

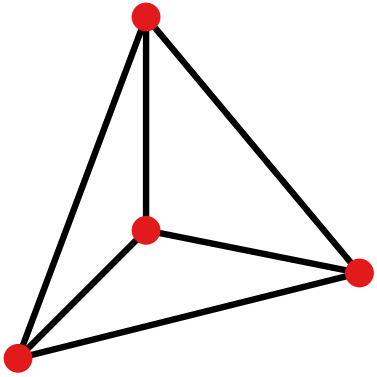
Hypergraph Representation via Axis-Aligned Point-Subspace Cover

Oksana Firman

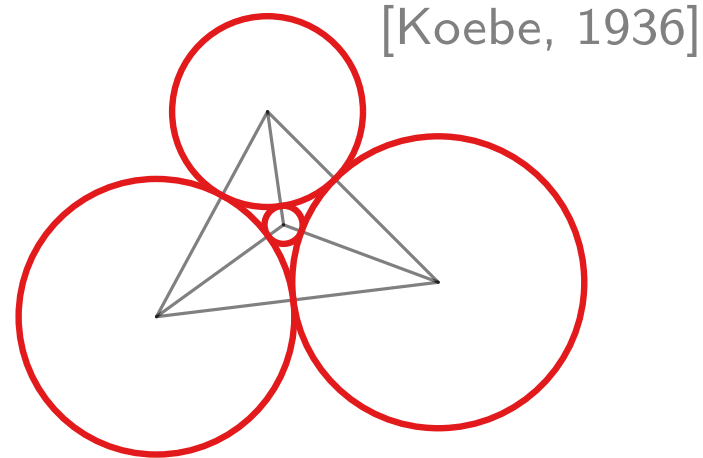
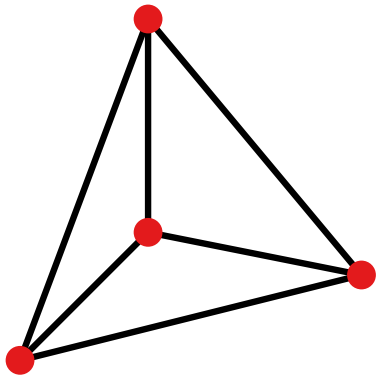
Joachim Spoerhase

Julius-Maximilians-Universität Würzburg, Germany

Geometric Representation of (Hyper)graphs

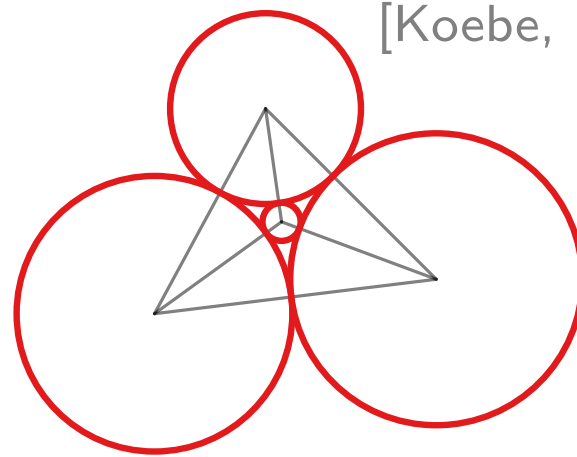
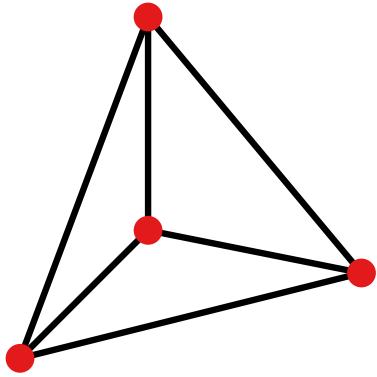


Geometric Representation of (Hyper)graphs



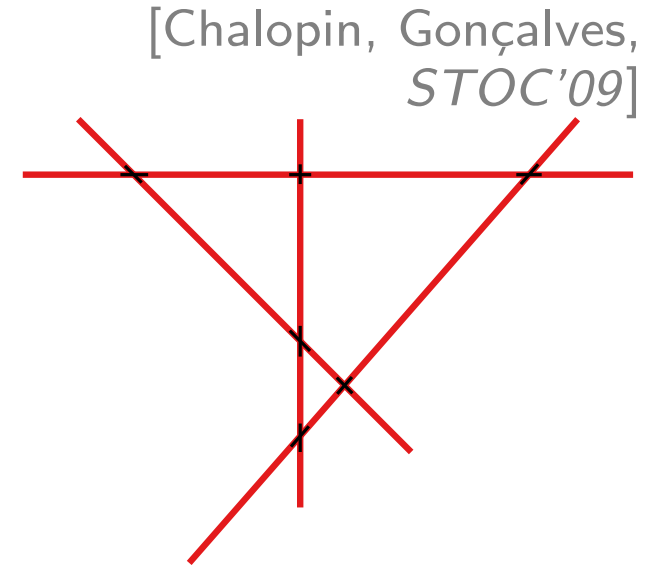
contact representation
by discs

Geometric Representation of (Hyper)graphs



[Koebe, 1936]

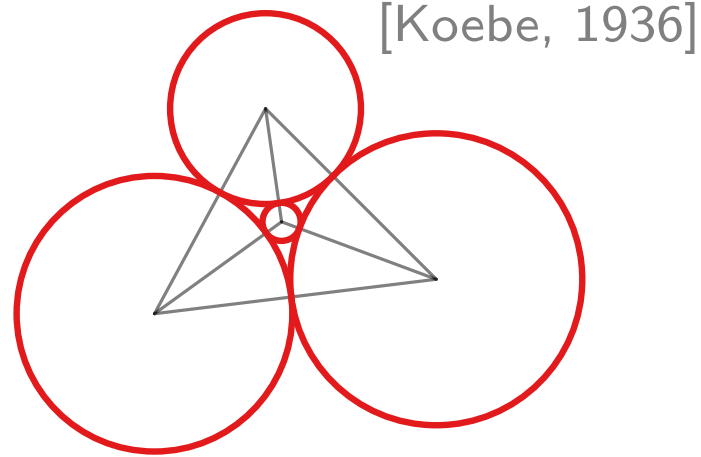
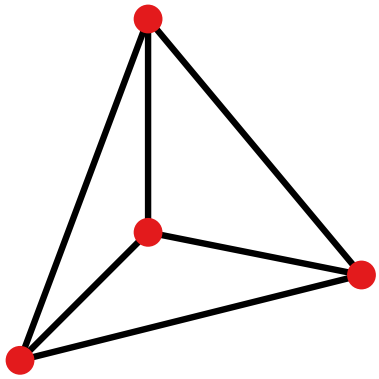
contact representation
by discs



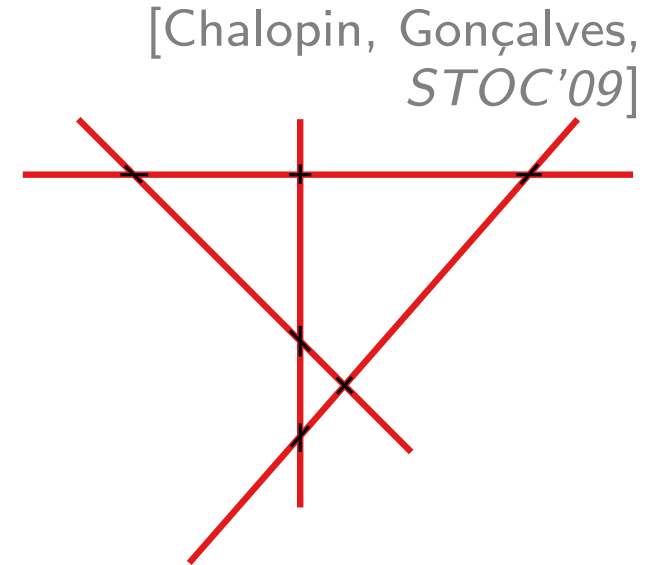
[Chalopin, Gonçalves,
STOC'09]

intersection representation
by segments

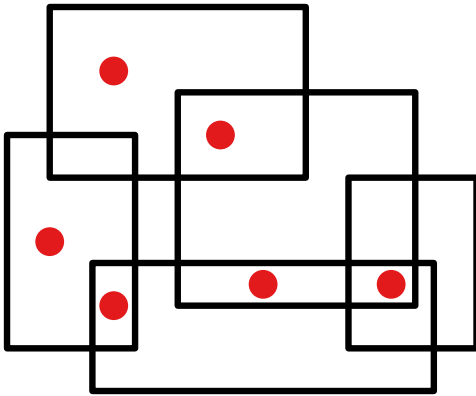
Geometric Representation of (Hyper)graphs



contact representation
by discs

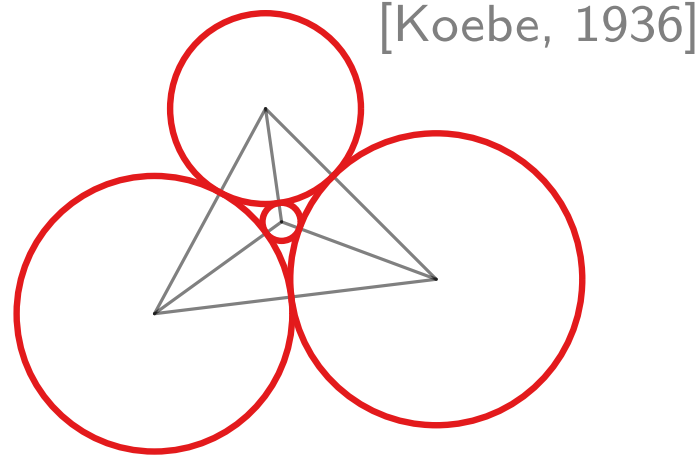
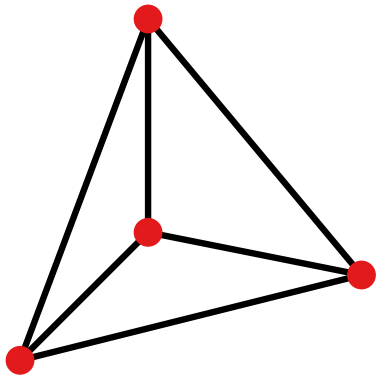


intersection representation
by segments

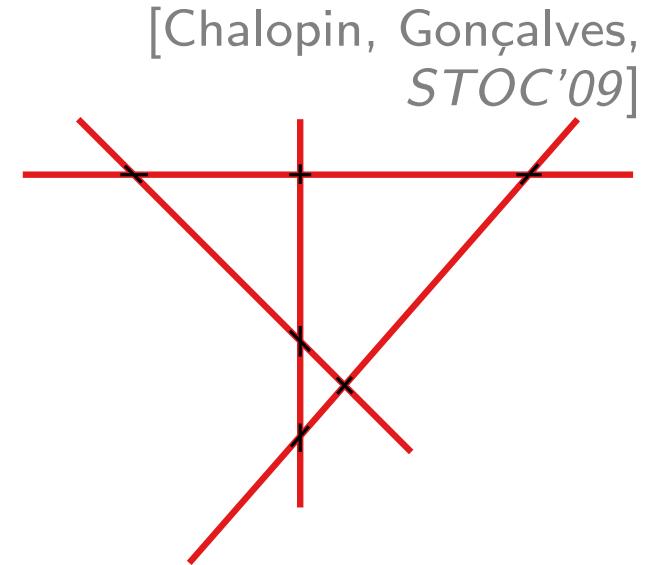


covering representation
points by rectangles

Geometric Representation of (Hyper)graphs



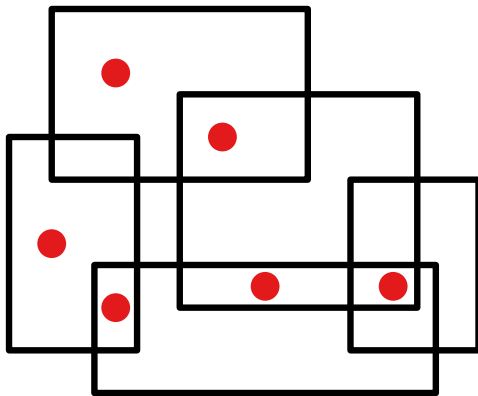
[Koebe, 1936]



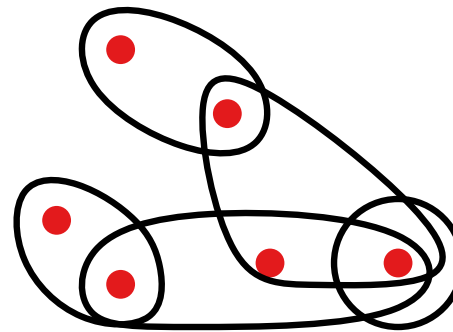
[Chalopin, Gonçalves,
STOC'09]

contact representation
by discs

intersection representation
by segments

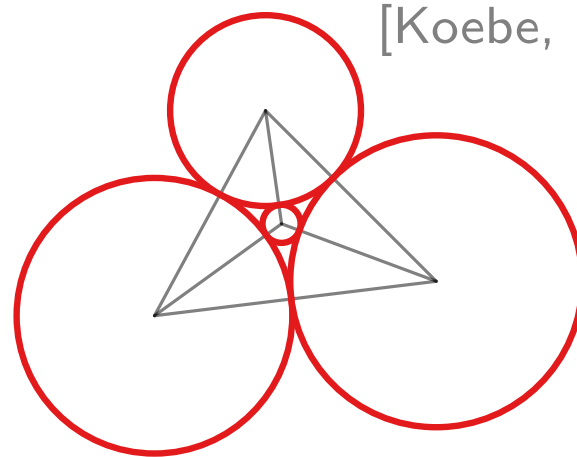
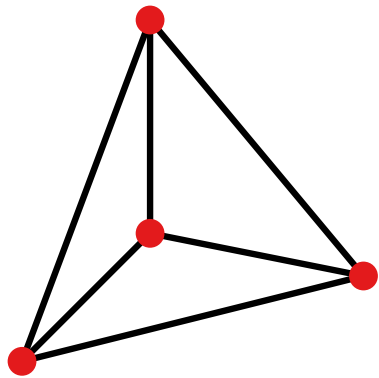


covering representation
points by rectangles



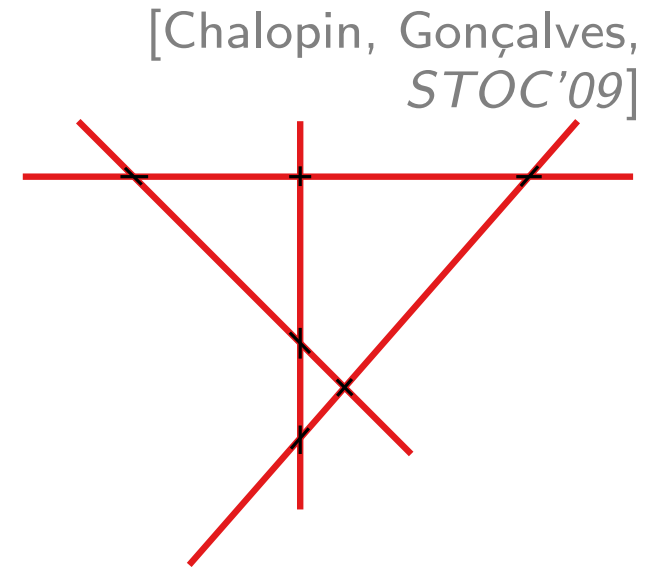
points \rightarrow vertices
covering objects \rightarrow hyperedges

Geometric Representation of (Hyper)graphs



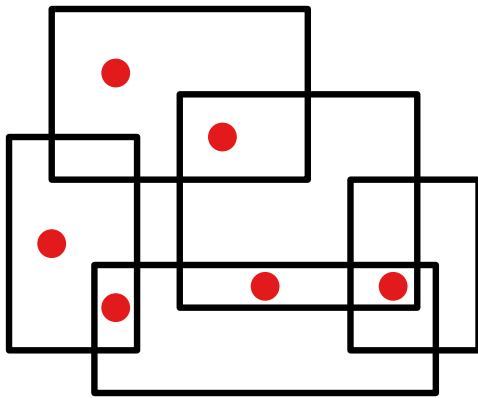
[Koebe, 1936]

contact representation
by discs

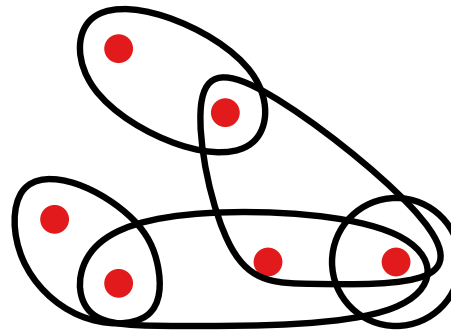


[Chalopin, Gonçalves,
STOC'09]

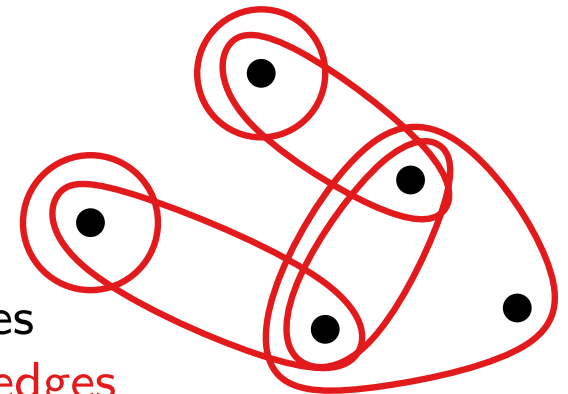
intersection representation
by segments



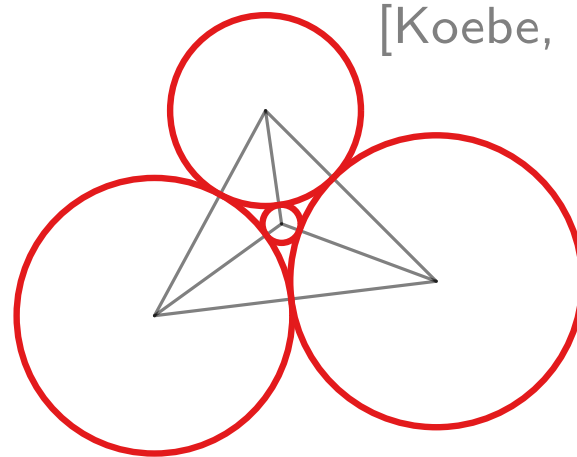
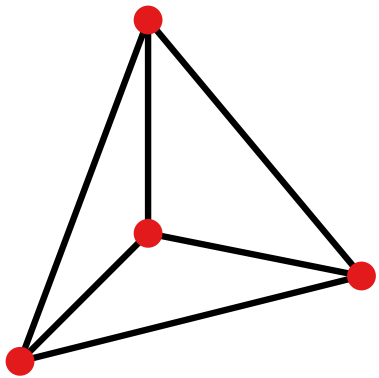
covering representation
points by rectangles



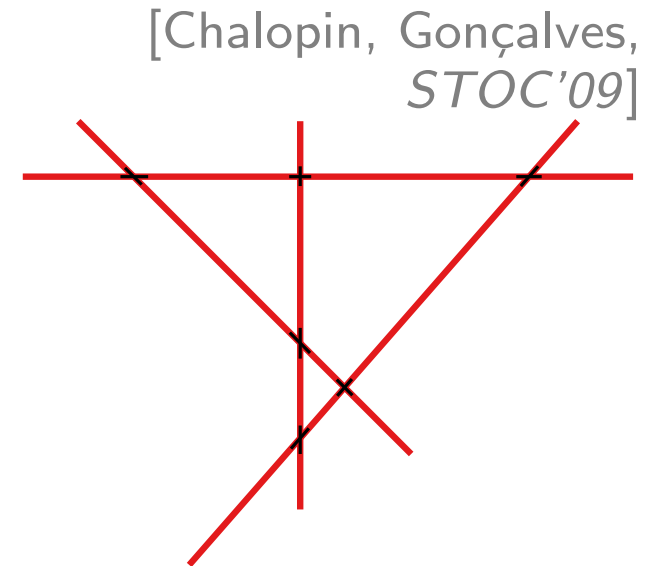
points \bowtie vertices
covering objects \bowtie hyperedges



Geometric Representation of (Hyper)graphs



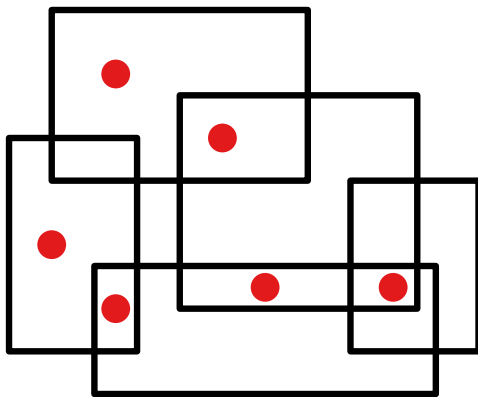
[Koebe, 1936]



[Chalopin, Gonçalves,
STOC'09]

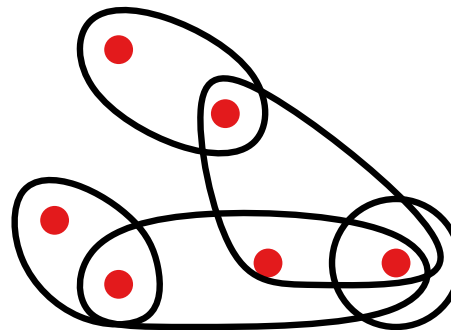
contact representation
by discs

intersection representation
by segments



covering representation

points by rectangles

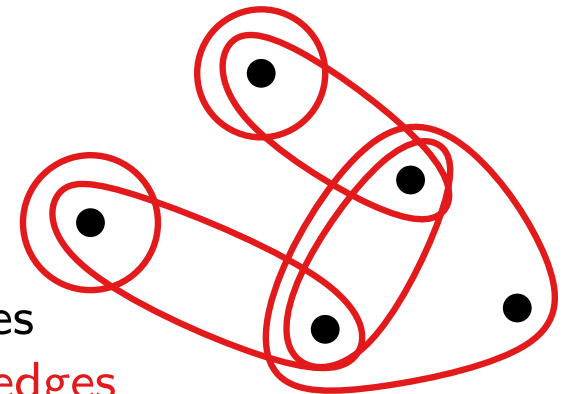


points
covering objects



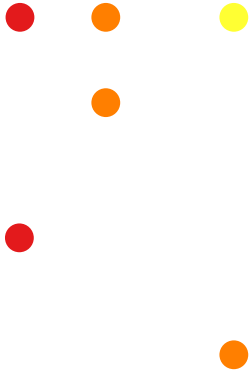
vertices

hyperedges



Point Line Cover

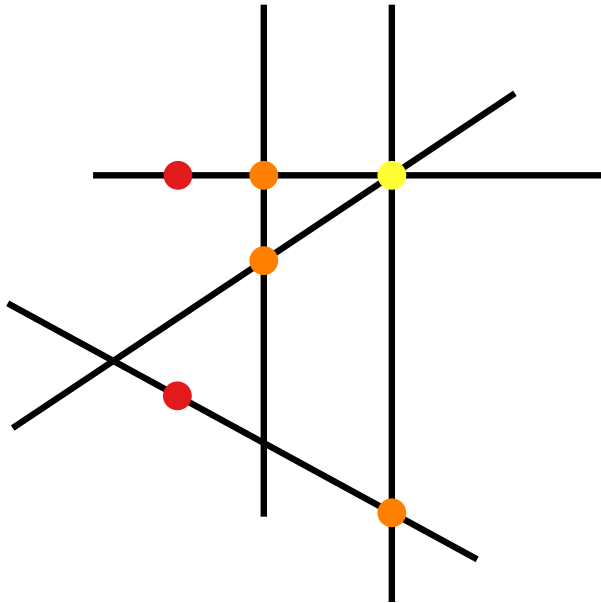
set of points P in 2D



Point Line Cover

set of points P in 2D

set of lines

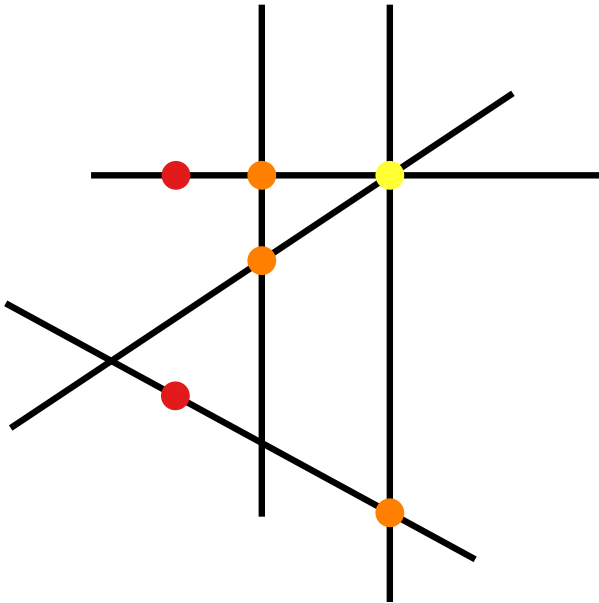


Point Line Cover

set of points P in 2D

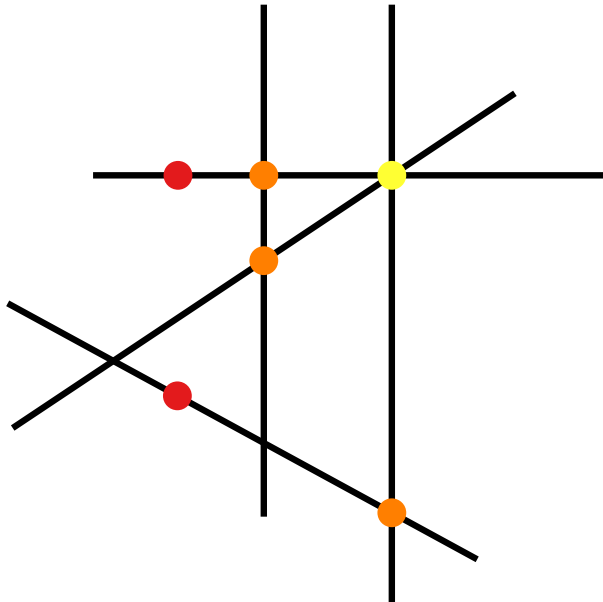
set of lines

Hypergraph representation



Point Line Cover

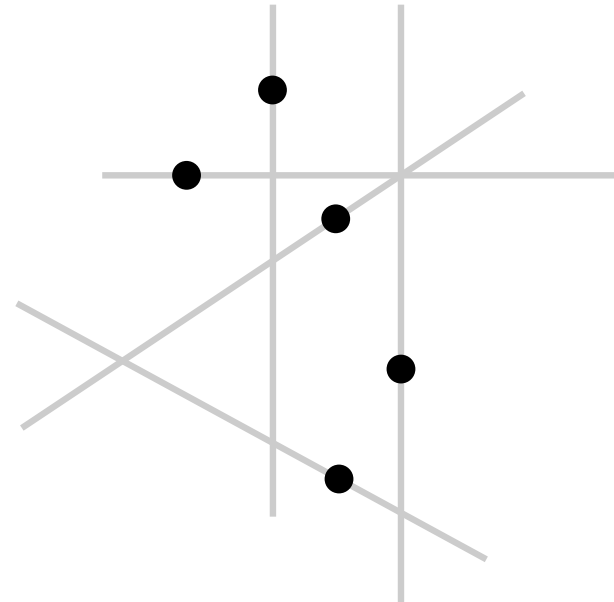
set of points P in 2D



vertices

set of lines

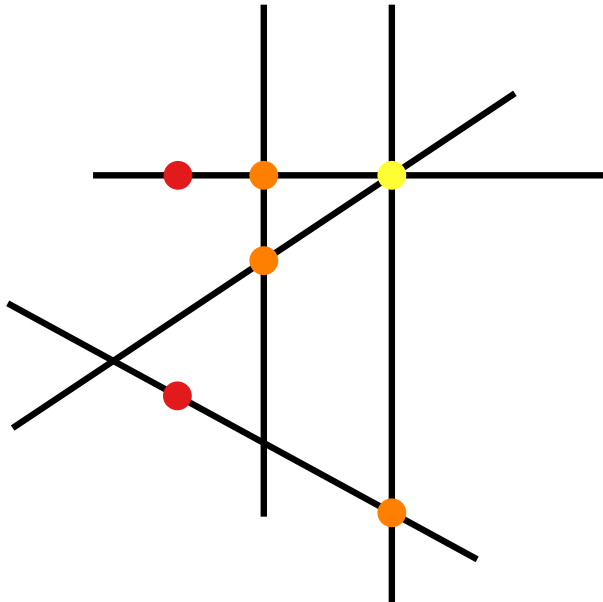
Hypergraph representation



Point Line Cover

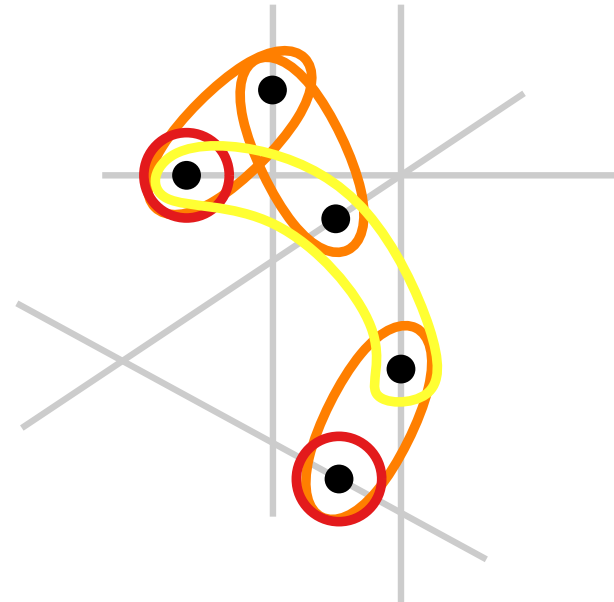
hyperedges

set of points P in 2D



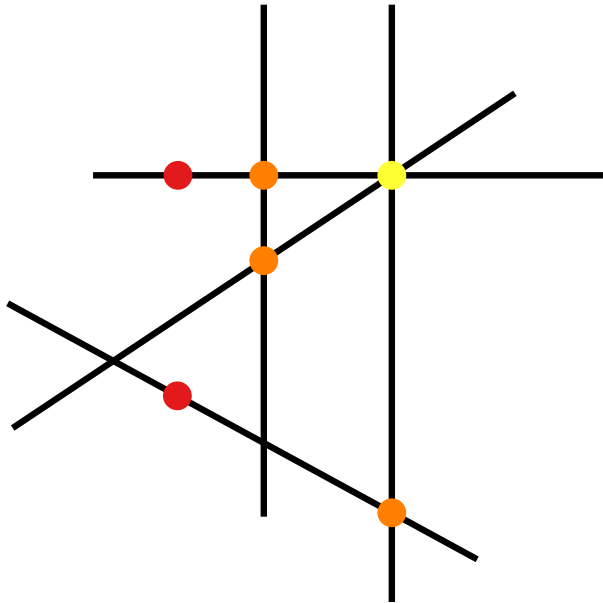
set of lines

Hypergraph representation



Point Line Cover – Motivation

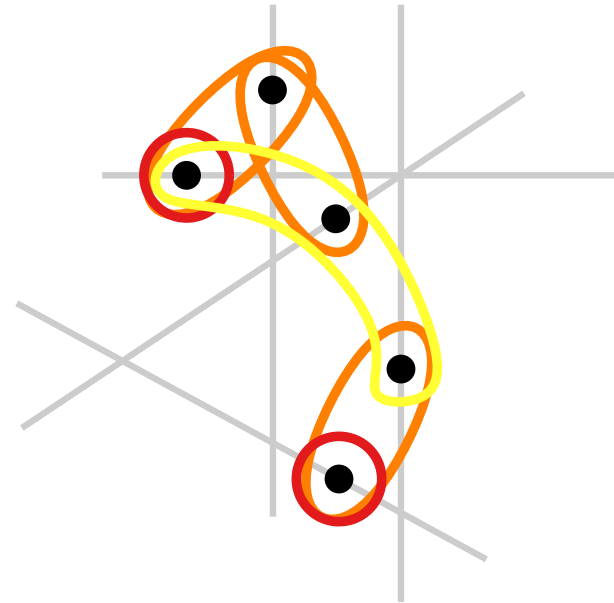
set of points P in 2D



point line cover instances

set of lines

Hypergraph representation



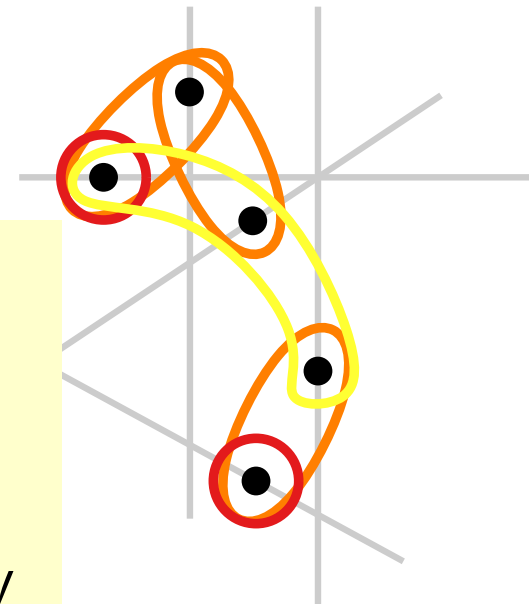
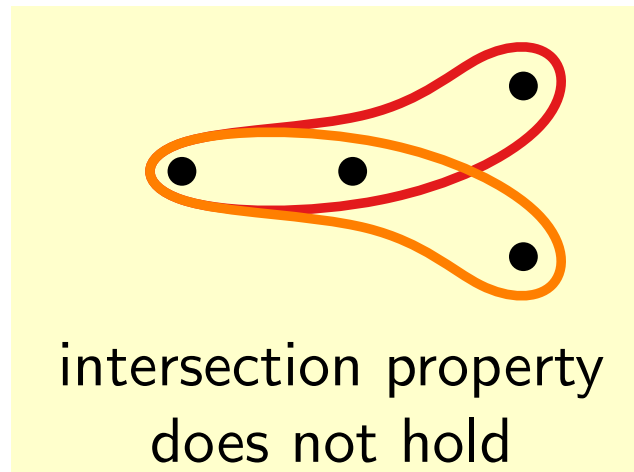
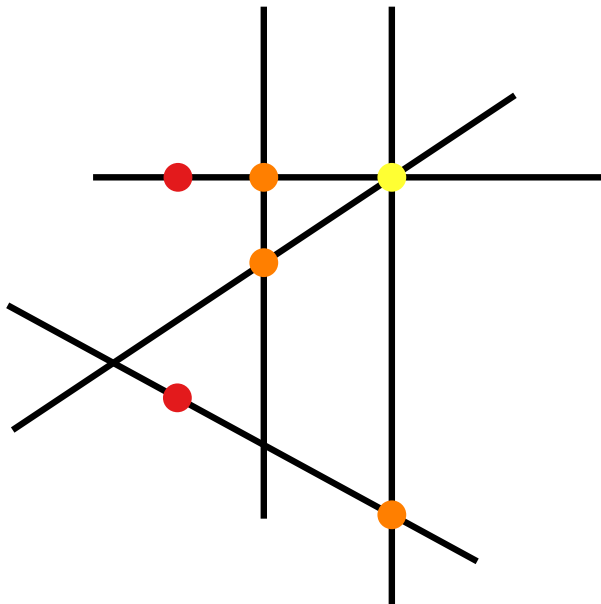
\subset general hypergraphs

Point Line Cover – Motivation

set of points P in 2D

set of lines

Hypergraph representation



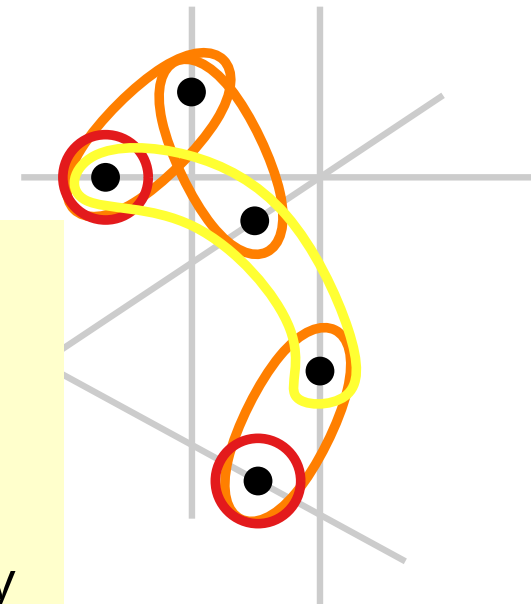
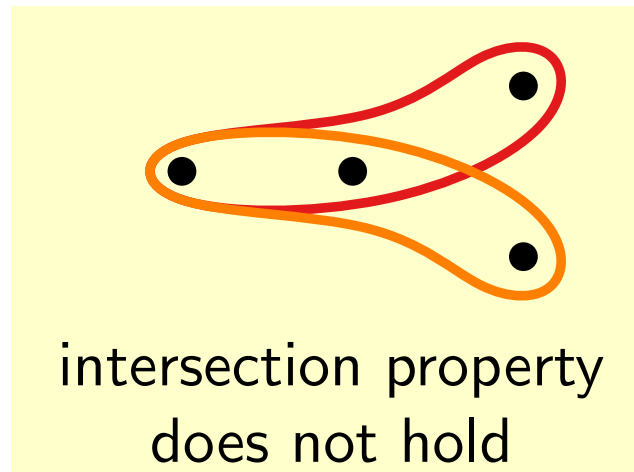
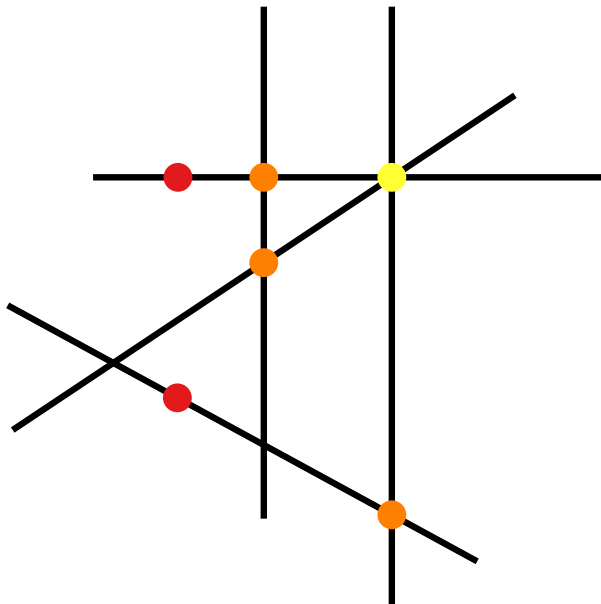
point line cover instances \subset general hypergraphs

Point Line Cover – Motivation

set of points P in 2D

set of lines

Hypergraph representation

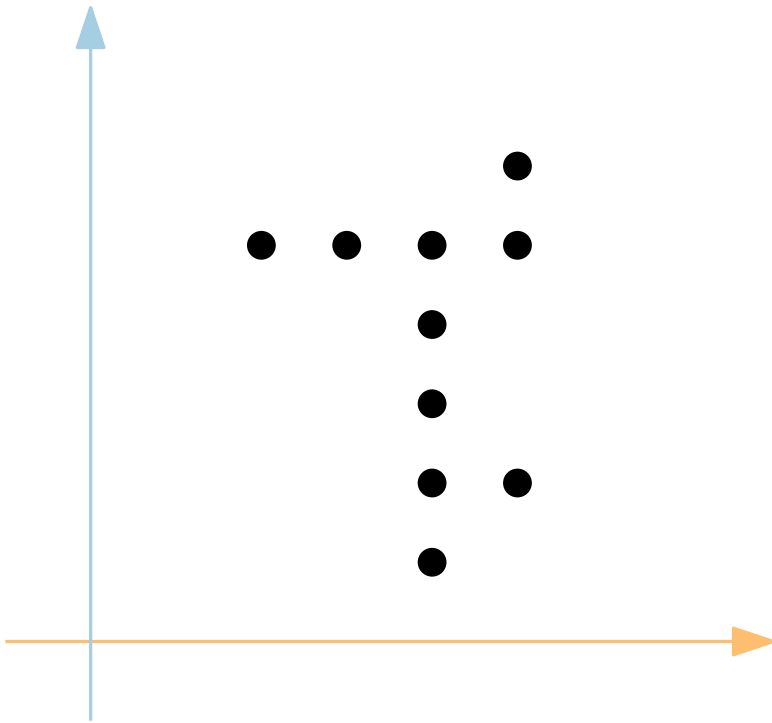


point line cover instances \subset general hypergraphs

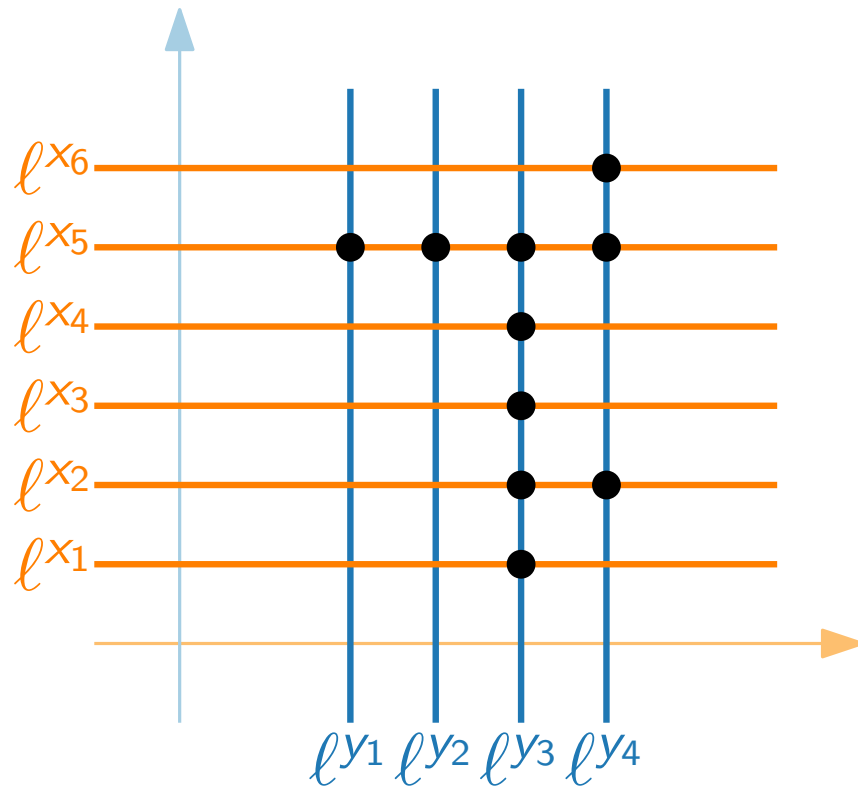
Is there a simple combinatorial characterization?

Open Q.
[Kumar, Ramesh,
ICALP 2000]

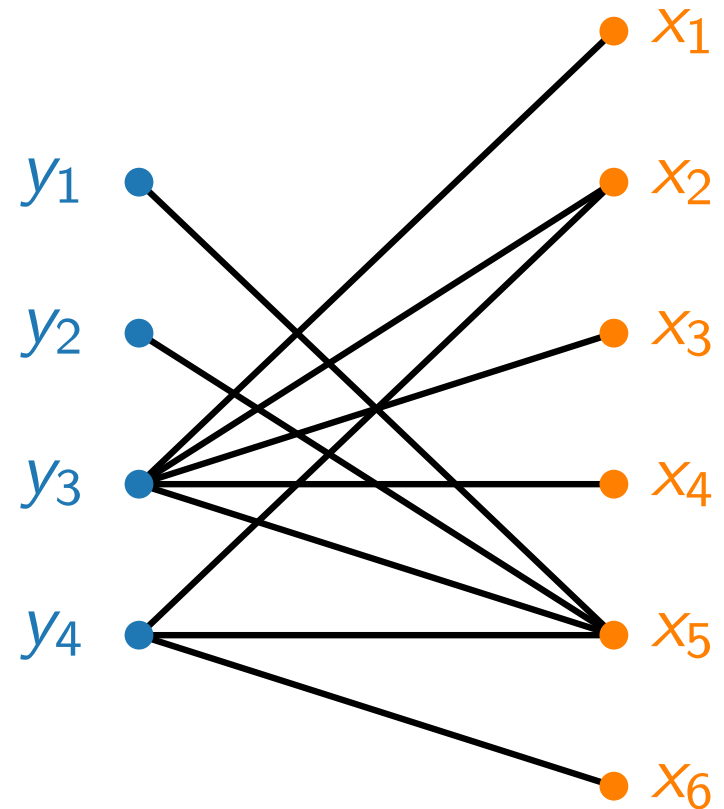
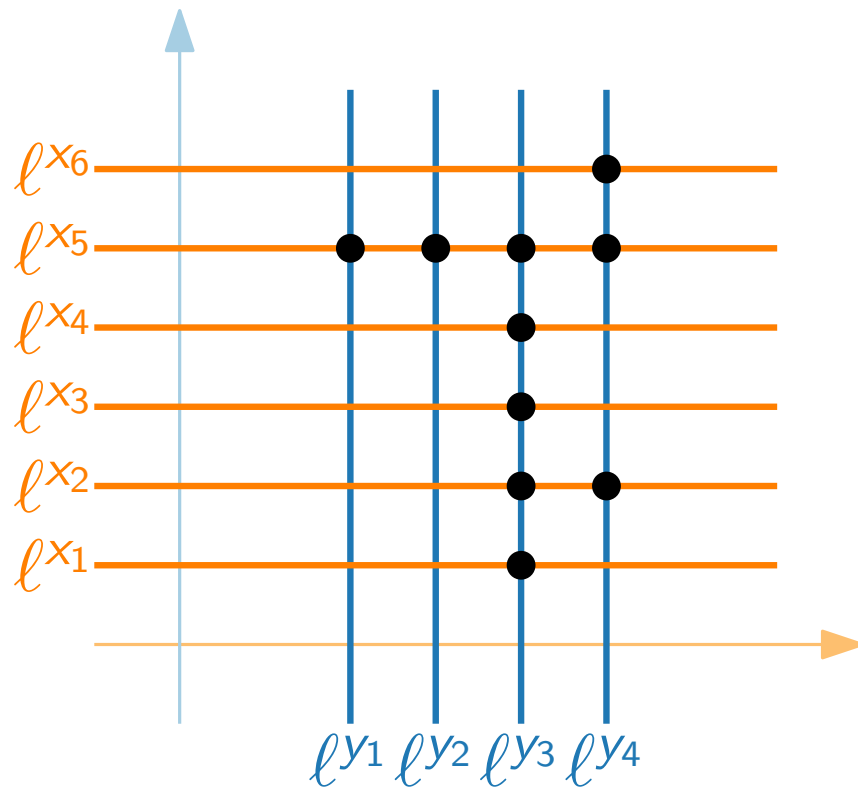
Axis-Aligned Point Line Cover in 2D



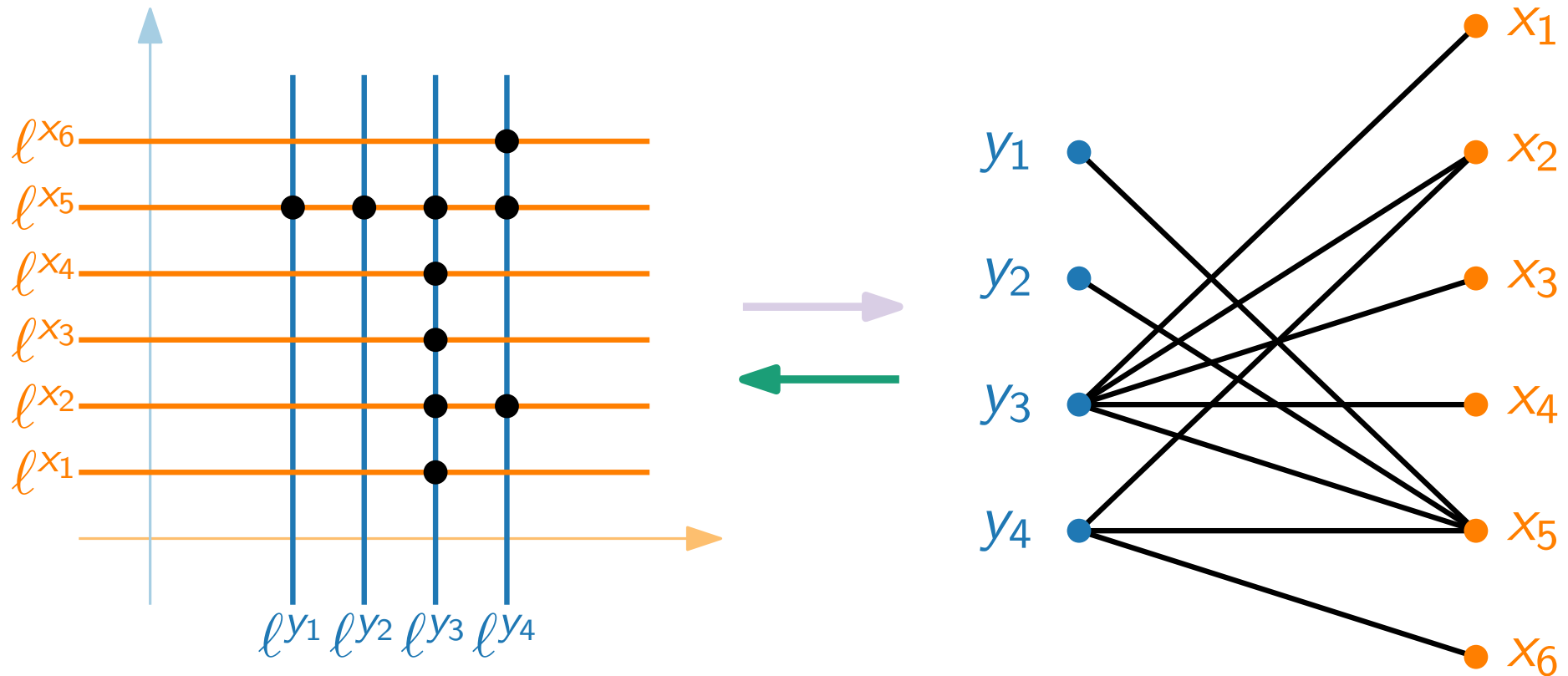
Axis-Aligned Point Line Cover in 2D



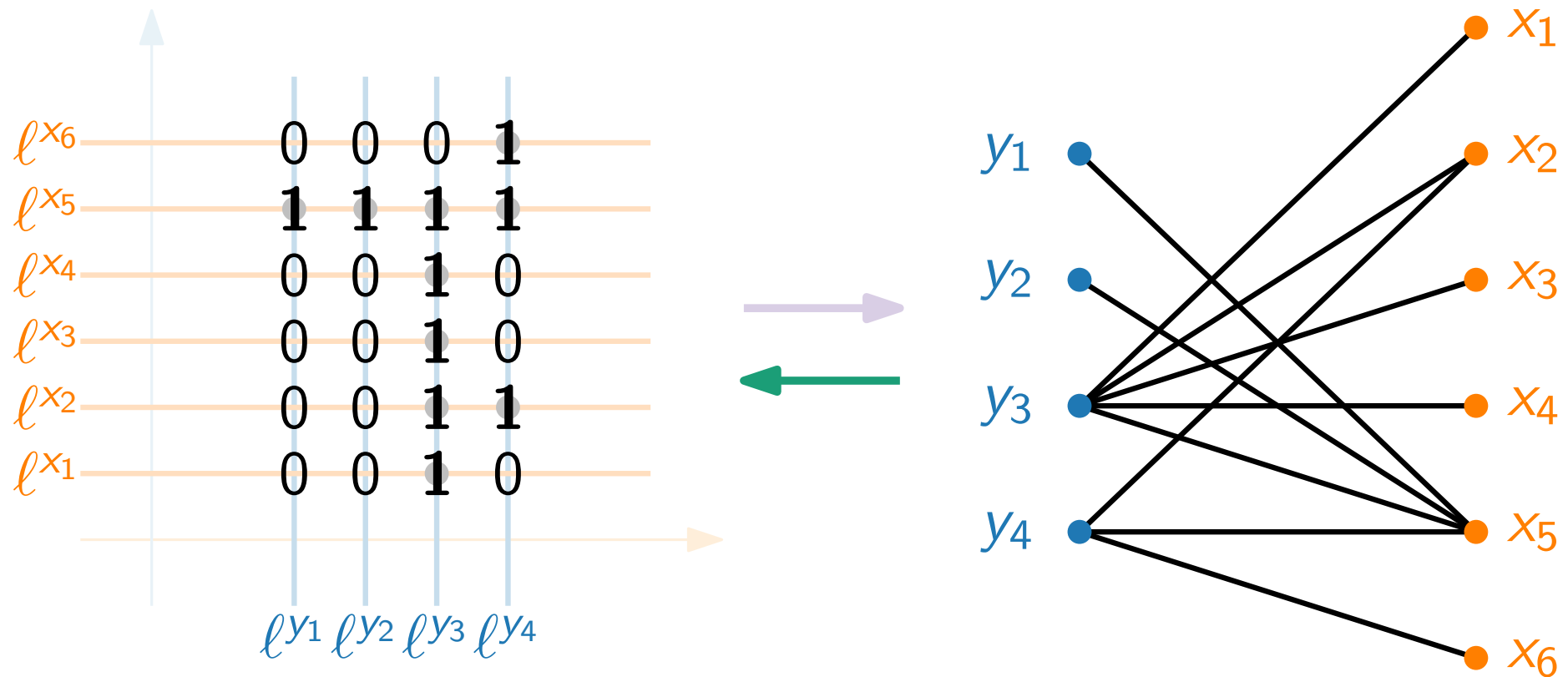
Axis-Aligned Point Line Cover in 2D



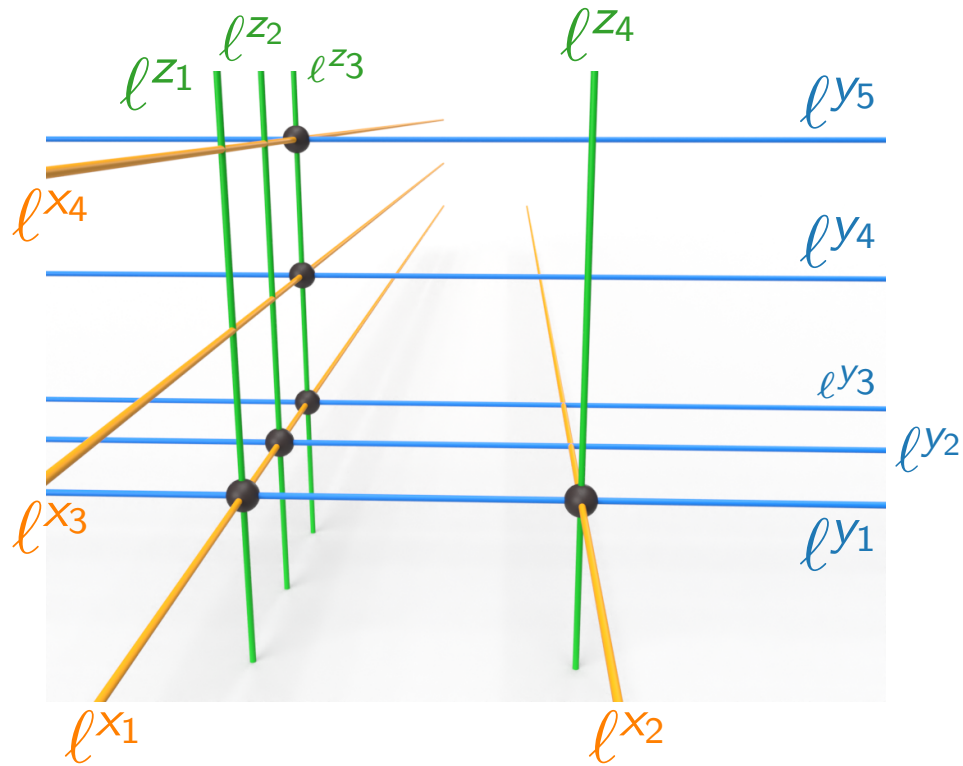
Axis-Aligned Point Line Cover in 2D



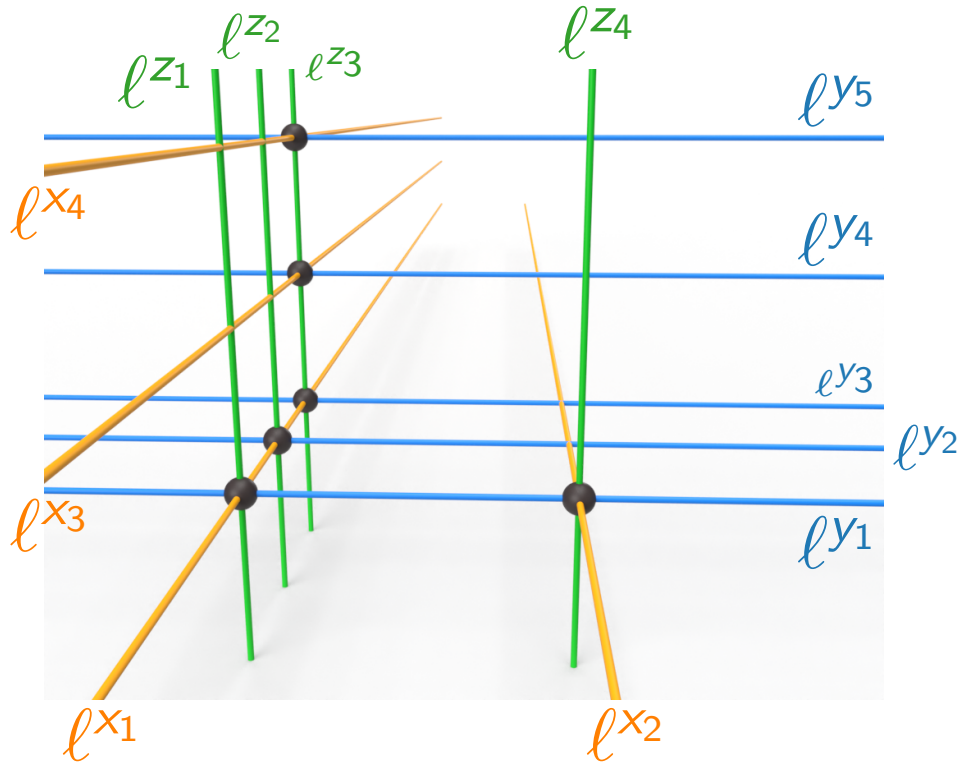
Axis-Aligned Point Line Cover in 2D



Axis-Aligned Point Line Cover in 3D

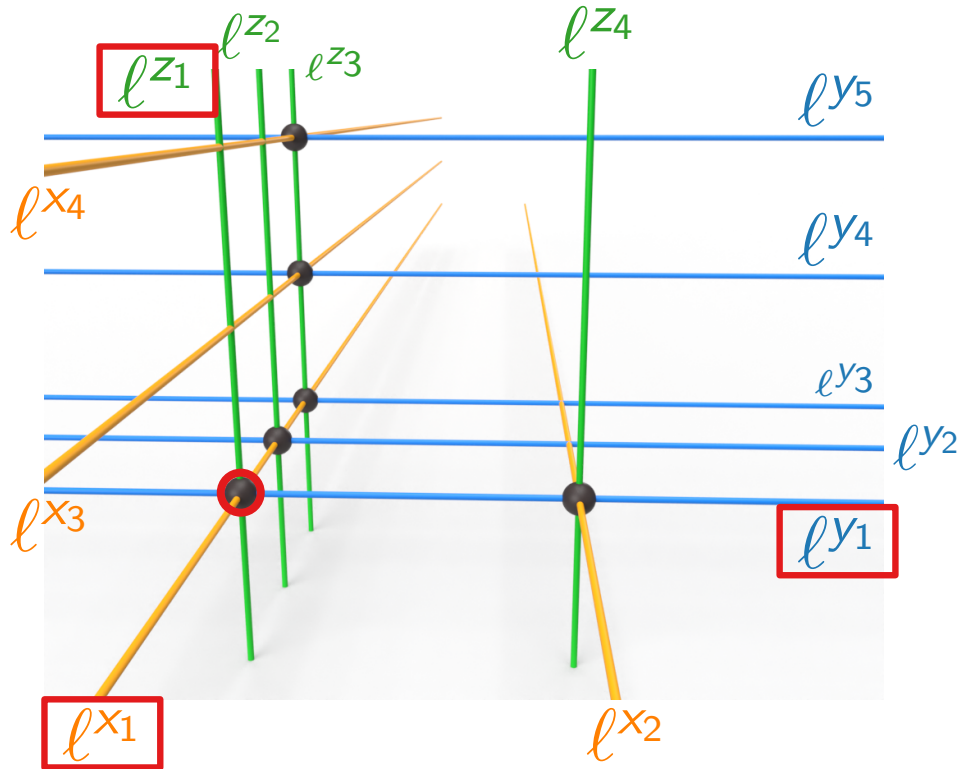


Axis-Aligned Point Line Cover in 3D



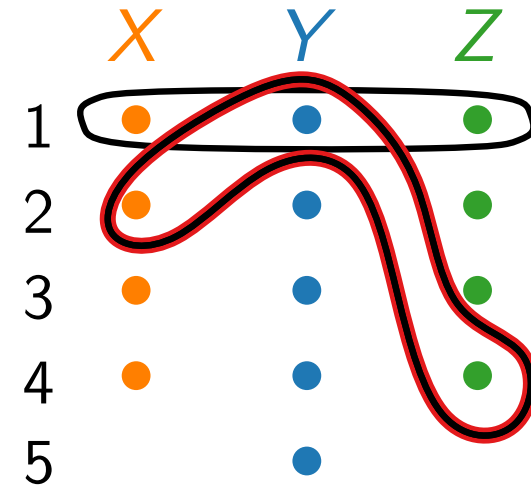
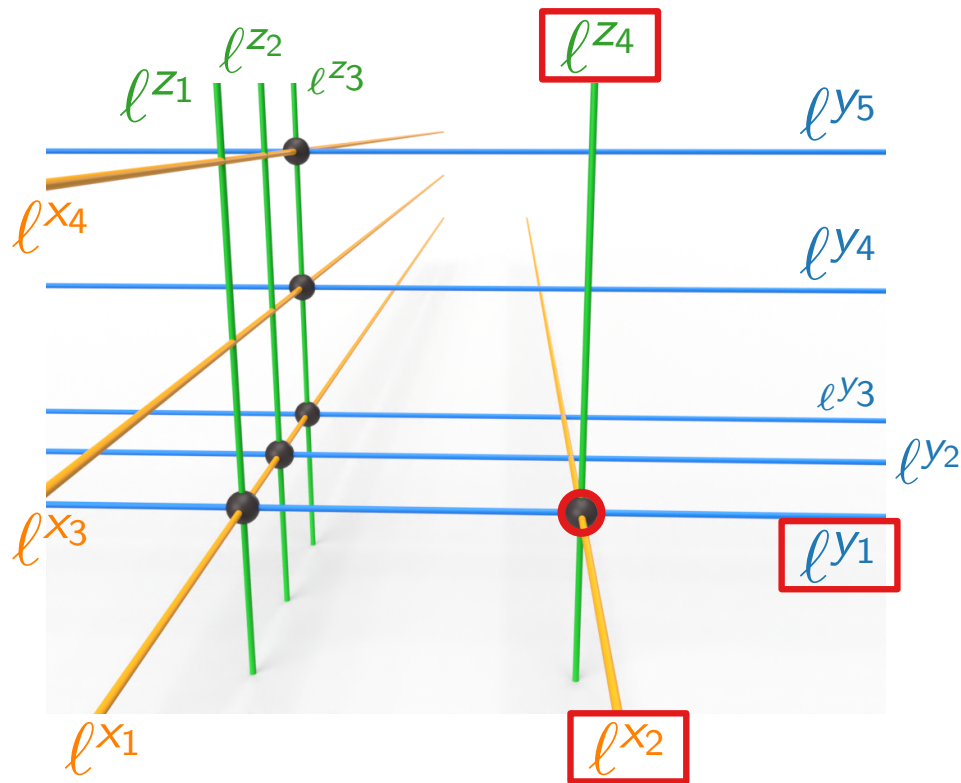
	X	Y	Z
1	●	●	●
2	●	●	●
3	●	●	●
4	●	●	●
5		●	

Axis-Aligned Point Line Cover in 3D

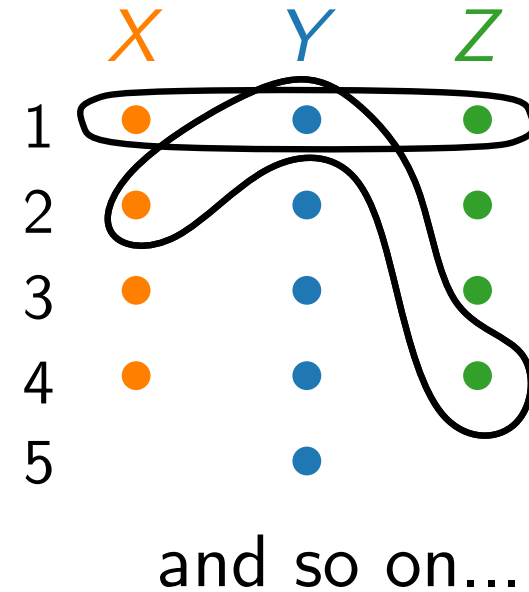
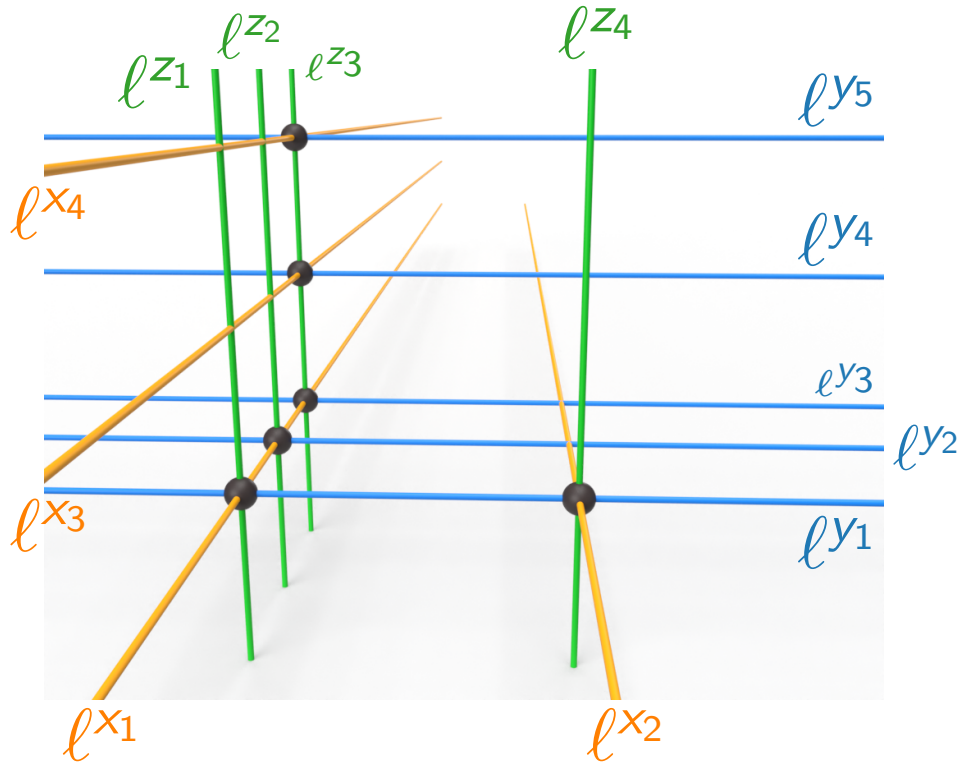


	X	Y	Z
1			
2			
3			
4			
5			

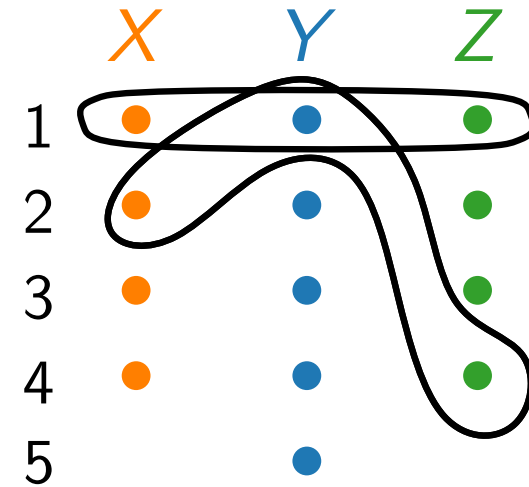
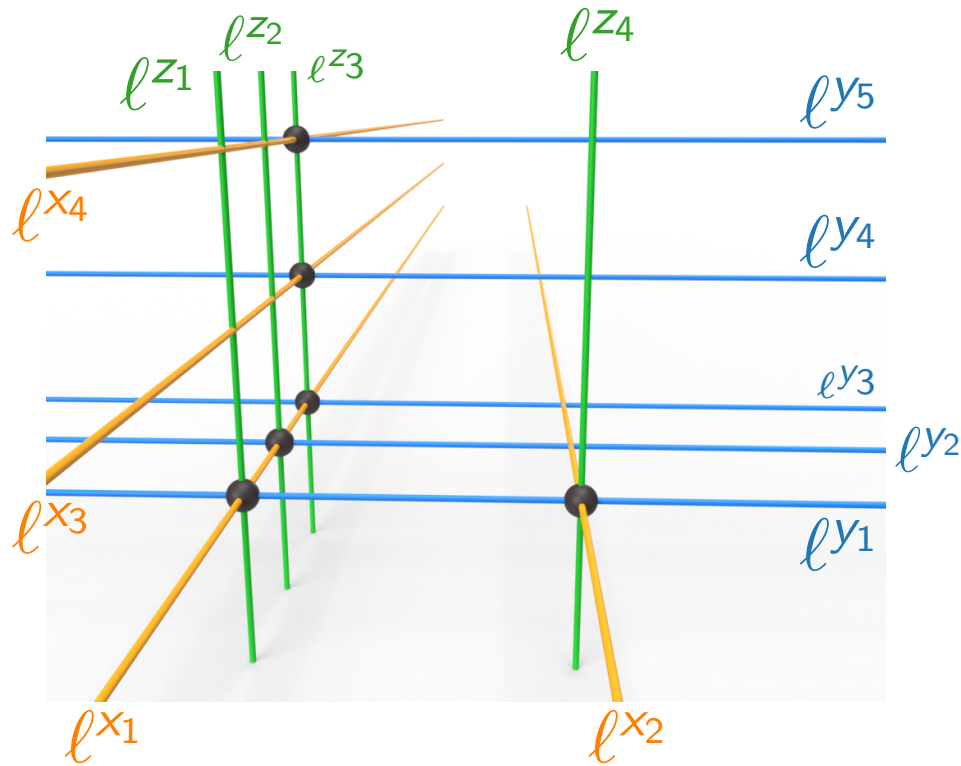
Axis-Aligned Point Line Cover in 3D



Axis-Aligned Point Line Cover in 3D



Axis-Aligned Point Line Cover in 3D



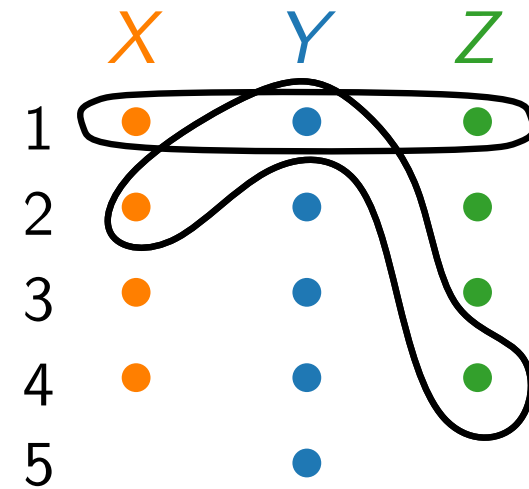
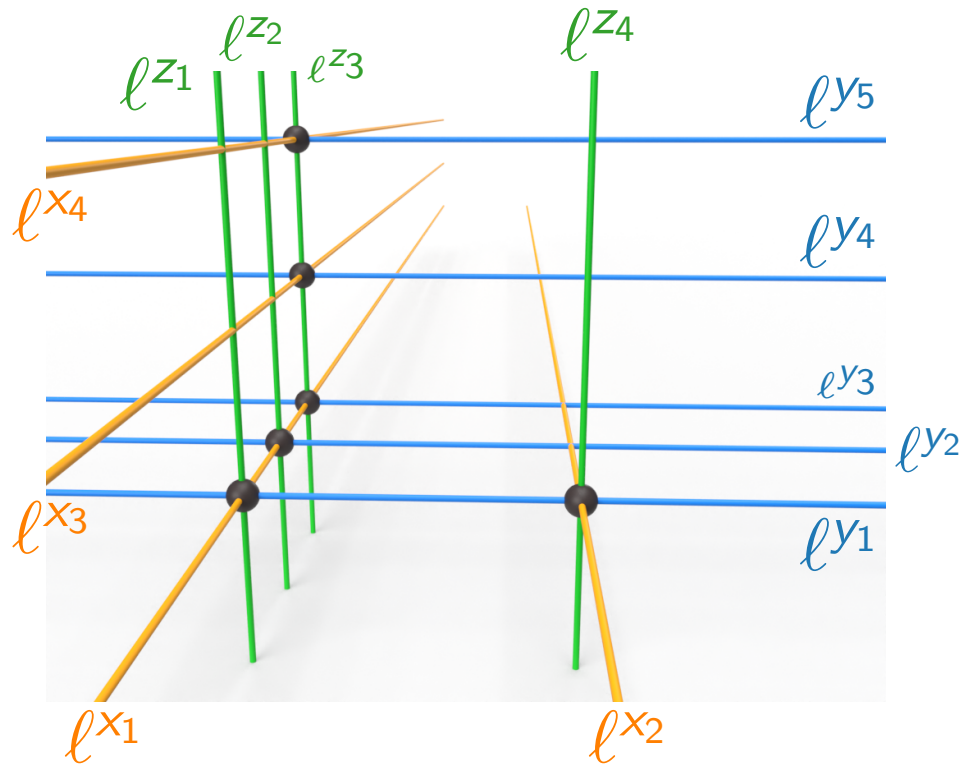
and so on...

(every hyperedge has
exactly 3 vertices,
one from each group)

3-uniform

3-partite

Axis-Aligned Point Line Cover in 3D



and so on...

(every hyperedge has
exactly 3 vertices,
one from each group)

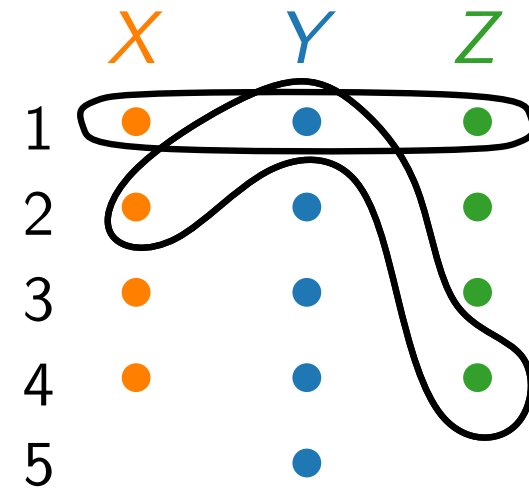
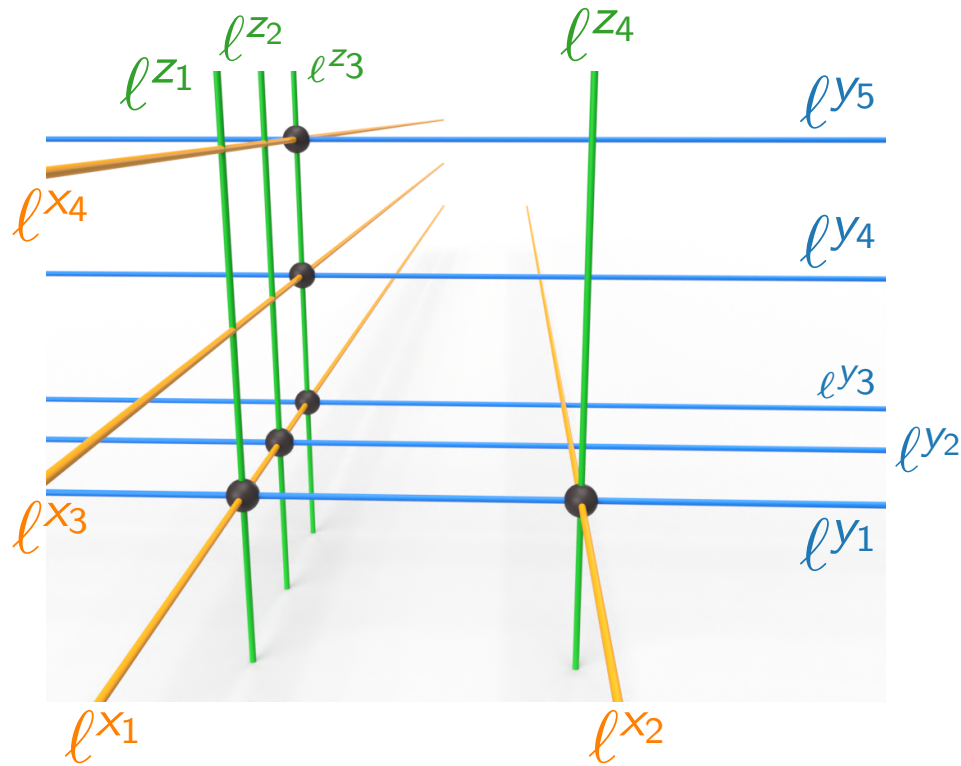
3-uniform

easy to check

3-partite

NP-hard

Axis-Aligned Point Line Cover in 3D



(every hyperedge has
exactly 3 vertices,
one from each group)

3-uniform

easy to check

3-partite

NP-hard

3-hypergraph

Representable Hypergraphs

axis-aligned point line cover instance



k -hypergraph

k -partite and k -uniform

Representable Hypergraphs

axis-aligned point line cover instance



k -hypergraph

k -partite and k -uniform

Representable Hypergraphs

axis-aligned point line cover instance

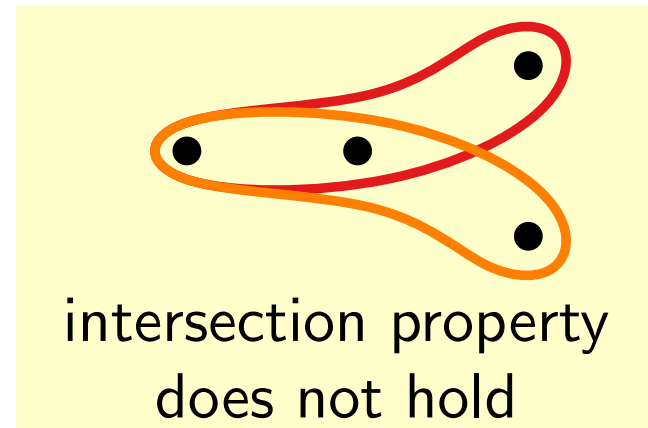


k -hypergraph

k -partite and k -uniform

No

exception: 2D



intersection property
does not hold

Representable Hypergraphs

axis-aligned point line cover instance

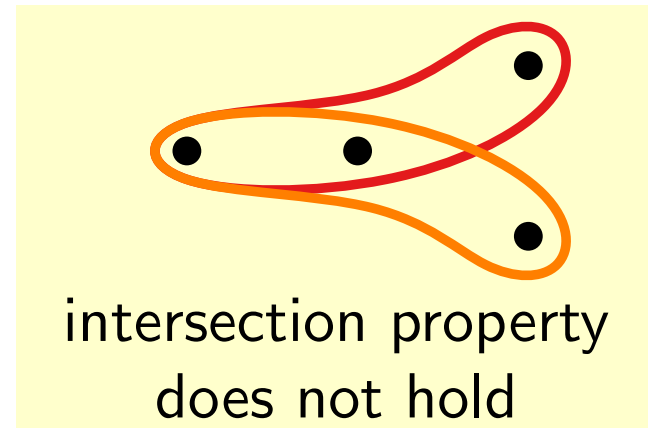


k -hypergraph

k -partite and k -uniform

No

exception: 2D



intersection property
does not hold

Which k -hypergraphs can be *represented*
via axis-aligned point line cover instances?

Paths

Notation.

$[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$

A hypergraph $G = (V, E)$

$V = V_1 \cup \dots \cup V_k$

Paths

Notation.

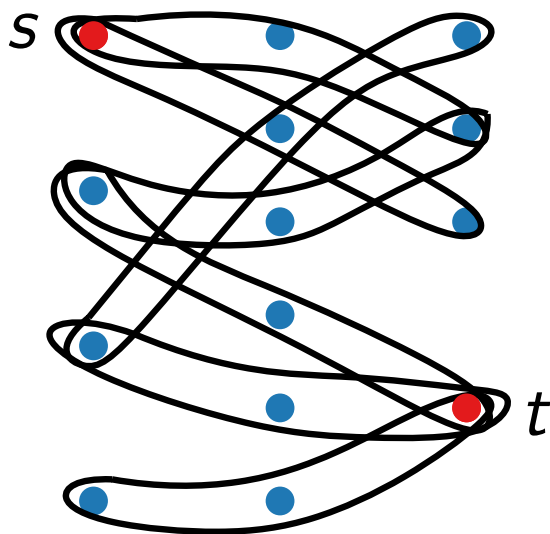
$[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$

A hypergraph $G = (V, E)$

$V = V_1 \cup \dots \cup V_k$

Def.

Let $s, t \in V$. An *s - t path* is a sequence of vertices $s = v_1, \dots, v_r = t$ such that $\forall i \in [r - 1]$ v_i and v_{i+1} belong to the same edge.



Paths

Notation.

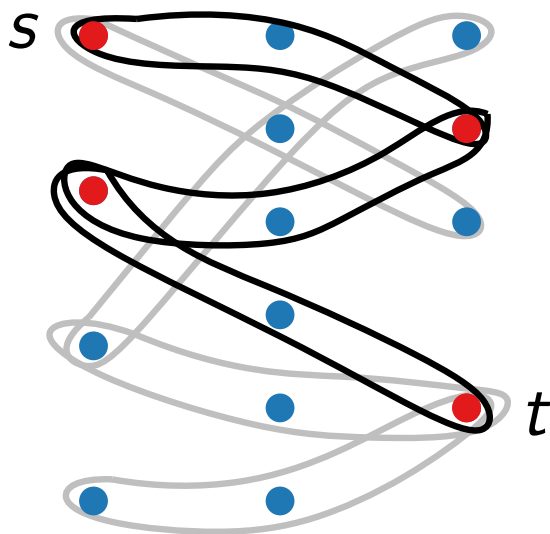
$[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$

A hypergraph $G = (V, E)$

$V = V_1 \cup \dots \cup V_k$

Def.

Let $s, t \in V$. An *s - t path* is a sequence of vertices $s = v_1, \dots, v_r = t$ such that $\forall i \in [r - 1]$ v_i and v_{i+1} belong to the same edge.



Paths

Notation.

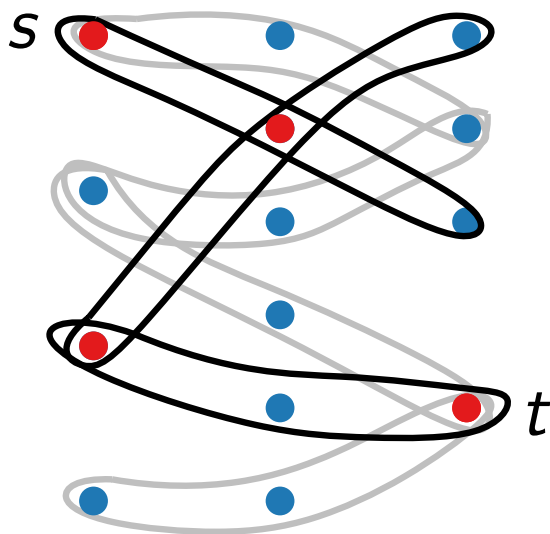
$[k] = \{1, \dots, k\}$ for $k \in \mathbb{N}$

A hypergraph $G = (V, E)$

$V = V_1 \cup \dots \cup V_k$

Def.

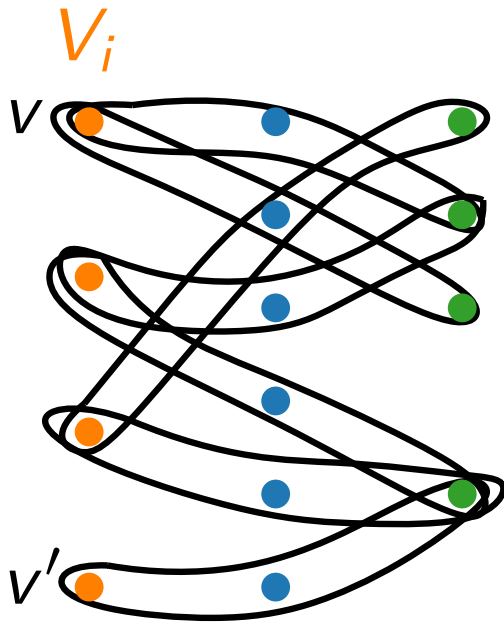
Let $s, t \in V$. An *s - t path* is a sequence of vertices $s = v_1, \dots, v_r = t$ such that $\forall i \in [r - 1]$ v_i and v_{i+1} belong to the same edge.



Separability – Key Property

Def. *Vertex separability*

For a given k -hypergraph G two distinct vertices v and v' from the same group V_i where $i \in [k]$ are *separable* if there exists $j \in [k]$ with $j \neq i$ such that every v - v' path contains a vertex in V_j .

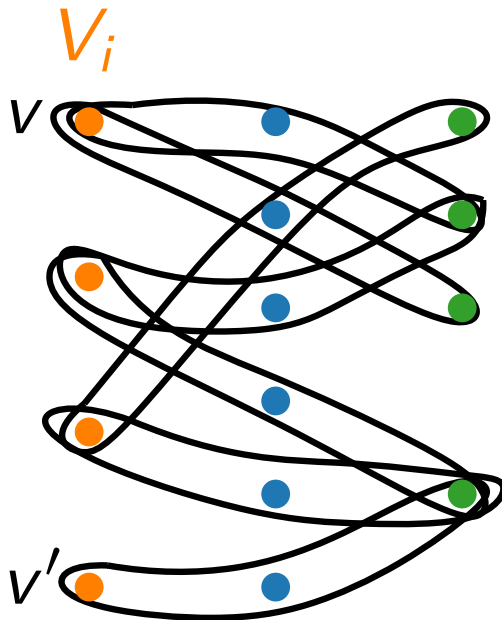


Separability – Key Property

Def. *Vertex separability*

For a given k -hypergraph G two distinct vertices v and v' from the same group V_i where $i \in [k]$ are *separable* if there exists $j \in [k]$ with $j \neq i$ such that every v - v' path contains a vertex in V_j .

(Informally, removing V_j from the vertex set and from the edges separates v and v' .)

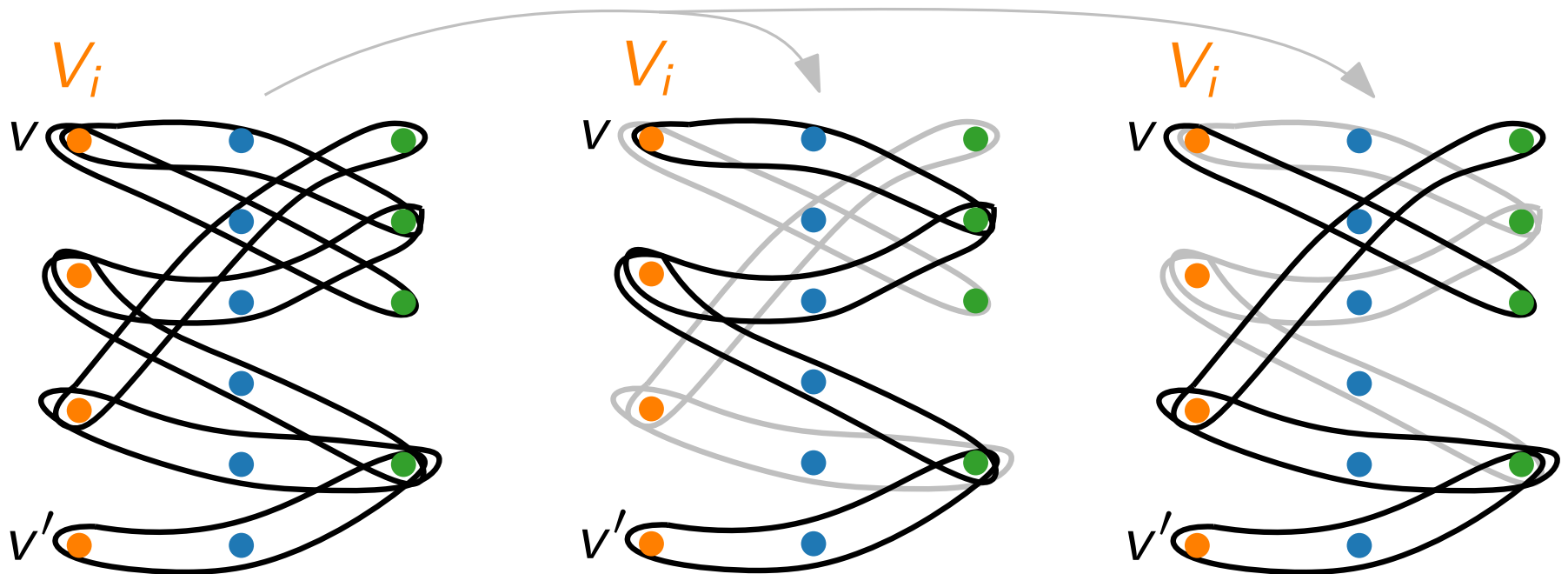


Separability – Key Property

Def. *Vertex separability*

For a given k -hypergraph G two distinct vertices v and v' from the same group V_i where $i \in [k]$ are *separable* if there exists $j \in [k]$ with $j \neq i$ such that every v - v' path contains a vertex in V_j .

(Informally, removing V_j from the vertex set and from the edges separates v and v' .)

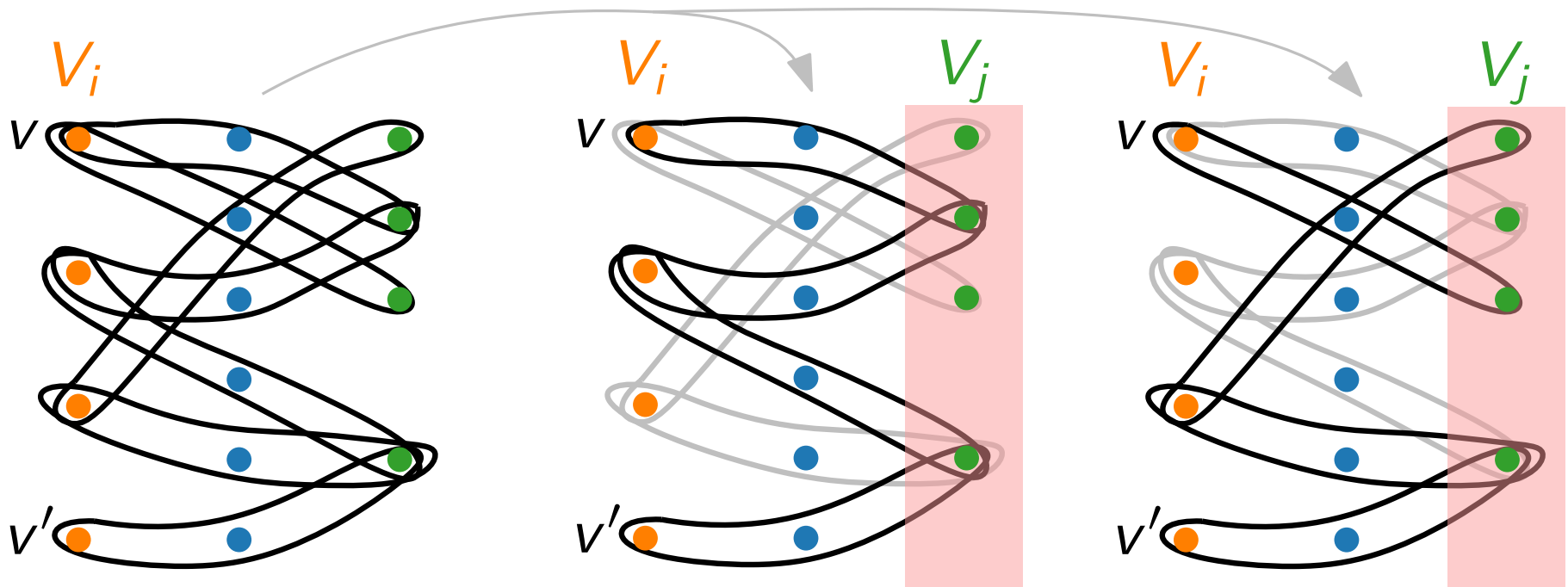


Separability – Key Property

Def. *Vertex separability*

For a given k -hypergraph G two distinct vertices v and v' from the same group V_i where $i \in [k]$ are *separable* if there exists $j \in [k]$ with $j \neq i$ such that every v - v' path contains a vertex in V_j .

(Informally, removing V_j from the vertex set and from the edges separates v and v' .)

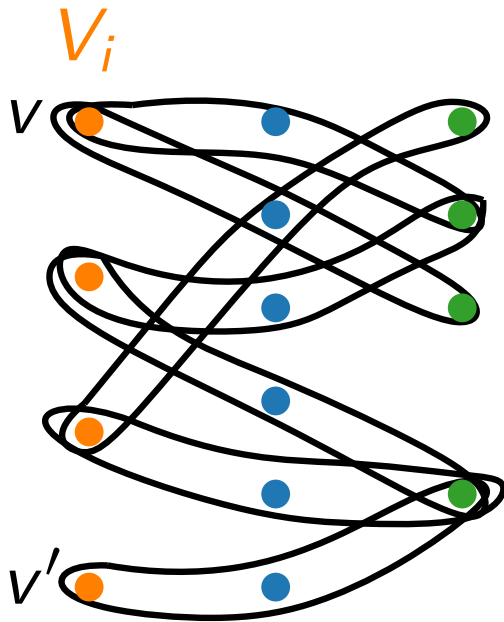


Separability – Key Property

Def. *Vertex separability*

For a given k -hypergraph G two distinct vertices v and v' from the same group V_i where $i \in [k]$ are *separable* if there exists $j \in [k]$ with $j \neq i$ such that every v - v' path contains a vertex in V_j .

(Informally, removing V_j from the vertex set and from the edges separates v and v' .)



A k -hypergraph is called *vertex separable* if every two vertices from the same group are separable.

Main Result

Theorem

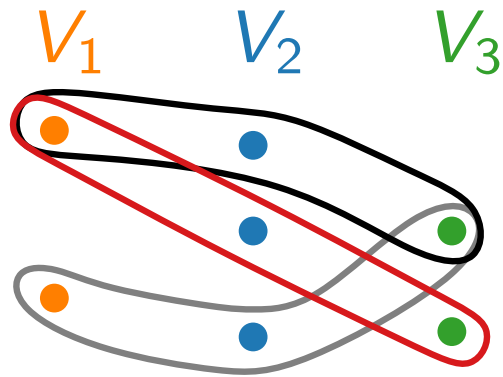
A k -hypergraph G is representable if and only if it is vertex separable.

Main Result – Construction

Theorem

A k -hypergraph G is representable if and only if it is vertex separable.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}



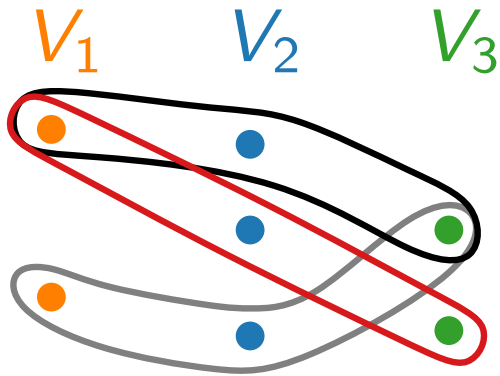
Main Result – Construction

Theorem

A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.



Main Result – Construction

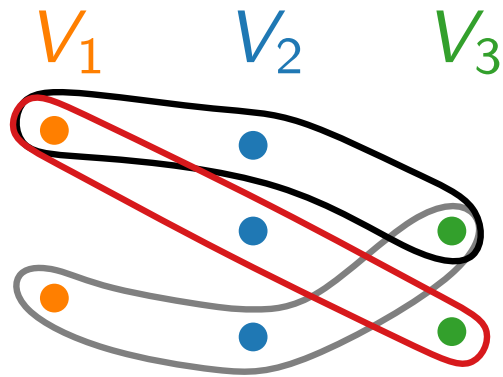
Theorem

A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i



G_1



G_2



G_3



Main Result – Construction

Theorem

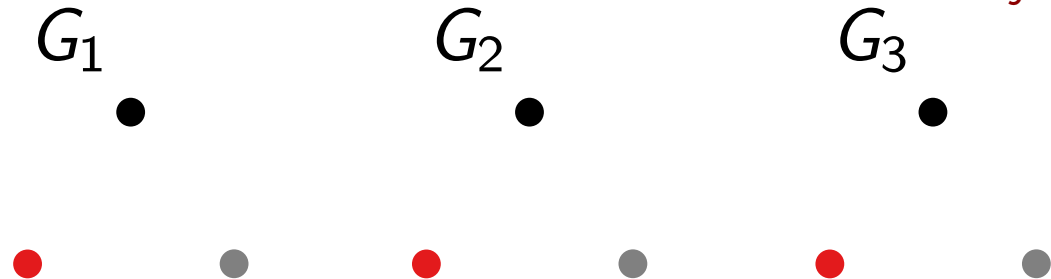
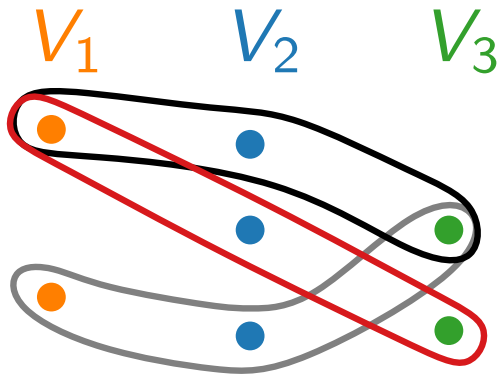
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

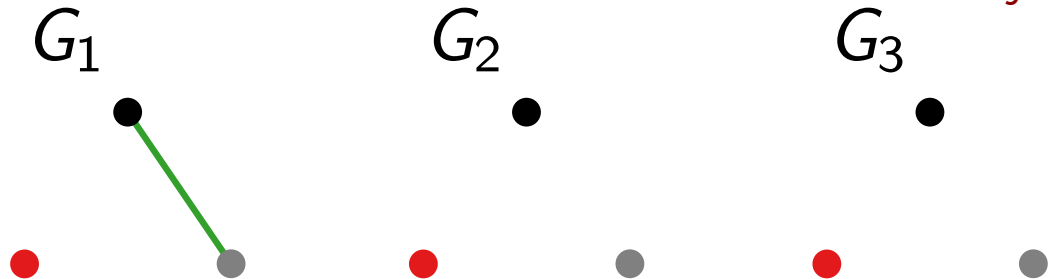
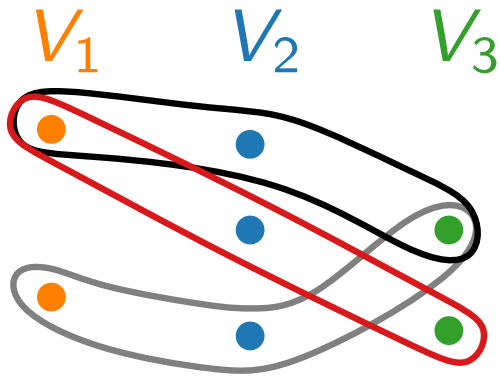
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

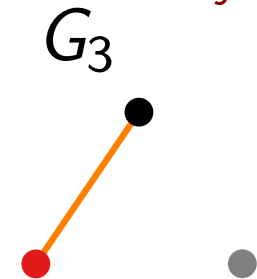
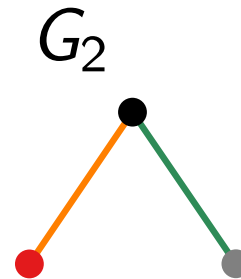
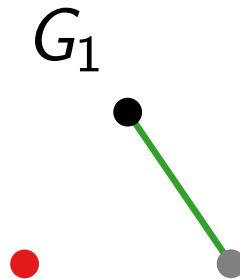
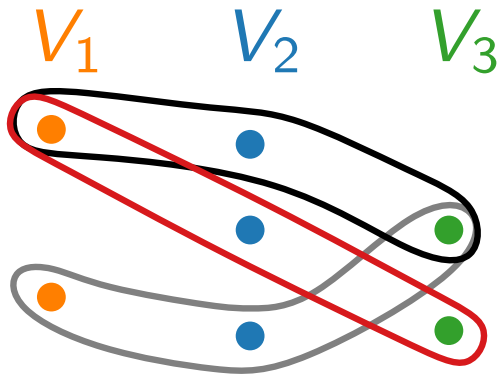
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

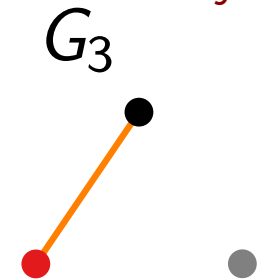
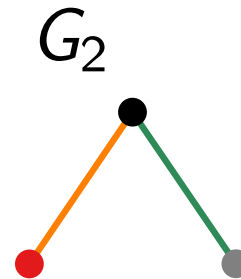
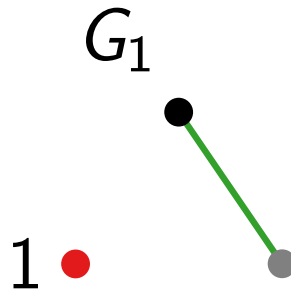
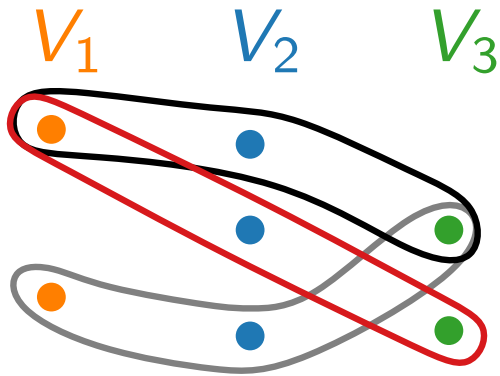
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

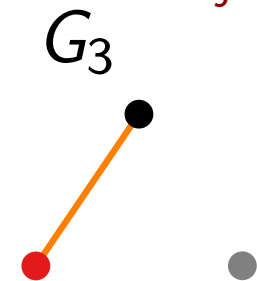
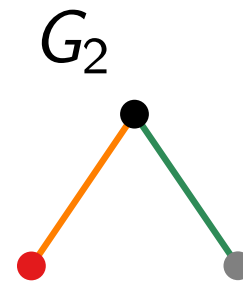
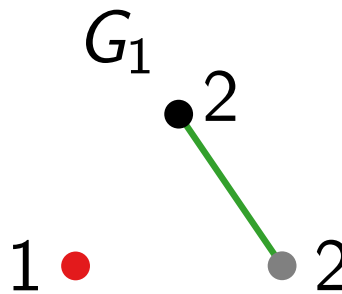
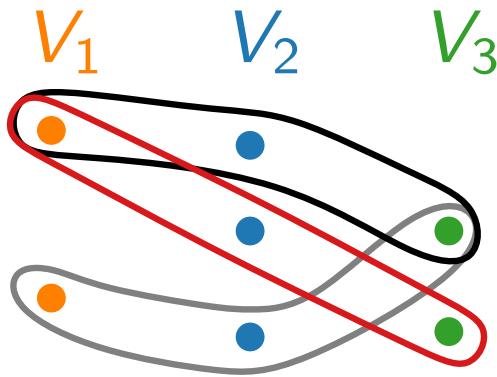
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

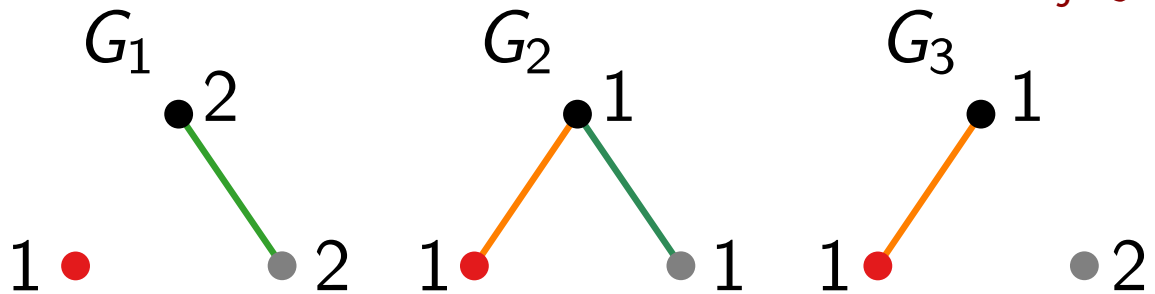
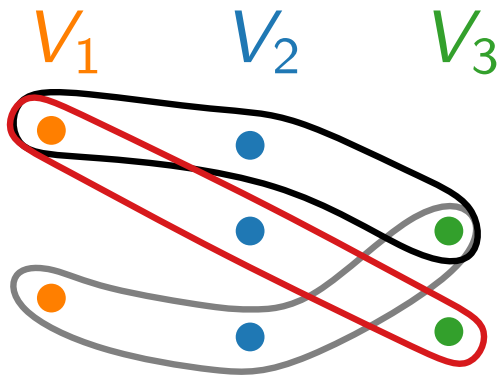
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

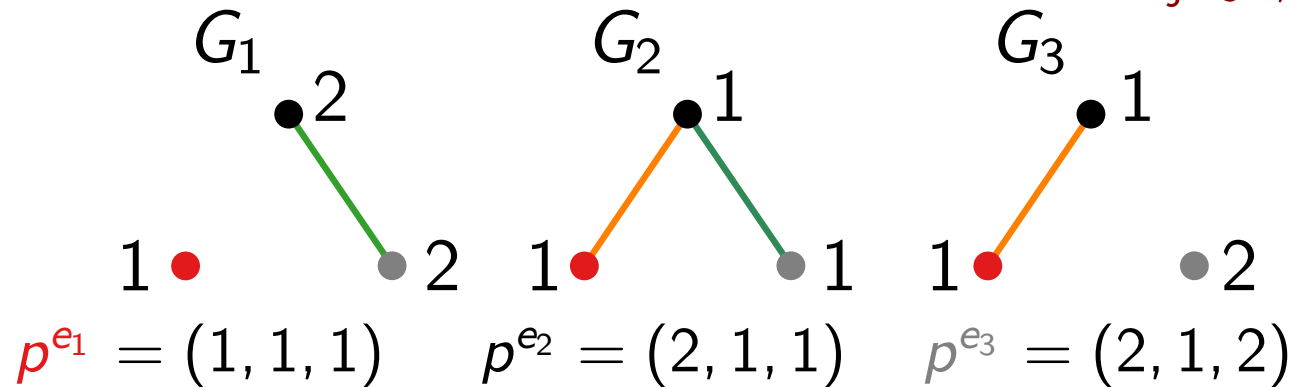
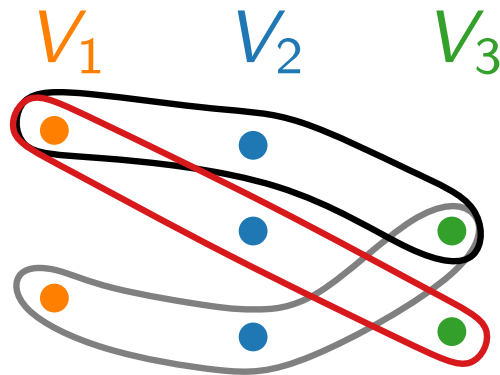
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

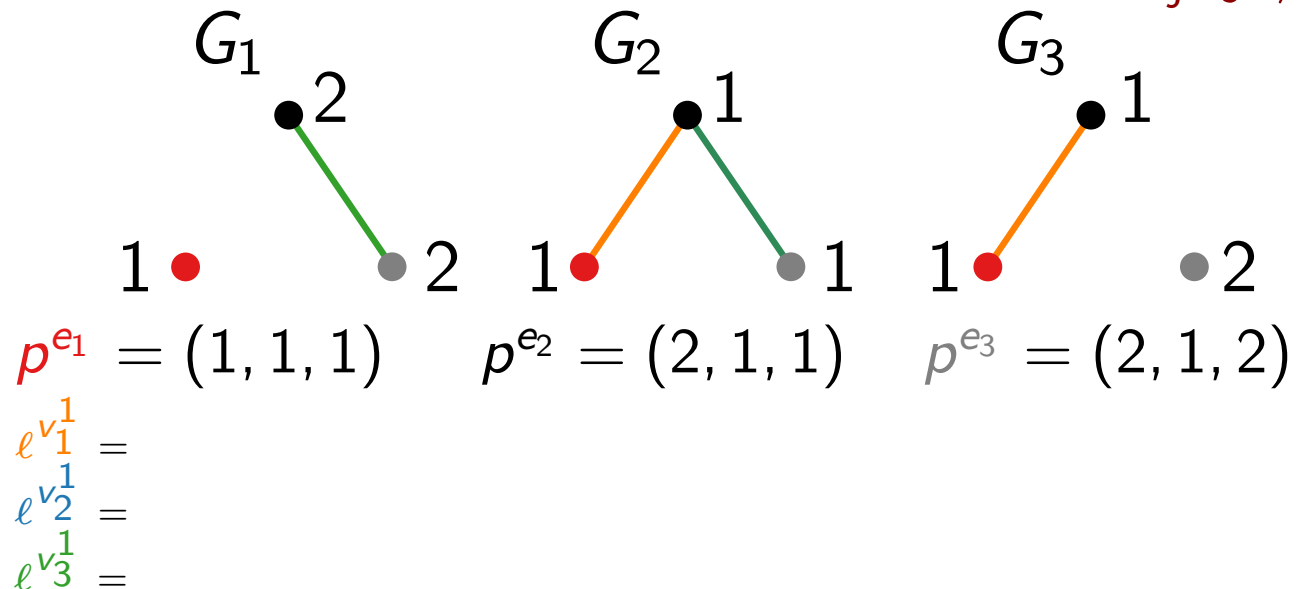
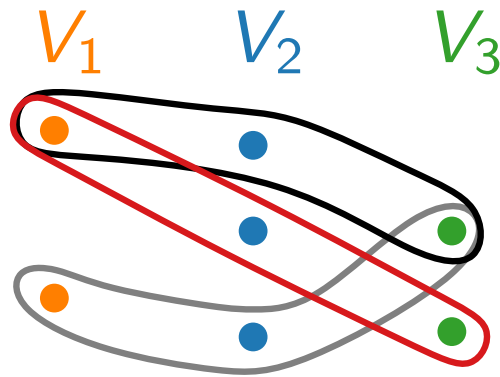
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

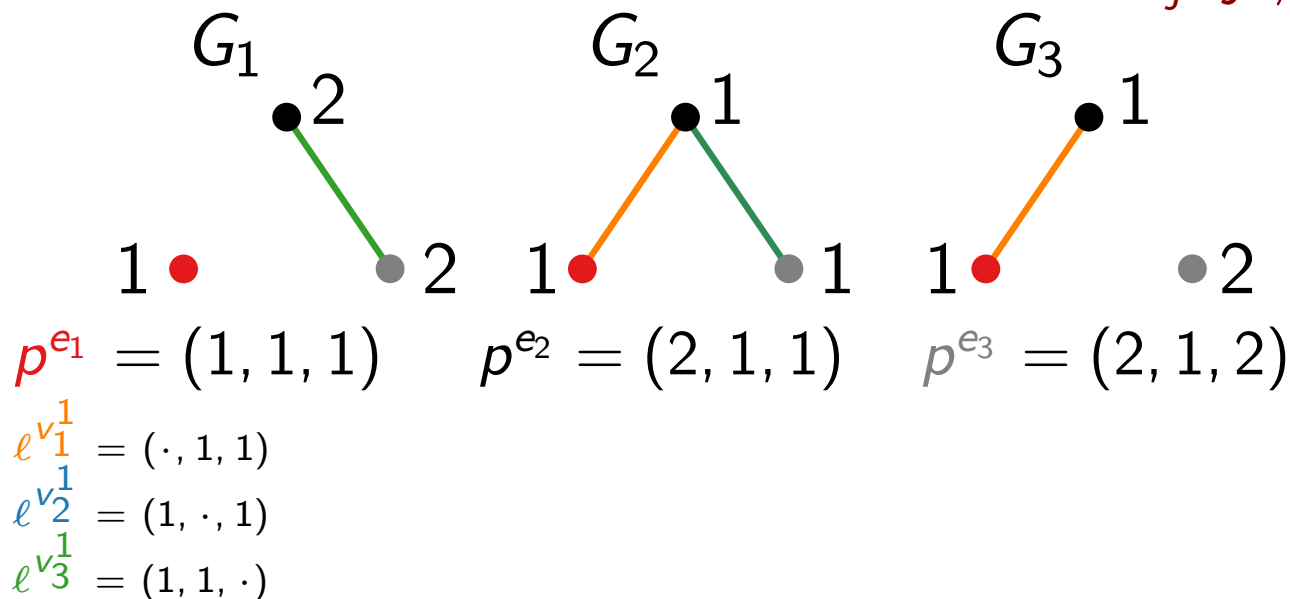
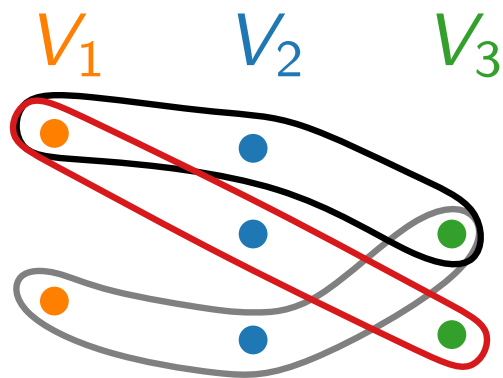
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

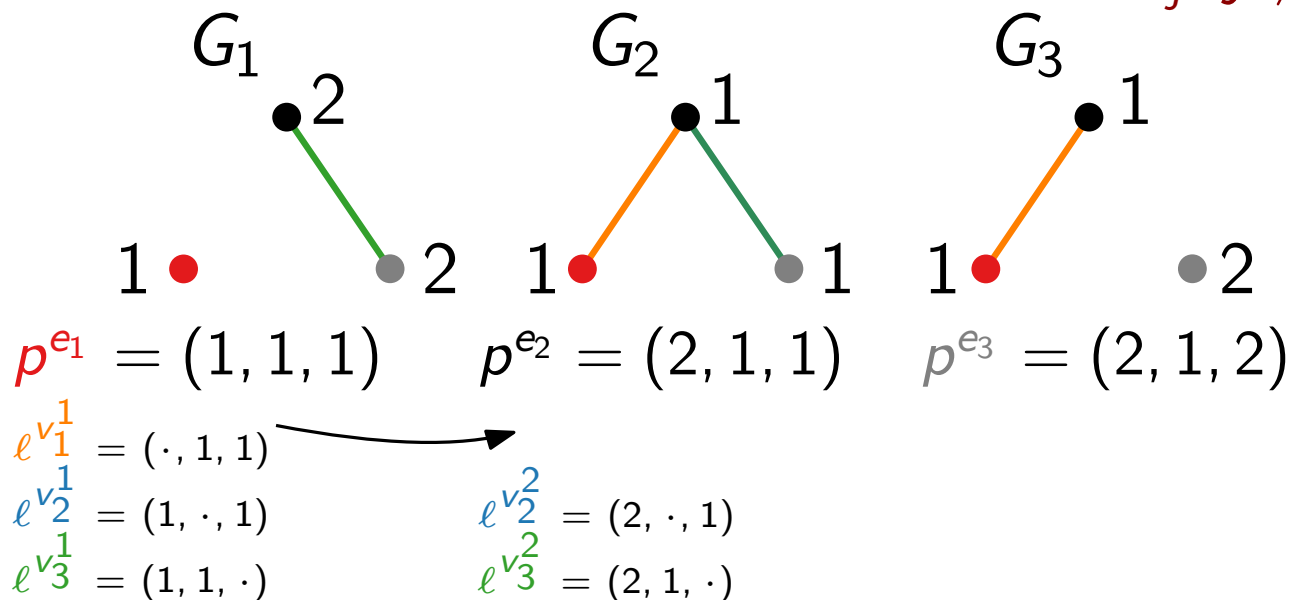
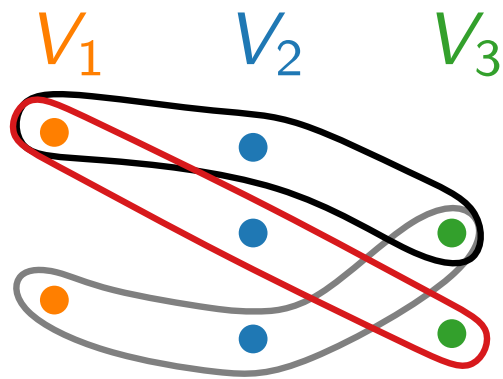
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

Theorem

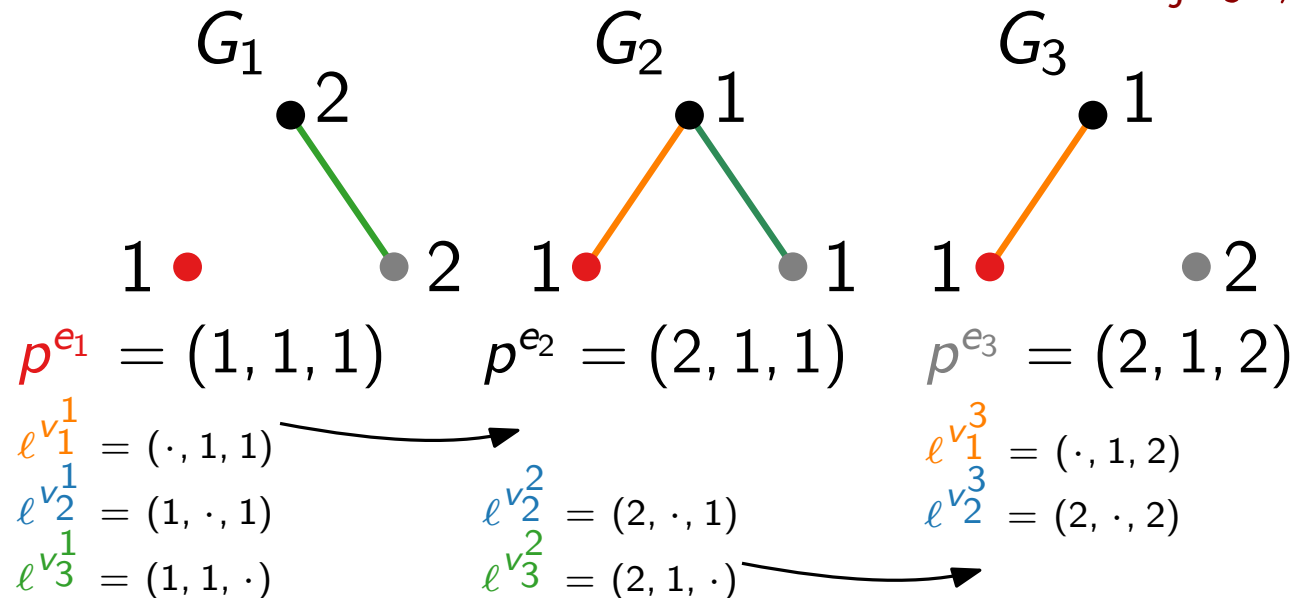
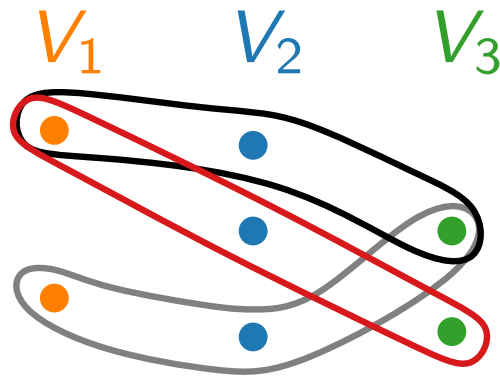
A k -hypergraph G is **representable** if and only if it is **vertex separable**.

hyperedge $e \rightarrow$ point p^e
vertex $v_i \rightarrow$ line ℓ^{v_i}

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i

e and e' from G_i are adjacent iff they have a common vertex in $V_j, j \neq i$



Main Result – Construction

vertex separable



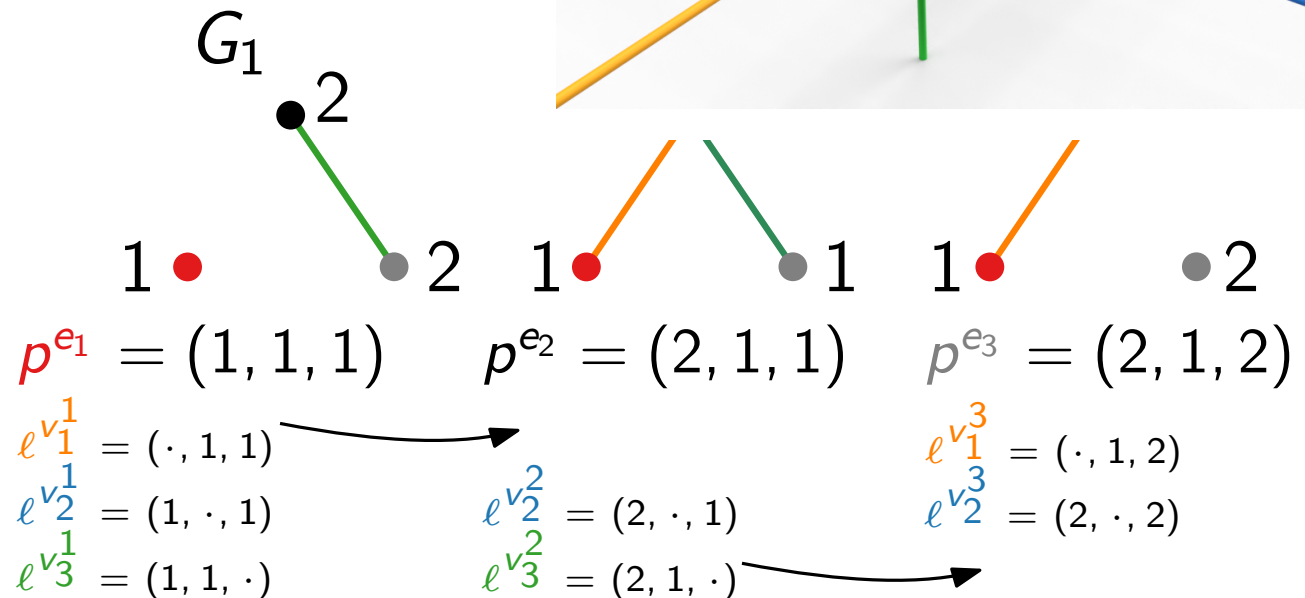
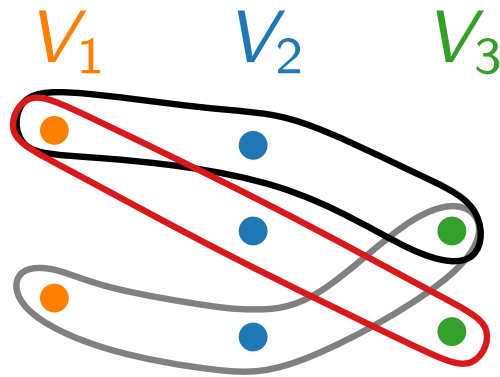
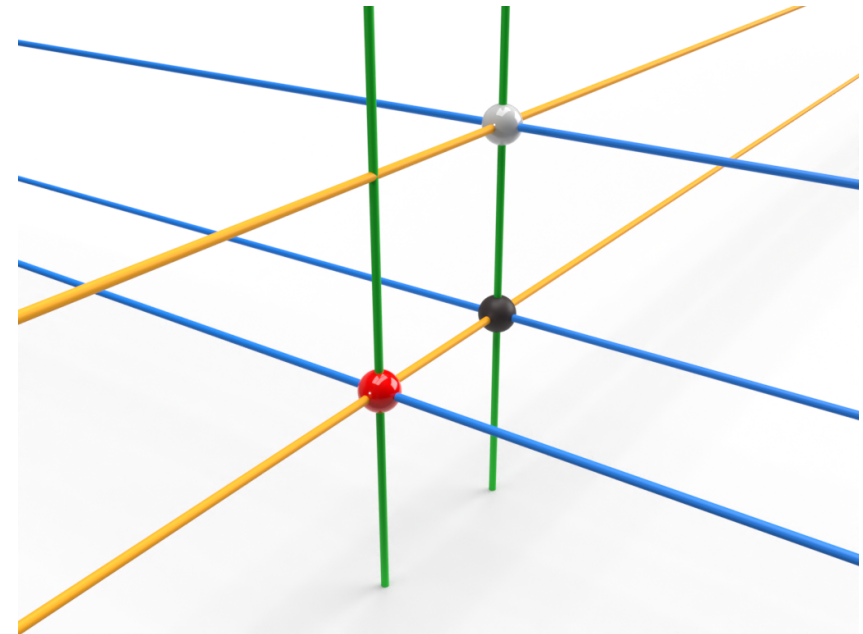
there is a representation

Theorem

A k -hypergraph G is **representable** if and only if it is **vertex separable**.

For each group V_i we use an auxiliary graph G_i that gives us the i -th coordinate for the points and the lines.

hyperedge in $G \rightarrow$ vertex in G_i



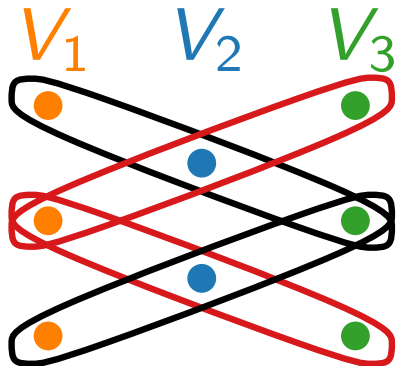
Proof – Part 2

vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.



Proof – Part 2

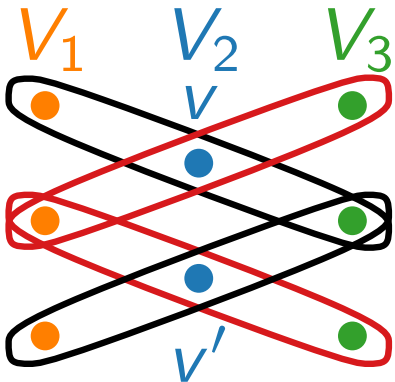
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;



Proof – Part 2

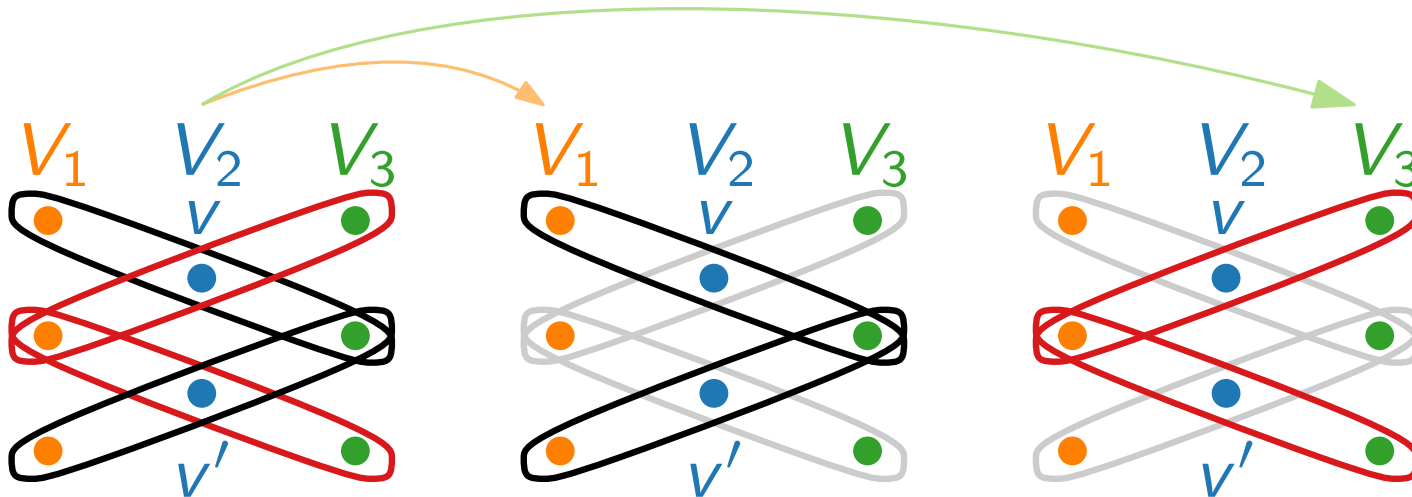
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;
- for each group V_j with $j \neq i$, there exists a v - v' path $v = v_1, \dots, v_r = v'$ such that $v_t \notin V_j$ for each $t \in [r]$;



Proof – Part 2

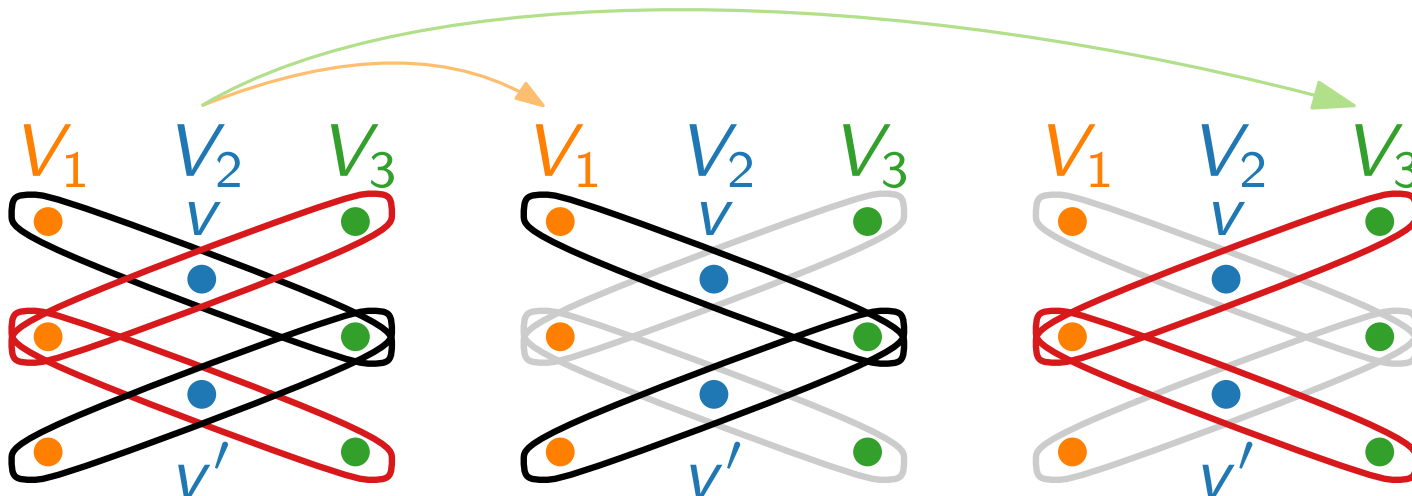
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;
- for each group V_j with $j \neq i$, there exists a v - v' path $v = v_1, \dots, v_r = v'$ such that $v_t \notin V_j$ for each $t \in [r]$;
- all lines ℓ^{v_t} with $t \in [r]$ that represent the vertices v_1, \dots, v_r lie on the same hyperplane H_j perpendicular to the x_j -axis;



Proof – Part 2

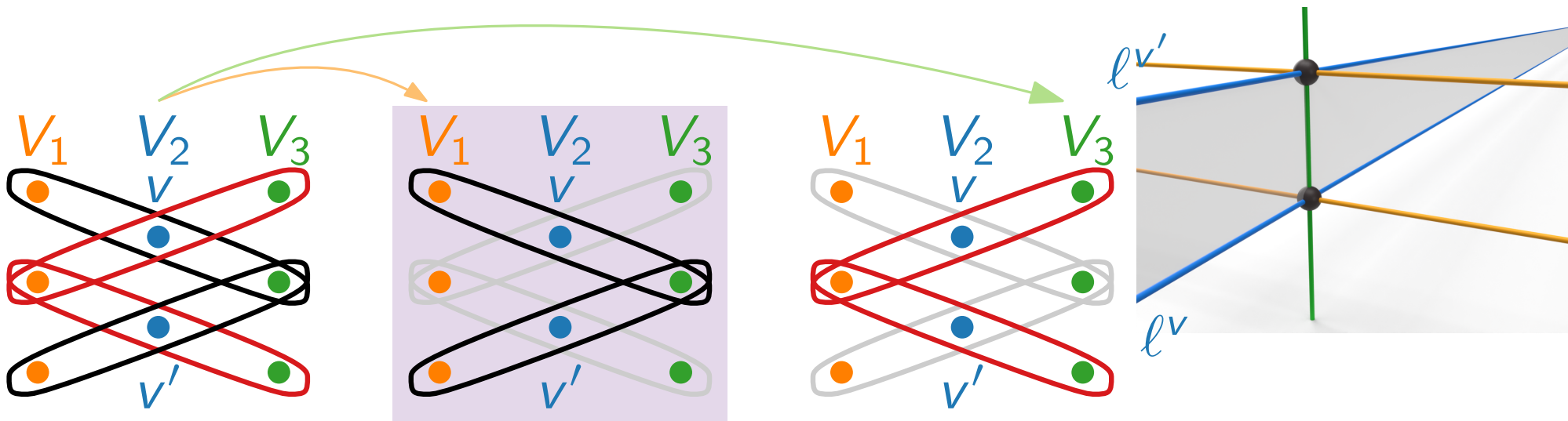
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;
- for each group V_j with $j \neq i$, there exists a v - v' path $v = v_1, \dots, v_r = v'$ such that $v_t \notin V_j$ for each $t \in [r]$;
- all lines ℓ^{v_t} with $t \in [r]$ that represent the vertices v_1, \dots, v_r lie on the same hyperplane H_j perpendicular to the x_j -axis;



Proof – Part 2

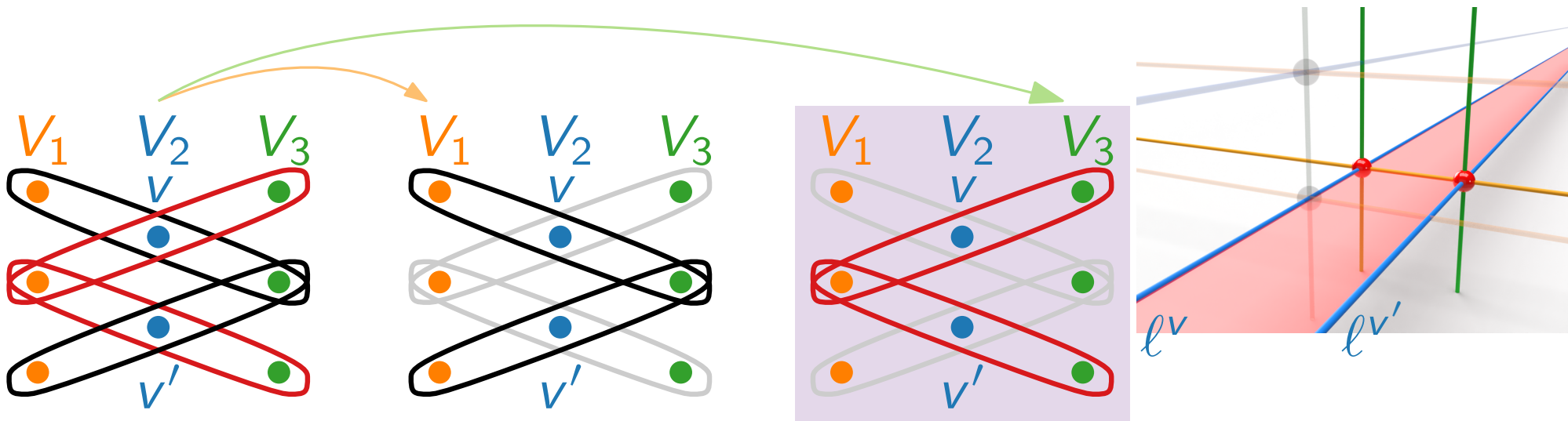
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;
- for each group V_j with $j \neq i$, there exists a v - v' path $v = v_1, \dots, v_r = v'$ such that $v_t \notin V_j$ for each $t \in [r]$;
- all lines ℓ^{v_t} with $t \in [r]$ that represent the vertices v_1, \dots, v_r lie on the same hyperplane H_j perpendicular to the x_j -axis;



Proof – Part 2

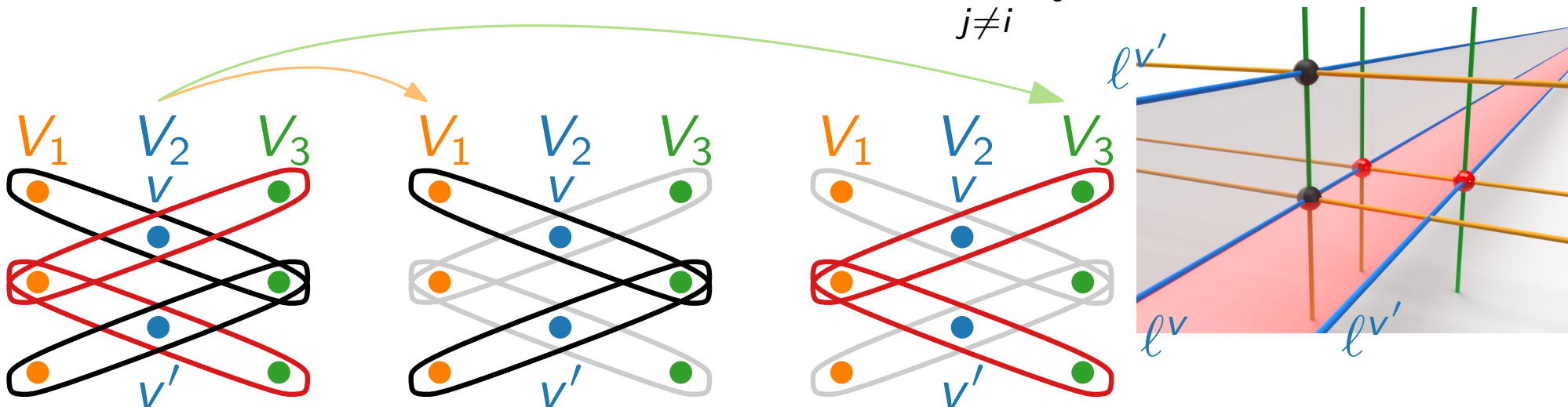
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;
- for each group V_j with $j \neq i$, there exists a v - v' path $v = v_1, \dots, v_r = v'$ such that $v_t \notin V_j$ for each $t \in [r]$;
- all lines ℓ^{v_t} with $t \in [r]$ that represent the vertices v_1, \dots, v_r lie on the same hyperplane H_j perpendicular to the x_j -axis;
- the lines ℓ^v and $\ell^{v'}$ lie in the intersection $\bigcap_{j \neq i} H_j$.



Proof – Part 2

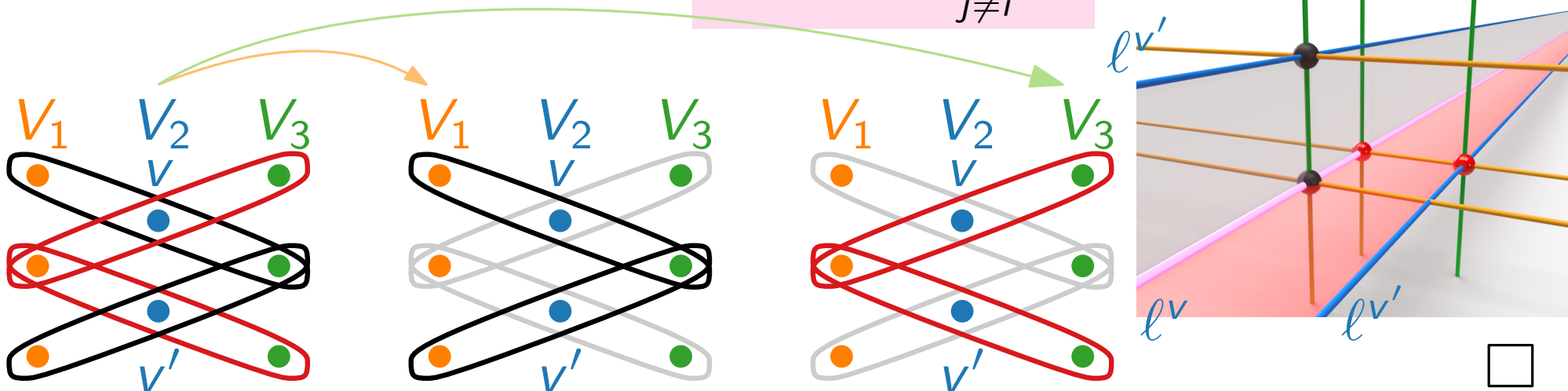
vertex separable



there is a representation

Assume that G is not vertex separable but it has a point line cover representation.

- it contains at least two distinct vertices v and v' from the same group V_i that are not separable;
- for each group V_j with $j \neq i$, there exists a v - v' path $v = v_1, \dots, v_r = v'$ such that $v_t \notin V_j$ for each $t \in [r]$;
- all lines ℓ^{v_t} with $t \in [r]$ that represent the vertices v_1, \dots, v_r lie on the same hyperplane H_j perpendicular to the x_j -axis;
- the lines ℓ^v and $\ell^{v'}$ lie in the intersection $\bigcap_{j \neq i} H_j$.



Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional
Covering objects	lines
Representable hypergraphs	vertex separable

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional
Covering objects	lines
Representable hypergraphs	vertex separable
	polynomial recognition algorithm

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional	
Covering objects	lines	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	all
polynomial recognition algorithm		

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional	
Covering objects	lines	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	all
<p>polynomial recognition algorithm</p> <p>similar to representation of bipartite graphs in 2D</p>		

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional		
Covering objects	lines	ℓ -dimensional subspaces $2 \leq \ell \leq (d - 2)$	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	generalized vertex separable	all
polynomial recognition algorithm			

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional		
Covering objects	lines	ℓ -dimensional subspaces $2 \leq \ell \leq (d - 2)$	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	generalized vertex separable	all
	polynomial recognition algorithm	polynomial for a fixed d	

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional		
Covering objects	lines	ℓ -dimensional subspaces $2 \leq \ell \leq (d - 2)$	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	generalized vertex separable	all
polynomial recognition algorithm		polynomial for a fixed d	What about non-constant d ?

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional		
Covering objects	lines	ℓ -dimensional subspaces $2 \leq \ell \leq (d - 2)$	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	generalized vertex separable	all
	polynomial recognition algorithm	polynomial for a fixed d	What about non-constant d ?

- Design *improved* algorithms for vertex separable hypergraphs (e.g vertex cover, matching) parameterized by ℓ and d

Further Results & Open Questions

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional		
Covering objects	lines	ℓ -dimensional subspaces $2 \leq \ell \leq (d - 2)$	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	generalized vertex separable	all
	polynomial recognition algorithm	polynomial for a fixed d	What about non-constant d ?

- Design *improved* algorithms for vertex separable hypergraphs (e.g vertex cover, matching) parameterized by ℓ and d
- Relation to other graph classes

Further Results & Open Questions

Thank you!

- Generalization to ℓ -dimensional subspace, $\ell < d$

Space	d -dimensional		
Covering objects	lines	ℓ -dimensional subspaces $2 \leq \ell \leq (d - 2)$	$(d - 1)$ -dimensional subspaces
Representable hypergraphs	vertex separable	generalized vertex separable	all
	polynomial recognition algorithm	polynomial for a fixed d	What about non-constant d ?

- Design *improved* algorithms for vertex separable hypergraphs (e.g vertex cover, matching) parameterized by ℓ and d
- Relation to other graph classes