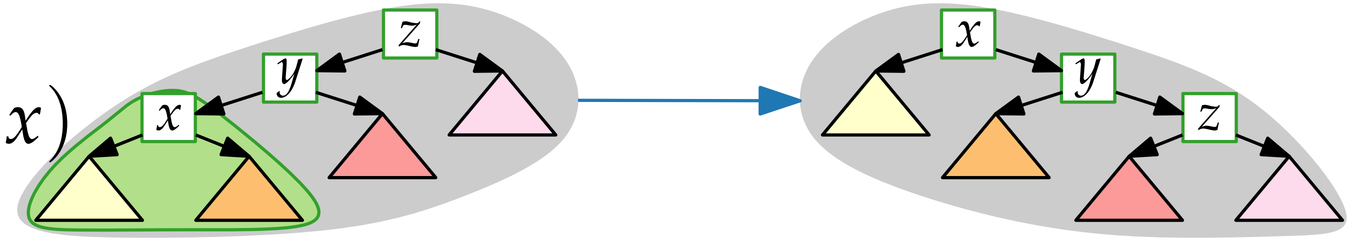

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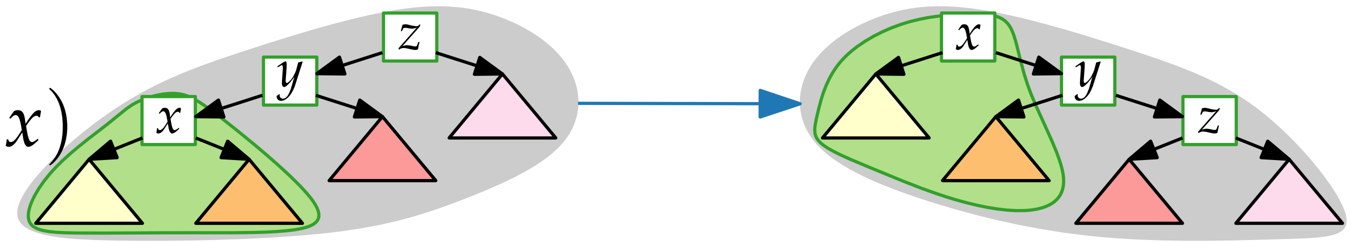

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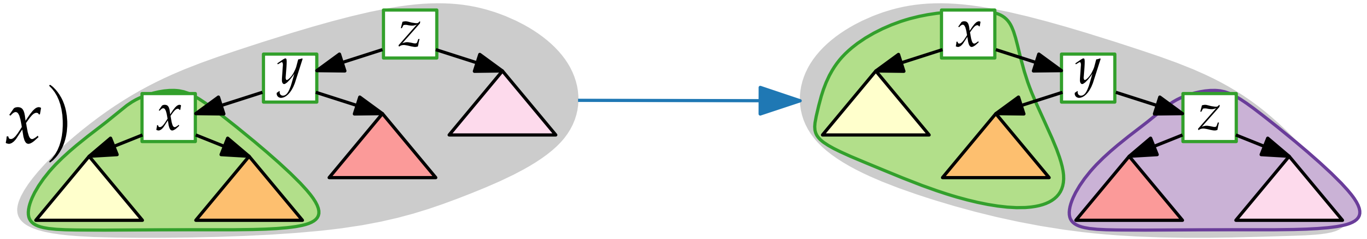

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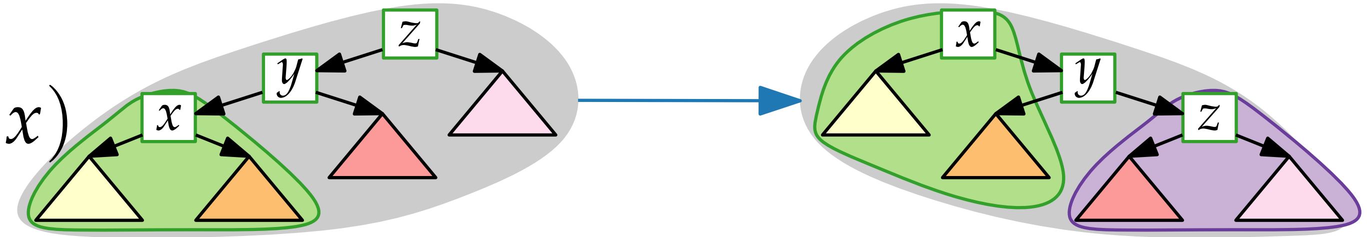

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$s(x) + s_+(z) \leq s_+(x)$

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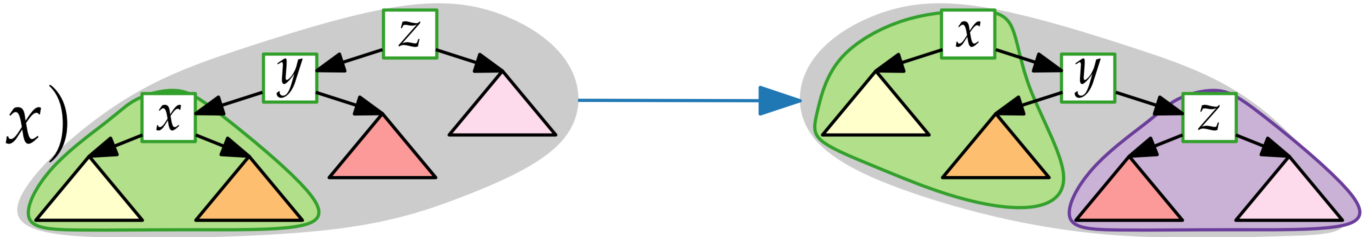

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(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x)
\end{aligned}
$$

$s(x) + s_+(z) \leq s_+(x) \Rightarrow \log s(x) + \log s_+(z)$

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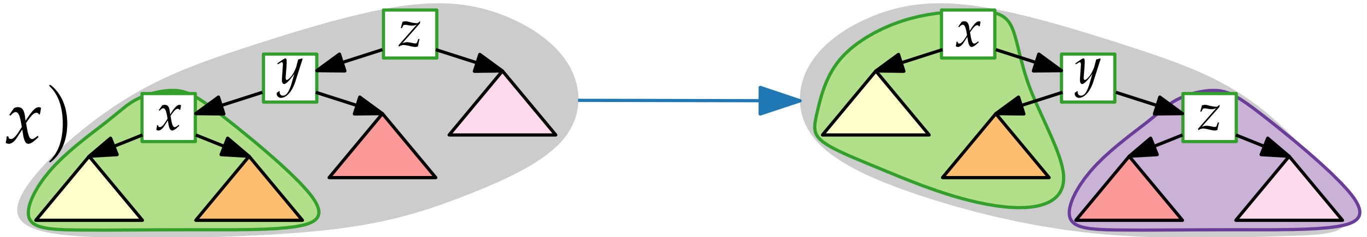

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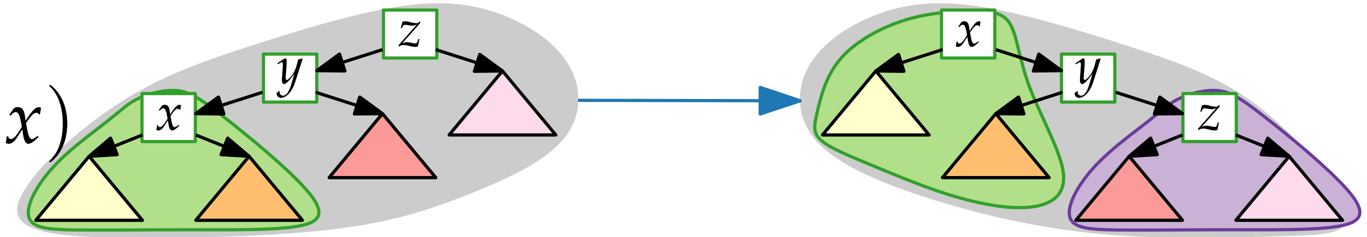

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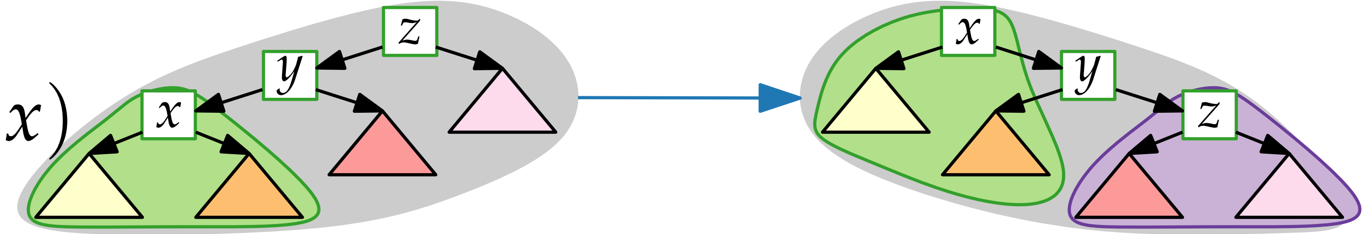

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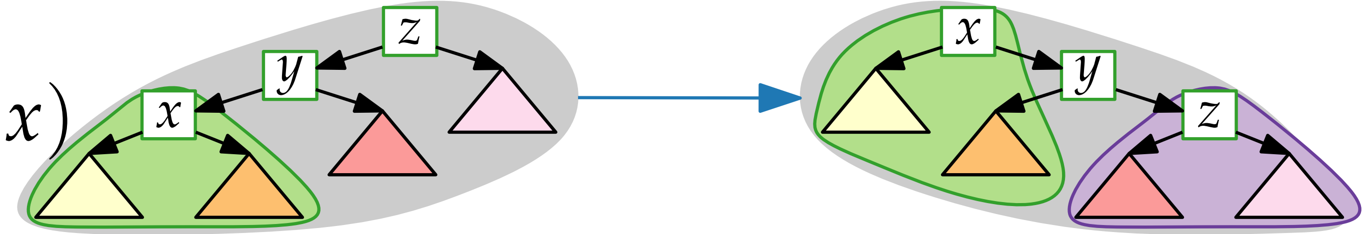

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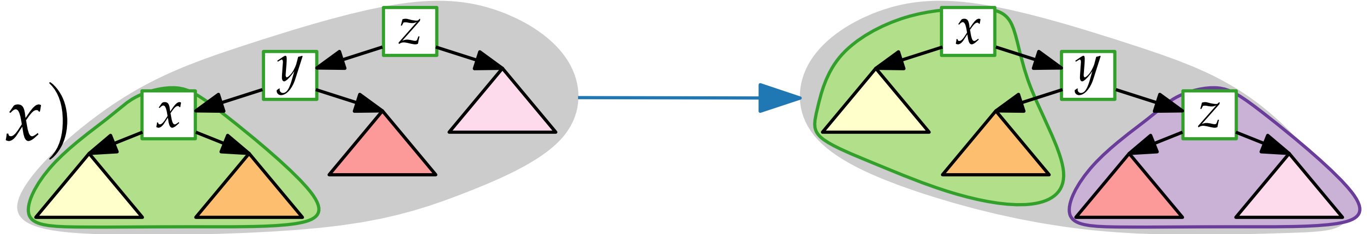

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(s_+(y) \leq s_+(x)) \quad &\leq \log s_+(x) + \log s_+(z) - 2\log s(x) \\
&\leq 3\log s_+(x) - 3\log s(x) - 2
\end{aligned}$$

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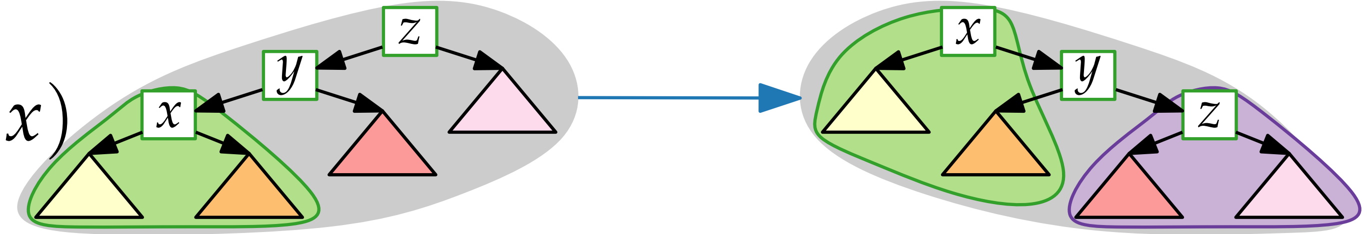

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(s_+(x) = s(z)) \quad & = \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) \\
(s(x) \leq s(y)) \quad & \leq \log s_+(y) + \log s_+(z) - 2\log s(x) \\
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& \leq 3\log s_+(x) - 3\log s(x) - 2 \quad \checkmark
\end{aligned}
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Proof. / Left-Left(x)

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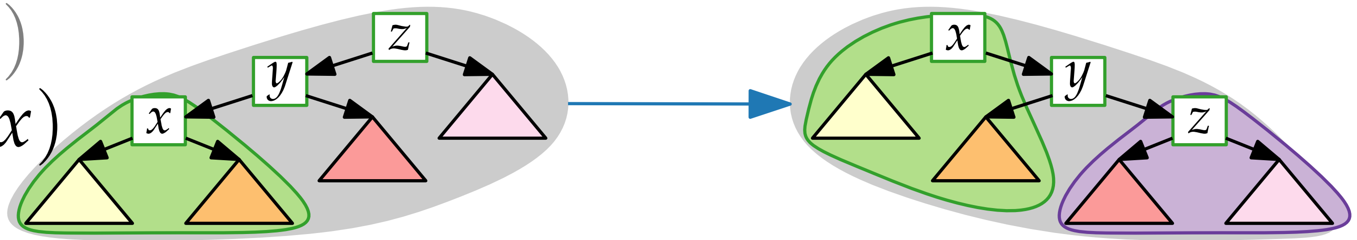

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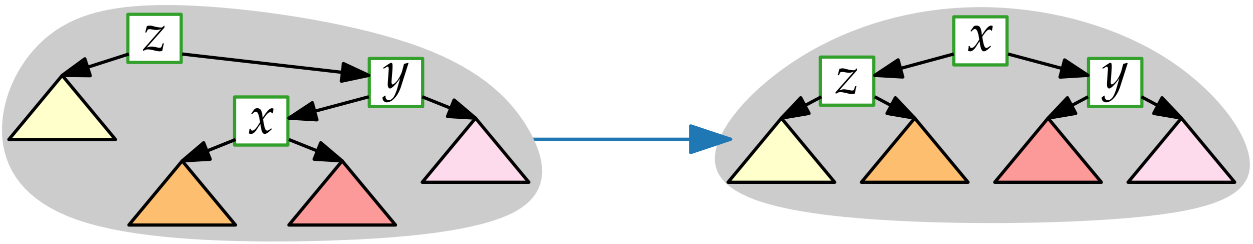

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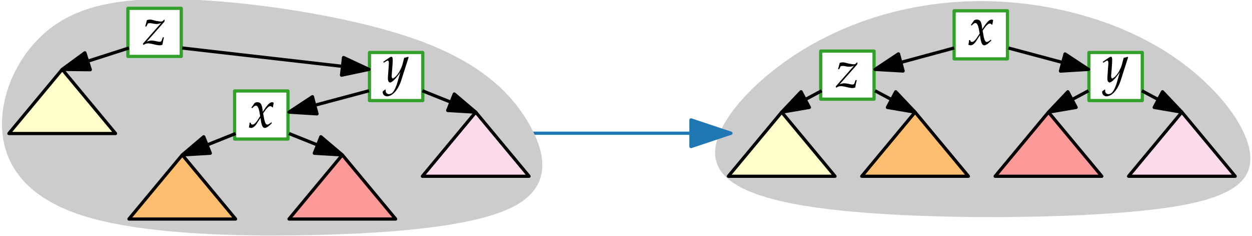

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z)$
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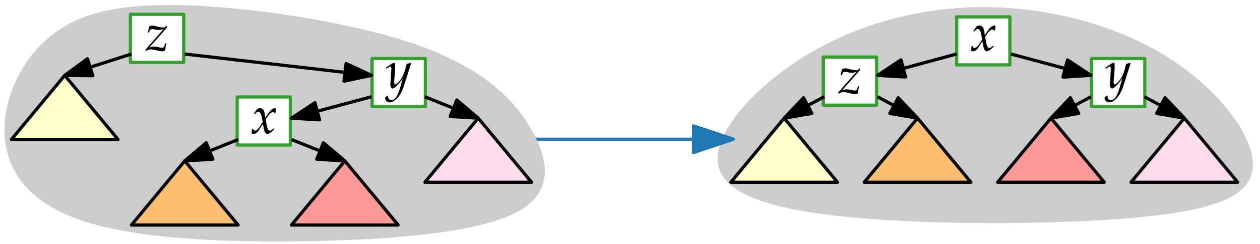

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$(s_+(x) = s(z))$ $= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

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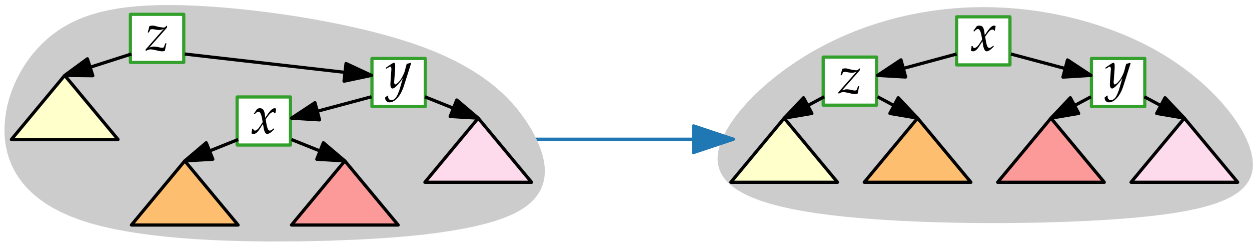

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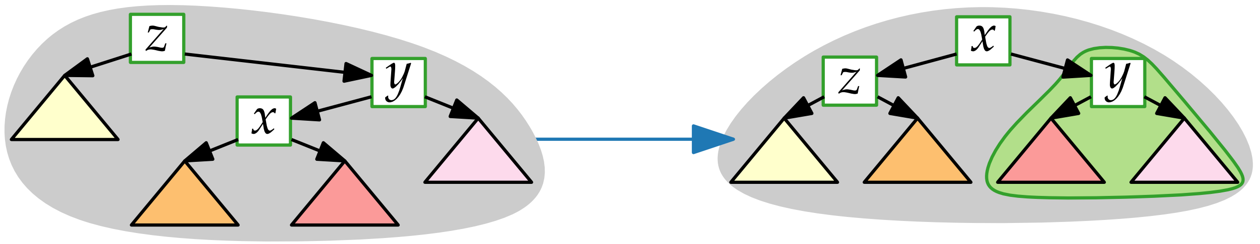

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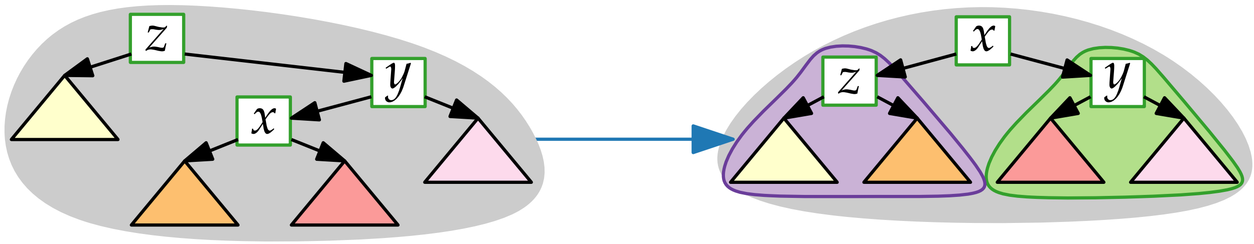

pot. change $= \log s_+(x) + \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y) - \log s(z)$

$(s_+(x) = s(z))$ $= \log s_+(y) + \log s_+(z) - \log s(x) - \log s(y)$

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Consider any rotation; $s(x)$ before rotation, $s_+(x)$ afterwards

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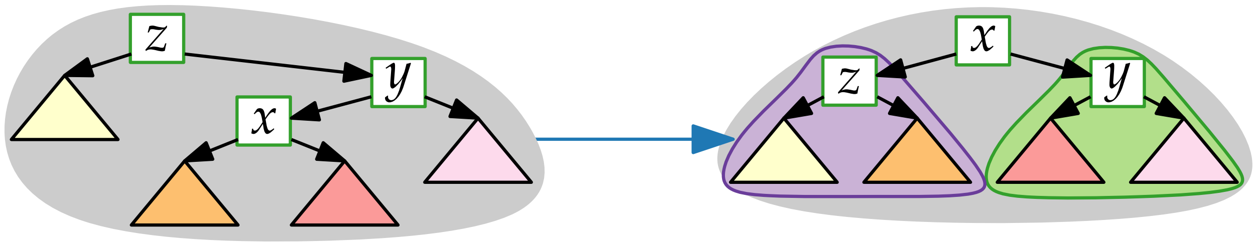

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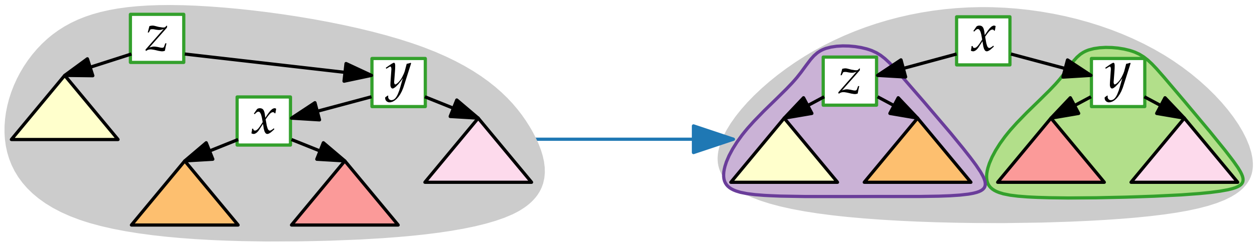

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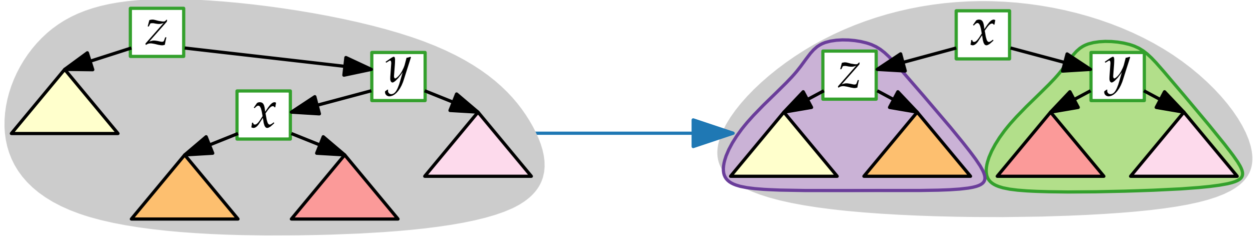

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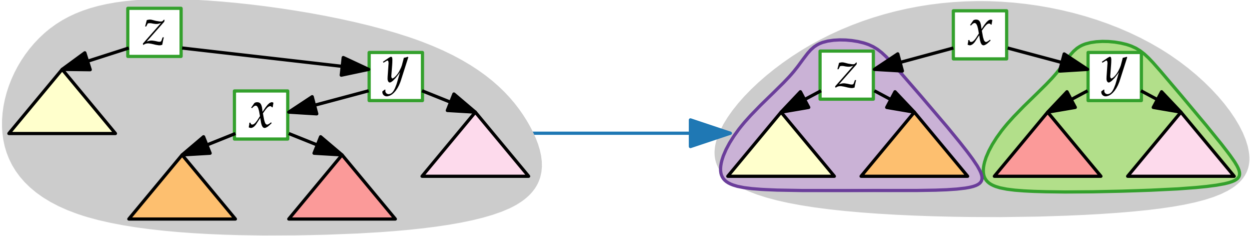

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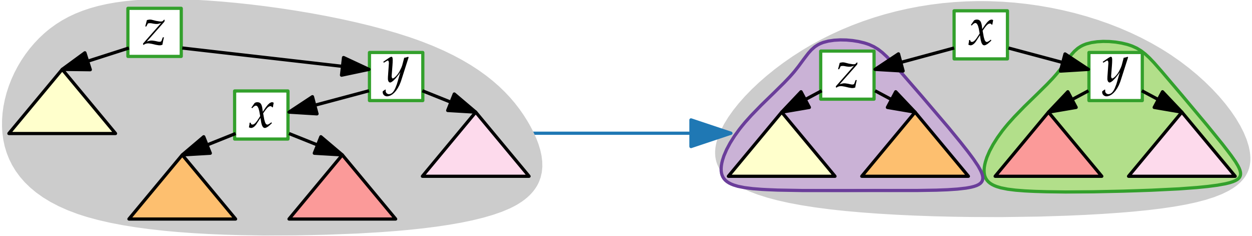

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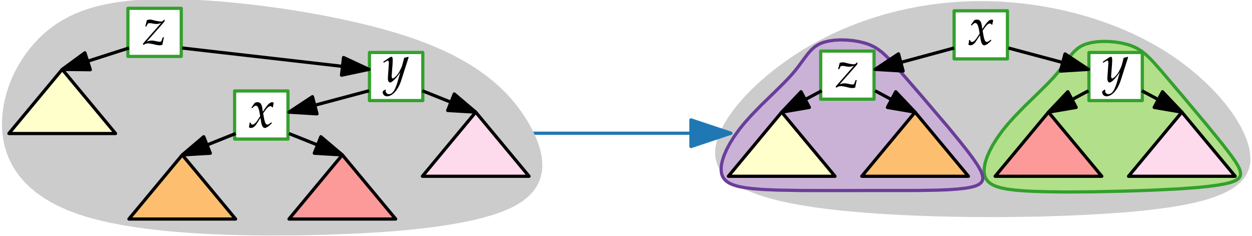

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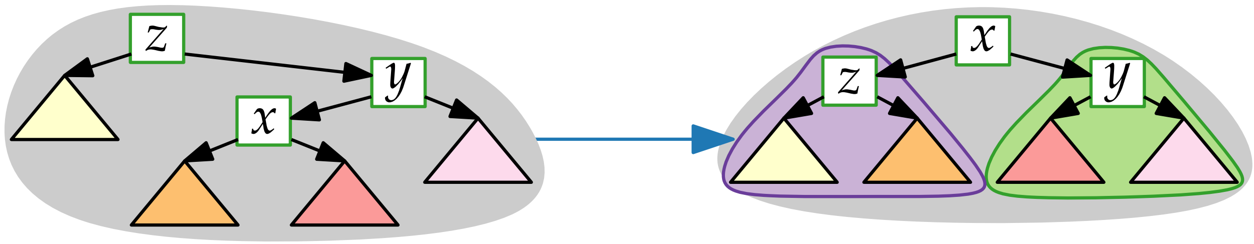

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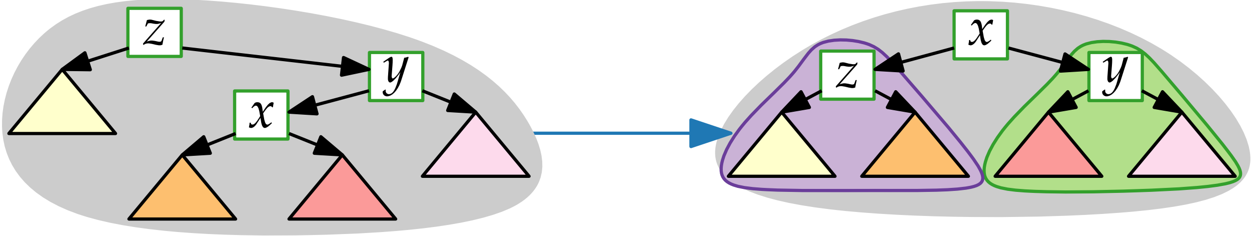

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Lemma. After a single rotation, the potential increases by $\leq 3\,(\log s_+(x) - \log s(x))$.
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$$= 3\,(\log s_{k+1}(x) - \log s(x)) - 2k$$

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\text{root!} \quad & = 3\,(\log s_{k+1}(x) - \log s(x)) - 2k \\
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$2k+1$ rotations $\Rightarrow$ (amort.) cost

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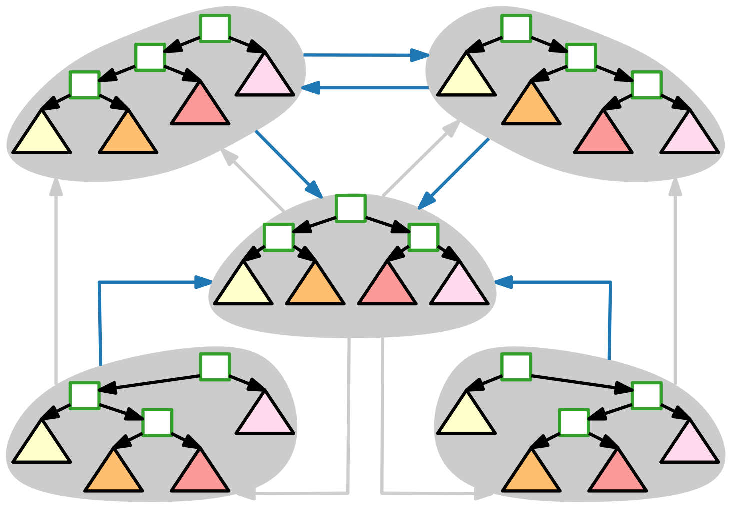
$2k+1$ rotations $\Rightarrow$ (amort.) cost $\leq 1 + 3\log(W/w(x))$ $\square$

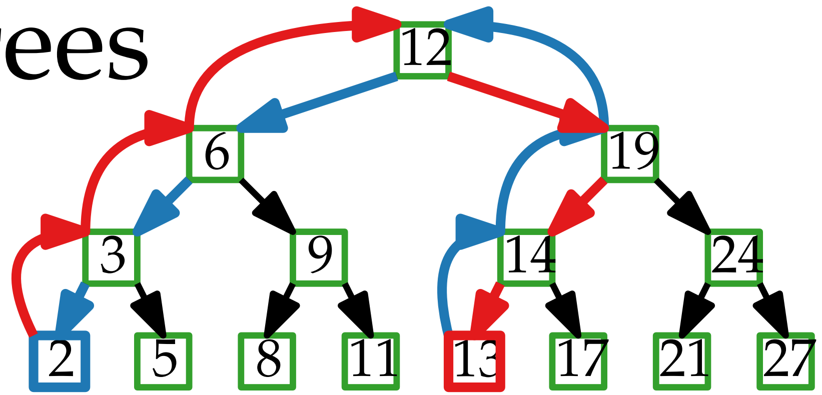
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Optimal Binary Search Trees

Splay Trees





Philipp Kindermann · WS20

Part VI:
Properties of Splay Trees

All These Properties...

Balanced:	Queries take (amort.) $O(\log n)$ time
Entropy:	Queries take expected $O(1 + H)$ time
Dynamic Finger:	Queries take $O(\log \delta_i)$ time (δ_i: rank diff.)
Working Set:	Queries take $O(\log t)$ time (t: recency)
Static Optimality:	Queries take (amort.) $O(\mathrm{OPT}_S)$ time.

... is there one BST to rule them all?

Yes!

Querying a Sequence

Let S be a sequence of queries.

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How can we bound $\Phi_0 - \Phi_{|S|}$?

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$\Rightarrow$ as long as every key is queried at least once, it doesn't change the asymptotic running time.

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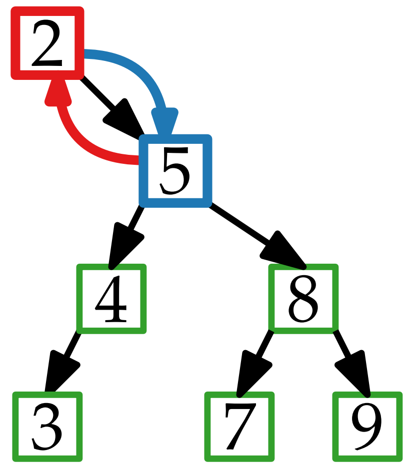

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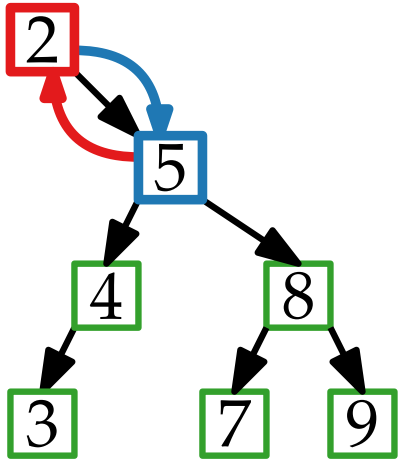

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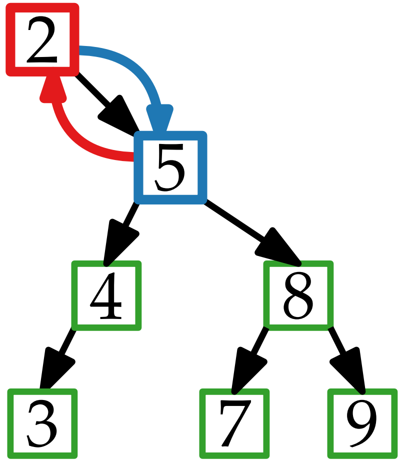

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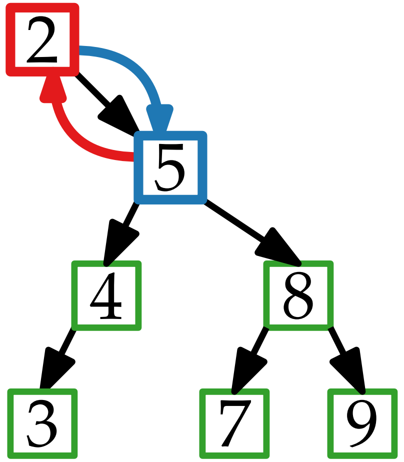

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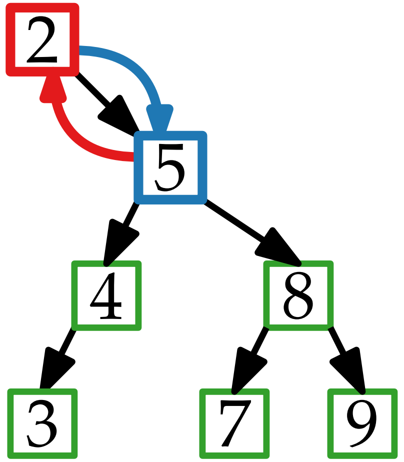

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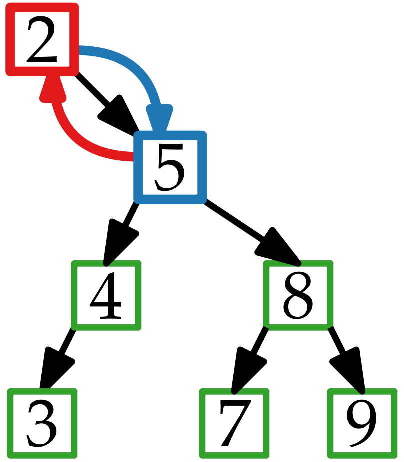

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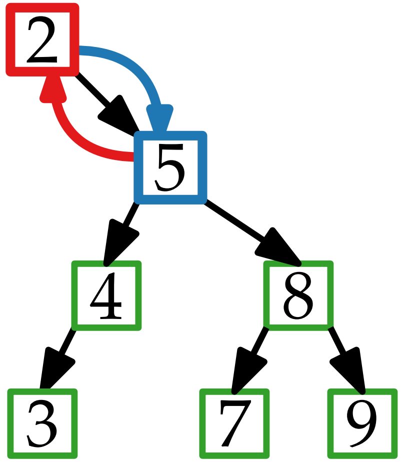

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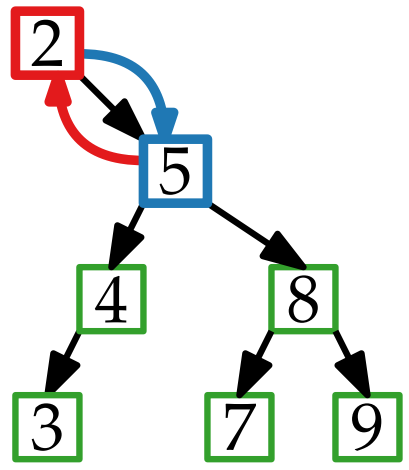

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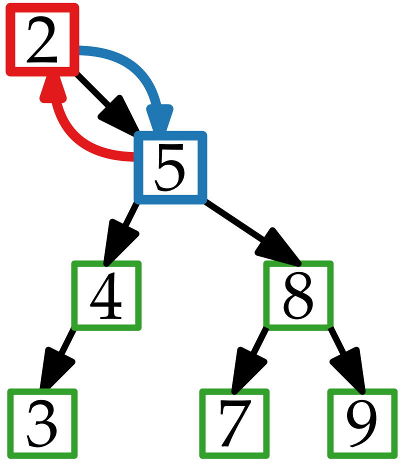

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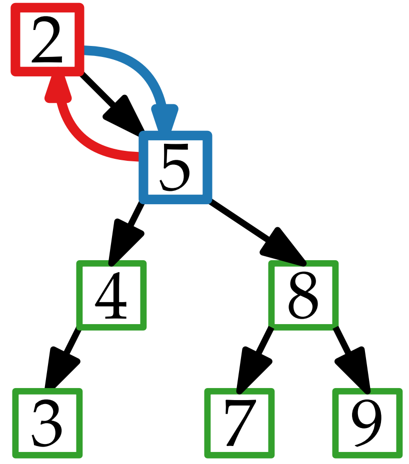

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Theorem. Splay Trees are statically optimal.

Proof. Let f_i be the number of items on path to x_i in T^*.
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Let T_S^* be the *optimal* static tree with the shortest query time OPT_S for S.

e.g. $S = 2, 5, 2, 5, 2, \ldots, 5$
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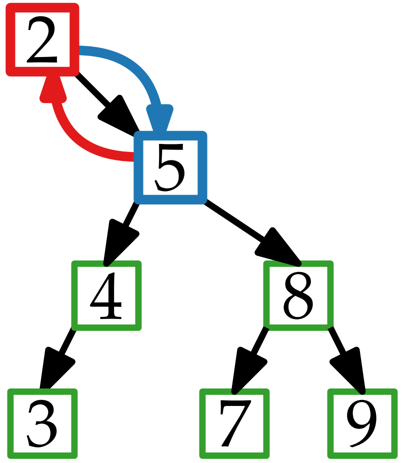

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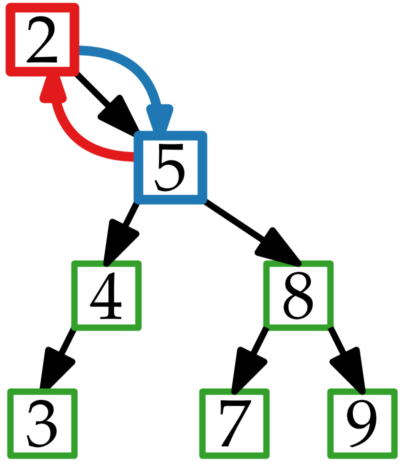

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Conjecture. Splay Trees are dynamically optimal.